FOURIER INTEGRAL APPROACH TO THE STUDY OF TRANSMISSION LINE PHENOMENA

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FOURIER INTEGRAL APPROACH TO THE STUDY OF TRANSMISSION LINE PHENOMENA

This work, as the title suggests, gives the Fourier approach to the study of transmission line phenomena. Although the field of steadistate performance is well developed, very little information is available on the transient performance. The first work was done along this line probably by Heaviside who developed a short cut for evaluating Fourier's integral.

The scope of this work is only to lay a foundation for future detailed study by means of a basic approach through the application of Fourier's integral. The general theory is to take the general transmission line set up and by applying Fourier's integral develope relations for voltage and current in terms of the driving force or charge and the frequency, time, and distance from the point of application. It is shown that the resulting current and voltage are traveling waves, travelling both to the right and to the left of the point of driving force application.

This work is extended through the derivation of the general relations for voltage and current and the application of one special case as an example. It will be interesting to note in this application the special limitations used. There is no end to the number of special cases which might be applied.

The writer believes that an extended study based upon this basic approach will make great strides toward the understanding of transients on transmission lines resulting from lightning and static electricity charges.

A transmission line may be considered as a series of four-terminal networks with the following constants:

L - Inductance per mile along the line measured in henries per mile.

R - Resistance per mile along the line measured in ohms per mile.

G - Leakage conductance across the line measured in mhos per mile.

C - Capacity per mile across line measured in farads per mile.

These constants may be set up in the following T-section:

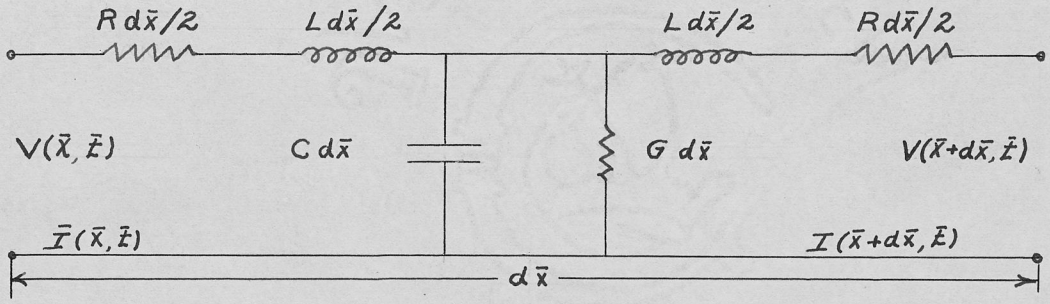


Figure 1

Thus at the beginning of this network of incremental length, $d\bar{x}$, the voltage and current are functions of the distance, $\bar{x}$, and time, $\bar{t}$, and at the other end of the network they are functions of $\bar{x}$ and $d\bar{x}$, and of $\bar{t}$.

Both of these relations may be expanded by Taylor's series as follows:

$$\bar{I}(\bar{x}+d\bar{x},t) = \bar{I}(\bar{x},\bar{t}) + \left[\frac{\partial \bar{I}(\bar{x},\bar{t})}{\partial \bar{x}}\right] d\bar{x} + \ldots \qquad (1)$$

$$V(\bar{x}+d\bar{x},\bar{t}) = V(\bar{x},\bar{t}) + \left[\frac{\partial V(\bar{x},\bar{t})}{\partial \bar{x}}\right] d\bar{x} + \ldots \qquad (2)$$

These relations are purely mathematical. The terms following the second are so comparatively small they are neglected. $\left[\frac{\partial \bar{I}(\bar{x},\bar{t})}{\partial \bar{x}}\right]d\bar{x}$ represents the current bipassed by the incremental section of line and is equal to $-\Delta \bar{I}$; likewise $\left[\frac{\partial V(\bar{x},\bar{t})}{\partial \bar{x}}\right]d\bar{x}$ is the voltage drop along the section and is equal to $-\Delta V$.

From a physical concept the following relations are obtained:

$$\Delta I = G\,d\bar{x}\,V(\bar{x},\bar{t}) + C\,d\bar{x}\,\frac{\partial V(\bar{x},\bar{t})}{\partial \bar{t}} \qquad (3)$$

$$\Delta V = R\,d\bar{x}\,I(\bar{x},\bar{t}) + L\,d\bar{x}\,\frac{\partial I(\bar{x},\bar{t})}{\partial \bar{t}} \qquad (4)$$

$$[I = GV,\ I = C\,dV/dt,\ V = RI,\ V = L\,dI/dt]$$

$V(\bar{x},\bar{t})$ and $\bar{I}(\bar{x},\bar{t})$ are not exactly correct values for the points of bi-pass in the T - section, differing by $d\bar{x}$ which would introduce $(d\bar{x})^2$. Since $d\bar{x}$ itself is incremental, $(d\bar{x})^2$ would be smaller and is therefore disregarded.

Combining (1) and (3) and cancelling $d\bar{x}$ and rearranging:

$$\Delta \bar{I} = G\,d\bar{x}\,V(\bar{x},\bar{t}) + C\,d\bar{x}\,\frac{\partial V(\bar{x},\bar{t})}{\partial \bar{t}} = -\frac{\partial \bar{I}(\bar{x},\bar{t})\,d\bar{x}}{\partial \bar{x}}$$

$$-\frac{\partial \bar{I}(\bar{x},\bar{t})}{\partial \bar{x}} = G\,V(\bar{x},\bar{t}) + C\,\frac{\partial V(\bar{x},\bar{t})}{\partial \bar{t}} \qquad (5)$$

Combining (2) and (4) and cancelling $d\bar{x}$ and rearranging:

$$\Delta V = R\,d\bar{x}\,\bar{I}(\bar{x},\bar{t}) + L\,d\bar{x}\,\frac{\partial \bar{I}(\bar{x},\bar{t})}{\partial \bar{t}} = -\frac{\partial V(\bar{x},\bar{t})\,d\bar{x}}{\partial \bar{x}}$$

$$-\frac{\partial V(\bar{x},\bar{t})}{\partial \bar{x}} = R\,\bar{I}(\bar{x},\bar{t}) + L\,\frac{\partial \bar{I}(\bar{x},\bar{t})}{\partial \bar{t}} \qquad (6)$$

Let the variables in equations (5) and (6) be expressed in terms of factors of proportionality:

$$I(x,t) = \sqrt{\frac{L}{C}}\ \bar{I}(\bar{x},\bar{t})$$

$$x = R\sqrt{\frac{C}{L}}\ \bar{x}$$

$$t = R/L\ \bar{t}$$

Then by cogwheel differentiation the following relations are obtained:

(1) $$\frac{\partial V}{\partial \bar{x}} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial \bar{x}} = \frac{\partial V}{\partial x}\cdot R\sqrt{\frac{C}{L}}$$

(2) $$\frac{\partial \bar{I}}{\partial \bar{t}} = \frac{\partial \bar{I}}{\partial t}\cdot\frac{\partial t}{\partial \bar{t}} = \frac{\partial I}{\sqrt{\frac{L}{C}}\,\partial t}\cdot\frac{R}{L} = \frac{R}{L}\cdot\frac{1}{\sqrt{\frac{L}{C}}}\frac{\partial I}{\partial t}$$

(3) $$\frac{\partial \bar{I}}{\partial \bar{x}} = \frac{\partial \bar{I}}{\partial x}\cdot\frac{\partial x}{\partial \bar{x}} = \frac{1}{\sqrt{\frac{L}{C}}}\frac{\partial I}{\partial x}\times R\sqrt{\frac{C}{L}} = \frac{RC}{L}\frac{\partial I}{\partial x}$$

(4) $$\frac{\partial V}{\partial \bar{t}} = \frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial \bar{t}} = \frac{\partial V}{\partial t}\cdot\frac{R}{L}$$

