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I am submitting herewith a dissertation written by Douglas S. Daniel entitled "Uniqueness and Summability of Two-Dimensional Walsh Series and Their Generalizations." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

William R. Wade, Major Professor

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UNIQUENESS AND SUMMABILITY OF TWO-DIMENSIONAL WALSH SERIES AND THEIR GENERALIZATIONS

A Dissertation Presented

for the

Doctor of Philosophy Degree

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Douglas S. Daniel

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DEDICATION

This paper is dedicated to the greatest influences in my life:

my parents, D.C. (Jim), Jr. and Susan Daniel,

whose guidance and continued encouragement made this moment possible.

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ABSTRACT

This paper explores a variety of questions associated with two-dimensional Vilenkin series and their special case two-dimensional Walsh series. Some of the results are an extension of known results of the one-dimensional case to those of two-dimensions. These theorems concern the uniqueness of two-dimensional Walsh series, and a Tauberian theorem, which shows the relationship between the summability and convergence of square partial sums of two-dimensional Walsh series. The final results generalize known theorems concerning the summability of two-dimensional Vilenkin-Fourier series of unbounded type. In all cases, the sequence $p = (p_k)$ is the generating sequence for the Vilenkin group G_p ($p_k = 2$ in the Walsh case), and the sequence (P_k) is given by $P_0 = 1$ and $P_k = p_0 p_1 \cdots p_k - 1$.

Fundamental real analysis and measure theory are vital to the techniques used. First, a two-dimensional quasi-measure is defined and then each two-dimensional Vilenkin series is shown to have an associated two-dimensional quasi-measure. Using this last fact and newly defined two-dimensional derivates, a two dimensional variant is found to the classical result that differentiable functions with negative derivatives are decreasing. This leads to a uniqueness result which says that if a two-dimesional Walsh series, S, satisfy both a two-dimensional CS-condition for all $x \in \mathsf{G}^2$ and the condition

$$\lim_{n\to\infty} S_{2^n,2^n}(x) = 0$$

for all but countably many $x \in \mathsf{G}^2$, then S is the zero series.

Next the dyadic square partial sums are found to be very good and finite or very bad and infinite, which leads to a Tauberian style theorem. It says that on

a measurable subset of $[0,1)^2$ a two-dimensional Walsh series, S, with bounds on certain Cesàro means for each x in the subset, then there is a function f such that

$$\lim_{n\to\infty} S_{2^n,2^n}(x) = f(x)$$

for almost every x in the measurable set.

The final set of results deals with growth estimates for two-dimensional Vilenkin-Fourier series of unbounded type. Under the condition on the sequence (p_k) known as δ-strongly quasibounded, a certain maximal summability operator is bounded.

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Chapter 1

Background

This dissertation can be divided into three parts, each of which addresses a fundamental problem concerning two-dimensional Vilenkin series or their special case, two-dimensional Walsh series. The first part (Chapters 2-3) will culminate in a result concerning uniqueness of two-dimensional Walsh series, that is, given a two-dimensional Walsh series S, under what conditions is S guaranteed to be Sf, the two-dimensional Walsh-Fourier series of some f in $L^1([0,1)^2)$. The second part (Chapter 4) contains a deep beautiful Tauberian theorem, that is, a result which shows that under mild conditions, summability and convergence of dyadic square partial sums of two-dimensional Walsh series are equivalent. The third part (Chapters 5-7) will deal with the problem of estimating the growth of Cesàro means of two-dimensional Vilenkin-Fourier series of unbounded type.

The purpose of this first chapter is to introduce enough notation and jargon to understand and make sense out of the ideas mentioned in the previous paragraph. We shall denote the set of *non-negative integers* by $\mathbb{N}$, the set of *positive integers*

by $\mathbb{P}$, and the set of *dyadic rationals* in the unit interval $[0,1)$ by $\mathbb{Q}$. In particular, each element in $\mathbb{Q}$ has the form $p/2^n$ for some $n, p \in \mathbb{N}$ and $0 \leq p < 2^n$.

A *compact abelian group* is an abelian group $(\mathsf{G}, +)$ which is also a compact Hausdorff space in which the map $(x, y) \to x + y$ of $\mathsf{G} \times \mathsf{G} \to \mathsf{G}$ and the map $x \to -x$ of $\mathsf{G} \to \mathsf{G}$ are both continuous. A *Vilenkin group* G is an infinite, compact, 0-dimensional, metrizable, abelian group. In this dissertation we shall study the harmonic analysis of Vilenkin groups and their special case the Walsh Group. The following is a brief description of multiplicative Vilenkin groups, which lead to Vilenkin systems.

Let $p = \{p_k |\ k \in \mathbb{N}\}$ be a (infinite) sequence of natural numbers such that $p_k \geq 2$ for each k. Construct a compact abelian group G_p associated with each sequence p, as follows. First let Z_{p_k} be the *cyclic group of order* $\{p_k\}$, that is, Z_{p_k} is the set $\{0, 1, \ldots, p_k - 1\}$, where the group operation is mod p_k addition and every subset is open. (Note that the Haar measure on Z_{p_k} assigns to each singleton the value $1/p_k$.) The group G_p is defined to be the direct product of the compact groups Z_{p_k} over k. On G_p use coordinate-wise addition as the group operation and the product topology. By the Tychonoff product theorem, G_p is a compact group. Therefore, G_p is compact abelian group. Multiplicative Vilenkin groups have two important special cases. The first is the *q-adic case* for which there is a fixed prime q such that $p_k = q$ for each k. The other case is the most basic case called the *dyadic case* where $p_k = 2$ for each k. This last case leads to the Walsh system. Call the associated compact abelian group of the Walsh system, G.

A Vilenkin group G is said to be *bounded* if

$$\limsup_{n\to\infty} p_k < \infty.$$

Otherwise, it is of *unbounded type*. The unbounded case has many idiosyncracies, and can have different results from those of the dyadic, p-adic, and bounded cases. In fact, some theorems are false for the unbounded case that are true for the other three (see Wade [13]).

From the above description, it is evident that the multiplicative Vilenkin group G_p generated from the sequence p consists of the sequences $x = (x_0, x_1, \dots)$, where each x_i is an integer such that $0 \leq x_i < p_k$ for each $i \in \mathsf{N}$. In particular, the dyadic group is made up of sequences in which each element is either 0 or 1. Addition of elements $x, y \in \mathsf{G}_p$ is obtained by adding the $k-$th elements of x and y modulo p_k and is denoted by $\dot{+}$. (Addition by the additive inverse, that is, subtraction, is denoted by $\dot{-}$.) The zero element of G_p is $0 = (0, 0, \dots)$. Hence the addition of x and y is the sequence

$$x \dot{+} y := \{x_k \oplus y_k \mid x, y \in \mathsf{G}_p, k \in \mathbb{N}\},$$

where $x_k \oplus y_k = x_k + y_k \pmod{p_k}$. *Intervals* for the group are defined as

$$I_n(x) := \{y \in \mathsf{G}_p \mid x_k = y_k \quad \text{for all} \quad k = 0, 1, \dots, n-1\}.$$

These intervals form a base for the product topology and are both open and closed in G_p. When no confusion will arise, we shall suppress the subscript p and denote a general Vilenkin group by G.

The group G_p is metrizable. Let $P_0 = 1$ and $P_n = p_0 p_1 \dots p_{n-1}$, for $n \in \mathbb{P}$.

Define the distance between the elements $x, y \in \mathsf{G}_p$ by

$$\rho(x,y) := \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{P_{k+1}}.$$

The topology induced by this metric coincides with the topology on G_p and $I_n(x)$ is a basis for the topology. Furthermore, it can be shown that the product measure for G_p is the normalized Haar measure μ for G_p. (See Vilenkin [11])

A *character* on G_p is a continuous, complex-valued map ψ such that

$$|\psi(x)| = 1 \text{ and } \psi(x \dot{+} y) = \psi(x)\psi(y)$$

for each $x, y \in \mathsf{G}_p$. The set of all continuous characters of G_p forms the *dual group* $\widehat{\mathsf{G}_p}$, where the addition of characters is defined by

$$\psi_1(x) + \psi_2(x) = \psi_1(x)\psi_2(x)$$

for $x \in \mathsf{G}_p$ and $\psi_1(x), \psi_2(x) \in \widehat{\mathsf{G}_p}$. An important fact is that since G_p is compact and metrizable, then $\widehat{\mathsf{G}_p}$ is countable (see Rudin [6] p.38).

Vilenkin discovered that the characters of G_p can be ordered in a manner similar to that of Paley's earlier ordering of the Walsh functions (see Wade [12]) using functions on G_p analogous to the Rademacher functions (Schipp, Wade, Simon [8]). Vilenkin first used the well known fact that each non-negative n has a unique representation of the form

$$n = \sum_{k=0}^{\infty} n_k P_k$$

where each $n_k \in \mathbb{N}$ such that $0 \leq n_k < p_k$ and P_k is as above. Using this representation one can define the characters of G_p by

$$\psi_n(x) := \prod_{k=0}^{\infty} exp\left(\frac{2\pi i x_k n_k}{p_k}\right)$$

where x_k is the $k-$th coordinate of $x \in \mathsf{G}_p$. It is not difficult to see that this system is orthonormal in the sense that, for any $\psi_i, \psi_k \in \widehat{\mathsf{G}_p}$, we have

$$\int_{\mathsf{G}_p} \psi_i(x)\psi_k(x)\, d\mu(x) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{otherwise} \end{cases}$$

This system is called the *Vilenkin system* for p, and when $p_k = 2$ for each k, this system is called the *Walsh system.*

Now, it is sometimes convenient to work with a more concrete situation. A Vilenkin group G_p can be identified with the interval $[0, 1)$. Given $x \in \mathsf{G}_p$ with coordinates x_k define a map $\phi(x) : \mathsf{G}_p \to [0, 1)$ by

$$\phi(x) = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}.$$

This map is 1-1, onto, and continuous, except at a countable subset, which corresponds to the $p-$adic rationals in [0,1]. This countable subset will behave differently under ϕ because each p-adic rational has more than two expansions: one expansion ends in zeros, and another ends with each coordinate being $p_n - 1$ for all $n \geq M$ for some $M \in \mathbb{N}$. If we restrict the map to G_p minus the expansions of the $p-$adic rationals that don't end in zeros, then we get a map that is 1-1, onto, and continuous as desired. The inverse of ϕ exists and is onto the restriction defined above. Therefore, ϕ allows one to go back and forth between G_p and $[0, 1)$, using which ever formulation is most convenient for the problem at hand.

The map ϕ can also be used to pull concepts from $[0, 1)$ back to G_p and vice versa. For example, in the Walsh case, a function f is said to be *W-continuous* on $[0, 1)$ if $f \circ \phi$ is continuous on G_p. Notice once and for all that every continuous function is W-continuous, but not conversely.

Let $C(\mathsf{G}_p)$ represent the collection of functions $f : \mathsf{G}_p \to \mathbb{R}$ which are continuous from the p-adic topology on G_p to the usual topology on $\mathbb{R}$. Let C_W represent the collection of functions $g : [0,1) \to \mathbb{R}$ which are continuous at every dyadic irrational, continuous from the on $[0,1)$, and have a finite limit from the left on $(0,1]$ in the usual topology. Call the map $f \to f \circ \phi$ the canonical isomorphism.

This map ϕ is a generalization of the associated map that Fine [1] developed for the dyadic case. Note that for a system defined by a sequence p, ϕ takes $I_n(0)$ to the interval $[0, 1/P_n)$, and $I_n(x)$ to the interval of length $1/P_n$ that contains $\phi(x)$, that is, the interval

$$I(q,n) := \left[\frac{q}{P_n}, \frac{q+1}{P_n}\right)$$

where $0 \leq q < P_n$ and $q \leq \phi(x)P_n < q+1$. Sometimes the notation may be less ambiguous if $I(q, P_n)$ is used for $I(q,n)$. In an abuse of this notation if $x \in [0,1)$ then call $I(q,n)$, $I_n(x)$ or $I_{P_n}(x)$.

Given $f \in L^1(\mathsf{G})$, then *Vilenkin-Fourier coefficients* of f will be denoted by

$$\widehat{f}(k) := \int_{\mathsf{G}} f(x)\overline{\psi_k(x)}d\mu(x) \qquad \text{for } k \in \mathbb{N}.$$

Denote the *Vilenkin-Fourier series* of f evaluated at $x \in \mathsf{G}$ by

$$(Sf)(x) := \sum_{k=0}^{\infty} \widehat{f}(k)\psi_k(x)$$

whose *partial sums* are denoted by

$$(S_n f)(x) := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k(x) \qquad \text{for } n \in \mathbb{N}.$$

Denote the *Cesàro means*, of order n, of the Vilenkin-Fourier series of f evaluated at $x \in \mathsf{G}$ by

$$\sigma_n f(x) := \frac{1}{n}\sum_{k=1}^{n} S_n(x) = \sum_{k=0}^{n-1}\left(1 - \frac{k}{n}\right)\widehat{f}(k)\psi_k(x).$$

Define the *convolution* of two functions f and g on the group G by

$$f * g := \int_{\mathsf{G}} f(x-y)g(y)d\mu(y).$$

The *Vilenkin-Dirichlet kernel* is defined by

$$D_n(x) := \sum_{k=0}^{n-1} \psi_k(x),$$

and the *Vilenkin-Fejér kernel* is defined by

$$K_n(x) := \sum_{k=1}^{n} D_k(x) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \psi_k(x),$$

where $n \in \mathbb{P}$. It is easy to see that

$$(S_n f)(x) = f * D_n(x),$$

and

$$\sigma_n f(x) = f * K_n(x)$$

for $n \in \mathbb{P}$.

In this work, *two dimensional Vilenkin series* are of particular interest. We note without proof that the previous definitions and obsevations can be extended to double Vilenkin series by making obvious modifications. We shall use these below as needed. Here is some notation which will fascilitate those modifications.

Let $p = \{p_k\}_{k=0}^{\infty}$ and $q = \{q_k\}_{l=0}^{\infty}$ be two sequences of natural numbers with $p_n \geq 2$ and $q_n \geq 2$ for each n. Then set $P_n := p_0 p_1 \dots p_{n-1}$ and $Q_n := q_0 q_1 \dots q_{n-1}$ where the empty product is defined to be 1. The *two dimensional Vilenkin system* associated with these sequences is the system $(\psi_{n,m} | n, m \in \mathbb{N})$ defined on $[0,1)^2$ as $\psi_{n,m}(x_1, x_2) = \psi_n(x_1)\theta_m(x_2)$, where $\psi_k(x)$ is really $\psi_k \circ \phi^{-1}(x)$ and where $\theta_j(x)$ is

really $\theta_j \circ \phi^{-1}(x)$, again through a bit of an abuse of notation. When $p_k = q_k = 2$ for every k, then the system is called the two dimensional Walsh system, and use ω in the place of ψ.

Define the *two dimensional Vilenkin-Fourier coefficients* of $f \in L^1([0,1))$ to be

$$\widehat{f}(k,l) = \int \int_{[0,1)^2} f(x_1,x_2)\psi_{k,l}(x_1,x_2)d(x_1,x_2).$$

Denote the *two dimensionalVilenkin-Fourier series* of f by

$$Sf := \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \widehat{f}(k,l)\psi_{k,l}$$

whose *rectangular partial sums* are denoted by

$$S_{n,m}f := \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \widehat{f}(k,l)\psi_{k,l}(x) \qquad \text{for } n,m \in \mathbb{N}.$$

Denote the *rectangular Cesàro means* of the Vilenkin-Fourier series of f by

$$\sigma_{n,m}f := \frac{1}{nm}\sum_{k=1}^{n}\sum_{l=0}^{m} S_{n,m} = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\left(1-\frac{k}{n}\right)\left(1-\frac{k}{m}\right)\widehat{f}(k,l)\psi_{k,l}.$$

Any further notation and concrete results concerning these concepts will be introduced as needed.

Chapter 2

Two Dimensional Quasimeasures

Uniqueness of Walsh series is intimately connected with the theory of quasimeasures (see Wade-Yoneda [15]). Thus we begin our study of uniqueness of two-dimensional Walsh series with a chapter on two-dimensional quasimeasures.

An *algebra* of sets on X is a nonempty subcollection of $\mathcal{P}(\mathrm{X})$ which is closed under finite unions and complements. Let $\mathcal{A}$ denote the algebra of sets generated by the dyadic squares in $[0,1)^2$, that is,

$$\mathcal{A} = \{A = \bigcup_{i=0}^{n} Q_i : n \in \mathbb{N} \text{ and } Q_i \text{ are dyadic squares in } [0,1)^2\}.$$

Define a *two dimensional quasimeasure* to be a real valued set function which is finitely additive on $\mathcal{A}$. Clearly, every finite Borel measure on $[0,1)^2$ is a two dimensional quasimeasure, but not every two dimensional quasimeasure on $[0,1)^2$ is a Borel measure. Denote the collection of two dimensional quasimeasures by QM2.

Definition 2.1. Let $\nu \in$ QM2 and define integration of a simple function with

respect to ν as follows:

$$\int_{\mathrm{E}} C\, d\nu = C \int_{[0,1)^2} \chi_{\mathrm{E}}\, d\nu := C\nu(\mathrm{E}) \text{ for } \mathrm{E} \in A \text{ and } C \text{ constant.}$$

Definition 2.2. Define the *Walsh-Fourier-Stieljes coefficients* of ν by

$$\widehat{\nu}(i,j) = \int_{[0,1)^2} \omega_{i,j}(x_1, x_2)\, d\nu$$

for $(i,j) \in \mathbb{N}^2$, where $\omega_{i,j}(x_1,x_2) = \omega_i(x_1)\omega_j(x_2)$ are the two dimensional Walsh functions. Notice, since $\omega_{i,j}$'s are constant on dyadic squares that this integral is defined.

The Walsh-Fourier-Stieljes series $S\nu$ of $\nu \in \mathsf{QM2}$ will be denoted by

$$S\nu(x) := \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \widehat{\nu}(k,l)\omega_{k,l}(x)$$

for $x = (x_1, x_2) \in [0,1)^2$. For each pair $(n,m) \in \mathbb{N}^2$, define rectangular partial sums of $S\nu$ by

$$(S_{n,m}\nu)(x) := \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \widehat{\nu}(k,l)\omega_{k,l}(x).$$

We now prove that the function $\nu \longrightarrow \widehat{\nu}$ is a 1-1 function from QM2 onto ℓ^0. First, observe that for each $x \in [0,1)^2$

$$(S_{n,m}\nu)(x) = \int_{[0,1)^2} D_{n,m}(x \dot{+} t) d\nu(t),$$

where $D_{n,m}(x) := \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \omega_{n,m}(x)$ and $D_{n,m}(x_1,x_2) = D_n(x_1)D_m(x_2)$. Thus by the Paley Lemma

$$S_{2^n,2^m}\nu(x) = 2^n 2^m \nu(I_n(x_1) \times I_m(x_2))$$

for $x = (x_1,x_2) \in [0,1)^2$ and $(n,m) \in \mathbb{N}^2$. In particular, $S_{2^n,2^n}\nu(x) = 2^{2n}\nu(\mathcal{I}_n(x))$, where $\mathcal{I}_n(x) = I_n(x_1) \times I_n(x_2)$. It follows that the map $\widehat{\ }$ is 1-1 and linear on QM2.

To show that this map is onto, let $\{a_{k,l} \mid (k,l) \in \mathbb{N}^2\} \in \ell^0$ and consider the two dimensional Walsh polynomials

$$S_{2^n,2^n}(x) = \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} a_{k,l}\omega_{k,l}(x)$$

for each $n \in \mathbb{N}$ and $x \in [0,1)^2$. Define ν on A by

$$\nu(\mathcal{I}) := \lim_{n\to\infty} \int_{\mathcal{I}} S_{2^n,2^n}(x)dx$$

where $\mathcal{I}$ is a dyadic square. This limit exists because $\int_{\mathcal{I}} \omega_{i,j}(x)dx = 0$ whenever $i \geq 2^{n_0}$ or $j \geq 2^{n_0}$ and $|\mathcal{I}| = 2^{-2n_0}$ for some $n_0 \in \mathbb{N}$. Hence,

$$\nu(\mathcal{I}) = \int_{\mathcal{I}} S_{2^{n_0},2^{n_0}}(x)\ dx$$

for $|\mathcal{I}| = 2^{-2n_0}$.

Now given $\mathcal{J} \in A$, there exists $\{\mathcal{I}_i\}_{i=1}^d$ a set of mutually disjoint dyadic squares such that $\mathcal{J} = \bigcup_{i=1}^d \mathcal{I}_i$, where $|\mathcal{I}_i| = 2^{-2n_i}$ for some $n_i \in \mathbb{N}$. Let $n_* \geq \max_i\{n_i\}$. By this fact and the definition of integration with respect to ν,

$$\nu(\mathcal{J}) = \int_{\mathcal{J}} S_{2^{n_*},2^{n_*}}(x)dx.$$

Hence, it follows that ν is a two dimensional quasimeasure.

