July 23, 1956

To the Graduate Council:

I am submitting herewith a thesis written by William C. McCorkle, Jr., entitled "The Lateral Vibrations of an Axially Rotating Bar". I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

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THE LATERAL VIBRATIONS OF AN AXIALLY ROTATING BAR

A THESIS

Submitted to
The Graduate Council
of
The University of Tennessee
in
Partial Fulfillment of the Requirements
for the degree of
Doctor of Philosophy

---

by

William C. McCorkle, Jr.

August 1956

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CHAPTER I

INTRODUCTION

It has long been known that the rotational motion of a bar or shaft at certain speeds is capable of exciting lateral vibrations of the shaft. Almost invariably these vibrations are undesirable or even destructive, and for this reason considerable attention has been given to their study. The subject has a fairly extensive literature dating back to 1869 when Rankine[1,2] first treated the stability of a shaft, running true and in perfect balance, against small disturbances. His results were erroneous because he assumed that the plane of vibration always rotated with the shaft, which is equivalent to neglecting the Coriolis force. Amazingly, this assumption has proved irresistible to many authors since his time, dating well into the present century.

When an unbalanced shaft is running at a constant speed near one of its critical speeds, as the speeds which cause very large oscillation are termed, the steady state

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[1] W. J. M'Q Rankine, The Engineer, xxvii (1869), p. 219 ff.

[2] The papers of D. Robertson on the subject contain a fairly complete historical summary and bibliography of the work done through 1935 on both whirling and transient whirling at constant speed. cf. D. Robertson, Phil. Mag. 20, (1935), pp. 823-825 and 21, (1936), pp. 495-500, and The Engineer, clvi, pp. 152-153, clviii, pp. 179-181 and pp. 213-214, (1933).

motion is called whirling. Exactly at the critical speed, no steady state motion exists unless there is friction. If the frictional forces are large enough in relation to the unbalanced forces, the shaft may not be destroyed, but generally operation very near this point is avoided. When the shaft is disturbed from its steady state whirl, the subsequent motion, which decays because of friction to the steady state, is termed the transient whirl.

As might be expected, the critical speeds occur at the natural frequencies of lateral vibration of the shaft. This discovery led to more intensive investigation of the lateral vibrations of a bar with the aim of predicting the natural frequencies from the configuration and other physical parameters of the bar. Many aspects of whirling are now well understood, and most engineering texts on vibrations give good treatments of the constant speed behavior of shafts and methods for predicting the critical speeds of many types of configurations of rotating machinery[3] under various types of constraints.

Rankine's old theory erroneously predicted a critical condition at $1/\sqrt{2}$ times the natural frequency and also predicted that the shaft would be unstable against small oscillations at rotational speeds higher than critical. For

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[3]See, for example, S. Timoshenko, Vibration Problems in Engineering (D. Van Nostrand & Co., New York, 1928), p. 60 ff. and p. 159 ff.

a long time engineers believed this true, but the development of the De Laval turbine in 1889, which operated far above its first critical speed, showed this to be false. Thus, while a shaft may develop vibrations at or near the rotational speeds corresponding to the natural vibration frequencies, it runs smoothly and stably when considerably away from these frequencies, and designers try to construct rotating machinery to operate in these safe regions. Nevertheless, when starting up or slowing down, the rotational speeds must pass through any critical speeds which are lower than the operating speed. This led to the investigation of the behavior during acceleration through a critical speed. Lewis[4] treated this problem, and succeeded in solving it analytically for the case of constant acceleration in 1932. Since Lewis' results are somewhat hard to apply, Baker[5] also treated the problem with a type of automatic computer in 1939. Baker's treatment also assumed constant acceleration, but he mentioned the desirability of examining the case for constant torque, for which the equations were unknown.

Nature of the Problem

It is not hard to see that constant acceleration may not represent the actual case. In order for the angular acceleration to remain constant, the driving torque for the

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[4]F. M. Lewis, Trans. ASME, 54 (1932) APM-54-84.

[5]J. G. Baker, Trans. ASME, 61 (1939) APM-61-145.

system must change to compensate for the varying load caused by the growth and decay of the lateral vibrations when the rotational speed is passing through the neighborhood of a critical speed. The vibrations, as they grow, present a load to the driving mechanism which is beyond that necessary to maintain a constant acceleration, and consequently the acceleration will decrease during this period if the mechanism providing the power for acceleration is characterized by a constant torque output or is one that varies slowly with the speed, as do nearly all prime movers. When the resonance is passed, the vibrations tend to decay, returning their energy and momentum to the pure rotational motion and causing an acceleration above that which the external torque alone could provide. The reverse process occurs during constant deceleration. The approximation in which a constant acceleration replaces a constant torque is good only if the total inertial mass of the system is large compared to that which may perform lateral vibrations.

The purpose of this work is to investigate the problem of acceleration through a resonant speed using an arbitrary external torque rather than a constant acceleration.

Scope of the Work

It will be seen in Chapter II that the determination of the correct equations of motion is in itself a fairly long process, and this is probably the reason the problem has not been approached from this direction previously. In

this chapter, the equations of motion are obtained in a fairly general form. Chapter III is devoted to the adaptation of the basic equations obtained in Chapter II to some particular cases and to the solution of some of these cases. It will be seen that one important case turns out to be soluble analytically. This case, which is the frictionless, torque free motion in the neighborhood of a critical speed, was actually the problem which prompted this investigation. The case with constant torque was treated with an analogue computer.

In Chapter IV the conclusions and results are reviewed, and some heretofore unexplained experimental results are discussed in the light of the results of this investigation.

So far the terms bar and shaft have been employed. Neither of these is quite satisfactory, since both imply a more or less regular or uniform long object, whereas it is desirable to use a broader concept which can include such things as shaft and pulley assemblies, armatures, turbines, and even more irregularly shaped objects which have one moment of inertia which is small compared to the other two. The term "rotor" is used in this sense in some of the literature and will be employed for that purpose here.

CHAPTER II

DERIVATION OF THE EQUATIONS OF MOTION

The general nature of the problem has been outlined in Chapter I. The problem of the rotation-vibration interaction of a rotor is a special case of the general problem of the coupled bending-torsional vibrations of a bar; therefore, the equations which govern this type of motion will be obtained and adapted to the present case.

In forming these equations, an attempt will be made to be as general as possible in an extensive sense, but less important intensive refinements will usually be passed over with qualitative observations on the result of their inclusion. Thus, the theory is made applicable to a rotor of arbitrary shape, but such things as the kinetic energy associated with the deformation of a cross section are neglected since they contribute nothing essential and usually complicate the analysis far out of proportion to their effect.

In obtaining the equation for the simple lateral vibration of a bar, D'Alembert's principle as applied to a continuous system is usually employed. This procedure requires knowledge of all the forces which act on an element of the bar, and might be risky in the present case, since the forces become much more complicated with the introduction of two more degrees of freedom. As remarked in Chapter I, the history of the problem of the whirling of shafts records

many errors in obtaining the equations of motion, chiefly because of difficulty with the rotational inertia forces. Thus a method is sought which does not involve knowing explicitly the internal forces.

The Euler-Lagrange differential equations extended to a continuous system provide such a method, since it is necessary to know, in addition to the external forces, only the Lagrangian density function for the system, which is generally much easier to discover than the internal force system. However, it appears that no one has obtained these equations for a system in which the potential energy density is a function of the second spatial derivative of the displacement coordinate. Rayleigh[1] and Love[2] both treat such systems, but both apply a variational principle directly to the Lagrangian density of their particular problems without bothering to obtain the general Euler-Lagrange equations for such systems. Their procedure could be followed here, but instead, for the sake of both generality and completeness, and because it turns out to be easier, the Euler-Lagrange equations are determined from Hamilton's principle. Three coordinates are involved (actually, of course, three infinite sets, since the bar is a continuous system), and the variational procedure would have to be repeated for each, whereas obtaining the

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[1]Lord Rayleigh, The Theory of Sound (Macmillan and Co., London, 2nd ed. 1937), p. 258, v. 1.

[2]A. E. H. Love, The Mathematical Theory of Elasticity (Dover Publications, New York, 4th ed. 1944), p. 430.

Euler-Lagrange equations does it once for all. Thus the Euler-Lagrange equations are derived first, then the Lagrangian density, and finally the equations of motion which result from substituting the Lagrangian density into the Euler-Lagrange equations.

The Euler-Lagrange Equations

Hamilton's principle for a conservative, holonomic, continuous system takes the form[3]

$$\delta \int_{t_1}^{t_2} \int_V \mathcal{L}\, dv\, dt = 0 , \tag{II-1}$$

where δ is the standard variation symbol,[4] $\mathcal{L}$ is the Lagrangian density and the volume integration is taken over the equilibrium configuration of the system. The endpoints in time, t_1 and t_2, are not permitted to vary. The system is taken from some fixed configuration at t_1 to some other fixed configuration at t_2 and hence the configuration variation must vanish at these time endpoints and at the corresponding space endpoints in configuration space. This does not mean that the configuration variation must vanish at the

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[3] H. Goldstein, Classical Mechanics (Addison-Wesley Press, Inc., Cambridge, Mass., 1951), p. 380.

[4] For a discussion of the variation procedure and the properties of δ the reader is referred to any of the standard works on advanced dynamics or the calculus of variations. See, for example, ibid., Chapter II.

spatial integration boundaries, and in fact, as will be seen later, important boundary conditions are obtained by paying careful attention to this point.[5] The Lagrangian density is the excess of a kinetic energy density $\mathcal{T}$ over a potential energy density $\mathcal{V}$. For the type of system we contemplate, two dimensions will be negligibly small compared to the third, which we take in the z- direction of a Cartesian system. Thus, the volume integration in (II-1) becomes, for our system, a line integration over the length ℓ of the rotor. We want to perform the variation in (II-1) for a function $\mathcal{L}$ of the quantities q_α'', q_α', q_α, $\dot{q}_\alpha'$, $\dot{q}_\alpha$, z and t, where primes denote differentiation with respect to z, including both implicit and explicit dependence on z, and a dot denotes the same for time. The quantities $q_\alpha = q_\alpha(z,t)$ denote the displacements of points along the rotor from some initial or equilibrium configuration. The α labels the type of displacement, and it is assumed the various types are independent. Thus

$$\mathcal{T} - \mathcal{V} = \mathcal{L}(q_\alpha'', q_\alpha', q_\alpha, \dot{q}_\alpha', \dot{q}_\alpha, z, t) . \tag{II-2}$$

If some of the forces are non-conservative, they can

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[5] The author's treatment is similar to that found in Love (op. cit., pp. 165-6). The difference is that the treatment here is for only one dimension, but is extended to include the dependence of $\mathcal{L}$ on the second spatial derivatives of the coordinates and on certain other terms. A more general type of generalized force is also included. The treatment of Goldstein (op.cit., pp. 351-2) uses a type of variation which vanishes at the spatial integration boundaries and is therefore too restricted to be useful here.

be included as well, since Hamilton's principle applies also to non-conservative systems. The principle then is stated

$$\int_{t_1}^{t_2}\int_0^{\ell}\left\{\delta\mathcal{L}+\sum_{\alpha=1}^{N}\left(Q_{\alpha 1}\,\delta q_\alpha+Q_{\alpha 2}\,\delta(q_\alpha')+Q_{\alpha 3}\,\delta(q_\alpha'')\right)\right\}dz\,dt=0\,, \tag{II-3}$$

where $Q_{\alpha 1}$, $Q_{\alpha 2}$, $Q_{\alpha 3}$ are generalized non-conservative force densities associated with the displacement coordinates q_α and their spatial derivatives. To write them in this way takes account of the fact that some generalized forces, such as those associated with bending an twisting, produse changes in the derivatives of the coordinates, rather than directly affecting the coordinates themselves.