Substituting these values in equations (5) and (6):

(5)- $$-\frac{RC}{L}\frac{\partial I(x,t)}{\partial x} = G\,V(x,t) + \frac{RC}{L}\frac{\partial V(x,t)}{\partial t}$$

$$-\frac{\partial I(x,t)}{\partial x} = \frac{GL}{RC}\,V(x,t) + \frac{\partial V(x,t)}{\partial t} \qquad (5a)$$

(6)- $$-R\sqrt{\frac{C}{A}}\frac{\partial V(x,t)}{\partial x} = \frac{R}{\sqrt{\frac{L}{C}}}I(x,t) + \frac{A}{A}R\frac{1}{\sqrt{\frac{L}{C}}}\frac{\partial I(x,t)}{\partial t}$$

$$-\frac{\partial V(x,t)}{\partial x} = I(x,t) + \frac{\partial I(x,t)}{\partial t} \qquad (6a)$$

Putting GL/RC equal to k, the equations take the form:

$$-\frac{\partial I(x,t)}{\partial x} = kV(x,t) + \frac{\partial V(x,t)}{\partial t} \qquad (7)$$

$$-\frac{\partial V(x,t)}{\partial x} = I(x,t) + \frac{\partial I(x,t)}{\partial t} \qquad (8)$$

Taking the partial derivative of each equation with respect to both x and t:

$$-\frac{\partial^2 I(x,t)}{\partial x^2} = k\frac{\partial V(x,t)}{\partial x} + \frac{\partial^2 V(x,t)}{\partial t\,\partial x} \qquad (A)$$

$$-\frac{\partial^2 V(x,t)}{\partial t\,\partial x} = \frac{\partial I(x,t)}{\partial t} + \frac{\partial^2 I(x,t)}{\partial t^2} \qquad (B)$$

$$-\frac{\partial^2 I(x,t)}{\partial x\,\partial t} = k\frac{\partial V(x,t)}{\partial t} + \frac{\partial^2 V(x,t)}{\partial t^2} \qquad (C)$$

$$-\frac{\partial^2 V(x,t)}{\partial x^2} = \frac{\partial I(x,t)}{\partial x} + \frac{\partial^2 I(x,t)}{\partial t\,\partial x} \qquad (D)$$

Combining (A) and (B) and (8):

$$-\frac{\partial^2 I(x,t)}{\partial x^2} = k\frac{\partial V(x,t)}{\partial x} + \frac{\partial^2 V(x,t)}{\partial t\,\partial x}$$

$$-\frac{\partial^2 I(x,t)}{\partial x^2} = -k\left[I(x,t) + \frac{\partial I(x,t)}{\partial t}\right] + \left[-\frac{\partial I(x,t)}{\partial t} - \frac{\partial^2 I(x,t)}{\partial t^2}\right]$$

$$\frac{\partial^2 I(x,t)}{\partial x^2} = \frac{\partial^2 I(x,t)}{\partial t^2} + (k+1)\frac{\partial I(x,t)}{\partial t} + kI(x,t) \qquad (9)$$

Combining (C) and (D) and (7):

$$-\frac{\partial^2 V(x,t)}{\partial x^2} = \frac{\partial I(x,t)}{\partial x} + \frac{\partial^2 I(x,t)}{\partial t\,\partial x}$$

$$-\frac{\partial^2 V(x,t)}{\partial x^2} = -k\,V(x,t) - \frac{\partial V(x,t)}{\partial t} - k\frac{\partial V(x,t)}{\partial t} - \frac{\partial^2 V(x,t)}{\partial t^2}$$

$$+\frac{\partial^2 V(x,t)}{\partial x^2} = \frac{\partial^2 V(x,t)}{\partial t^2} + (k+1)\frac{\partial V(x,t)}{\partial t} + k\,V(x,t) \qquad (10)$$

Equations (9) and (10) can be recognized as differential equations of the same form and some class of functions that might describe V and also I.

In order to separate the variables let:

$$V(x,t) = X_1(x)T_1(t) \qquad (11)$$

$$I(x,t) = X_2(x)T_2(t) \qquad (12)$$

Substituting (11) in (10); dividing result by $X_1(x)T_1(t)$:

$$T_1(t)\frac{d^2X_1(x)}{dx^2} = X_1(x)\frac{d^2T_1(t)}{dt^2} + (k+1)X_1(x)\frac{dT_1(t)}{dt} + kX_1(x)T_1(t) \qquad (11a)$$

$$\frac{1}{X_1(x)}\frac{d^2X_1(x)}{dx^2} = \frac{1}{T_1(t)}\left[\frac{d^2T_1(t)}{dt^2} + (k+1)\frac{dT_1(t)}{dt} + kT_1(t)\right] \qquad (11b)$$

The left hand side does not depend on t and the right does not depend on x. With x fixed time can change but the right hand side must be constant in value and likewise with t fixed. Therefore, each side may be equated to an identical constant:

$$\frac{1}{X_1(x)}\frac{d^2X_1(x)}{dx^2} = -\gamma^2 \qquad (11c)$$

$$\frac{1}{T_1(t)}\left[\frac{d^2T_1(t)}{dt^2} + (k+1)\frac{dT_1(t)}{dt} + kT_1(t)\right] = -\gamma^2 \qquad (11d)$$

These equations will be recognized as equations with exponential solutions:

$$(11c) \quad -\frac{d^2X_1(x)}{dx^2} + \gamma^2 X_1(x) = 0$$

$$X_1(x) = \varepsilon^{j\gamma x}$$

Checking this solution:

$$\frac{dX_1(x)}{dx} = j\eta\, \mathcal{E}^{j\eta x}$$

$$\frac{d^2X_1(x)}{dx^2} = -\eta^2 \mathcal{E}^{j\eta x} = -\eta^2 X_1(x)$$

$$\therefore X_1(x) = A_1 \mathcal{E}^{j\eta x}$$

There is also another solution:

$$X_1(x) = B_1 \mathcal{E}^{-j\eta x}$$

$$\therefore X_1(x) = A_1 \mathcal{E}^{j\eta x} + B_1 \mathcal{E}^{-j\eta x} \qquad (11c_1)$$

If $X_1(x)$ is of this general type it will satisfy (11c).

Considering (11d):

$$\frac{d^2T_1(t)}{dt^2} + (k+1)\frac{dT_1(t)}{dt} + (k+\eta)^2 T_1(t) = 0$$

This also has an exponential solution; try:

$$T_1(t) = \mathcal{E}^{j\omega t} \qquad (11d_1)$$

Substituting in (11d):

$$-\omega^2 \mathcal{E}^{j\omega t} + (k+1)j\omega \mathcal{E}^{j\omega t} + (k+\eta^2)\mathcal{E}^{j\omega t} = 0$$

$$-\omega^2 + (k+1)j\omega + k = -\eta^2$$

$$j\eta = \sqrt{-\omega^2 + j(k+1)\omega + k}$$

$$j\eta = \sqrt{(k+j\omega)(1+j\omega)}$$

Since $V(x,t) = X_1(x)T_2(t)$

$$\therefore V(x,t) = [A_1 \varepsilon^{j\eta x} + B_1 \varepsilon^{-j\eta x}]\, \varepsilon^{j\omega t}$$

The voltage relation is then found by taking a rich summation from $\omega = -\infty$ to $\omega = \infty$:

$$V(x,t) = \int_{\omega=-\infty}^{\omega=\infty} [A_1(\omega)\, \varepsilon^{j\eta x} + B_1(\omega)\, \varepsilon^{-j\eta x}]\, \varepsilon^{j\omega t}\, d\omega \qquad (13)$$

In which equation $A_1(\omega)$ and $B_1(\omega)$ are frequency functions.