Now fix $(i,j) \in \mathbb{N}^2$. By definition, $\widehat{\nu}(i,j) = \int_{[0,1)^2} \omega_{i,j}(x)d\nu$, but $\omega_{i,j}(x) = \sum_{n=1}^d b_n\chi_{\mathcal{I}_n}(x)$ for $\{\mathcal{I}_n\}_{n=1}^d$ a set of dyadic squares and $b_n \in \{1,-1\}$. Choose

$n_* \geq \max_l\{n_l, i, j\}$ where $|\mathcal{I}_l| = 2^{-2n_l}$. Hence,

$$\begin{aligned}
\widehat{\nu}(i,j) &= \int_{[0,1)^2} \sum_{n=1}^{d} b_n \chi_{\mathcal{I}_n}(x)\, d\nu \\
&= \sum_{n=1}^{d} b_n \int_{[0,1)^2} \chi_{\mathcal{I}_n}(x)\, d\nu = \sum_{n=1}^{d} b_n \int_{\mathcal{I}_n} d\nu \\
&= \sum_{n=1}^{d} b_n \nu(\mathcal{I}_n) \\
&= \sum_{n=1}^{d} b_n \int_{\mathcal{I}_n} S_{2^{n_*},2^{n_*}}(x)\, dx \text{ (since } n_* \geq \max_l\{n_l\}) \\
&= \int_{[0,1)^2} \sum_{n=1}^{d} b_n \chi_{\mathcal{I}_n}(x) S_{2^{n_*},2^{n_*}}(x)\, dx \\
&= \int_{[0,1)^2} \omega_{i,j}(x) S_{2^{n_*},2^{n_*}}(x)\, dx \\
&= a_{i,j} \text{ by the orthogonality of the } \omega_{i,j}\text{'s.}
\end{aligned}$$

Summarizing these results, given any two dimensional Walsh series $S(x) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{k,l}\omega_{k,l}(x)$, there is a unique $\nu \in \mathsf{QM2}$ such that $\widehat{\nu}(k,l) = a_{k,l}$ and $S_{2^n,2^n}(x) = 2^{2n}\nu(\mathcal{I}_n(x))$ for each $n \in \mathbb{N}$ and $x \in [0,1)^2$. We shall call ν the two dimensional quasimeasure associated with the two dimensional Walsh series $S(x)$.

Chapter 3

QM2 and Derivates

The connection between uniqueness of Walsh series and quasimeasures comes about through the following concepts.

Definition 3.1. The *upper binary derivate* of $\nu \in \mathsf{QM2}$ at a point $x \in [0,1)^2$ is defined by

$$(\overline{D}\nu)(x) := \limsup_{n\to\infty} \frac{\nu(\mathcal{I}_n(x))}{|\,\mathcal{I}_n(x)\,|}.$$

Definition 3.2. The *lower binary derivate* of $\nu \in \mathsf{QM2}$ at a point $x \in [0,1)^2$ is defined by

$$(\underline{D}\nu)(x) := \liminf_{n\to\infty} \frac{\nu(\mathcal{I}_n(x))}{|\,\mathcal{I}_n(x)\,|}.$$

A two dimensional quasimeasure ν is said to have a *binary derivate* if

$$(\overline{D}\nu)(x) := (\underline{D}\nu)(x),$$

in which case this common value will be denoted by $(D\nu)(x)$.

We begin this chapter with the following technical lemma.

Lemma 3.3. *If for some $\varepsilon > 0$, a measurable set $E_0 \subseteq [0,1)^2$, a real number r, and a dyadic square Q_0, the following three conditions hold*

$$| E_0 \cap \mathrm{Q}_0 | > (1-\varepsilon) | \mathrm{Q}_0 | \tag{3.1}$$

$$\nu(\mathrm{R}) > 0 \text{ for all dyadic squares satisfying } \mathrm{R} \subset \mathrm{Q}_0, \mathrm{R} \cap E_0 \neq \emptyset, \tag{3.2}$$

and

$$\overline{D}\nu(x) > r \text{ where } x \in E_0, \tag{3.3}$$

then

$$\nu(\mathrm{Q}_0) > r(1-2\varepsilon) | \mathrm{Q}_0 | . \tag{3.4}$$

Proof. Let $\varepsilon > 0$ and $x \in E_0$, and suppose without loss of generality that E_0 lies in the interior of Q_0. By (3.3) there is a dyadic square $\mathrm{Q}_x \subset \mathrm{Q}$ containing x such that $\nu(\mathrm{Q}_x) > r | \mathrm{Q}_x |$. (Note that $| \mathrm{Q}_x |$ can be made arbitrarily small by (3.3)).

Now recall Vitali's Covering Lemma (see Folland [2]): If $E \subset \mathbb{R}^2$ and $\mathcal{Q}$ is a family of cubes such that each $x \in E$ is contained in members of $\mathcal{Q}$ of arbitrarily small diameter, then there exists a (finite or infinite) disjoint sequence $\{\mathrm{Q}_j\} \subset \mathcal{Q}$ such that $| (E \setminus \bigcup_{j=1}^{\infty} \mathrm{Q}_j) | = 0$.

Hence by (3.1) and Vitali's Covering Lemma there exists a non-overlapping sequence of dyadic squares $\mathrm{Q}_1, ..., \mathrm{Q}_n$ such that

$$\nu(\mathrm{Q}_i) > r | \mathrm{Q}_i | \text{ for } \mathrm{Q}_i \cap E \neq \emptyset \text{ and } i \in \{1, ..., n\}, \tag{3.5}$$

and

$$\sum_{i=1}^{n} | \mathrm{Q}_i | > (1-\varepsilon) | \mathrm{Q}_0 | . \tag{3.6}$$

Let $\mathcal{Q}_1 = \{\mathrm{Q}_i\}_{i=1}^n$ and let $\mathcal{Q}_2$ denote the collection of dyadic squares such that

(i) $\mathrm{Q} \subset \mathrm{Q}_0$ such that $\mathrm{Q} \cap E_0 \neq \emptyset$, and

(ii) at least one of the quarters of Q does not contain any dyadic squares from $\mathcal{Q}_1$.

Choose an integer n_0 so large that

$$| Q_i | > 2^{-2n_0} \tag{3.7}$$

for each i and consider the collection of dyadic squares

$$\tilde{\mathcal{Q}} = \{Q \in \mathcal{Q}_1 \cup \mathcal{Q}_2 : |Q| \geq 2^{-2n_0}\}.$$

We claim that $\tilde{\mathcal{Q}}$ covers Q_0. Suppose to the contrary that some point x of Q_0 is not covered by $\tilde{\mathcal{Q}}$, i.e. $x \notin Q$ for every $Q \in \tilde{\mathcal{Q}}$. Now since Q_0 is a dyadic square there exists R_0, a dyadic square, such that $x \in R_0 \subset Q_0$ where $| R_0 | = 2^{-2n_0}$. Notice that $R_0 \not\subseteq Q$ for every $Q \in \tilde{\mathcal{Q}}$. Indeed, if there is a $Q \in \tilde{\mathcal{Q}}$ such that $R_0 \subseteq Q$, then $x \in Q$ which is contrary to supposition. Since $|R_0| = 2^{-2n_0}$, then by (3.7) $Q_i \not\subseteq R_0$ for every i. Hence R_0 does not contain any element from $\tilde{\mathcal{Q}}_1$. Also since these are all dyadic squares, if the intersection of any two of the dyadic squares is non-empty, then one of the squares must contain the other. By choice, $R_0 \not\subseteq Q$ for every $Q \in \tilde{\mathcal{Q}}$, and $R_0 \not\subseteq Q_i$ for every i. Therefore $R_0 \notin \tilde{\mathcal{Q}}_2$. However, R_0 does not contain any element of $\tilde{\mathcal{Q}}_1$ and thus satisfies (ii) of $\tilde{\mathcal{Q}}_2$. Therefore, R_0 cannot satisfy condition (i) of $\tilde{\mathcal{Q}}_2$ and so $R_0 \cap E_0 = \emptyset$.

Now $R_0 \subseteq R_1$ for some R_1 such that $|R_1| = 2^{-2(n_0-1)}$, but $R_1 \not\subseteq Q \in \tilde{\mathcal{Q}}$ for every Q. If there is a $Q \in \tilde{\mathcal{Q}}$ such that $R_1 \subseteq Q$ then $x \in Q$ contrary to supposition. Since R_0 and R_1 are dyadic squares, then R_0 must represent one quarter of R_1. Since R_0 does not contain any element from $\mathcal{Q}_1$, then R_1 satisfies condition (ii) for $\mathcal{Q}_2$. However, $R_1 \notin \tilde{\mathcal{Q}}$, so $R_1 \notin \mathcal{Q}_2$. Hence, R_1 must not satisfy condition (i)

for $\mathcal{Q}_2$, and so $R_1 \cap E_0 = \emptyset$. This is important, because $Q_i \cap E_0 \neq \emptyset$, so $Q_i \not\subseteq R_1$ for every i. If there is an $i \leq n$ such that $Q_i \subseteq R_1$, then $R_1 \cap E_0 \neq \emptyset$, which is contrary to the previous sentence. Hence one can inductively choose an R_2, R_3, and so on such that $R_0 \subset R_1 \subset R_2 \subset \cdots$ with $|R_j| = 4|R_{j-1}|$ and $R_j \cap E_0 = \emptyset$ for each $j > 0$. However, $R_j \subseteq Q_0$, so there exists a j such that $R_j = Q_0$. Thus $Q_0 \cap E_0 = \emptyset$, which is a contradiction.

Therefore, $\tilde{\mathcal{Q}}$ covers Q_0.

Now let $A = \cup_{i=1}^n Q_i$ and recall by Vitali's Covering Lemma the Q_i's are pairwise disjoint. Hence (3.5) implies that $\nu(A) > r|A|$. Moreover, since $\tilde{\mathcal{Q}}$ covers Q_0 the set $(Q_0 \setminus A)$ consists of a finite number of non-overlapping dyadic squares from $\mathcal{Q}_2$. By the definition of $\mathcal{Q}_2$ and (3.1) it follows that $\nu(Q_0 \setminus A) \geq 0$. Consequently, $\nu(Q_0) = \nu(A) + \nu(Q_0 \setminus A) \geq \nu(A) \geq r|A|$.

Now since $|A| = \sum_{i=1}^n |Q_i|$, it follows from (3.6) that

$$\varepsilon|Q_0| > |Q_0| - \sum_{i=1}^{n} |Q_i| = |Q_0| - |A| = |Q_0 \setminus A|.$$

In fact, $|Q_0 \setminus A| < \varepsilon|Q_0|$.

On the other hand, $\varepsilon|Q_0| + |A| > |Q_0| = |Q_0 \setminus A| + |A| > |Q_0 \setminus A| + (1-\varepsilon)|Q_0| \geq (1-\varepsilon)|Q_0|$. So, $|A| > (1-2\varepsilon)|Q_0|$. Since, $\nu(Q_0) > r|A|$, we conclude that

$$\nu(Q_0) > r(1-2\varepsilon)|Q_0|$$

and (3.4) is established and the lemma is proved. □

Theorem 3.4. *Let $\nu \in$ QM2. Then except on a subset of $[0,1)^2$ of measure zero*

either

$$\underline{D}\nu(x) = -\infty \text{ and } \overline{D}\nu(x) = +\infty,$$

or

$$\underline{D}\nu(x) = \overline{D}\nu(x) \text{ and is finite.} \tag{3.8}$$

Proof. By symmetry, we need only to show that (3.8) holds for almost every $x \in (0,1)^2$ satisfying $\underline{D}\nu(x) > -\infty$.

Fix $\varepsilon > 0$ and consider the set

$$\begin{aligned} E :&= \{x \in (0,1)^2 \mid \overline{D}\nu(x) > \underline{D}\nu(x) > -\infty\} \\ &= \bigcup_{r\in\mathbb{Q}}\bigcup_{k\in\mathbb{Z}}\{x \in (0,1)^2 \mid \overline{D}\nu(x) - \underline{D}\nu(x) > r \ , \ k\varepsilon \le \underline{D}\nu(x) < (k+1)\varepsilon\}. \end{aligned}$$

It remains to show that E is of Lebesgue measure zero.

Suppose not. Since E is obviously measurable, then we are supposing that $|E| > 0$. Then for some rational $r > 0$ and for some integer k, the Lebesgue measure of the set $\{x \in (0,1)^2 \mid \overline{D}\nu(x) - \underline{D}\nu(x) > r \ , \ k\varepsilon \le \underline{D}\nu(x) < (k+1)\varepsilon\}$ is also positive.

By replacing $\nu(\mathcal{I})$ by $\tilde{\nu}(\mathcal{I}) = \nu(\mathcal{I}) - k\varepsilon|\mathcal{I}|$ (a new quasimeasure) we may suppose that there is a number $\eta > 0$ and a measurable set E_0, with $|E_0| > 0$, such that

$$\overline{D}\tilde{\nu}(x) > r, \ 0 < \underline{D}\tilde{\nu}(x) < 2\varepsilon \ (x \in E_0) \text{ and} \tag{3.9}$$

$$\tilde{\nu}(\mathcal{I}) > 0 \text{ for all dyadic rectangles satisfying } E_0 \cap \mathcal{I} \neq \emptyset, |\mathcal{I}| < \eta. \tag{3.10}$$

Let $x \in E_0$ be a point of density of E_0. Then there exists a dyadic square $\mathcal{J}_0$ with $|\mathcal{J}_0| < \eta$ such that (3.1) from the previous lemma holds. Since (3.9) and

(3.10) imply (3.2) and (3.3) from the lemma, it follows from (3.4) that $\tilde{\nu}(\mathcal{J}_0) > r(1-2\epsilon)|\mathcal{J}_0|$. Moreover, in view of (3.9) one can shrink $\mathcal{J}_0$, if necessary so that $\tilde{\nu}(\mathcal{J}_0) < 2\varepsilon|\mathcal{J}_0|$. Consequently, $2\varepsilon|\mathcal{J}_0| > r(1-2\epsilon)|\mathcal{J}_0|$ holds for every $\varepsilon > 0$. In particular, let $\varepsilon \to 0$ to verify $r \leq 0$. This is a contradiction. Hence $|E_0| = 0$.

These last few paragraphs imply that $\overline{D}\tilde{\nu}(x) = \underline{D}\tilde{\nu}(x)$ for almost every x that satisfies $\underline{D}\tilde{\nu}(x) > -\infty$. Therefore, it remains to show that $\overline{D}\tilde{\nu}(x)$ is finite. However, if $\overline{D}\tilde{\nu}(x) = +\infty$ on a set E_0 of positive measure, then (3.3) holds everywhere. Hence by (3.4) we can choose a sufficiently small, fixed, dyadic square $\mathcal{J}_0$ such that $\nu(\mathcal{J}_0) > r > (1-2\varepsilon)|\mathcal{J}_0|$. (The conditions for Lemma 3.3 are met by $\mathcal{J}_0$ since $|E_0| > 0$, $\mathcal{J}_0$ has a point of density, and $\overline{D}\tilde{\nu}(x) = +\infty$). In particular, if we let $r \to +\infty$ we see that $\nu(\mathcal{J}_0) = +\infty$. This contradicts the fact that ν is finite valued.

The theorem is proved. □

Let $\mathbb{Q}$ denote the dyadic rationals in $[0,1)$. Given an integer $n > 0$ and a point $x \in [0,1)^2$ we shall denote the dyadic square of area 2^{-2n} which contains x by $\mathcal{I}_n(x)$. Thus,

$$\mathcal{I}_n(x) := \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right) \times \left[\frac{q}{2^n}, \frac{q+1}{2^n}\right)$$

where $x = (x_1, x_2)$ and $p, q \in \mathbb{N}$ are chosen such that $p \leq 2^n x_1 < p+1$ and $q \leq 2^n x_2 < q+1$.

Now define the following three variants of $\mathcal{I}_n(x)$.

Definition 3.5. Define the set $\mathcal{I}'_n(x)$ as follows:

$$\mathcal{I}'_n(x) = \emptyset, \text{ if } \{x\} \cap (\{0\} \times [0,1)) \neq \emptyset \text{ or } x = (0, x_2).$$
$$\mathcal{I}'_n(x) = \left[\frac{p-1}{2^n}, \frac{p}{2^n}\right) \times \left[\frac{q}{2^n}, \frac{q+1}{2^n}\right) \text{ otherwise.}$$

This set is the dyadic square that is directly left of $\mathcal{I}_n(x)$, which lies in $[0,1)^2$.

Definition 3.6. Define the set $\mathcal{I}_n^*(x)$ as follows:

$$\mathcal{I}_n^*(x) = \emptyset, \text{ if } \{x\} \cap ([0,1) \times \{0\}) \neq \emptyset \text{ or } (x_1, 0).$$
$$\mathcal{I}_n^*(x) = \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right) \times \left[\frac{q-1}{2^n}, \frac{q}{2^n}\right) \text{ otherwise.}$$

This set is the dyadic square that is directly below $\mathcal{I}_n(x)$, which lies in $[0,1)^2$.

Definition 3.7. Define the set $\mathcal{I}_n^\sharp(x)$ as follows:

$$\mathcal{I}_n^\sharp(x) = \emptyset, \text{ if } x = (x_1, 0) \text{ or } x = (0, x_2).$$
$$\mathcal{I}_n^\sharp(x) = \left[\frac{p-1}{2^n}, \frac{p}{2^n}\right) \times \left[\frac{q-1}{2^n}, \frac{q}{2^n}\right) \text{ otherwise.}$$

This set is the dyadic square that is to the left and below $\mathcal{I}_n(x)$, which lies in $[0,1)^2$.

Also define the following two hybrid lower derivates.

Definition 3.8.

$$(\underline{D}'\nu)(x) = \liminf_{n\to\infty} 2^{2n}\nu(\mathcal{I}_n'(x)) \text{ for } x \in [0,1)^2.$$

Definition 3.9.

$$(\underline{D}^*\nu)(x) = \liminf_{n\to\infty} 2^{2n}\nu(\mathcal{I}_n^*(x)) \text{ for } x \in [0,1)^2.$$

Corresponding hybrid upper derivates can be defined in a similar manner.

These definitions will be used in the following theorem, which can be thought of as a QM2 variant of the classical result that a differentiable function with a negative derivative is decreasing.

Theorem 3.10. *Let ν be* QM2 *and E be a countable subset of $[0,1)^2$. If*

$$\liminf_{n\to\infty} \nu(\mathcal{I}'_n(x)) \leq 0 \text{ for } x \in (\mathbb{Q} \times [0,1)), \tag{3.11}$$

$$\liminf_{n\to\infty} \nu(\mathcal{I}^*_n(x)) \leq 0 \text{ for } x \in ([0,1) \times \mathbb{Q}), \tag{3.12}$$

$$\liminf_{n\to\infty} \nu(\mathcal{I}^\sharp_n(x)) \leq 0 \text{ for } x \in \mathbb{Q}^2, \tag{3.13}$$

$$\liminf_{n\to\infty} \nu(\mathcal{I}_n(x)) \leq 0 \text{ for } x \in [0,1)^2, \tag{3.14}$$

$$(\underline{D}\nu)(x) \leq 0 \text{ for } x \in [0,1)^2 \setminus E, \tag{3.15}$$

$$(\underline{D}'\nu)(x) \leq 0 \text{ for } x \in (\mathbb{Q} \times [0,1)) \setminus E, \tag{3.16}$$

and

$$(\underline{D}^*\nu)(x) \leq 0 \text{ for } x \in ([0,1) \times \mathbb{Q}) \setminus E, \tag{3.17}$$

then $\nu(\mathcal{I}) \leq 0$ for every dyadic square $\mathcal{I} \in [0,1)^2$.

Proof. First assume without loss of generality, that $\mathbb{Q}^2 \subseteq E$. Moreover, by considering $\nu - \varepsilon m$ for some $\varepsilon > 0$ and m Lebesgue measure, we may also suppose that (3.15), (3.16), and (3.17) are all strict inequalities.

Suppose for purposes of contradiction that there exists a dyadic square $\mathcal{I}_0$ such that $\nu(\mathcal{I}_0) > 0$. Choose $p \in \mathbb{N}$ such that $|\mathcal{I}_0| = 2^{-2p}$. We shall construct a nested sequence $\mathcal{K}_0 \supset \mathcal{K}_1 \supset \cdots$, where the $\mathcal{K}_i$'s satisfy $\mathcal{K}_0 = \mathcal{I}_0$, $|\mathcal{K}_i| = 2^{-2(p+i)}$ and $\nu(\mathcal{K}_i) > 0$ for every $i \in \mathbb{N}$.

First, let $\mathbb{Q}^2 \cup E = \{y_1, y_2, ...\}$, where the y_i's are points in $[0,1)^2$, and proceed by induction. Suppose that $\mathcal{K}_n$ has been chosen. Divide $\mathcal{K}_n$ into 4 quarters as dyadic squares. Call these quarters $\mathcal{J}^n_1, \mathcal{J}^n_2, \mathcal{J}^n_3$, and $\mathcal{J}^n_4$. Hence, $|\mathcal{J}^n_i| = \frac{1}{4}|\mathcal{K}_n| = 2^{-2(p+n+1)}$ for each i. Let $\mathcal{J}^n_1$ be the lower left quarter, $\mathcal{J}^n_2$ be the upper left quarter, $\mathcal{J}^n_3$ be the lower right quarter, and $\mathcal{J}^n_4$ be the upper right quarter.

Now $\nu(\mathcal{K}_n) = \nu(\mathcal{J}_1^n) + \nu(\mathcal{J}_2^n) + \nu(\mathcal{J}_3^n) + \nu(\mathcal{J}_4^n)$. Thus if $\nu(\mathcal{K}_n) > 0$, then $\nu(\mathcal{J}_i^n) > 0$ for some $i \in \{1, 2, 3, 4\}$.

