

Mathematics Education from a Mathematicians Point of View

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Dedication

This is dedicated to Robert Simpson, my father, who gave me the encouragement to pursue this degree, and Nan Simpson, my mother, who gave me the encouragement to finish.

Abstract

This study has been written to illustrate the development from early mathematical learning (grades 3-8) to secondary education regarding the Fundamental Theorem of Arithmetic and the Fundamental Theorem of Algebra. It investigates the progression of the mathematics presented to the students by the current curriculum adopted by the Rhea County School System and the mathematics academic standards set forth by the State of Tennessee.

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List of Notation

FToA	Fundamental Theorem of Arithmetic
FTA	Fundamental Theorem of Algebra
$\stackrel{\text{def}}{=}$	Is defined as
mk or $m \times k$	m times k
$\frac{m}{n}$ or $m \div n$	m divided by n
GCF	Greatest common factor
LCM	Least common multiple
$\left(\frac{k}{l}\right)^*$	Negative $\frac{k}{l}$, or $-\frac{k}{l}$
$\mathbb{N}$	The set of Natural Numbers
$\mathbb{W}$	The set of Whole Numbers
$\mathbb{Z}$	The set of Integers
$\mathbb{Q}$	The set of Rational Numbers
$\mathbb{R} - \mathbb{Q}$	The set of Irrational Numbers
$\mathbb{R}$	The set of Real Numbers
$\mathbb{I}$	The set of Imaginary Numbers
$\mathbb{C}$	The set of Complex Numbers

Chapter 1: Introduction

The Fundamental Theorem of Arithmetic and The Fundamental Theorem of Algebra and their consequences are an essential part of the mathematics taught in elementary, middle, and high schools. The applications that arise from the Fundamental Theorem of Arithmetic alone can be seen through out most of school mathematics, such as the Greatest Common Factor (GCF), the Least Common Multiple (LCM), adding and subtracting fractions, and simplifying radicals. This study will look at the progression of mathematical principles from basic arithmetic to the solutions of algebraic problems that are related to the Fundamental Theorem of Arithmetic and the Fundamental Theorem of Algebra as they are presented to students from the third to the twelfth grades. Specifically, we will look at how the standards adopted by the Tennessee department of Education and the current textbooks being used by students correlate to the “basic characteristics of mathematics” (Wu, 2009). Dr. Hung-Hsi Wu states:

All in all, a mathematician approaching . . . school mathematics cannot help but be struck by the total absence of the characteristic features of mathematics: precise definitions as a starting point, logical progression from topic to topic, and most importantly, explanations that accompany each step. (Wu, 2011)

Mathematical discoveries throughout history require an exacting level of precision and proof to be accepted by the mathematical world as truth. As teachers and students of mathematics, we should strive to meet those same mathematical characteristics within the classroom. Therefore, this thesis examines whether the mathematical topics leading up to the Fundamental Theorem of Arithmetic and the

Fundamental Theorem of Algebra and their consequences presented to Tennessee public school students follow Wu's characteristics of mathematics: precise definitions that remain consistent throughout the curriculum and across grades, sound reasoning and examples, and logical progressions through topics (Wu, 2011).

There are other parameters that affect mathematics education not evaluated in this study, however, they are worth mentioning. One parameter is the pedagogical content knowledge of mathematics teachers. Are teachers prepared with a firm mathematical knowledge in order to "customize" it for the K-12 student "without sacrificing mathematical integrity" (Wu, 2006). Another factor is the question of state and standardized assessments. Are there "mathematically incorrect" items on these tests and are the test scores being misinterpreted (Wu, 2006)? Student readiness can also be a factor. Are some students pushed from grade-to-grade, being misplaced in the mathematics class in which they find themselves? These parameters could each be a study within themselves.

Chapter 2: Historical Background

Although the Fundamental Theorem of Arithmetic (FToA) is not formally stated in the textbooks reviewed for this study nor is it part of Tennessee's state curriculum standards, the elements from this essential theorem (e.g., the concepts of prime and composite numbers and the existence of prime factorization) are taught, particularly in the fourth and fifth grades. A student usually will not see this theorem formally stated unless they continue with their mathematics education into college level or beyond, as undergraduate and graduate students endeavoring to study number theory will study this theorem and its consequences. The undergraduate textbook, *Elementary Number Theory*, states the FToA as, "any integer n greater than 1 has a factorization into primes. This factorization is unique up to the order of the factors" (Eynden, 2006).

The FToA, also known as the Unique Factorization Theorem, has a somewhat ambiguous history. Credit for the FToA is often given to Euclid in his *Elements* Book IX. Euclid's proposition 14 of Book IX is as follows:

If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it.

(Heath)

Book IX proposition 14, as stated by Howard Eves, "is equivalent to the important *fundamental theorem of arithmetic*, namely that *any integer greater than 1 can be expressed as the product of primes in one, and essentially only one, way*" (Eves, 1969).

Euclid doesn't actually state that every integer can be written as a product of primes (Collison, 1980), but he does give us propositions in *Elements* VII, specifically

proposition 30 and proposition 31, regarding prime and composite numbers making it possible to prove this.

Euler and Legendre both presented the idea of representation in their works from the late 18th and early 19th centuries; however, neither of them discussed uniqueness. So, we turn to Carl Friedrich Gauss and his *Disquisitiones Arithmeticae*, which translates to Arithmetical Inquiry, of 1801. Prior to stating the FToA, Gauss presents what amounts to Euclid's proposition 30, "If neither a nor b can be divided by a prime number p , the product ab cannot be divided by p " (Gauss, 1966). He then proceeds to give credit to Euclid for proving it in *Elements*. Gauss states that he included this article and its proof because "by this very simple case we can more easily understand the nature of the method which will be used later for solving much more difficult problems." Gauss's Theorem 16 says, "*A composite number can be resolved into prime factors in only one way*" (Gauss, 1966). The FToA, as given by Gauss, focuses on the uniqueness of factorization. He states,

It is clear from elementary considerations that any composite number can be resolved into prime factors, but it is tacitly supposed and generally without proof that this cannot be done in many various ways. (Gauss, 1966)

Gauss, then, goes on to prove the uniqueness of factorization by contradiction.

The Fundamental Theorem of Algebra (FTA) as stated for an Algebra 2 student is, "if $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly n roots, including multiple and complex roots" (Algebra 2, 2012). However, the proof for this theorem is not presented to students in a typical high school mathematics class. "The theorem really is a monster to prove in full generality, for it requires some sophisticated

preliminary results about complex functions. Clearly a complete proof *is* beyond the reach of elementary mathematics” (Dunham, 1991). *Algebra & Trigonometry*, a textbook for high school seniors in an advanced algebra course states “the proof is beyond the scope of this book” (Sullivan, 2012) and hence a typical advanced algebra course.

Historically, solving polynomials has been the impetus for much of the achievements made in both ancient and modern algebra. Dating back to around 250 AD a significant Greek work written by Diophantus of Alexandria, *Arithmetica*, contained not only the use of symbolic notation for “the unknown, powers of the unknown up through the sixth, subtraction, equality, and reciprocals” (Eves, 1969) but solutions of equations. However, Diophantus only recognized “positive rational answers, and was, in most cases, satisfied with only one answer to a problem” (Eves, 1969). Around 830 AD an exposition on algebra was written by the most famous Arabian mathematicians of the time, Al-Khowarizi. Within this work, Al-Khowarizi solved linear and quadratic equations. He, like Diophantus, also only allowed positive real roots for solutions. His works, however, “exerted tremendous influence in Europe . . . in the twelfth century” (Eves, 1969). It was not until the 16th century that the idea occurred for the need of negative and complex numbers to “effect complete solutions” (Newman, 1956). Girolamo Cardan (1501-1576), an Italian mathematician, presented this idea with his work *Ars Magna* (1545). “He opened up the general theory of the cubic and quartic equations, by discussing how many roots an equation may have” (Newman, 1956).

In 1629 the FTA is first stated. Albert Girard, a Flemish mathematician, theorized “an equation of degree n always has n solutions in the domain of the complex numbers” (Ben-Menahem, 2009). And in 1749 Leonhard Euler makes an attempt at a proof of the

FTA. However, Euler's proof is not complete. Euler's proof does not prove the full generality of the theorem, but only proves the theorem "for polynomials up to the sixth degree" (Kline, 1972). The first serious attempt to prove the FTA, though, is by Jean D'Alembert in 1746. He too only partially proved it (Kline, 1972).

Again, we turn to Carl Friedrich Gauss to see the first rigorous proof of this theorem. At the age of twenty-one with his doctoral dissertation Gauss presented his first proof of this theorem (Rice & Scott, 2005). He began by reviewing and criticizing earlier attempts. "With his uncompromising demand for logical and mathematical rigor" (Newman, 1956), Gauss proved the existence of a root and then he proved that " n^{th} degree polynomial can be expressed as a product of linear and quadratic factors with real coefficients" (Kline, 1972). It was with Gauss' fourth proof in which he allowed "the coefficients of the polynomial to be complex numbers" (Kline, 1972). "Gauss viewed the fundamental theorem of algebra as a great and worthy project indeed" (Dunham, 1991).

Chapter 3: Literature Review

Education reform has been deliberated long before the twentieth century. However, it came into national prominence in 1957 when the Soviet Union launched Sputnik 1 into orbit. This event “played a significant role in education reform” (Bybee, 1997). From the late 50’s “New Math” to the Common Core Math Standards of 2010, the colossal pendulum of mathematics education continues to swing. To truly improve mathematics education, mathematicians and educators need to work together (Wu, 2006). Good mathematics education should be universal. We should look at views from mathematicians as well as educators as we evaluate the accepted standards and the textbooks offered to our students.

It is a widely studied belief that children develop their reasoning skills in a certain chronological order. As Harry Beilin states, “No one has influenced developmental psychology more than Jean Piaget (1896-1980)” (Beilin, 1992). Piaget believed that a child’s way of thinking is far different from the way in which an adult thinks. He defined four stages of cognitive development: sensorimotor (0-2 years), preoperational (2-7 years), concrete operational (7-11 years), and formal operational (11 years-adulthood) (Woolfolk, 1998).

In the field of education, Jean Piaget and his work on children’s cognitive development, Ojose (2008) notes, has “garnered much attention”. However, some say that he underestimated children’s thinking and some say he overestimated their thinking. While Piaget’s theory has seen some criticism, it has been given much attention within the mathematics and education communities (Ojose, 2008).

Students in late elementary to early middle school would be considered to be in the concrete operational stage. During the concrete operational stage, students develop “an understanding of three basic aspects of reasoning: identity, compensation, and reversibility” (Woolfolk, 1998). A student in this stage can comprehend that things can be “changed or transformed” (Woolfolk, 1998) and still maintain the key properties with which it began. They begin to understand that the operation addition can be reversed with subtraction and the operation multiplication with division. The thought process of the concrete operational student is “complete and very logical” (Woolfolk, 1998). However, this logic remains tied to tangible experiences. These students need a hands-on approach in their learning, as abstract ideas and thinking still remain difficult (Ojose, 2008). The use of manipulatives can give students the concrete experiences needed in order to “explore concepts such as place value and arithmetical operations” (Ojose, 2008).

According to Ojose (Ojose, 2008), students do not always progress at the same rate and “some will remain at the concrete operational stage throughout their school years” (Woolfolk, 1998), though students in junior and senior high school should generally be at the formal operational stage. During this stage, their thought process matures having the capability of using inductive and deductive reasoning. These students are able to think hypothetically and analyze elements of a problem systematically. Ojose (2008) states that at the formal operational stage a student develops a proficiency in abstract reasoning using symbols to generalize logical arguments.

“Communication is an essential part of mathematics and mathematics education” (NCTM, 2000). The Communication Standard for prekindergarten through twelfth grade

in *Principles and Standards for School Mathematics* states that a student should develop the ability to “communicate their mathematical thinking coherently” and “use the language of mathematics to express mathematical ideas precisely” (NCTM, 2000).

Mathematics is a language with syntax and vocabulary (Bruun, Diaz, & Dykes, 2015) and it is “inherently precise” (Wu, 2011). Hence, the language used when teaching mathematics must also exhibit precision. Students in the early grades, though, will use a more informal language when communicating their mathematical understanding.

Dunston and Tyminski (Dunston & Tyminski, 2013) contend that in the middle grades (grades 5-9) is where a significant portion of mathematical language is developed. As students progress in school they “should understand the role of mathematical definitions” (NCTM, 2000) and by high school the importance of and the use of mathematical definitions “should become pervasive” (NCTM, 2000). As mathematical language is limited largely to the classroom, teaching the mathematical vocabulary is essential for the development of a students “conceptual” understanding of mathematics (Dunston & Tyminski, 2013). Dr. Wu maintains that at the forefront, precise definitions are “the foundation for all reasoning and discussions” (Wu, 2011) in mathematics. It is with precise definitions that “rigorous reasoning now becomes possible” (Wu, 2011).