Repeating this procedure by substituting (12) in (9) and dividing by $X_2(x)T_2(t)$:

$$(12) - I(x,t) = X_2(x)T_2(t)$$

$$(9) - -\frac{\partial^2 I(x,t)}{\partial x^2} = \frac{\partial^2 I(x,t)}{\partial t^2} + (k+1)\frac{\partial I(x,t)}{\partial t} + k I(x,t)$$

$$T_2(t)\frac{d^2X_2(x)}{dx^2} = X_2(x)\frac{d^2T_2(t)}{dt^2} + (k+1)X_2(x)\frac{dT_2(t)}{dt} + k X_2(x) T_2(t) \qquad (12a)$$

$$\frac{1}{X_2(x)}\frac{d^2X_2(x)}{dx^2} = \frac{1}{T_2(t)}\left[\frac{d^2T_2(t)}{dt^2} + (k+1)\frac{dT_2(t)}{dt} + k T_2(t)\right] \qquad (12b)$$

As before set each side equal the constant:

$$\frac{1}{X_2(x)}\frac{d^2X_2(x)}{dx^2} = -\eta^2 \qquad (12c)$$

$$\frac{1}{T_2(t)}\left[\frac{d^2T_2(t)}{dt^2} + (k+1)\frac{dT_2(t)}{dt} + k T_2(t)\right] = -\eta^2 \qquad (12d)$$

As shown in the case of $X_1(x)$ and $T_1(t)$:

$$X_2(x) = A_2 \varepsilon^{j\eta x} + B_2 \varepsilon^{-j\eta x} \qquad (12e)$$

$$T_2(t) = \varepsilon^{j\omega t}, \qquad j\eta = \sqrt{(R+j\omega)(1+j\omega)} \qquad (12d_1)$$

$$\therefore I(x,t) = X_2(x)\,T_2 t = [A_2 \varepsilon^{j\eta x} + B_2 \varepsilon^{-j\eta x}]\varepsilon^{j\omega t}$$

$$I(x,t) = \int_{\omega=-\infty}^{\omega=\infty} [A_2(\omega)\,\varepsilon^{j\eta x} + B_2(\omega)\,\varepsilon^{-j\eta x}]\varepsilon^{j\omega t}\,d\omega \qquad (14)$$

These relations for V(x,t) and I(x,t) are restricted by the boundary conditions:

1. V(0,t) = f(t) - description of home end voltage
2. $\lim_{x\to\infty} I(x,t) \to 0$ - current at infinite distance

From inspection of the general Fouier's Integral for f(t):

$$f(t) = \frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\varepsilon^{-j\omega\xi}\,d\xi \int_{\omega=-\infty}^{\omega=\infty} \varepsilon^{j\omega t}\,d\omega$$

it can be readily seen that the expression for $V(0,t) = f(t) = \int_{\omega=-\infty}^{\omega=\infty} [A_1(\omega) + B_1(\omega)]\varepsilon^{j\omega t}\,d\omega$ is a Fourier's Integral in which $[A_1(\omega) + B_1(\omega)] = \frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\varepsilon^{-j\omega\xi}\,d\xi$, in which f( ) is the known voltage at the home end.

It is desirable to remove the four unknown constants, $A_1(\omega)$, $B_1(\omega)$, $A_2(\omega)$, and $B_2(\omega)$. First go back to the general equations (7) and (8) before the variables were separated and restating (13) and (14):

$$(7) \quad -\frac{\partial I(x,t)}{\partial x} = RV(x,t) + \frac{\partial V(x,t)}{\partial t}$$

$$(8) \quad -\frac{\partial V(x,t)}{\partial x} = I(x,t) + \frac{\partial I(x,t)}{\partial t}$$

$$(13) \quad V(x,t) = \int_{\omega=-\infty}^{\omega=\infty} [A_1(\omega)\varepsilon^{j\eta x} + B_1(\omega)\varepsilon^{-j\eta x}]\varepsilon^{j\omega t}\,d\omega$$

$$(14) \quad I(x,t) = \int_{\omega=-\infty}^{\omega=\infty} [A_2(\omega)\varepsilon^{j\eta x} + B_2(\omega)\varepsilon^{-j\eta x}]\varepsilon^{j\omega t}\,d\omega$$

Taking the partial derivatives of (13) and (14) as needed for substitution in (7) and (8) and substituting; transposing terms to the right side of the equations and collecting terms; the following relations are obtained:

$$(7)\; j\eta \int_{\omega=-\infty}^{\omega=\infty} [A_2(\omega)\varepsilon^{j\eta x} - B_2(\omega)\varepsilon^{-j\eta x}]\varepsilon^{j\omega t}\,d\omega = R\int_{\omega=-\infty}^{\omega=\infty} [A_1(\omega)\varepsilon^{j\eta x} + B(\omega)\varepsilon^{-j\eta x}]\varepsilon^{j\omega t}\,d\omega + j\int_{\omega=-\infty}^{\omega=\infty} [A_1(\omega)\varepsilon^{j\eta x} + B_1(\omega)\varepsilon^{-j\eta x}]\omega\,\varepsilon^{j\omega t}\,d\omega$$

$$(7_1)\int_{\omega=-\infty}^{\omega=\infty} \left\{ [(R+j\omega)A_1(\omega) + j\eta A_2(\omega)]\varepsilon^{j\eta x} + [(R+j\omega)B(\omega) - j\eta B_2(\omega)]\varepsilon^{-j\eta x} \right\} \varepsilon^{j\omega t}\,d\omega = 0$$

$$(8)\; -j\eta \int_{\omega=-\infty}^{\omega=\infty} [A_1(\omega)\varepsilon^{j\eta x} - B_1(\omega)\varepsilon^{-j\eta x}]\varepsilon^{j\omega t}\,d\omega = \int_{\omega=-\infty}^{\omega=\infty} [A_2(\omega)\varepsilon^{j\eta x} + B_2(\omega)\varepsilon^{-j\eta x}]\varepsilon^{j\omega t}\,d\omega + j\int_{\omega=-\infty}^{\omega=\infty} [A_2(\omega)\varepsilon^{j\eta x} + B_2(\omega)\varepsilon^{-j\eta x}]\varepsilon^{j\omega t}\,\omega\,d\omega$$

$$(8_1)\int_{\omega=-\infty}^{\omega=\infty} \left\{ [(1+j\omega)A_2(\omega) + j\eta A_1(\omega)]\varepsilon^{j\eta x} + [(1+j\omega)B_2(\omega) - j\eta B_1(\omega)]\varepsilon^{-j\eta x} \right\} \varepsilon^{j\omega t}\,d\omega = 0$$

In order that (7_1) and (8_1) are satisfied for all values of x and t, the coefficienty must equal zero. Therefore:

$$(1+j\omega)A_2(\omega) + j\eta A_1(\omega) = 0$$

$$(1+j\omega)B_2(\omega) - j\eta B_1(\omega) = 0$$

$$(R+j\omega)A_1(\omega) + j\eta A_2(\omega) = 0$$

$$(R+j\omega)B_1(\omega) - j\eta B_2(\omega) = 0$$

$$A_2(\omega) = \frac{-j\eta}{1+j\omega} A_1(\omega) \qquad - \qquad A_2(\omega) = -\frac{k+j\omega}{j\eta} A_1(\omega)$$

$$\therefore \frac{-j\eta}{1+j\omega} = -\frac{k+j\omega}{j\eta} \quad ; \quad j\eta = \sqrt{(1+j\omega)(k+j\omega)}$$

$$B_2(\omega) = \frac{j\eta}{1+j\omega} B_1(\omega) \qquad - \qquad B_2(\omega) = \frac{k+j\omega}{j\eta} B_1(\omega)$$

$$\therefore \frac{j\eta}{1+j\omega} = \frac{k+j\omega}{j\eta} \quad ; \quad j\eta = \sqrt{(1+j\omega)(k+j\omega)}$$

$$A_2(\omega) = \frac{-j\eta}{1+j\omega} A_1(\omega) \quad \text{if} \quad j\eta = \sqrt{(1+j\omega)(k+j\omega)}$$

$$B_2(\omega) = \frac{j\eta}{1+j\omega} B_1(\omega) \quad \text{if} \quad j\eta = \sqrt{(1+j\omega)(k+j\omega)}$$

Since jn was assigned this value on a preceeding page, these relations between $A(\omega)$ and $B(\omega)$ will satisfy the conditions.