Let $k \geq 1$ be the smallest integer so that $y_k \in \overline{\mathcal{K}_n}$, closed in the usual topology. If y_k is the center point of $\mathcal{K}_n$, then set $\mathcal{K}_{n+1} = \mathcal{J}_i^n$ such that i is the smallest and $\nu(\mathcal{J}_i^n) > 0$. If y_k is not a center point, but rather a boundary point of the closures of two $\mathcal{J}_i^n$'s, set $\mathcal{K}_{n+1} = \mathcal{J}_i^n$ such that i is the smallest, $\nu(\mathcal{J}_i^n) > 0$, and $y_k \notin \overline{\mathcal{J}_i^n}$. If no such $\mathcal{J}_i^n$ exists, set $\mathcal{K}_{n+1} = \mathcal{J}_i^n$ such that i is the smallest, $\nu(\mathcal{J}_i^n) > 0$. If y_k is not a center point or a boundary point of two of the $\overline{\mathcal{J}_i^n}$'s, then $y_k \in \overline{\mathcal{J}_j^n}$ for some $j \in \{1, 2, 3, 4\}$. Set $\mathcal{K}_{n+1} = \mathcal{J}_i^n$ where $\nu(\mathcal{J}_i^n) > 0$, i is smallest, and $i \neq j$. If $\nu(\mathcal{J}_i^n) \leq 0$ for all $i \neq j$, set $\mathcal{K}_{n+1} = \mathcal{J}_j^n$.

Now by construction, $|\mathcal{K}_{n+1}| = 2^{-2(p+n+1)}$, , $\nu(\mathcal{K}_{n+1}) > 0$, and $\mathcal{K}_n \supset \mathcal{K}_{n+1}$, for each n as desired.

Since $\cap_{n=0}^{\infty} \overline{\mathcal{K}_n} \neq \emptyset$ by the finite intersection property, let $x \in \cap_{n=0}^{\infty} \overline{\mathcal{K}_n}$, where $x = (x_1, x_2) \in [0, 1]^2$.

This leads to the following seven possibilities.

Case 1. Suppose $x \notin E$, $x_1 \notin \mathbb{Q}$, and $x_2 \notin \mathbb{Q}$. Then $\mathcal{K}_n = \mathcal{I}_{n+p}(x)$ for every $n \in \mathbb{N}$. By hypothesis, $(\underline{D}\nu)(x) < 0$. However,

$$(\underline{D}\nu)(x) = \liminf_{n\to\infty} \frac{\nu(\mathcal{I}_n(x))}{|\mathcal{I}_n(x)|} = \liminf_{n\to\infty} \frac{\nu(\mathcal{K}_n(x))}{|\mathcal{K}_n(x)|} \geq 0,$$

since $\nu(\mathcal{K}_n(x)) > 0$ and $|\mathcal{K}_n(x)| > 0$ for all $n \in \mathbb{N}$. Hence a contradiction. This case does not occur.

Case 2. Suppose that $x \notin E$, $x_1 \in \mathbb{Q}$, but $x_2 \notin \mathbb{Q}$. Then x is eventually a right side point or a left side point of the $\mathcal{K}_n$'s. We consider these two subcases separately:

(a) x is a left side point of $\mathcal{K}_n$. Hence $\mathcal{K}_n = \mathcal{I}_{n+p}(x)$ for every n. This leads to the same contradiction as in Case 1.

(b) x is a right side point of $\mathcal{K}_n$. Hence $\mathcal{K}_n = \mathcal{I}'_{n+p}(x)$ for n eventually large enough.

$$(\underline{D}'\nu)(x) = \liminf_{n\to\infty} \frac{\nu(\mathcal{I}'_n(x))}{|\mathcal{I}'_n(x)|} = \liminf_{n\to\infty} \frac{\nu(\mathcal{K}_n(x))}{|\mathcal{K}_n(x)|} \geq 0,$$

since $\nu(\mathcal{K}_n(x)) > 0$ and $|\mathcal{K}_n(x)| > 0$ for all $n \in \mathbb{N}$. However, by hypothesis $(\underline{D}'\nu)(x) < 0$. Therefore, a contradiction is reached.

Since a contradiction was reached in both (a) and (b), then Case 2 does not occur either.

Case 3. Suppose that $x \notin E$, $x_1 \notin \mathbb{Q}$, but $x_2 \in \mathbb{Q}$. Then x is eventually a top side or a bottom side point of the $\mathcal{K}_n$'s.

(a) x is a bottom side point of $\mathcal{K}_n$. Hence, $\mathcal{K}_n = \mathcal{I}_{n+p}(x)$ for every n. This leads to the same contradiction as in Case 1.

(b) x is a top side point of $\mathcal{K}_n$. Hence, $\mathcal{K}_n = \mathcal{I}^*_{n+p}(x)$ eventually for every n large enough.

$$(\underline{D}^*\nu)(x) = \liminf_{n\to\infty} \frac{\nu(\mathcal{I}^*_n(x))}{|\mathcal{I}^*_n(x)|} = \liminf_{n\to\infty} \frac{\nu(\mathcal{K}_n(x))}{|\mathcal{K}_n(x)|} \geq 0,$$

since $\nu(\mathcal{K}_n(x)) > 0$ and $|\mathcal{K}_n(x)| > 0$ for all $n \in \mathbb{N}$. However, by hypothesis $(\underline{D}^*\nu)(x) < 0$. Therefore, a contradiction is reached.

Since a contradiction was reached in both (a) and (b), then Case 3 does not occur either.

Case 4. Suppose that now $x \in E$, but $x_1 \notin \mathbb{Q}$ and $x_2 \notin \mathbb{Q}$. Hence, $x \in \mathcal{K}_n$ for every n. Therefore, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $\nu(\mathcal{K}_{n+1}) > 0$ and

$\nu(\mathcal{J}_i^n) \leq 0$ for each i such that $\mathcal{J}_i^n \neq \mathcal{K}_{n+1}$ by construction. Hence, $\nu(\mathcal{K}_{n+1}) > \nu(\mathcal{K}_n)$ for each $n \geq N$, and $\mathcal{K}_n = \mathcal{I}_{n+p}(x)$ for each n. Therefore,

$$\liminf_{n\to\infty} \nu(\mathcal{I}_n) = \liminf_{n\to\infty} \nu(\mathcal{K}_n) \geq \nu(\mathcal{K}_N) = \nu(\mathcal{I}_{N+p}(x)) > 0.$$

This is a contradiction to (3.14). Therefore, Case 4 does not occur.

Case 5. Suppose that $x \in E$ and $x_1 \in \mathbb{Q}$, but $x_2 \notin \mathbb{Q}$. Therefore, there is an $N \in \mathbb{N}$ such that x is a left side point or a right side point of $\mathcal{K}_n$ for each $n \geq N$.

(a) x is eventually a left side point, i.e. there is an $N \in \mathbb{N}$ such that x is a left side point of $\mathcal{K}_n$ for each $n \geq N$. Then $\nu(\mathcal{J}_3^n) \leq 0$ and $\nu(\mathcal{J}_4^n) \leq 0$ for each $\mathcal{K}_n$ such that $n \geq N$, by construction. Hence $\mathcal{K}_{n+1} = \mathcal{J}_i^n$ for each $n \geq N$ and for some $i \in \{1, 2\}$. Since $x_2 \notin \mathbb{Q}$, then $x \in \mathcal{J}_1^n$ or $x \in \mathcal{J}_2^n$. Therefore, $\nu(\mathcal{J}_i^n) \leq 0$ for $x \notin \mathcal{J}_i^n$ by construction. Hence, $\nu(\mathcal{K}_{n+1}) > \nu(\mathcal{K}_n)$ for each $n \geq N$ and so a contradiction occurs, as in Case 4.

(b) x is eventually a right side point, i.e. there is an $N \in \mathbb{N}$ such that x is a right side point of $\overline{\mathcal{K}_n}$ for each $n \geq N$. Therefore, $\nu(\mathcal{J}_1^n) \leq 0$ and $\nu(\mathcal{J}_2^n) \leq 0$ for each $\mathcal{K}_n$ such that $n \geq N$, by construction. Also, x is only in $\overline{\mathcal{J}_j^n}$ for some $j \in \{3, 4\}$ for every $n \geq N$. In fact, $\nu(\mathcal{J}_i^n) \leq 0$ for every $i \neq j$ and $n \geq N$. However, $\mathcal{K}_n = \mathcal{I}'_{n+p}(x)$ for each $n \geq N$. This statement leads to the following statement.

$$\liminf_{n\to\infty} \nu(\mathcal{I}'_n(x)) = \liminf_{n\to\infty} \nu(\mathcal{K}_n) \geq \nu(\mathcal{K}_N) > 0.$$

This statement is a contradiction of 3.11.

Since both (a) and (b) lead to a contradiction, Case 5 does not occur either.

Case 6. Suppose that $x \in E$ and $x_2 \in \mathbb{Q}$, but $x_1 \notin \mathbb{Q}$. Therefore, there is an $N \in \mathbb{N}$ such that x is a bottom side point or a top side point of $\mathcal{K}_n$ for each $n \geq N$.

(a) x is eventually a bottom side point, i.e. there is an $N \in \mathbb{N}$ such that x is a bottom side point of $\mathcal{K}_n$ for each $n \geq N$. Then $\nu(\mathcal{J}_2^n) \leq 0$ and $\nu(\mathcal{J}_4^n) \leq 0$ for each $\mathcal{K}_n$ such that $n \geq N$, by construction. Hence $\mathcal{K}_{n+1} = \mathcal{J}_i^n$ for each $n \geq N$ and for some $i \in \{1, 3\}$. Since $x_1 \notin \mathbb{Q}$, then $x \in \mathcal{J}_1^n$ or $x \in \mathcal{J}_3^n$. Therefore, $\nu(\mathcal{J}_i^n) \leq 0$ for $x \notin \mathcal{J}_i^n$ by construction. Hence, $\nu(\mathcal{K}_{n+1}) > \nu(\mathcal{K}_n)$ for each $n \geq N$ and so a contradiction occurs, as in Case 4.

(b) x is eventually a top side point, i.e. there is an $N \in \mathbb{N}$ such that x is a top side point of $\overline{\mathcal{K}_n}$ for each $n \geq N$. Therefore, $\nu(\mathcal{J}_2^n) \leq 0$ and $\nu(\mathcal{J}_4^n) \leq 0$ for each $\mathcal{K}_n$ such that $n \geq N$, by construction. Also, x is only in $\overline{\mathcal{J}_j^n}$ for some $i \in \{1, 3\}$ for every $n \geq N$. In fact, $\nu(\mathcal{J}_i^n) \leq 0$ for every $i \neq j$ and $n \geq N$. However, $\mathcal{K}_n = \mathcal{I}_{n+p}^*(x)$ for each $n \geq N$. This statement leads to the following statement.

$$\liminf_{n\to\infty} \nu(\mathcal{I}_n^*(x)) = \liminf_{n\to\infty} \nu(\mathcal{K}_n) \geq \nu(\mathcal{K}_N) > 0.$$

This statement is a contradiction of 3.12.

Since both (a) and (b) lead to a contradiction, Case 6 does not occur either.

Case 7. Suppose that $x \in \mathbb{Q}^2$. Hence, there is an $N \in \mathbb{N}$ such that x is a corner point of $\mathcal{K}_n$ for each $n \geq N$.

(a) x is a bottom left corner of $\mathcal{K}_n$ for each $n \geq N$. Therefore, $\mathcal{K}_n = \mathcal{I}_{n+p}(x)$, and by construction $\nu(\mathcal{J}_i^n) \leq 0$ for each $i \neq 1$. Hence, $\nu(\mathcal{K}_{n+1}) \geq \nu(\mathcal{K}_n) > 0$ for

each $n \geq N$. Therefore,

$$\liminf_{n\to\infty} \nu(\mathcal{I}_n(x)) = \liminf_{n\to\infty} \nu(\mathcal{K}_n) \geq \nu(\mathcal{K}_N) > 0.$$

This is a contradiction to (3.14).

(b) x is a top left corner of $\overline{\mathcal{K}_n}$ for each $n \geq N$. Hence, $\mathcal{K}_{n+1} = \mathcal{J}_2^n$ for each $n > N$, and so $\mathcal{K}_n = \mathcal{I}^*_{n+p}(x)$ for each $n > N$. Since $x \in \overline{\mathcal{J}_2^n}$ for every $n \geq N$, then, by construction, $\nu(\mathcal{J}_i^n) \leq 0$ for each $i \neq 2$. Hence, $\nu(\mathcal{K}_{n+1}) \geq \nu(\mathcal{K}_n) > 0$ for each $n \geq N$.

$$\liminf_{n\to\infty} \nu(\mathcal{I}^*_n(x)) = \liminf_{n\to\infty} \nu(\mathcal{K}_n) \geq \nu(\mathcal{K}_N) > 0.$$

This is a contradiction to (3.12).

(c) x is a bottom right corner of $\overline{\mathcal{K}_n}$ for each $n \geq N$. Hence, $\mathcal{K}_{n+1} = \mathcal{J}_3^n$ for each $n > N$, and so $\mathcal{K}_n = \mathcal{I}'_{n+p}(x)$ for each $n > N$. Since $x \in \overline{\mathcal{J}_3^n}$ for every $n \geq N$, then, by construction, $\nu(\mathcal{J}_i^n) \leq 0$ for each $i \neq 3$. Hence, $\nu(\mathcal{K}_{n+1}) \geq \nu(\mathcal{K}_n) > 0$ for each $n \geq N$.

$$\liminf_{n\to\infty} \nu(\mathcal{I}'_n(x)) = \liminf_{n\to\infty} \nu(\mathcal{K}_n) \geq \nu(\mathcal{K}_N) > 0.$$

This is a contradiction to (3.11).

(d) x is a top right corner of $\overline{\mathcal{K}_n}$ for each $n \geq N$. Hence, $\mathcal{K}_{n+1} = \mathcal{J}_4^n$ for each $n > N$, and so $\mathcal{K}_n = \mathcal{I}^\sharp_{n+p}(x)$ for each $n > N$. Since $x \in \overline{\mathcal{J}_4^n}$ for every $n \geq N$, then, by construction, $\nu(\mathcal{J}_i^n) \leq 0$ for each $i \neq 4$. Hence, $\nu(\mathcal{K}_{n+1}) \geq \nu(\mathcal{K}_n) > 0$ for each $n \geq N$.

$$\liminf_{n\to\infty} \nu(\mathcal{I}^\sharp_n(x)) = \liminf_{n\to\infty} \nu(\mathcal{K}_n) \geq \nu(\mathcal{K}_N) > 0.$$

This is a contradiction to (3.13).

Since, (a),(b),(c), and (d) all lead to contradictions, then Case 7 does not occur.

All seven of the possible cases for x lead to contradictions. Therefore, the supposition that $\nu(\mathcal{I}_0) > 0$ must be false. Therefore, $\nu(\mathcal{I}) \leq 0$ for every dyadic square $\mathcal{I} \in [0,1)^2$. The theorem is true, as promised. □

Notice that a similar theorem, where $\geq$ is substituted for $\leq$, can be proved in a very similar manner through the same methods.

At last, we are prepared to use the ideas developed in Chapter 2 and Theorem 3.10 to obtain an answer to the question of uniqueness of two-dimensional Walsh series.

Theorem 3.11. *If S is a two dimensional Walsh Series satisfying*

$$\lim_{n\to\infty} 2^{-2n} S_{2^n,2^n}(x) = 0$$

for each $x \in \mathsf{G}^2$ and if

$$\lim_{n\to\infty} S_{2^n,2^n}(x) = 0$$

for all but countably many $x \in \mathsf{G}^2$, then S is the zero series, that is, the coefficients of S are identically zero.

Proof. First, let S be as in the hypotheses, and then notice that

$$\lim_{n\to\infty} S_{2^n,2^n}(x) = \limsup_{n\to\infty} S_{2^n,2^n}(x) = \liminf_{n\to\infty} S_{2^n,2^n}(x) = 0$$

for every $x \in \mathsf{G}^2$, which implies that

$$\limsup_{n\to\infty} S_{2^n,2^n}(x) \geq 0 \text{ and } \liminf_{n\to\infty} S_{2^n,2^n}(x) \leq 0$$

for every $x \in \mathsf{G}^2$.

Now consider the function

$$T_{2^n,2^n}(x_1,x_2) := S_{2^n,2^n}(\phi(x_1),\phi(x_2)),$$

for each $n \in \mathbb{N}$ and $(x_1,x_2) \in [0,1)^2$. $T_{2^n,2^n}$ is a two-dimensional Walsh series on $[0,1)^2$ with the same coefficients as $S_{2^n,2^n}$ for each n. Also let

$$g(x_1,x_2) := \lim_{n\to\infty} 2^{-2n} T_{2^n,2^n}(x_1,x_2),$$

for each $n \in \mathbb{N}$ and $(x_1,x_2) \in [0,1)^2$. By hypothesis, $g(x_1,x_2) \equiv 0$ for every $(x_1,x_2) \in [0,1)^2$. In fact, g is in class C_W in each coordinate. Hence, $g(x_1 - 1/2^n, x_2) = 0$ for each $n \in \mathbb{P}$,$x_1 \in \mathbb{Q}\setminus\{0\}$ and $x_2 \in [0,1)$, $g(x_1, x_2 - 1/2^n) = 0$ for each $n \in \mathbb{P}$,$x_2 \in \mathbb{Q}\setminus\{0\}$ and $x_1 \in [0,1)$, and $g(x_1 - 1/2^n, x_2 - 1/2^n) = 0$ for each $n \in \mathbb{P}$, $x_1 \in \mathbb{Q}\setminus\{0\}$ and $x_2 \in \mathbb{Q}\setminus\{0\}$.

Now recall from Chapter 2: If $x = (x_1,x_2) \in [0,1)^2$ and for some two-dimensional quasimeasure ν and for some $0 < t_n \le 1/2^n$ for each n, then

$$\begin{aligned}
\nu(\mathcal{I}'_n(x)) &= \int_{\mathcal{I}'_n(x)} d\nu \\
&= \lim_{m\to\infty} \int_{\mathcal{I}'_n(x)} T_{2^m,2^m}(u_1,u_2)\, d(u_1,u_2) \\
&= \int_{\mathcal{I}'_n(x)} T_{2^n,2^n}(u_1,u_2)\, d(u_1,u_2) \\
&= T_{2^n,2^n}(x_1 - t_n, x_2) \cdot |\mathcal{I}'_n(x)| \\
&= \begin{cases} 0, & \text{when } |\mathcal{I}'_n(x)| = 0 \\ 2^{-2n} T_{2^n,2^n}(x_1 - t_n, x_2), & \text{otherwise} \end{cases}
\end{aligned}$$

If we let $t_n = 1/2^n$ for this equation, then

$$\nu(\mathcal{I}'_n(x)) = \begin{cases} 0, & \text{when } |\mathcal{I}'_n(x)| = 0 \\ 2^{-2n} T_{2^n,2^n}(x_1 - 1/2^n, x_2), & \text{otherwise} \end{cases}$$

In fact, similarly

$$\nu(\mathcal{I}_n^*(x)) = \begin{cases} 0, \text{ when } |\mathcal{I}_n'(x)| = 0 \\ 2^{-2n}T_{2^n,2^n}(x_1, x_2 - t_n) = 2^{-2n}T_{2^n,2^n}(x_1, x_2 - 1/2^n), \end{cases}$$

$$\nu(\mathcal{I}_n^\sharp(x)) = \begin{cases} 0, \text{ when } |\mathcal{I}_n'(x)| = 0 \\ 2^{-2n}T_{2^n,2^n}(x_1 - t_n, x_2 - t_n) = 2^{-2n}T_{2^n,2^n}(x_1 - 1/2^n, x_2 - 1/2^n), \end{cases}$$

and

$$\nu(\mathcal{I}_n(x)) = 2^{-2n}T_{2^n,2^n}(x_1, x_2)$$

for appropriate x_1 and x_2. Hence, conditions 3.11, 3.12, 3.13, and 3.14 of Theorem 3.10 are all satisfied.

Now consider the function

$$h(x_1, x_2) := \lim_{n\to\infty} T_{2^n,2^n}(x_1, x_2)$$

for each $n \in \mathbb{N}$ and $(x_1, x_2) \in [0,1)^2$. By hypothesis, $h((x_1, x_2) \equiv 0$ for all but a countable set of points in $[0,1)^2$. Call this set E. In fact, by the equations for ν above and the definitions of the derivates from earlier in the chapter, we get:

$$(\underline{D}\nu)(x_1, x_2) = h((x_1, x_2)$$

$$(\underline{D}'\nu)(x_1, x_2) = \begin{cases} h(x_1 - t_n, x_2) \\ 0, \text{ when } |\mathcal{I}_n'(x)| = 0 \end{cases}$$

and

$$(\underline{D}^*\nu)(x_1, x_2) = \begin{cases} h(x_1, x_2 - t^n) \\ 0, \text{ when } |\mathcal{I}_n'(x)| = 0 \end{cases}$$

where t_n is chosen in each case so that $(x_1 - t_n, x_2) \in [0,1)^2 \setminus E$, $(x_1, x_2 - t_n) \in [0,1)^2 \setminus E$, and the equations for $\mathcal{I}_n(x)$, $\mathcal{I}_n^*(x)$, and $\mathcal{I}_n'(x)$ are true as above. Hence,

conditions 3.15, 3.16, and 3.17 of Theorem 3.10 are all satisfied by these equations and the hypothesis. Hence, all of the conditions for Theorem 3.10 are true and the conclusion holds. Therefore, $\nu(\mathcal{I}) \leq 0$ for every dyadic square $\mathcal{I}$. However,

$$\nu(\mathcal{I}) = 2^{-2n} T_{2^n,2^n}(x_1, x_2)$$

for some $n \in \mathbb{N}$ such that $|\mathcal{I}| = 2^{-2n}$ and any $(x_1, x_2) \in \mathcal{I}$. Hence, $T_{2^n,2^n}(x_1, x_2) \leq 0$ for any $(x_1, x_2) \in \mathcal{I}$. Since, $\mathcal{I}$ was arbitrary, then the last inequality hold for all $n \in \mathcal{N}$ and all $(x_1, x_2) \in [0, 1)^2$.