The variation (II-3) is thus

$$\sum_{\alpha=1}^{N}\int_{t_1}^{t_2}\int_0^{\ell}\left\{\left(\frac{\partial\mathcal{L}}{\partial q_\alpha''}+Q_{\alpha 3}\right)\delta(q_\alpha'')+\frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha'}\,\delta(\dot{q}_\alpha')\right.$$

$$\left.+\left(\frac{\partial\mathcal{L}}{\partial q_\alpha'}+Q_{\alpha 2}\right)\delta(q_\alpha')+\frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha}\,\delta(\dot{q}_\alpha)+\left(\frac{\partial\mathcal{L}}{\partial q_\alpha}+Q_{\alpha 1}\right)\delta q_\alpha\right\}dz\,dt=0\,. \tag{II-4}$$

It is necessary to express the terms in the integrand as a multiple of δq_α, and we do this by means of a series of integrations by parts, using the property of δ of commutation with partial derivatives. The z-integration of the first term in the integrand of (II-4) may be written as

$$\int_0^{\ell}\left(\frac{\partial\mathcal{L}}{\partial q_\alpha''}+Q_{\alpha 3}\right)\delta q_\alpha''\,dz=\left[\left(\frac{\partial\mathcal{L}}{\partial q_\alpha''}+Q_{\alpha 3}\right)\delta q_\alpha'-\left(\frac{\partial\mathcal{L}}{\partial q_\alpha''}+Q_{\alpha 3}\right)'\delta q_\alpha\right]_0^{\ell}$$

$$+ \int_0^l \left(\frac{\partial \mathcal{L}}{\partial q_\alpha''} + Q_{\alpha 3}\right)'' \delta q_\alpha \, dz \, .$$

Similarly, the other term not involving time derivatives may be written in the form

$$\int_0^l \left(\frac{\partial \mathcal{L}}{\partial q_\alpha'} + Q_{\alpha 2}\right) \delta(q_\alpha') \, dz = \left[\left(\frac{\partial \mathcal{L}}{\partial q_\alpha'} + Q_{\alpha 2}\right) \delta q_\alpha\right]_0^l - \int_0^l \left(\frac{\partial \mathcal{L}}{\partial q_\alpha'} + Q_{\alpha 2}\right)' \delta q_\alpha \, dz \, .$$

For those terms involving time derivatives we recall that the variation δq_α must vanish at the endpoints of time for all z; hence $\delta q_\alpha'$ also vanishes at the time endpoints. Thus

$$\int_{t_1}^{t_2} \int_0^l \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha'} (\delta \dot{q}_\alpha') \, dz \, dt = 0 - \int_{t_1}^{t_2} \left\{ \left[\left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{q}_\alpha'}\right) \delta q_\alpha \right]_0^l - \int_0^l \left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{q}_\alpha'}\right)' \delta q_\alpha \, dz \right\} dt ,$$

and

$$\int_{t_1}^{t_2} \int_0^l \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} (\delta \dot{q}_\alpha) \, dz \, dt = 0 - \int_{t_1}^{t_2} \int_0^l \left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{q}_\alpha}\right) \delta q_\alpha \, dz \, dt \, .$$

If the above results are collected, we find that the variation (II-3) is now

$$\sum_{\alpha=1}^{N} \int_{t_1}^{t_2} \left\{ \left[\left(\frac{\partial \mathcal{L}}{\partial q_\alpha''} + M_\alpha\right) \delta q_\alpha' - \left(\left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{q}_\alpha'}\right) + \left(\frac{\partial \mathcal{L}}{\partial q_\alpha''}\right)' - \left(\frac{\partial \mathcal{L}}{\partial q_\alpha'}\right) + F_\alpha\right) \delta q_\alpha \right]_0^l \right.$$

$$\left. + \int_0^l \left[\left(\frac{\partial \mathcal{L}}{\partial q_\alpha''}\right)'' - \left(\frac{\partial \mathcal{L}}{\partial q_\alpha'}\right)' + \left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{q}_\alpha'}\right)' - \frac{\partial \mathcal{L}}{\partial q_\alpha} + Q_\alpha \right] \delta q_\alpha \, dz \right\} dt = 0 , \tag{II-5}$$

where we have put

$$M_\alpha = Q_{\alpha 3}\ , \quad F_\alpha = Q'_{\alpha 3} + Q_{\alpha 2}\ , \quad Q_\alpha = Q_{\alpha 1} + Q'_{\alpha 2} + Q''_{\alpha 3}\ .$$

The q_α have been assumed to be independent; hence each term of the summation must vanish separately. Since the interior and boundary values of δq_α are arbitrary, and at the boundary δq_α and $\delta q'_\alpha$ may be independently specified, the integral and integrated parts must vanish separately. Thus we obtain the following four end conditions:

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \ddot{q}_\alpha} + M_\alpha &= 0 \\ \left(\frac{\partial \mathcal{L}}{\partial \dot{q}'_\alpha}\right)^{\cdot} + \left(\frac{\partial \mathcal{L}}{\partial q''_\alpha}\right)' - \frac{\partial \mathcal{L}}{\partial q'_\alpha} + F_\alpha &= 0 \end{aligned}\right\} \quad x = \begin{matrix} 0 \\ l \end{matrix} \qquad \text{(II-6)}$$

in order that the integrated terms of (II-5) vanish, and the Euler-Lagrange equation

$$\left(\frac{\partial \mathcal{L}}{\partial q''_\alpha}\right)'' - \left(\frac{\partial \mathcal{L}}{\partial q'_\alpha}\right)' + \left(\frac{\partial \mathcal{L}}{\partial \dot{q}'_\alpha}\right)' - \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)^{\cdot} + \frac{\partial \mathcal{L}}{\partial q_\alpha} + Q_\alpha = 0 \qquad \text{(II-7)}$$

in order that the integral vanish. We have permitted variations of the boundaries, which leads to the requirement that the end conditions (II-6) be satisfied. These conditions state that the terminal values of certain quantities must be equal to the terminal forces and moments acting on the bar. If we had not permitted variations of a coordinate or its derivative at the boundaries, we would have required that $\delta q_\alpha = \delta q'_\alpha = 0$ at the ends and could not obtain equation

(II-6). However, $\delta q_\alpha = \delta q_\alpha' = 0$ implies that q_α and q_α' are constant at the ends, which are sometimes convenient boundary conditions. Other combinations in which either δq_α or $\delta q_\alpha'$ vanish on the boundaries are possible. We see that Hamilton's principle not only yields the differential equations governing the rotor, but also yields the boundary conditions to meet specified terminal torques and moments, the conditions for fixed boundaries, and even the condition for a combination of the two types of end conditions, i.e., the two types resulting from having either a variation vanish or the coefficient of the variation vanish at the boundary.

The boundary conditions (II-6) and the Euler-Lagrange equation (II-7) will serve to produce the equations of motion and the terminal conditions from the Lagrangian density.

The Lagrangian Density

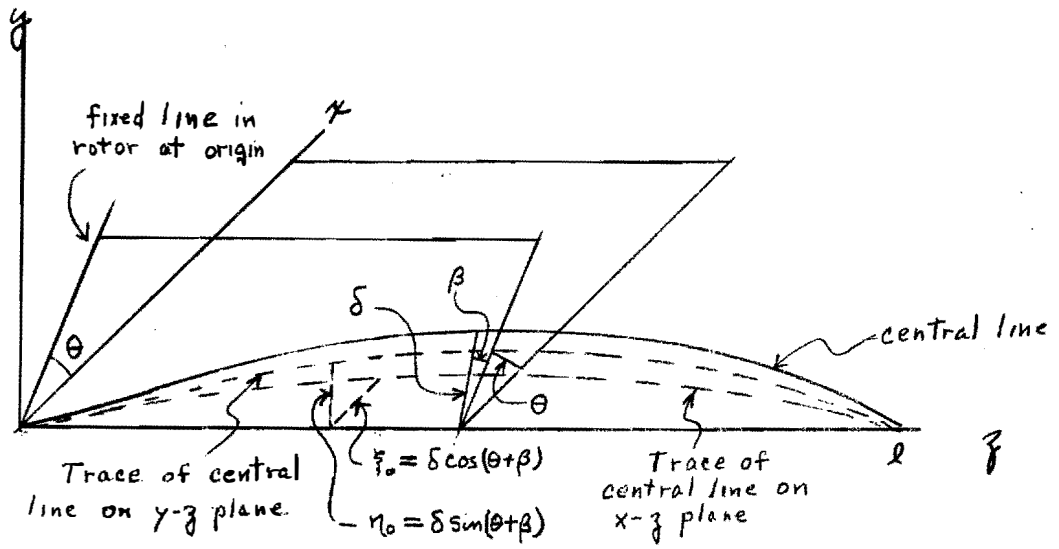
It is necessary to define precisely the coordinate system in order to proceed with the development of the kinetic and potential energy densities.

Coordinate System

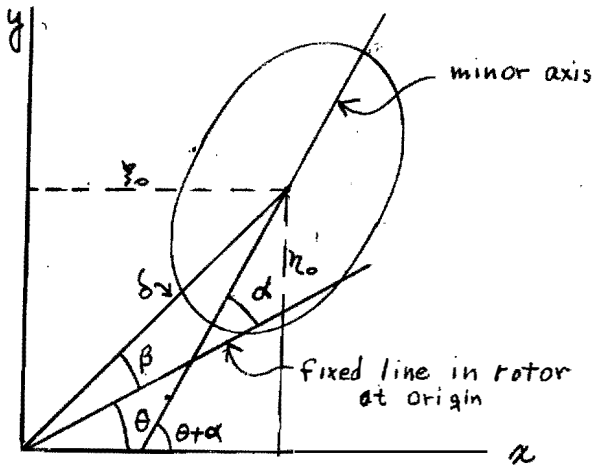
The rotor is assumed to lie with one end at the origin of a right-handed Cartesian inertial system and to extend a distance l along the z-axis. The line passing through the centroids of the cross sections of the rotor is called the central line, and is not in general straight, although it will be assumed to be nearly so. If the vibrations of the

free rotor are to be considered, it is necessary to assume further that the axis of the minimum moment of inertia of the rotor considered as a rigid body lies parallel to the z-axis. The reason for this is that if rotation about the z-axis is permitted and the body is not rotating about one of the principal axes, the precessional motion of the free rigid body will take place in addition to the motion of interest, and this will complicate matters beyond the scope of this work. If the kinetic and potential energy densities were rigorously formed, however, this complication would of course include itself in the equations of motion. If the ends are constrained in some fashion, as by bearings, this trouble does not arise whether or not the principal axis lies along the z-axis.

Figure 1 illustrates the coordinate system. Displacements of the central line in the x- and y-directions are given by $\xi(z,t)$ and $\eta(z,t)$, respectively. If the rotor is naturally bent, the equilibrium central line displacements are denoted by ξ_0 and η_0. With respect to areal moments of inertia, each cross section will have two principal axes lying in the section. Rotation of the section is measured from the xz-plane to the minor principal axis and is denoted by $\theta(z,t)+\alpha(z)$. The angle α measures the rotation of the minor axis along the rotor in the equilibrium state with respect to some line fixed in the rotor, say the minor axis at the origin. Thus α measures the natural twist of the rotor. In the equilibrium state, or in rotation where no twisting occurs, all cross sections have the same



(a) Coordinates of central line of rotor, in equilibrium position.



(b) Coordinates of cross section of rotor, in equilibrium position.