The expressions for voltage and current are now of the form containing the two unknown constants $A_1(\omega)$ and $B_1(\omega)$:

$$V(x,t) = \int_{\omega=-\infty}^{\omega=\infty} [A_1(\omega)\, \varepsilon^{j\eta x} + B_1(\omega)\, \varepsilon^{-j\eta x}]\, \varepsilon^{j\omega t}\, d\omega \qquad (15)$$

$$I(x,t) = \int_{\omega=-\infty}^{\omega=\infty} \frac{j\eta}{1+j\omega} [B_1(\omega)\, \varepsilon^{-j\eta x} - A_1(\omega)\, \varepsilon^{j\eta x}]\, \varepsilon^{j\omega t}\, d\omega \qquad (16)$$

where $j\eta = \sqrt{(j\omega+1)(j\omega+k)}$, $k = GL/RC$

(For very high frequencies, $j\eta = \sqrt{j\omega^2} = j\omega$)

jn is a complex propogation constant, say τ, consisting of a real or dampening part and a j part causing waves:

$$j\eta = \tau = \alpha + j\beta$$

In (15) the value, $\alpha + j\beta$, can be substituted for $j\eta$:

$$A_1(\omega)\, d\omega\, \varepsilon^{j(\omega t + \eta x)} + B_1(\omega)\, d\omega\, \varepsilon^{j(\omega t - \eta x)}$$

$$\varepsilon^{j(\omega t + \eta x)} = \varepsilon^{j\omega t + \alpha x + j\beta x} = \varepsilon^{\alpha x}\varepsilon^{j\omega\left(t + \frac{x}{\omega/\beta}\right)}$$

$$\varepsilon^{j(\omega t - \eta x)} = \varepsilon^{j\omega t - \alpha x - j\beta x} = \varepsilon^{-\alpha x}\varepsilon^{j\omega\left(t - x/\omega/\beta\right)}$$

In these relations the first terms, $\varepsilon^{\alpha x}$ and $\varepsilon^{-\alpha x}$, are the real attenuation part. The latter terms can be recognized to be of the form, $\varepsilon^{j\omega\left(t \pm \frac{x}{v}\right)}$, which signifies traveling waves to the right with the negative sign and to the left with the positive sign.

It is necessary to remove the constants $A_1(\omega)$ and $B_1(\omega)$. Considering the boundary conditions:

$V(x,t) = f(t)$, when x is zero (home end)

$I(x,t) = V(x,t)/Z$, when x is 1 (load end).

Again stating the general Fourier's Integral for f(t):

$$f(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} f(\xi)\,d(\xi) \int_{\omega=-\infty}^{\omega=\infty} \varepsilon^{j\omega(t-\xi)}\,d\omega = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} f(\xi)\,\varepsilon^{-j\omega\xi} \int_{\omega=-\infty}^{\omega=\infty} \varepsilon^{j\omega t}\,d\omega$$

$$V(0,t) = \int_{\omega=-\infty}^{\omega=\infty} [A_1(\omega) + B_1(\omega)]\,\varepsilon^{j\omega t}\,d\omega = f(t), \quad [\varepsilon^0 = 1]$$

To satisfy this condition, $V(0,t) = f(t)$, ($A_1(\omega)$ $B_1(\omega)$) must be chosen to make $V(0,t) = f(t)$:

$$\therefore [A_1(\omega) + B_1(\omega)] = \frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\,\varepsilon^{-j\omega\xi}\,d\xi$$

$$I(l,t) = \frac{V(l,t)}{Z} = \int_{\omega=-\infty}^{\omega=\infty} \left[\frac{A_1(\omega)\,\varepsilon^{j\eta l} + B_1(\omega)\,\varepsilon^{-j\eta l}}{Z}\right] \varepsilon^{j\omega t}\,d\omega$$

But from (16):

$$I(l,t) = \int_{\omega=-\infty}^{\omega=\infty} \left[\frac{j\eta}{j\omega + 1}\left(-A_1(\omega)\,\varepsilon^{j\eta l} + B_1(\omega)\,\varepsilon^{-j\eta l}\right)\varepsilon^{j\omega t}\,d\omega\right]$$

Therefore, $A_1(\omega)$ and $B_1(\omega)$ must be chosen to make the two relations for I(1,t) identical; equating coefficients in the brackets:

$$\frac{A_1(\omega)\,\varepsilon^{j\eta\ell}+B_1(\omega)\,\varepsilon^{-j\eta\ell}}{Z}=\frac{j\eta}{j\omega+1}\left[B_1(\omega)\,\varepsilon^{-j\eta\ell}-A_1(\omega)\,\varepsilon^{j\eta\ell}\right]$$

$$A_1(\omega)\,\varepsilon^{j\eta\ell}\left[\frac{1}{Z}+\frac{j\eta}{j\omega+1}\right]=B_1(\omega)\,\varepsilon^{-j\eta\ell}\left[-\frac{1}{Z}+\frac{j\eta}{j\omega+1}\right]$$

$$A_1(\omega)\,\varepsilon^{j\eta\ell}\left[1+j(\omega+\eta Z)\right]=B_1(\omega)\,\varepsilon^{-j\eta\ell}\left[-1+j(-\omega+\eta Z)\right]$$

$$A_1(\omega)=B_1(\omega)\,\varepsilon^{-2j\eta\ell}\left(\frac{-1+j(\eta Z-\omega)}{1+j(\eta Z+\omega)}\right) \qquad (17)$$

$$B_1(\omega)=A_1(\omega)\,\varepsilon^{2j\eta\ell}\left(\frac{1+j(\eta Z+\omega)}{-1+j(\eta Z-\omega)}\right) \qquad (18)$$

From the other boundary condition:

$$\left[A_1(\omega)+B_1(\omega)\right]=\frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\,\varepsilon^{-j\omega\xi}\,d\xi$$

$$(17)\quad A_1(\omega)+B_1(\omega)=B_1(\omega)\left[1+\varepsilon^{-2j\eta\ell}\left(\frac{-1+j(\eta Z-\omega)}{1+j(\eta Z+\omega)}\right)\right]$$

$$=\frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\,\varepsilon^{-j\omega\xi}\,d\xi$$

$$B_1(\omega)=\frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\,\varepsilon^{-j\omega\xi}\,d\xi \Bigg/\left[1+\varepsilon^{-2j\eta\ell}\left(\frac{-1+j(\eta Z-\omega)}{1+j(\eta Z+\omega)}\right)\right] \qquad (19)$$

$$(18)\quad A_1(\omega)+B_1(\omega)=A_1(\omega)\left[1+\varepsilon^{2j\eta\ell}\left(\frac{1+j(\eta Z+\omega)}{-1+j(\eta Z-\omega)}\right)\right]$$

$$=\frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\,\varepsilon^{-j\omega\xi}\,d\xi$$

$$A_1(\omega)=\frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=+\infty} f(\xi)\,\varepsilon^{-j\omega\xi}\,d\xi \Bigg/\left[1+\varepsilon^{2j\eta\ell}\left(\frac{1+j(\eta Z+\omega)}{-1+j(\eta Z-\omega)}\right)\right] \qquad (20)$$