Using the alternate version of Theorem 3.10,which has $\geq$ in the place of $\leq$, we can also prove that $\nu(\mathcal{I}) \geq 0$ for every dyadic square $\mathcal{I}$, and hence $T_{2^n,2^n}(x_1, x_2) \geq 0$ for any $n \in \mathcal{N}$ and all $(x_1, x_2) \in [0, 1)^2$. Combining this inequality with the one derived above, we conclude that

$$T_{2^n,2^n}(x_1, x_2) = 0$$

for any $n \in \mathcal{N}$ and all $(x_1, x_2) \in [0, 1)^2$. Therefore, every coefficient of T must be equal to zero. Since the coefficients of T and S are the same, then S must be the zero series. The theorem is proved. □

If a two dimensional Walsh series satisfies the first hyposthesis of this last theorem (Theorem 3.11), namely, if

$$\lim_{n\to\infty} 2^{-2n} S_{2^n,2^n}(x) = 0,$$

holds for all $x \in \mathsf{G}^2$, then we say that it satisfies the *C-S condition.* Notice that the C-S condition is satisfied, by any two dimnesional Walsh series whose coefficients tend to zero. Indeed, if

$$S := \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} \omega_k \omega_l$$

then

$$2^{-2n}|S_{2^n,2^n}| \leq 2^{-2n} \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} |a_{k,l}|.$$

Hence, if $a_{k,l} \to 0$ as $k, l \to \infty$, then $2^{-2n} S_{2^n,2^n} \to 0$, as $n \to \infty$ uniformly on G^2. In particular, the two dimensional Walsh-Fourier series of an $f \in L^1(\mathsf{G}^2)$ satisfies the C-S condition.

Chapter 4

A Tauberian Theorem for Cesàro Summability of Double Walsh Series

There will be two main results in this chapter that will be based upon the theory that was developed in Chapters 2 and 3. The first shows that dyadic square partial sums of double Walsh series are either very bad or very good.

Theorem 4.1. *Let E be a measurable subset of $[0,1)^2$ and S be a two dimensional Walsh series. Then except on a subset of E of Lebesgue measure zero, either*

$$\limsup_{n\to\infty} S_{2^n,2^n}(x) = +\infty \text{ and } \liminf_{n\to\infty} S_{2^n,2^n}(x) = -\infty, \text{ or}$$

$$\lim_{n\to\infty} S_{2^n,2^n}(x) \text{ exists and is finite for } x \in E$$

In particular, if $\liminf_{n\to\infty} S_{2^n,2^n}(x) > -\infty$ $(x \in E)$, then $S_{2^n,2^n}$ converges almost everwhere on E, as $n \to \infty$, to some finite limit.

Proof. Let $\nu \in \mathsf{QM2}$ such that $S = S\nu$. Theorem 3.4 says that

$$\overline{D}\nu(x) = +\infty \text{ and } \underline{D}\nu(x) = -\infty,$$

$$\overline{D}\nu(x) = \underline{D}\nu(x) \text{ is finite}$$

for almost every $x \in [0,1)^2$.

However, recall

$$S_{2^n,2^n}(x) = 2^{2n}\nu(\mathcal{I}^n(x)) = \frac{\nu(\mathcal{I}^n(x))}{|(\mathcal{I}^n(x))|},$$

where $|(\mathcal{I}^n(x))| = 2^{-2n}$.

Hence the theorem is proved. □

In order to prove the next theorem two definitions will be needed. First, recall the definition of the *two dimensional Cesàro means.*

$$\sigma_{n,m}(x) = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} S_{i,j}(x) \text{ for } x \in [0,1)^2.$$

Definition 4.2. Define the *first hybrid Cesáro mean* to be

$$\sigma_n^1(x) := \frac{1}{n}\sum_{i=1}^{n} S_{i,n}(x) \text{ for } x \in [0,1)^2.$$

Definition 4.3. Define the *second hybrid Cesáro mean* to be

$$\sigma_n^2(x) := \frac{1}{n}\sum_{j=1}^{n} S_{n,j}(x) \text{ for } x \in [0,1)^2.$$

Before we state the next main result is, we shall prove the following technical lemma.

Lemma 4.4. *Let K be a subset of $[0,1)^2$ of positive Lebesgue measure, and let $x = (x_1, x_2)$ be a dyadic irrational point of density for K, that is, both coordinates*

are dyadic irrational points of density in $[0,1)$ *for* $x \in K$. *If* $p_{m,1}$ *and* $p_{m,2}$ *are integers such that* $0 \leq p_{m,i} \leq 2^m$ *for each* $i \in \{1,2\}$ *and* $m \in \mathbb{N}$ *such that* $\{x = (x_1, x_2)\} = \cap_{m=0}^{\infty}(I(p_{m,1}, m) \times (I(p_{m,2}, m))$, *then there is a number* M_0 *and three sequences* $t_{m,i} \in [0,1)^2$ *for* $i \in \{1,2,3\}$ *where* $t_{m,i} := t_{m,i}(x) \to (0,0)$ *for each* i *as* $m \to \infty$ *such that if* $m > M_0$ *and* $2^m \leq n \leq 2^{m+1}$, *then*

(i) $x + t_{m,i} \in K$,

(ii) x, $x + t_{m,1}$, $x + t_{m,2}$, *and* $x + t_{m,3}$ *are in different quarters of* $(I(p_{m,1}, m) \times (I(p_{m,2}, m))$, *and*

(iii)

$$\frac{1}{4}(\sigma_{n,n}(x) + \sum_{i=1}^{3}\sigma_{n,n}(x + t_{m,i})) = \frac{2^{2m}}{n^2}\sigma_{2^m,2^m}(x) + \frac{(n-2^m)^2}{n^2}S_{2^m,2^m}(x) + \frac{(n-2^m)(2^m)}{n^2}(\sigma^1_{2^m}(x) + \sigma^2_{2^m}(x))$$

Proof. Fix $m > 0$. Let $b_{m,1}$ be the midpoint of $I(p_{m,1}, m)$, and let $b_{m,2}$ be the midpoint of $I(p_{m,2}, m)$. Next set $a_{m,j} := b_{m,j} - x_j$, $j = 1, 2$.

If $a_{m,1} > 0$, then x_1 is in the left half of $I(p_{m,1}, m)$. If $a_{m,1} < 0$, then x_1 is in the right half of $I(p_{m,1}, m)$. If $a_{m,2} > 0$, then x_2 is in the bottom half of $I(p_{m,2}, m)$. If $a_{m,2} > 0$, then x_2 is in the top half of $I(p_{m,2}, m)$.

If $a_{m,1} > 0$, then set

$$J^{(1)}_{m,1} = (p_{m,1}2^{-m}, b_{m,1}),$$
$$J^{(2)}_{m,1} = (p_{m,1}2^{-m}, (p_{m,1}+1)2^{-m}), \text{and}$$
$$J^{(3)}_{m,1} = (b_{m,1}, (p_{m,1}+1)2^{-m}).$$

If $a_{m,1} < 0$, then set

$$J_{m,1}^{(1)} = (b_{m,1}, (p_{m,1}+1)2^{-m}),$$
$$J_{m,1}^{(2)} = (p_{m,1}2^{-m}, (p_{m,1}+1)2^{-m}), \text{and}$$
$$J_{m,1}^{(3)} = (p_{m,1}2^{-m}, b_{m,1}).$$

If $a_{m,2} > 0$, then set

$$J_{m,2}^{(1)} = (p_{m,2}2^{-m}, b_{m,2}),$$
$$J_{m,2}^{(2)} = (p_{m,2}2^{-m}, (p_{m,2}+1)2^{-m}), \text{and}$$
$$J_{m,2}^{(3)} = (b_{m,2}, (p_{m,2}+1)2^{-m}).$$

If $a_{m,2} < 0$, then set

$$J_{m,2}^{(1)} = (b_{m,2}, (p_{m,2}+1)2^{-m}),$$
$$J_{m,2}^{(2)} = (p_{m,2}2^{-m}, (p_{m,2}+1)2^{-m}), \text{and}$$
$$J_{m,2}^{(3)} = (p_{m,2}2^{-m}, b_{m,2}).$$

Now since x is an interior point of $(I(p_{m,1}, m) \times I(p_{m,2}, m))$ for each m, it is in one of the four quarters. For each m one can choose $\{J_m^{(i)}\}_{i=1}^{7}$ to represent sets that are the quarters or unions of the quarters of $(I(p_{m,1}, m) \times I(p_{m,2}, m))$ open in the usual topology.

Let $J_m^{(1)}$ be the quarter containing x open in the usual topology. Let $J_m^{(2)}$ be the quarter directly above or below $J_m^{(1)}$ open in the usual topology. Let $J_m^{(3)} = J_m^{(1)} \cup J_m^{(2)}$. Let $J_m^{(4)}$ be the quarter directly right or left of $J_m^{(1)}$. Let $J_m^{(5)} = J_m^{(1)} \cup J_m^{(4)}$. Let $J_m^{(6)}$ be the quarter diagonally opposite to $J_m^{(1)}$, and lastly, let $J_m^{(7)}$ be the union of the four corners each open in the usual topology.

For example, if $a_{m,1} > 0$ and $a_{m,2} > 0$, then $J_m^{(1)}$ be the lower left quarter of $(I(p_{m,1}, m) \times I(p_{m,2}, m))$ open in the usual topology. $J_m^{(2)}$ is the upper left quarter of $(I(p_{m,1}, m) \times I(p_{m,2}, m))$ open in the usual topology. $J_m^{(4)}$ is the lower right quarter of $(I(p_{m,1}, m) \times I(p_{m,2}, m))$ open in the usual topology. $J_m^{(6)}$ is the upper right quarter of $(I(p_{m,1}, m) \times I(p_{m,2}, m))$ open in the usual topology, and the other sets are defined, as above. (Note that all of these sets are nicely shrinking with respect to $B(x, (\sqrt{2})2^{-m}) \cap [0,1)^2$.)

These definitions mean that $J_m^{(2)}, J_m^{(4)}, J_m^{(6)}$, and x all belong to different quarters of $(I(p_{m,1}, m) \times I(p_{m,2}, m))$. Also,

$$|J_m^{(1)}| = |J_m^{(2)}| = \frac{1}{2}|J_m^{(3)}| = \frac{1}{2^{2(m+1)}}.$$

$$|K \cap J_m^{(2)}| = |K \cap J_m^{(3)}| - |K \cap J_m^{(1)}|$$

$$\begin{aligned}
\lim_{m\to\infty} \frac{|K \cap J_m^{(2)}|}{|J_m^{(2)}|} &= \lim_{m\to\infty} \left(\frac{|K \cap J_m^{(3)}| - |K \cap J_m^{(1)}|}{|J_m^{(2)}|} \right) \\
&= \lim_{m\to\infty} \left(\frac{|K \cap J_m^{(3)}|}{2^{-2(m+1)}} - \frac{|K \cap J_m^{(1)}|}{2^{-2(m+1)}} \right) \\
&= \lim_{m\to\infty} \left(\frac{2|K \cap J_m^{(3)}|}{|J_m^{(3)}|} - \frac{|K \cap J_m^{(1)}|}{|J_m^{(1)}|} \right) = 1
\end{aligned}$$

This last line happens to be true, since $x = (x_1, x_2)$ is a point of density for K, that is

$$\lim_{m\to\infty} \frac{|K \cap J_m^{(3)}|}{|J_m^{(3)}|} = 1 = \lim_{m\to\infty} \frac{|K \cap J_m^{(1)}|}{|J_m^{(1)}|}.$$

In a similar fashion,

$$|J_m^{(1)}| = |J_m^{(4)}| = \frac{1}{2}|J_m^{(5)}| = \frac{1}{2^{2(m+1)}}.$$

$$|K \cap J_m^{(4)}| = |K \cap J_m^{(5)}| - |K \cap J_m^{(1)}|$$

$$\lim_{m\to\infty}\frac{|K\cap J_m^{(4)}|}{|J_m^{(4)}|} = \lim_{m\to\infty}\left(\frac{|K\cap J_m^{(5)}|-|K\cap J_m^{(1)}|}{|J_m^{(4)}|}\right)$$
$$= \lim_{m\to\infty}\left(\frac{|K\cap J_m^{(5)}|}{2^{-2(m+1)}}-\frac{|K\cap J_m^{(1)}|}{2^{-2(m+1)}}\right)$$
$$= \lim_{m\to\infty}\left(\frac{2|K\cap J_m^{(5)}|}{|J_m^{(5)}|}-\frac{|K\cap J_m^{(1)}|}{|J_m^{(1)}|}\right)=1$$

This last line happens to be true, since $x=(x_1,x_2)$ is a point of density for K, that is

$$\lim_{m\to\infty}\frac{|K\cap J_m^{(5)}|}{|J_m^{(5)}|}=1=\lim_{m\to\infty}\frac{|K\cap J_m^{(1)}|}{|J_m^{(1)}|}.$$

Now show that $\lim_{m\to\infty}\frac{|K\cap J_m^{(6)}|}{|J_m^{(6)}|}=1$.

$$|J_m^{(6)}|=|J_m^{(1)}|=|J_m^{(2)}|=|J_m^{(4)}|=\frac{1}{4}|J_m^{(7)}|=\frac{1}{2^{2(m+1)}}$$

$$|K\cap J_m^{(6)}|=|K\cap J_m^{(7)}|-|K\cap J_m^{(1)}|-|K\cap J_m^{(2)}|-|K\cap J_m^{(4)}|$$

$$\lim_{m\to\infty}\frac{|K\cap J_m^{(6)}|}{|J_m^{(6)}|} = \lim_{m\to\infty}\left(\frac{|K\cap J_m^{(7)}|}{|J_m^{(6)}|}-\frac{|K\cap J_m^{(1)}|}{|J_m^{(6)}|}-\frac{|K\cap J_m^{(2)}|}{|J_m^{(6)}|}-\frac{|K\cap J_m^{(4)}|}{J_m^{(6)}|}\right)$$
$$= \lim_{m\to\infty}\left(\frac{|K\cap J_m^{(7)}|}{2^{2(m+1)}}-\frac{|K\cap J_m^{(1)}|}{2^{2(m+1)}}-\frac{|K\cap J_m^{(2)}|}{2^{2(m+1)}}-\frac{|K\cap J_m^{(4)}|}{2^{2(m+1)}}\right)$$
$$= \lim_{m\to\infty}\left(\frac{4|K\cap J_m^{(7)}|}{|J_m^{(7)}|}-\frac{|K\cap J_m^{(1)}|}{|J_m^{(1)}|}-\frac{|K\cap J_m^{(2)}|}{|J_m^{(2)}|}-\frac{|K\cap J_m^{(4)}|}{|J_m^{(4)}|}\right)$$
$$=1$$

Again, this last statement is true, since x is a point of density for K and by the previous results.

Hence, now

$$\lim_{m\to\infty}\frac{|K\cap J_m^{(1)}|}{|J_m^{(1)}|}=\lim_{m\to\infty}\frac{|K\cap J_m^{(2)}|}{|J_m^{(2)}|}=\lim_{m\to\infty}\frac{|K\cap J_m^{(4)}|}{|J_m^{(4)}|}=\lim_{m\to\infty}\frac{|K\cap J_m^{(6)}|}{|J_m^{(6)}|}=1.$$

In particular, there is $M_0 \in \mathbb{N}$ large enough so that $K \cap J_m^{(2)} \neq \emptyset$, $K \cap J_m^{(4)} \neq \emptyset$, and $K \cap J_m^{(6)} \neq \emptyset$ for all $m > M_0$.

Let $m > M_0$ and choose 3 numbers $t_{m,i}$ for $i \in \{1, 2, 3\}$ such that $x + t_{m,1} \in K \cap J_m^{(2)}$, $x + t_{m,2} \in K \cap J_m^{(4)}$, and $x + t_{m,3} \in K \cap J_m^{(6)}$. Since, $|J_m^{(2)}| \to 0$, $|J_m^{(4)}| \to 0$, and $|J_m^{(6)}| \to 0$, as $m \to \infty$, then $t_{m,i} \to 0$ and $x + t_{m,i} \to x$ for each i as $m \to \infty$.

Moreover, $x, x + t_{m,1}, x + t_{m,2}$, and $x + t_{m,3}$ all belong to different quarters of $(I(p_{m,1}, m) \times I(p_{m,2}, m))$. Therefore, conditions (i) and (ii) are satisfied.

Now fix $2^m \leq n \leq 2^{m+1}$ and consider $2^m < k \leq 2^{m+1}$. Then,

$$\begin{aligned} S_{k,k} &= \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} a_{i,j}\omega_{i,j} \\ &= \sum_{i=0}^{2^m-1} \sum_{j=0}^{2^m-1} a_{i,j}\omega_{i,j} + \sum_{i=2^m}^{k-1} \sum_{j=2^m}^{k-1} a_{i,j}\omega_{i,j} + \sum_{i=0}^{2^m-1} \sum_{j=2^m}^{k-1} a_{i,j}\omega_{i,j} + \sum_{i=2^m}^{k-1} \sum_{j=0}^{2^m-1} a_{i,j}\omega_{i,j} \end{aligned}$$

Now, if $r, l > 2^m$, then

$$S_{r,l} = \sum_{i=0}^{2^m-1} \sum_{j=0}^{2^m-1} a_{i,j}\omega_{i,j} + \sum_{i=2^m}^{r-1} \sum_{j=2^m}^{l-1} a_{i,j}\omega_{i,j} + \sum_{i=0}^{2^m-1} \sum_{j=2^m}^{l-1} a_{i,j}\omega_{i,j} + \sum_{i=2^m}^{r-1} \sum_{j=0}^{2^m-1} a_{i,j}\omega_{i,j}.$$

If $r > 2^m$ and $l \leq 2^m$, then

$$S_{r,l} = \sum_{i=0}^{2^m-1} \sum_{j=0}^{l-1} a_{i,j}\omega_{i,j} + \sum_{i=2^m}^{r-1} \sum_{j=0}^{l-1} a_{i,j}\omega_{i,j}.$$

If $r \leq 2^m$ and $l > 2^m$, then

$$S_{r,l} = \sum_{i=0}^{r-1} \sum_{j=0}^{2^m-1} a_{i,j}\omega_{i,j} + \sum_{i=0}^{r-1} \sum_{j=2^m}^{l-1} a_{i,j}\omega_{i,j}.$$

Also,

$$\begin{aligned}\sigma_{n,n} &= \frac{1}{n^2}\sum_{r=1}^{n}\sum_{l=1}^{n} S_{r,l} \\ &= \frac{1}{n^2}\sum_{r=1}^{2^m}\sum_{l=1}^{2^m} S_{r,l} + \frac{1}{n^2}\sum_{r=2^m+1}^{n}\sum_{l=2^m+1}^{n} S_{r,l} + \frac{1}{n^2}\sum_{r=1}^{2^m}\sum_{l=2^m+1}^{n} S_{r,l} + \frac{1}{n^2}\sum_{r=2^m+1}^{n}\sum_{l=1}^{2^m} S_{r,l} \\ &= \frac{1}{n^2}\sigma_{2^m,2^m} + \frac{1}{n^2}\sum_{r=2^m+1}^{n}\sum_{l=2^m+1}^{n} S_{r,l} + \frac{1}{n^2}\sum_{r=1}^{2^m}\sum_{l=2^m+1}^{n} S_{r,l} + \frac{1}{n^2}\sum_{r=2^m+1}^{n}\sum_{l=1}^{2^m} S_{r,l}\end{aligned}$$

Recall that ω_j is constant on $I(p_{m,1}, m)$ and $I(p_{m,2}, m)$ for $0 \leq j < 2^m$. Also recall that ω_j changes sign on $I(p_{m,1}, m)$ and $I(p_{m,2}, m)$ for each j such that $2^m \leq j < 2^{m+1}$. Hence,

$$\sigma_{2^m,2^m}(x + t_{m,i}) = \sigma_{2^m,2^m}(x) \text{ for each } i,$$

and

$$S_{2^m,2^m}(x + t_{m,i}) = S_{2^m,2^m}(x) \text{ for each } i.$$

Recall that $\omega_{i,j}(x) = \omega_i(x_1)\omega_j(x_2)$ for $x = (x_1, x_2)$. If $2^m < i, j \leq k$ and given $x, x + t_{m,i}$ for each i, then it follows that

$$\omega_{i,j}(x) = -\omega_{i,j}(x + t_{m,1}), \qquad \omega_{i,j}(x + t_{m,1}) = -\omega_{i,j}(x + t_{m,3}),$$

$$\omega_{i,j}(x) = -\omega_{i,j}(x + t_{m,2}), \qquad \omega_{i,j}(x + t_{m,1}) = \omega_{i,j}(x + t_{m,2}),$$

$$\omega_{i,j}(x) = -\omega_{i,j}(x + t_{m,3}), \qquad \omega_{i,j}(x + t_{m,2}) = -\omega_{i,j}(x + t_{m,3}).$$

Next, show condition (iii) holds. So using these facts above,

$$\begin{aligned}
&\sigma_{n,n}(x)+\sigma_{n,n}(x+t_{m,1})+\sigma_{n,n}(x+t_{m,2})+\sigma_{n,n}(x+t_{m,3})\\
&\quad=\frac{4\cdot 2^{2m}}{n^2}\sigma_{2^m,2^m}(x)+\frac{4}{n^2}\sum_{r=2^m+1}^{n}\sum_{l=2^m+1}^{n}S_{2^m,2^m}(x)\\
&\qquad+\frac{4}{n^2}\sum_{r=1}^{2^m}\sum_{l=2^m+1}^{n}\left[\sum_{i=0}^{r-1}\sum_{j=0}^{2^m-1}a_{i,j}\omega_{i,j}(x)\right]\\
&\qquad+\frac{4}{n^2}\sum_{r=2^m+1}^{n}\sum_{l=1}^{2^m}\left[\sum_{i=0}^{2^m-1}\sum_{j=0}^{l-1}a_{i,j}\omega_{i,j}(x)\right]\\
&\quad=\frac{4\cdot 2^{2m}}{n^2}\sigma_{2^m,2^m}(x)+\frac{4(n-2^m)^2}{n^2}S_{2^m,2^m}(x)\\
&\qquad+\frac{4(n-2^m)2^m}{n^2}\left(\frac{1}{2^m}\sum_{r=1}^{2^m}S_{r,2^m}(x)\right)+\frac{4(n-2^m)2^m}{n^2}\left(\frac{1}{2^m}\sum_{l=1}^{2^m}S_{2^m,l}(x)\right)\\
&\quad=\frac{4\cdot 2^{2m}}{n^2}\sigma_{2^m,2^m}(x)+\frac{4(n-2^m)^2}{n^2}S_{2^m,2^m}(x)\\
&\qquad+\frac{4(n-2^m)2^m}{n^2}\left(\frac{1}{2^m}\sum_{r=1}^{2^m}S_{r,2^m}(x)+\frac{1}{2^m}\sum_{l=1}^{2^m}S_{2^m,l}(x)\right)\\
&\quad=\frac{4\cdot 2^{2m}}{n^2}\sigma_{2^m,2^m}(x)+\frac{4(n-2^m)^2}{n^2}S_{2^m,2^m}(x)\\
&\qquad+\frac{4(n-2^m)(2^m)}{n^2}\left(\sigma^1_{2^m}(x)+\sigma^2_{2^m}(x)\right)
\end{aligned}$$

So, dividing both sides of this last equation by four gives

$$\begin{aligned}
\frac{1}{4}(\sigma_{n,n}(x)+\sum_{i=1}^{3}\sigma_{n,n}(x+t_{m,i}))=&\frac{2^{2m}}{n^2}\sigma_{2^m,2^m}(x)+\frac{(n-2^m)^2}{n^2}S_{2^m,2^m}(x)\\
&+\frac{(n-2^m)(2^m)}{n^2}(\sigma^1_{2^m}(x)+\sigma^2_{2^m}(x)),
\end{aligned}$$

and (iii) is true. □

Now we are prepared to prove the following Tauberian style theorem.