In the pursuit of ease and simplicity, students too often use memorized tricks and procedures to perform mathematics. However, this approach often leads to misconceptions (Gojak, 2013) and “gaps between the how and why” (Chang, 2013). The National Board for Professional Teaching Standards (NBPTS) emphasize, “...that mastering mathematical facts and procedures is only part of what it means to learn mathematics” (NBPTS, 2010). The Reasoning and Proof Standard for prekindergarten

through twelfth grade in *Principles and Standards for School Mathematics* states, that students should be provided with the means to “recognize reasoning and proof of fundamental aspects of mathematics” (NCTM, 2000).

Chang argues there are four areas of mathematics in which reasoning is implemented in students learning: numbers and symbols, properties and rules, procedures, and methods. Furthermore, when reasoning is skipped in the pursuit of simplicity, misconceptions and mistakes can be made (Chang, 2013). Students might make common errors when simplifying numerical or algebraic expressions if they lack an understanding of the connection between the definitions “of the defined entities with the basic operations on the defined entities” (Chang, 2013). They might misuse or wrongly apply properties or rules if they lack an understanding of the connection between the “definition of properties with the use of properties or rules” (Chang, 2013). Dr. Wu states, “reasoning disarms students’ perception that mathematics is a game played with rules they know nothing about” (Wu, 2011).

“From children’s earliest experiences with mathematics, it is important to help them understand that assertions should always have reasons” (NCTM, 2000). According to Chang, students who make the connections among the definitions, rules and procedures are able to better “make sense” of those methods and procedures (Chang, 2013). Teachers must provide students the experiences with which to make these connections (Gojak, 2013). Concrete examples should always be accompanied with sound reasoning (Wu, 2011). *Principles and Standards for School Mathematics* states that concrete models can help students make sense of the mathematical ideas they encounter. The use of algebra tiles and other manipulatives, when used appropriately,

present students with “opportunities to connect their work to the concepts” (Gojak, 2013). The use of metaphors and analogies, solely to explain a concept thought of as too abstract for students to understand otherwise, creates an ambiguity many students find difficult to hurdle as they progress in their study of mathematics. Everything a student needs to know about a concept should be found within its definition (Wu, 2011). Therefore, one “should be able to formulate logical arguments in terms of the original definition” when using manipulatives, metaphors, analogies, and examples (Wu, 2011).

The Connection Standard for prekindergarten through twelfth grade in *Principles and Standards for School Mathematics* underscores that mathematical ideas “interconnect and build on one another to produce a coherent whole” (NCTM, 2000). Students should be made aware of the fact that what they are learning in one lesson leads to what they will learn in the following lessons. Mathematical ideas should be introduced with intention. Dr. Wu states that there is a “structure of mathematics” (Wu, 2011). Mathematical ideas are not just thrown together at random or considered “isolated pieces of information” (Gojak, 2013). There must be a systematic, logical order, in which definitions, rules, and procedures are presented from simple to complex (Wu, 2011). Gojak emphasizes the importance of reflecting on how concepts “connect across grade levels” (Gojak, 2013). She states that considering how a mathematical idea “originates, extends, and connects with other concepts across the grades” help to “develop a deeper understanding” among teachers as well as students (Gojak, 2013).

The Common Core State Standards Initiative (CCSSI), established in 2009 by the governors and state commissioners of education from 48 states, the District of Colombia, and two territories of the United States, was a state-led effort. These standards took the

best state standards already in use and the experience of teachers and content experts into consideration and developed a set of grade-by-grade standards that encompassed a set of college and career ready standards, which had been developed beforehand to “address what students are expected to know and understand by the time they graduate from high school” (CCSSI, 2010). In June 2010 the final Common Core State Standards (CCSS) were released. The Tennessee Department of Education adopted these standards in 2010 with full implementation by the 2013-14 school year (CCSSI, 2010).

Due to ample political upheaval and charges of federal overreach, Gov. Bill Haslam announced a review of Tennessee’s current Mathematics and English Language Arts standards (Balakit, 2016). Subsequently, by the spring of 2016 Tennessee Department of Education replaced CCSS with Tennessee Academic Standards (TNAS). Beginning with CCSS, the current state standards, the State Board of Education organized a review consisting of two phases of public feedback. The TNAS were designed from the reviews of the public feedback and revisions made by a committee appointed by the Governor, Lt. Governor, and Speaker of the House of Representatives. The first year of implementation and assessment is scheduled for the 2017-18 school year (Math and English Language Arts, 2016).

Common Core State Standards for Mathematics (CCSSM) as well as the new TNAS include the Standards for Mathematical Content and Standards for Mathematical Practice (SMP). The Standards for Mathematical Content are a “balanced combination of procedure and understanding” while the eight SMPs describe ways in which developing students should “engage” with mathematics as they “grow in mathematical maturity and expertise throughout the elementary, middle and high school years” (CCSSI, 2010).

They are “mathematical ways of thinking” (Otten & De Araujo, 2015).

TNAS (Tennessee Math Standards, 2016) lists and describes the same SMPs as CCSSM (CCSSI, 2010). The first three SMPs are about reasoning. A skillful mathematics student will persevere in their problem solving and continually question the reasoning they are using (SMP1). They should be able to “reason abstractly and quantitatively” (SMP2), and be able to determine whether the arguments of others are logical and be able to improve on them (SMP3). SMP4 is to model with mathematics. Applying “the mathematics [students] know to solve problems...”. SMP5 is to “use appropriate tools strategically”. SMP6 identifies the need for precision. “Mathematically proficient” students should use accurate definitions when discussing their reasoning and by high school they should “make explicit use of definitions.” Seeing and using the structure in mathematics (SMP7) and “looking for regularity in repeated reasoning” (SMP8) describe the necessity of having intention behind each skill taught and coherence as a student progresses through each topic.

TNAS also lists four literacy skills for mathematical proficiency: “reading, vocabulary, speaking and listening, and writing” (Tennessee Math Standards, 2016). These skills support SMPs 1, 2, 3, 4, and 6. They encapsulate the characteristic of precision in mathematics and strong reasoning skills. They describe the necessity for precise definitions with mathematical vocabulary and clarity when communicating ones ideas to others.

Wu maintains that the Common Core Mathematics Standards (CCMS), with its inevitable flaws, “emphasizes what counts the most in mathematics education” (Wu, 2011). Wu says that from grade to grade there is “logical continuity”, reasoning pervades

throughout the presentation of each topic, and “mathematical clarity and precision is restored to school mathematics” (Wu, 2011). While the TNAS are a set of “Tennessee-specific standards” (Balakit, 2016), Balakit (Balakit, 2016) and Tatter (Tatter, 2016) both state that TNAS has preserved its Common Core roots.

Chapter 4: Methodology

Students enrolled in the Tennessee public school system are expected to be proficient in the mathematics standards set forth by the Tennessee Department of Education through each grade level. High school graduation requirements, as stated by Tennessee's Department of Education, are such that a student is required to be enrolled in four school years of mathematics. These courses include Algebra 1, Geometry, Algebra 2 and a fourth math course, which must be above the level of Algebra 1 (TN Department of Education, 2014). Students should be able to "use their math education when the need arises" and be prepared to "pursue post-secondary education opportunities" (Tennessee Math Standards, 2016).

There are clearly a plethora of mathematical topics to which students are introduced throughout their educational journey. This study merely examines the topics leading up to and the consequences of the FToA and the FTA as presented by the standards and the textbooks. This thesis was designed to look at the progression of mathematical principles that apply to the FToA and the FTA and how they are presented to the Third through Twelfth grade student. For the purposes of discussing these theorems, their concepts and consequences, the reviewer assessed the following textbooks currently adopted by the Rhea County Department of Education:

enVision Math Tennessee, 3rd Grade. Pearson Education, Inc.

enVision Math Tennessee, 4th Grade. Pearson Education, Inc.

enVision Math Tennessee, 5th Grade. Pearson Education, Inc.

Math Connects, Course 1. The McGraw-Hill Companies, Inc.

Math Connects, Course 2. The McGraw-Hill Companies, Inc.

Math Connects, Course 3. The McGraw-Hill Companies, Inc.

Algebra 1. Pearson Education, Inc.

Algebra 2. Pearson Education, Inc.

Algebra and Trigonometry (Ninth Edition ed.). Pearson Education

Every school district has a certain autonomy in choosing the textbooks they wish to provide for their students. Textbooks differ from publisher to publisher and may or may not raise the same issues with definitions, examples, or order of topics, etc. This reviewer selected to limit the evaluation to the textbooks currently approved for use by the Rhea County Department of Education.

Educational standards are adopted on a state-by-state basis. Many states are still using CCSS as the mathematical standards expected from their students and teachers. However, as a result of Tennessee replacing CCSS with TNAS, the reviewer elected to use the newly adopted TNAS that will be implemented in the 2017-18 school year.

TNAS Standards for Mathematical Practices asserts that “being successful in mathematics requires the development of approaches, practices, and habits of mind” as one develops “mathematical fluency and conceptual understanding” (Tennessee Math Standards, 2016). Within this list of 8 practices, the attributes of precision and sound reasoning can be found. Dr. Wu affirms these traits, “coherence, precision, and reasoning” as “a prerequisite to making school mathematics learnable” (Wu, 2009).

Wu maintains in *How Mathematical Can Contribute to K-12 Mathematics Education* that students need “unambiguous definitions” which tell students what each mathematical idea is exactly, but is also appropriate for the grade level of the student.

These definitions should “not leave room for misunderstanding or guesswork” (Wu, 2009). He also states that students “need to be exposed to...*mathematical* connections” (Wu, 2006). The mathematical ideas taught in the school curriculum are not a fragmented collection of facts and procedures. They “form a whole tapestry where each item exists as part of a larger design” (Wu, 2009). These topics should be introduced in a logical order, “progressing from simple to complex” (Wu, 2011). “Some concepts and skills must precede others because logical reasoning demands it” (Wu, 2011). And, it is reasoning that “is the power that enables us to move from one step to the next” (Wu, 2009).

This thesis will evaluate whether the mathematical topics leading up to the FToA and the FTA and their consequences presented to Tennessee public school students follow Wu’s characteristics of mathematics. Therefore, this evaluation uses Wu’s philosophy of the attributes of mathematics as the criteria from which each topic is to be measured.

The method utilized for analyzing each grade range (3rd – 5th, 6th – 8th, and 9th – 12th grades) is first to review the state mathematics standards with regard to the topics relating to the FToA and the FTA. Secondly, the reviewer will evaluate textbook descriptions, definitions, and algorithms with respect to the following tenets of Dr. Wu.

- Statements and definitions given in a clear and precise way.
- Sound reasoning and examples accompanying each topic.
- Coherence as one progresses from topic to topic.

Finally, the reviewer will discuss how well the content of the textbooks adheres to the standards and how well it supports Dr. Wu’s theory.

Chapter 5: Curriculum Review

Grades 3, 4, and 5

State Standard Requirements

As a student enters third grade, they should be proficient in adding and subtracting single digit numbers 1 - 20, knowing all single digit sums and the correlating subtraction facts from memory, and have the ability to add and subtract within 100 (Tennessee Math Standards, 2016). This forms a foundation on which to build an understanding of multiplication and division. Realizing that some teachers begin teaching multiplication and some students have the capability to grasp the concept before grade three, new Tennessee Academic Mathematics Standards state that it is in third grade that a student should develop an understanding of multiplication and division of whole numbers. They should also “develop an understanding of fractions as numbers” (Content standard 3.NF.A.2 & 3), in particular unit fractions (3.NF.A.1). Third grade students should develop strategies for multiplication and division within 100. In fact, by the end of third grade, a student should know their multiplication and related division facts of two one-digit numbers by memory (3.OA.C.7) (Tennessee Math Standards, 2016).

In the fourth grade curriculum, the ideas and concepts introduced in the prior year begin to expand. The fourth grade student should have certain fluency with their multiplication, i.e., quick and accurate computation, with not only two two-digit numbers but with up to four-digits multiplied by a one-digit number (4.NBT.B.5). They will

develop an understanding of dividing to find quotients involving four-digit dividends and one-digit divisors (4.NBT.B.6) (Tennessee Math Standards, 2016). The concept of a fraction is extended with the understanding of fraction equivalence (4.NF.A.1). Students will become aware that they can “compose and decompose fractions in different ways” (Tennessee Math Standards, 2016). By the end of fourth grade, they should have a firm grasp of adding and subtracting fractions with like denominators (4.NF.B.3) and should be able to “apply and extend previous understandings of multiplication as repeated addition to multiply a whole number by a fraction (4.NF.B.4)” (Tennessee Math Standards, 2016). As a result, students gain a familiarity with factors and multiples. Finally, the idea of prime and composite numbers are introduced, and by the end of fourth grade, a student should have the ability to determine whether a whole number between 1 and 100 is either prime or composite. This knowledge of factors and multiples allows them to “recognize that a whole number is a multiple of each of its factors (4.OA.B.4)” (Tennessee Math Standards, 2016).