Therefore the general equations for voltage and current are:

$$V(x,t) = \int_{\omega=-\infty}^{\omega=\infty} [A_1(\omega)\varepsilon^{j\eta x} + B_1(\omega)\varepsilon^{-j\eta x}]\varepsilon^{j\omega t} d\omega \qquad (21)$$

$$I(x,t) = \int_{\omega=-\infty}^{\omega=\infty} \frac{j\eta}{j\omega+1} [B_1(\omega)\varepsilon^{-j\eta x} - A_1(\omega)\varepsilon^{j\eta x}]\varepsilon^{j\omega t} d\omega \qquad (22)$$

$$A_1(\omega) = \frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\varepsilon^{-j\omega\xi} d\xi \Big/ \left[1+\varepsilon^{2j\eta x}\left(\frac{1+j(\eta Z+\omega)}{-1+j(\eta Z-\omega)}\right)\right] \qquad (23)$$

$$B_1(\omega) = \frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\varepsilon^{-j\omega\xi} d\xi \Big/ \left[1+\varepsilon^{-2j\eta x}\left(\frac{-1+j(\eta Z-\omega)}{1+j(\eta Z+\omega)}\right)\right] \qquad (24)$$

$f(\xi)$ = Applied Voltage.

$j\eta = \sqrt{(k+j\omega)(1+j\omega)}$

$k = GL/RC$

The simplification of these equations is left to the applying of special cases consisting of particular conditions.

Before applying special cases to these relations, it will be well to consider the following theorem:

$$f(t) = \frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\,d\xi \int_{\omega=-\infty}^{\omega=\infty} \varepsilon^{j\omega(t-\xi)} d\omega$$

$$\psi(t) = \frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\,d(\xi) \int_{\omega=-\infty}^{\omega=\infty} F(\omega)\varepsilon^{j\omega(t-\xi)} d\omega$$

It is permissible to differentiate $\psi(t)$ repeatedly under both signs of integration, <u>provided</u> the result is evaluated by one of the following three ways:

(a) As a Rieman (definite) integral, if that interpretation

is possible.

(b) As a sum of two contour integrals (in complex plane) when $F(\omega)$ vanishes at infinity.

(c) As the sum of two contour integrals $\pm$ an auxiliary function which may be obtained by expanding $F(\omega)$ in descending powers of $j\omega$ ignoring the negative powers and replacing each $(j\omega)^s$ by the $\frac{d^s(f(t))}{dt^s}$.

As an example of the theorem consider a simple circuit containing a battery supply, V, and an impedance, R and C..

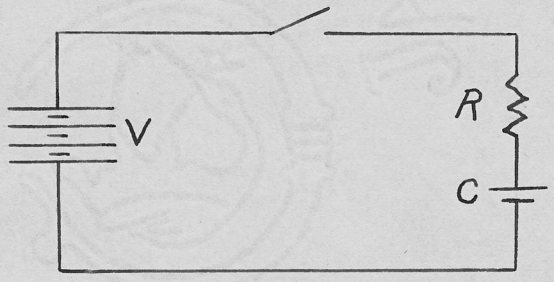


Fig. 2

Solving this by the classical method:

$$Ri + q/C = V$$

$$R\frac{di}{dt} + \frac{dq/dt}{C} = 0$$

$$dq/dt = i$$

$$\therefore R\frac{di}{dt} + \frac{i}{C} = 0$$

Put: $i = A\varepsilon^{\beta t}$

$$\frac{di}{dt} = A\beta\varepsilon^{\beta t}$$

$$RA\beta\varepsilon^{\beta t} + A\varepsilon^{\beta t}/C = 0$$

$$R\beta + \frac{1}{C} = 0$$

$$\beta = -\frac{1}{RC}$$

When t is zero and the switch is closed:

$$Ri + q/C = V$$

$$Ri + 0 = V$$

$$i = V/R$$

$$i = A\varepsilon^{\beta t} = A\varepsilon^{0} = A = V/R$$

$$\therefore\ i = A\varepsilon^{\beta t} = V/R\ \varepsilon^{(\frac{1}{RC})t} = V/R\ \varepsilon^{-\frac{t}{RC}}$$

Solving this circuit by means of Fourier's Integral:

$$V = \int_{\omega=-\infty}^{\omega=\infty} S(\omega)\,\varepsilon^{j\omega t}\,d\omega\ ,$$

$$S(\omega) = \frac{V}{2\pi}\int_{\xi=0}^{\xi=\infty} \varepsilon^{-j\omega\xi}\,d\xi$$

V vanishes for $\xi < 0$, therefore lower limit is 0.

Considering incremental values:

$$e(\omega) = S(\omega)\,d\omega\,\varepsilon^{j\omega t}$$

$$i_{(e(\omega))} = \frac{e(\omega)}{Z} = \frac{e\omega}{R - \frac{j}{\omega C}} = \frac{S(\omega)\,d\omega^{j\omega t}}{R - j/\omega C}$$

$$i_T(t) = \sum i_{e\omega}\,d\omega = \int_{\omega=-\infty}^{\omega=\infty} \frac{S(\omega)\,\varepsilon^{j\omega t}\,d\omega}{R - j/\omega C}$$

$$= \frac{1}{2\pi}\int_{\xi=0}^{\xi=\infty} (\psi)\,d\xi \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{R - j/\omega C}\,\varepsilon^{j\omega(t-\xi)}\,d\omega$$

It can be easily recognized that V is $f(\varphi)$ and $(1/R - j/\omega C)$ is the $F(\omega)$ in the Fourier's Integral stated in the theorem. Solving the last integral by contour integration; multiplying by ω/R:

$$\mathcal{I} = \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{R - \frac{1}{j\omega C}} \mathcal{E}^{j\omega(t-\varphi)}\, d\omega = \oint \frac{\omega/R}{\omega - \frac{j}{RC}} \mathcal{E}^{j\omega(t-\varphi)}\, d\omega$$

There is a pole in the complex ω plane at $\omega = j/RC$, for at this value the coefficient goes to infinity. If $(t - \varphi)$ is positive, then:

$\mathcal{I}$ = 2πj x sum of residues

$$\text{Residue} = (j/RC)/R\,\mathcal{E}^{j\cdot\frac{j}{RC}(t-\varphi)} = \frac{j}{R^2C}\,\mathcal{E}^{\frac{-t-\varphi}{RC}}$$

$$\mathcal{I} = -2\pi/R^2C\;\mathcal{E}^{\frac{-t-\varphi}{RC}}$$

If $(t - \varphi)$ is negative there is no pole, and $\mathcal{I}$ is zero.

Substituting back in $i_T(t)$ the value of the last integral:

$$i_T(t) = \frac{V}{2\pi}\int_{\varphi=0}^{\varphi=t} \mathcal{I}\, d\varphi = \frac{V}{2\pi}\int_{\varphi=0}^{\varphi=t} \frac{-2\pi}{R^2C}\,\mathcal{E}^{\frac{-t-\varphi}{RC}}\, d\varphi + \frac{V}{2\pi}\int_{\varphi=t}^{\varphi=\infty} \text{Zero}\; d\varphi$$

$$\therefore\; i_T(t) = -\frac{V}{R^2C}\,RC\,\mathcal{E}^{\frac{-t-\varphi}{RC}}\Big]_{\varphi=0}^{\varphi=t} = -\frac{V}{R} + \frac{V}{R}\mathcal{E}^{\frac{-t}{RC}}$$

It is noted that this answer varies from that found by the classical method of solution. Thus, Theorems (a) and (b) have not solved the problem and (c) must be applied. This is what was expected since $F(\omega)$ does not vanish at $\omega = \infty$.