Theorem 4.5. *If* $E \subset [0,1)^2$ *is measurable and* $\limsup_{n\to\infty} |\sigma_{2^n,2^n}(x)| < \infty$, $\limsup_{n\to\infty} |\sigma^1_{2^n}(x)| < \infty$, *and* $\limsup_{n\to\infty} |\sigma^2_{2^n}(x)| < \infty$ *for* $x \in E$, *then there is a finite valued measurable function* f *such that*

$$\lim_{n\to\infty} S_{2^n,2^n}(x) = f(x)$$

for almost every $x \in E$.

Proof. First suppose that $|E| > 0$ and let $\varepsilon > 0$, $y \in E$, and set

$$F(y) := \liminf_{n\to\infty} \sigma_{2^n,2^n}(y)$$

By hypothesis, F is finite valued on E. Moreover, if $F_m(y) := min\{F(y), \sigma_{2^m+1,2^m+1}(y)\}$, then $F_m(y) \to F(y)$ almost everywhere on E, as $m \to \infty$. Hence, by Egoroff's Theorem and Lusin's Theorem, there is a perfect set $K \subseteq E$ where $|K| > |E| - \varepsilon$ such that F is continuous on K and $F_m \to F$ uniformly on K, as $m \to \infty$.

Thus for any fixed $x \in K$, there is $x + t_{m,i} \in K$ and $t_{m,i} \to 0$ for each $i \in \{1,2,3\}$ by Lemma 4.4. Choose m large enough so that for each i

$$\begin{aligned}\sigma_{2^{m+1},2^{m+1}}(x + t_{m,i}) &\geq F_m(x + t_{m,i}) \\ &\geq F(x + t_{m,i}) - \frac{\varepsilon}{2} \\ &\geq F(x) - \varepsilon.\end{aligned}$$

In particular,

$$\sigma_{2^{m+1},2^{m+1}}(x + t_{m,i}) \geq \liminf_{n\to\infty} \sigma_{2^n,2^n}(x) - \varepsilon$$

for each i and m large enough.

By (iii) of Lemma 4.4,

$$\frac{1}{4}(\sigma_{n,n}(x)+\sum_{i=1}^{3}\sigma_{n,n}(x+t_{m,i}))=\frac{2^{2m}}{n^2}\sigma_{2^m,2^m}(x)+\frac{(n-2^m)^2}{n^2}S_{2^m,2^m}(x)$$
$$+\frac{(n-2^m)(2^m)}{n^2}(\sigma^1_{2^m}(x)+\sigma^2_{2^m}(x)),$$

for $2^m \leq n \leq 2^{m+1}$. Letting $n = 2^{m+1}$ makes this equation

$$\frac{1}{4}(\sigma_{2^{m+1},2^{m+1}}(x)+\sum_{i=1}^{3}\sigma_{2^{m+1},2^{m+1}}(x+t_{m,i}))=\frac{1}{4}\sigma_{2^m,2^m}(x)+\frac{1}{4}S_{2^m,2^m}(x)$$
$$+\frac{1}{4}(\sigma^1_{2^m}(x)+\sigma^2_{2^m}(x)).$$

Multiply this equation by 4, and the last inequality above. It follows that, for large values of m we get

$$\sigma_{2^{m+1},2^{m+1}}(x)+3\liminf_{n\to\infty}\sigma_{2^n,2^n}(x)-3\varepsilon \leq \sigma_{2^m,2^m}(x)+S_{2^m,2^m}(x)$$
$$+\sigma^1_{2^m}(x)+\sigma^2_{2^m}(x),$$

and in fact,

$$4\liminf_{n\to\infty}\sigma_{2^n,2^n}(x)-4\varepsilon \leq \sigma_{2^m,2^m}(x)+S_{2^m,2^m}(x)+\sigma^1_{2^m}(x)+\sigma^2_{2^m}(x).$$

Now, by hypothesis and again for m sufficiently large this inequality becomes,

$$4\liminf_{n\to\infty}\sigma_{2^n,2^n}(x)-4\varepsilon \leq \limsup_{n\to\infty}\sigma_{2^n,2^n}(x)+S_{2^m,2^m}(x)+\limsup_{n\to\infty}\sigma^1_{2^m}(x)+\limsup_{n\to\infty}\sigma^2_{2^m}(x)+3\varepsilon.$$

Since $|K| > |E| - \varepsilon$ and ε is arbitrary, we have by the hypotheses that

$$\liminf_{m\to\infty}S_{2^m,2^m}(x) > -\infty$$

for almost every $x \in E$. Therefore the hypotheses of Theorem 4.1 are satisfied, and so there is a finite valued function f such that $S_{2^m,2^m}(x) \to f(x)$ almost everywhere on E as $m \to \infty$.

The theorem is proved.

□

Chapter 5

Summability Background

The purpose of this chapter is provide some basic background that will be needed to prove the theorems in Chapters 6 and 7.

A Vilenkin *rectangle* of order (n, m) is a rectangle of the form

$$\begin{aligned} R &:= [kP_n^{-1}, (k+1)P_n^{-1}) \times [lQ_m^{-1}, (l+1)Q_m^{-1}) \\ &= I(k, P_n) \times I(l, Q_n), \end{aligned}$$

where k and l are integers such that $0 \leq k < P_n$ and $0 \leq l < Q_m$. The $\sigma-$algebra created by all Vilenkin rectangles of order (n, m) will be denoted by $\mathcal{F}_{n,m}$. Given a rectangle $R \in \mathcal{F}_{n,m}$ and $r \in \mathbb{N}$, the *r-fold expansion* of R will be a Vilenkin rectangle $R^r \in \mathcal{F}_{n-r,m-r}$ such that $R \subset R^r$.

Given $0 < p < \infty$, then a *p-atom* on $[0, 1)^2$ is a bounded measurable function a on $[0, 1)^2$ that satisfies the following conditions:

(i) a is of *mean zero*, that is,

$$\int\int_{[0,1)^2} a(x_1, x_2) d(x_1, x_2) = 0,$$

(ii) there is a Vilenkin rectangle $R \in \mathcal{F}_{k,k}$ such that $\| a \|_\infty \leq |R|^{-\frac{1}{p}}$,

(iii) a is supported on R.

A *Vilenkin martingale* is a sequence of integrable functions $(f_{n,m} : n, m \in \mathbb{N})$ such that each $f_{n,m}$ is $\mathcal{F}_{n,m}$ measurable, and the conditional expectation operator with respect to these $\sigma-$algebras satisfies $E_{k,l}(f_{n,m}) = f_{k,l}$ for $k \leq n$ and $l \leq m$. It is well known that the partial sums $S_{P_n,Q_m} f$ of the two dimensional Vilenkin-Fourier series of any $f \in L^1([0,1)^2)$ is a Vilenkin martingale.

Using Vilenkin martingales and quadratic variation instead of distributions and the maximal operator, Weisz [16] introduced Vilenkin Hardy-Lorentz spaces, $H_{p,q}([0,1^2)$, for each pair $0 < p, q < \infty$. These spaces reduce to the regular Vilenkin Hardy spaces for $p = q$. The only thing that is needed about these Hardy-Lorentz spaces is the following extrapolation theorem (Wiesz [16]):

Lemma 5.1. *Suppose that T is a sublinear operator defined on Vilenkin martinglaes and $0 < \delta \leq 1$. There exists an $r \in \mathbb{N}$ and a constant $C_{p,r}$, which depends only on p and r such that*

$$\int \int_{[0,1)^2 \setminus R^r} |Ta|^p d(x_1, x_2) \leq C_{p,r}$$

hold for all p-atoms a. Moreover, if T is bounded from $L_\infty([0,1)^2)$ to $L_\infty([0,1)^2)$, then T is of weak type $(1,1)$ on $L_1([0,1)^2)$ and for every $\delta < p \leq 1$, $0 < q \leq \infty$ there is a constant $C_{p,r}$ which depends only on p and q such that

$$\| Tf \|_{p,q} \leq C_{p,r} \| f \|_{H_{p,q}([0,1)^2)} .$$

As was mentioned in the introduction, the theory of bounded Vilenkin systems is quite different from the unbounded case. In order to get some positive results

for the unbounded case, Wade [14], working the growth of Cesàro means of two dimensional Vilenkin-Fourier series, found it necessary to limit the growth of the Vilenkin parameters using the following concept.

Definition 5.2. Given $0 < \delta \leq 1$, call a sequence $(p_n; n \in \mathbb{N})$ or the Vilenkin system that it generates $\delta-$*quasibounded*, if there is a positive number M such that

$$M_n := \sum_{j=0}^{n-1} \left(\frac{p_j}{p_{j+1} \cdots p_n} \right)^{\delta} \leq M$$

for all $n \in \mathbb{N}$. A 1-quasibounded system will be called *quasibounded.*

Clearly, if $\delta > \mu$, then a $\mu-$quasibounded system is $\delta-$quasibounded, but not conversely. In fact, all $\delta-$quasibounded systems are quasibounded.

We have found it necessary to introduce another measurement to control the growth of the Vilenkin parameters which is slightly more stringent than $\delta-$quasiboundedness.

Definition 5.3. Given $0 < \delta \leq 1$, call a sequence $(p_n; n \in \mathbb{N})$ or the Vilenkin system that it generates $\delta-$*strongly quasibounded*, if there is a positive number K such that

$$K_n := \sum_{l=0}^{n-1} \left(\frac{p_l(n+2-l)}{p_{l+1} \cdots p_n} \right)^{\delta} \leq K$$

for all $n \in \mathbb{N}$ A 1-quasibounded system will be called *strongly quasibounded.*

Clearly, if $\delta > \mu$, then a $\mu-$strongly quasibounded system is $\delta-$strongly quasibounded, but not conversely. In fact, all $\delta-$strongly quasibounded systems are strongly quasibounded. Also, notice that since $n+2-l \geq 1$, then every $\delta-$strongly quasibounded system is $\delta-$quasibounded.

Now this definition begs the question as to whether any such sequences exist. Notice that if $p_n = O(1)$ as $n \to \infty$, then since $p_n \geq 2$,

$$K_n \leq C \sum_{l=0}^{\infty} \frac{(l+2)^{\delta}}{2^{\delta l}} < \infty.$$

The finiteness condition comes from the ratio test. Thus any bounded Vilenkin system is $\delta-$strongly quasibounded for all $0 < \delta \leq 1$.

Chapter 6

The Vilenkin-Fejer Kernel

Let $p = \{p_i\}_{i=0}^{\infty}$ be a sequence that will produce a Vilenkin system, and $P_n = p_0 p_1 ... p_{n-1}$. First recall, if ψ_k is the kth Vilenkin function for some Vilenkin system, then

$$D_n := \sum_{k=0}^{n-1} \psi_k \qquad \text{and} \qquad K_n := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \psi_k$$

represent the one-dimensional Vilenkin-Dirichlet kernel and the one-dimensional Vilenkin-Fejer kernel, respectively. Next recall that

$$D_{P_n}(x) \equiv P_n \chi_{[0,P_n-1)}(x) := \begin{cases} P_n, & 0 \le x < {P_n}^{-1} \\ 0, & {P_n}^{-1} \le x < 1 \end{cases}$$

The following lemmas will be crucial to our main estimates. The first two come from (Simon [9])

Lemma 6.1. *Let* $n = \sum_{\alpha=0}^{\infty} n_\alpha P_\alpha$ *for* $n \in \mathbb{N}$, *where* $0 \le n_\alpha < p_\alpha$ *and* $n_\alpha \in \mathbb{N}$.

Also let

$$\mathsf{A}_\alpha^{(j)} := \{l | l, j \in \mathbb{N},\ j \cdot P_\alpha \leq l < (j+1)P_\alpha,\ \textit{and}\ 1 \leq j < p_\alpha\},$$

and

$$\mathsf{B}_\alpha^{(j)}(n) := \{k | k \in \mathbb{N},\ k = n \oplus l,\ \textit{and}\ l \in \mathsf{A}_\alpha^{(j)}\} = n \oplus \mathsf{A}_\alpha^{(j)}.$$

Then it is true that

$$\{0, 1, 2, \ldots, n-1\} = \cup_{\alpha=0}^{\infty}\{\cup_{k=0}^{n_\alpha - 1}\mathsf{B}_\alpha^{(p_\alpha - n_\alpha + k))}(n)\}.$$

Proof. It is evident that $\mathsf{A}_\alpha^{(j)} \cap \mathsf{A}_{\alpha'}^{(j')} = \emptyset$, $\mathsf{B}_\alpha^{(j)}(n) \cap \mathsf{B}_{\alpha'}^{(j')}(n) = \emptyset$, where $(\alpha, j) \neq (\alpha', j')$. Designate by $[\mathsf{A}]$ the number of elements in the set A. Hence, we get

$$[\mathsf{A}_\alpha^{(j)}] = [\mathsf{B}_\alpha^{(j)}(n)] = P_\alpha,$$

and

$$[\cup_{\alpha=0}^{\infty}\{\cup_{k=0}^{n_\alpha - 1}\mathsf{B}_\alpha^{(p_\alpha - n_\alpha + k))}(n)\}] = \sum_{\alpha=0}^{\infty}\sum_{k=0}^{n_\alpha - 1} P_\alpha = \sum_{\alpha=0}^{\infty} n_\alpha P_\alpha = n.$$

On the basis of this equation it is enough to show that if $t \in \cup_{\alpha=0}^{\infty}\{\cup_{k=0}^{n_\alpha - 1}\mathsf{B}_\alpha^{(p_\alpha - n_\alpha + k))}(n)\}$, then $0 \leq t \leq n-1$. For each such t there is $l \in \mathbb{N}$ such that $t = n \oplus l$ and $l \in \mathsf{A}_\alpha^{(j)}$, where $j = p_\alpha - n_\alpha + k$, $0 \leq k \leq n_\alpha - 1$, and $\alpha, j, t\ in \mathbb{N}$. First,

$$l = \sum_{\mu=0}^{\alpha-1} l_\mu P_\mu + j \cdot P_\alpha,$$

then we get

$$t = \sum_{\mu=0}^{\alpha-1}[n_\mu + l_\mu \pmod{p_\mu}]P_\mu + [n_\alpha + j \pmod{p_\alpha}]P_\alpha + \sum_{\mu=\alpha+1}^{\infty} n_\mu P_\mu.$$

But $[n_\alpha + j \pmod{p_\alpha}] = k \leq n_\alpha - 1$, therefore it is true that

$$\begin{aligned} t &\leq \sum_{\mu=0}^{\alpha-1}(p_\mu - 1)P_\mu + (n_\alpha - 1)P_\alpha + \sum_{\mu=\alpha+1}^{\infty} n_\mu P_\mu \\ &= -1 + n_\alpha P_\alpha + \sum_{\mu=\alpha+1}^{\infty} n_\mu P_\mu \\ &\leq \sum_{\mu=0}^{\infty} n_\mu P_\mu - 1 = n - 1. \end{aligned}$$

Therefore, $0 \leq t \leq n-1$, and the lemma is proved. □

The next lemma comes from the same work by Simon. It uses Lemma 6.1 to get a powerful decomposition for the Vilenkin Dirichlet kernels D_n in terms of the positive, locally supported D_{P_α}'s.

Lemma 6.2. *If $n = \sum_{\alpha=0}^{\infty} n_\alpha P_\alpha$ where $0 \leq n_\alpha < p_\alpha$ and $n_\alpha \in \mathbb{N}$, then*

$$D_n(x) = \psi_n(x) \sum_{\alpha=0}^{\infty} \left(\sum_{j=p_\alpha - n_\alpha}^{p_\alpha - 1} (r_\alpha(x))^j \right) D_{P_\alpha}(x), \ (x \in \mathsf{G}_p).$$

In fact, if there is $s \in \mathbb{N}$ so that $P_{s-1} \leq n < P_s$, then $n_\alpha = 0$ for every $\alpha \geq s$ and

$$D_n(x) = \psi_n(x) \sum_{\alpha=0}^{s-1} \left(\sum_{j=p_\alpha - n_\alpha}^{p_\alpha - 1} (r_\alpha(x))^j \right) D_{P_\alpha}(x), \ (x \in \mathsf{G}_p).$$

Proof. By Lemma 6.1, it follows that

$$\begin{aligned} D_n(x) = \sum_{t=0}^{n-1} \psi_t(x) &= \sum_{t \in \cup_{\alpha=0}^{\infty} \left(\cup_{k=0}^{n_\alpha - 1} \mathsf{B}_\alpha^{(p_\alpha - n_\alpha + k)}(n) \right)} \psi_t(x) \\ &= \sum_{\alpha=0}^{\infty} \sum_{j=p_\alpha - n_\alpha}^{p_\alpha - 1} \sum_{l \in \mathsf{A}_\alpha^{(j)}} \psi_{n \oplus l}(x) \\ &= \psi_n(x) \sum_{\alpha=0}^{\infty} \sum_{j=p_\alpha - n_\alpha}^{p_\alpha - 1} \sum_{l \in \mathsf{A}_\alpha^{(j)}} \psi_l(x). \end{aligned}$$

However,

$$\sum_{l\in A_\alpha^{(j)}} \psi_l(x) = \sum_{k=0}^{P_\alpha-1} \psi_{jP_\alpha\oplus k}(x) = \psi_{jP_\alpha}(x) \sum_{k=0}^{P_\alpha-1} \psi_k(x)$$

$$= \psi_{jP_\alpha}(x) D_{P_\alpha}(x) = (r_\alpha(x))^j D_{P_\alpha}(x).$$

Therefore, Lemma 6.2 is proved. □

The next lemmas and theorems come as a result of initial formulas presented by Pál and Simon [5], from which the following definition comes.

Definition 6.3. The function $f \in L^1(\mathsf{G}_p)$ has a *strong derivative* $df \in L^1(\mathsf{G}_p)$, if

$$\lim_{n\to\infty} \| df - d_n f \|_1 = 0,$$

where

$$(d_n f)(x) = \sum_{j=0}^{n-1} P_j \sum_{k=0}^{p_j-1} k \cdot {p_j}^{-1} 1 \sum_{l=0}^{p_j-1} (exp\frac{-2\pi i l k}{p_j}) f(x \dot{+} l e_j)$$

for $x \in \mathsf{G}_p$, $n \in \mathbb{P}$, and $e_j := (0, 0, \ldots, 0, \overbrace{1}^{j}, 0, \ldots) \in \mathsf{G}_p$.

From this definition, $d\psi_n = n\psi_n$ is easily verified.

