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Dean of The Graduate School

A THEORETICAL MECHANICS ANALYSIS OF SHOOTING BASKETBALL

**A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville**

**Terry L. Dickson
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I am grateful to my late father who instilled in me a strong motivation for the pursuit of knowledge and continuing education. I would also like to acknowledge my late father-in-law, who had a favorite saying that, in spite of politics, personalities, and institutional hierarchies, sooner or later, someone had better know something. I suppose their influences explain why I started work toward this degree when I was 45 years old. I would also like to acknowledge my older brother who set a very good example for me, in a time and place that it would have been very easy to sell myself short. I am also grateful to my wife, Maureen, for being patient while I developed this analysis and wrote this thesis. During the fall of 1998 and winter of 1999, I would occasionally hibernate with my computer on weekends at our cabin on the river to develop and document this work. On several occasions, during these periods of seclusion, a very sweet lady, Mrs. Von Little, a good neighbor and a retired teacher, would voluntarily bring me a very great home-cooked meal.

I would like to acknowledge two of my colleagues at Oak Ridge National Laboratory. Angie Raby was a tremendous help in getting this thesis into the required format. John Bryson was also a big help in producing some of the illustrative figures. I am grateful to the Oak Ridge National Laboratory for financing this graduate degree.

I initially considered writing this thesis on a job-related topic. I have been a member of the Computational Physics and Engineering

Division at the Oak Ridge National Laboratory for over ten years, so there is certainly no shortage of interesting and challenging mathematical topics. While I consider the work that I perform at Oak Ridge National Laboratory both important and interesting, I chose to do this thesis on a non job-related topic, according to the old saying that one should broaden their horizon if given the opportunity. The context of this development is basketball; however, this type of analysis has other more deadly-serious applications such as calculating the trajectory of a three-dimensional projectile in a gravitational field required for it to intersect a stationary target.

I have enjoyed working with Dr. Steve Serbin as my major advisor in the Department of Mathematics at the University of Tennessee. I have had Steve as an instructor for several different courses through the years, primarily concerned with numerical analysis. Steve was also on my masters committee in 1978 when I received my Master of Science Degree in Engineering Science and Mechanics.

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ABSTRACT

The branch of mathematical analysis that involves the determination of trajectories (flight paths) of projectiles is known as ballistics. Historically, ballistics has been applied to designing weapons of war. The shape of the trajectory of a projectile is uniquely determined by the velocity and angle at which it is launched. In this thesis, ballistics is applied to determine if the trajectory of a basketball shot from a particular location, relative to the rim, will be successful, i.e., will pass through the rim. The determination of whether a basketball shot launched from a particular location is successful depends on relationships between the shape of the trajectory and geometrical constraints imposed by the spherical shape of the ball and the circular shape of the rim.

A computational methodology is developed to generate the solution of all trajectories for which a basketball released from the hand of the shooter at a particular location would be successful. Numerical results are presented that illustrate the application of this computational methodology for a sampling of cases. These include the case where gravity is the only force acting on the projectile and also the case where the drag force due to air resistance is included in the formulation.

1.0 Introduction

My interest in this topic originated with a book I purchased at the Smithsonian Institute in 1986 entitled "Newton at the Bat." The book is a collection of essays about the physics of various sports. The essays are directed at the layman and therefore are primarily descriptive and qualitative, rather than quantitative and full of equations. In any case, there is an essay in this book entitled "In Search of the Perfect Jumpshot." Being a former basketball player, I found the essay to be quite interesting, although I thought that the author had drawn general conclusions from what I perceived to be a rather simplified analysis. I decided that a more rigorous analysis of this topic was challenging and interesting enough mathematically to be worthy of some intensive analysis. The requirement to write a thesis to complete the requirements for this degree provided the opportunity to pursue this topic.

A basketball shot is analyzed by considering the basketball to be a projectile moving in a gravitational field. The trajectory (flight path) of a projectile in a gravitational field is a parabola that resides in a two-dimensional vertical plane. The variables that determine the shape of the trajectory in the vertical plane are the launch velocity (V_0) and a launch angle (θ). θ is the angle between the velocity vector (with which the projectile is launched) and any horizontal plane (in this context any horizontal plane parallel to the gym floor). Therefore, each pair of V_0 and θ determine uniquely the

shape of a particular trajectory. α is an angle that defines which vertical plane (perpendicular to the gym floor) that the trajectory resides in, relative to some reference vertical plane.

An objective of this thesis is to develop a computational methodology to solve for all trajectories, each uniquely determined by a pair of V_0 and θ , that result in a successful shot launched from a specified distance from the front of the rim, some height above the floor, and at an angle α . A successful shot is defined as one that has a trajectory such that the basketball successfully passes through the rim on the downward part of the trajectory, i.e., that part of the trajectory after the maximum height has been achieved. Success in these analyses does not include bank shots or cases where the ball hits the rim, bounces around, and finally falls through the rim.

It will be shown that all pairs of V_0 and θ that produce trajectories which result in success for a shot launched from a specified distance and height, and in a vertical plane defined by α , when plotted in the θ - V_0 plane, can be defined by two curves; a lower-bound curve and an upper-bound curve. All ordered pairs of (θ, V_0) that reside on these curves or inside of the region encompassed by these two curves result in a successful shot as previously defined. In the context of this thesis, the lower-bound curve, the upper-bound curve and the region in the θ - V_0 plane encompassed between these two curves are collectively referred to as the solution envelope. All pairs of (θ, V_0) that reside inside of the solution envelope result in a

successful shot whereas all pairs that reside outside of the solution envelope result in an unsuccessful shot. The determination of the lower-bound and upper-bound curves will be discussed in detail.

This analysis is necessarily developed incrementally using models of increasing realism and subsequently increasing complexity. The generation of the rigorous solutions for envelope the lower-bound launch velocities V_0 and launch angles θ that result in a successful shot includes the repeated solution of a system of nonlinear algebraic equations.

After the analysis that determines the solution envelope for a successful shot, there is a discussion regarding the definition of the best shot. There is also a discussion of possible directions for further development.

2.0 Problem Definition - Modeling Assumptions and Nomenclature

This analysis neglects air resistance, the effects of the Coriolis effect due to the earth's rotation and any three-dimensional rigid-body dynamic effects due to the rotation of the basketball [1]. The effect of the earth's rotation would be infinitesimal. The frictional (drag) force due to air resistance is proportional to the square of the velocity of the projectile [2]; however, initially the drag force will be neglected. Since this analysis does not include bank shots, neglecting the rotation of the basketball should introduce a very small amount of error into the analysis.

Figures 1-2 illustrate some of the variables involved in the analysis of the trajectory of a basketball shot launched from a specified distance (d) from the front of the rim and a specified height h above a horizontal plane representing the gym floor. Figure 1 illustrates the origin of the global coordinate system. The launch location is the location at which the ball is released from the hand of the shooter. From the launch location, gravity is the only force acting on the ball. $x=0$, $y=h$, and $z=0$ at the launch location. The top of the rim is ten feet above the gym floor.

The nomenclature illustrated in figures 1 and 2 is as follows:
 x is the global horizontal cartesian coordinate direction along a straight line from the launch location to the center of the rim as illustrated in figures 1 and 2. The global x axis is always along a straight line from the launch location to the center of the rim,

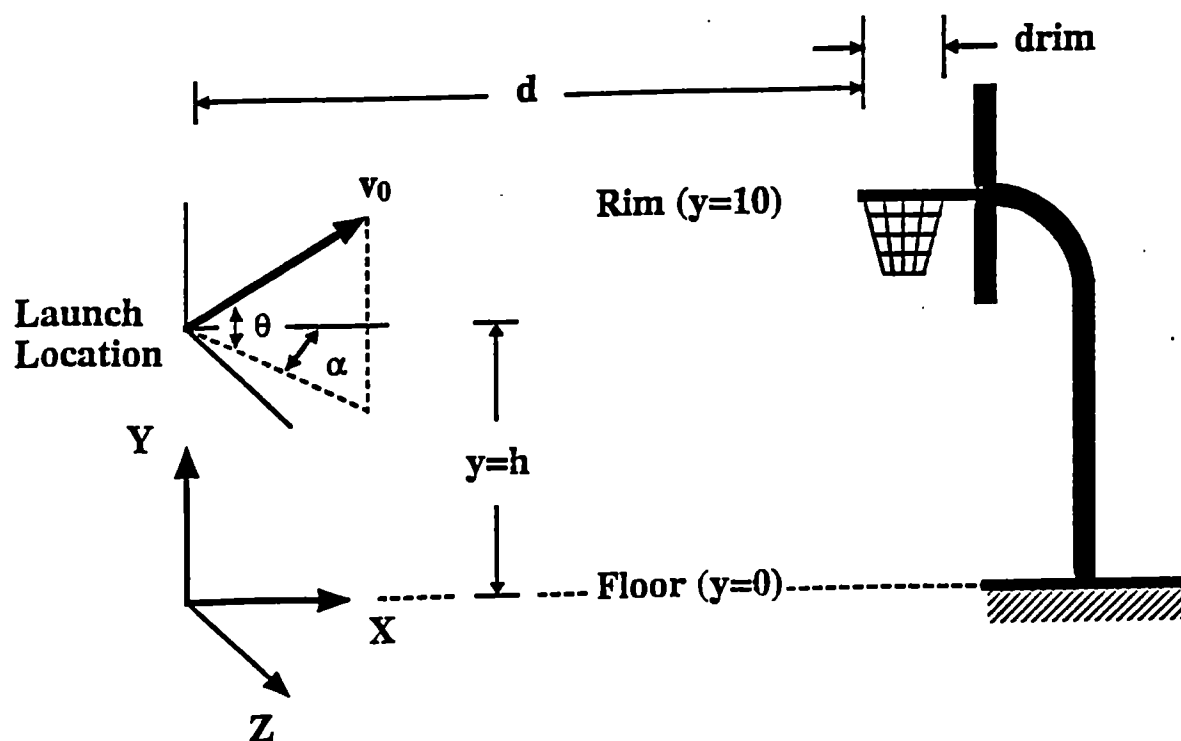


Figure 1. Illustration of global variables involved in trajectory analysis.

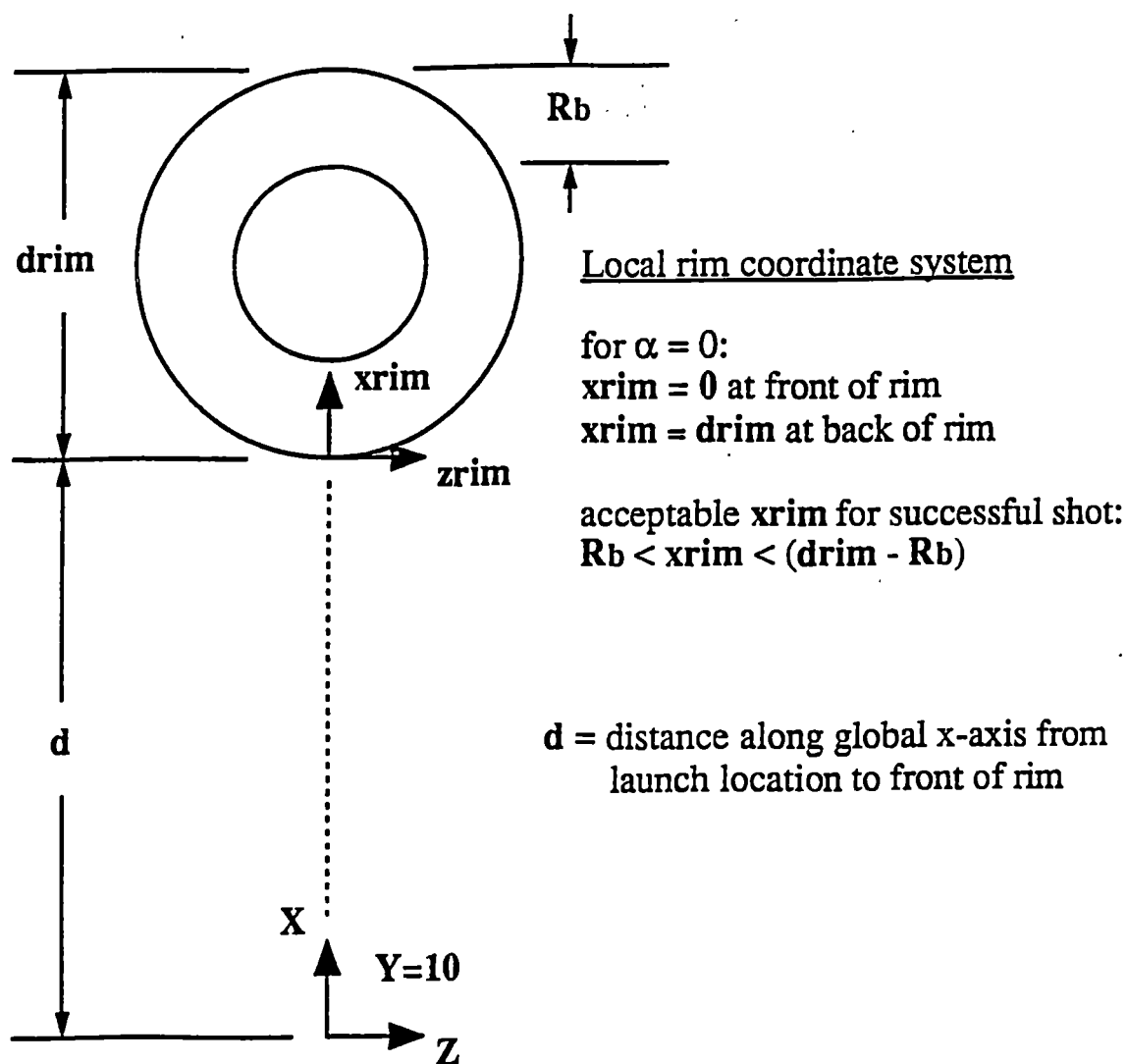


Figure 2. Illustration of local rim coordinate system relative to the global coordinate system.

regardless of the angular position on the shooting perimeter from where the shot is launched.

z is the global cartesian coordinate direction that forms a right-handed coordinate system as illustrated in figure 1.

y is the global vertical cartesian coordinate direction. $y=0$ at the horizontal plane representing the gym floor and $y=h$ at the launch location. The rim lies in the $x-z$ plane at $y=10$.

d is the horizontal distance from the launch location to the front of the rim measured along the global x axis.

h is the height above the gym floor at which the ball is launched.

V_0 is the magnitude of the initial velocity vector at which the ball is launched.

θ is the angle between the initial velocity vector and the horizontal $x-z$ plane.

α is the angle between the initial velocity vector and the vertical $x-y$ plane. α is a measure of the deviation of the flight path of the ball from the global x cartesian axis.

d_{rim} is the diameter of the rim and R_b is the radius of the ball. d_{rim} and R_b are assumed to be 1.6 and 0.375 feet, respectively.

Figure 2 illustrates a local coordinate system (x_{rim}, z_{rim}) for the rim relative to the previously defined global coordinate system.

Figure 2 is a view looking down at the top of the rim. The rim resides in a horizontal x-z plane (parallel to the gym floor) ten feet above the floor ($y=10$). This local coordinate system is introduced because it is easier to quantify the geometry of the rim in this local coordinate system as opposed to the global coordinate system previously defined, which had its origin at the launch location. The origin of this local coordinate system is at the front of the rim. The variable x_{rim} is oriented in the same direction as the global x-axis and varies from the front of the rim where $x_{rim}=0$ to the back of the rim where $x_{rim}=d_{rim}$. It should also be noted that $z_{rim}=0$ at the front of the rim.

The inner circle illustrated in figure 2 has a radius of $(d_{rim}/2)-R_b$, where d_{rim} is the inner diameter of the rim. For a shot to be successful, the centroid of the ball must pass through the rim on or inside of this inner circle in the x-z plane at $y=10$. For the case where the flight path of the ball is along the global x axis ($\alpha=0$), the values of x_{rim} for which the centroid of the ball can successfully pass through the rim (the x-z plane at $y=10$) are:

$$R_b \leq x_{rim} \leq (d_{rim} - R_b) \quad (1)$$

When the centroid of the ball is in the interval $(0 \leq x_{rim} < R_b)$, the front of rim would interfere with the trajectory of the spherical ball. In other words, the shot would be short. When the centroid of the ball is in the interval $((d_{rim}-R_b) < x_{rim} < d_{rim})$, the back of the rim would interfere with the trajectory of the spherical ball. In

other words, the shot would be long. In either of these cases, the interference of the flight path of the ball and the rim would result in the spherical ball being deflected and consequently an unsuccessful shot.