Expanding $F(\omega)$; descending power of ωj :

$$F(\omega) = \frac{1}{R - \frac{1}{j\omega C}} = \frac{1}{R + \frac{1}{j\omega C}} = C\left(\frac{1}{RC + \frac{1}{j\omega}}\right)$$

$$= C\left(\frac{1}{RC} - \frac{1}{R^2C^2}\cdot\frac{1}{j\omega} + \frac{1}{R^3C^3}\left(\frac{1}{j\omega}\right)^2 + \cdots\right.$$

$$= C\left(\frac{1}{RC}\left(\frac{1}{j\omega}\right)^0 - \frac{1}{R^2C^2}\cdot\left(\frac{1}{j\omega}\right)^1 - \frac{1}{R^3C^3}\cdot\left(\frac{1}{j\omega}\right)^2 + \cdots\right.$$

All the terms after the first will be negative either from their position in the series or from containing $(j)^2$. As Theorem (c) states the negative powers are to be ignored and the term $C(1/RC(j\omega)^0 = 1/R\ (j\omega)^0$. The zero derivative of f(t) is V. Therefore, to the sum of the contour integrals must be added V/R.

$$i_T(t) = -\frac{V}{R} + \frac{V}{R}\mathcal{E}^{-\frac{t}{RC}} + \frac{V}{R} = \frac{V}{R}\mathcal{E}^{-\frac{t}{RC}}$$

This value for $i_T(t)$ checks the value previously found and confirms the three parts of the Theorem.

APPLICATION TO SPECIAL CASE

Consider a condition in which there is no echo; i.e., no wave traveling to the left. Referring to the discussion on the traveling wave effect, it will be noted that the term containing $A_1(\omega)$ is the wave traveling to the left. This may be removed by making $A_1(\omega)$ equal to zero. Repeating (20):

$$A_1(\omega) = \frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\mathcal{E}^{-j\omega\xi}\,d\xi \Big/ \left[1 + \mathcal{E}^{2j\eta x}\left(\frac{1+j(\eta Z+\omega)}{-1+j(\eta Z-\omega)}\right)\right]$$

$A_1(\omega)$ will be zero if the term $-1 + j(\eta Z-\omega) = 0$

$$-1 + j\eta Z - j\omega = 0$$

$$j\eta Z = 1 + j\omega$$

$$Z = \frac{1 + j\omega}{j\eta} = Z_o$$

Z_o is the reciprocal of $j\eta/(1 + j\omega)$ in I(x,t), (22)

This is the impedance necessary to make $A_1(\omega)$ equal zero and to remove the wave traveling to the left. This is

called the iterative impedance, and is sometimes called the surge, echo, or characteristic impedance. The same effect of Z can be obtained by making the line infinitely long; however, in this case this infinite length is replaced by an equivalent impedance, Z_o. The equation for voltage becomes:

$$V(x,t) = \int_{\omega=-\infty}^{\omega=\infty} [0 + B_1(\omega)\, \varepsilon^{-j\eta x}]\, \varepsilon^{j\omega t}\, d\omega$$

$$B_1(\omega) = \frac{1}{2\pi} \int_{\varphi=-\infty}^{\varphi=\infty} f(\varphi)\, \varepsilon^{-j\omega\varphi}\, d\varphi \Big/ [1 + \varepsilon^{-2j\eta x}(0)] = \frac{1}{2\pi} \int_{\varphi=-\infty}^{\varphi=\infty} f(\varphi)\, \varepsilon^{-j\omega\varphi}\, d\varphi$$

$$\therefore V(x,t) = \frac{1}{2\pi} \int_{\varphi=-\infty}^{\varphi=\infty} f(\varphi)\, d\varphi \int_{\omega=-\infty}^{\omega=\infty} \varepsilon^{j\omega(t-\varphi)} \cdot \varepsilon^{-j\eta x}\, d\omega$$

$$I(x,t) = \frac{1}{2\pi} \int_{\varphi=-\infty}^{\varphi=\infty} f(\varphi)\, d\varphi \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{Z_o}\, \varepsilon^{j\omega(t-\varphi)} \cdot \varepsilon^{-j\eta x}\, d\omega$$

It can be readily seen that I(x,t) is of the form of $\psi(t)$ in the Theorem and $1/Z_o$ is the term $F(\omega)$. In the voltage equation $\varepsilon^{-j\eta x}$ is the term $F(\omega)$. $\varepsilon^{-j\eta x}$ not only does not vanish at $\omega=\infty$, but also becomes exponentially infinite. Since there would be an infinite number of $\frac{d^s f(t)}{dt^s}$. It is therefore necessary to rewrite V(x,t):

$$V(x,t) = \frac{1}{2\pi} \int_{\varphi=-\infty}^{\varphi=\infty} f(\varphi)\, d\varphi \int_{\omega=-\infty}^{\omega=\infty} [\varepsilon^{j\omega x - j\eta x}]\, \varepsilon^{j\omega(t-x-\varphi)}\, d\omega$$

In this relation $F_V(\omega) = \varepsilon^{j\omega x - j\eta x}$. Since $\eta \to \omega$ as $\omega \to \infty$ (from relations on page 10, for high frequencies $j\eta = \sqrt{(j\omega)^2} = j\omega$), $F(\omega)$ does not become exponentially infinite nor does it vanish.

$$F_I(\omega) = \frac{1}{Z_o} [\varepsilon^{j\omega x - j\eta x}]$$

$$I(x,t) = \frac{1}{2\pi}\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\,d\xi \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{Z_0}\left[\varepsilon^{j\omega x - j\eta x}\right] \varepsilon^{j\omega(t-x-\xi)}\,d\omega$$

$$Z_0 = \frac{j\omega+1}{\Gamma} = \frac{j\omega+1}{j\eta} = \frac{j\omega+1}{\alpha+j\beta} = \frac{j\omega+1}{\sqrt{(j\omega+1)(j\omega+k)}}$$

From this value of Z_0 it is seen that there are no first order poles and therefore no residues and Theorem (b) will not work. Therefore Theorem (c) is used in the solution.

$V(x,t) = \psi_V(t-x)$ + conditions of Theorem (c)

$I(x,t) = \psi_I(t-x)$ + conditions of Theorem (c)

$\psi(t-x)$ is to be evaluated by contour integration.

$$F_V(\omega) = \varepsilon^{j\omega x - j\eta x} = \varepsilon^{j\omega x - x\sqrt{(j\omega+1)(j\omega+k)}} = \varepsilon^{x\left(j\omega - \sqrt{(j\omega+1)(j\omega+k)}\right)}$$

$$F_V(\omega) = \varepsilon^{x\left(j\omega - \sqrt{(j\omega)^2+(k+1)j\omega}\right)}, \quad \text{(Disregard } k^2 \text{ as } \omega\to\infty\text{)}$$

Expand $\left[(j\omega)^2+(k+1)j\omega\right]^{\frac{1}{2}}$ by the binomial theorem:

$$(a+b)^n = a^n + na^{n-1}b + n(n-1)a^{n-2}\frac{b^2}{2!} + \cdots$$

$$F_V(\omega) = \varepsilon^{x\left(j\omega - \left[j\omega + \frac{1}{2}(j\omega)^{-1}(k+1)j\omega + \cdots\right]\right)}$$

The remaining terms in this expansion are negligible as $j\omega$ becomes infinite:

$$\therefore F_V(\omega) = \varepsilon^{\frac{-x(k+1)}{2}}(j\omega)^0 + \text{ negative powers of } (j\omega)$$

Then, $V(x,t) = \psi_V(t-x) + \varepsilon^{\frac{-x(k+1)}{2}} f(t-x)$

As, $\omega\to\infty$, $Z_0 \longrightarrow j\omega/j\omega = 1$

$$\therefore I(x,t) = \psi_I(t-x) + \varepsilon^{\frac{-x(k+1)}{2}} f(t-x)$$

Before proceeding with the evaluations it will be well to discuss the three kinds of singularities:

(1) Poles

a. The first order, $1/(Z-A)$, gives a residue, the pole being at $Z = A$.

b. The higher integral orders, $1/(Z-A)^n$, does not give residues.