Figure 1. The coordinate system

value of θ . It will also be convenient to consider ξ_o and η_o in a similar way. These quantities rotate with θ and their angular dependence is most simply given by a polar form $\xi_o = \delta \cos(\theta+\beta)$, $\eta_o = \delta \sin(\theta+\beta)$, where $\delta = (\xi_o^2 + \eta_o^2)^{1/2}$ and β measures the angle of δ to the same fixed line in the rotor described above.

This completes the assignment of coordinates, but the reader is cautioned that the above system, however simple, may be deceptive. For example, apparent rotation of the rotor can occur, by virtue of the ξ and η motion, while θ remains constant. The angle of rotation of the central line about the z-axis is $\tan^{-1} \eta/\xi$ while the rotation of the cross section is denoted by θ .

Potential Energy Density

Potential energy is stored in the rotor by bending, twisting and shearing deformation. The last is usually neglected because the bending moment at any point is the integral of moments generated by forces acting all along the rotor, whereas the lateral shear deformation is caused by the derivatives of these forces at the point considered. The former may be expected to be much larger than the latter unless the forces change sign rapidly along the rotor, indicating a more sinuous central line and a higher mode of oscillation. Timoshenko[6] has treated the effect of shear, and from this it

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[6]S. Timoshenko, Vibration Problems in Engineering (D. Van Nostrand Co. Inc., New York, 3rd ed. 1955), pp. 334-335.

can be concluded that for the present purposes this effect is negligible unless the wavelength of vibration along the rotor approaches the thickness of the rotor. A method for including it will be indicated.

Bending potential energy density. If the rotor is considered to be made up of a bundle of filaments, and if no part is deformed beyond its elastic limit, the filament which lies along the central line will be unextended and is called the neutral filament. In fact, there will be a surface of neutral filaments, and the line lying in this surface and perpendicular to the central line is called the bending axis at that point. The curvature vector[7] of the central line due to bending lies along this axis. The other filaments are stretched or compressed in proportion to their distance away from or toward the center of curvature of the neutral surface, and this allows the curvature of the central line to be used to measure the bending energy density. With the aid of Figure 2 on the following page, the relation between bending moment and curvature is discovered. Consider a wafer-like slice of the rotor of thickness ds. Let y denote the distance from the neutral surface to an arbitrary filament, which is stretched an amount $d(ds)$, and let ρ be the radius of curvature of the neutral surface.

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[7]The curvature vector is often defined differently. This definition is the one used by Love. A. E. H. Love, op. cit., p. 382.

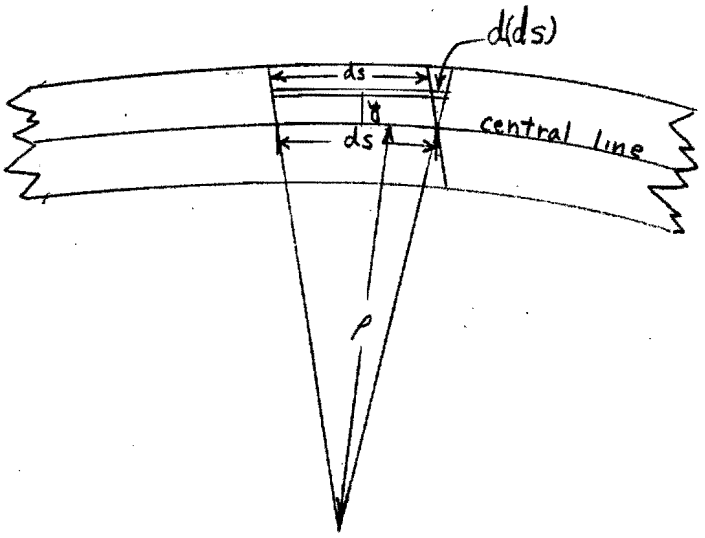


Figure 2. Analysis of bending

Then $$\frac{y}{\rho} = \frac{d(ds)}{ds} .$$

The quantity $\frac{d(ds)}{ds}$ is the extension. If F is force per unit area, e the extension, and E Young's modulus, then Hookes' law states

$$F = eE = \frac{y}{\rho} E .$$

If dA is the cross section of the filament, the elemental bending moment is

$$dM = F y \, dA ,$$

and

$$M = \frac{E}{\rho} \int_A y^2 \, dA ,$$

where A is the cross section of the rotor. The quantity

$\int_A y^2 dA$ is just the squared radius of gyration k about the bending axis multiplied by the area; hence

$$\vec{M} = EAk^2\left(\frac{\vec{1}}{\rho}\right),$$

(II-8)

where the superscript arrow denoted a vector quantity. The bending moment and curvature are directed along the bending axis. Although it will not matter in this case, the sense of $\left(\frac{\vec{1}}{\rho}\right)$ is chosen so that, for successive points along the central line, the tangent to the central line rotates so as to drive a right-handed screw in the direction of $\left(\frac{\vec{1}}{\rho}\right)$.

If the central line has a slight natural curvature $\left(\frac{\vec{1}}{\rho_0}\right)$, the bending moment is proportional to the difference in the original and deformed curvatures:

$$\vec{M} = EAk^2\left(\left(\frac{\vec{1}}{\rho}\right) - \left(\frac{\vec{1}}{\rho_0}\right)\right).$$

(II-9)

Since for any given element the deformation is proportional to the bending moment, the work done per unit length, or energy density associated with bending the rotor from the equilibrium to the final curvature, is just

$$V_B = \frac{1}{2}EAk^2\left|\left(\frac{\vec{1}}{\rho}\right) - \left(\frac{\vec{1}}{\rho_0}\right)\right|^2.$$

(II-10)

The quantity EAk^2 is called the stiffness. It is the actual measured stiffness, which will be denoted by S, except

where A or k^2 change abruptly. Then the above analysis obviously does not hold, for the rotor cannot be considered as a bundle of continuous filaments at such a point. The actual stiffness S will change smoothly across a discontinuity in EAk^2, assuming the value of EAk^2 again at a little distance to each side, according to St. Venant's principle.[8] A more exact expression could be obtained by solving the fundamental equations of elasticity for the appropriate boundary conditions, but this is another problem. The stiffness can be obtained in several ways, and S will be used to denote its actual value.

If the central line is projected on the xz-plane, the curvature of the projection will be the y-component of the curvature vector. From elementary kinematics its value is exactly $\xi''/(1+\xi'^2)^{3/2}$. Similarly the projection on the yz-plane has curvature $\eta''/(1+\eta'^2)^{3/2}$ which is the negative x-component of the curvature. If the slope of the central line is small, then the denominators are very close to unity, and the curvature is very nearly

$$\left(\frac{1}{\rho}\right) = -\eta''\,\vec{i} + \xi''\,\vec{j}\,,$$

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[8] St. Venant's principle states that local stresses in a continuous medium which result from points of attachment or similar sources quickly smooth out and have negligible effect on the stresses distant from the source. A.E.H. Love, ibid., p. 131.

where $\vec{\imath}$ and $\vec{\jmath}$ are the usual Cartesian unit vectors. Similarly,

$$\left(\frac{\vec{1}}{\rho_0}\right) = -\eta_0''\,\vec{\imath} + \xi_0''\,\vec{\jmath}\,,$$

and the bending energy density becomes

$$V_B = \tfrac{1}{2} S\left\{(\xi''-\xi_0'')^2+(\eta''-\eta_0'')^2\right\}. \tag{II-11}$$

An expression for the stiffness about an arbitrary bending axis is needed. S transforms like the moment of inertia and in a given cross section has the same principal axes. S_1 is taken as the value of the stiffness about the minor principal axis. It will be convenient to use $S_2 = (1+\varepsilon)S_1$ for the other principal axis in the plane, where ε measures what is called the skew stiffness, or deviation from symmetrical stiffness.

The principal axes of the stiffness tensor are assumed known in the body-fixed system, and the components in the inertial system are found by operating on this diagonal tensor with the rotation matrix which rotates the coordinate system with axes along the principal axes to one with axes along the inertial axes. Thus

$$S_1\begin{pmatrix}\cos(\theta+\alpha) & -\sin(\theta+\alpha)\\ \sin(\theta+\alpha) & \cos(\theta+\alpha)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1+\varepsilon\end{pmatrix}\begin{pmatrix}\cos(\theta+\alpha) & \sin(\theta+\alpha)\\ -\sin(\theta+\alpha) & \cos(\theta+\alpha)\end{pmatrix} = S_1\begin{pmatrix}1+\varepsilon\sin^2(\theta+\alpha) & -\varepsilon\sin(\theta+\alpha)\cos(\theta+\alpha)\\ -\varepsilon\sin(\theta+\alpha)\cos(\theta+\alpha) & 1+\varepsilon\cos^2(\theta+\alpha)\end{pmatrix} \tag{II-12}$$

is the desired stiffness tensor.

In order to obtain the stiffness about the bending axes, the above tensor must be contracted twice by the unit vector in the direction of the bending axis. The curvature vector due to bending has this direction, and normalization gives the desired unit vector. Hence

$$S = \frac{S_1}{(\xi''-\xi_0'')^2+(\eta''-\eta_0'')^2}\begin{pmatrix}-(\eta''-\eta_0'') & (\xi''-\xi_0'')\end{pmatrix}\begin{pmatrix}1+\varepsilon\sin^2(\theta+\alpha) & -\varepsilon\sin(\theta+\alpha)\cos(\theta+\alpha)\\ -\varepsilon\sin(\theta+\alpha)\cos(\theta+\alpha) & 1+\varepsilon\cos^2(\theta+\alpha)\end{pmatrix}\begin{pmatrix}-\eta''-\eta_0''\\ \xi''-\xi_0''\end{pmatrix},$$

(II-13)

where the sum of squares in the demoninator results from the normalization. If the indicated matrix multiplications are performed, (II-13) becomes

$$S = S_1\,\frac{(\xi''-\xi_0'')^2+(\eta''-\eta_0'')^2+\varepsilon\left((\xi''-\xi_0'')\cos(\theta+\alpha)+(\eta''-\eta_0'')\sin(\theta+\alpha)\right)^2}{(\xi''-\xi_0'')^2+(\eta''-\eta_0'')^2}.$$

(II-14)

This is inserted into V_B, (II-11), which then reads

$$V_B = \frac{1}{2}S_1\left\{(\xi''-\xi_0'')^2+(\eta''-\eta_0'')^2+\varepsilon\left((\xi''-\xi_0'')\cos(\theta+\alpha)+(\eta''-\eta_0'')\sin(\theta+\alpha)\right)^2\right\}.$$

(II-15)

In terms of the polar forms of ξ_0 and η_0 which will be convenient later,

$$V_B = \frac{1}{2} S_1 \left\{ \left[\xi - \delta \cos(\theta+\beta) \right]''^2 + \left[\eta - \delta \sin(\theta+\beta) \right]''^2 \right.$$

$$\left. + \varepsilon \left[\left(\xi - \delta \cos(\theta+\beta) \right)'' \cos(\theta+\alpha) + \left(\eta - \delta \sin(\theta+\beta) \right)'' \sin(\theta+\alpha) \right]^2 \right\}.$$

(II-16)

Twisting potential energy density. For twisting, consider a cylindrical shell element of length dz shown in Figure 3.