For the case where $\alpha \neq 0$, the values of x_{rim} for which the centroid of the ball could successfully pass through the rim would become smaller and therefore more restrictive. This will be considered in detail later in this thesis.

In figure 3, the variables ϕ_1 and ϕ_2 are introduced. The definition of these variables allows relationships to be developed between variables that determine the trajectory of the ball (V_0 and θ) and the variables that express the geometry of the ball (R_b) and rim (d_{rim}). There are certain relationships between the trajectory variables and the rim and ball variables that must be satisfied for a shot to successfully pass through the rim. These variables and relationships will be discussed below.

Figure 3 illustrates the position of the ball at two locations along the downward part of the trajectory. The second location of the ball in figure 3 corresponds to where the centroid of the ball is successfully passing through the x-z plane containing the rim at the coordinates $x = d + x_{rim}$, $y = 10$. ϕ_2 is defined as the angle between the velocity vector of the centroid of the ball and the horizontal x-z plane at this location. The negative of the tangent of ϕ_2 is equivalent to the derivative of the trajectory evaluated at this point,

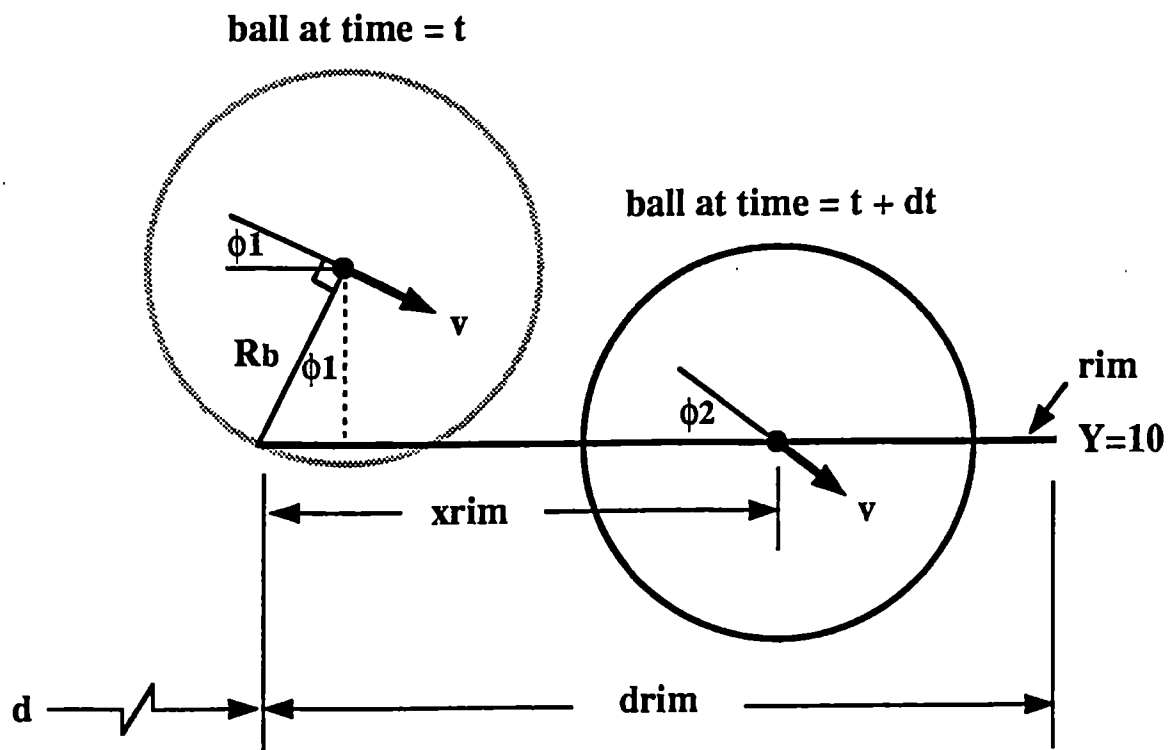


Figure 3. Illustration of ϕ_1 and ϕ_2 corresponding to the flattest trajectory successful shot for a specific allowable value of x_{rim}

since the trajectory has a negative slope. For each value of x_{rim} satisfying equation 1, there is some limiting minimum value of ϕ_2 for which the ball can successfully pass through the rim. For any value of ϕ_2 less than this limiting value, the trajectory would be short, i.e., the ball would be deflected by the front of the rim. For any value of ϕ_2 equal to or greater than this limiting value, the shot would be successful. The solution of V_0 and θ corresponding to this limiting minimum value of ϕ_2 will be referred to as the flattest trajectory successful shot and will be discussed in detail. It will be shown that the solution for the flattest trajectory successful shot is required to accomplish the objective of defining the solution envelope of trajectories as previously defined.

The need to analyze the flattest trajectory successful shot illustrated in figure 3 requires defining the variable ϕ_1 . The first location of the ball in figure 3 corresponds to an earlier time during the flight of the ball where the ball is at a higher point on its downward trajectory. For each value of x_{rim} satisfying equation 1 and its limiting minimum value of ϕ_2 corresponding to the flattest trajectory successful shot, there is a corresponding value of ϕ_1 as illustrated in figure 3. ϕ_1 is defined as the angle between the velocity vector of the centroid of the ball and the horizontal x-z plane at the location where the ball is touching the front of the rim but has no component of velocity along the radius vector between the centroid of the ball and the front of the rim (in the plane containing

the flight path of the ball). This location will be referred to as the limiting front rim location. Therefore, from the limiting front rim location at which ϕ_1 is defined, the ball can successfully complete its trajectory and pass through the rim without being deflected by the front of the rim. The global coordinates of the centroid of the ball at this location are $x = d + R_b \sin(\phi_1)$, $y = l_0 + R_b \cos(\phi_1)$.

3.0 Trajectory Analysis for Two-Dimensional Model

Recalling from chapter 1, the solution envelope was defined as all ordered pairs of (θ, V_0) that reside on the lower-bound curve or upper-bound curve or inside of the region in the θ - V_0 plane encompassed by these two curves. The objective now is to determine the upper-bound and lower-bound curves. The first analysis will be for the two-dimensional model where $\alpha = 0$, i.e., the flight path of the ball is assumed to remain in the vertical plane that contains the launch location and the center of the rim.

The strategy is to derive models of increasing complexity as follows:

- (1) Derive two equations of which the simultaneous solution will yield the V_0 and θ for an arbitrary value of ϕ_2 , without any consideration as to whether the value of ϕ_2 is greater than or equal to the limiting minimum value of ϕ_2 corresponding to the flattest trajectory successful shot. For values of ϕ_2 greater than or equal to the limiting value of ϕ_2 , these equations yield valid solutions of V_0 and θ that result in a successful shot. It will be shown that these equations can be applied to generate the upper-bound curve to the solution envelope.
- (2) Use a simplified method to determine the limiting minimum value of ϕ_2 that corresponds to the flattest trajectory successful shot. Using this simplified method to approximate the limiting minimum value of ϕ_2 , derive two equations of which the simultaneous

solution will yield an approximate solution of V_0 and θ corresponding to the flattest trajectory successful shot.

- (3) Use a rigorous method to remove the simplifying assumption previously made in the determination of the flattest trajectory successful shot. It will be shown that these equations can be applied to generate the lower-bound curve to the solution envelope.

The analysis of the trajectory for the two-dimensional model begins with the application of a pair of differential equations expressing Newton's second law of motion. F is the force vector acting on a particle of mass m at time t .

Since air resistance is being disregarded, there are no forces acting on the ball in the x direction; therefore, applying Newton's second law [3,4] in the x direction results in:

$$\sum F_x = m \frac{d^2x}{dt^2} = 0 \quad (2)$$

Integrating equation 2 and applying the initial condition that $dx/dt = V_0 \cos(\theta)$ when evaluated at $t=0$ results in a constant velocity in the x direction that is equal to the initial launch velocity in the x direction.

$$\frac{dx}{dt} = V_0 \cos(\theta) \quad (3)$$

Integrating equation 3 and applying the initial condition that $x = 0$ when evaluated at $t=0$ results in an equation for x as a function of time (t):

$$x(t) = (V_o \cos(\theta)) t \quad (4)$$

The force of gravity is acting on the ball in the y direction; therefore, applying Newton's law in the y direction results in:

$$\sum F_y = m \frac{d^2 y}{dt^2} = -mg \quad \frac{d^2 y}{dt^2} = -g \quad (5)$$

where g is the acceleration due to gravity [3,4]. Integrating equation 5 and applying the initial condition that $dy/dt = V_o \sin(\theta)$ when evaluated at $t=0$ results in an equation for the velocity in the y direction as a function of t:

$$\frac{dy}{dt} = V_o \sin(\theta) - gt \quad (6)$$

Integrating equation 6 and applying the initial condition that $y=h$ when evaluated at $t=0$ results in an equation for y as a function of t as follows:

$$y(t) = (V_o \sin(\theta)) t - \frac{gt^2}{2} + h \quad (7)$$

An equation for y as a function of x can be obtained by solving for t in equation 4 and substituting it in equation 7 as follows:

$$y(x) = x \tan(\theta) - \{g x^2 / 2V_o^2 \cos^2(\theta)\} + h \quad (8)$$

Equation 8 is a quadratic equation in x; therefore, the trajectory is parabolic. Solving equation 8 for x results in the following two solutions for x as a function of y:

$$x(y) = \frac{V_o^2 \cos(\theta) \sin(\theta)}{g} - \frac{V_o \cos(\theta)}{g} \{V_o^2 \sin^2(\theta) - 2g(y-h)\}^{1/2} \quad (9a)$$

$$x(y) = \frac{V_o^2 \cos(\theta) \sin(\theta)}{g} + \frac{V_o \cos(\theta)}{g} \{V_o^2 \sin^2(\theta) - 2g(y-h)\}^{1/2} \quad (9b)$$

The trajectory of the ball consists of two parts; the upward trajectory of the ball from the launch location up to the maximum height and the downward trajectory from the maximum height. Equation 9a is applicable to the upward trajectory and equation 9b is applicable to the downward trajectory. By definition, a successful shot passes through the rim on its downward trajectory; therefore, equation 9b is applicable for this analysis.

As illustrated in figure 3, for a successful shot, the centroid of the ball is at the coordinates $y=10$ and $x = x_{rim} + d$ when the centroid of the ball passes through the x-z plane at $y=10$, where x_{rim} satisfies equation 1. Substituting these values into equation 9b yields the following equation:

$$\frac{V_o^2 \cos(\theta) \sin(\theta)}{g} + \frac{V_o \cos(\theta)}{g} \{V_o^2 \sin^2(\theta) - 2g(10-h)\}^{1/2} - (x_{rim}+d)=0 \quad (10)$$

Since there are two unknowns (V_o and θ), another equation is required. The tangent of the angle (with the horizontal x-z plane) at which the velocity vector of the centroid of the ball passes through the coordinates ($x = x_{rim} + d$, $y=10$), previously defined as ϕ_2 , corresponds to the derivative of the trajectory evaluated at $x=x_{rim}+d$, $y=10$. The derivative of trajectory in equation 8 evaluated at $x = x_{rim} + d$ is as follows:

$$\frac{dy}{dx} = \tan(\theta) - \{g (x_{rim} + d) / V_o^2 \cos^2(\theta)\} \quad (11)$$

The derivative of the trajectory will be a negative value on the downward part of the trajectory; therefore, it can be equated to the negative of the tangent of the angle of the velocity vector with the x-z plane at y=10 resulting in the following equation:

$$\tan(\theta) - \{g (xrim + d) / Vo^2 \cos^2(\theta)\} + \tan(\phi_2) = 0 \quad (12)$$

The simultaneous solution of equations 10 and 12 provides a solution of Vo and θ for a successful shot from a distance of d and a height of h for a given acceptable value of xrim from equation 1 and an arbitrary angle ϕ_2 . The arbitrary value of ϕ_2 gives no consideration to the fact that there is some minimum value of ϕ_2 , corresponding to the flattest trajectory shot, below which the shot would be unsuccessful.

Proceeding to solve equations 10 and 12 simultaneously by substitution, equation 10 can be manipulated to solve for Vo^2 as follows:

$$Vo^2 = \frac{(g / 2) (xrim + d)^2}{\{\sin(\theta) \cos(\theta) (xrim + d) - \cos^2(\theta) (10 - h)\}} \quad (13)$$

Substituting equation 13 into equation 12 and solving for θ yields a solution for the launch angle as follows:

$$\theta = \tan^{-1} \left[\frac{(xrim+d)\tan(\phi_2)+2(10-h)}{xrim+d} \right] \quad (14)$$

Equation 14 solves for the value of θ as a function of the spatial variables d, h, xrim, and for an arbitrary value of ϕ_2 . The resulting

value of θ can be substituted into equation 13 to solve for the magnitude of the launch velocity V_0 .

This solution of V_0 and θ generated by equations 13 and 14 is a mathematical solution for an arbitrary value of ϕ_2 . Thus far, this general solution treats the basketball as a particle, when in reality the ball is a sphere of radius R_b ; however, in this two-dimensional development, the ball may be considered as a circular disk. As previously discussed, there is some minimum value of ϕ_2 , corresponding to the flattest trajectory successful shot, for which values of ϕ_2 less than this minimum value would result in the shot falling short, i.e., would result in interference between the front of the rim and the spherical ball resulting in an unsuccessful shot. For values of ϕ_2 greater than the limiting value of ϕ_2 corresponding to the flattest trajectory successful shot, there is no interference between the spherical ball and the rim; therefore, the solution of V_0 and θ generated by equations 13 and 14 will result in a successful shot for acceptable values of x_{rim} from equation 1, except for the case where $\phi_2 = 90$ degrees. For this particular case, the solution requires an infinite velocity. The cosine terms in the denominator of equation 13 cause V_0 to tend to infinity as θ approaches 90 degrees.

Consideration will now be given to the determination of the limiting value of ϕ_2 corresponding to the flattest trajectory shot. Figure 3 illustrates the flattest trajectory for a specified acceptable value of x_{rim} from equation 1.