Lemma 6.4. *For $k \in \mathbb{N}$ and $x \in \mathsf{G}_p$, let r_j be the function defined by*

$$r_j(x) := exp\left(\frac{2\pi i x_j}{p_j}\right).$$

Then

$$K_n(x) = \frac{1}{n}\psi_n(x) \sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} \sum_{j=p_i-n_i}^{p_i-1} \Big((r_i(x))^j \Big[n_\alpha D_{P_i}(x)$$

$$- \frac{1}{p_\alpha} \sum_{k=0}^{p_\alpha-1} k \sum_{l=0}^{p_\alpha-1} \overline{(r_\alpha(l e_\alpha))^k} (r_i(l e_\alpha))^j \psi_n(l e_\alpha) D_{P_i}(x \dot{+} l e_\alpha) \Big] \Big).$$

Notice that this will lead to the following inequality:

$$|K_n(x)| \leq \frac{1}{n}\sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} n_i \Big[n_\alpha D_{P_i}(x) - \frac{1}{p_\alpha}\sum_{k=0}^{p_\alpha-1} k \sum_{l=0}^{p_\alpha-1} D_{P_i}(x \dot{+} l e_\alpha) \Big].$$

Proof. Let $n \in \mathbb{P}$ be an arbitrary fixed number and let $s \in \mathbb{P}$ be the natural number which satisfies $P_{s-1} \leq n < P_s$. Then from the last definition,

$$K_n = D_n - \frac{1}{n} d D_n = D_n - \frac{1}{n} d_s D_n$$

or

$$nK_n = nD_n - d_s D_n.$$

Hence,

$$\begin{aligned}
nK_n(x) &= n\psi_n(x) \sum_{i=0}^{s-1} \sum j = p_i - n_i^{p_i-1} (r_i(x))^j D_{P_i}(x) \\
&\quad - \sum_{\alpha=0}^{s-1} P_\alpha \sum_{k=0}^{p_\alpha-1} \frac{k}{p_\alpha} \sum_{l=0}^{p_\alpha-1} \overline{(r_\alpha(le_\alpha))^k} D_n(x \dot{+} l e_\alpha) \\
&= n\psi_n(x) \sum_{i=0}^{s-1} \sum_{j=p_i-n_i}^{p_i-1} (r_i(x))^j D_{P_i}(x) \\
&\quad - \sum_{\alpha=0}^{s-1} P_\alpha \sum_{k=0}^{p_\alpha-1} \frac{k}{p_\alpha} \sum_{l=0}^{p_\alpha-1} \overline{(r_\alpha(le_\alpha))^k} \psi_n(x \dot{+} l e_\alpha) \sum_{i=0}^{s-1} \sum_{j=p_i-n_i}^{p_i-1} (r_i(x \dot{+} l e_\alpha))^j D_{P_i}(x \dot{+} l e_\alpha) \\
&= \psi_n(x) \sum_{i=0}^{s-1} \sum_{j=p_i-n_i}^{p_i-1} (r_i(x))^j \cdot B_i,
\end{aligned}$$

where

$$B_i = nD_{P_i}(x) - \sum_{\alpha=0}^{s-1} P_\alpha \sum_{k=0}^{p_\alpha-1} \frac{k}{p_\alpha} \sum_{l=0}^{p_\alpha-1} \overline{(r_\alpha(le_\alpha))^k} \psi_n(le_\alpha)(r_i(le_\alpha))^j D_{P_i}(x \dot{+} l e_\alpha).$$

Recall that Paley's lemma says that $D_{P_i}(x) = P_i$, if $x \in \mathrm{supp} D_{P_i}$, and $D_{P_i}(x) = 0$, if $x \notin \mathrm{supp} D_{P_i}$. Hence, if $0 \leq i < \alpha$, then $D_{P_i}(x \dot{+} l e_\alpha) = D_{P_i}(x)$ and $r_i(le_\alpha) = 1$.

Again, if $0 \leq i < \alpha$, then by Paley's Lemma, $B_i = 0$ if $x \notin \text{supp}D_{P_i}$. For $n = \sum_{\alpha=0}^{s-1} n_\alpha P_\alpha$, $0 \leq i < \alpha$, and $x \in \text{supp}D_{P_i}$, then

$$B_i = \sum_{\alpha=0}^{s-1} P_\alpha \left[n_\alpha P_i - \sum_{k=0}^{p_\alpha - 1} \frac{k}{p_\alpha} \sum_{l=0}^{p_\alpha - 1} \overline{(r_\alpha(le_\alpha))^k} \psi_n(le_\alpha) P_i \right]$$

otherwise.

We shall show that $B_i = 0$. Indeed, first show by definition

$$\begin{aligned} B_i &= \sum_{\alpha=0}^{s-1} P_\alpha P_i \left[n_\alpha - \sum_{k=0}^{p_\alpha - 1} \frac{k}{p_\alpha} \sum_{l=0}^{p_\alpha - 1} \overline{(r_\alpha(le_\alpha))^k} \psi_n(le_\alpha) \right] \\ &= \sum_{\alpha=0}^{s-1} P_\alpha P_i \left[n_\alpha - \sum_{k=0}^{p_\alpha - 1} \frac{k}{p_\alpha} \sum_{l=0}^{p_\alpha - 1} exp(\frac{-2\pi i l k}{p_\alpha}) exp(\frac{2\pi i l n_\alpha}{p_\alpha}) \right] \\ &= \sum_{\alpha=0}^{s-1} P_\alpha P_i \left[n_\alpha - \sum_{k=0}^{p_\alpha - 1} \frac{k}{p_\alpha} \sum_{l=0}^{p_\alpha - 1} exp(\frac{-2\pi i l (n_\alpha - k)}{p_\alpha}) \right] \end{aligned}$$

However,

$$\begin{aligned} & n_\alpha - \sum_{k=0}^{p_\alpha - 1} \frac{k}{P_\alpha} \sum_{l=0}^{p_\alpha - 1} exp(\frac{-2\pi i l (n_\alpha - k)}{p_\alpha}) \\ &= n_\alpha - n_\alpha - \sum_{\substack{k \neq \alpha \\ 0 \leq k \leq p_\alpha - 1}} \frac{k}{p_\alpha} \sum_{l=0}^{p_\alpha - 1} exp(\frac{-2\pi i l (n_\alpha - k)}{p_\alpha}) \\ &= - \sum_{\substack{k \neq \alpha \\ 0 \leq k \leq p_\alpha - 1}} \frac{k}{p_\alpha} \sum_{l=0}^{p_\alpha - 1} exp(\frac{-2\pi i l (n_\alpha - k)}{p_\alpha}) \\ &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} nK_n(x) = \psi_n(x) \sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} \sum_{j=p_i - n_i}^{p_i - 1} & ((r_i(x))^j [n_\alpha D_{P_i}(x) \\ & - \frac{1}{p_\alpha} \sum_{k=0}^{p_\alpha - 1} k \sum_{l=0}^{p_\alpha - 1} \overline{(r_\alpha(le_\alpha))^k} (r_i(le_\alpha))^j \psi_n(le_\alpha) D_{P_i}(x \dot{+} le_\alpha)]) \end{aligned}$$

and the lemma is proved. □

The first application of Lemma 6.4 is an estimate of the $L_1[0,1)$ norm of $K_n(x)$ for a $\delta-$strong quasibounded Vilenkin system.

Lemma 6.5. *If $\{p_n\}_{n=0}^{\infty}$ is a $\delta-$strong quasibounded Vilenkin system for some $0 < \delta \leq 1$ and if $\beta_s = \sup_{0\leq i<s} p_i$, then*

$$\| K_n(x) \|_1 = O(\beta_s^3)$$

as $n \to \infty$ for $P_{s-1} \leq n < P_s$.

Proof. By Lemma 6.4

$$\begin{aligned} n|K_n(x)| &\leq \sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} \sum_{j=p_i-n_i}^{p_i-1} \left[n_\alpha D_{P_i} + \frac{1}{p_\alpha} \sum_{k=0}^{p_\alpha-1} k \sum_{l=0}^{p_\alpha-1} D_{P_i}(x \dot{+} l e_\alpha) \right] \\ &\leq \sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} n_i \left[n_\alpha D_{P_i} + \frac{1}{p_\alpha} \sum_{k=0}^{p_\alpha-1} k \sum_{l=0}^{p_\alpha-1} D_{P_i}(x \dot{+} l e_\alpha) \right] \end{aligned}$$

Therefore,

$$\begin{aligned}
\| K_n(x) \|_1 &\leq \frac{1}{n}\sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} n_i \left[n_\alpha + \frac{1}{p_\alpha}\sum_{k=0}^{p_\alpha-1} k \sum_{l=0}^{p_\alpha-1} 1 \right] \\
&\leq \frac{1}{n}\sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} p_i \left[p_\alpha + \sum_{k=0}^{p_\alpha-1} \frac{k}{p_\alpha} p_\alpha \right] \\
&\leq \frac{1}{n}\sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} p_i \left[p_\alpha + \sum_{k=0}^{p_\alpha-1} k \right] \\
&\leq \frac{1}{n}\sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} p_i \left[p_\alpha + \sum_{k=0}^{p_\alpha-1} (p_\alpha - 1) \right] \\
&\leq \frac{1}{n}\sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} p_i \left[p_\alpha + p_\alpha(p_\alpha - 1) \right] \\
&= \frac{1}{n}\sum_{\alpha=0}^{s-1} P_\alpha \sum_{i=\alpha}^{s-1} p_i (p_\alpha^2) \\
&= \frac{1}{n}\sum_{\alpha=0}^{s-1} P_{\alpha+1} \sum_{i=\alpha}^{s-1} p_i p_\alpha \\
&\leq \beta_s \sum_{\alpha=0}^{s-1} \frac{P_{\alpha+1}}{P_{s-1}} \sum_{i=\alpha}^{s-1} p_\alpha \\
&= \beta_s \sum_{\alpha=0}^{s-1} \frac{P_{\alpha+1}}{P_{s-1}} p_\alpha (s-\alpha) \\
&= \beta_s \left(p_{s-1}^2 + 2p_{s-2} + \sum_{l=0}^{s-3} \frac{p_\alpha (s-l)}{p_{l+1}\cdots p_{s-2}} \right)
\end{aligned}$$

Now since $\{p_n\}_{n=0}^{\infty}$ is a $\delta-$strong quasibounded Vilenkin system, then it is a strong quasibounded Vilenkin system. So there is an $M \in \mathbb{N}$ such that

$$\sum_{l=0}^{s-3} \frac{p_\alpha (s-l)}{p_{l+1}\cdots p_{s-2}} \leq M$$

for every $s \in \mathbb{N}$. Also, $p_{s-1}^2 \leq \beta_s^2$, $2p_{s-1} \leq \beta_s^2$, and $M \leq \beta_s^2$, for each s.

Therefore,

$$\| K_n(x) \|_1 = O(\beta_s^3)$$

and the lemma is proved. □

Chapter 7

Summability Estimates of Two Dimensional Vilenkin-Fourier Series

Cesàro summability of Vilenkin-Fourier series for systems of bounded type is fairly well understood. Building upon the earlier results of Schipp, Pál and Simon [5] have proved that for the one-dimensional case that if $f \in L_1([0,1))$, then $\sigma_n f \to f$ almost everywhere on $[0,1)$. This is not the case for Vilenkin systems of unbounded type. Indeed, from earlier results of Price, Tsutserova [10] proved that for any unbounded Vilenkin system, there are continuous functions for which $\sigma_{P_n} f$ diverges at any prescribed point. For the two-dimensional case, Móricz, Schipp, and Wade [4] proved that if $f \in L_1([0,1)^2)$ and $p_n \equiv q_n \equiv 2$ for all $n \in \mathbb{N}$, then $\sigma_{P_n,Q_m} f \to f$ almost everywhere on $[0,1)^2$, provided that $|n-m| \leq \alpha$ for some $\alpha > 0$. Weisz [17] extended this result to bounded Vilenkin systems and

also proved in this case that $\sigma_{n,m}f \to f$ almost everywhere on $[0,1)^2$, provided that $\alpha^{-1} < n/m < \alpha$ for some $\alpha > 0$. None of these results extends to unbounded Vilenkin systems.

In this chapter, we will use the ideas developed in Chapters 5 and 6 to obtain estimates on the Cesàro means of certain classes of two-dimensional Vilenkin-Fourier series of unbounded type. We begin this by letting $(p_k : k \in \mathbb{N})$ and $(q_l : l \in \mathbb{N})$ represent two unbounded sequences of integers with each element less than or equal to 2. Let D_n and K_n represent the two-dimensional Vilenkin-Dirichlet and two-dimensional Vilenkin-Fejér kernels, respectively depending upon the sequences above. Let $\dot{+}$ and $\dot{-}$ represent the addition and subtraction inherited from the underlying Vilenkin group structure, for example, if $x = \sum_{k=o}^{\infty} x_k P_k^{-1}$ and $y = \sum_{k=o}^{\infty} y_k P_k^{-1}$, then

$$x \dot{+} y := \sum_{k=o}^{\infty} (x_k \oplus y_k) P_k^{-1},$$

where $\oplus$ is as defined in Chapter 1. From previous results one can see that

$$(\sigma_{n,m}f)(x_1,x_2) \equiv \int\int_{[0,1)^2} f(t,u) K_n(x_1 \dot{-} t) K_m(x_2 \dot{-} u)\, d(t,u).$$

Wade [14] made the first progress on this problem, in particular with the following result. Fix $\alpha > 0$ and choose $r \in \mathbb{N}$ such that $r - 1 < \alpha \leq r$. For each Vilenkin martingale $f = (f_{n,m} : n,m \in \mathbb{N})$ and each $0 < p \leq 1$, let

$$\sigma_{\alpha,p}f := \sup_{|n-m| \leq \alpha} \frac{|\sigma_{P_n,Q_m}f|}{\beta_{n,m}^{1/p}, \beta_{n+r,m+r}^{2r/p}},$$

where for every pair $k,l \in \mathbb{N}$, $\beta_{k,l} := \max\{p_0,\ldots,p_{k-1},q_0,\ldots,q_{l-1}\}$. Maximal operators of this type were first discussed for the Walsh case in Móricz, Schipp, and Wade [4].

Wade [14] obtained the following estimate.

Theorem 7.1. *Suppose that* $\{p_k\}$ *and* $\{q_l\}$ *are* $\delta-$*quasibounded for some* $0 < \delta \leq 1$. *Then* $\sigma_{\alpha,\delta}$ *is of weak type* $(1,1)$ *on* $L_1([0,10^2)$, *and for every* $\delta \leq p \leq 1$, $0 < q \leq \infty$ *there is a constant* $C_{p,q}$ *which depends only on* p *and* q *such that*

$$\| \sigma_{\alpha,\delta} f \|_{p,q} \leq C_{p,r} \| f \|_{H_{p,q}([0,1)^2)} .$$

This result gives estimates for the maximal operator associated with Cesàro means of the form σ_{P_n,Q_m}. The next result gives estimates for the maximal operator associated with Cesàro means of the form $\sigma_{n,m}$. To obtain positive results, we must increase the weights used in the denominator. Accordingly, fix $\alpha > 0$ and choose $r \in \mathbb{P}$ such that $r - 1 < \alpha \leq r$. For each Vilenkin martingale $f = (f_{n,m} : n, m \in \mathbb{N})$ and each $0 < \delta \leq 1$, let

$$\sigma_\alpha^\delta f := \sup_{2^{-\alpha} \leq n/m \leq 2^\alpha} \frac{|\sigma_{n,m} f|}{\beta_{m,n}^{4/\delta}, \beta_{m_r,n_r}^{2r/\delta}},$$

where for each $n, m \in \mathbb{N}$ there exists $s, d \in \mathbb{N}$ such that $P_{s-1} \leq n < P_s$, $Q_{d-1} \leq m < Q_d$,

$$\beta_{m,n} := \max\{p_0, \ldots, p_{s-1}, q_0, \ldots, q_{d-1}\},$$

and

$$\beta_{m_r,n_r} := \max\{p_0, \ldots, p_{s-1+r}, q_0, \ldots, q_{d-1+r}\}.$$

Theorem 7.2. *Fix* $\alpha > 0$ *and choose* $r - 1 < \alpha \leq r$. *Suppose that* $(p_k : k \in \mathbb{N}) = (q_l : l \in \mathbb{N})$ *is a* $\delta-$*semiquasibounded sequence for some* $0 < \delta \leq 1$. *Then* σ_α^δ *is of weak type* $(1,1)$ *on* $L_1([0,1)^2$, *and for every* $\delta \leq 1$ *and* $0 < q \leq \infty$ *there exists a constant* $C_{1,q}$ *which depends only on* p *and* q *such that*

$$\| \sigma_\alpha^\delta f \|_{1,q} \leq C_{1,q} \| f \|_{H_{1,q}([0,1)^2)} .$$

Proof. It is clear by hypothesis that Lemma 6.5 is true and hence $\| K_n(x) \|_1 = O(\beta_s^3)$ is true for every n. Recall that

$$(\sigma_{n,m} f)(x,y) \equiv \int\int_{[0,1)^2} f(t,u) K_n(x \dot{-} t) K_m(y \dot{-} u) d(t,u). \tag{7.1}$$

Combining these facts with Lemma (5.1) we shall prove the theorem. Indeed, by the lemma, it suffices to show that σ_α^δ is 1-quasilocal for each $\delta \leq 1$, that is, there is a constant $C_{1,r}$ such that if a is a 1-atom supported on some Vilenkin "square" $R = I \times J$, then

$$\int\int_{[0,1)^2 \setminus R^r} |(\sigma_\alpha^\delta a)(x_1,x_2)| \, d(x_1,x_2) \leq C_{1,r}. \tag{7.2}$$

Since R is a Vilenkin square, choose $K \in \mathbb{N}$ such that $|I| = P_K^{-1} = Q_K^{-1} = |J|$. If $m, n < P_K = Q_K$, then both ψ_n and ψ_m are constant on I and J. Since a is of mean zero and has support on $R = I \times J$, it follows that $\widehat{a}(n,m) = 0$. Thus we may suppose that $m \geq Q_K$ or $n \geq P_K = Q_K$. With r as in the hypothesis, if $n \geq P_K$, then by condition $2^{-\alpha} \leq n/m \leq 2^\alpha$,

$$m \geq \frac{n}{2^\alpha} \geq \frac{P_K}{2^\alpha} \geq \frac{P_K}{2^r} \geq \frac{P_K}{p_{K-r} \cdots p_{K-1}} = P_{K-r} = Q_{k-r}.$$

Hence, both $n, m \geq P_{K-r} = Q_{K-r}$, for both $n \geq P_K$ and $m \geq Q_K$.

To prove equation (7.2), notice first that $[0,1)^2 \setminus R^r$ can be broken into eight pieces: two horizontal pieces $([0,1) \setminus I^r) \times J^r$; two vertical pieces $I^r \times ([0,1) \setminus J^r)$; and four rectangular pieces $([0,1) \setminus I^r) \times ([0,1) \setminus J^r)$.

Step 1. We will show the estimate for integrating over the horizontal pieces $([0,1) \setminus I^r) \times J^r$. First let $(x_1,x_2) \in ([0,1) \setminus I^r) \times J^r$. By given results $\| K_m \|_1 \leq C\beta_{m,-1}^3$ for some $C \in \mathbb{N}$ and $Q_{d-1} \leq m < Q_d$. Since a is supported on $R = I \times J$

and $|a| \le |R|^{-1}$, it follows from (7.1) that

$$\begin{aligned}|(\sigma_{n,m}a)(x_1,x_2)| &\le \int_J\int_I |a(t,u)||K_n(x_1\dot{-}t)||K_m(x_2\dot{-}u)|\,dt\,du \\ &\le C\beta_{m,-1}^3|R|^{-1}\int_I |K_n(x_1\dot{-}t)|\,dt\end{aligned}$$

Recall that $n = \sum_{\theta=0}^{\infty} n_\theta P_{\theta+1}$ for $0 \le n_\theta < p_\theta$. Using the estimate for $K_n(x)$ from Lemma 6.4, we can get an estimate for $|K_n(x)|$. This estimate can be substituted into the last inequality with the following result.

$$\begin{aligned}|(\sigma_{n,m}a)(x_1,x_2)| &\le C\beta_{m,-1}^3|R|^{-1}\int_I \frac{1}{n}\sum_{\gamma=0}^{s-1}P_\gamma\sum_{i=\gamma}^{s-1}n_i\Big[n_\gamma D_{P_i}(x_1\dot{-}t) \\ &\qquad - \frac{1}{p_\gamma}\sum_{k=0}^{p_\gamma-1}k\sum_{l=0}^{p_\gamma-1}D_{P_i}(x_1\dot{+}lP_{\gamma+1}^{-1}\dot{-}t)\Big]\,dt \\ &\le C\beta_{m,-1}^3|R|^{-1}\int_I \frac{1}{n}\sum_{\gamma=0}^{s-1}P_\gamma\sum_{i=\gamma}^{s-1}p_i\Big[p_\gamma D_{P_i}(x_1\dot{-}t) \\ &\qquad - \frac{1}{p_\gamma}\sum_{k=0}^{p_\gamma-1}k\sum_{l=0}^{p_\gamma-1}D_{P_i}(x_1\dot{+}lP_{\gamma+1}^{-1}\dot{-}t)\Big]\,dt \\ &\le C\beta_{m,-1}^3|R|^{-1}\int_I \frac{1}{n}\sum_{\gamma=0}^{s-1}P_\gamma\sum_{i=\gamma}^{s-1}p_i\Big[p_\gamma D_{P_i}(x_1\dot{-}t) \\ &\qquad - p_\gamma\sum_{l=0}^{p_\gamma-1}D_{P_i}(x_1\dot{+}lP_{\gamma+1}^{-1}\dot{-}t)\Big]\,dt.\end{aligned}$$

Many of the terms are zero, if i is too large. Indeed, since $x_1 \notin I^r$ and D_{P_i} is supported on $[0, P_i^{-1})$, if $i \ge \gamma \ge K - r$, then both $D_{P_i}(x_1\dot{-}t)$ and $D_{P_i}(x_1\dot{+}lP_{\gamma+1}^{-1}\dot{-}t)$ are identically zero for $t \in I$ and $0 \le l \le p_\gamma$, which is because under these conditions x_1 and $x_1\dot{+}lP_{\gamma+1}^{-1}$ both lie outside I^r, so the coefficients of

t cannot cancel all of the "lower order" coefficients of x_1 or $x_1 \dot{+} lP_{\gamma+1}^{-1}$. Thus,

$$|(\sigma_{n,m}a)(x_1,x_2)| \le C\beta_{m,-1}^3 |R|^{-1} \int_I \frac{1}{n} \sum_{\gamma=0}^{K-r-1} P_\gamma \sum_{i=\gamma}^{s-1} p_i \Big[p_\gamma D_{P_i}(x_1 \dot{-} t) - p_\gamma \sum_{l=0}^{p_\gamma - 1} D_{P_i}(x_1 \dot{+} lP_{\gamma+1}^{-1} \dot{-} t) \Big] \, dt. \tag{7.3}$$

Next the terms of this estimate are zero, if x_1 is too near to I or too far from I. To see this, notice that since Lebesgue measure is translation invariant with respect to $\dot{-}$. Suppose without loss of generality that $I = [0, P_K^{-1})$. Since $D_{P_i} = P_i \chi[0, P_i^{-1})$ for every $i \in \mathbb{N}$, we have

$$\int_I D_{P_i}(x_1 \dot{+} lP_{\gamma+1}^{-1} \dot{-} t) \, dt = \frac{P_i}{P_K} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_i^{-1})(x_1)}(x_1)$$

for $\gamma \le i \le K-1$ and $x \notin I^r$, where $\xi_{\gamma l} = (p_\gamma - l)P_{\gamma+1}^{-1}$. Also,

$$\int_I D_{P_i}(x_1 \dot{-} t) \, dt = \frac{P_i}{P_K} \chi_{[0, P_i^{-1})}(x_1),$$

where $i \le K-1$ and $x_1 \notin I^r$. To show that this last statement is true, let $y \cong (y_1, y_2, \dots)$, where $y = \sum_{i=1}^{\infty} y_1 P_i^{-1}$ for each $y \in [0,1)$. Hence if $x_1 \dot{-} t \in [0, P_i^{-1})$, then $x_1 \dot{-} t \cong (0, 0, \dots, \alpha_i, \alpha_{i+1}, \dots)$. However, $t \in [0, P_i^{-1})$ and $i < K$, so $1/P_i > 1 > P_K$ and $t \cong (0, 0, \dots, 0, \beta_K, \dots)$. Therefore, $x_1 \cong (0, 0, \dots, 0, \alpha_i, \dots)$ and hence, $x_1 \in [0, P_i^{-1})$.