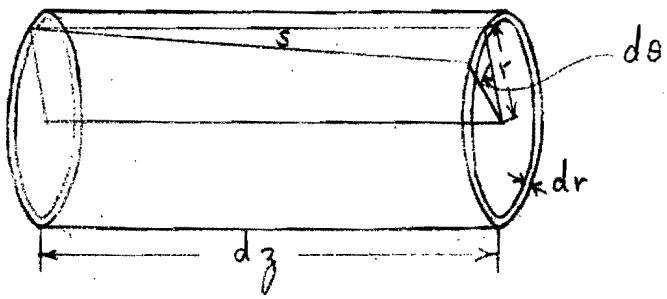


Figure 3. Analysis of torsion

One end of the cylindrical shell is twisted through an angle $d\theta$ with respect to the other end. For shearing deformation, the relation is that the angle of shear (s) times the shear modulus (μ) is equal to the shearing force per unit cross section (F_s). For the element in Figure 3, the net shearing force acting to shear one end relative to the other is $F_s' dz$, hence

$$F_s' dz = \mu s .$$

Also we note that $s\, dz = r d\theta$, hence

$$F_s' dz = \mu r \theta' .$$

The elemental twisting moment is

$$dM_T = F' r\, dz\, dA = \mu \theta' r^2 dA \ ,$$

where dA is the area of the end of the element. Thus

$$M_T = \mu \theta' \int_A r^2 \, dA \ .$$

The integral is just the areal moment of inertia about the perpendicular to the section at the centroid. Since

$$\int_A r^2 dA = \int_A (x^2+y^2)\, dA = \frac{1}{E}(S_1 + S_2) \ ,$$

it follows that

$$M_T = \chi \theta' S_1 (2+\varepsilon) \ , \qquad \text{(II-17)}$$

where $\chi = \mu/E$.

If the cross section is not nearly circular, then the above analysis fails. This happens because of the implicit assumption that the rotation of one cross section relative to another is pure transverse shear at every point, which is the case only if the cross section is bounded by concentric circles. In general, the deformation at any point due to twisting of the rotor is compounded of a shearing between contiguous cross sections and a shearing between contiguous

longitudinal sections, and the cross section is distorted into a curved surface. The problem of finding the exact deformation for a given shape of cross section is known as the torsion problem for that shape, and is generally a considerable problem in itself. As in the case of bending, the actual twisting stiffness will be denoted by T, and

$$M_T = T\theta' . \tag{II-18}$$

Again, since each element is deformed in proportion to M_T,

$$U_T = \frac{1}{2} T \theta'^2 . \tag{II-19}$$

In spite of the fact that the derivation of twisting stiffness holds only for nearly circular sections, it does show the relative magnitude of bending and twisting stiffness coefficients. Thus

$$T \cong 2 S_1 M/E , \tag{II-20}$$

and this information will be useful in estimating the degree to which lateral and torsional vibrations affect each other.

Inclusion of shear. Because the lateral forces change along the rotor, the cross sections also undergo small lateral shearing motions with respect to each other, and the slope of the central line is changed at a given point beyond that due to bending alone. The lateral displacement at this point is the sum of three parts. Consider the displacement in the η direction:

$$\xi = \xi_o + \xi_B + \xi_S ,$$

where ξ_o and ξ_B denote the equilibrium and bending displacements already considered, and ξ_S denotes the extra displacement due to shear. As before, the derivative of the lateral force gives the shear angle, which is just ξ_S' . Thus,

$$F' = KA\mu\, \xi_S' ,$$

where K is a constant depending on the shape of the section and A is the area. Thus, if η_S' is included also,

$$V_S = \tfrac{1}{2} KA\mu \left[(\xi_S')^2 + (\eta_S')^2 \right] . \qquad \text{(II-21)}$$

This has the effect of introducing another displacement coordinate. Where previously $\xi - \xi_o$ was used, it would be replaced by $\xi - \xi_o - \xi_S$ if shear were included. The coordinates ξ_B and ξ_S are independent, and each will have its equation of motion, which will have both coordinates in it. However, the only observable displacement is still ξ and thus ξ_S and ξ_B must be combined by proper manipulation of the equations to obtain ξ alone. This complication is ultimately the reason shear is neglected in most cases, although it is the largest of the group of small effects, being several times larger[9] than kinetic energy corrections which will be

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[9] S. Timoshenko, op. cit., p. 231.

discussed below.

Kinetic Energy Density

First, an elemental cross section has the kinetic energy associated with the translations of its centroid. This will be principally in the x- and y-directions, since the longitudinal motion associated with slight bending is extremely small. In addition, there is the very small kinetic energy associated with the distortion of the cross section in bending and twisting in which the section both changes its shape and becomes non-planar.[10] Finally, there is the rotation of the cross section about a line through its centroid approximately parallel to the z-axis. Because the rotor is bending and rotating simultaneously, the section will perform a slight wobbling motion which has a somewhat complex kinetic energy. However, the latter can be obtained by writing the inertia tensor for a cross section in terms of the inertial system components and contracting this tensor twice with the angular velocity with components in the same system. This has been done. To the first order in ξ' and η', which are assumed very small, the inertia tensor is, in the inertial system,

$$\rho k^2 \begin{pmatrix} 1+\varepsilon\sin^2(\theta+\alpha) & -\varepsilon\sin(\theta+\alpha)\cos(\theta+\alpha) & \xi'-\varepsilon[\xi'(1-\sin^2(\theta+\alpha))+\eta'\sin(\theta+\alpha)\cos(\theta+\alpha)] \\ & 1+\varepsilon\cos^2(\theta+\alpha) & \eta'+\varepsilon[\eta'(1-\cos^2(\theta+\alpha))+\xi'\sin(\theta+\alpha)\cos(\theta+\alpha)] \\ \text{(Symmetric)} & & 2+\varepsilon \end{pmatrix},$$

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[10] A. E. H. Love, op. cit., p. 430.

where ρ is the linear density of the rotor. This tensor must now be contracted twice with the angular velocity, $\vec{\omega} = (-\dot{\eta}' \quad \dot{\xi}' \quad -\dot{\theta})$. The result is the rotational kinetic energy:

$$\frac{1}{2} I \omega^2 = \rho k^2 \left(\dot{\eta}'^2 + \dot{\xi}'^2 + 2\xi' \dot{\eta}' \dot{\theta} - 2\eta' \dot{\xi}' \dot{\theta} \right.$$

$$\left. + 2\dot{\theta}^2 + \varepsilon \dot{\theta}^2 + \varepsilon \cdot \text{coefficients of } \varepsilon \right). \qquad \text{(II-22)}$$

All of the terms which result, except $\rho k^2 (2+\varepsilon) \dot{\theta}^2$, are very small since they are assumed to be of the second order of small quantities compared with ξ and η . Their interest lies in the fact that the coupling between the various equations, as will be apparent shortly, depends in the potential energy on ξ_o, η_o and ε . If these all vanish, as for a straight circular rotor, then the only coupling terms between the vibration and rotation are the small gyroscopic terms $2\xi' \dot{\eta}' \dot{\theta}$ and $2\eta' \dot{\xi}' \dot{\theta}$ above. It is commonly thought that if any rotor is run at or near the speed of the angular frequency of its natural vibration, large vibrations will develop. This is not so if the rotor is perfectly symmetric, and in that case the rotor will perform rotation and vibration very nearly independently, coupled only by the very small terms above. If it is not vibrating initially, none will develop since even this small coupling vanishes with the vibration. A rotor must possess some asymmetry to couple the rotation with the vibration, and in actual practice it is very difficult to

eliminate all asymmetry.

There is one case of practical importance in which the gyroscopic coupling is appreciable. If a flywheel is fixed to one end of a shaft, and the bearings supporting the shaft are some distance away, then the wobbling of the flywheel due to bending of the shaft can introduce strong gyroscopic terms. The equations for this case can be obtained by inclusion of the gyroscopic terms. This case was treated for the case of constant speed by Lees.[11] We neglect the slight wobbling effect (which in the unrotated bar is called the effect of rotary inertia because of the slight rotation of the section about the bending axis), and the total kinetic energy density becomes

$$\mathcal{T} = \tfrac{1}{2}\rho\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \tfrac{1}{2} I \dot{\theta}^2, \tag{II-23}$$

where I is the moment of inertia of the cross section per unit length about the perpendicular axis of the section. The complete Lagrangian density is expressed by $\mathcal{L} = \mathcal{T} - \mathcal{V}_B - \mathcal{V}_T$, or

$$\mathcal{L} = \tfrac{1}{2}\rho(\dot{\xi}^2 + \dot{\eta}^2) + \tfrac{1}{2} I \dot{\theta}^2 - \tfrac{1}{2} S_1 \Big\{ (\xi' - \delta\cos(\theta+\varphi))''^2 + (\eta' - \delta\sin(\theta+\varphi))''^2 + \varepsilon\left[(\xi' - \delta\cos(\theta+\varphi))''\cos(\theta+\alpha) + (\eta' - \delta\sin(\theta+\varphi))''\sin(\theta+\alpha)\right]^2 \Big\} - \tfrac{1}{2} T \theta'^2. \tag{II-24}$$

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[11] S. Lees, Phil. Mag. (6) 45 (1923), pp. 689-714.

The necessary Lagrangian density has now been obtained, and it remains only to substitute this into the Euler-Lagrange equation previously derived in order to obtain the equations of motion.

Equations of Motion

By inspection of the Lagrangian density, equation (II-24), and from the Euler-Lagrange equations (II-7), the equations of motion in terms of the coordinates are seen to be:

$$\left(\frac{\partial \mathcal{L}}{\partial \xi''}\right)'' - \left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{\xi}}\right) = -Q_\xi \ ;$$

$$\left(\frac{\partial \mathcal{L}}{\partial \eta''}\right)'' - \left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{\eta}}\right) = -Q_\eta \ ;$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)' + \left(\frac{\partial \mathcal{L}}{\partial \theta''}\right)'' - \left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{\theta}}\right) = Q_\theta \ , \qquad \text{(II-25)}$$

where we have dropped terms such as $\frac{\partial \mathcal{L}}{\partial \xi}$, which have not been included in the Lagrangian density.

The first two equations, which govern the lateral vibrations, suggest a formulation using a complex dependent variable when the differentiation is carried out:

$$\frac{\partial \mathcal{L}}{\partial \xi''} = -S_1 \Big\{ \big(\xi - \delta\cos(\theta+\beta)\big)'' + \varepsilon\Big[\big(\xi - \delta\cos(\theta+\beta)\big)''\cos(\theta+\alpha) + \big(\eta - \delta\sin(\theta+\beta)\big)''\sin(\theta+\alpha)\Big]\cos(\theta+\alpha)\Big\} \ ;$$

$$\frac{\partial \mathcal{L}}{\partial \eta''} = -S_1\Big\{\big(\eta - \delta\sin(\theta+\beta)\big)'' + \varepsilon\Big[\big(\xi - \delta\cos(\theta+\beta)\big)''\cos(\theta+\alpha) + \big(\eta - \delta\sin(\theta+\beta)\big)''\cos(\theta+\alpha)\Big]\sin(\theta+\alpha)\Big\} \;;$$

$$\left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{\xi}}\right) = \rho\,\ddot{\xi} \;; \quad \left(\frac{\partial \dot{\mathcal{L}}}{\partial \dot{\eta}}\right) = \rho\,\ddot{\eta} \;.$$

(II-26)

Let $\zeta = \xi + i\eta$ and combine equations (II-26) to obtain

$$\frac{\partial \mathcal{L}}{\partial \xi''} + i\frac{\partial \mathcal{L}}{\partial \eta''} = -S_1\Big\{\big(\zeta - \delta e^{i(\theta+\beta)}\big)'' + \varepsilon/2\Big[\big(\zeta^* - \delta e^{-i(\theta+\beta)}\big)'' e^{i(\theta+\alpha)} + \big(\zeta - \delta e^{i(\theta+\beta)}\big)'' e^{-i(\theta+\alpha)}\Big]e^{i(\theta+\alpha)}\Big\}. \qquad \text{(II-27)}$$

where ζ^* is used to denote the complex conjugate of ζ. Further simplification is obtained by the substitution

$$\underline{\underline{\Xi}} = \zeta - \delta e^{i(\theta+\beta)}. \qquad \text{(II-28)}$$

Note that

$$\dot{\underline{\underline{\Xi}}} = \dot{\zeta} - i\,\delta\,\dot{\theta}\,e^{i(\theta+\beta)}$$

and

$$\ddot{\Xi} = \ddot{\zeta} + \delta\left(\dot{\theta}^2 - i\ddot{\theta}\right)e^{i(\theta+\beta)} .$$

The lateral motion equation is

$$\left(-S_1\left\{\Xi'' + \varepsilon/2\left(\Xi^{*\prime\prime}e^{i(\theta+\alpha)} + \Xi''e^{-i(\theta+\alpha)}\right)e^{i(\theta+\alpha)}\right\}\right)''$$

$$-\rho\left(\ddot{\Xi} - \delta\left(\dot{\theta}^2 - i\ddot{\theta}\right)e^{i(\theta+\beta)}\right) = -Q_\Xi(\zeta,t),$$

or

$$\left(S_1(1+\varepsilon/2)\,\Xi_1'''\right)'' + \rho\,\ddot{\Xi}$$

$$= \rho\,\delta\left(\dot{\theta}^2 - i\ddot{\theta}\right)e^{i(\theta+\beta)} - \left(S_1\varepsilon/2\,\Xi^{*\prime\prime}e^{2i(\theta+\alpha)}\right)'' + Q_\Xi(\zeta,t). \qquad \text{(II-29)}$$

In the inertial system the coordinate Ξ measures the deviation of the central line from its equilibrium position. For example, if $\Xi \equiv 0$, and $\dot{\theta} \neq 0$, the equilibrium central line is merely rotating without bending or twisting.