As an initial approximation to solving for the flattest trajectory successful shot, a simplifying assumption is made that the limiting angle of $\phi_2 = \phi_1$, as illustrated in figure 4. This assumption ignores the amount that the slope (derivative) of the trajectory changes while the centroid of the ball travels from $x = d + R_b \sin(\phi_1)$ to $x = x_{rim} + d$. From figure 4 and the assumption that $\phi_2 = \phi_1$, the limiting minimum value of ϕ_2 for the flattest trajectory successful shot is:

$$\phi = \phi_2 = \phi_1 = \sin^{-1}(R_b/x_{rim}) \quad (15)$$

Two equations are required to generate a solution for V_o and θ for the flattest trajectory successful shot if it is released from a specified distance d , a height of h , and passes through the rim at a specified acceptable value of x_{rim} from equation 1. Again, equation 9b is applicable when it is evaluated at $x = x_{rim} + d$, $y=10$, which results in equation 10. The tangent of the angle between the velocity vector and the x - z plane at $x=d+R_b\sin(\phi_1)$ is equal to the negative of the derivative of the trajectory since it is on the downward part of the trajectory; therefore, another equation can be obtained as follows:

$$\tan(\theta) - \frac{g(d + R_b \sin(\phi))}{V_o^2 \cos^2(\theta)} + \tan(\phi) = 0 \quad (16)$$

Substituting equation 15 into equation 16 results in the following equation:

$$\tan(\theta) - \frac{g(d + (R_b^2/x_{rim}))}{V_o^2 \cos^2(\theta)} + \tan(\sin^{-1}(R_b / x_{rim})) = 0 \quad (17)$$

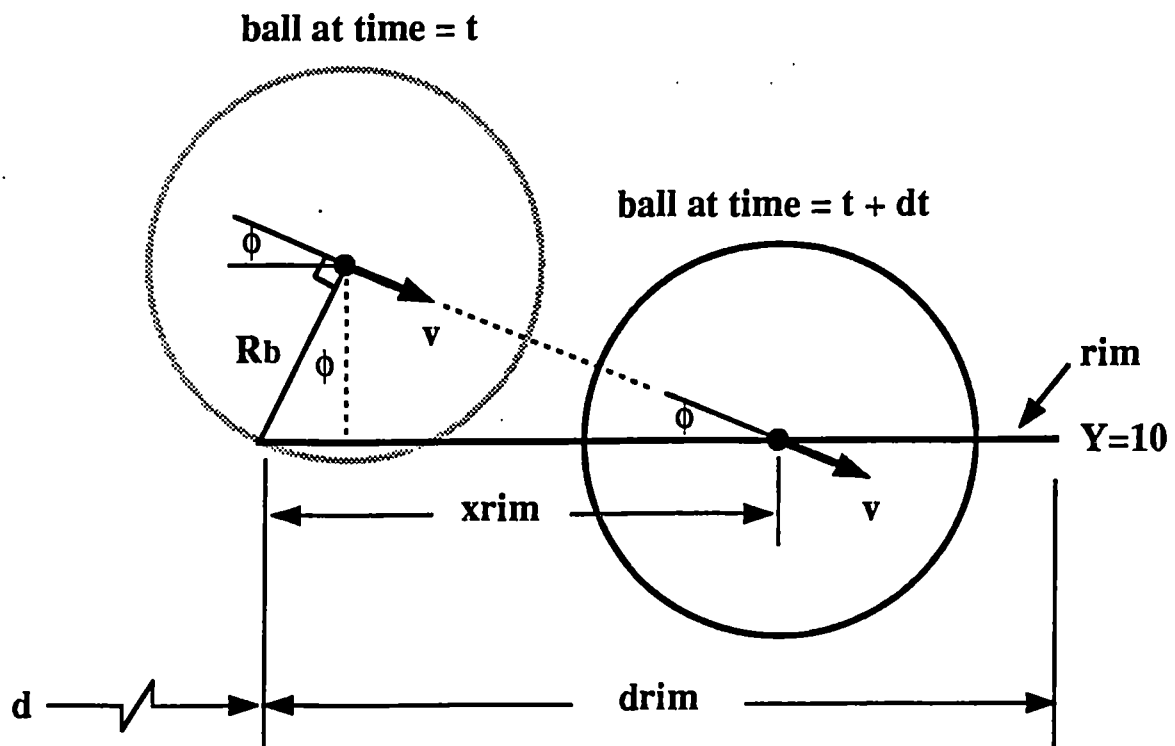


Figure 4. Illustration of simplifying assumption that $\phi_1 = \phi_2$.

Substituting equation 13 for V_o^2 into equation 17 and solving for θ yields the following solution for θ .

$$\theta = \tan^{-1} \left[\frac{2(d + R_b \sin(\phi))(10 - h) + (x_{rim} + d)^2 \tan(\phi)}{2(d + R_b \sin(\phi))(x_{rim} + d) - (x_{rim} + d)^2} \right] \quad (18)$$

Equation 18 solves for the value of θ as a function of the spatial variables d , h , x_{rim} , as did equation 14; however equation 18 also includes a parameter related to the geometry of the ball, i.e., R_b . The resulting value θ can be substituted into equation 13 to solve for the magnitude of the launch velocity V_o . These values of V_o and θ would produce the flattest trajectory successful shot for a shot launched from a specified distance from the front of the rim (d), a specified vertical distance above the gym floor (h), and an acceptable value of x_{rim} from equation 1. This solution includes the assumption that $\phi = \phi_1 = \phi_2$.

4.0 Relaxation of the Assumption that $\phi_1 = \phi_2$ for the Two-Dimensional Model

A model of increasing complexity and realism for determining the flattest trajectory successful shot is to remove the assumption that $\phi_1 = \phi_2$, i.e., to include the change in trajectory that occurs when the ball moves from the limiting front rim location as previously defined ($x = d + Rb \sin(\phi_1)$, $y = 10 + Rb \cos(\phi_1)$) to the location where the centroid of the ball passes through the rim ($x = x_{rim} + d$, $y = 10$). This is illustrated in figure 3. The change in the trajectory is:

$$\Delta\phi = \phi_2 - \phi_1 \quad (19)$$

Removing the assumption that $\phi_1 = \phi_2$ means equation 15 is no longer valid; therefore, ϕ_1 and ϕ_2 must be treated as unknown variables. The model now has four unknowns rather than two as in the simplified case; therefore, four equations will be required to solve for the four unknowns (V_0 , θ , ϕ_1 , ϕ_2). This will require the simultaneous solution of a system of four nonlinear algebraic equations.

Two equations can be obtained by applying equation 9b with the appropriate boundary conditions. Equation 10, which is an application of equation 9b, evaluated with $x = x_{rim} + d$, $y = 10$, is still valid; however, in this case, equation 10 requires rearranging to avoid potential numerical difficulties associated with taking the square root of a negative number during the iterative numerical solution of the system of nonlinear algebraic equations. Rearranging equation 10 to remove the square root results in the following equation:

$$\begin{aligned}
& -\{(x_{rim}+d)-(V_o^2 \cos(\theta) \sin(\theta)/g)\}^2 + \\
& (V_o^2 \cos^2(\theta)/g^2) (V_o^2 \sin^2(\theta) - 2g(10-h)) = 0
\end{aligned} \tag{20}$$

Equation 9b can also be applied at the limiting front rim location $x = d + R_b \sin(\phi_1)$ when $y = 10 + R_b \cos(\phi_1)$. Again, the equation is rearranged to avoid potential numerical difficulties associated with the square root resulting in the following equation:

$$\begin{aligned}
& -\{(d+R_b \sin(\phi_1)-(V_o^2 \cos(\theta) \sin(\theta)/g)\}^2 + \\
& (V_o^2 \cos^2(\theta)/g^2) (V_o^2 \sin^2(\theta) - 2g(10+R_b \cos \phi_1 - h)) = 0
\end{aligned} \tag{21}$$

The derivative of the trajectory previously derived in equation 8 evaluated at the limiting front rim location ($x = d + R_b \sin(\phi_1)$, $y = 10 + R_b \cos(\phi_1)$) can be equated to the negative of the tangent of ϕ_1 resulting in the following equation:

$$\tan(\theta) - \frac{g(d + R_b \sin(\phi_1))}{V_o^2 \cos^2(\theta)} + \tan(\phi_1) = 0 \tag{22}$$

Thus far, there are three equations (20, 21, and 22) relating V_o , θ , and ϕ_1 . Another equation is required to relate the fourth variable ϕ_2 to the other three variables. This can be done by the application of equation 19. Solving equation 22 for ϕ_1 results in the following equation:

$$\phi_1 = -\tan^{-1} \left[\tan(\theta) - \frac{g(d + R_b \sin(\phi_1))}{V_o^2 \cos^2(\theta)} \right] \tag{23}$$

Applying equation 8 again to determine the derivative of the trajectory at the location where the centroid of the ball successfully

passes through the rim ($x = d + x_{rim}$, $y=10$) and equating it to the negative of the tangent of ϕ_2 results in the following equation:

$$\tan(\theta) - \frac{g(x_{rim} + d)}{Vo^2 \cos^2(\theta)} + \tan(\phi_2) = 0 \quad (24)$$

Solving equation 24 for ϕ_2 results in the following equation

$$\phi_2 = -\tan^{-1} \left[\tan(\theta) - \frac{g(x_{rim} + d)}{Vo^2 \cos^2(\theta)} \right] \quad (25)$$

Now equation 19 can be applied by subtracting equation 23 from equation 25 resulting in the desired fourth algebraic equation which relates all four unknown variables as follows:

$$\begin{aligned} \phi_2 - \phi_1 = & -\tan^{-1} \left[\tan(\theta) - \frac{g(x_{rim} + d)}{Vo^2 \cos^2(\theta)} \right] + \\ & \tan^{-1} \left[\tan(\theta) - \frac{g(d + Rb \sin(\phi_1))}{Vo^2 \cos^2(\theta)} \right] \end{aligned} \quad (26a)$$

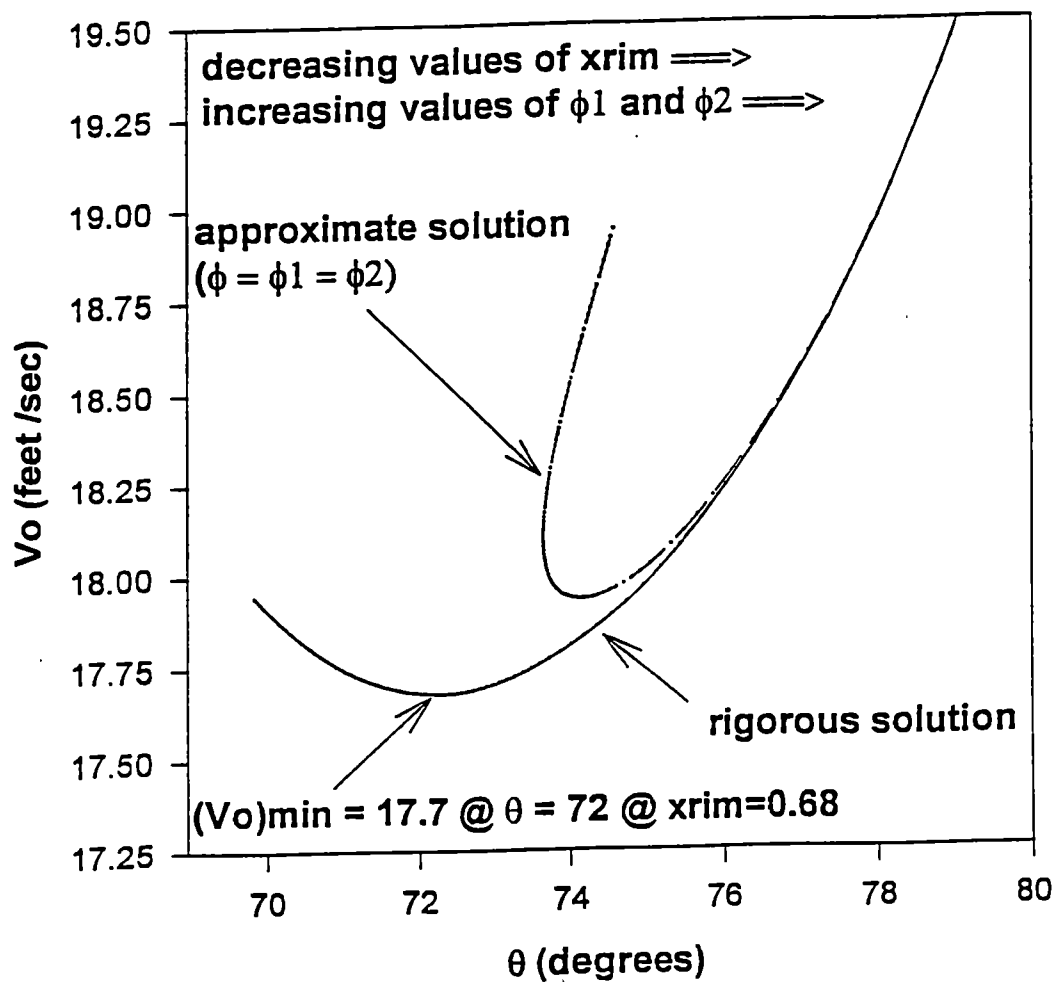
$$\begin{aligned} \phi_2 - \phi_1 + \tan^{-1} \left[\tan(\theta) - \frac{g(x_{rim} + d)}{Vo^2 \cos^2(\theta)} \right] - \\ \tan^{-1} \left[\tan(\theta) - \frac{g(d + Rb \sin(\phi_1))}{Vo^2 \cos^2(\theta)} \right] = 0 \end{aligned} \quad (26b)$$

Equations 20,21,22, and 26b form a system of four nonlinear algebraic equations with four unknowns. Solution of these equations yields the Vo , θ , ϕ_1 , and ϕ_2 , for the flattest trajectory successful shot from a specified distance (d) and height (h) such that the centroid of the ball successfully passes through the rim at the coordinates $x=x_{rim}+d$, $y=10$, and also passes through the limiting front rim location $x = d + Rb \sin(\phi_1)$, $y=10 + Rb \cos(\phi_1)$, such that the spherical ball is not deflected by the front of the rim.

The Newton-Raphson numerical technique [5-7] is used to solve for the values the V_o , θ , ϕ_1 , and ϕ_2 that simultaneously satisfy equations 20, 21, 22, and 26b. This system of equations must be solved for selected discrete values of x_{rim} that satisfy equation 1. The solutions for V_o and θ obtained from equations 13 and 18, using the simplified assumption that $\phi_1 = \phi_2$, are used as starting guesses in the application of the Newton Raphson numerical method. The value of ϕ obtained from equation 15 is used as the starting guess for ϕ_1 . A value of $\phi_1 = 0.01$ radian is used as the starting guess for ϕ_2 . This insures that the starting guess vector is reasonably close to the solution. This scheme for generating a starting guess vector has resulted in convergence for all cases examined thus far. In the implementation of the Newton-Raphson algorithm, the partial derivative of each of the four equations with respect to each variable is calculated numerically using the finite difference method [5]. Each variable is perturbed by 0.01 when calculating the partial derivative of each equation with respect to each of the four variables.

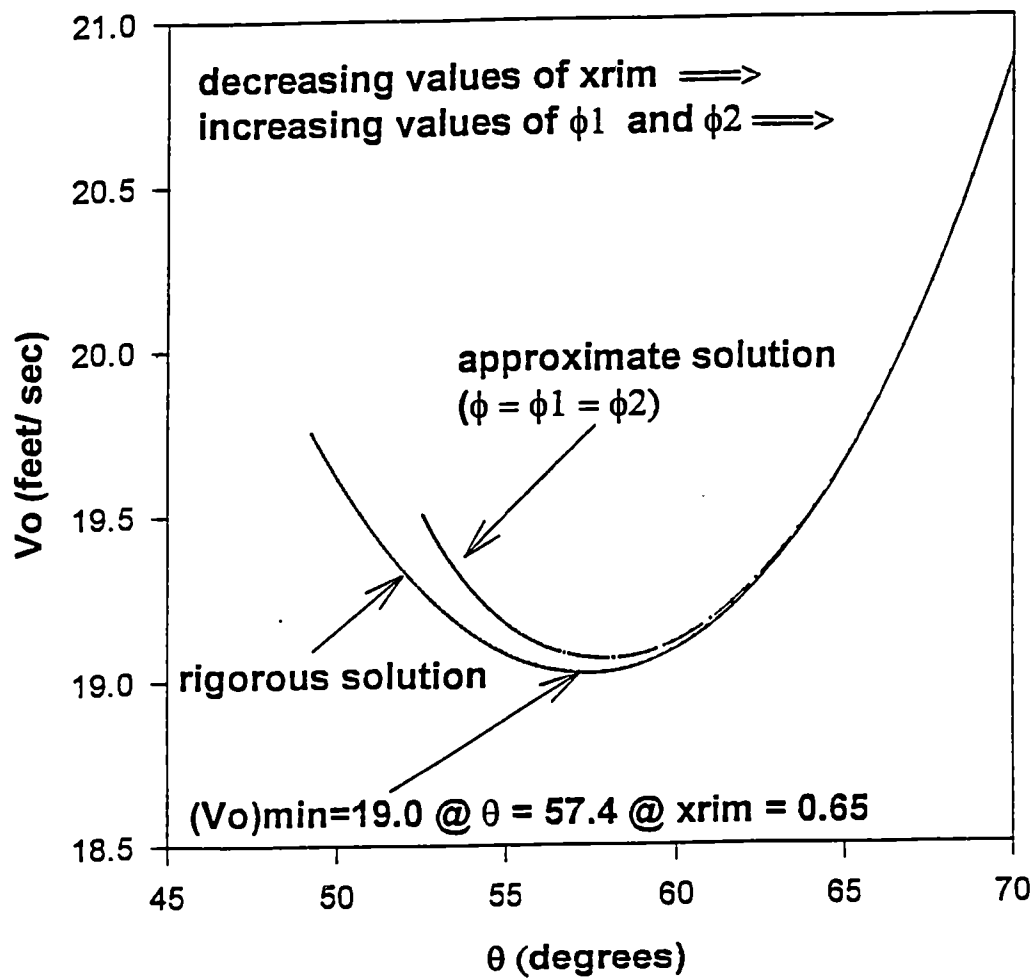
If insurmountable convergence problems had been encountered using the Newton-Raphson scheme, we would have implemented the method of steepest descent to solve the system of nonlinear algebraic equations, since the convergence of this scheme is not dependent on the starting guess vector being within a certain proximity to the solution vector [5-7].