The fifth grade student will “finalize their understanding of multi-digit addition, subtraction, multiplication, and division with whole numbers” (Tennessee Math Standards, 2016). Upon completion of fifth grade, students should “fluently multiply multi-digit whole numbers (up to three-digit by four-digit factors) using appropriate strategies and algorithms (5.NBT.B.5)” (Tennessee Math Standards, 2016). “For the first time, [fifth grade] students will develop an understanding of fractions as division” (Tennessee Math Standards, 2016), e.g., $\frac{a}{b} = a \div b$ (5.NF.B.3). They will “apply...their understanding of multiplication and division to multiply and divide fractions (5.NF.B.4 &

5)”, limited to division of unit fractions by non-zero whole numbers and visa versa (Tennessee Math Standards, 2016).

Textbook Review

Multiplication is first introduced to a student in the third grade. Dr. Wu introduces the concept of multiplication as shorthand for repeated addition”, such as $2 \times 6 \stackrel{\text{def}}{=} 6 + 6$ and $4 \times 3 \stackrel{\text{def}}{=} 4 + 4 + 4$. Euclid in his *Elements* Book VII defined multiplication in this way. *Elements* Book VII Definition 15 says, “A number is said to *multiply* a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced” (Heath). The third grade student should already have a concrete idea of the concept of addition. Therefore, if m and k are *whole numbers*, then Dr. Wu defines mk (or $m \times k$) as:

$$mk = \begin{cases} 0 & \text{if } m = 0, \\ \underbrace{k + k + \cdots + k}_m & \text{if } m \neq 0. \end{cases}$$

This definition, though, will need modification as more of the number system is introduced. The term mk is referred to as the **product** of m and k , and m or k are **factors** of mk . This definition clearly states that any number multiplied by zero is defined as zero and mk is m copies of k when m is not zero (Wu, *Understanding Numbers in Elementary School Mathematics*, 2011).

The first encounter with multiplication for a third grader defines the operation of multiplication as repeated addition as well as being the operation that “gives the total number when you join equal groups” (enVision Math Tennessee, 3rd Grade, 2012). Fourth grade *enVision Math* also gives this double definition for multiplication. Children in the third and fourth grades still need some kind of concrete experience with which to

associate their reasoning. Counting manipulatives, or counters, serve this purpose. The use of manipulatives with third and fourth grade students may be the reason for this double meaning. Therefore, the third grade student is given this definition from *enVision Math*, with the example shown in figure 1.

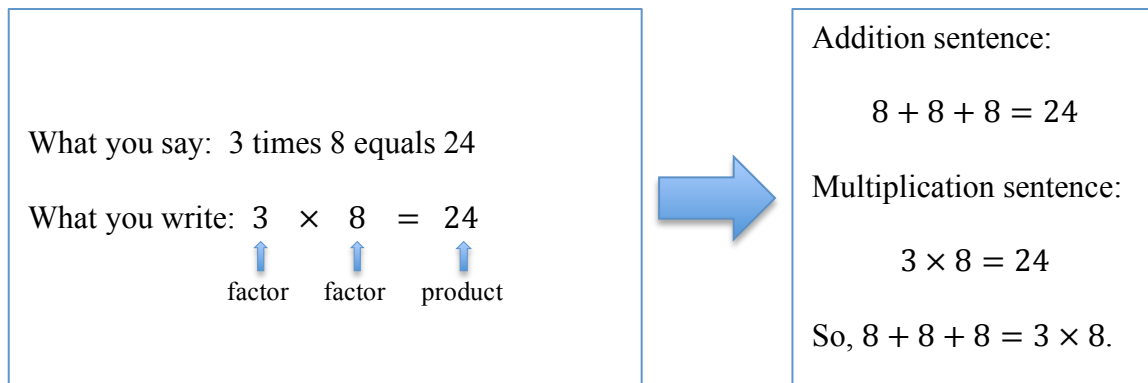


Figure 1 – Multiplication as Repeated Addition (enVision Math Tennessee, 3rd Grade, 2012)

The practice exercises could create a bit of confusion for the student. Within a single exercise they are to write three different sentences to describe equal groups of objects: an addition sentence, a multiplication sentence, and a “groups of” sentence, e.g., 2 groups of 4; however, there is no example shown in these textbooks for a “groups of” sentence and by the fifth grade textbook that phrase is gone all together. The use of an array or area model is also used to show multiplication, which is the idea of multiplication as “groups of” in a geometric way. This method would be helpful when showing the distribution property, as well as the concept of area. In the experience of this reviewer, children at this age are very literal. It is enough for the student to write the addition and multiplication sentences without using the “groups of” portion of the exercise.

It is this first encounter in the third grade that the terms **factor** and **product** are defined. The definitions for these terms presented by the textbook *enVision Math* is in concert with the previously stated definition. As for the term **multiples**, it is defined as “the products of a number and other whole numbers” (enVision Math Tennessee, 3rd Grade, 2012). Even though it is not expected for a third grader to have the ability to fluently multiply numbers greater than 100 by single-digit whole numbers, it is expected for them to have strategies for this operation. As with single-digit multiplication, the third and fourth grade student begins multi-digit multiplication using manipulatives. An example given is to build an array to show 4×24 by using place-value blocks to break 24 into tens and ones, $24 = 2 \text{ tens } 4 \text{ ones}$. Then $4 \times 2 \text{ tens}$ is $4 \times 20 = 80$, $4 \times 4 \text{ ones}$ is $4 \times 4 = 16$, and $80 + 16 = 96$. 80 and 16 are **partial products** because “they are parts of the product” (enVision Math Tennessee, 3rd Grade, 2012). Students can visually see from the place-value blocks the idea of the expansion of a number, which will be the

What You Show	What You Write
Find 3×46. Use place value blocks to make an array for 3×46 $46 = 4 \text{ tens } 6 \text{ ones}$ Multiply tens: $3 \times 40 = 120$ Multiply ones: $3 \times 6 = 18$	$\begin{array}{r} 46 \\ \times 3 \\ \hline 18 \\ + 120 \\ \hline 138 \end{array}$ Write the partial products. Line up the ones, tens, and hundreds. Add

Figure 2 - Expanded Multiplication Algorithm (enVision Math Tennessee, 3rd Grade, 2012)

basis for the multiplication algorithm. Also, this demonstrates to students the distributive property. They can see that $4 \times 24 = 4 \times (20 + 4) = (4 \times 20) + (4 \times 4) = 80 + 16 = 96$. The multiplication algorithm relates to this type of number expansion and is presented by example in the third and fourth grade textbooks as in figure 2.

Algorithms are meant to simplify our computations. They are “a finite sequence of explicitly defined, step-by-step computational procedures which ends in a clearly defined outcome” (Wu, Understanding Numbers in Elementary School Mathematics, 2011). When larger computations are presented, such as $34,578 \times 56,890$, some shortcuts are necessary. The involuntary reflex of pulling out the calculator, that most students tend to acquire, may relieve the anxiety of such a computation, however “without a firm grasp of the place value of our numeral system and the logical underpinning of the algorithms” (Wu, Understanding Numbers in Elementary School Mathematics, 2011) they will not be able to identify mistakes. Dr. Wu has a “leitmotif” for the standard algorithm, which is “to perform a computation with numbers, break it down into several steps so that each step (when suitably interpreted) is a computation involving only a single digit,” hence, the need to memorize the multiplication table at least up to the 9’s. In the sense of simplification students are shown a short cut version of the expanded multiplication algorithm using the notion of regrouping. The example shown in *enVision Math, Third Grade* begins by using manipulatives and stating that $3 \times 6 = 18$. Eighteen can be regrouped as 1 ten and 8 ones. Figure 3 shows the step-by-step example for multiplying a three-digit number by a one-digit number that is given to the third grader and to the fourth grader. By the fifth grade, the use of manipulatives is a bygone tool. The short-cut given in figure 3 for multiplying multi-digit numbers by a

Find 8×605 .

Use what you know about multiplying 2-digit number for 3-digit numbers.

Step 1	Step 2	Step 3
Multiply the ones. $8 \times 5 \text{ ones} = 40 \text{ ones}$ Regroup.	Multiply the tens. $8 \times 0 \text{ tens} = 0 \text{ tens}$ Add the regrouped tens.	Multiply the hundreds. $8 \times 6 \text{ hundreds} = 48 \text{ hundreds}$
$\begin{array}{r} 4 \\ 605 \\ \times 8 \\ \hline 0 \end{array}$	$\begin{array}{r} 4 \\ 605 \\ \times 8 \\ \hline 40 \end{array}$	$\begin{array}{r} 4 \\ 605 \\ \times 8 \\ \hline 4,840 \end{array}$

Figure 3 - Regroup to Multiply (enVision Math Tennessee, 4th Grade, 2012)

one-digit number is combined with the expanded multiplication algorithm in figure 2 in order to multiply arbitrary whole numbers by multi-digit numbers. Figure 4 is an example of the process of the expanded multiplication algorithm. In essence, the explanation is: multiply by the ones and regroup if necessary, then multiply by the tens and regroup if necessary, and finally add the partial products.

Directly after multiplication facts, in each of the textbooks, third, fourth, and fifth grade, the meaning of division is given. This is the logical progression as “division is an alternate, but equivalent, way of expressing multiplication” (Wu, Understanding Numbers in Elementary School Mathematics, 2011). Dr. Wu’s definition is as follows:

$$\left\{ \begin{array}{l} \text{The **division of } m \text{ by } n, m \div n, \text{ is} \\ \text{the whole number } k \text{ that satisfies } m = kn. \end{array} \right.**$$

The number k is called the **quotient** of the division $m \div n$, n the **divisor**, and m the **dividend**. Because of the preceding definition of multiplication as repeated addition, the

Find 12×389 .

Step 1

Multiply the ones, and regroup if necessary.

$$\begin{array}{r} 11 \\ 389 \\ \times 12 \\ \hline 778 \end{array}$$

2×9 ones = 18 ones *or* 1 ten *and* 8 ones

2×8 tens = 16 tens

16 tens + 1 ten = 17 tens

17 tens = 1 hundred 7 tens

2×3 hundreds = 6 hundreds

6 hundreds + 1 hundred = 7 hundreds

Step 2

Multiply the tens, and regroup if necessary.

$$\begin{array}{r} 389 \\ \times 12 \\ \hline 778 \\ + 3890 \end{array}$$

10×9 ones = 90 ones

10×8 tens = 80 tens

or 8 hundreds

10×3 hundreds = 30 hundreds

or 3 thousand

Step 3

Add the partial products.

$$\begin{array}{r} 389 \\ \times 12 \\ \hline 778 \\ + 3890 \\ \hline 4,668 \end{array}$$

Figure 4 - Multiplication Algorithm for Multiplying an Arbitrary Number by a Multi-Digit Number (enVision Math Tennessee, 5th Grade, 2012)

explanation of “sharing” or “how many objects are in each group” as used in the third and fourth grade textbooks is logical. Dr. Wu states in *Understanding Numbers in Elementary School Mathematics*, “because $m = kn = n + n + \cdots + n$ (k times), the division $m \div n = k$ has the following interpretation: $m \div n$ is the number of groups if m objects are partitioned into equal groups of n objects.”

Prime and composite numbers are introduced briefly in the fourth grade textbooks and again in more detail in the fifth grade textbook. The definitions given are: a prime number is a “whole number greater than 1 that has exactly two factors, 1 and itself,” and a composite number is a “whole number greater than 1 that has more than two factors” (enVision Math Tennessee, 4th Grade, 2012). The example for the fourth grade student uses an array of buttons to show that 5 is a prime number and 6 is a composite number. Five can only be grouped by five ones or one group of five. Hence, five has only two factors, 1 and 5. Six can be grouped as one group of six or two groups of three. Therefore, 1, 2, 3, and 6 are factors of six. The text proceeds to state that 1 is special. One is neither prime nor composite. The reason for this exclusion is not given. Students are supposed to just accept this fact and go on with their day. However, we should be asking why 1 is so special. As mathematicians we need be interested in the reasoning on which our assertions are founded. So, let us discuss the reason 1 is neither prime nor composite. Simply put, 1 has exactly one factor. It doesn’t fit the definition given for prime or composite numbers. Another reason, and maybe the most interesting reason is that excluding 1 from the set of primes makes the statement of the FToA possible, specifically the uniqueness of prime decomposition.