For the θ equation, the derivatives $\frac{\partial \mathcal{L}}{\partial \theta}$, $\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)'$, and $\left(\frac{\partial \mathcal{L}}{\partial \theta''}\right)''$ are needed. The first is

$$\frac{\partial \mathcal{L}}{\partial \theta} = -S_1\left\{\left(\xi - \delta\cos(\theta+\beta)\right)''\left(\delta\sin(\theta+\beta)\right)''\right.$$

$$\left.-\left(\eta - \delta\sin(\theta+\beta)\right)''\left(\delta\cos(\theta+\beta)\right)'' + \varepsilon\left[\left(\xi - \delta\cos(\theta+\beta)\right)''\cos(\theta+\alpha)\right.\right.$$

$$+\left(\eta-\delta\sin(\theta+\beta)\right)''\sin(\theta+\alpha)\Big]\cdot\Big[-\left(\xi-\delta\cos(\theta+\beta)\right)''\sin(\theta+\alpha)$$

$$+\left(\eta-\delta\sin(\theta+\beta)\right)''\cos(\theta+\alpha)+\left(\delta\sin(\theta+\beta)\right)''\cos(\theta+\alpha)$$

$$-\left(\delta\cos(\theta+\beta)\right)''\sin(\theta+\alpha)\Big]\Big\}.$$

In terms of $\Xi' = \zeta - \delta e^{i(\theta+\beta)}$, this can be written in the form

$$\frac{\partial \mathcal{L}}{\partial \theta} = -S_1\Big\{(1+\varepsilon/2)\Big[\operatorname{Im}\Xi^{*''}\left(\delta e^{i(\theta+\beta)}\right)''\Big]$$

$$+\varepsilon/2\operatorname{Im}\Big[\Xi''\left(\Xi+\delta e^{i(\theta+\beta)}\right)''e^{-2i(\theta+\alpha)}\Big]\Big\}. \tag{II-30}$$

$\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)'$ and $\left(\frac{\partial \mathcal{L}}{\partial \theta''}\right)''$ are very complicated terms. This can be seen by partially performing the indicated differentiations. For $\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)'$ the result is

$$\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)' = -\Big[2S_1\Big\{\delta'\operatorname{Im}\Xi''e^{-i(\theta+\beta)}-\delta(\theta+\beta)'\operatorname{Re}\Xi''e^{-i(\theta+\beta)}$$

$$+\varepsilon\left[\operatorname{Re}\Xi''e^{-i(\theta+\beta)}\right]\left(\delta'\sin(\alpha-\beta)-\delta(\theta+\beta)'\cos(\alpha-\beta)\right)-T\theta'\Big\}\Big]'. \tag{II-31}$$

For $\left(\frac{\partial \mathcal{L}}{\partial \theta''}\right)''$ the result is

$$\left(\frac{\partial \mathcal{L}}{\partial \theta''}\right)'' = \left[-S_1\delta\left\{\mathrm{Im}\,\Xi''e^{-i(\theta+\beta)} + \varepsilon\left(\mathrm{Re}\,\Xi''e^{-i(\theta+\beta)}\right)\left(\cos(\theta+\beta) - \sin(\theta+\beta)\right)\right\}\right]''. \tag{II-32}$$

Fortunately both the terms $\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)'$ and $\left(\frac{\partial \mathcal{L}}{\partial \theta''}\right)''$, which contain much of the torsion flexure interaction between higher modes of oscillation of both types, can be integrated as they stand in the cases of interest here, and the remaining differentiation need not be explicitly carried out.

The θ equation, including $\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = I\dot{\theta}$, can therefore be written in the following form:

$$-S_1\left\{(1+\varepsilon/2)\left[\mathrm{Im}\,\Xi^{*\prime\prime}(\delta e^{i(\theta+\beta)})'' + \varepsilon/2\,\mathrm{Im}\left[\Xi''(\Xi + \delta e^{i(\theta+\beta)})''e^{-2i(\theta+\alpha)}\right]\right\}\right. + \left[2S_1\left\{\delta'\mathrm{Im}\,\Xi''e^{-i(\theta+\beta)} - \delta(\theta+\beta)'\mathrm{Re}\,\Xi''e^{-i(\theta+\beta)} + \varepsilon\left[\mathrm{Re}\,\Xi''e^{-i(\theta+\beta)}\right]\times\right.\right.$$
$$\left.\left.\left(\delta'\sin(\alpha-\beta) - \delta(\theta+\beta)'\cos(\alpha-\beta)\right)\right\} - T\theta'\right]' + \left[S_1\delta\left\{\mathrm{Im}\,\Xi''e^{-i(\theta+\beta)} + \varepsilon\left(\mathrm{Re}\,\Xi''e^{-i(\theta+\beta)}\right)\left(\cos(\theta+\beta) - \sin(\theta+\beta)\right)\right\}\right]'' - I\ddot{\theta} = -Q_\theta(\zeta,t). \tag{II-33}$$

The equations of motion have now been obtained in a fairly general form, neglecting only the following very small quantities: the correction due to lateral shearing motion, the contribution to the rotation of the cross section caused by the bending motion, the kinetic energy associated with the deformation of the cross section during bending, and the longitudinal motion.

It will be convenient to work with dimensionless quantities, so the following transformations are made:

$$\begin{aligned} \Xi &= \ell W \\ \zeta &= \ell u \\ \omega t &= \tau \\ S_1 &= S_0 K(u) \\ \rho &= \rho_0 D(u) , \end{aligned} \qquad \text{((II-34)}$$

where the value of ω is left to be chosen later, and S_0 and ρ_0 denote, respectively, the stiffness and density at some chosen reference point on the rotor, and $K(u)$ and $D(u)$ express their variation along the rotor. The lateral motion equation (II-29) becomes

$$\left(K(u)(1+\varepsilon/2)W''\right)'' + b^4\omega^2 D(u)\ddot{W} =$$

$$b^4\omega^2\Delta(u)D(u)\left(\dot{\theta}^2 - i\ddot{\theta}\right)e^{i(\theta+\beta)} - \left(K(u)\varepsilon/2\, W^{*\prime\prime}e^{2i(\theta+\alpha)}\right)'' + f(u,\tau) ,$$

(II-35)

where primes and dots now mean differentiation with respect to u and τ, and

$$b^4 = \frac{\rho_0}{S_0}\,\ell^4$$

$$\Delta(u) = \frac{\delta(u)}{\ell}$$

$$f(u,\tau) = \frac{\ell^3}{S_0}\,Q_w(u,\tau)\,. \tag{II-36}$$

In terms of the dimensionless variables, the θ equation, (II-33), becomes

$$-K\left\{(1+\varepsilon/2)\left[\mathrm{Im}\,W^{*\prime\prime}\left(\Delta e^{i(\theta+\beta)}\right)''\right] + \varepsilon/2\,\mathrm{Im}\left[W''\left(W + \Delta e^{i(\theta+\beta)}\right)'' e^{-2i(\theta+\alpha)}\right]\right\} + \left[2K\left\{\Delta'\,\mathrm{Im}\,W'' e^{-i(\theta+\beta)} - \Delta(\theta+\beta)'\,\mathrm{Re}\,W'' e^{-i(\theta+\beta)} + \varepsilon\left[\mathrm{Re}\,W e^{-i(\theta+\beta)}\right]\times\left(\Delta'\sin(\alpha-\beta) - \Delta(\theta+\beta)'\cos(\alpha-\beta)\right)\right\} - B\theta'\right]' + \left[K\Delta\left\{\mathrm{Im}\,W'' e^{-i(\theta+\beta)} + \varepsilon\left(\mathrm{Re}\,W'' e^{-i(\theta+\beta)}\right)\left(\cos(\theta+\beta) - \sin(\theta+\beta)\right)\right\}\right]'' - C\ddot{\theta} = -g(u,\tau)\,, \tag{II-37}$$

where

$$B(u) = \frac{\ell^2 T(u)}{S_o}$$

$$C(u) = \frac{\ell^2 I(u) \omega^2}{S_o}$$

$$g(u,\tau) = \frac{\ell^2}{S_o} Q_\theta(u,\tau) \, .$$

(II-38)

The dimensionless equations (II-35) and (II-37) can now serve as a starting point for the examination of the particular cases which initiated this investigation. This will be done in the next chapter.

CHAPTER III

SOLUTION OF THE EQUATIONS OF MOTION FOR CERTAIN SPECIAL CASES

It is apparent upon inspection of the general equations (II-35) and (II-37) that little useful information can be obtained from them in their present form, and their application to the particular problems will require considerable reduction and adaptation. One advantage of having such general equations is, of course, that they cover a very wide range of problems, and often serve to unify apparently diverse concepts. A second advantage is that one can readily compare the sizes of various terms for a particular problem and thus more readily determine which ones can safely be neglected than if approximate equations were derived from less general assumptions. When the problem which initiated this research was first considered, it was thought that torsional vibrations might be important. Therefore terms depending on twisting were included in the potential energy to see if perhaps they could be treated without excessive difficulty. Equations (II-35) and (II-37) show that the torsional motion may be quite important and may interact with the lateral motion to a considerable extent because of the extensive mixing of the lateral and torsional displacement coordinates. Moreover, θ often occurs as an exponent, and the complications which arise because of this when θ is a function

of u as well as τ, while not insurmountable, nevertheless present such formidable difficulties as to justify seeking some method for eliminating the u dependence of θ from consideration in our treatment. A method which suggests itself is to integrate the θ equation over u, which eliminates the troublesome terms $\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)'$ and $\left(\frac{\partial \mathcal{L}}{\partial \theta''}\right)''$ displayed in equations (II-31) and (II-32), and then to make the approximation of replacing $\theta(u,\tau)$ by $\theta_{ave}(\tau)$ everywhere. We discuss the validity of this below.

Conditions for Ignoring Torsional Motion

The procedure indicated above can be justified most easily on the basis of physical reasoning from what is known about coupled vibrating systems in general.