Figures 5a-c illustrate the comparison of flattest trajectory successful shot solutions for V_0 and θ for the approximate (simplified) and rigorous solutions for various values of d . The interval of acceptable values of x_{rim} was equally divided into 100 discrete points. Equations 13 and 18 were solved for each of the 100 discrete points to generate the approximate solutions whereas the nonlinear system of algebraic equations 20-22, and 26b was solved 100 times to generate each of the rigorous solution curves. The curves illustrate that the accuracy of the approximate solution increases with increasing distance (d). In figure 5c, for $d=10$, there is a small difference between the two solutions; however; in figure 5a, for $d=3$, there is a dramatic difference between the two methods of solution. It is noted in these illustrations that the minimum value of θ corresponds to the maximum value of x_{rim} .

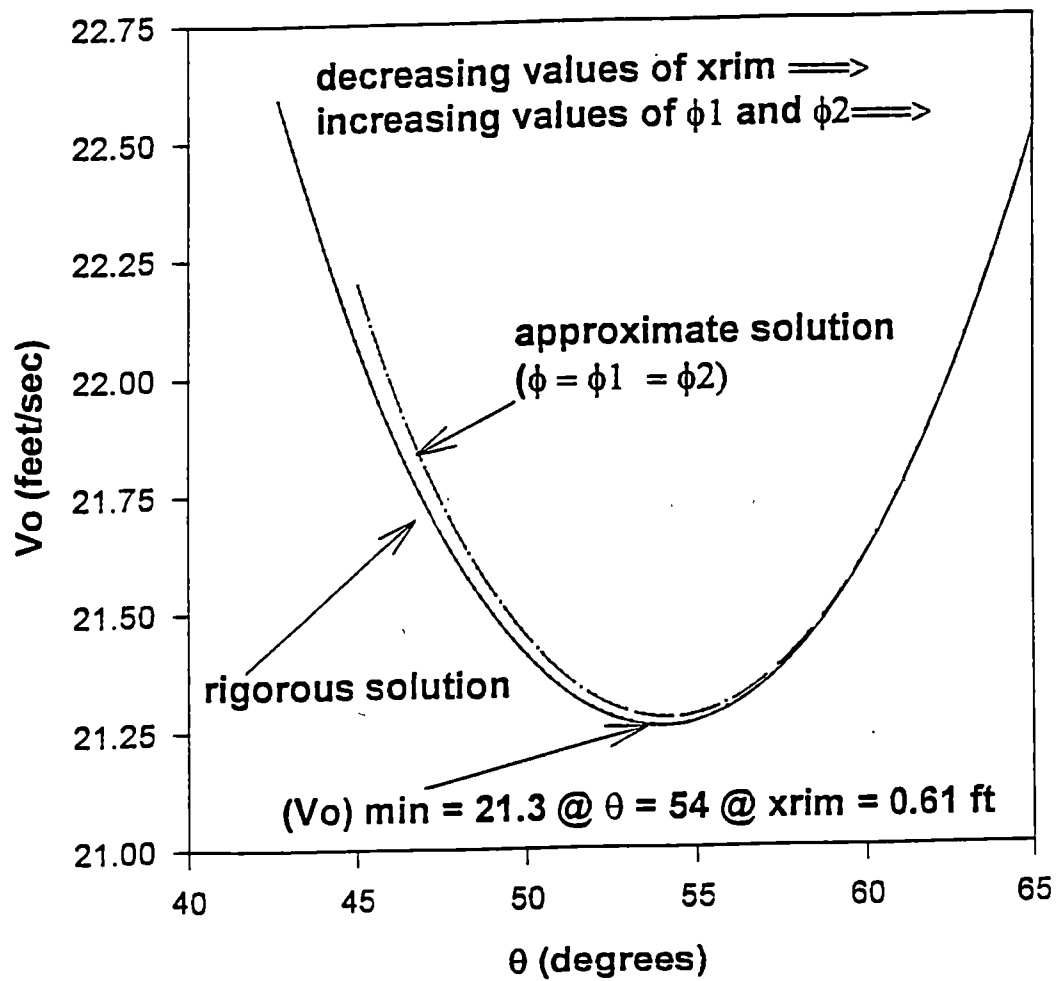


a. Solutions of V_o and θ for $d=3$ and $h=6$

Figure 5. Comparison of flattest trajectory successful shot solutions between the simplified and rigorous models for various values of d and h .



b. Solutions of V_o and θ for $d=7$ and $h=7$,
Figure 5 continued.



c. Solutions of V_o and θ for $d=10$ and $h=7$,
Figure 5 continued.

5.0 Determination of the Solution Envelope of V_0 and θ for Successful Shot for the Two-Dimensional Model

Figure 6 illustrates solutions of V_0 and θ for ϕ_2 greater than or equal to the limiting minimum value for the flattest trajectory successful shot, calculated for various values of x_{rim} , for $d=10$ and $h=7$. The top curve is for $x_{rim} = d_{rim} - R_b = 1.225$ feet, which is the maximum allowable value of x_{rim} for which the ball could successfully pass through the rim. The other curves are generated for smaller acceptable values of x_{rim} . The left-most point on each of these curves, corresponding to the minimum value of θ (and also the limiting minimum value of ϕ_2) corresponds to the flattest trajectory successful shot.

It can be concluded from the above illustration that the upper-bound curve of the solution envelope of V_0 and θ for a successful shot corresponds to solutions for the maximum acceptable value of $x_{rim} = d_{rim} - R_b$ in equation 1 for values of ϕ_2 greater than or equal to the limiting minimum value of ϕ_2 (for the flattest trajectory shot). It can also be concluded that the lower-bound curve of the solution envelope of V_0 and θ is the locus of points for the flattest trajectory successful shots corresponding to each of the acceptable values of x_{rim} that satisfy equation 1. A more detailed discussion regarding the solution envelope for a successful shot follows.

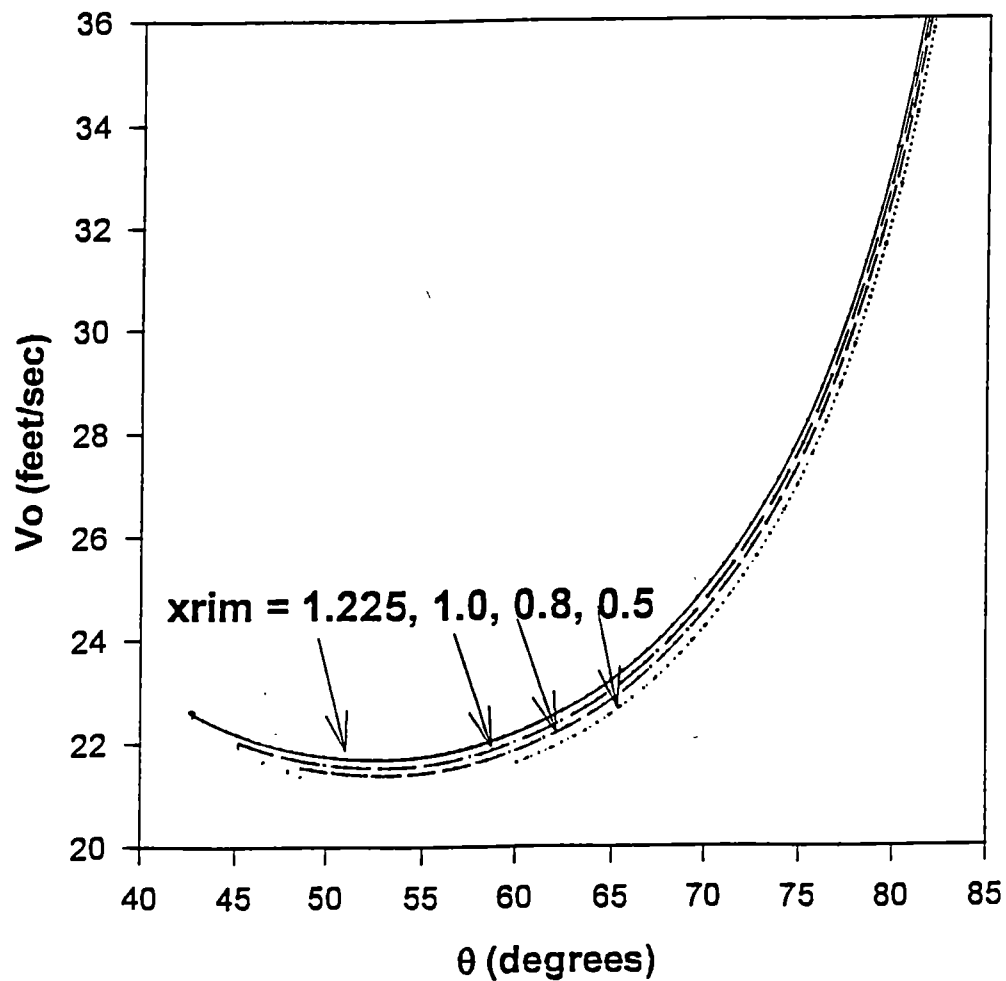


Figure 6. Illustration of solutions of V_o and θ for discrete acceptable values of x_{rim} for values of $\phi_2 \geq$ limiting minimum value corresponding to the flattest trajectory successful shot for $d=10$ and $h=7$.

Figure 7 illustrates the envelope of solutions of V_0 and θ for a successful shot for the conditions illustrated in figure 6, i.e., a shot launched from $d=10$ feet and $h=7$ feet. The envelope consists of two parts:

- (1) The upper-bound curve in figure 7 is as discussed above. It corresponds to solutions for the maximum acceptable value of $x_{rim} = d_{rim} - R_b = 1.225$ feet in equation 1 with values of ϕ_2 ranging from the minimum value of ϕ_2 (for the flattest trajectory shot) to the maximum value of ϕ_2 at which the maximum height of the trajectory (y_{max}) is less than some designated value. $y_{max}=25$ was arbitrarily chosen; however, higher values would be of little practical interest because rapidly increasing values required for V_0 and θ would result in poor accuracy and subsequently a low probability of actually executing a successful shot. The solution corresponding to the minimum value of θ on this curve was generated using the Newton-Raphson algorithm to solve the four nonlinear algebraic equations 20-22, and 26b for the maximum acceptable value of $x_{rim}=d_{rim}-R_b$. The remaining points on the curve were solved using equations 13 and 14 for increasing values of ϕ_2 up to where the maximum height of the trajectory exceeds 25 feet. For this curve, increasing values of θ correspond to increasing values of ϕ_2 .
- (2) The lower-bound curve in figure 7 corresponds to solutions of V_0 and θ for the flattest trajectory successful shots for acceptable

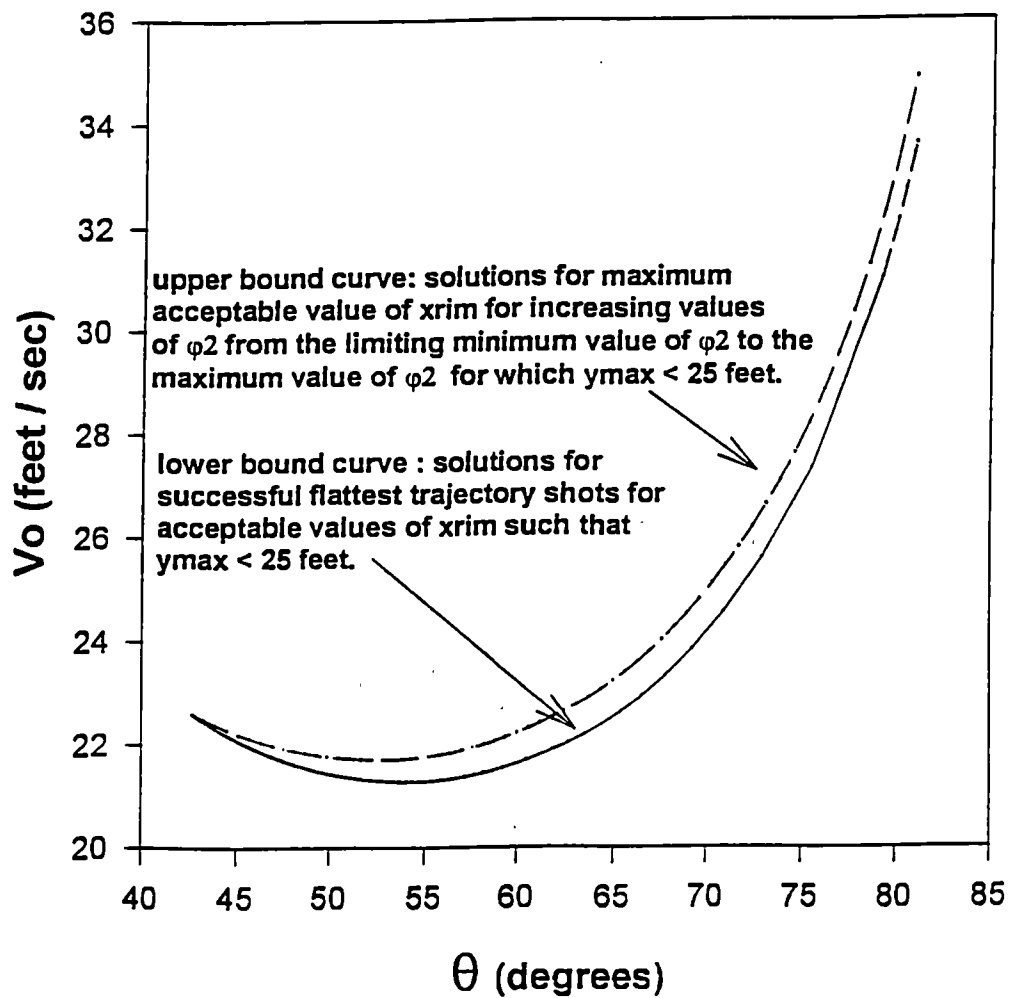


Figure 7. Illustration of solution envelope for all successful shots for $d=10$, $h=7$, $y_{\max}=25$.

values of x_{rim} , and a maximum trajectory height less than the maximum allowable value y_{max} . The solution at the minimum value of θ corresponds to the maximum allowable value of $x_{rim} = d_{rim} - R_b$. Increasing values of θ correspond to the flattest trajectory successful shot at decreasing allowable values of x_{rim} in equation. For selected discrete acceptable values of x_{rim} , the Newton-Raphson algorithm solves the four nonlinear algebraic equations 20-22, and 26b. Each solution yields values for V_o and θ as well as ϕ_1 and ϕ_2 .

The lower-bound curve and the upper-bound curve coincide at a single point in the envelope of acceptable values of V_o and θ , i.e., at the minimum value of θ in the solution envelope. In both cases, this point represents the solution for the flattest trajectory successful shot at the maximum acceptable value of $x_{rim} = (d_{rim} - R_b)$. If there is no restriction of y_{max} imposed on the solution envelope, the two curves would also converge toward the single point where $\theta = 90$ degrees and V_o tends to infinity. For the upper-bound curve, convergence toward this point occurs as the increasing value of ϕ_2 approaches 90 degrees. For the lower-bound curve, convergence toward this point occurs when determining the flattest trajectory shot as x_{rim} approaches the minimum allowable value of x_{rim} in equation 1, i.e., R_b , where ϕ_2 would have to be 90 degrees for a successful shot.

All points that are inside of the solution envelope correspond to solutions V_o and θ for values of x_{rim} in the interval

$R_b < x_{rim} < (d_{rim} - R_b)$ for values of ϕ_2 greater than the limiting minimum value corresponding to the flattest trajectory successful shot.

As discussed above, for the analysis to be of practical interest, solutions should be limited to those that might be expected to have a reasonable probability of being successfully executed. For the specified problem discussed above ($d=10$ and $h=7$), one mathematical solution is $V_o=111.06$ feet per second and $\theta=89.16$ degrees. The trajectory of this shot would reach a height of 198.63 feet. It is a mathematical solution; however, it is not a practical solution with regard to successfully executing it.