Now, fourth graders are tasked with identifying prime and composite numbers. A

hundreds board is a good way to show a young student how to find all the prime numbers from one to one hundred, using the sieve of Eratosthenes. Fifth graders are taught to use divisibility rules to help them decide if a number is prime or composite. If a number is composite, they are shown how to write it as a product of prime factors or **prime factorization** with the use of a **factor tree**, “a diagram that shows the prime factorization of a composite number” (enVision Math Tennessee, 5th Grade, 2012). They offer two ways to construct a factor tree as shown in figure 5. Then again, a more logical way to construct prime factorizations may be to see if the smallest prime number, 2, is a factor first, since it is rather easy to divide by 2. Continue in this way until 2 no longer is a factor. Then look at 3 and so forth until the last “branch” contains all prime numbers. However, the FToA states that prime factorization is “unique up to the order of factors” (Eynden, 2006), which implies that the method of arriving at prime factorization is not important. There are more ways than one to get from point A to point B. Some ways just happen to be easier than others.

Even though the FToA is never mentioned in the reviewed curriculum, the existence portion is shown in figure 5. Why, then, should we be concerned with the uniqueness part? For the fifth grade student, the consequences of unique prime factorizations provide a new way to search for greatest common divisors (GCD) and least common multiples (LCM), which provide the skill set for adding and subtracting fractions.

Write 24 as a product of prime factors.

One Way

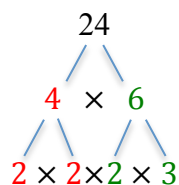
Step 1

Find a factor pair for 24



Step 2

Write each factor that is not prime as a product of primes numbers.

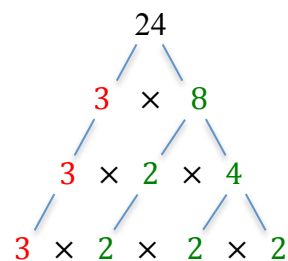


Last “branch” of the tree contains all prime numbers.

So, $24 = 2 \times 2 \times 2 \times 3$.

Another Way

You can use a different factor pair for 24



Continue until all “branches” end in prime numbers.

Figure 5 - Factor Tree (enVision Math Tennessee, 4th Grade, 2012)

The study of fractions is impacted by prime factorization. Therefore, Wu's definition of a fraction is included.

Let k, l be whole numbers with $l > 0$. Then $\frac{1}{l}$ is by definition one part when the unit is divided into l equal parts, and $\frac{k}{l}$ is by definition (the totality of) k of these parts. (Wu, Understanding Numbers in Elementary School Mathematics, 2011)

Although fractions are not essential to proving or using the FToA or the FTA, they are important to the development of the complex number system.

Evaluation

Students in these middle elementary grades are fully enveloped in Piaget's "concrete operational" stage. They have the ability to develop a complete and very logical system of thinking. However, their reasoning is tied to concrete objects (Woolfolk, 1998). Therefore, teachers should strive to give their students, even in these early elementary grades, precise definitions for the concepts being introduced that are logically sound and that remain true throughout. Using more complicated or undefined words in order to define new concepts will only confuse and frustrate a student trying to learn and master a new skill. As stated earlier, counting manipulatives, or counters, serve as a hands-on visual for students in these early grades. This correlates to Wu's idea of examples with sound reasoning accompanying each topic.

In the discussion of multiplication as repeated addition, students are asked to write "groups of" sentences in the exercises. However, there are no examples available in the text for students to patterns their practice from. Although, this seems not to fit the tenets of Wu as statements given in a clear and precise way, this reviewer feels that this is

grade appropriate terminology. By fifth grade, “groups of” disappears from the text, which is about the time students are beginning to experience fractions.

Fourth grade students are introduced to prime and composite numbers. The text *enVision Math, Fourth Grade* states that 1 is special, i.e., neither prime nor composite. This reviewer believes there should be an explanation alongside this statement. This does not meet the standard of a clear and precise definition.

The textbooks reviewed for grades 3, 4, and 5 follow the standards set by the State of Tennessee. In general, with the exceptions given above, the concepts are introduced in a logical, grade appropriate order with precise and clear definitions. Therefore, this adheres to the basic characteristics of mathematics upheld by Wu.

Grades 6, 7, and 8

State Standard Requirements

The middle school student will learn significant amounts of algebra as well as expand their earlier knowledge of multiplication and division to multi-digit and non-negative rational numbers. The sixth grade student should reach fluency with multi-digit division using the standard algorithm (6.NS.B.2) as well as understanding how to solve problems involving the multiplication and division of fractions (Tennessee Math Standards, 2016). Expressions involving whole number exponents are introduced to the sixth grade student. They will be asked to write and evaluate these types of expressions (6.EE.A.1); they will be introduced to the concept of the greatest common factor (GCF) of two whole numbers less than or equal to 100 and the least common multiple (LCM) of two whole numbers less than or equal to 12 (6.NS.B.4); they will also be introduced to

the common properties of numbers, namely the distributive property (6.EE.A.3). This allows the student to generate equivalent expressions such as $2(4 + 5x) = 8 + 10x$. The TNAS state that a student exiting the sixth grade should be capable of reading, writing, and evaluating expressions containing variables (6.EE.A.2). Using non-negative rational numbers, the sixth grade student should be able to reason and solve one-variable, one-step equations (6.EE.B.7) and understand the variable can represent an unknown number or any number in a specified set (6.EE.B.6).

Seventh grade students will broaden their repertoire of addition, subtraction, multiplication, and division “to all rational numbers, maintaining the properties of operation and the relationships between addition and subtraction, and multiplication and division” (7.NS.A.1-3) (Tennessee Math Standards, 2016). Applying these properties, the seventh grade student should have a competent understanding of the rules for adding, subtracting, multiplying, and dividing with negative numbers. They will learn to add, subtract, factor, and expand linear expressions with rational coefficients (7.EE.A.1). Students in this grade will have the ability to simplify numerical expressions involving rational numbers, evaluate algebraic expressions involving rational values for coefficients and/or variables, and solve a linear equation with rational coefficients (7.EE.A.3). Progressing from solving the one-step equations preformed in sixth grade to solving two-step equations.

The eighth grader will continue to refine their sixth and seventh grade numeric and algebraic skills. Upon transitioning out of eighth grade, students are expected to fluently “know and apply the properties of integer exponents...” (8.EE.A.1) (Tennessee Math Standards, 2016). They will progress from solving two-step equations to solving

multiple-step equations. The eighth grade student will develop an understanding of the definition of a function (8.F.A.1); however, the use of function notation is not required until ninth grade Algebra 1. A function of the form $y = mx + b$ will be defined as a linear function (8.EE.B.6, 8.F.A.3). They will learn to solve a linear equation in one variable (8.EE.C.7), as well as, analyze the graph.

Textbook Review

Students have had experience with multiplying and dividing decimals and fractions since fifth grade. Furthermore, ratios are defined for the sixth grade student to be “a comparison of two quantities by division” (Math Connects, Course 1, 2012). This textbook proceeds to explain that a rational number is “any number that can be written as a fraction” with no mention of zeros in the denominator. It’s not until seventh grade that students are told the following:

A rational number is a number that can be expressed as a ratio of two integers expressed as a fraction, in which the denominator is not zero. Common fractions, terminating and repeating decimals, percents, and integers are all rational numbers. (Math Connects, Course 2, 2012)

Students are told in third grade “you cannot divide any number by 0” (enVision Math Tennessee, 3rd Grade, 2012). Moreover, as early as fourth grade, students see fractions as division. The clause “denominator is not zero” should not be a foreign concept to a sixth grade student. The definitions we present to students need and should remain the same from when we first pose them. This clause needs to be included from the start. The definition presented to the seventh grade student in *Math Connects, Course 2* is sufficient

and should be accessible to the sixth grade student and should be offered to them. The definition of a rational number could even be simplified to “the set all fractions and their mirror reflections with respect to 0, i.e., the numbers $\frac{m}{n}$ and $(\frac{k}{l})^*$ for all whole numbers k, l, m, n ($l \neq 0, n \neq 0$)” (Wu, 2011). A mirror reflection, $(\frac{k}{l})^*$, with respect to zero of a number is the additive inverse. Of course, whole numbers are included with this definition since they are included in the set of fractions. Several lessons later, *Math Connects, Course 1* illustrates for the sixth grade student positive rational numbers with a number line. The definition from *Math Connects, Course 2* is accompanied not only with a number line, but, with a Venn diagram. This is instrumental in giving students a concrete visual for comparing and ordering this new set of numbers. The Venn diagram gives them a visual model of how each number set builds on the preceding number set. The Venn diagram presented in *Math Connects, Course 2* is illustrated in figure 6. A similar diagram is presented in *Math Connects, Course 3*. Zero has been included in the

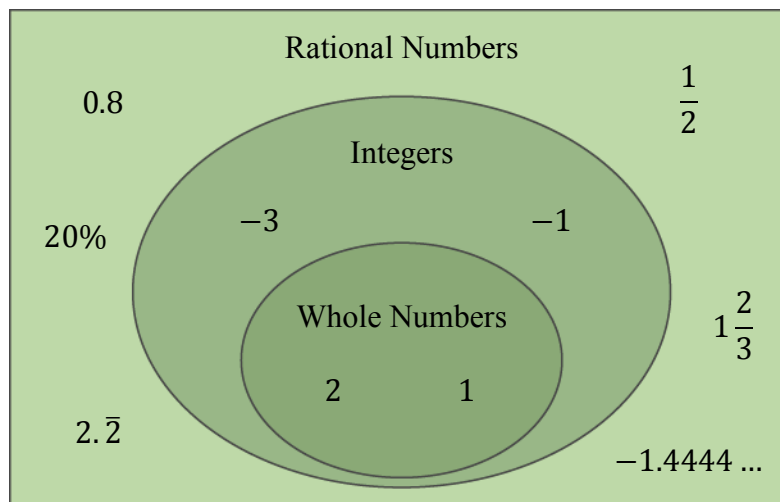


Figure 6 - Venn Diagram of the Rational Numbers
(Math Connects, Course 2, 2012)

set of whole numbers since they were defined for the third grade student. If the conventions that started with third grade students are to continue, zero should be in its appropriate place in the Venn diagram. Without zero, the set of whole numbers becomes what is generally understood to be the set of natural numbers.

Students as early as fifth grade are introduced to the set of integers, equations, and graphs of equations. However, more than likely, it is when they arrive in the sixth grade that they begin to experience these concepts with more detail. The first explanation given to the fifth grade student concerning the integers is that “integers are the whole numbers and their opposites; 0 is its own opposite” (enVision Math Tennessee, 5th Grade, 2012). This explanation is accompanied with a number line demonstrating these facts: -5 and +5 are the same distance from 0, -2 is the opposite of +2, and +4 is the opposite of -4. The sixth and seventh grade student are given a compatible definition; “Numbers like -2 and 2 are called integers. An integer is any number from the set $\{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$, where $\dots$ means *continues without end*” (Math Connects, Course 1, 2012). The number line accompanying this definition is:

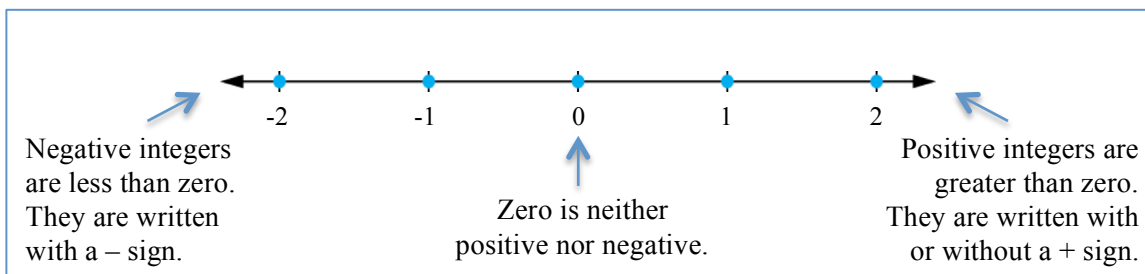


Figure 7 - Number Line with the Integers (Math Connects, Course 1, 2012)

So, the number system has been broadened to a two-sided number line.

Irrational numbers are introduced to the eighth grade student, completing the set of real numbers. *Math Connects, Course 3* defines an irrational number as “numbers that are not rational” and “a number that cannot be expressed as the ratio $\frac{a}{b}$, where a and b are integers and $b \neq 0$.” The examples given are: $\sqrt{2} \approx 1.414213562 \dots$ and $-\sqrt{3} \approx -1.732050807 \dots$, along with a Venn diagram illustrating the set real numbers (Math Connects, Course 3, 2012).

The properties of operations in figure 8 are presented to the eighth grade student. Students as early as third grade have been introduced to the idea of some of these properties without the use of formal language. Sixth grade students are presented with the following properties: commutative and associative properties for both addition and multiplication, distributive property, identities for both addition and multiplication. In addition to these, the seventh grade student is presented with the additive inverse. Within the eighth grade curriculum, all of the properties in figure 8 will be studied. However, closure is treated as an enrichment activity only.