Examination of equations (II-35) and (II-37) shows that they reduce to independent equations for lateral and torsional motion when the parameter functions $\Delta(u)$ and $\mathcal{E}(u)$ vanish everywhere. However, as Δ and $\mathcal{E}$ depart from zero, the modes of vibration associated with the natural frequencies of both types of motion begin to interact with each other, and therefore the pure lateral modes begin to involve torsional motion, and vice versa for the torsional modes. If the coupling is weak, which means Δ and $\mathcal{E}$ very small everywhere compared to unity, most of the modes of the two types of vibration will preserve to a high degree their essential torsional or lateral character. Only when two

modes of different type have their frequencies close together in the uncoupled state would the small coupling be expected to alter their shape or shift their frequencies appreciably. It will be seen later that we can consider the behavior of the rotor at each resonant speed separately, and therefore we need only to know whether or not the particular mode considered has appreciable torsional motion associated with it.

There is no approximation involved in averaging over the θ equation (II-37). The approximation consists of substituting $\theta_{av}(\tau)$ for $\theta(u,\tau)$, and this will be permissible if θ does not vary from its average position more than a few degrees along the entire rotor. The greater the excursion of θ from the average position, the less exact is the analysis. Physically, the approximation is equivalent to neglecting the energy of the torsional vibrations, and this approximation will be a good one if the torsional vibration energy is negligibly small compared to the energies of the rotational and lateral motion. Thus we restrict the analysis which follows to modes of oscillation which are essentially lateral in character, which will usually be assured by having $\Delta(u)_{max}$ and $\mathcal{E}(u)_{max}$ small compared to unity.

The cases in which Δ and $\mathcal{E}$ are large do, however, constitute a very important class of problems. For example, the wing of a multi-engine aircraft is characterized by large values of Δ at the engine locations, and the vibrational modes of this structure involve extensive lateral and

torsional motions. Certain types of crankshafts and turbine blades pose similar problems. In principle, all can be treated with equations (II-35) and (II-37).

By averaging the torsional motion over the length, and replacing $\theta(u,\tau)$ by $\theta_{ave}(\tau)$, all torsion-bending interaction is removed from consideration except that of bending with the rotation of the rotor as a whole. The result on equation (II-37) of this averaging by integration over u is

$$\int_0^1 -K\Big\{(1+\varepsilon/2)\,[\mathrm{Im}\, W^{*\prime\prime}(\Delta e^{i(\theta+\beta)})''] + \varepsilon/2\,\mathrm{Im}\big[W''(W+\Delta e^{i(\theta+\beta)})'' e^{-2i(\theta+\alpha)}\big]\Big\}\,du$$

$$-\int_0^1 C\ddot{\theta}\,du = -G_s(\tau) - G_T(\tau)\,, \qquad \text{(III-1)}$$

where we have put $G_s(\tau)$ equal to the integral over the external torque density $g(u,\tau)$, and $G_T(\tau)$ represents the terminal torques arising from the integration of $\left(\frac{\partial \mathcal{L}}{\partial \theta}\right)'$ and $\left(\frac{\partial \mathcal{L}}{\partial \theta''}\right)''$. These terms, when integrated over u, become $\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right)$ and $\left(\frac{\partial \mathcal{L}}{\partial \theta''}\right)'$ evaluated at the ends of the rotor, and inspection of the second condition of (II-6) shows they must represent terminal rotational torques. The quantities $G_s(\tau)$ and $G_T(\tau)$ are just space average torques per unit length. Also, note that

$$\int_0^1 C\ddot{\theta}\,du = \ddot{\theta}_{ave}\int_0^1 C\,du = \ddot{\theta}_{ave}\frac{\omega^2 b^4}{\rho_0 l^2} I_{ave} ,$$

(III-2)

where we have used (II-36) and (II-38) in the expression for $C(u)$. Because the limits of integration are constant, $(\ddot{\theta}_{ave}) = \ddot{\theta}_{ave}$. Hereafter we use θ where we mean θ_{ave} , and drop the subscript. The θ equation (III-1) becomes, using (III-2),

$$\ddot{\theta} + R_1\int_0^1 K\left\{(1+\varepsilon/2)\left[\mathrm{Im}\,W^{*\prime\prime}\left(\Delta e^{i(\theta+\beta)}\right)''\right.\right.$$

$$\left.\left. + \varepsilon/2\,\mathrm{Im}\left[W''\left(W+\Delta e^{i(\theta+\beta)}\right)''\bar{e}^{2i(\theta+\alpha)}\right]\right\}du$$

$$- R_1\left(G_S(I) + G_T(I)\right) = 0 ,$$

(III-3)

where

$$R_1 = \frac{\rho_0 l^2}{\omega^2 b^4 I_{ave}} .$$

(III-4)

The lateral motion equation (II-34) is unchanged by the substitution of θ_{ave} for θ .

Reduction to Ordinary Differential Equations

The lateral motion equation, (II-35), can now be separated

in the usual fashion[1] into ordinary differential equations.

The Lateral Vibrations

The homogeneous part of the lateral motion equation (II-35) is

$$\left(K\left(1+\varepsilon/2\right)W''\right)'' + b^4\omega^2 D\,\ddot{W} = 0$$

The solution to the homogeneous equation can be made a little more general by including in the homogeneous equation a term representing a frictional force proportional to the lateral velocity. The equation will still be soluble providing we are willing to accept a distribution of the force along the rotor which is proportional to the mass distribution. This, of course, is not strictly true, but the frictional force does tend to vary in this fashion. The constant f_o is the friction force taken at some convenient reference point. We therefore take the homogeneous equation to be

$$\left(K\left(1+\varepsilon/2\right)W''\right)'' + 2f_o\,\omega D\dot{W} + b^4\omega^2 D\ddot{W} = 0 \qquad \text{(III-5)}$$

We assume $W = U(\omega)H(\tau)$, and (III-5) becomes

$$\left(K(1+\varepsilon/2)U''\right)''H + 2f_o\,\omega DU\dot{H} + b^4\omega^2 DU\ddot{H} = 0$$

Now we divide by DUH, and obtain

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[1]See J. D. Tamarkin and W. Feller, Partial Differential Equations (Brown University, Providence, R. I., 1941), p. 88.

$$\frac{(K(1+\varepsilon/2)U'')''}{DU} = -\frac{2f_0\omega\dot{H} + b^4\omega^2\ddot{H}}{H} . \qquad \text{(III-6)}$$

Since each side of (III-6) is a function of a single different independent variable, the equation can be satisfied only if each side is equal to the same constant. We choose this constant to be λ^4, and (III-6) splits into

$$(K(1+\varepsilon/2)U'')'' - \lambda^4 DU = 0 , \qquad \text{(III-7)}$$

and

$$H + \frac{2f_0\omega\dot{H} + b^4\omega^2\ddot{H}}{\lambda^4} = 0 . \qquad \text{(III-8)}$$

Equation (III-7) and the boundary conditions (II-6) constitute an eigenvalue problem. Solutions, which are termed eigenfunctions, exist only for certain values of λ, which are called eigenvalues. Because the operator $(K(1+\varepsilon/2)U'')''$ is in the self-adjoint form, and D is positive for $0 \le u \le 1$, the values of λ are both real and positive.[2]

W is a complex quantity, and the four boundary conditions expressed by (II-6) hold for each component, hence there will be a different set of eigenvalues and eigenfunctions for each direction of deflection, if the boundary conditions for the two directions of deflection are different. We shall assume that the boundary conditions are the same for each

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[2]Ibid, p. 72.

component of deflection since it is only a slightly restrictive assumption, and allows considerable simplification of the analysis. In this way we can assume the u-dependent part of W to be real, and the τ-dependent part complex.

The case in which D, K and $\mathcal{E}$ are constants is the familiar one of the uniform bar. In this case the eigenfunctions are linear combinations of the circular and hyperbolic sines and cosines and, therefore, relatively simple. The general case in which D and $K(1+\epsilon/2)$ are arbitrary non-negative functions with no zeroes or singularities for $0 \leq u \leq 1$ is, of course, much more difficult. The problem of the determination of the eigenfunctions of equation (III-7) is one which has received much attention because of the numerous engineering applications, and several good methods exist for approximating the eigenfunctions to any desired accuracy. When the problem differs only slightly from one whose solutions are known, then the method due to Rayleigh[3] can be used which obtains the desired eigenfunctions as an expansion in terms of the eigenfunctions of the soluble problem. This procedure when adapted to quantum mechanics is familiar as the Schrödinger perturbation method. A method developed by W. T. White[4] appears to be very powerful when the functions D and $K(1+\epsilon/2)$ are more general.

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[3]Lord Rayleigh, The Theory of Sound (Macmillan & Co., London, 1877), p. 87 ff. v. 1.

[4]W. T. White, J. Franklin Inst. 245, pp. 25-36; pp. 117-134 (1948).

In what follows it will be assumed that the eigenfunctions have been obtained to sufficient accuracy for the requirements of the problem considered and that they form a complete orthonormal set with a weighting function. Let the eigenvalues be denoted by λ_n, the eigenfunctions by U_n, and the weighting function is the previously defined density function, $D(u)$. Assume that there is only one eigenfunction for each λ_n. The orthonormality is expressed by the condition

$$\int_0^1 D(u)\, U_n(u)\, U_m(u)\, du = \delta_{nm} , \qquad \text{(III-9)}$$

where δ_{nm} is zero if $n \neq m$ and one if $n = m$. The four boundary conditions (II-6) determine only the ratios of the four constants of integration which result when equation (III-7) is integrated, therefore the additional condition (III-9) is required to fix the amplitudes of the eigenfunctions.

For each value of λ_n there exists a τ-dependent homogeneous equation (III-8). Adding a subscript n to (III-8), we obtain

$$H_n + \left(2 f_0 \omega \dot{H}_n + b^4 \omega^2 \ddot{H}_n\right) \bar{\lambda}_n^4 = 0 . \qquad \text{(III-10)}$$

This is just the equation for a damped harmonic oscillator, and the solution is

$$H_n = e^{-\mu_n \tau}\left(A_n e^{i \sigma_n \tau} + B_n e^{-i \sigma_n \tau}\right), \qquad \text{(III-11)}$$

where

$$\nu_n = \frac{f_o}{b^4 \omega}$$

$$\sigma_n = \frac{\sqrt{\lambda_n^4 b^4 - f_o^2}}{b^4 \omega} \quad . \tag{III-12}$$

Recall that we have not fixed the value of ω . It will be convenient when considering the behavior of H_n in the neighborhood of the n^{th} frequency, σ_n , to take $\omega = \sqrt{\lambda_n^4 b^4 - f_o^2}/b^4$, which makes $\sigma_n = 1$. We can then write

$$H = e^{-\nu\tau}\left(Ae^{i\tau} + Be^{-i\tau}\right) , \tag{III-13}$$

where we have dropped the subscript with the understanding that σ is always chosen as stated above for the resonance being considered. This means time is measured in units of the natural angular period of the mode of vibration considered.

The inhomogeneous equations can now be treated. We assume

$$W = \sum_m U_m(u)\, T_m(\tau) \quad . \tag{III-14}$$

where the U_m are as before, but a different time dependence is assumed. The function $f(u,\tau)$, which represents now all the external force density except the linear damping term included in the homogeneous equation, must be

expanded in a series of U_m with time-dependent coefficients. Thus

$$f(u,\tau) = \sum U_m(u)\, h_m(\tau) . \tag{III-15}$$

The function $\Delta e^{i\beta}$ which locates the equilibrium central line can be expanded in the following fashion:

$$\Delta e^{i\beta} = \sum_m \Delta_m e^{i\beta_m} U_m(u) , \tag{III-16}$$

where $\Delta_m e^{i\beta_m}$ is the complex expansion coefficient.