Also,

$$\int_I D_{P_i}(x_1 \dot{-} t) \, dt = 0$$

if $i \ge K$ and $[0, P_i^{-1}) \subset [0, P_K^{-1})$ for $x_1 \notin I^r$.

$$\int_I D_{P_i}(x_1 \dot{+} lP_{\gamma+1}^{-1} \dot{-} t) \, dt = \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_i^{-1})}(x_1)$$

for $i \geq K$ and $\gamma \leq K-r-1$. This last statement is true, since if $x_1 \dot{+} lP_{\gamma+1}^{-1} \dot{-} t \in [0, P_i^{-1})$, then $x_1 \dot{+} lP_{\gamma+1}^{-1} \dot{-} t \cong (0,0,\ldots,0,\alpha_i,\ldots)$, but $t \cong (0,0,\ldots,\gamma_K,\ldots,\gamma_i,\ldots)$ and $lP_{\gamma+1}^{-1} \cong (0,0,\ldots,0,l,0,\ldots)$ for $\gamma \leq K-r-1$. Therefore, $x_1 \cong (0,0,\ldots,p_\gamma - l,0,0,\ldots,\gamma_K,\ldots,\gamma_{i-1},\beta_i,\ldots)$ and $x_1 \in [\xi_{\gamma l}, \xi_{\gamma l} + P_K^{-1})$.

Therefore, (7.3) becomes

$$|(\sigma_{n,m}a)(x_1,x_2)| \leq C\beta_{m,-1}^3 |R|^{-1} \frac{1}{n} \sum_{\gamma=0}^{K-r-1} P_\gamma \sum_{i=\gamma}^{K-1} p_i p_\gamma \frac{P_i}{P_K} \Big(\chi_{[0,P_i^{-1})}(x_1) + \sum_{l=0}^{p_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_i^{-1})}(x_1) \Big)$$
$$+ C\beta_{m,-1}^3 |R|^{-1} \frac{1}{n} \sum_{\gamma=0}^{K-r-1} P_\gamma \sum_{i=K}^{s-1} p_i p_\gamma \Big(\sum_{l=0}^{p_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_K^{-1})}(x_1) \Big)$$

Now since $n \geq P_{s-1} \geq P_{K-r}$,

$$|(\sigma_{n,m}a)(x_1,x_2)| \leq C\beta_{m,-1}^3 |R|^{-1} \frac{1}{P_{K-r}} \sum_{\gamma=0}^{K-r-1} P_{\gamma+1} \sum_{i=\gamma}^{K-1} \frac{P_{i+1}}{P_K} \Big(\chi_{[0,P_i^{-1})}(x_1) + \sum_{l=0}^{p_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_i^{-1})}(x_1) \Big)$$
$$+ C\beta_{m,-1}^3 |R|^{-1} \frac{1}{P_{s-1}} \sum_{\gamma=0}^{K-r-1} P_{\gamma+1} \sum_{i=K}^{s-1} p_i \Big(\sum_{l=0}^{p_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_K^{-1})}(x_1) \Big)$$

Now recall that $|R|^{-1} = P_K Q_K$ and see that

$$\frac{P_K Q_K}{P_{K-r} Q_{K-r}} = p_{K-r} \cdots p_{K-1} q_{K-r} \cdots q_{K-1} \leq \beta_{n_r,m_r}^{2r}$$

since $s-1 \geq K-r$ and $d-1 \geq K-r$. Also,

$$\beta_{n_r,m_r}^{2r} \leq \beta_{n_r,m_r}^{2r/\delta}$$

since $1/\delta \geq 1$. Also, $\beta_{m,-1}^3 \leq \beta_{m,n}^3 \leq \beta_{m,n}^{3/\delta}$.

Hence,

$$\frac{|(\sigma_{n,m}a)(x_1,x_2)|}{\beta_{m,n}^{4/\delta}\beta_{n_r,m_r}^{2r/\delta}} \le \mathrm{A}+\mathrm{B},$$

where

$$\mathrm{A} = C\frac{P_{K-r}Q_{K-r}}{\beta_{m,n}^{1/\delta}}\sum_{\gamma=0}^{K-r-1}\frac{P_{\gamma+1}}{P_{K-r}}\sum_{i=\gamma}^{K-1}\frac{P_{i+1}}{P_K}\Big(\chi_{[0,P_i^{-1})}(x_1)+\sum_{l=0}^{p_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+P_i^{-1})}(x_1)\Big),$$

and

$$\mathrm{B} = C\frac{P_{K-r}Q_{K-r}}{\beta_{m,n}^{1/\delta}P_{s-1}}\sum_{\gamma=0}^{K-r-1}P_{\gamma+1}\sum_{i=K}^{s-1}p_i\Big(\sum_{l=0}^{p_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+P_K^{-1})}(x_1)\Big).$$

Therefore, we consider the following integral:

$$\int_{[0,1)\setminus I_r}\int_{J_r}\mathrm{A}\,dx_1\,dx_2.$$

Since, $|J^r| = Q_{K-r}^{-1}$, then

$$\begin{aligned}\int_{[0,1)\setminus I_r}\int_{J_r}\mathrm{A}\,dx_1\,dx_2 &\le \frac{C}{\beta_{n,m}^{1/\delta}}\sum_{\gamma=0}^{K-r-1}P_{\gamma+1}\sum_{i=\gamma}^{K-1}\frac{P_{i+1}}{P_K}\Big[\int_{[0,1)}\Big(\chi_{[0,P_i^{-1})}(x_1)\\ &\qquad+\sum_{l=0}^{p_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+P_i^{-1})}(x_1)\Big)\,dx_1\Big]\\ &\le \frac{C}{\beta_{n,m}^{1/\delta}}\sum_{\gamma=0}^{K-r-1}P_{\gamma+1}\sum_{i=\gamma}^{K-1}\frac{P_{i+1}}{P_K}\Big[\frac{p_\alpha+1}{P_i}\Big]\\ &= \frac{C}{\beta_{n,m}^{1/\delta}}\sum_{\gamma=0}^{K-r-1}P_{\gamma+1}\sum_{i=\gamma}^{K-1}\frac{p_i}{P_K}(p_\alpha+1)\end{aligned}$$

However, $(p_i/\beta_{n,m}^{1/\delta}) \leq 1$, and so

$$\begin{aligned}\int_{[0,1)\setminus I_r}\int_{J_r} \mathrm{A}\, dx_1\, dx_2 &\leq C\sum_{\gamma=0}^{K-r-1}\Big(\frac{P_{\gamma+1}}{P_K}(p_\alpha+1)\Big)(K-\gamma)\\ &\leq C\sum_{\gamma=0}^{K-r-1}\Big(\frac{p_\alpha+1}{p_{\alpha+1},\cdots p_{K-1}}\Big)(K-\gamma)\\ &\leq C\sum_{\gamma=0}^{K-2}\Big(\frac{p_\alpha+1}{p_{\alpha+1},\cdots p_{K-1}}\Big)(K-\gamma)\\ &\leq C\sum_{\gamma=0}^{K-2}\Big(\frac{p_\alpha}{p_{\alpha+1},\cdots p_{K-1}}\Big)(K-\gamma)\\ &\qquad + C\sum_{\gamma=0}^{K-2}\Big(\frac{1}{p_{\alpha+1},\cdots p_{K-1}}\Big)(K-\gamma) < \infty\end{aligned}$$

The last inequality is true, since $(p_i : i \in \mathbb{N})$ is $\delta-$strongly quasibounded and $p_i \geq 2$ for every i.

Again $|J^r| = Q_{K-r}^{-1}$, and so

$$\begin{aligned}\int_{[0,1)\setminus I_r}\int_{J_r} \mathrm{B}\, dx_1\, dx_2 &\leq C\frac{P_{K-r}}{\beta_{m,n}^{1/\delta}P_{s-1}}\sum_{\gamma=0}^{K-r-1}P_{\gamma+1}\sum_{i=K}^{s-1}p_i\Big(\int_{[0,1)}\sum_{l=0}^{p_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+P_K^{-1})}(x_1)\,dx_1\Big)\\ &\leq C\frac{P_{K-r}}{\beta_{m,n}^{1/\delta}P_{s-1}}\sum_{\gamma=0}^{K-r-1}P_{\gamma+1}\sum_{i=K}^{s-1}p_i\Big(\frac{p_\alpha}{P_K}\Big)\end{aligned}$$

Since $\frac{p_i}{\beta_{m,n}^{1/\delta}} \leq 1$ and $\frac{P_{K-r}}{P_K} \leq 1$, then

$$\begin{aligned}
\int_{[0,1)\setminus I_r} \int_{J_r} \mathrm{B}\, dx_1\, dx_2 &\leq C \sum_{\gamma=0}^{K-r-1} \frac{P_{\gamma+1}}{P_{s-1}} \sum_{i=K}^{s-1} p_\alpha \\
&\leq C \sum_{\gamma=0}^{K-r-1} \frac{P_{\gamma+1}}{P_{s-1}} \sum_{i=\gamma}^{s-1} p_\alpha \\
&= C \sum_{\gamma=0}^{K-r-1} \frac{P_{\gamma+1}}{P_{s-1}} (p_\alpha)(s-\gamma) \\
&= C \sum_{\gamma=0}^{K-r-1} \Big(\frac{p_\alpha}{p_{\alpha+1}\cdots p_{s-2}}\Big)(s-\gamma) \\
&\leq C \sum_{\gamma=0}^{K-2} \Big(\frac{p_\alpha}{p_{\alpha+1}\cdots p_{s-2}}\Big)(s-\gamma) < \infty
\end{aligned}$$

This last inequality is true by the δ–strongly quasibounded condition.

Therefore there is a constant $M_{1,r}$ such that

$$\int_{[0,1)\setminus I^r} \int_{J^r} |(\sigma_\alpha^\delta a)(x_1,x_2)|\, d(x_1,x_2) \leq M_{1,r}.$$

Step 2. We will show the estimate for integrating over the vertical pieces $I^r \times ([0,1) \setminus J^r)$. This step will follow in almost exactly the same manner as Step 1. However, it will take place over the horizontal set of values and coordinates $q = (q_i : i \in \mathbb{N})$. First let $(x_1, x_2) \in I^r \times ([0,1) \setminus J^r$. By given results $\| K_n \|_1 \leq C\beta_{-1,n}^3$ for some $C \in \mathbb{N}$ and $P_{s-1} \leq n < P_s$. Since a is supported on $R = I \times J$ and $|a| \leq |R|^{-1}$, it follows from (7.1) that

$$\begin{aligned}
|(\sigma_{n,m}a)(x_1,x_2)| &\leq \int_J \int_I |a(t,u)||K_n(x_1 \dot{-} t)||K_m(x_2 \dot{-} u)|\, dt\, du \\
&\leq C\beta_{-1,n}^3 |R|^{-1} \int_I |K_m(x_2 \dot{-} u)|\, du
\end{aligned}$$

Recall that $m = \sum_{\theta=0}^{\infty} m_\theta Q_{\theta+1}$ for $0 \leq m_\theta < q_\theta$. Using the estimate for $K_m(x)$ from Lemma 6.4, we can get an estimate for $|K_m(x)|$. This estimate can be

substituted into the last inequality with the following result.

$$\begin{aligned}
|(\sigma_{n,m}a)(x_1,x_2)| &\le C\beta_{-1,n}^3|R|^{-1}\int_J \frac{1}{m}\sum_{\gamma=0}^{d-1} Q_\gamma \sum_{i=\gamma}^{d-1} m_i \Big[m_\gamma D_{Q_i}(x_2 \dot{-} u) \\
&\qquad - \frac{1}{q_\gamma}\sum_{k=0}^{q_\gamma-1} k \sum_{l=0}^{q_\gamma-1} D_{Q_i}(x_2 \dot{+} lQ_{\gamma+1}^{-1} \dot{-} u)\Big]\, du \\
&\le C\beta_{-1,n}^3|R|^{-1}\int_J \frac{1}{m}\sum_{\gamma=0}^{d-1} Q_\gamma \sum_{i=\gamma}^{d-1} q_i \Big[q_\gamma D_{Q_i}(x_2 \dot{-} u) \\
&\qquad - \frac{1}{q_\gamma}\sum_{k=0}^{q_\gamma-1} k \sum_{l=0}^{q_\gamma-1} D_{Q_i}(x_2 \dot{+} lQ_{\gamma+1}^{-1} \dot{-} u)\Big]\, du \\
&\le C\beta_{-1,n}^3|R|^{-1}\int_J \frac{1}{m}\sum_{\gamma=0}^{d-1} Q_\gamma \sum_{i=\gamma}^{d-1} q_i \Big[q_\gamma D_{Q_i}(x_2 \dot{-} u) \\
&\qquad - q_\gamma \sum_{l=0}^{q_\gamma-1} D_{Q_i}(x_2 \dot{+} lQ_{\gamma+1}^{-1} \dot{-} u)\Big]\, du.
\end{aligned}$$

Again, many of the terms are zero, if i is too large. Indeed, since $x_2 \notin J^r$ and D_{Q_i} is supported on $[0, Q_i^{-1})$, if $i \ge \gamma \ge K - r$, then both $D_{Q_i}(x_2 \dot{-} u)$ and $D_{Q_i}(x_2 \dot{+} lQ_{\gamma+1}^{-1} \dot{-} u)$ are identically zero for $u \in J$ and $0 \le l \le q_\gamma$, which is because under these conditions x_2 and $x_2 \dot{+} lQ_{\gamma+1}^{-1}$ both lie outside J^r, so the coefficients of u cannot cancel all of the "lower order" coefficients of x_2 or $x_2 \dot{+} lQ_{\gamma+1}^{-1}$. Thus,

$$\begin{aligned}
|(\sigma_{n,m}a)(x_1,x_2)| &\le C\beta_{-1,n}^3|R|^{-1}\int_J \frac{1}{m}\sum_{\gamma=0}^{K-r-1} Q_\gamma \sum_{i=\gamma}^{d-1} q_i \Big[q_\gamma D_{Q_i}(x_2 \dot{-} u) \\
&\qquad - q_\gamma \sum_{l=0}^{q_\gamma-1} D_{Q_i}(x_2 \dot{+} lQ_{\gamma+1}^{-1} \dot{-} u)\Big]\, du. \qquad (7.4)
\end{aligned}$$

Next the terms of this estimate are zero, if x_2 is too near to J or too far from J. To see this, notice that since Lebesgue measure is translation invariant with respect to $\dot{-}$. Suppose without loss of generality that $J = [0, Q_K^{-1})$. Since

$D_{Q_i} = Q_i \chi[0, Q_i^{-1})$ for every $i \in \mathbb{N}$, we have

$$\int_J D_{Q_i}(x_2 \dot{+} l Q_{\gamma+1}^{-1} \dot{-} u)\, du = \frac{Q_i}{Q_K} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + Q_i^{-1})(x_2)}(x_2)$$

for $\gamma \leq i \leq K-1$ and $x_2 \notin J^r$, where $\xi_{\gamma l} = (q_\gamma - l) Q_{\gamma+1}^{-1}$. Also,

$$\int_J D_{Q_i}(x_2 \dot{-} u)\, du = \frac{Q_i}{Q_K} \chi_{[0, Q_i^{-1})}(x_2),$$

where $i \leq K-1$ and $x_1 \notin J^r$. To show that this last statement is true, let $y \cong (y_1, y_2, \dots)$, where $y = \sum_{i=1}^{\infty} y_1 Q_i^{-1}$ for each $y \in [0,1)$. Hence if $x_2 \dot{-} u \in [0, Q_i^{-1})$, then $x_1 \dot{-} u \cong (0, 0, \dots, \alpha_i, \alpha_{i+1}, \dots)$. However, $u \in [0, Q_i^{-1})$ and $i < K$, so $1/Q_i > 1 > Q_K$ and $u \cong (0, 0, \dots, 0, \beta_K, \dots)$. Therefore, $x_2 \cong (0, 0, \dots, 0, \alpha_i, \dots)$ and hence, $x_2 \in [0, Q_i^{-1})$.

Also,

$$\int_J D_{Q_i}(x_2 \dot{-} u)\, du = 0$$

if $i \geq K$ and $[0, Q_i^{-1}) \subset [0, Q_K^{-1})$ for $x_2 \notin J^r$.

$$\int_J D_{Q_i}(x_2 \dot{+} l Q_{\gamma+1}^{-1} \dot{-} u)\, du = \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + Q_i^{-1})}(x_2)$$

for $i \geq K$ and $\gamma \leq K - r - 1$. This last statement is true, since if $x_2 \dot{+} l Q_{\gamma+1}^{-1} \dot{-} u \in [0, Q_i^{-1})$, then $x_2 \dot{+} l Q_{\gamma+1}^{-1} \dot{-} u \cong (0, 0, \dots, 0, \alpha_i, \dots)$, but $u \cong (0, 0, \dots, \gamma_K, \dots, \gamma_i, \dots)$ and $l Q_{\gamma+1}^{-1} \cong (0, 0, \dots, 0, l, 0, \dots)$ for $\gamma \leq K - r - 1$. Therefore, $x_2 \cong (0, 0, \dots, q_\gamma - l, 0, 0, \dots, \gamma_K, \dots, \gamma_{i-1}, \beta_i, \dots)$ and $x_2 \in [\xi_{\gamma l}, \xi_{\gamma l} + Q_K^{-1})$.

Therefore, (7.4) becomes

$$|(\sigma_{n,m}a)(x_1,x_2)| \le C\beta_{-1,n}^3|R|^{-1}\frac{1}{m}\sum_{\gamma=0}^{K-r-1}Q_\gamma\sum_{i=\gamma}^{K-1}q_iq_\gamma\frac{Q_i}{Q_K}\Big(\chi_{[0,Q_i^{-1})}(x_2) + \sum_{l=0}^{q_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+Q_i^{-1})}(x_2)\Big)$$
$$+ C\beta_{-1,n}^3|R|^{-1}\frac{1}{m}\sum_{\gamma=0}^{K-r-1}Q_\gamma\sum_{i=K}^{d-1}q_iq_\gamma\Big(\sum_{l=0}^{q_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+Q_K^{-1})}(x_2)\Big)$$

Now since $m \ge Q_{d-1} \ge Q_{K-r}$,

$$|(\sigma_{n,m}a)(x_1,x_2)| \le C\beta_{-1,n}^3|R|^{-1}\frac{1}{Q_{K-r}}\sum_{\gamma=0}^{K-r-1}Q_{\gamma+1}\sum_{i=\gamma}^{K-1}\frac{Q_{i+1}}{Q_K}\Big(\chi_{[0,Q_i^{-1})}(x_2) + \sum_{l=0}^{q_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+Q_i^{-1})}(x_2)\Big)$$
$$+ C\beta_{-1,n}^3|R|^{-1}\frac{1}{Q_{d-1}}\sum_{\gamma=0}^{K-r-1}Q_{\gamma+1}\sum_{i=K}^{d-1}q_i\Big(\sum_{l=0}^{q_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+Q_K^{-1})}(x_2)\Big)$$

Now recall that $|R|^{-1} = P_KQ_K$ and observe

$$\frac{P_KQ_K}{P_{K-r}Q_{K-r}} = p_{K-r}\cdots p_{K-1}q_{K-r}\cdots q_{K-1} \le \beta_{n_r,m_r}^{2r}$$

since $s-1 \ge K-r$ and $d-1 \ge K-r$. Also,

$$\beta_{n_r,m_r}^{2r} \le \beta_{n_r,m_r}^{2r/\delta}$$

since $1/\delta \ge 1$. Also, $\beta_{-1,n}^3 \le \beta_{m,n}^3 \le \beta_{m,n}^{3/\delta}$.