The third grade student is presented with numerical expressions and the definition of an equation. Appearing in the fourth and fifth grade curriculum are the definitions for algebraic expressions and the idea of a solution to an algebraic equation. However, *Principles and Standards for School Mathematics* states:

In the middle grades, students should work more frequently with algebraic symbols than in lower grades...Students should develop an initial understanding of several different meanings and uses of variables through representing quantities in a variety of problem situations. (NCTM, 2000)

<i>Associative property of addition</i>	$(a + b) + c = a + (b + c)$
<i>Commutative property of addition</i>	$a + b = b + a$
<i>Additive identity (property of 0)</i>	$a + 0 = 0 + a = a$
<i>Associative property of multiplication</i>	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
<i>Commutative property of multiplication</i>	$a \cdot b = b \cdot a$
<i>Multiplicative identity (property of 1)</i>	$a \cdot 1 = 1 \cdot a = a$
<i>Distributive property</i>	$a \cdot (b + c) = a \cdot b + a \cdot c$
<i>Inverse property for addition</i>	$a \cdot a^{-1} = a^{-1} \cdot a = 1$
<i>Inverse property for multiplication</i>	$a + (-a) = (-a) + a = 0$
<i>Closure under addition</i>	If a and b are in a set, $a + b$ is in that set.
<i>Closure under multiplication</i>	If a and b are in a set, $a \cdot b$ is in that set.

Figure 8 – Properties of Operations

The TNAS also list to “write, read, and evaluate expressions in which letters stand for numbers” as a standard for sixth grade and not before, so we will review these here.

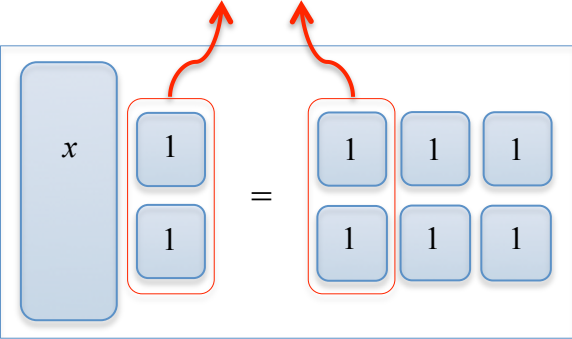
An expression, numerical or algebraic, contains at least one operation. Where a numerical expression is a combination of numbers and operations, e.g., $6 + 4 \cdot 3$, an algebraic expression includes variables, “a symbol, usually a letter, used to represent a number” (Math Connects, Course 1, 2012). *Math Connects, Course 2* describes a variable as “a symbol that represents an unknown quantity.” The concept of variables is one that tends to be mystifying to students. Here are some examples of how “variables” are used (Math Connects, Course 1, 2012):

- 1) Evaluate $n + 3$ if $n = 4$.
- 2) The commutative property, $a \cdot b = b \cdot a$ for all real numbers.
- 3) Solve $2x + 3 = 9$.

All three examples above illustrate the symbolic notation used in these middle grades. The idea that these letter symbols represent something unknown is incorrect. The first example states n stands for the number 4, the second quantifies a and b as standing for all real numbers, and finally, to solve the linear equation $2x + 3 = 9$ means to find all the numbers x within a given set that make this statement true. The important fact is that each example quantifies the symbol being used. The statement $a \cdot b = b \cdot a$ “is meaningless” if we do not define a and b (Wu, 2014). A **variable** is simply a placeholder. It is a placeholder for a particular quantity or number from a specified set which makes a mathematical statement true. In the equation $2x + 3 = 9$ we will call 2 the **coefficient** of the variable x , because 2 is “the number by which x is multiplied” (Math Connects, Course 2, 2012).

As stated, to solve an equation is to find all the numbers x within a given set which satisfy the given equation. The sixth grade student is told that “in algebra, an *equation* is like a balance. The weights on each side of the scale are equal. The expressions on each side of an equation are equal” (Math Connects, Course 1, 2012). “An equation is a mathematical sentence showing two expressions are equal” (Math Connects, Course 2, 2012). Throughout the curriculum, the importance of maintaining equality and checking solutions is constant. Figure 9 is an example from *Math Connects, Course 1* presented to the sixth grade student for how to solve an equation.

Solve the equation $x + 2 = 6$ by subtracting 2 from each side of this equation.



$$\begin{array}{rcl} x + 2 & = & 6 \\ -2 & = & -2 \\ \hline x & = & 4 \end{array}$$

The equations $x + 2 = 6$ and $x = 4$ are **equivalent equations** because they have the same solution, 4.

Figure 9 - Equivalent Equations (Math Connects, Course 1, 2012)

Students naturally progress from one-step equations to multi-step equations in a logical order. The sixth grade student learns the methods for solving one-step equations, or equations involving one operation and they are introduced to two-step equations, those that contain two operations. Multi-step equations appear in the eighth grade text. With

this comes a new process, combining like terms.

The sixth grade student solving one and two-step equations is presented with the concepts of “properties of equality” and the idea of inverse operations “which *undo* each other” (Math Connects, Course 1, 2012), e.g., to solve an addition equation, use subtraction. These are introduced to them as a way to solve equations. The “properties of equality” are: the subtraction property of equality, the addition property of equality, the division property of equality, and the multiplication of equality. The meaning is “if you (apply an operation to) each side of an equation by the same number, the two sides remain equal” (Math Connects, Course 1, 2012). The method of solving one and two-step equations is to apply the additive inverse or multiplicative inverse of a specified number, or element of a set, to each side of the equation. Students are told to “*undo* the operations in reverse order of the order of operations” (Math Connects, Course 2, 2012). But, what exactly does it mean to *undo* an operation? By applying the additive or multiplicative inverse, you maybe *undoing* an operation, e.g., $+2$ *undoes* -2 . However, in the sense of being precise and maintaining a convention from the inception of a definition throughout, we need to find an appropriate approach for teaching the correct definitions for the processes we are asking of our students at their level of understanding. The “properties of equality” best describe the idea of applying the additive and multiplicative inverse. There is really no need to obscure things with the notion of *undoing* an operation.

The efficient method for solving one and two-step equations would be using inverse operation; however, with multi-step equations the need arises to expand our strategies and use the distribution property and grouping like terms. For example,

$3(5x + 2) - 3x = 7$. We could add $3x$ to both sides of the equation first, divide each side by 3, then subtract 2 from each side, etc. Our goal is to find the value of x that makes our mathematical sentence true in an efficient manner. If we continued our computation, we would eventually arrive at the correct number value; however, we would have to sludge through some messy computations when we could have solved it in four steps:

$$3(5x + 2) - 3x = 7$$

$$15x + 6 - 3x = 7 \text{ (Distribute the 3)}$$

$$12x + 6 = 7 \text{ (Group like terms)}$$

$$12x = 1 \text{ (Apply the additive inverse of 6)}$$

$$x = \frac{1}{12} \text{ (Apply the multiplicative inverse of 12).}$$

This equation employs the processes of grouping like terms and the distributive property which are integral to solving multi-step equations. “When addition or subtraction signs separate an algebraic expression into parts, each part is called a **term**” (Math Connects, Course 3, 2012). “**Like terms** contain the same variables to the same powers” (Math Connects, Course 3, 2012). A term without a variable is defined to be a constant and “constant terms are also like terms” (Math Connects, Course 3, 2012). These definitions are presented in the *Math Connects, Course 2*, with the exclusion of the phrase “to the same powers.” Seventh grade students have had experience with exponents; therefore, this phrase should be included when this definition is initially presented to the seventh grade student.

“If we may use an analogy, using numbers to describe the phenomena around us is akin to using snapshots to describe an evolving situation...Functions are the videos in

the mathematical world...To describe the changes in both the human and natural worlds, we need functions” (Wu, 2011). Functions are defined for the sixth grade student, however it is not expected for the student to fully understand the definition until eighth grade (Tennessee Math Standards, 2016). A function is defined to be “a relation in which each member of the input is paired with *exactly* one member of the output” and a function rule is “the operation(s) preformed on the input value to get the output value” (Math Connects, Course 2, 2012). Input values are a member of the **domain** and output values are members of the **range** (Math Connects, Course 2, 2012). This is demonstrated with the illustration of a “function machine”, as in figure 10.

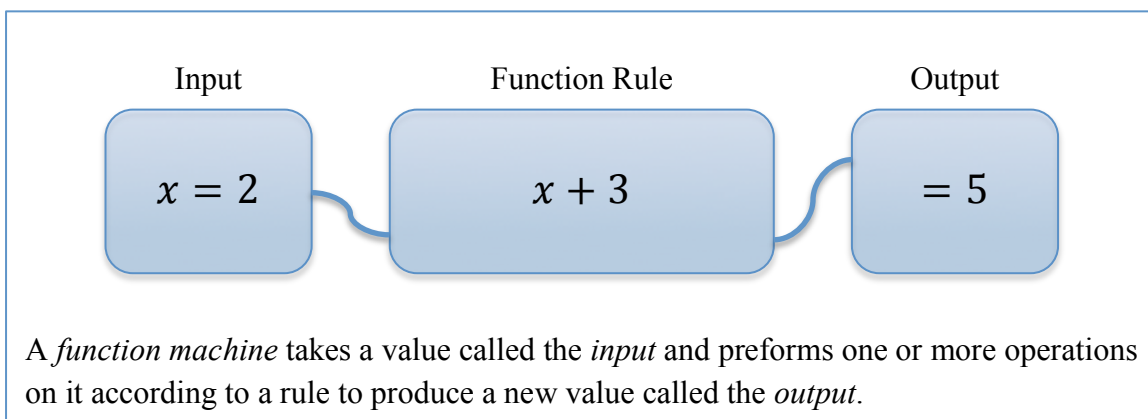


Figure 10 - Function Machine (Math Connects, Course 2, 2012)

The formal definition for a function one finds in Advanced Algebra is: “Let X and Y be two nonempty sets. A **function** from X into Y is a relation that associates with each element of X exactly one element of Y ” (Sullivan, 2012). This definition is consistent with the definition from sixth grade; however, the distinction between a function and an equation is not made in the middle grades. The sixth grade textbook

states: “A linear function is a function whose graph is a line. The function is then called a *linear equation*” (Math Connects, Course 1, 2012). Moreover, “you can use an equation to represent a function” (Math Connects, Course 1, 2012). This could contribute to a misconception that all equations are functions and visa versa. *Math Connects, Course 2* when introducing function notation says, “A function that is written as an equation can also be written in a form called **function notation**. Consider the equation $y = 4x$.” In function notation it is written $f(x) = 4x$. “The variable y and $f(x)$ both represent the dependent variable” (Math Connects, Course 2, 2012). On the surface this does not seem terribly inaccurate. The idea that y and $f(x)$ can be freely interchanged is pervasive in middle and high school classrooms. Consider the equation $y = \pm\sqrt{1-x^2}$, this is a perfectly good equation that graphs the unit circle, yet it cannot be a function. It can be easily checked that $f(0) = \pm 1$. This equation does not satisfy the definition of a function. Middle and high school students are normally just asked to observe if it passes the *vertical line test* to determine if it is a function. Students should understand, however, that the vertical line test is just a visible illustration of the relation between the x -values and y -values on a graph. They need to understand why the vertical line test demonstrates whether or not a relation is a function.

There is a difference between a function and an equation. An equation is simply an equality of two quantities or expressions. Where as a function is a defined relationship between two sets. For middle and early high school students these sets are usually represented by the Real numbers. A function can be defined by an expression or equation, by a graph, or not in symbols at all. For example, the function $f(w)$ assigns to each day of the week whether or not a phone alarm turns on. If it is Saturday and

Sunday your phone alarm does not turn on and the other days of the week it turns on at 6am. As Dr. Wu states:

At the moment, many middle and high school teachers resist the teaching of this concept. For example, if a function $f(x) = 3x^2 + 1$ is given, they would ignore the meaning of f as an assignment $x \mapsto 3x^2 + 1$ and simply look at the *equation* $y = 3x^2 + 1$ and its graph, and ask students to memorize that the latter satisfies the *vertical line rule*. (Wu, 2011)

When there is a need to simplify the idea of a mathematical concept, care should be taken to maintain the precision necessary for mathematical definitions.

The eighth grade student is presented with the definition of a linear function as “a function in which the graph of the solutions form a line. Therefore, $y = x + 2$ is a *linear function*” (Math Connects, Course 3, 2012). All functions constitute a line, however, linear functions are straight lines. A proper definition which should be offered to students is; linear functions are “defined on $\mathbb{R}$ of the form $f(x) = ax + b$, for all x in $\mathbb{R}$, where a and b are fixed numbers” (Wu, 2011). If $a = 0$, then $f(x) = b$ is a constant function. If $b = 0$, then $f(x) = ax$ is a linear function without a constant term. This type of linear function relates to problems dealing with proportion or constant speed (Wu, 2011).