(2) Essential singularities

a. $\varepsilon^{1/z}$ has an essential singularity at origin.

b. $\varepsilon^{\frac{1}{Z-A}}$ has an essential singularity at $Z = A$. One would run into trouble with this kind of singularity and should regroup the relations to get around this type.

(3) Branch points

Branch points have two values of for every point on the contour integral which are on different Rieman surfaces. These occur for the function $\omega = Z^{\frac{1}{2}}$.

The form, $\frac{1}{Z_0} = \frac{\sqrt{(j\omega+1)(j\omega+R)}}{(j\omega+1)} = \sqrt{\frac{(j\omega+1)(j\omega+R)}{(j\omega+1)^2}} = \sqrt{\frac{j\omega+R}{j\omega+1}}$, suggests two branch points for which there is no general method of treatment, each case having to be considered separately.

Consider: $I(x,t) = \psi(t-x) + \varepsilon^{-(R+1)\frac{x}{2}} f(t-x)$

Solving the last part of the Fourier's expression for $\psi(t-x)$ by contour integration:

$$\delta_I(t-x-\xi) = \frac{1}{2\pi} \oint \frac{1}{Z_0} \varepsilon^{j\omega(t-\xi) - j\eta x} d\omega$$

The occurrence of the radical: $j\eta = \sqrt{(j\omega+1)(j\omega+R)} = \sqrt{(j\omega)^2+(1+R)j\omega+R}$

$= \sqrt{(j\omega)^2+(1+R)j\omega+(\frac{1}{4}+\frac{R}{2}+\frac{R^2}{4})-(\frac{1}{4}-\frac{R}{2}+\frac{R^2}{4})}$, suggests the use of the

$= \sqrt{(j\omega+\frac{1+R}{2})^2-(\frac{1-R}{2})^2}$

usual trigonometric substitution; i.e., $j\omega + \frac{1+k}{2} = \frac{1-k}{2}\sin\theta$:

$$\therefore j\eta = \sqrt{\left(\frac{1-k}{2}\right)^2\sin^2\theta - \left(\frac{1+k}{2}\right)^2} = \left(\frac{1-k}{2}\right)\sqrt{\sin^2\theta - 1} = \frac{1+k}{2}\, j\cos\theta$$

$$j\omega + \frac{1+k}{2} = \frac{1-k}{2}\sin\theta$$

$$j\omega + 1 = \frac{1-k}{2}\sin\theta - \frac{1+k}{2} + 1 = \frac{1-k}{2}(\sin\theta + 1)$$

$$\therefore \frac{1}{Z} = \frac{j\eta}{j\omega+1} = \frac{\frac{1-k}{2}\, j\cos\theta}{\frac{1-k}{2}(\sin\theta+1)} = \frac{j\cos\theta}{\sin\theta+1} = \frac{j\cos\theta}{\sin\theta+1}\cdot\frac{1-\sin\theta}{1-\sin\theta}$$

$$= j\,\frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} = \frac{j(1-\sin\theta)}{\cos\theta}$$

$$d\omega = \frac{1-k}{2j}\cos\theta\, d\theta$$

Substituting the above values in the expression for $\mathcal{J}_I(t-x-\varphi)$

$$\therefore \mathcal{J}_I(t-x-\varphi) = \frac{1}{2\pi}\int_{\theta=?}^{\theta=2\pi+?} j\,\frac{1-\sin\theta}{\cos\theta}\,\mathcal{E}^{\left[\frac{1-k}{2}\sin\theta - \frac{1+k}{2}\right](t-\varphi) - j\frac{1-k}{2}\cos\theta\cdot x}\,\frac{1-k}{2j}\cos\theta\, d\theta$$

$$= \frac{1-k}{4\pi}\,\mathcal{E}^{(\varphi-t)\frac{1+k}{2}}\int_{\theta=?}^{\theta=2\pi+?}(1-\sin\theta)\,\mathcal{E}^{-j\left(\frac{1-k}{2}\right)\{j(t-\varphi)\sin\theta + x\cos\theta\}}\, d\theta$$

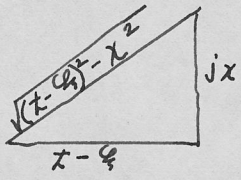
$$= \frac{1-k}{4\pi}\,\mathcal{E}^{(\varphi-t)\frac{1+k}{2}}\int_{\theta=?}^{\theta=2\pi+?}(1-\sin\theta)\,\mathcal{E}^{-j\frac{1-k}{2}\sqrt{x^2+(t-\varphi)^2}\left\{\sin\theta\,\frac{j(t-\varphi)}{\sqrt{x^2+(t-\varphi)^2}} + \cos\theta\,\frac{x}{\sqrt{x^2+(t-\varphi)^2}}\right\}}\, d\theta$$

$$= \frac{1-k}{4\pi}\,\mathcal{E}^{(\varphi-t)\frac{1+k}{2}}\int_{\theta=?}^{\theta=2\pi+?}(1-\sin\theta)\,\mathcal{E}^{-j\frac{1-k}{2}\sqrt{x^2-(t-\varphi)^2}\left\{\sin\theta\,\frac{t-\varphi}{\sqrt{(t-\varphi)^2-x^2}} - \cos\theta\,\frac{jx}{\sqrt{(t-\varphi)^2-x^2}}\right\}}\, d\theta$$

$$\mathcal{L}_I(t-x-\xi) = \frac{1-k}{4\pi}\,\mathcal{E}^{(\xi-t)\frac{1+k}{2}} \int_{\theta=?}^{\theta=2\pi+?} (1-\sin\theta)\,\mathcal{E}^{-j\frac{1-k}{2}\sqrt{x^2-(t-\xi)^2}\{\sin\theta\cos\epsilon-\cos\theta\sin\epsilon\}}\,d\theta$$

$$= \frac{1-k}{4\pi}\,\mathcal{E}^{(\xi-t)\frac{1+k}{2}} \int_{\theta=?}^{\theta=2\pi+?} (1-\sin\theta)\,\mathcal{E}^{-j\frac{1-k}{2}\sqrt{x^2-(t-\xi)^2}\sin(\theta-\epsilon)}$$

Where ϵ is a complex angle such that $\tan\epsilon = \frac{jx}{t-\xi}$



Let, $\sigma = \frac{1-k}{2}\sqrt{x^2-(t-\xi)^2}$

$$\therefore \mathcal{L}_I(t-x-\xi) = \frac{1-k}{4\pi}\,\mathcal{E}^{(\xi-t)\frac{1+k}{2}} \int_{\theta=?}^{\theta=2\pi+?} (1-\sin\theta)\,\mathcal{E}^{-j\sigma\sin(\theta-\epsilon)}$$

Now choose ? to be ϵ and substitute $\phi = \theta - \epsilon$:

$$\therefore \mathcal{L}_I(t-x-\xi) = \frac{1-k}{4\pi}\,\mathcal{E}^{(\xi-t)\frac{1+k}{2}} \int_{\theta=\epsilon}^{\theta=2\pi+\epsilon} (1-\sin\theta)\,\mathcal{E}^{-j\sigma\sin(\theta-\epsilon)}\,d\theta$$

$$= \frac{1-k}{4\pi}\,\mathcal{E}^{(\xi-t)\frac{1+k}{2}} \int_{\phi=0}^{\phi=2\pi} (1-\sin\theta)\,\mathcal{E}^{-j\sigma\sin\phi}\,d\phi$$

In the above expressions the path of integration runs between any two values of θ that differ by exactly 2π, so these limits can be chosen as ϵ and $\epsilon+2\pi$. Upon introducing the new variable $\phi = \theta - \epsilon$, the limits are from 0 to 2π.