The maximum height of the trajectory can easily be calculated by first determining the value of x at which the maximum height will be reached. This is done by determining the location at which the derivative of the trajectory is equal to zero. This can be done by differentiating equation 8 with respect to x , equating the derivative to 0, and solving for x as follows:

$$x_{ymax} = \frac{V_o^2 \sin(\theta) \cos(\theta)}{g} \quad (27)$$

where x_{ymax} is the value of x when y is a maximum. The maximum value of y may be calculated simply by substituting the value of x_{ymax} into equation 8, which yields the following:

$$y_{max} = \frac{V_o^2 \sin^2(\theta)}{2g} + h \quad (28)$$

A numerical integration scheme similar to the trapezoid rule [7] was developed to calculate the area inside of the solution envelope.

Figure 8 is a plot that compares the area inside of the solution envelopes for a free throw with $d=15$, $h=7$, and a three point shot with $d=18$, $h=7$, while imposing a maximum trajectory height of $y_{\max}=18$. The value of $y_{\max}=18$ is arbitrary; however, the intent is to limit the solution envelope to solutions of V_0 and θ that have a chance of actually being executed. The area of the solution envelopes are 13.3 and 11.1, respectively. The size of the solution envelope is reduced by approximately 17 per cent when the launch location is moved from $d=15$ to $d=18$.

Figure 9 is a plot of the area inside of the solution envelopes as a function of d for arbitrary values $y_{\max} = 18, 25$, and 50 . It should be noted that the Newton-Raphson algorithm to solve the system of nonlinear equations had to be solved thousands of times to generate the data contained in figure 9. The rate at which the size of the solution envelope decreases with respect to d declines with increasing d . It appears that the rate at which the solution envelope decreases with respect to distance is not sensitive to the value of $y_{\max}$ imposed on the solutions.

Figure 10 is a plot of the area inside of the solution envelope as a function of d for various values of h . In this case, an arbitrary value of $y_{\max} = 18$ was imposed on the solution envelope. The area of the solution envelope is quite sensitive to h for smaller values of d , however, the sensitivity decreases for increasing values of d . In practical terms, this means that taller players definitely have an

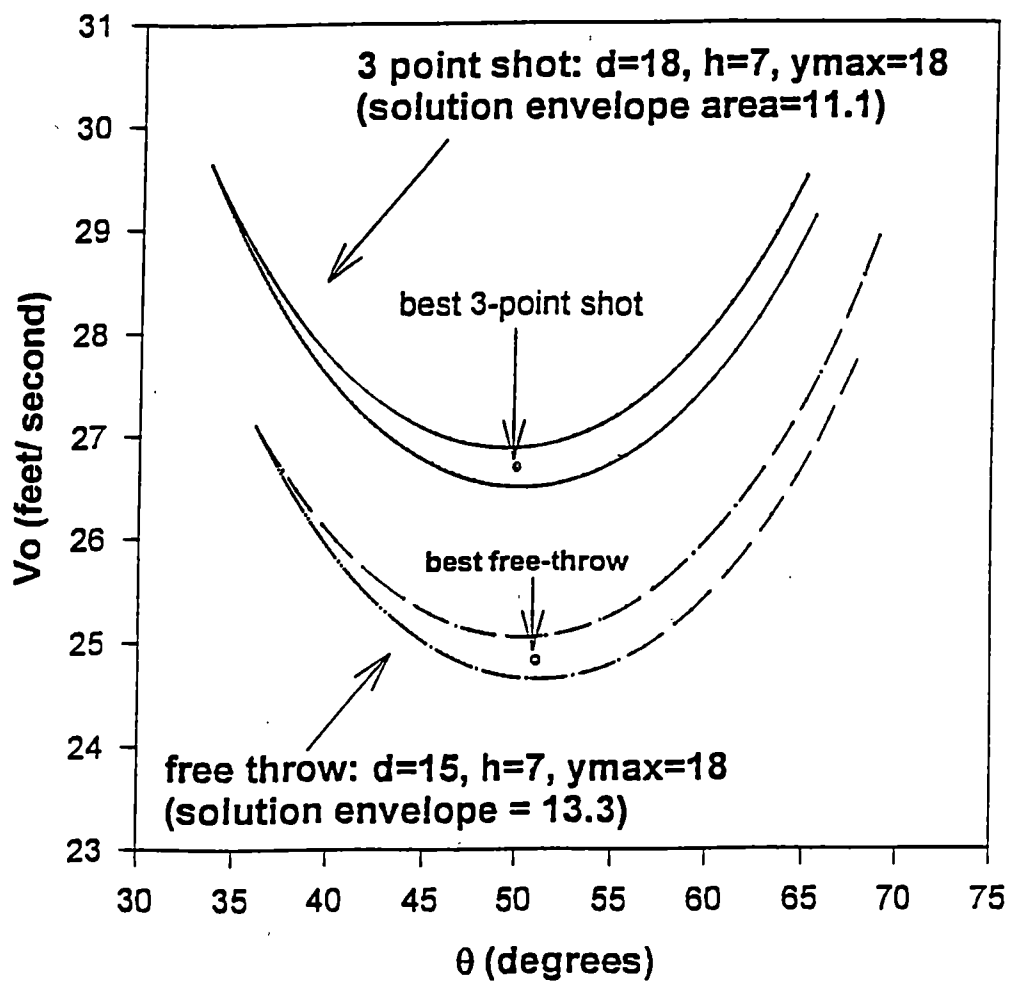


Figure 8. Comparison of solution envelopes for free throw and three-point shot for various values of y_{max} .

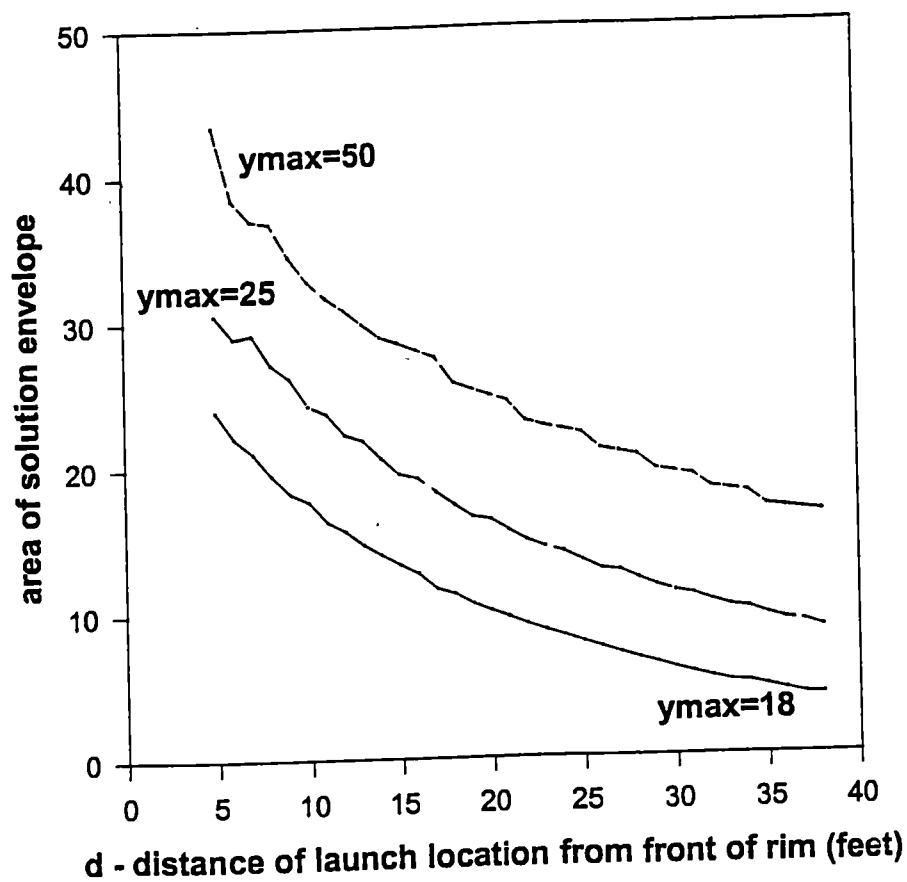


Figure 9. Area of solution envelope as a function of d for various values of y_{max}

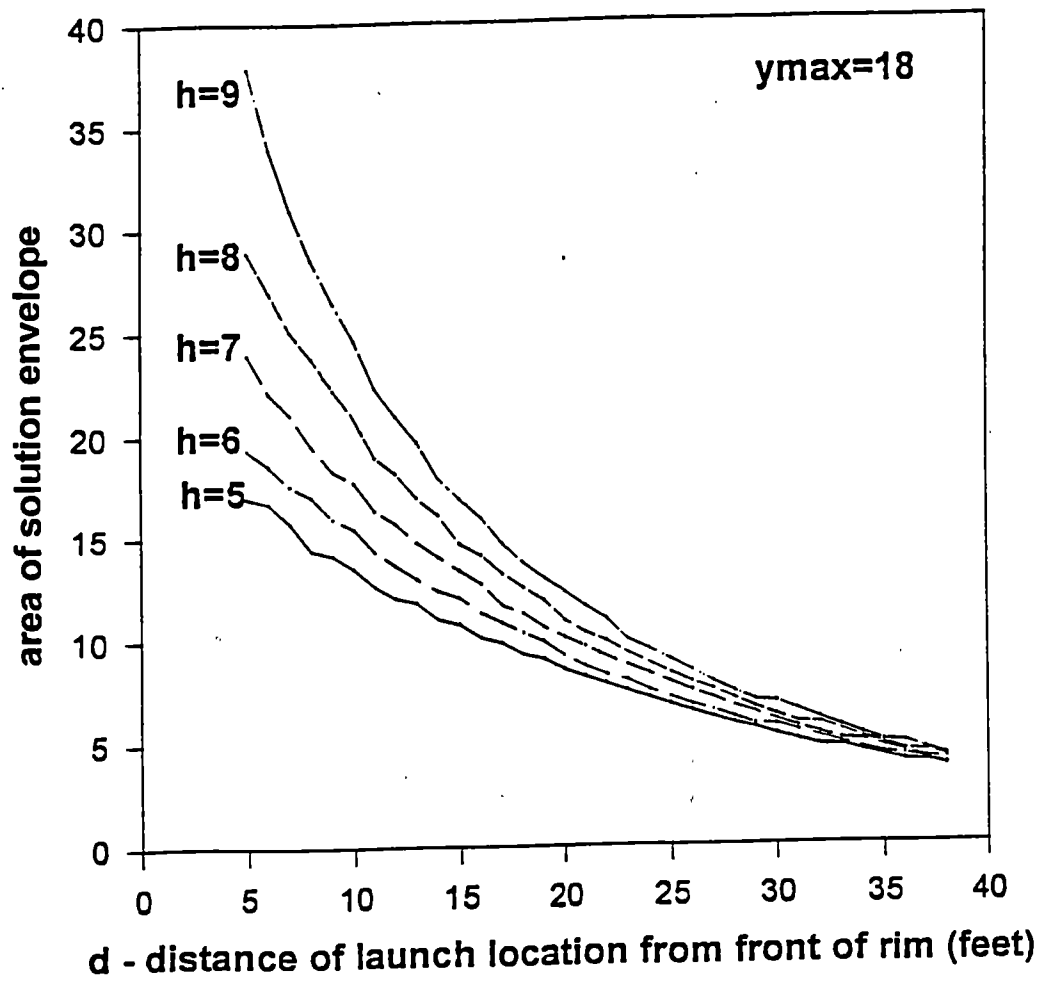


Figure 10. Area of solution envelope as a function of d for various values of h for $y_{\max}=18$

advantage when shooting from close range; however, the advantage diminishes as the distance from the rim increases.

6.0 Three Dimensional Model

A model of increasing complexity is a three-dimensional model in which the flight path of the ball is allowed to deviate from the global x axis, i.e., allowed to deviate from the vertical x-y plane that contains the launch location of the ball and the center of the rim. In other words α is not restricted to zero, as was the case in the two-dimensional model. As previously discussed, for the case where $\alpha \neq 0$, the range of values of x_{rim} for which the centroid of the ball could successfully pass through the rim becomes smaller and therefore more restrictive. This is illustrated in figure 11.

The previous two-dimensional model considered the two-dimensional aspects of the ball and treated the rim as one-dimensional. The trajectory was defined in terms of two coordinates x and y. The three-dimensional model considers the three-dimensional aspects of the ball and treats the rim as two-dimensional. The y dimension of the rim is not relevant. The trajectory of the ball is defined in terms of three coordinates x, y, and z.

Again, the objective is to determine the solution envelope of V_0 and θ that result in a successful shot; however, in the three-dimensional case, a specified shot of d, h, and α , will be considered rather than a shot involving only d and h as was the case for the two-dimensional model. As in the two-dimensional case, a range of acceptable values of x_{rim} must be found such that the centroid of the spherical ball with radius R_b can successfully pass through the rim in

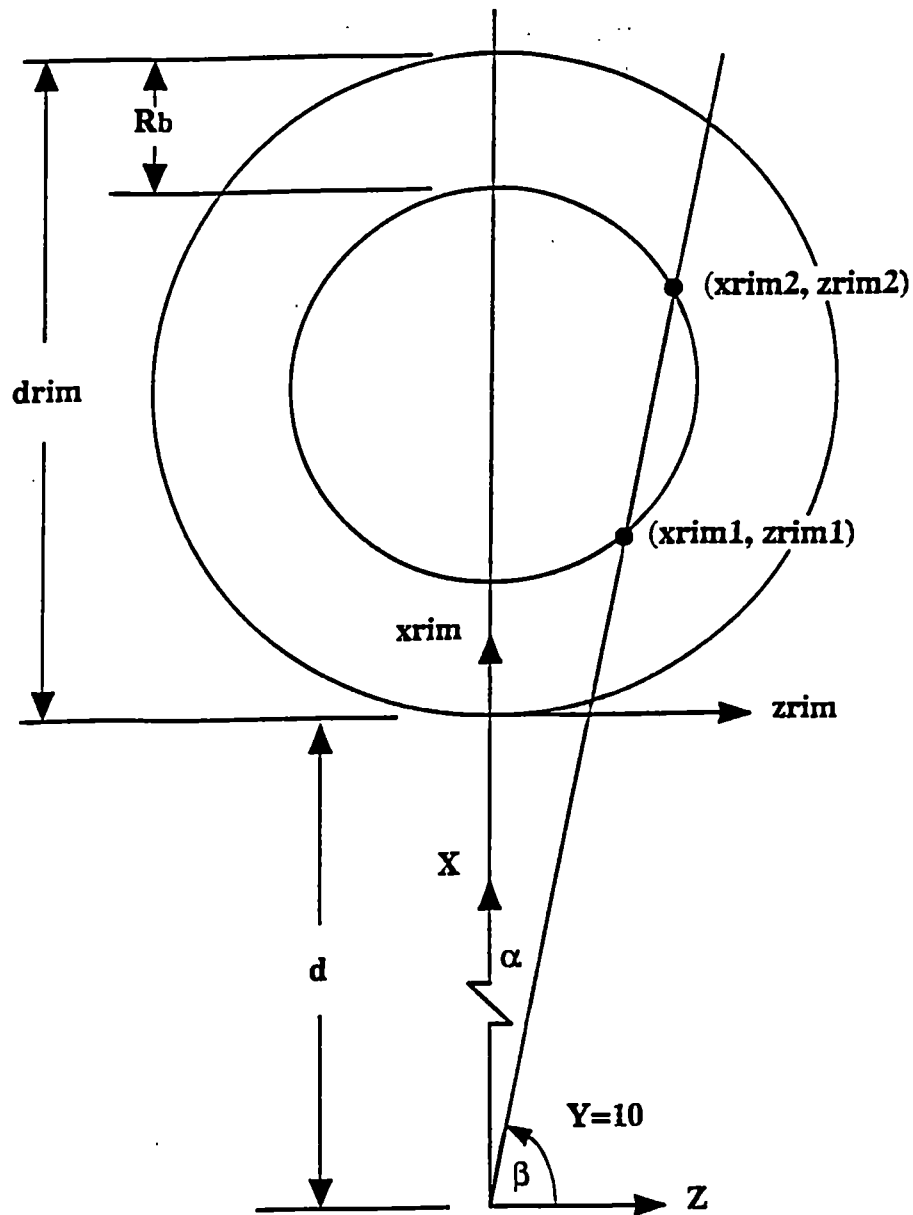


Figure 11. Illustration of acceptable values of x_{rim} as a function of α for the three-dimensional model

the x-z plane where $y=10$. In the three-dimensional case, the acceptable values of x_{rim} are dependent on α .