Hence,

$$\frac{|(\sigma_{n,m}a)(x_1,x_2)|}{\beta_{m,n}^{4/\delta}\beta_{n_r,m_r}^{2r/\delta}} \le \mathrm{A}+\mathrm{B},$$

where

$$\mathrm{A} = C\frac{P_{K-r}Q_{K-r}}{\beta_{m,n}^{1/\delta}}\sum_{\gamma=0}^{K-r-1}\frac{Q_{\gamma+1}}{Q_{K-r}}\sum_{i=\gamma}^{K-1}\frac{Q_{i+1}}{Q_K}\Big(\chi_{[0,Q_i^{-1})}(x_2)+\sum_{l=0}^{q_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+q_i^{-1})}(x_2)\Big),$$

and

$$\mathrm{B} = C\frac{P_{K-r}Q_{K-r}}{\beta_{m,n}^{1/\delta}Q_{d-1}}\sum_{\gamma=0}^{K-r-1}Q_{\gamma+1}\sum_{i=K}^{d-1}q_i\Big(\sum_{l=0}^{q_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+Q_K^{-1})}(x_2)\Big).$$

Therefore, we consider the following integral:

$$\int_{I_r}\int_{[0,1)\setminus J_r}\mathrm{A}\,dx_1\,dx_2.$$

Since, $|I^r| = P_{K-r}^{-1}$, then

$$\begin{aligned}\int_{I_r}\int_{[0,1)\setminus J_r}\mathrm{A}\,dx_1\,dx_2 &\le \frac{C}{\beta_{n,m}^{1/\delta}}\sum_{\gamma=0}^{K-r-1}Q_{\gamma+1}\sum_{i=\gamma}^{K-1}\frac{Q_{i+1}}{Q_K}\Big[\int_{[0,1)}\Big(\chi_{[0,Q_i^{-1})}(x_2)\\ &\qquad+\sum_{l=0}^{q_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+Q_i^{-1})}(x_2)\Big)\,dx_2\Big]\\ &\le \frac{C}{\beta_{n,m}^{1/\delta}}\sum_{\gamma=0}^{K-r-1}Q_{\gamma+1}\sum_{i=\gamma}^{K-1}\frac{Q_{i+1}}{Q_K}\Big[\frac{q_\alpha+1}{Q_i}\Big]\\ &= \frac{C}{\beta_{n,m}^{1/\delta}}\sum_{\gamma=0}^{K-r-1}Q_{\gamma+1}\sum_{i=\gamma}^{K-1}\frac{q_i}{Q_K}(q_\alpha+1)\end{aligned}$$

However, $(q_i/\beta_{n,m}^{1/\delta}) \le 1$, and so

$$\begin{aligned}\int_{I_r}\int_{[0,1)\setminus J_r}\mathrm{A}\,dx_1\,dx_2 &\le C\sum_{\gamma=0}^{K-r-1}\Big(\frac{Q_{\gamma+1}}{Q_K}(q_\alpha+1)\Big)(K-\gamma)\\ &\le C\sum_{\gamma=0}^{K-r-1}\Big(\frac{q_\alpha+1}{q_{\alpha+1},\dots q_{K-1}}\Big)(K-\gamma)\\ &\le C\sum_{\gamma=0}^{K-2}\Big(\frac{q_\alpha+1}{q_{\alpha+1},\dots q_{K-1}}\Big)(K-\gamma)\\ &\le C\sum_{\gamma=0}^{K-2}\Big(\frac{q_\alpha}{q_{\alpha+1},\dots q_{K-1}}\Big)(K-\gamma)\\ &\qquad+C\sum_{\gamma=0}^{K-2}\Big(\frac{1}{q_{\alpha+1},\dots q_{K-1}}\Big)(K-\gamma)<\infty\end{aligned}$$

The last inequality is true, since $(q_i : i \in \mathbb{N})$ is $\delta-$strongly quasibounded and $q_i \geq 2$ for every i.

Again $|I^r| = P_{K-r}^{-1}$, so

$$\int_{I_r}\int_{[0,1)\setminus J_r} \mathrm{B}\, dx_1\, dx_2 \leq C\frac{Q_{K-r}}{\beta_{m,n}^{1/\delta} Q_{d-1}} \sum_{\gamma=0}^{K-r-1} Q_{\gamma+1} \sum_{i=K}^{d-1} q_i \Big(\int_{[0,1)} \sum_{l=0}^{q_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + Q_K^{-1})}(x_2)\, dx_2 \Big)$$
$$\leq C\frac{Q_{K-r}}{\beta_{m,n}^{1/\delta} Q_{d-1}} \sum_{\gamma=0}^{K-r-1} Q_{\gamma+1} \sum_{i=K}^{d-1} q_i \Big(\frac{q_\alpha}{Q_K} \Big)$$

Since $\frac{q_i}{\beta_{m,n}^{1/\delta}} \leq 1$ and $\frac{Q_{K-r}}{Q_K} \leq 1$, then

$$\begin{aligned}
\int_{I_r}\int_{[0,1)\setminus J_r} \mathrm{B}\, dx_1\, dx_2 &\leq C \sum_{\gamma=0}^{K-r-1} \frac{Q_{\gamma+1}}{Q_{d-1}} \sum_{i=K}^{d-1} q_\alpha \\
&\leq C \sum_{\gamma=0}^{K-r-1} \frac{Q_{\gamma+1}}{Q_{d-1}} \sum_{i=\gamma}^{d-1} q_\alpha \\
&= C \sum_{\gamma=0}^{K-r-1} \frac{Q_{\gamma+1}}{Q_{d-1}} (q_\alpha)(d-\gamma) \\
&= C \sum_{\gamma=0}^{K-r-1} \Big(\frac{q_\alpha}{q_{\alpha+1}\cdots q_{d-2}} \Big)(d-\gamma) \\
&\leq C \sum_{\gamma=0}^{K-2} \Big(\frac{q_\alpha}{q_{\alpha+1}\cdots q_{d-2}} \Big)(d-\gamma) < \infty
\end{aligned}$$

This last inequality is true by the $\delta-$strongly quasibounded condition.

Therefore there is a constant $M_{2,r}$ such that

$$\int_{I^r}\int_{[0,1)\setminus J^r} |(\sigma_\alpha^\delta a)(x_1, x_2)|\, d(x_1, x_2) \leq M_{2,r}.$$

Step 3. In this last step we will develop an estimate for integrating over the four rectangular pieces, $([0,1)\setminus I_r)\times([0,1)\setminus J_r)$. Since a is supported on $R = I \times J$

and $|a| \le |R|^{-1}$, it follows from (7.1) that

$$\begin{aligned}|(\sigma_{n,m}a)(x_1,x_2)| &\le \int_J\int_I |a(t,u)||K_n(x_1\dot{-}t)||K_m(x_2\dot{-}u)|\,dt\,du \\ &\le C|R|^{-1}\int_J |K_n(x_1\dot{-}t)|\,dt \int_I |K_m(x_2\dot{-}u)|\,du\end{aligned}$$

Recall that $m = \sum_{\theta=0}^{\infty} m_\theta Q_{\theta+1}$ and $n = \sum_{\theta=0}^{\infty} n_\theta P_{\theta+1}$ for $0 \le m_\theta < q_\theta$ and $0 \le n_\theta < p_\theta$. Using the decompositions of $K_m(x)$ and $K_n(x)$ given by Lemma 6.4, we can get an estimate for $|K_m(x)|$ and $|K_n(x)|$. These estimates can be substituted into the last inequality with the following result.

$$\begin{aligned}|(\sigma_{n,m}a)(x_1,x_2)| \le C|R|^{-1}\Big(&\int_I \frac{1}{n}\sum_{\gamma=0}^{s-1} P_\gamma \sum_{i=\gamma}^{s-1} n_i\Big[n_\gamma D_{P_i}(x_1\dot{-}t) \\ &- \frac{1}{p_\gamma}\sum_{k=0}^{p_\gamma-1} k \sum_{l=0}^{p_\gamma-1} D_{P_i}(x_1\dot{+}lP_{\gamma+1}^{-1}\dot{-}t)\Big]\,dt\Big) \\ &\Big(\int_J \frac{1}{m}\sum_{\gamma=0}^{d-1} Q_\gamma \sum_{i=\gamma}^{d-1} m_i\Big[m_\gamma D_{Q_i}(x_2\dot{-}u) \\ &- \frac{1}{q_\gamma}\sum_{k=0}^{q_\gamma-1} k \sum_{l=0}^{q_\gamma-1} D_{Q_i}(x_2\dot{+}lQ_{\gamma+1}^{-1}\dot{-}u)\Big]\,du\Big) \\ \le C|R|^{-1}\Big(&\int_I \frac{1}{n}\sum_{\gamma=0}^{s-1} P_\gamma \sum_{i=\gamma}^{s-1} p_i\Big[p_\gamma D_{P_i}(x_1\dot{-}t) \\ &- p_\gamma\sum_{k=0}^{p_\gamma-1} k \sum_{l=0}^{p_\gamma-1} D_{P_i}(x_1\dot{+}lP_{\gamma+1}^{-1}\dot{-}t)\Big]\,dt\Big) \\ &\Big(\int_J \frac{1}{m}\sum_{\gamma=0}^{d-1} Q_\gamma \sum_{i=\gamma}^{d-1} q_i\Big[q_\gamma D_{Q_i}(x_2\dot{-}u) \\ &- q_\gamma\sum_{k=0}^{q_\gamma-1} k \sum_{l=0}^{q_\gamma-1} D_{Q_i}(x_2\dot{+}lQ_{\gamma+1}^{-1}\dot{-}u)\Big]\,du\Big).\end{aligned}$$

As in Steps 1 and 2, many of the terms are zero, if i is too large. Thus,

$$|(\sigma_{n,m}a)(x_1,x_2)| \le C|R|^{-1}\Big(\int_I \frac{1}{n}\sum_{\gamma=0}^{K-r-1} P_\gamma \sum_{i=\gamma}^{s-1} p_i \Big[p_\gamma D_{P_i}(x_1 \dot{-} t) - p_\gamma \sum_{l=0}^{p_\gamma-1} D_{P_i}(x_1 \dot{+} lP_{\gamma+1}^{-1} \dot{-} t)\Big]\, dt\Big) \Big(\int_J \frac{1}{m}\sum_{\gamma=0}^{K-r-1} Q_\gamma \sum_{i=\gamma}^{d-1} q_i \Big[q_\gamma D_{Q_i}(x_2 \dot{-} u) - q_\gamma \sum_{l=0}^{q_\gamma-1} D_{Q_i}(x_2 \dot{+} lQ_{\gamma+1}^{-1} \dot{-} u)\Big]\, du\Big). \tag{7.5}$$

As in Steps 1 and 2, the terms of this estimate are zero if x_1 is too close to I or too far from I, or if x_2 is too close to J or too far from J. Suppose without loss of generality that $I = [0, P_K^{-1})$. Since $D_{P_i} = P_i\chi[0, P_i^{-1})$ for every $i \in \mathbb{N}$, we have

$$\int_I D_{P_i}(x_1 \dot{+} lP_{\gamma+1}^{-1} \dot{-} t)\, dt = \frac{P_i}{P_K}\chi_{[\xi_{\gamma l}, \xi_{\gamma l}+P_i^{-1})(x_1)}(x_1)$$

for $\gamma \le i \le K-1$ and $x \notin I^r$, where $\xi_{\gamma l} = (p_\gamma - l)P_{\gamma+1}^{-1}$. Also,

$$\int_I D_{P_i}(x_1 \dot{-} t)\, dt = \frac{P_i}{P_K}\chi_{[0,P_i^{-1})}(x_1),$$

where $i \le K-1$ and $x_1 \notin I^r$. Similarly as before, suppose without loss of generality that $J = [0, Q_K^{-1})$. Since $D_{Q_i} = Q_i\chi[0, Q_i^{-1})$ for every $i \in \mathbb{N}$, we have

$$\int_J D_{Q_i}(x_2 \dot{+} lQ_{\gamma+1}^{-1} \dot{-} u)\, du = \frac{Q_i}{Q_K}\chi_{[\xi_{\gamma l}, \xi_{\gamma l}+Q_i^{-1})(x_2)}(x_2)$$

for $\gamma \le i \le K-1$ and $x_2 \notin J^r$, where $\xi_{\gamma l} = (q_\gamma - l)Q_{\gamma+1}^{-1}$. Also,

$$\int_J D_{Q_i}(x_2 \dot{-} u)\, du = \frac{Q_i}{Q_K}\chi_{[0,Q_i^{-1})}(x_2),$$

where $i \le K-1$ and $x_1 \notin J^r$.

Also as in Step 1,

$$\int_I D_{P_i}(x_1 \dot{-} t)\, dt = 0$$

if $i \geq K$ and $[0, P_i^{-1}) \subset [0, P_K^{-1})$ for $x_1 \notin I^r$.

$$\int_I D_{P_i}(x_1 \dot{+} l P_{\gamma+1}^{-1} \dot{-} t)\, dt = \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_i^{-1})}(x_1)$$

for $i \geq K$ and $\gamma \leq K - r - 1$. Similarly as in Step 2.,

$$\int_J D_{Q_i}(x_2 \dot{-} u)\, du = 0$$

if $i \geq K$ and $[0, Q_i^{-1}) \subset [0, Q_K^{-1})$ for $x_2 \notin J^r$.

$$\int_J D_{Q_i}(x_2 \dot{+} l Q_{\gamma+1}^{-1} \dot{-} u)\, du = \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + Q_i^{-1})}(x_2)$$

for $i \geq K$ and $\gamma \leq K - r - 1$.

Therefore, 7.5 becomes

$$\begin{aligned}
|(\sigma_{n,m}a)(x_1, x_2)| \leq C|R|^{-1} \Big[\frac{1}{n} \sum_{\gamma=0}^{K-r-1} P_\gamma \sum_{i=\gamma}^{K-1} p_i p_\gamma \frac{P_i}{P_K} \Big(\chi_{[0, P_i^{-1})}(x_1) \\
+ \sum_{l=0}^{p_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_i^{-1})}(x_1) \Big) \\
+ \frac{1}{n} \sum_{\gamma=0}^{K-r-1} P_\gamma \sum_{i=K}^{s-1} p_i p_\gamma \Big(\sum_{l=0}^{p_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_K^{-1})}(x_1) \Big) \Big] \\
\Big[\frac{1}{m} \sum_{\gamma=0}^{K-r-1} Q_\gamma \sum_{i=\gamma}^{K-1} q_i q_\gamma \frac{Q_i}{Q_K} \Big(\chi_{[0, Q_i^{-1})}(x_2) \\
+ \sum_{l=0}^{q_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + Q_i^{-1})}(x_2) \Big) \\
+ \frac{1}{m} \sum_{\gamma=0}^{K-r-1} Q_\gamma \sum_{i=K}^{d-1} q_i q_\gamma \Big(\sum_{l=0}^{q_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + Q_K^{-1})}(x_2) \Big) \Big].
\end{aligned}$$

Now since $n \geq P_{s-1} \geq P_{K-r}$ and $m \geq Q_{d-1} \geq Q_{K-r}$,

$$
\begin{aligned}
|(\sigma_{n,m}a)(x_1,x_2)| \leq C|R|^{-1}\Big[&\frac{1}{P_{K-r}}\sum_{\gamma=0}^{K-r-1} P_{\gamma+1}\sum_{i=\gamma}^{K-1}\frac{P_{i+1}}{P_K}\Big(\chi_{[0,P_i^{-1})}(x_1) \\
&\qquad + \sum_{l=0}^{p_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+P_i^{-1})}(x_1)\Big) \\
&+ \frac{1}{P_{s-1}}\sum_{\gamma=0}^{K-r-1} P_{\gamma+1}\sum_{i=K}^{s-1} p_i\Big(\sum_{l=0}^{p_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+P_K^{-1})}(x_1)\Big)\Big] \\
\Big[&\frac{1}{Q_{K-r}}\sum_{\gamma=0}^{K-r-1} Q_{\gamma+1}\sum_{i=\gamma}^{K-1}\frac{Q_{i+1}}{Q_K}\Big(\chi_{[0,Q_i^{-1})}(x_2) \\
&\qquad + \sum_{l=0}^{q_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+Q_i^{-1})}(x_2)\Big) \\
&+ \frac{1}{Q_{d-1}}\sum_{\gamma=0}^{K-r-1} Q_{\gamma+1}\sum_{i=K}^{d-1} q_i\Big(\sum_{l=0}^{q_\gamma-1}\chi_{[\xi_{\gamma l},\xi_{\gamma l}+Q_K^{-1})}(x_2)\Big)\Big].
\end{aligned}
$$

Now recall that $|R|^{-1} = P_K Q_K$ and observe

$$\frac{P_K Q_K}{P_{K-r}Q_{K-r}} = p_{K-r}\cdots p_{K-1}q_{K-r}\cdots q_{K-1} \leq \beta_{n_r,m_r}^{2r}$$

since $s-1 \geq K-r$ and $d-1 \geq K-r$. Also,

$$\beta_{n_r,m_r}^{2r} \leq \beta_{n_r,m_r}^{2r/\delta}$$

since $1/\delta \geq 1$.

Hence,

$$\frac{|(\sigma_{n,m}a)(x_1,x_2)|}{\beta_{m,n}^{4/\delta}\beta_{n_r,m_r}^{2r/\delta}} \leq C\frac{P_{K-r}Q_{K-r}}{\beta_{m,n}^{4/\delta}}\Big(\mathrm{A}+\mathrm{B}\Big)\Big(\mathrm{C}+\mathrm{D}\Big),$$

where

$$\mathrm{A} = \sum_{\gamma=0}^{K-r-1} \frac{P_{\gamma+1}}{P_{K-r}} \sum_{i=\gamma}^{K-1} \frac{P_{i+1}}{P_K} \Big(\chi_{[0,P_i^{-1})}(x_1) + \sum_{l=0}^{p_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_i^{-1})}(x_1) \Big),$$

$$\mathrm{B} = \frac{1}{P_{s-1}} \sum_{\gamma=0}^{K-r-1} P_{\gamma+1} \sum_{i=K}^{s-1} p_i \Big(\sum_{l=0}^{p_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + P_K^{-1})}(x_1) \Big),$$

$$\mathrm{C} = \sum_{\gamma=0}^{K-r-1} \frac{Q_{\gamma+1}}{Q_{K-r}} \sum_{i=\gamma}^{K-1} \frac{Q_{i+1}}{Q_K} \Big(\chi_{[0,Q_i^{-1})}(x_2) + \sum_{l=0}^{q_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + q_i^{-1})}(x_2) \Big),$$

and

$$\mathrm{D} = \frac{1}{Q_{d-1}} \sum_{\gamma=0}^{K-r-1} Q_{\gamma+1} \sum_{i=K}^{d-1} q_i \Big(\sum_{l=0}^{q_\gamma - 1} \chi_{[\xi_{\gamma l}, \xi_{\gamma l} + Q_K^{-1})}(x_2) \Big).$$

Next we consider the following integral:

$$\frac{P_{K-r}}{\beta_{m,n}^{1/\delta}} \int_{[0,1)\setminus I_r} (\mathrm{A} + \mathrm{B})\, dx_1.$$

However, from Step 1 we know that this integral is bounded by $M_{1,r}$. Hence,

$$\frac{P_{K-r}}{\beta_{m,n}^{1/\delta}} \int_{[0,1)\setminus I_r} (\mathrm{A} + \mathrm{B})\, dx_1 \leq M_{1,r}.$$

Next consider the other integral:

$$\frac{Q_{K-r}}{\beta_{m,n}^{1/\delta}} \int_{[0,1)\setminus J_r} (\mathrm{C} + \mathrm{D})\, dx_2.$$

However, from Step 2 we know that this integral is bounded by $M_{2,r}$. Hence,

$$\frac{Q_{K-r}}{\beta_{m,n}^{1/\delta}} \int_{[0,1)\setminus J_r} (\mathrm{C} + \mathrm{D})\, dx_2 \leq M_{2,r}.$$

Therefore, since $1/\beta_{m,n}^{2/\delta} \leq 1$

$$\int_{[0,1)\setminus I_r} \int_{[0,1)\setminus J_r} |(a_\alpha^\delta)(x_1, x_2)|\, dx_1\, dx_2 \leq C(M_{1,r} \times M_{2,r}) \leq C_{1,r},$$

for some $C_{1,r}$. In fact, $C_{1,r}$ will work as a bound in all three estimates.

Now that we have established the estimates from Steps 1, 2, and 3, it follows that σ_α^δ is 1-quasilocal for each $\delta \leq 1$ and by Lemma (5.1) the theorem is proved.

□

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VITA

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