The last expression is to be solved by Sommerfeld's definite integral representation of whole order Bessel functions. The following statement of three identities must be noted carefully:

$$(1)\quad J_n(\zeta) = \frac{1}{2\pi}\int_{\phi=0}^{\phi=2\pi} \varepsilon^{j(n\phi - \zeta\sin\phi)}\, d\phi$$

$$(2)\quad J_{-1}(\zeta) = -J_1(\zeta)$$

$$(3)\quad \frac{d J_n(\zeta)}{d\zeta} = \frac{n}{\zeta} J_n(\zeta) - J_{n+1}(\zeta)$$

By closely comparing these identities with the expression for $\partial_I(t-x-\eta)$ the following expression can be obtained:

$$\partial_I(t-x-\eta) = \frac{1-k}{8\pi j}\varepsilon^{(1+k)(\eta-t)\frac{1}{2}}\left\{2j\int_{\phi=0}^{\phi=2\pi}\varepsilon^{-j\sigma\phi}\,d\phi - \varepsilon^{j\epsilon}\int_{\phi=0}^{\phi=2\pi}\varepsilon^{j(\sigma\phi-\sigma\sin\phi)}\,d\phi + \varepsilon^{-j\epsilon}\int_{\phi=0}^{\phi=2\pi}\varepsilon^{j(-\phi-\sigma\sin\phi)}\,d\phi\right\}$$

$$\partial_I(t-x-\eta) = \frac{1-k}{2}\varepsilon^{(1+k)(\eta-t)\frac{1}{2}}\left\{J(\sigma) + j\cos\epsilon\, J_1(\sigma)\right\} \quad (1)+(2)$$

$$= \frac{d}{d\eta}\left\{\varepsilon^{(1+k)(\eta-t)\frac{1}{2}} J_0(\sigma)\right\} - \chi\varepsilon^{(1+k)(\eta-t)\frac{1}{2}} J(\sigma)$$

Having thus obtained an expression for $\partial_I(t-x-\eta)$ the following expressions can be stated as general relations:

$$I(x,t) = \psi_I(t-x) + \varepsilon^{-\left(\frac{k+1}{2}\right)x} f(t-x)$$

$$\psi_I(t-x) = \int_{\eta=0}^{\eta=t-x} f(\eta)\,\partial_I(t-x-\eta)\,d\eta$$

Before going further there must be more limitations on the problem. In addition to the fact that there is no echo, the applied voltage be a Heaviside voltage; i.e., before t=0 f(t) is zero and at t=0 and thereafter f(t) is 1. This is known as a Heaviside generator. I(x,t) is zero when t is less than x and the upper limit is therefore t-x for the values of t greater than x.

$$I(x,t) = \varepsilon^{-\frac{(k+1)x}{2}} \cdot 1 + \int_{\varphi=0}^{\varphi=t-x} \mathcal{J}_I(t-x-\varphi)\, d\varphi$$

$$= 2\varepsilon^{-\frac{(k+1)x}{2}} - \varepsilon^{-\frac{(k+1)t}{2}} \cdot \int_0 \frac{1-k}{2} \sqrt{x^2-t^2} - k \int_{\varphi=0}^{\varphi=t-x} \varepsilon^{(\varphi-t)(1+k)\frac{1}{2}} \mathcal{J}_0\left[\frac{1-k}{2}(x^2-(\varphi-t)^2)\right] d\varphi$$

Another restriction is put on the problem at this point in that the line is distortionless; that is, k = GL/RC = 1. Therefore with k = 1 and since $\mathcal{J}_0(0)=1$:

$$I(x,t) = 2\varepsilon^{-(k+1)\frac{x}{2}} - \varepsilon^{-(k+1)\frac{t}{2}} \cdot 1 - k \int_{\varphi=0}^{\varphi=t-x} \varepsilon^{(\varphi-t)(1+k)\frac{1}{2}} \cdot 1 \cdot d\varphi$$

$$= 2\varepsilon^{-(k+1)\frac{x}{2}} - \varepsilon^{-(k+1)\frac{t}{2}} - \frac{2k}{1+k}\left[\varepsilon^{(\varphi-t)(1+k)\frac{1}{2}}\right]_{\varphi=0}^{\varphi=t-x}$$

$$= 2\varepsilon^{-(k+1)\frac{x}{2}} - \varepsilon^{-(k+1)\frac{t}{2}} - \frac{2k}{1+k}\varepsilon^{-\frac{x}{2}(1+k)} + \frac{2k}{1+k}\varepsilon^{-\frac{t}{2}(1+k)}$$

$k = 1$

$$\therefore I(x,t) = 2\varepsilon^{-x} - \varepsilon^{-t} - \varepsilon^{-x} + \varepsilon^{-t} = \varepsilon^{-x}$$

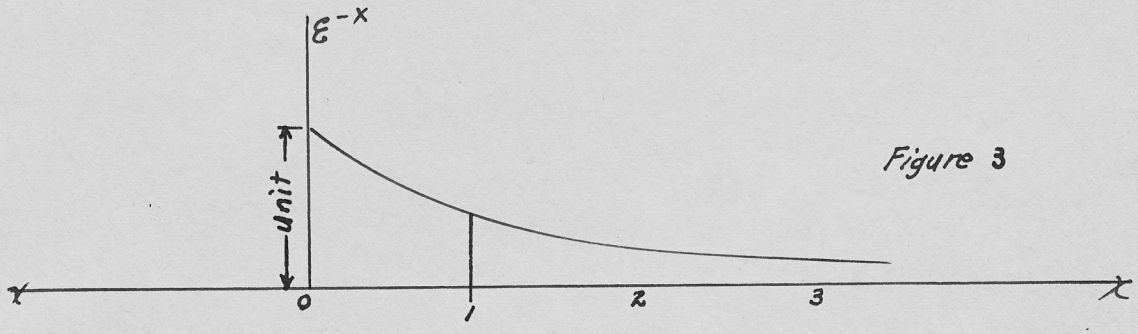


Figure 3

Restating the limitations in this special case:

1. No echo; $Z = Z_0$.
2. Heaviside voltage applied; up to $t = 0$, $f(t)$ is zero, and at $t = 0$ and thereafter $f(t)$ is 1.
3. $I(x,t)$ is zero when t is less than x, but at $t = x$ it is ε^{-x} (As found on page 25).
4. The line is distortionless; $k = GL/RC = 1$.

From the result, $I(x,t) = \varepsilon^{-x}$, it would seem that the result is independent of t and only depends on x. However, as discussed in limitation (3) before t is equal to x there is no current, but at t equal to x and thereafter the current is equal to ε^{-x}. Therefore, at $x = 1$ there is no current until t becomes equal to 1, after which the current takes the value $1/\varepsilon$. Also at $x = 2$ no current is present until t is 2 and then the value is $1/\varepsilon^2$.

This relation for $I(x,t)$ therefore is a traveling wave, made by ε^{-x} passing to the right in the vertical position (distortionless) but diminishing in magnitude as x increases.

APPROVAL

The foregoing thesis is hereby approved as a creditable study of an engineering subject, carried out and presented in a manner sufficiently satisfactory to warrant its acceptance as a prerequisite to the degree for which it has been submitted. It is to be understood that by this approval the undersigned does not necessarily endorse or approve any statement made, opinions expressed, or conclusions drawn therein, but approves the thesis only for the purpose for which it is submitted.

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