The same strategy will be applied as was utilized in the development of the two-dimensional models, i.e., the derivation of models of increasing complexity. A requirement for all equations derived for the three-dimensional models is that they must reduce to the corresponding counterpart equations for the two-dimensional models when $\alpha=0$.

Clearly, the three-dimensional model requires more complex geometrical relationships than the two-dimensional model. The following analyses are a derivation of geometrical relationships required in the trajectory analysis for the three-dimensional models.

7.0 Determination of Acceptable Values of xrim for the Three-Dimensional Model

As defined earlier, xrim is a variable in the local coordinate system for the rim. xrim is along the global x-axis and varies from xrim=0 at the front of the rim to xrim=drim at the back of the rim.

For the two dimensional case, the acceptable values of xrim for a successful shot, as expressed in equation 1, were relatively simple; however, for the three-dimensional case, the acceptable values of xrim are also a function of α . As in the two-dimensional case, the basketball can successfully pass through the rim only if the centroid of the ball passes through the x-z plane at y=10 on or inside of the inner circle that has a radius of $R_{rim}-R_b$. Figure 11 illustrates that when the flight path of the ball deviates from the global x axis by an angle α , the ball must pass through the x-z plane at y=10 between the local coordinates (xrim1,zrim1) and (xrim2,zrim2). Otherwise, there would be interference between the spherical ball and the rim, in which case the ball would be deflected resulting in an unsuccessful shot.

For a specified shot with variables d, h, and α , a determination must be made of the acceptable coordinates in the local xrim, zrim coordinate system in the x-z plane at y=10 for which the shot can be successful, i.e., (xrim1,zrim1) and (xrim2,zrim2) as illustrated in figure 11. This is a problem in analytical geometry consisting of the solution of the coordinates where a circle and a straight line intersect. The straight line originates from the launch location. The

circle is the inside circle of radius $R_r - R_b$ as illustrated in figure 11. In figure 11, β is the complementary angle of α . For a coordinate system with its origin at the center of the rim, an equation for the straight line originating from the launch location can be written as:

$$x = z \tan \beta - (R_r + d) \quad (29a)$$

$$z = (x + R_r + d) \cot(\beta) \quad (29b)$$

Since α and β are complementary, $\cot(\beta) = \tan(\alpha)$. The equation for the straight line may therefore be written in terms of α as follows:

$$z = (x + R_r + d) \tan(\alpha) \quad (29c)$$

For the same coordinate system with its origin at the center of the rim, an equation for the inside circle in figure 11 with a radius of $R_r - R_b$ can be written in terms of x and z as follows:

$$x^2 + z^2 = (R_r - R_b)^2 \quad (30a)$$

$$z = ((R_r - R_b)^2 - x^2)^{1/2} \quad (30b)$$

To determine the coordinates where the straight line and circle intersect, equations 29c and 30b are equated as follows:

$$(x + R_r + d) \tan(\alpha) = ((R_r - R_b)^2 - x^2)^{1/2} \quad (31)$$

After squaring both sides of this equation, it is quadratic in x . Solving equation 31 for x results in the following two solutions for the x coordinates where the inside circle in figure 11 and straight line intersect:

$$x_{1,2} = \frac{-(R_r + d) \tan^2(\alpha) \pm \{\tan^2(\alpha) [R_b^2 - 2R_r R_b - 2R_r d - d^2] + [R_b^2 - 2R_r R_b + R_r^2]\}^{1/2}}{1 + \tan^2(\alpha)} \quad (32a)$$

$$x_2 = -\frac{(R_r + d) \tan^2(\alpha) - \{\tan^2(\alpha) [R_b^2 - 2R_r R_b - 2R_r d - d^2] + [R_b^2 - 2R_r R_b + R_r^2]\}^{1/2}}{1 + \tan^2(\alpha)} \quad (32b)$$

These coordinates are relative to the coordinate system with its center at the center of the rim. These coordinates can be transformed to the local xrim coordinate system which, by definition, has its origin at the front of the rim (with $\alpha = 0$) as follows:

$$x_{rim1} = R_r + x_1 \quad (33a)$$

$$x_{rim2} = R_r + x_2 \quad (33b)$$

Therefore, for the three-dimensional model, the acceptable range of values of xrim where the ball can successfully pass through the x-z plane at $y=10$ are:

$$(R_r + x_1) \leq x_{rim} \leq (R_r + x_2) \quad (33c)$$

where x_1 and x_2 are as defined in equations 32a and 32b, respectively. This is analogous to the acceptable range of values in equation 1 for the two-dimensional model.

Equations 33a and 33b reduce to $x_{rim1} = R_b$ and $x_{rim2} = d_{rim} - R_b$ for $\alpha=0$, respectively, which is compatible with the acceptable values of xrim in equation 1 for the two-dimensional model.

The question arises: for a shot of specified d , what is the maximum value of α for which that shot can be successful? As α increases, there is some maximum value of α , beyond which the straight line would fail to intersect the circle. In this case equations 32a and 32b would not yield real solutions (the discriminant would be less than zero). For the maximum value of α which does not result in a negative discriminant, the discriminant would be equal to zero. In

this case, there is a double root $(Rr-d)/\tan^2(\alpha)$. Geometrically, for this case, the straight line intersects the circle at a single point of tangency.

The value of α which causes the discriminant in equations 32 to go to 0 can be determined by equating the two terms in the discriminant and solving for α as follows:

$$\tan^2(\alpha) [Rb^2 - 2RrRb - 2Rrd - d^2] - (Rr^2 - 2RrRb - Rb^2) = 0 \quad (34)$$

Solving equation 34 for α yields the maximum value of α that would permit a successful shot from a specified distance as follows:

$$\alpha_{\max} = \tan^{-1} \left[\frac{(Rr - Rb)}{(d^2 + 2RrRb + 2Rrd - Rb^2)^{1/2}} \right] \quad (35)$$

This is the angle at which the straight line (flight path) and circle of radius $Rr - Rb$ intersect at a point of tangency. In this case $x_{rim1} = x_{rim2} = (Rr-d)/\tan^2(\alpha)$. Therefore for a specified shot from a distance d , acceptable values of α are as follows:

$$-\alpha_{\max} \leq \alpha \leq \alpha_{\max} \quad (36)$$

For a shot of specified d and subsequently an acceptable value of α specified in equation 36, the range of values of x_{rim} for which a shot can be successful without being deflected by the rim are as specified in equations 33c. This completes the determination of acceptable values of x_{rim} for the three-dimensional case.

There are some additional geometrical relationships required for the three-dimensional model; therefore, before proceeding to the

analysis of the trajectory in the three-dimensional model, these additional geometrical relationships will be derived.

8.0 Derivation of Additional Geometrical Relationships Required for the Three-Dimensional Model

Figure 12 illustrates the intersection of the trajectory of the centroid of the ball and the front of the rim in the x-z plane at a height of $y=10$ for some acceptable value of α from equation 36. x^* is the coordinate in the global x coordinate system of the intersection as follows:

$$x^* = d + \Delta x \quad (37)$$

where Δx is the coordinate in the local xrim coordinate system of the intersection of the flight path of the ball and the front of the rim. From figure 12, the following geometrical relationship can be derived:

$$\tan \alpha = z_{rim} / (d + \Delta x) ; z_{rim} = (d + \Delta x) \tan \alpha \quad (38)$$

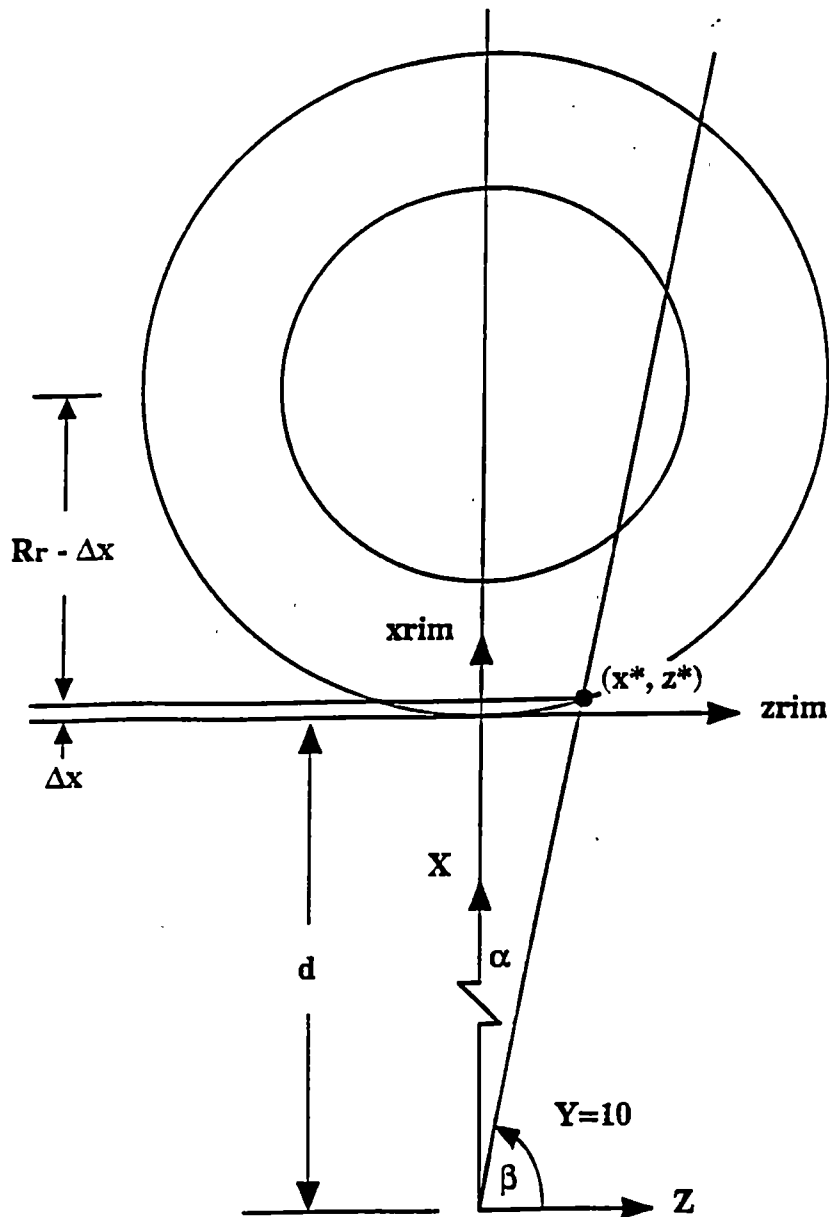
where z_{rim} is the z coordinate in the local xrim coordinate system of the intersection of the flight path and the front of the rim. Application of the Pythagorean theorem in figure 12 results in the following equation:

$$(Rr - \Delta x)^2 + (z_{rim})^2 = Rr^2 \quad (39)$$

Substitution of z_{rim} from equation 38 into equation 39 and expansion of the result produces the following equation:

$$\Delta x^2 (1 + \tan^2 \alpha) + \Delta x (2d \tan^2 \alpha - 2Rr) + d^2 \tan^2 \alpha = 0 \quad (40)$$

Equation 40 is quadratic in Δx . Solving equation 40 results in the following solutions:



$$x^* = d + \frac{(Rr - \tan^2(\alpha)) - (Rr^2 - \tan^2(\alpha)) (d^2 + 2dRr)^{1/2}}{1 + \tan^2(\alpha)}$$

Figure 12. Illustration of x^* , global x coordinate of the intersection of the trajectory with the front of the rim ($x^* = d + \Delta x$)

$$\Delta x = \frac{(Rr - \tan^2(\alpha)) + \{Rr^2 - \tan^2(\alpha)(d^2 + 2dRr)\}^{1/2}}{1 + \tan^2(\alpha)} \quad (41a)$$

$$\Delta x = \frac{(Rr - \tan^2(\alpha)) - \{Rr^2 - \tan^2(\alpha)(d^2 + 2dRr)\}^{1/2}}{1 + \tan^2(\alpha)} \quad (41b)$$

Equation 41b yields the relevant solution since it reduces to $\Delta x=0$ for $\alpha = 0$. Substituting equation 41b into 37 yields an equation for x^* for a specified d and α as follows:

$$x^* = d + \frac{(Rr - \tan^2(\alpha)) - \{Rr^2 - \tan^2(\alpha)(d^2 + 2dRr)\}^{1/2}}{1 + \tan^2(\alpha)} \quad (42)$$

For $\alpha = 0$, equation 42 reduces to the two-dimensional case where $x^* = d$.

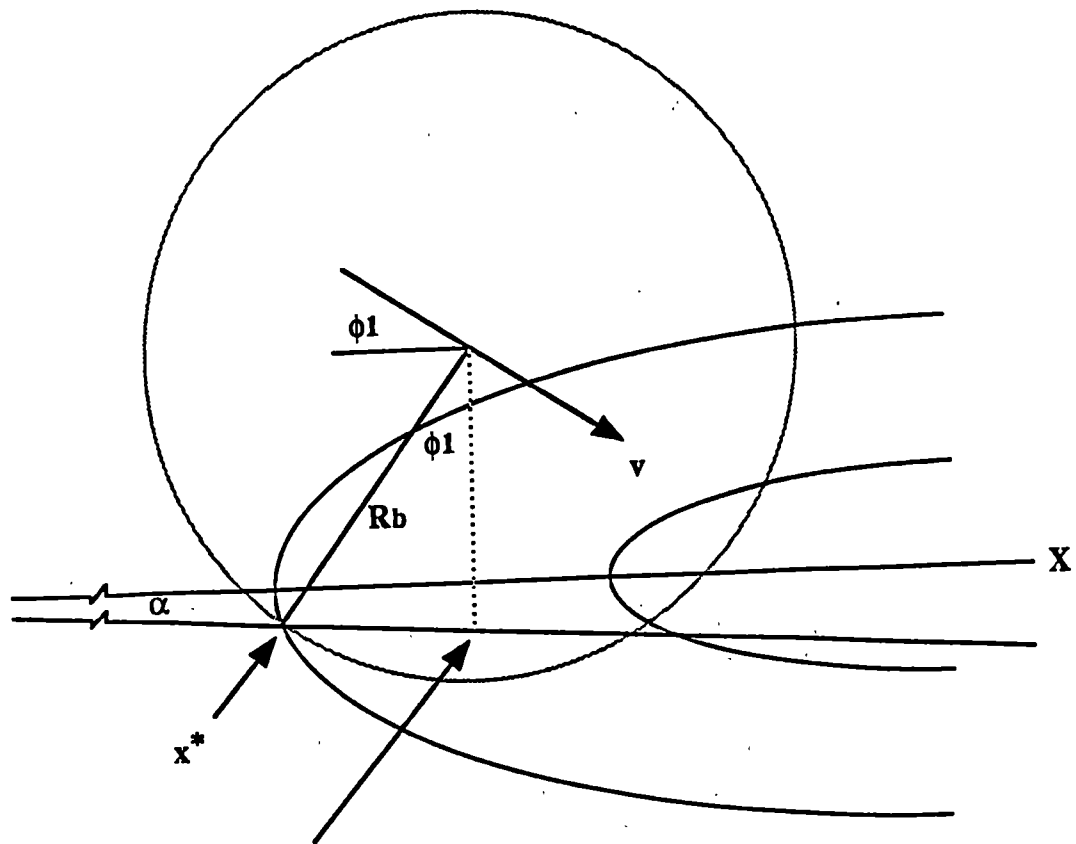
In figure 13, x^{**} is the x coordinate in the global coordinate system of the centroid of the basketball when the ball is in the limiting front rim location and is equal to:

$$x^{**} = x^* + Rb \sin(\phi_1) \cos(\alpha) \quad (43a)$$

$$x^{**} = d + \frac{(Rr - \tan^2(\alpha)) - \{Rr^2 - \tan^2(\alpha)(d^2 + 2dRr)\}^{1/2}}{1 + \tan^2(\alpha)} + Rb \sin(\phi_1) \cos(\alpha) \quad (43b)$$

For $\alpha = 0$, equation 43b reduces to the two-dimensional case where $x^{**} = d + Rb \sin(\phi_1)$.