Evaluation

Students in grades 6, 7, and 8 should be in the Piaget’s formal operational stage. These students should be able to think hypothetically, analyze problems systematically and use symbols to generalize arguments (Ojose, 2008). Ojose maintains that students do

always progress at the same rate, so some of the students in this grade level will remain in the Piaget's concrete operational stage.

Concerning the introduction of rational numbers, there seems to be a problem with coherence and definitions as it applies to division by zero. Third grade students are told you cannot divide by zero. However, the sixth grade text gives an incomplete definition of a fraction, as there is no mention of zeros as a denominator. The seventh grade text then uses this definition of a fraction as part of the definition of a rational number. This is clearly an inconsistency with the definitions of fractions and rational numbers. Moreover, the Venn diagram in *Math Connects, Course 2* and *Math Connects, Course 3* appears to exclude zero from the whole numbers. This exclusion defies precision and coherence.

In the discussion involving variables, the reviewer feels the definitions given in each text lacks clarity as well as coherence from grade to grade. This is not about grade appropriateness. A proper and precise definition should be given from the introduction of the concept and maintained throughout.

When the concept of solving equations is introduced, it is presented in a logical progression from one-step to multi-step equations as the student progresses from seventh to eighth grade. The students are told the process of solving equations is to *undo* the operations (Math Connects, Course 2, 2012). This terminology may be grade appropriate. However, it is unclear and should be accompanied by more sound reasoning.

The definition of a linear function as presented to the eighth grade student is "a function in which the graph of the solutions form a line" (Math Connects, Course 3,

2012). This definition is far from precise. Lines can be curvy and crooked. The definition for a linear function should include the words *straight line*.

As students in these grades can be in two different developmental stages and are being presented with introductory algebra. It is important to have a good foundation on which their high school mathematics will be built. The concepts presented to these students are grade appropriate and follow the standards set by the State of Tennessee. Most of the material reviewed appears clear and precise, has examples with sound reasoning accompanying each topic and is coherent. Concepts such as the closure property are offered as enrichment. Students that are in the formal operational stage should be able to make sense of these more abstract topics giving them the opportunity to expand their knowledge.

Grades 9, 10, 11, and 12

State Standard Requirements

Most students entering ninth grade will be enrolled in an Algebra 1 class. The Algebra 1 student should begin to recognize that “polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication (A1.A.APR.A.1) (Tennessee Math Standards, 2016).” These students will learn to interpret different types of expressions, both numerical and polynomial in one variable. Identifying different ways of writing them, e.g., $4^2 - x^4$ as $(2^2)^2 - (x^2)^2$, and understanding that expressions such as these have a certain structure and can be factored, e.g., as a difference of squares (A1.A.SSE.A.2). Factoring quadratic and cubic polynomials to “reveal the zeros of the function it defines” (A1.A.SSE.B.3); using the

zeros of the function to create a rough graph of the function defined by the polynomial; and recognizing cases in which the “quadratic formula gives complex solutions” (A1.A.REI.B.3b) (Tennessee Math Standards, 2016) are all introduced within the Algebra 1 curriculum. Not only will these students be factoring quadratic and cubic equations in order to identify the zeros (A1.A.APR.B.2, A1.F.IF.C.7), they will be presented with the quadratic formula derived through the concept of completing the square (A1.A.SSE.B.3b). The methods used for solving quadratic equations in an Algebra 1 course involve taking square roots, completing the square, the quadratic formula, and factoring. By the conclusion of Algebra 1, a student should have the skills needed to inspect a quadratic equation, recognizing which is the most efficient method for solving it (A1.A.REI.A.1), and although students are not required to write a complex solution, they are required to understand that some quadratic equations have roots “that have...nonzero imaginary parts” (A1.A.REI.B.3b).

Algebra 1 students are only expected to know the real number system. The idea of the complex number system and the form of a complex number, $a \pm bi$ where a and b are real, are introduced within the curriculum of Algebra 2 (A2.N.CN.A.1). Students in an Algebra 2 course will be honing their skills of solving quadratic equations from Algebra 1 with the addition of complex numbers. They will be solving quadratic equations with real coefficients that have complex solutions (A2.N.CN.B.3, A2.A.REI.B.3). Algebra 2 students are required to write complex solutions as $a \pm bi$ where a and b are real (A2.A.REI.B.3). Understanding how to identify the zeros of a polynomial when “suitable factorizations are available” is a skill standard from Algebra 1 that is now extended to higher order polynomials, e.g., cubic and quartic polynomials

(A2.A.APR.A.2). Learning and applying the Remainder Theorem is part of the Algebra 2 curriculum standards (A2.A.APR.A.1); and while it is not an Algebra 2 standard, the FTA is finally introduced.

The Pre-Calculus curriculum standards require the student to know the FTA and be able to show that it is true for quadratic polynomials (P.N.CN.B.7). Polynomial identities are to be extended to the complex numbers (P.N.CN.B.6). These students are expected to identify the real zeros of a function and understand the relationship they have with the x-intercepts of the graph of a function (P.F.IF.A.4). In addition, they should be able to analyze a function, including polynomial functions, and identify the distinguishing properties of that function from a table, its graph, or an equation (P.F.IF.A).

Textbook Review

As early as fifth grade, students are presented with the concept of exponents and exponential notation. They are instructed: “the exponent is the number that tells how many times the base is used as a factor” (enVision Math Tennessee, 5th Grade, 2012), e.g., $5 \times 5 \times 5 = 5^3$ where 5 is the base, 5^3 is the exponential notation, and $5 \times 5 \times 5$ is the expanded form. Seventh grade students experiment with the product and quotient properties of exponents with the same base and eighth grade students are introduced to the power property of exponents and the definitions of zero and negative exponents.

Math Connects, Course 3 gives these definitions in sentence form, “algebra” or symbolic form, and with a numerical example. Figure 11 is an example of the **product property of exponents** from *Math Connects, Course 3*. The following properties are illustrated in

Product of Powers

$$3^2 \cdot 3^4 = \underbrace{(3 \cdot 3)}_{2 \text{ factors}} \cdot \underbrace{(3 \cdot 3 \cdot 3 \cdot 3)}_{4 \text{ factors}} \text{ or } 3^6$$

6 factors

Words To multiply powers with the same base, add their exponents.

Examples

Numbers

$$2^4 \cdot 2^3 = 2^{4+3} \text{ or } 2^7$$

Algebra

$$a^m \cdot a^n = a^{m+n}$$

Figure 11 - Product Property of Exponents (Math Connects, Course 3, 2012)

the same way. They are presented in their symbolic form only.

Quotient property of exponents: $\frac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$.

Power property of exponents: $(a^m)^n = a^{m \cdot n}$ and $(ab)^n = a^n b^n$.

Negative exponents: $x^{-n} = \frac{1}{x^n}$, $x \neq 0$.

Zero exponents: $x^0 = 1$, $x \neq 0$.

Each property is offered with a one-step proof similar to the one from figure 11, one that an eighth grade student should be able to understand. However, the zero and negative exponent property are unaccompanied by an explanation for why these properties hold. Figure 12 demonstrates a short example articulated in a similar way as the product property illustrated in figure 11 using the product and quotient properties of exponents to illustrate that the zero and negative property of exponents hold. These properties are going to provide a shortcut to factoring equations and polynomials, e.g., identifying different ways of writing numerical and polynomial expressions and factoring out common factors as in

$4x^5 - 24x^3 + 8x = 4x(x^4) + 4x(-6x^2) + 4x(2) = 4x(x^4 - 6x^2 + 2)$ (Algebra 1, 2012).

<u>Zero Property of Exponents</u>	<u>Negative Property of Exponents</u>
$\frac{a^2}{a^2} = 1$ (Inverse property of multiplication)	$x^4 x^{-6} = x^{4+(-6)} = x^{-2}$ (Product property)
$\frac{a^2}{a^2} = a^{2-2} = a^0$ (Quotient property)	$x^4 x^{-6} = \frac{x^4}{x^6} = \frac{\cancel{x^4}}{\cancel{x^4} x^2} = \frac{1}{x^2}$ (Cancellation laws)
So, $a^0 = 1$	So, $x^{-2} = \frac{1}{x^2}$

Figure 12 - Zero Property and Negative Property of Exponents

“In high school, students should build on their prior knowledge, learning more-varied and more-sophisticated problem-solving techniques...Moreover, their understanding of the properties of those functions will give them insights into the phenomena being modeled” (National Council of Teachers of Mathematics, 2000). A typical ninth grader is beyond the need to connect their reasoning to a concrete situation. They should be developing the capacity for more abstractness in their thinking.

At the beginning stage of algebra, your students are put in touch with the idea of *generality* for the first time, namely, the first identity is not just the equality of the expressions $(x + y)^2$ and $x^2 + 2xy + y^2$ for certain numbers x and y , but that it is *valid for all numbers x and y* (Wu, 2014).

Algebra 1 students begin learning the properties of polynomials by working with polynomial expressions. Polynomial expressions are defined to be “a monomial or a sum of monomials” with the standard form being such that the degrees of its monomial terms

decrease from left to right (Algebra 1, 2012). The degree of the monomial term is “the sum of the exponents of its variables. The degree of a nonzero constant is 0. Zero has no degree” (Algebra 1, 2012). Algebra 2 and Advanced Algebra students are presented with the formal definitions of a polynomial,

an algebraic expression of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where $a_n, a_{n-1}, \dots, a_1, a_0$ are constants called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is called the **leading coefficient**, $a_n x^n$ is called the **leading term**, and n is the **degree** of the polynomial. (Sullivan, 2012)

And a polynomial function, a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and $a_n, a_{n-1}, \dots, a_1, a_0$ are real numbers (Algebra 2, 2012). Examples of algebraic expressions not classified as a polynomial are not provided in the Algebra 1 and Algebra 2 textbooks; however $\frac{1}{x}$ and $\frac{x^2+1}{x+5}$ are provided as examples in the textbook *Algebra and Trigonometry*. The first one is not a polynomial because $\frac{1}{x} = x^{-1}$ has an exponent that is a negative integer and the second has a denominator with degree greater than 0.

Before discussing factoring polynomial functions we must understand the operations on polynomials. We can add and subtract polynomials by adding or subtracting like terms. Examples of these operations from *Algebra 1* are:

$$\begin{aligned} & (-7.1x^2 - 180x + 5800) + (21x^2 - 140x + 1900) \\ &= (-7.1x^2 + 21x^2) + (-180x - 140x) + (5800 + 1900) \end{aligned}$$

$$= 13.9x^2 - 320x + 7700,$$

and

$$\begin{aligned} & (x^3 - 3x^2 + 5x) - (7x^3 + 5x^2 - 12) \\ &= x^3 - 3x^2 + 5x - 7x^3 - 5x^2 + 12 \\ &= (x^3 - 7x^3) + (-3x^2 - 5x^2) + 5x + 12 \\ &= -6x^3 - 8x^2 + 5x + 12. \end{aligned}$$

Also vertical addition and subtraction are illustrated, by vertically lining up the like terms in each polynomial and adding the coefficients; or in the case of subtraction, changing the sign of each coefficient in the second polynomial and then adding the coefficients as shown in figure 13.

$\begin{array}{r} x^3 - 3x^2 + 5x \\ -(7x^3 + 5x^2 - 12) \\ \hline x^3 - 3x^2 + 5x \\ -7x^3 - 5x^2 + 12 \\ \hline -6x^3 - 8x^2 + 5x + 12 \end{array}$	Line up like terms. Then add the opposite of each term in the polynomial being subtracted.
---	--

Figure 13 - Vertical Subtraction of Polynomial Expressions (Algebra 1, 2012)

Just as with factoring numbers, understanding the property of polynomial multiplication is essential for factoring polynomials. A formal definition for multiplying two polynomials as stated by Dr. Wu is:

Let $g(X) = a_2X^2 + a_1X + a_0$ and $h(X) = b_3X^3 + b_2X^2 + b_1X + b_0$, where the a_i 's and b_j 's are numbers, and $a_2 \neq 0, b_3 \neq 0$. Then by definition, their product

$$\begin{aligned} \text{is } g(X)h(X) = (a_2b_3)X^5 + (a_1b_3 + a_2b_2)X^4 + (a_0b_3 + a_1b_2 + a_2b_1)X^3 \\ + (a_0b_2 + a_1b_1 + a_2b_0)X^2 + (a_0b_1 + a_1b_0)X + a_0b_0. \end{aligned}$$

In other words, the product of two polynomial forms is obtained by multiplying out all possible terms and then collecting like terms by their powers. (Wu, 2011)

A nice property we see from multiplying two polynomials is that the degree of the resulting polynomial is the sum of the degrees of the polynomial factors.