The complete lateral motion equation (II-35) can now be written:

$$\sum_m \left\{ \left(K(1+\varepsilon/2)\, U_m''\right)'' T_m + 2f_0\omega D U_m \dot{T}_m + b^4\omega^2 D U_m \ddot{T}_m \right\}$$

$$= \sum_m \left\{ \lambda_m^4 D \Delta_m e^{i\beta_m} U_m \omega^2 (\dot{\theta}^2 - i\ddot{\theta})\, e^{i\theta} \right.$$

$$\left. -\left(K(1+\varepsilon/2)\, U_m'' e^{2i(\theta+\alpha)}\right)'' T_m^* + U_m h_m(\tau) \right\} . \tag{III-17}$$

If we multiply (III-17) by U_m and integrate over u, we obtain, using (III-7) and (III-9),

$$T_n + \lambda_n^{-4} \left(2f_0\omega \dot{T} + b^4\omega^2 \ddot{T}_n\right)$$

$$= \Delta_n e^{i\beta_n} \omega^2 \left(\dot\theta^2 - i\ddot\theta\right) e^{i\theta} + \lambda_n^{-4} \sum_m \left(-J_{nm} T_m^* e^{2i\theta} + N_{nm}\, l_m(\tau)\right),$$

(III-18)

where J_{nm} and N_{nm} are defined by

$$J_{nm} = \int_0^1 U_n \left(K \varepsilon/2\, U_m'' e^{2i\alpha}\right)'' du,$$

$$N_{nm} = \int_0^1 U_n U_m \, du. \qquad \text{(III-19)}$$

Note that, if ε and α are constants,

$$J_{nm} = \lambda_n^4 \frac{\varepsilon/2}{1+\varepsilon/2} e^{2i\alpha} \delta_{nm} \qquad \text{(III-20)}$$

and, if $D(u)$ is a constant,

$$N_{nm} = \delta_{nm} \qquad \text{(III-21)}$$

Solutions of the homogeneous part of (III-18) are the $H_n(\tau)$, and we can write the complete solution as an integral.[5] If two independent solutions, y_1 and y_2, of $\ddot y + f(t)\dot y + g(t)\, y = 0$ are known, then the solution of $\ddot y + f(t)\dot y + g(t) y = S(t)$ is

$$y = y_1 \int_c^t \frac{S y_2}{y_1 \dot y_2 - y_2 \dot y_1}\, dx - y_2 \int_c^t \frac{S y_1}{y_1 \dot y_2 - \dot y_1 y_2}\, dx, \qquad \text{(III-22)}$$

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[5] See H. and B. Jeffreys, Methods of Mathematical Physics (Cambridge University Press: Cambridge, 1950), p. 494.

where the lower limit is an arbitrary constant which introduces a constant multiple of the complementary function. If we apply (III-22) to (III-18) we obtain

$$T_n = \sigma_n^{-1}\Delta_n e^{-\nu_n\tau+i\beta_n}\omega^2\int_0^\tau\left(\dot\theta^2_{(x)}-i\ddot\theta_{(x)}\right)e^{\nu_n x+i\theta(x)}\sin\sigma_n(\tau-x)\,dx$$

$$+\sigma_n^{-1}\lambda_n^{-4}e^{-\nu_n\tau}\sum_m\left\{-J_{nm}\int_0^\tau e^{\nu_n x+2i\theta(x)}T_m^*\sin\sigma_n(\tau-x)\,dx\right.$$

$$\left.+H_{nm}\int_0^\tau e^{i\theta(x)+\nu_n x}h_m(x)\sin\sigma_n(\tau-x)\,dx\right\}+\quad H_n\,,$$

(III-23)

where we have written explicitly the complementary function H_n necessary to meet arbitrary initial conditions.

Further reduction of (III-23) is possible by observing that $-\left(e^{i\theta}\right)^{\cdot\cdot}=\left(\dot\theta^2-i\ddot\theta\right)e^{i\theta}$. We perform two integrations by parts on the term containing $\left(\dot\theta^2-i\ddot\theta\right)e^{i\theta}$, and obtain, if we set $\theta(0)=0$ and $\dot\theta(0)=\dot\theta_0$,

$$-\int_0^\tau\frac{d^2}{dx^2}\left(e^{i\theta(x)}\right)e^{\nu_n x}\sin\sigma_n(\tau-x)\,dx$$

$$=-\sigma_n e^{i\theta+\nu_n\tau}+\sigma_n\cos\sigma_n\tau+\left(-\nu_n+i\dot\theta_0\right)\sin\sigma_n\tau$$

$$+\int_0^\tau e^{i\theta+\nu_n x}\left[\left(\sigma_n^2-\nu_n^2\right)\sin\sigma_n(\tau-x)-2\nu_n\sigma_n\cos\sigma_n(\tau-x)\right]dx\,.$$

(III-24)

Parts of (III-24) belong to the solution of the homogeneous equation, but we retain them since they meet initial conditions we shall need later. Substituting (III-23) in (III-22), we obtain:

$$T_n = -\Delta_n e^{i\beta_n}\left\{e^{i\theta} + e^{-\nu_n\tau}\left[\cos\sigma_n\tau + \sigma_n^{-1}(-\nu_n + i\dot{\theta}_0)\sin\sigma_n\tau\right]\right\}$$

$$+ \Delta_n e^{i\beta_n} e^{-\nu_n\tau}\int_0^\tau e^{i\theta + \nu_n x}\left[(\sigma_n^2 - \nu_n^2)\sin\sigma_n(\tau-x) - 2\nu_n\sigma_n\cos\sigma_n(\tau-x)\right]dx$$

$$+ \sigma_n^{-1}\lambda_n^4 e^{-\nu_n\tau}\sum_m\left\{-J_{nm}\int_0^\tau e^{\nu_n x + 2i\theta_0 x} T_m^* \sin\sigma_n(\tau-x)\,dx\right.$$

$$\left. + N_{mn}\int_0^\tau e^{\nu_n x} l_m(x)\sin\sigma_n(\tau-x)\,dx\right\} + H_n \quad . \tag{III-25}$$

If the forcing applied to the lateral motion has a frequency range which is narrow compared to the spacing of the natural frequencies of the rotor , and the $J_{nm}\,(n\neq m)$ are sufficiently small, only one of the T_n will be large compared with the others, and it will be that one for the mode whose natural frequency is near the forcing frequency. The $J_{nm}\,(n\neq m)$ represent the coupling between the various modes T_n and, if sufficiently large, prevent the excitation of one mode without exciting others appreciably. It is not sufficient that $\varepsilon \ll 1$; inspection of (III-19) shows that $\varepsilon' \ll 1$ and $\varepsilon'' \ll 1$ are required as well if $J_{nm}\,(n\neq m)$ are to be very small.

Assuming the above restrictions are satisfied, we can write, as a good approximation,

$$T = -\Delta_c\left(e^{i\theta} + e^{-\nu\tau}\left[\cos\tau + (-\nu + i\dot{\theta}_0)\sin\tau\right]\right)$$

$$+\Delta_c e^{-\nu\tau}\int_0^\tau e^{i\theta+\nu x}\left[(1-\nu^2)\sin(\tau-x)-2\nu\cos(\tau-x)\right]dx$$

$$+\lambda^{-4}e^{-\nu\tau}\int_0^\tau\left(-J e^{2i\theta}T^* + N h(x)\right)e^{\nu x}\sin(\tau-x)\,dx + H,$$

(III-26)

where we have used our freedom in the choice of ω in (III-12) to make $\sigma_n = 1$ and have dropped the subscript n, so that (III-26) is applicable to any resonance, provided the quantities formerly labeled with n are chosen with the correct values for the resonance considered. The subscript c on the constant Δ_c is to avoid confusion with the $\Delta(u)$ function used previously. We have also set $\beta = 0$, which is permissible since we consider each resonance separately and their relative phases are unimportant.

Recall that T is not the coefficient of the eigenfunction for the observed motion, but only for the deviation from the equilibrium configuration. This fact results from substituting $\xi = \zeta - \delta e^{i(\alpha+\beta)}$ into equation (II-27). If the first term on the right is (III-26) is transferred to the left side, we have

$$\ell^{-1}(\bar{\xi}+\bar{\eta}) = \bar{T} = T + \Delta_c e^{i\theta},$$

(III-27)

which is the time-dependent coefficient which, when multiplied by the proper eigenfunction, gives the actual displacement of the rotor as a function of u and τ. The ℓ^{-1} arises

from the non-dimensionalization of the original equations in Chapter II, while $\bar{\zeta}$ and $\bar{\eta}$ are the time dependent parts of the ζ and η used in Chapter II.

Equation (III-25) gives the general response of the rotor to rotation which occurs near the resonance considered, plus that which arises from forcing by other sources denoted by $h(\tau)$, and includes the effect of skew stiffness. An interesting consequence of skew stiffness which has been known for some time is that it causes a resonance response for a rotation speed of one half the resonance frequency, if there is a constant space-fixed lateral force, such as gravity. Physically the explanation is simple and consists of the observation that the rotor, because of the rotation, presents a varying stiffness to this force, and the variation has a period half that of the rotation period. Hence a resonance response occurs at half the usual critical speed when the above conditions are met.

This behavior can also be deduced from equation (III-25). For simplicity, suppose $\Delta_c = 0$ and $H h(\tau)$ is constant. Expand T in powers of $\lambda^{-4} J = \nu$, which is a quantity of the order of ε, from equation (III-20). Thus

$$T = T_0 + T_1 \nu + T_2 \nu^2 + \cdots \tag{III-28}$$

Substituting this in equation (III-26), we obtain an equation for each power of ν:

$$T_0 = e^{-\nu\tau} Nk \int_0^\tau e^{\nu x} \sin(\tau - x)\, dx + H$$

$$T_1 = -e^{-\nu\tau} Nk \int_0^\tau T_0^* e^{\nu x + 2i\theta} \sin(\tau - x)\, dx$$

$$T_2 = \ldots, \text{ etc.}$$

(III-29)

If the integral in the first equation is evaluated,

$$T_0 = \frac{Nk}{1+\nu^2} + \frac{Nk}{1+\nu^2}(\cos\tau - \nu \sin\tau) e^{-\nu\tau} + H,$$

(III-30)

where everything except the constant belongs to the complementary function. If this is substituted in the T_1 equation, the constant introduces the integral

$$K = -\frac{(Nk)^2}{1+\nu^2} e^{-\nu\tau} \int_0^\tau e^{\nu x + 2i\theta} \sin(\tau - x)\, dx .$$

(III-31)

If $\dot{\theta} = \frac{1}{2}$, and ν is sufficiently small, then the amplitude of K will become large. In the absence of friction, $\nu = 0$, the amplitude of K grows linearly with τ, which is characteristic of the amplitude growth of an undamped oscillator being forced at its natural frequency. Of course, this demonstrates only the initial behavior of T, and more terms must be computed if the motion is desired for a longer period.

The existence of whirling in certain cases for a rotational speed one half that expected from the natural frequency puzzled observers for some time until the correct physical explanation depending on skew stiffness and a constant lateral

force was advanced by Robertson.[6] F. M. Lewis, as mentioned in Chapter I, solved equation (III-26) for $\theta = k\tau^2$ and $J = H = 0$, which is the case of constant acceleration, but the equation including skew stiffness ($J \neq 0$) has not been previously reported in the literature as far as this writer can ascertain.