As in the two dimensional case, an analysis will be made that uses the simplifying assumption that $\phi = \phi_1 = \phi_2$; therefore, a relationship is required that solves for ϕ , analogous to the relationship in equation 15 for the two-dimensional case. A relationship for ϕ may be derived as follows:



$$x^{**} = x^* + Rb \sin(\phi1) \cos(\alpha)$$

$$x^{**} = d + \frac{(Rr - \tan^2(\alpha)) - (Rr^2 - \tan^2(\alpha)(d^2 + 2dRr))^{1/2}}{1 + \tan^2(\alpha)} + Rb \sin(\phi1) \cos(\alpha)$$

Figure 13. Illustration of x^{**} , global coordinate of the centroid of basketball where the ball just misses being deflected by the front of the rim (flattest trajectory successful shot).

$$\phi = \phi_1 = \phi_2 = \sin^{-1} (Rb / k) \text{ where } k = m - n \quad (44a)$$

$$\text{where } n = (d + \Delta x) \sec(\alpha) \text{ and } m = (d + x_{rim}) \sec(\alpha) \quad (44b)$$

Substituting Δx from equation 41b in equation 44b and simplifying yields the following solution for $\phi = \phi_1 = \phi_2$.

$$\phi = \sin^{-1} \left[\frac{Rb}{\sec(\alpha) \left\{ x_{rim} - \frac{(Rr - d \tan^2 \alpha) - [Rr^2 - \tan^2 \alpha (d^2 + 2dRr)]^{1/2}}{1 + \tan^2(\alpha)} \right\}} \right] \quad (45)$$

As required, this equation reduces to the two-dimensional counterpart in equation 15, i.e., $\phi = \sin^{-1} (Rb / x_{rim})$ when $\alpha = 0$.

This completes the derivation of geometrical relationships necessary for performing the three dimensional trajectory analysis.

9.0 Trajectory Analysis for the Three-Dimensional Model

As in the case of the two-dimensional model, the analysis begins with an application of differential equations expressing Newton's second law; however, in this case Newton's law is applied in all three cartesian coordinate directions.

For the x direction, as in equation 3, the velocity is constant and is equal to the initial launch velocity in the x direction. In this case, the component of velocity that projects into the horizontal x-z plane at y=h must be resolved into components in the x and z direction; therefore the constant velocity in the x direction is as follows:

$$\frac{dx}{dt} = V_o \cos(\theta) \cos(\alpha) \quad (46)$$

Integrating equation 46 and applying the initial condition that $x(t=0) = 0$ results in an equation for x as a function of t as follows:

$$x(t) = \{V_o \cos(\theta) \cos(\alpha)\} t \quad (47)$$

Similarly, there are no forces acting on the ball in the z direction; therefore, the velocity is constant and is equal to the initial launch velocity in the z direction as follows:

$$\frac{dz}{dt} = V_o \cos(\theta) \sin(\alpha) \quad (48)$$

Integrating equation 48 and applying the initial condition that $z(t=0) = 0$ results in an equation for z as a function of t as follows:

$$z(t) = \{V_o \cos(\theta) \sin(\alpha)\} t \quad (49)$$

Equations 6-7 are still applicable to determine the velocity and location of the ball in the y direction as a function of t. An equation for y as a function of x can be obtained by solving for t in equation 47 and substituting it into equation 7 as follows:

$$y(x) = x \tan(\theta) \sec(\alpha) - g x^2 / 2V_o^2 \cos^2(\theta) \cos^2(\alpha) + h \quad (50)$$

Equation 50 is a quadratic equation in x. Solving equation 50 for x results in the following two solutions for x as a function of y: function of y:

$$x(y) = \frac{V_o^2 \cos(\theta) \sin(\theta) \cos(\alpha)}{g} - \quad (51a)$$

$$\frac{V_o \cos(\theta) \cos(\alpha)}{g} \{V_o^2 \sin^2(\theta) - 2g(y-h)\}^{1/2}$$

$$x(y) = \frac{V_o^2 \cos(\theta) \sin(\theta) \cos(\alpha)}{g} + \quad (51b)$$

$$\frac{V_o \cos(\theta) \cos(\alpha)}{g} \{V_o^2 \sin^2(\theta) - 2g(y-h)\}^{1/2}$$

As in the two-dimensional model, only the second of these equations (51b) is applicable to the analysis of a successful shot on its downward trajectory.

For a successful shot, the centroid of the ball passes through the x-z plane at y=10 when x = xrim+d. Substituting these values into equation 51b yields the following equation:

$$\frac{V_o^2 \cos(\theta) \sin(\theta) \cos(\alpha)}{g} + \quad (51c)$$

$$\frac{V_o \cos(\theta) \cos(\alpha)}{g} \{V_o^2 \sin^2(\theta) - 2g(10-h)\}^{1/2} - (xrim+d) = 0$$

where xrim is an acceptable value as defined in equation 33c and α is an acceptable value as defined in equation 36.

Since there are two unknowns (V_o and θ), another equation is required. The tangent of the angle with the horizontal x-z plane at which the velocity vector of the centroid of the ball passes through the coordinates ($x=x_{rim}+d$, $y=10$), again designated as ϕ_2 , can be determined as the ratio of the magnitude of the velocity vector in the y direction to the magnitude of the velocity vector in the x-z plane. This is equivalent to the tangent of the angle that the velocity vector makes with the horizontal x-z plane at $y=10$, which is equivalent to the derivative of the trajectory at this point. This can be expressed as follows:

$$\left. \frac{dx}{dy} \right|_{x=x_{rim}+d} = \frac{V_y}{(V_x^2 + V_z^2)^{1/2}} \bigg|_{x=x_{rim}+d} \quad (52)$$

From equation 6:

$$V_y = \frac{dy}{dt} = V_o \sin(\theta) - gt \quad (53)$$

Solving for t from equation 47, evaluating at $x=x_{rim} + d$, and substituting into equation 53 yields an equation for V_y as follows:

$$V_y = V_o \sin(\theta) - \frac{g(x_{rim} + d)}{V_o \cos(\theta) \cos(\alpha)} \quad (54)$$

The denominator of equation 52 can be determined by substituting equations 46 and 48 into the denominator of equation 52 as follows:

$$(V_x^2 + V_z^2)^{1/2} = (V_o^2 \cos^2(\theta) \cos^2(\alpha) + V_o^2 \cos^2(\theta) \sin^2(\alpha))^{1/2} = V_o \cos(\theta) \quad (55)$$

The derivative of the trajectory evaluated at $x=x_{rim}+d$, $y=10$ may be written as equation 54 divided by equation 55. It may be equated to the negative of the tangent of ϕ_2 . After simplifying, the result is:

$$\tan(\theta) - \frac{g(x_{rim} + d)}{V_o^2 \cos(\theta) \cos(\alpha)} + \tan(\phi_2) = 0 \quad (56)$$

This equation reduces to the two-dimensional counterpart in equation 12 when $\alpha=0$.

The simultaneous solution of equations 51c and 56 provides a solution of V_o and θ for a successful shot from a distance of d , a height of h , and an arbitrary angle of ϕ_2 for an acceptable value of x_{rim} from equation 33c and an acceptable value of α from equation 36.

Proceeding to solve equations 51c and 56 simultaneously by substitution, equation 51c can be manipulated to solve for V_o^2 as follows:

$$V_o^2 = \frac{(g/2)(x_{rim} + d)^2}{\sin(\theta) \cos(\theta) \cos(\alpha) (x_{rim} + d) - \cos^2(\theta) \cos^2(\alpha) (10 - h)} \quad (57)$$

Equation 57 reduces to the two-dimensional counterpart in equation 13 when $\alpha=0$. Substituting equation 57 into equation 56 and solving for θ yields a solution for the launch angle θ as follows:

$$\theta = \tan^{-1} \left[\frac{\tan(\phi_2) (x_{rim} + d) + 2(10 - h) \cos(\alpha)}{(x_{rim} + d)} \right] \quad (58)$$

Equation 58 reduces to the two-dimensional counterpart in equation 14 when $\alpha=0$. Equation 58 solves for the value of θ as a function of d , h , x_{rim} , α , and ϕ_2 . The resulting value θ can be substituted into equation 57 to solve for the magnitude of the launch velocity.

Equations 57 and 58 yield successful solutions for V_o and θ as long as ϕ_2 is greater than the limiting minimum value of ϕ_2

corresponding to the flattest trajectory successful shot. Consideration will now be given to the determination of this limiting minimum value of ϕ_2 , making use of the previously derived geometrical relationships for the three-dimensional model.

As in the case of two dimensions, an analysis will be performed which makes the simplifying assumption that $\phi_1 = \phi_2$. Again, this solution has two unknowns (V_0 and θ) and will therefore require two equations. Equation 51c is one of those equations. Another equation can be derived by evaluating the derivative of the trajectory of the centroid of the ball at x^{**} .

The derivative of the trajectory at x^{**} can be evaluated as the ratio of the magnitude of the vertical component of the velocity to the magnitude of the component of the velocity vector in the x-z plane evaluated at $x=x^{**}$ as follows:

$$\left. \frac{dy}{dx} \right|_{x=x^{**}} = \frac{V_y}{[V_x^2 + V_z^2]^{1/2}} \bigg|_{x=x^{**}} \quad (59)$$

where x^{**} is as expressed in 43b and as illustrated in figure 12. Solving for t from equation 47, making use of equation 55, and simplifying results in the following equation:

$$\left. \frac{dy}{dx} \right|_{x=x^{**}} = \tan(\theta) - \frac{(g x^{**})}{V_0^2 \cos^2(\theta) \cos(\alpha)} \quad (60)$$

Equation 60 can be equated to the negative of the tangent of angle ϕ , where ϕ is as derived in equation 45, resulting in the following equation:

$$\tan(\theta) - \frac{(g x^{**})}{Vo^2 \cos^2(\theta) \cos(\alpha)} + \tan(\phi) = 0 \quad (61)$$

which is analogous to equation 17 for the two-dimensional model.

Substituting equation 57 for Vo^2 into equation 61 yields the following solution for the launch angle θ as follows:

$$\theta = \tan^{-1} \left[\frac{2x^{**} \cos(\alpha) (10 - h) + (xrim + d)^2 \tan(\phi)}{(xrim + d) (2x^{**} - (xrim + d))} \right] \quad (62)$$

where ϕ and x^{**} are as expressed in equation 45 and 43b, respectively.

Equation 62 solves for θ as a function of the spatial coordinates d , h , $xrim$, the angle α , and parameters related to the geometry of the ball and rim, i.e., Rb and Rr . Substitution of θ into equation 57 yields the solution for the magnitude of the launch velocity. Equations 57 and 62 are analogous to equations 13 and 18 for the two-dimensional model and reduce to these equations for the case when $\alpha=0$.

10.0 Relaxation of the Assumption that $\phi_1 = \phi_2$ for the Three-Dimensional Model

As in the two-dimensional model, an analysis will now be performed which removes the assumption that $\phi_1 = \phi_2$, i.e., to include the change in the trajectory that occurs when the centroid of the ball moves from the limiting front rim location ($x=x^{**}$, $y=10 + Rb\cos(\phi_1)$) to the location where the centroid of the ball passes through the x-z plane at $y=10$ ($x=x_{rim} + d, y=10$).

Two equations can be obtained by applying equation 51b at appropriate locations. Equation 51c, which is an application of equation 51b, evaluated at $x=x_{rim}+d$, $y=10$ is still valid. As in the two-dimensional case, equation 51c requires rearranging to remove the square root to avoid potential numerical difficulties during the iterative numerical solution of the system of nonlinear algebraic equations. Rearranging equation 51b to remove the square root results in the following equation:

$$-\{(x_{rim}+d) - (Vo^2 \cos(\theta) \sin(\theta) \cos(\alpha) / g)\}^2 + \\ (Vo^2 \cos^2(\theta) \cos^2(\alpha) / g^2) (Vo^2 \sin^2(\theta) - 2g(10-h)) = 0 \quad (63)$$

Equation 51b can also be applied to satisfy the conditions $x=x^{**}$ when $y=10+Rb\cos(\phi_1)$. Again, this equation is rearranged to avoid the potential numerical difficulties associated with the square root resulting in the following equation number 64:

$$-[x^{**} - (Vo^2 \cos(\theta) \sin(\theta) \cos(\alpha) / g)]^2 + \\ (Vo^2 \cos^2(\theta) \cos^2(\alpha) / g^2) (Vo^2 \sin^2(\theta) - 2g(10+Rb\cos(\phi_1)-h)) = 0 \quad (64)$$

reiterating that x^{**} is from equation 43b and the stipulation that values for α satisfy equation 36.

The method for calculating the trajectory derivative previously derived in equation 61 can be applied to the limiting front rim location ($x = x^{**}$, $y = 10 + Rb\cos(\phi_1)$). The result can be equated to the negative of the tangent of angle ϕ_1 resulting in the following equation:

$$\tan(\theta) - \frac{gx^{**}}{Vo^2 \cos^2(\theta) \cos(\alpha)} + \tan(\phi_1) = 0 \quad (65)$$

Solving equation 65 for ϕ_1 results in the following equation

$$\phi_1 = -\tan^{-1} \left[\tan(\theta) - \frac{gx^{**}}{Vo^2 \cos^2(\theta) \cos(\alpha)} \right] \quad (66)$$

Thus far, there are three equations (63, 64, and 66) relating Vo , θ , and ϕ_1 . Another equation is required to relate the fourth variable ϕ_2 to the other three variables. Following the same strategy used in the development of the two-dimensional model, the application of equation 19 in this case can be used to quantify the change in the derivative of the trajectory that occurs as the centroid of the ball moves from the limiting front rim location ($x=x^{**}$, $y=10 + Rb\cos(\phi_1)$) to the location where the ball passes through the rim in the x-z plane at $x=x_{rim}+d$, $y=10$.

Again, applying the method for calculating the trajectory derivative previously derived in equation 61 at the location where the centroid of the ball passes through the rim ($x = d + x_{rim}$, $y=10$) and

equating it the negative of the tangent of ϕ_2 results in the following equation:

$$\tan(\theta) - \frac{g(x_{rim}+d)}{V_o^2 \cos^2(\theta) \cos(\alpha)} + \tan(\phi_2) = 0 \quad (67)$$

Solving equation 67 for ϕ_2 results in the following equation:

$$\phi_2 = -\tan^{-1} \left[\tan(\theta) - \frac{g(x_{rim} + d)}{V_o^2 \cos^2(\theta) \cos(\alpha)} \right] \quad (68)$$

Applying equation 19 results in subtracting equation 66 from equation 68 as follows:

$$\begin{aligned} \phi_2 - \phi_1 = & -\tan^{-1} \left[\tan(\theta) - \frac{g(x_{rim} + d)}{V_o^2 \cos^2(\theta) \cos(\alpha)} \right] \\ & + \tan^{-1} \left[\tan(\theta) - \frac{gx^{**}}{V_o^2 \cos^2(\theta) \cos(\alpha)} \right] \end{aligned} \quad (69a)$$

$$\begin{aligned} \phi_2 - \phi_1 + \tan^{-1} \left[\tan(\theta) - \frac{g(x_{rim} + d)}{V_o^2 \cos^2(\theta) \cos(\alpha)} \right] - \\ \tan^{-1} \left[\tan(\theta) - \frac{gx^{**}}{V_o^2 \cos^2(\theta) \cos(\alpha)} \right] = 0 \end{aligned} \quad (69b)$$

Equations 63, 64, 66, and 69b form a system of four nonlinear algebraic equations with four unknowns. Solution of these equations yields the V_o , θ , ϕ_1 , and ϕ_2 , for the flattest trajectory successful shot from a specified distance (d) and height (h) and launch angle α such that the centroid of the ball passes through the rim in the x - z plane at the coordinates $x=x_{rim}+d$, $y=10$, and also passes through the limiting front rim location $x = x^{**}$, $y=10 + R_b \cos(\phi_1)$, such that the spherical ball is not deflected by the front of the rim.