A typical Algebra 1 student begins learning this property by multiplying a monomial with a binomial, a polynomial with two terms, or a trinomial, a polynomial with three terms, using the distribution property. It is presented geometrically as well as numerically. For example, consider the product $2x(3x + 1)$. Using the distributive property and the product property of exponents we obtain:

$$2x(3x + 1) = 2x \cdot 3x + 2x \cdot 1 = 6x^2 + 2x \text{ (Algebra 1, 2012).}$$

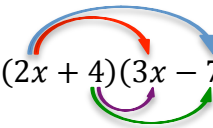
The Algebra 1 student is then introduced to multiplying two binomials. *Algebra 1* demonstrates multiplying two binomials using the distributive property, using F.O.I.L., and in a geometric model as in figure 14. Using a geometric model for squaring binomials is a good method for demonstrating the fact that $(a + b)^2 \neq a^2 + b^2$, a common mistake made by students thinking that the properties for exponents they learned for the set of Real numbers also hold for the set of polynomials (McKeague, 2011). This can be easily checked. If we were to substitute 1 for a , and 2 for b , we would have $(1 + 2)^2 \neq 1^2 + 2^2$ or $9 \neq 5$. Another method demonstrated in *Algebra 1* for multiplying a trinomial and a binomial is arranging them vertically and using a similar process to the multiplication algorithm learned for whole numbers. This method, the F.O.I.L. method, and the distribution method all illustrate the process of “multiplying out

Consider the product $(2x + 4)(3x - 7)$.

Using the Distributive Property

$$\begin{aligned}
 (2x + 4)(3x - 7) &= 2x(3x - 7) + 4(3x - 7) && \text{Distribute the second factor, } 3x - 7. \\
 &= 6x^2 - 14x + 4(3x - 7) && \text{Distribute } 2x. \\
 &= 6x^2 - 14x + 12x - 28 && \text{Distribute } 4. \\
 &= 6x^2 - 2x - 28 && \text{Combine like terms.}
 \end{aligned}$$

Using F.O.I.L



$$\begin{aligned}
 (2x + 4)(3x - 7) &= \overbrace{(2x)(3x)}^{\text{First}} + \overbrace{(2x)(-7)}^{\text{Outer}} + \overbrace{(4)(3x)}^{\text{Inner}} + \overbrace{(4)(-7)}^{\text{Last}} \\
 &= 6x^2 - 14x + 12x - 28 \\
 &= 6x^2 - 2x - 28
 \end{aligned}$$

The product is $6x^2 - 2x - 28$.

Using a Geometric Model

Consider the product $(2x + 1)(x + 2)$.

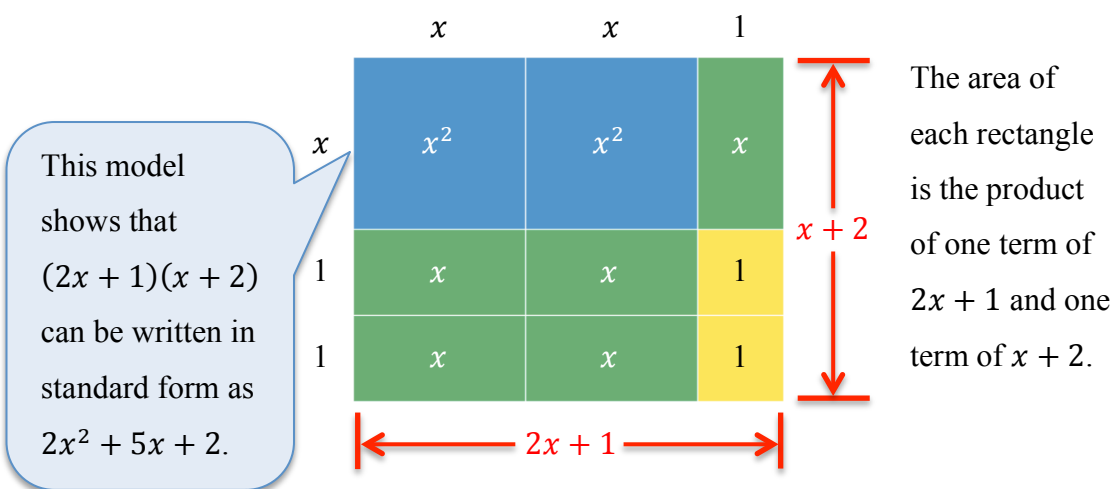


Figure 14 - Multiplication of Two Binomials (Algebra 1, 2012)

all possible terms and then collecting like terms by their powers” (Wu, 2011).

Factoring a polynomial is described to the Algebra 1 student to be a reversal of the multiplication process (Algebra 1, 2012). *Algebra 2* defines factoring as “rewriting an expression as a product of its factors.” *Algebra & Trigonometry* defines it as “expressing a given polynomial as a product of other polynomials”. It is in the pre-Calculus curriculum that the ideas of “prime factors” of polynomials and a polynomial that is “factored completely” are introduced.

To begin factoring a polynomial we need to observe if there is a common factor. If there is, we need to find the greatest common factor and use the distributive property for polynomials factorization. The **greatest common factor (GCF)** for a polynomial can be defined as the “largest monomial that evenly divides (is a factor of) each term of the polynomial” (McKeague, 2011). The Algebra 1 student is shown the example from figure 15. Using this method, which is an application of the Fundamental Theorem of

What is the GCF of the terms of $5x^3 + 25x^2 + 45x$?

List the prime factors of each term. Identify the factors common to all terms.

Think

Why use the factors 5 and x to form the GCF, but not 3?

Both 5 and x are factors of every term of the polynomial, but 3 is only a factor of the last term.

$$5x^3 = 5 \cdot x \cdot x \cdot x$$
$$25x^2 = 5 \cdot 5 \cdot x \cdot x$$
$$45x = 3 \cdot 3 \cdot 5 \cdot x$$

The GCF is $5 \cdot x$ or $5x$.

Remember to list only the prime factors of the variables

Figure 15 - Finding the GCF of a Polynomial (Algebra 1, 2012)

Arithmetic, the GCF can then be factored out of the polynomial. Consider the polynomial example $4x^2 + 20x - 56$. The GCF, 4, can be factored from each term resulting in $4(x^2) + 4(5x) + 4(-14) = 4(x^2 + 5x - 14)$, a factored form of the initial polynomial (Algebra 2, 2012). In order for this polynomial to be factored completely over the integers, we must factor the trinomial $x^2 + 5x - 14$.

Factoring polynomials of the form $ax^2 + bx + c$ is presented in two cases, when $a = 1$ and when $a \neq 1$. When $a = 1$, the quadratic $x^2 + bx + c$ is factored by the following method. First, list all the pairs of factors of c , identify the pair of factors whose sum equals b , and then rewrite the polynomial as a product of binomials in the form $(x + c_i)(x + c_j)$ where c_i and c_j is the factor pair of c whose sum equals b . For example, the factored form of the trinomial $x^2 + 2x - 15$ is determined by taking the factors -3 and 5 , since $-3 \cdot 5 = -15$ and $-3 + 5 = 2$, and rewriting the trinomial as $(x - 3)(x + 5)$ (Algebra 1, 2012). *Algebra 1* illustrates how this technique can be employed when: $b > 0$ and $c > 0$, $b < 0$ and $c > 0$, and in the case of our example when $c < 0$. The methods are the same regardless of the values of b and c . However, each method shortens the student's list of factors to: only positive factors, only negative factor, or one positive and one negative factor, respectively. The students are instructed to list all of the factor pairs and all of the summations of those factor pairs to find the one that works. Lastly, the students need to check their work by multiplying the polynomial factors, $(x - 3)(x + 5) = x^2 - 3x + 5x - 15 = x^2 + 2x - 15$.

In the case where $a \neq 1$, the trinomial $ax^2 + bx + c$ is factored by first determining the factor pairs of the product ac which have a sum equal to b . The trinomial is then rewritten by expanding the middle term, bx , by rewriting bx as a sum

of monomials, and then “factoring by grouping” the first two terms and the second two terms by factoring the GCF of each pair of terms. Then the distributive property is applied in order to have the trinomial factored to a product of binomials. Consider the example $3x^2 + 4x - 15$ (Algebra 1, 2012), $ac = -45$. $+9$ and -5 is the factor pair of -45 that add to be $+4$. The trinomial can then be rewritten as $(3x^2 - 5x) + (9x - 15)$. Factoring out the GCF of the first grouping, x , and the GCF of the second grouping, 3 , and using the distributive property results in $x(3x - 5) + 3(3x - 5) = (3x - 5)(x + 3)$. The methods of factoring $ax^2 + bx + c$ for the two cases, when $a = 1$ and when $a \neq 1$ are actually the same, because when $a = 1$, $ac = 1 \cdot c = c$.

Patterns for factoring special cases of polynomials are also pointed out for students beginning with the perfect square trinomial. A **perfect-square trinomial** is “of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$...it is the result of squaring a binomial” (Algebra 1, 2012). Figure 16 illustrates the example given in *Algebra 1* for factoring the

Algebra	For every real number a and b :
	$a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2$
Examples	For every real number a and b :
	$x^2 + 8x + 16 = (x + 4)(x + 4) = (x + 4)^2$ $4n^2 - 12n + 9 = (2n - 3)(2n - 3) = (2n - 3)^2$

Figure 16 - Factoring Perfect-Square Trinomials (Algebra 1, 2012)

perfect square trinomial. Students should recognize the different ways of expressing the terms in a polynomial as demonstrated in problem 1; “What is the factored form of $x^2 - 12x + 36$? $x^2 - 12x + 36 = x^2 - 12x + 6^2 = x^2 - 2(x)(6) + 6^2 = (x - 6)^2$ ” (Algebra 1, 2012).

Another special case introduced to the Algebra 1 student is factoring a difference of two squares. Factoring a difference or sum of two cubes would be introduced to the Algebra 2 student. A difference of two squares is defined “for all real numbers a and b : $a^2 - b^2 = (a + b)(a - b)$ ” (Algebra 1, 2012). An example presented in *Algebra 1* for students is illustrated in figure 17.

What is the factored form of $z^2 - 9$?

Think

Rewrite 9 as a square

Factor using the rule for a difference of two squares.

Check your answer by multiplying the factored form.

Write

$$z^2 - 9 = z^2 - 3^2$$

$$= (z + 3)(z - 3)$$

$$(z + 3)(z - 3) = z^2 - 3z + 3z - 9$$

$$= z^2 - 9$$

Figure 17 - Factoring a Difference of Two Squares (Algebra 1, 2012)

Factoring by grouping is introduced after factoring special cases. *Algebra 1* reminds students of a previous lesson involving factoring trinomials of the form $ax^2 + bx + c$. This technique is now given the name “factoring by grouping”. Algebra 1 students now see that they can extend this procedure for factoring polynomials of

higher degrees, such as cubic and quartic polynomials. For the Algebra 1 student, only numeric examples are given. A general rule for the process of factoring by grouping, though, is given to the Algebra 2 student. *Algebra 2* states the rule as, “ $ax + ay + bx + by = a(x + y) + b(x + y) = (a + b)(x + y)$.” The example that follows is:

$$x^3 + 2x^2 - 3x - 6 = x^2(x + 2) + (-3)(x + 2) = (x^2 - 3)(x + 2).$$

Algebra and Trigonometry describes the situation when a “common factor does not occur in every term of the polynomial, but in each of several groups of terms that together make up the polynomial.” When “the common factor can be factored out of each group by means of the distributive property”, the technique is called factoring by grouping. This textbook gives three examples of this process before reviewing factoring a trinomial.

Factoring the sum or difference of cubes is introduced in the Algebra 2 curriculum. The general rule presented to both the Algebra 2 student and the pre-Calculus student is: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$ and $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$ (*Algebra and Trigonometry*, 2012). Figure 18 is the illustration shown in

Here’s Why It Works Factoring $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$:

$$\begin{aligned} a^3 + b^3 &= a^3 + a^2b - a^2b + ab^2 - ab^2 + b^3 && \text{Add 0.} \\ &= a^2(a + b) - ab(a + b) + b^2(a + b) && \text{Factor out } a^2, -ab, \text{ and } b^2. \\ &= (a + b)(a^2 - ab + b^2) && \text{Factor out } (a + b). \end{aligned}$$

For $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, you can follow steps similar to those above, or you can factor $a^3 - b^3$ as the sum of cubes $a^3 + (-b)^3$.

Figure 18 – Factoring a sum or difference of cubes (*Algebra 2*, 2012)

Algebra 2 of why this technique works.