This completes the derivation of the equations required to generate the solution envelope of the V_o and θ for a shot launched from

specified variables d , h and α . As required, all of the equations for the three-dimensional model reduce to their two-dimensional counterparts for the case when $\alpha=0$. The same methodology would be used to generate the solution envelope for the three-dimensional model as was previously discussed for the two-dimensional model.

Figure 14 illustrates how the size of the solution envelope decreases from 11.8 for $\alpha=0$ to 2.1 for $\alpha=1.27$ degrees for a shot launched from $d=18$ and $h=7$ with $y_{\max}=18$ imposed on the solution envelope. It appears from figure 14 that the solution envelope for the two-dimensional model includes the solutions for all acceptable values of α in equation 36.

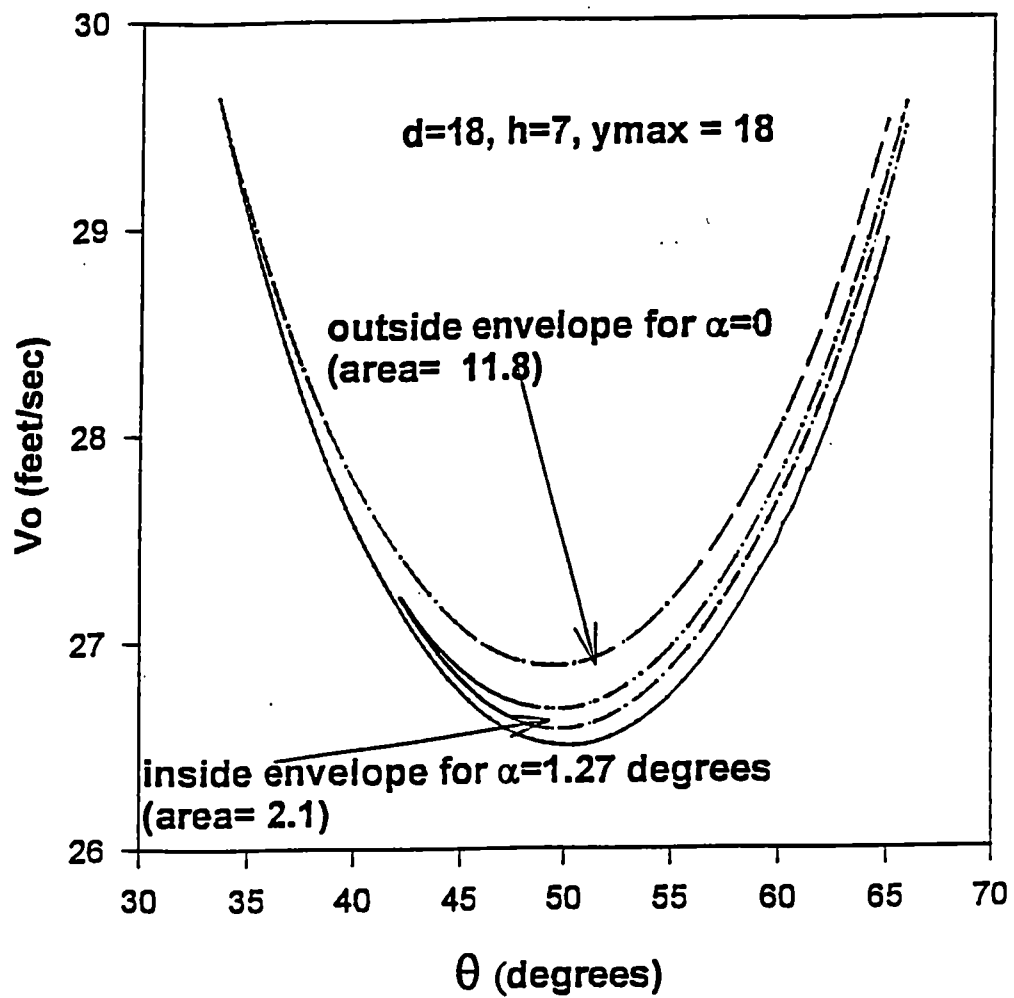


Figure 14. Illustration of solution envelopes for different values of α for $d=18$, $h=7$, $y_{max}=18$ for the three-dimensional model.

11.0 Discussion Regarding the Best Shot

Intuitively the best shot would certainly be one in which $\alpha=0$,

As discussed in reference 8, the best shot is considered to be the shot that has the largest margin of error to be successful. Reference 8 makes the qualitative statement that for a specified shot of d and h , there is a single value of θ that has the minimum V_0 . Reference 8 further concludes that the value of θ corresponding to the minimum V_0 is itself a minimum. The author in reference 8 refers to this value θ as the golden arc. He reasons that this is the best shot since it requires the smallest velocity and subsequently the smallest force; therefore, there is less chance of error in executing the minimum velocity shot.

The analyses performed in this thesis do not support the contention of reference 8 that the value of θ corresponding to the minimum V_0 is itself a minimum. As previously discussed and as illustrated in figures 5 and 6, the minimum value of θ in the solution envelope does not correspond to the minimum velocity. The minimum value of θ corresponds to the flattest trajectory successful shot for the maximum allowable value of $x_{rim} = d_{rim} - R_b$. This corresponds to the point where the upper and lower-bound solution envelope curves are coincident in figure 7.

It seems to this author that the conclusions reached in reference 8 were based on a simplified trajectory analysis that treated the ball as a particle and did not consider the three-dimensional aspects of the

ball, and therefore did not rigorously determine the flattest trajectory successful shot solution. Without the determination of the flattest trajectory successful shot solution, it is not possible to envelope the range of solutions of V_0 and θ .

These analyses support the contention that best shot would be the shot that has an average of the minimum velocities for the upper and lower-bound curves that envelope the solutions. This combination of V_0 and θ appears to have the largest margin for error for both the velocity and launch angle θ , i.e., the solution envelope encompasses the largest amount of perturbation about this point than any other point in the solution envelope.

Using the above logic, from figure 8, for a free throw, the best shot is $V_0=24.9$, $\theta=51$, $\alpha=0$. At this point, there is a margin for error of 0.5 feet per second for the velocity and a margin for error of nearly 10 degrees for θ . The best shot from the three-point distance is $V_0=26.7$, $\theta=49$, $\alpha=0$. At this point, there is a margin for error of 0.4 feet per second for the velocity and a margin for error of nearly 9 degrees for θ . Clearly, there is a considerably larger margin for error for θ than for V_0 where the shot will still be successful. As expected, the margin for error decreases as d increases.

12.0 Effect of Including Drag Force Due to Air Resistance

Thus far, the models have neglected the drag force due to air resistance. The drag force is a retarding force acting in the direction opposite the direction in which the projectile is moving assuming the motion is pure translation with no rotation. As discussed earlier, the retarding force due to air resistance is proportional to the square of the velocity in the velocity regime corresponding to a basketball shot.

The differential equations in equations 2 and 5 which did not include the drag force due to air resistance were second order linear differential equations. Inclusion of the drag force due to air resistance results in second order nonlinear differential equations. These nonlinear differential equations can be solved by making the appropriate transformations and applying the appropriate boundary conditions; however, the solutions are rather complicated. The differential equations and their solutions will not be included here; however, some numerical results generated from the solution of the nonlinear differential equations that included the drag force will be included.

Figure 15 illustrates the impact of including the drag force for a free-throw ($d=15$ feet) and a three-point shot ($d=18$ feet). In both cases, $h = 7$ feet. The curves in figure 15 are the upper-bound solutions to the solution envelope. As explained earlier, the upper-bound solutions correspond to solutions for the maximum acceptable

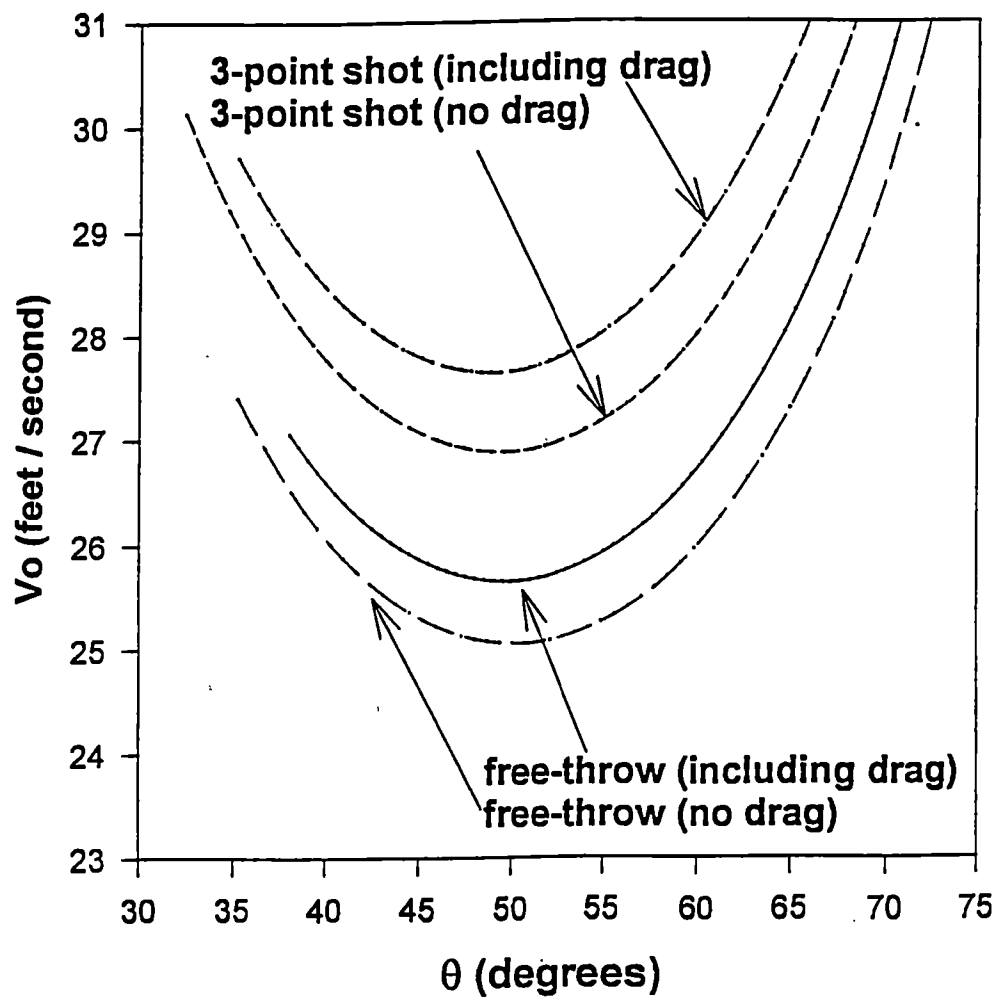


Figure 15. Impact of including drag on upper-bound solutions for free-throw ($d=15$) and 3-point shot ($d=18$) for $h=7$.

value of $x_{rim} = d_{rim} - R_b$ with values of ϕ_2 greater than or equal to the limiting minimum value. As illustrated in figure 15, including the drag force shifts the curves to higher launch velocities and launch angles. For the free throw, the required values of V_o for a successful shot were increased by 0.60 feet per second, or approximately 2.5%. For the three-point shot, the required values of V_o for a successful shot were increased by 0.77 feet per second, or approximately 2.9%.

13.0 Summary

Systems of equations were derived and a computational methodology was developed for determining the solution envelope of the launch velocity V_0 and the launch angle θ that result in a successful shot launched from a specified distance from the rim, height above the gym floor, and at an angle α with the vertical plane that contains the launch location and the center of the rim.

Equations were derived for models of increasing complexity. The determination of the lower-bound curve to the solution envelope for the flattest trajectory successful shot is generated by the repeated solution of a system of four nonlinear algebraic equations. The equations for the two-dimensional model are a special case of the more general three-dimensional model. As required, the equations for the three-dimensional model reduce to the system of equations for the two-dimensional model when $\alpha = 0$. The solution envelope generated using the two-dimensional model contains all solutions of V_0 and θ for which the shot would be successful.

14. Topics for Further Development

The size of the solution envelope is a measure of the chance of executing a successful specified shot from a particular launch location, but it obviously does not tell the entire story. Everyone familiar with basketball knows that the shooting percentage for free throws is considerably higher than for a three-point shot. A good shooter typically has shooting percentages of 75% or higher for free throws from a distance of 15 feet while the same shooter is considered a good outside perimeter shooter if they can shoot 40% from the three-point distance of 18 feet. The rate at which the solution envelope decreases with increasing distance does not alone account for difference in percentages. For example, as previously discussed and illustrated in figure 8, the size of the solution envelope decreased by 17 percent when the shooter moved from the free throw line to the three-point perimeter. According to this, an individual that was a 75% free throw shooter should be nearly a 60% shooter from the three-point perimeter, but we know from experience that this is not the case.

Another reason that the shooting percentage decreases with distance was alluded to earlier. As the required launch velocity increases, so does the required force and subsequently the chance of not properly executing the shot increases. Another way of expressing this concept is to simply state that the chance of error of the three launch variables (V_0 , θ , and α) increases with the amount of energy with which the shooter must use to launch a successful shot.

The quantification of the above concept could be realized by performing Monte Carlo probabilistic analyses in which the launch variables of V_0 , θ , and α for each shot from a specified location are stochastically sampled from Gaussian distributions [9]. The mean values of the three launch values could be considered as the best shot in the solution envelope and the standard deviation for each variable defined as some fractional part of the mean value. For each simulated shot, a determination could be made whether the shot is successful or not. The shooting percentage is simply the number of shots that are successful divided by the total number of simulated shots. It was originally intended to include this formulation as part of this analysis, but it was eventually realized that this is beyond the scope of this thesis.

A rigorous analysis of bank shots, even those that do not include the effects of the rotation of the ball would add another dimensional aspect to the three-dimensional analysis derived in this thesis. The geometrical relationship of the location of the shooter to the backboard and rim would have to be considered. There are angles between the shooter and the backboard that are more favorable, than others. Also, the three-dimensional rigid-body dynamic aspects of the rotation of the basketball would have a more significant effect on the solution since intuitively, it seems that the interaction of the rotating ball with the backboard could be significant in some cases. Basketball players intuitively know to impart a rotation on the ball

for some bank shots, sometimes referred to as putting english on the ball. In any case, a rigorous formulation of the generalized case for bank shots would be quite a mathematical challenge.

The biomechanics aspects of the shooter to optimize the probability of executing a particular shot would also lend itself to a challenging and interesting theoretical mechanics analysis.

Suffice it to say that the complete quantification of shooting basketball is not trivial.

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VITA

Terry Dickson was born in Nashville, Tennessee, on January 12, 1950. He attended public schools in Nashville and Dickson, Tennessee, graduating from Dickson High School in 1968. He graduated with a Bachelor of Science degree in Mechanical Engineering from the University of Memphis in 1972.

After working for the Louisville and Nashville Railroad for two years, he went to work in 1974 in the Engineering Analysis Division at the Tennessee Valley Authority in Knoxville, Tennessee, where his primary responsibilities were the application of software systems for thermal-hydraulic and structural mechanics analyses of nuclear power installations. During this period, he attended evening classes at the University of Tennessee, graduating with a Master of Science degree in Engineering Science and Mechanics in 1978.

From 1980-1988, he worked in Simulation Systems Division of the Arabian American Oil Company (ARAMCO) in Dhahran, Saudi Arabia, where his responsibilities were the development, validation, and implementation of computational methodologies into software systems for application to the optimization of reservoir production strategies and the design of oilfield surface facilities.

Since 1988, he has been working in the Computational Physics and Engineering Division at the Oak Ridge National Laboratory. His responsibilities have been the development, validation, and implementation of computational methodologies into software systems for

application to issues regarding the regulation of nuclear reactors. More specifically, his work has been focused in the development and application of computer codes designed to perform deterministic and probabilistic fracture analyses of aging and increasingly neutron-embrittled light-water nuclear reactor pressure vessels subjected to transient thermal-hydraulic loading conditions. He began taking classes in 1995 toward a Master of Science degree in Applied Mathematics, of which this thesis fulfills the final requirement.

Terry married Maureen Jackson in 1971. They have three children: Jeffrey, born in 1974, Cynthia, born in 1977, and Laura, born in 1979.