Early in the Algebra 1 curriculum, a solution of an equation with variables is defined as any value or values of the variables that make the equation true (Algebra 1, 2012). This definition remains constant in *Algebra 2*, and *Algebra and Trigonometry*. *Algebra and Trigonometry* states that these “values...that result in a true statement are called **solutions**, or **roots**, of the equation.” Upon discussing methods for solving quadratic equations, defined to be “an equation that can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$ ”, solutions and x -intercepts are equated with roots and zeros (Algebra 1, 2012). *Algebra 1* states, “the solutions of a quadratic equation and the x -intercepts of the graph of the related function are often called **roots of the equation** or **zeros of the function**.” *Algebra 2* clarifies the concept of the zero of a function, stating: “A value of x for which $f(x) = 0$ is a **zero of the function**.” *Algebra and Trigonometry* gives similar definitions. One definition is that if x is a real number that satisfies $f(x) = 0$, then x is a real zero of f ; and the other definition is that if x is a complex number that satisfies $f(x) = 0$, then x is a complex zero of f . The latter of the two is stated just prior to stating the FTA.

Completing the square is a concept introduced toward the end of the Algebra 1 curriculum. “You can change the expression $x^2 + bx$ into a perfect-square trinomial by adding $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$. This process is called completing the square” (Algebra 1, 2012). Both *Algebra 1* and *Algebra 2* model this concept using algebra tiles. *Algebra 2* continues by providing the statement $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$. From this method “a general formula for solving any quadratic equation” (Algebra and Trigonometry, 2012)

Here's Why It Works If you complete the square for the general equation $ax^2 + bx + c = 0$, you can derive the quadratic formula.

Step 1 Write $ax^2 + bx + c = 0$ so the coefficient of x^2 is 1.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{Divide each side by } a.$$

Step 2 Complete the square.

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \text{Subtract } \frac{c}{a} \text{ from each side.}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad \text{Add } \left(\frac{b}{2a}\right)^2 \text{ to each side.}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \quad \text{Write the left side as a square.}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \quad \text{Multiply } -\frac{c}{a} \text{ by } \frac{4a}{4a} \text{ to get common denominators.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \text{Simplify the right side.}$$

Step 3 Solve the equation for x .

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{Take the square root of each side.}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Simplify the right side.}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Subtract } \frac{b}{2a} \text{ from each side.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Simplify.}$$

Figure 19 – The Quadratic Formula derived by Completing the Square (Algebra 1, 2012)

may be obtained. The derivation of the Quadratic Formula is provided step-by-step in all three textbooks reviewed. Figure 19 shows the derivation presented in the text *Algebra 1*.

Algebra 1 students are only expected to recognize when a quadratic equation has no real solutions. They are not required “to write solutions for quadratic equations that have roots with nonzero imaginary parts” (Tennessee Math Standards, 2016). Hence, complex numbers are only defined in the curriculum for Algebra 2 students and beyond. *Algebra 2* defines a complex number to be of the form $a + bi$, where a and b are real numbers. “If $b = 0$, the number $a + bi$ is a real number. If $a = 0$ and $b \neq 0$, the number $a + bi$ is a **pure imaginary number**” (Algebra 2, 2012). *Algebra 2* points out that a is the real part and b is the imaginary part of a complex number. The Algebra 2 and pre-Calculus students are introduced to this new set of numbers that results from including the number i , defined as “the complex number whose square is -1, so $i^2 = -1$ and $i = \sqrt{-1}$ ” (Algebra 2, 2012).

Before presenting the FTA, both *Algebra 2* and *Algebra and Trigonometry* discusses the division algorithm for polynomials. The division algorithm for polynomials is presented to the Algebra 1 student on a basic introductory level. *Algebra 1* says that it is “similar to long division of real numbers. You write the answer as quotient + $\frac{\text{remainder}}{\text{divisor}}$.” Several examples in *Algebra 1* show each step as it repeats the process and how the final answer is written. The Algebra 2 student is presented with a slightly more formal definition.

You can divide polynomial $P(x)$ by polynomial $D(x)$ to get polynomial quotient $Q(x)$ and polynomial remainder $R(x)$. The result is $P(x) = D(x)Q(x) + R(x)$.

If $R(x) = 0$, then $P(x) = D(x)Q(x)$ and $D(x)$ and $Q(x)$ are factors of $P(x)$.

(Algebra 2, 2012)

The facts that $Q(x)$ and $R(x)$ are unique polynomial functions and “ $R(x)$ is either the zero polynomial or a polynomial of degree less than that of $D(x)$ ” (Algebra and Trigonometry, 2012) is only stated in the definition given to the pre-Calculus student. The proof for the division algorithm for polynomials is not given, even for the pre-Calculus student. However, this definition is used to prove the Remainder Theorem for both the Algebra 2 student and the pre-Calculus student.

At this point in the Algebra 2 curriculum, the FTA is introduced. “If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly n roots, including multiple and complex roots” (Algebra 2, 2012). *Algebra 2* lists two other “equivalent ways to state the FTA. You can use any one of these to prove the others”: “Every polynomial of degree $n \geq 1$ has n linear factors”, and “every polynomial function of degree $n \geq 1$ has at least one complex zero” (Algebra 2, 2012). *Algebra and Trigonometry* state the FTA as: “Every complex polynomial $f(x)$ of degree $n \geq 1$ has at least one complex zero” (Algebra and Trigonometry, 2012). The proof of the FTA is not given as the author of *Algebra and Trigonometry* maintains that it is “beyond the scope of this book” (Algebra and Trigonometry, 2012). The FTA, then, is used to prove:

Every complex polynomial $f(x)$ of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n)$$

where $a_n, r_1, r_2, \dots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may

repeat. (Algebra and Trigonometry, 2012)

Evaluation

According to Piaget's cognitive development stages, students in high school should be in the formal operational stage. However, through this reviewer's experience they are not all to that stage. Some of these students, especially early high school students, need a little more guidance in their abstract reasoning than others.

The high school level textbooks that were reviewed showed little variance from Wu's philosophies of mathematics education. These textbooks introduced concepts in a logical order with clear, precise statements and definitions. These textbooks showed coherence as one progressed from Algebra 1 to pre-Calculus. Also, these textbooks demonstrated sound reasoning and followed through with examples that reinforce the reasoning. The reviewer feels that of the many topics evaluated there were two of which that deserve additional emphasis: factoring trinomials and zeros of a function.

Concerning factoring trinomials of the form $ax^2 + bx + c$, the student has been previously taught the F.O.I.L. method for multiplying which is based on the distributive property. So, logically the reverse process, *un-F.O.I.L.-ing*, should be based on the distributive property, in other words, to expand the bx term and factor by grouping. It would be a more logical progression to teach this method for when $a = 1$. Therefore, making less confusion for the student when they are confronted with a trinomial such that $a \neq 1$.

Chapter 6: Conclusion

Mathematics, by nature, is precise, logical, and orderly. Wu contends mathematics education needs to follow the same principles. The scope of mathematics is broad and continually expanding. Therefore, this study limited the topics evaluated to the concepts relating to the Fundamental Theorem of Arithmetic and the Fundamental Theorem of Algebra.

The third grade student cannot understand the full complexity of theorems such as the FTA. However, they can, on their level, learn the foundations that will lead to using, then understanding, and eventually proving these theorems as they progress through their mathematical journey. Upon the completion of a student's journey through high school mathematics, the complex number system will have been fully developed as shown in figure 20. By using the complex number system, a student can fully utilize the FTA making it a perfect system for solving functions.

Dr. Wu, a mathematics professor for the University of California at Berkley, has written extensively on the subjects of mathematics education and professional development among mathematics teachers. His basic beliefs are that mathematicians and educators can come together to engineer a mathematics education that is both grade appropriate and maintains the basic characteristics of mathematics. Wu contends that these characteristics are: "precise definitions as a starting point, logical progression from topic to topic, and most importantly, explanations that accompany each step" (Wu, 2011).

Primarily, the standards reviewed are consistent with Wu's theories. TNAS starts with precise definitions. TNAS not only include standards for mathematical content,

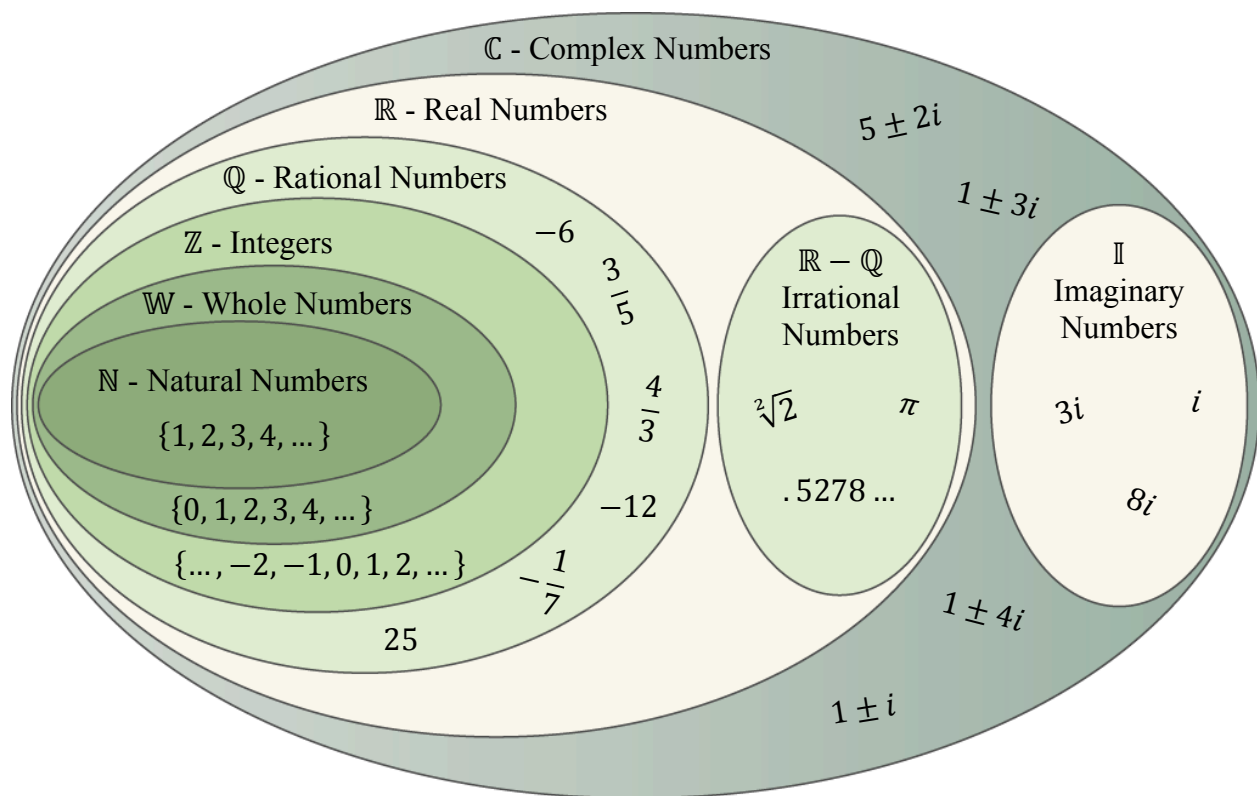


Figure 20 – The Set of Complex Numbers

there are eight standards for mathematical practices, which include precision and reasoning. The standards also show coherence in the progression from topic to topic and grade to grade. The standards are written in such a way that material presented for each grade is age appropriate.

Secondly, the textbooks reviewed, for the most part, followed the characteristics of mathematics set forth by Wu. The definitions were precise and in a logical progression from topic to topic. There were many examples accompanying each topic. These explanations were consistent from grade to grade. This study has noted in the evaluation sections the few areas in which improvement could be made.

- The text *enVision Math, Fourth Grade* states that 1 is special, i.e., neither prime nor composite with no explanation of why.

- Between the sixth and seventh grade texts, there is an inconsistency concerning the definitions of rational numbers regarding division by zero.

How interesting it is that these two anomalies relate to the numbers one and zero.

A major factor that has not been discussed in this study is teacher readiness. The teacher has the autonomy to interpret the text for the better or worse. A concept that may be unclear or imprecise in the text can be given clarity by a teacher with a deep understanding of the underlying principles of the subject. As a mathematician and a mathematics educator, this reviewer feels that Wu's premises are sound educational tactics that can lead to better textbooks and improving teaching skills.

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Vita

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She has been around mathematics education her entire life. Her father received his Doctorate degree in mathematics from the University of Tennessee the week prior to her birth and teaches mathematics on a college level. Her mother received her Master of Arts degree in mathematics a few years earlier and taught mathematics on a high school level. Sunny has been involved with teaching in many different aspects; teaching gymnastics while working on her bachelor’s degree, teaching fine hand sewing while working at Simply Sunshine, and while working on her Master of Science degree in mathematics at the University of Tennessee, Knoxville, she taught Algebra 1, Algebra 2, and Geometry for Rhea County Academy (2014-2015) in Dayton, TN. Currently, she is employed by the Rhea County Department of Education as a high school mathematics teacher.