The Rotational Motion

The integral over u in equation (III-3) can be evaluated with the aid of the lateral motion eigenfunctions. If two successive integrations by parts are performed on the integral, the necessary steps for the evaluation of the remaining integral will be more apparent. Thus the integral in (III-3) is readily seen to be

$$\int_0^1 K\left\{(1+\varepsilon/2)\left[\mathrm{Im}\, W^{*\prime\prime}(\Delta e^{i(\theta+\beta)})''\right] + \varepsilon/2\, \mathrm{Im}\left[W''(W+\Delta e^{i(\theta+\beta)})'' e^{-2i(\theta+\alpha)}\right]\right\} du$$

$$= \left[\mathrm{Im}\left\{(K(1+\varepsilon/2)W^{*\prime\prime})(\Delta e^{i(\theta+\beta)})' + \left[(K\varepsilon/2\, W'' e^{2i(\theta+\alpha)})(W+\Delta e^{i(\theta+\beta)})\right]\right\}\right]_0^1$$

$$- \left[\mathrm{Im}\left\{(K(1+\varepsilon/2)W^{*\prime\prime})'(\Delta e^{i(\theta+\beta)}) + \left[(K\varepsilon/2\, W'' e^{-2i(\theta+\alpha)})'(W+\Delta e^{i(\theta+\beta)})\right]\right\}\right]_0^1$$

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[6] D. Robertson, The Engineer, clvi, pp. 152-153, 179-181, 213-214, (1933).

$$+\int_0^1 \left[\mathrm{Im}\left\{ \left(K(1+\varepsilon/2)W^{*\prime\prime}\right)'' \Delta e^{i(\theta+\beta)} + \left(K\varepsilon/2\, W'' e^{-2i(\theta+u)}\right)'' \left(W+\Delta e^{i(\theta+\beta)}\right) \right\} \right] du. \tag{III-32}$$

The identification of the integrated terms in equation (III-32) is somewhat complicated. Since equation (III-32) governs the rotational motion, it is apparent that the integrated terms must represent rotational torques, but their sources and magnitudes are not immediately apparent. In the first integrated term in (III-32), $\left(W+\Delta e^{i(\theta+\beta)}\right)'$ can be identified, by referring to (II-28) and (II-34), as the derivative of the dimensionless complex displacement coordinate $(\xi+i\eta)\,\ell^{-1}$. The term $\left(\Delta e^{i(\theta+\beta)}\right)'$ is the derivative of the equilibrium displacement. These terms thus represent the slope of the central line at the ends for, respectively, a general displacement and the equilibrium displacement, and may be regarded as small angles. The sum of the coefficients of the of the terms $\left(W+\Delta e^{i(\theta+\beta)}\right)'$ and $\left(\Delta e^{i(\theta+\beta)}\right)'$ may be identified as $\left(\frac{\partial \mathcal{L}}{\partial \tau''}+i\frac{\partial \mathcal{L}}{\partial \nu''}\right)\ell^{-1}$ by referring to (II-27), (II-28) and (II-34). From the boundary conditions given by (II-6), it is seen that $\left(\frac{\partial \mathcal{L}}{\partial \tau''}+i\frac{\partial \mathcal{L}}{\partial \nu''}\right)\ell^{-1}$ is just the dimensionless complex terminal bending moment. It is thus readily seen that the product of the terminal bending moment and the terminal slope of the central line produces the component of the terminal bending moment which lies along the z-axis and therefore contributes to the rotational torque.

Examination of the second integrated term in (III-32) shows that it is obtained from the first simply by switching

the prime from the displacement terms to the moment terms. The derivative of the bending moment is the lateral force, as may be verified from (II-6), and this force, acting about the ζ-axis with the displacement as a moment arm, also contributes to the rotational torque.

The displacement and its derivative are assumed small, therefore both integrated terms are small. Moreover, they vanish entirely for the boundary conditions of greatest interest, namely the case of the free rotor, in which the terminal forces and moments vanish, and the case in which the ends are constrained by bearings, rigidly aligned along the ζ-axis, causing the terminal displacements and slopes to vanish. In any case, the torques arising from the integrated terms may be considered te be lumped into one quantity $G(\tau)$, which includes all the rotational torque from whatever sources, including the previously defined $G_s(\tau)$ and $G_r(\tau)$.

The integral on the right side of equation (III-32) can be evaluated. We note that

$$W = \sum_m U_m(u)\, T_m(\tau) ,$$

$$(K(1+\varepsilon/2)U_m'')'' = \lambda_m^4\, D\, U_m ,$$

$$\Delta e^{i\beta} = \sum_m \Delta_m e^{i\beta_m}\, U_m ;$$

and we make use of these relations to obtain

$$\int_0^1 K\left[(1+\varepsilon/2)\left[\mathrm{Im}\, W^{*\prime\prime}(\Delta e^{i(\theta+\beta)})''\right] + \varepsilon/2\, \mathrm{Im}\left[W'' e^{-2i(\theta+\alpha)}\left(W + \Delta e^{i(\theta+\alpha)}\right)''\right]\right] du$$

$$= \sum_m \left\{ \mathrm{Im}\left[\lambda_m^4 \Delta_m e^{i\beta_m} T_m^* e^{i\theta} \right. \right.$$

$$\left. \left. + \sum_n -J_{nm}\left(T_m T_n e^{-2i\theta} + T_m \Delta_n e^{i\beta_n} e^{i\theta} \right) \right] \right\},$$

(III-33)

where we have made use of equations (III-9) and (III-10). The complete θ equation, (III-3) is now

$$\ddot{\theta} + R_1 \sum_m \left\{ \mathrm{Im}\left[\lambda_m^4 \Delta_m e^{i\beta_m} T_m^* e^{i\theta} \right. \right.$$

$$\left. \left. + \sum_n -J_{mn}\left(T_m T_n e^{-2i\theta} + T_m \Delta_n e^{i\beta_n} e^{i\theta} \right) \right] \right\}$$

$$- R_1 G(\tau) = 0.$$

(III-34)

If we restrict the motion to the neighborhood of one resonance as was done for the lateral motion, equation (III-34) becomes

$$\ddot{\theta} + R_2 \left[\mathrm{Im}\left\{ \Delta_c T^* e^{i\theta} - J\left(T^2 e^{-2i\theta} + T \Delta_c e^{i\theta} \right) \right\} \right]$$

$$- R_1 G(\tau) = 0 \ , \qquad \text{(III-35)}$$

where

$$R_2 = \frac{\rho_0 \varrho^2 \lambda^4 b^4}{\left(\lambda_n^4 b^4 - f_0^2 \right) I_{ave}} \ . \qquad \text{(III-36)}$$

Equations (III-35) and (III-26) now constitute a set of ordinary non-linear integro-differential equations which, when solved simultaneously, yield both the rotation and lateral motion of a rotor in the neighborhood of a critical speed. The initial conditions are arbitrary, as are the end conditions with the restriction that they be the same in all planes containing the axis of rotation of the rotor, i.e., that they be axially symmetric. The accelerating torque and the lateral forces, may be arbitrarily specified, and the effect of skew stiffness is included.

The term arising from skew stiffness prevents reduction of equations (III-35) and (III-26) to a single equation. The expression for T, equation (III-26), cannot be solved explicitly for T because of J. Therefore if we assume $J = 0$, equation (III-35) can be reduced to an equation involving θ alone. In doing this we set $H = 0$, resulting in the initial conditions $T(0) = \dot{T}(0) = 0$, which will be useful later. Also we assume $N = 0$, which means no other non-conservative lateral forces are acting, except the friction previously included. By substituting equation (III-26) in equation (III-35) we obtain, after some manipulation:

$$\ddot{\theta} + \frac{R_2 \Delta_c^2 e^{-\nu\tau}}{2} \left\{ (\dot{\theta}_0 + \nu + 1)\sin(\theta - \tau) - (\dot{\theta}_0 + \nu - 1)\sin(\theta + \tau) \right.$$

$$\left. + \int_0^\tau e^{\nu\lambda} \left[(1-\nu^2)\left(\cos[(\theta(\tau) - \tau) - (\theta(\lambda) - \lambda)] - \cos[(\theta(\tau) - \tau) - (\theta(\lambda) - \lambda)]\right)\right.\right.$$

$$-2\nu\left(\sin[(\theta(\tau)-\tau)-(\theta(x)-x)]+\sin[(\theta(\tau)-\tau)-(\theta(x)-x)]\right]dx\Big\}$$

$$-R_1\,G(\tau)=0\ .$$

(III-37)

The terms in $\theta+\tau$ arise from the "mirror" resonance which occurs if the shaft is rotated at the same speed in the opposite direction. These terms should be neglected to be consistent with the neglect of other resonances. Equation (III-37) then becomes

$$\ddot{\theta}+R\,e^{-\nu\tau}\Big\{(\dot{\theta}_0+\nu+1)\sin(\theta-\tau)+\int_0^{\tau}e^{\nu x}\left((1-\nu^2)\cos[(\theta(\tau)-\tau)-(\theta(x)-x)]\right.$$

$$\left.-2\nu\sin[(\theta(\tau)-\tau)-(\theta(x)-x)]\right)dx\Big\}-M(\tau)=0\ ,$$

(III-38)

where

$$R=\frac{R_2\,\Delta_c^2}{2}=\frac{\rho_0\,l^2\lambda^4\,b^4\,\Delta_c^2}{2(\lambda^4b^4-f_0^2)\,I_{ave}}\ ,$$

$$M(\tau)=\frac{\rho_0\,l^2}{\omega^2b^4\,I_{ave}}\,G(\tau)\ .$$

(III-39)

Equation (III-38) describes the θ motion of a rotor with no skew stiffness initially rotating with angular velocity $\dot{\theta}_0$, with no initial lateral vibration (measured from the equilibrium configuration), under the action of a torque

function $M(\tau)$, and the lateral vibration damping represented by $e^{-\nu\tau}$. R measures the coupling between the rotation and vibration. Our choice of the time scale has placed the resonance at $\dot{\theta}=1$.

Equation (III-38) has been solved numerically for several values of R and $\dot{\theta}_0$, for $M=\nu=0$. It has also been solved on an analogue computer for a number of constant values of M for each of several values of R, all with $\nu=0$. The conservative case, $M=\nu=0$, is soluble analytically in terms of elliptic functions. These cases are discussed below, but first we convert equation (III-38) to a differential equation from its present integro-differential form. Let

$$S = \theta - \tau .$$

Then equation (III-38) becomes

$$\ddot{S} + R e^{-\nu\tau}\left\{(\dot{S}_0+\nu+2)\sin S + \int_0^\tau e^{\nu\chi}\left[(1-\nu^2)\cos(S(\tau)-S(\chi)) - 2\nu\sin(S(\tau)-S(\chi))\right]d\chi\right\} - M(\tau) = 0 . \tag{III-40}$$

Differentiation of equation (III-40) yields

$$\dddot{S} + \nu(\ddot{S}-M) + R\dot{S}e^{-\nu\tau}\left\{(\dot{S}_0+\nu+2)\cos S + \int_0^\tau e^{\nu\chi}\left[(1-\nu^2)(-\sin(S(\tau)-S(\chi)) - 2\nu\cos(S(\tau)-S(\chi))\right]d\chi\right\}$$

$$+ R(1-\nu^2) - \dot{M} = 0.$$

(III-41)

Differentiation of equation (III-41) yields

$$\dddot{S} + \nu(\ddot{S} - \dot{M}) - \left(\frac{\ddot{S}}{\dot{S}} - \nu\right)\left[\ddot{S} - \nu(\dot{S} - M) - \dot{M} + R(1-\nu^2)\right]$$

$$- R\dot{S}^2 e^{-\nu\tau}\left\{(\dot{S}_0 + \nu + 2)\sin S + \int_0^\tau e^{\nu x}\left[(1-\nu^2)\cos(S(\tau) - S(x))\right.\right.$$

$$\left.\left. - 2\nu \sin(S(\tau) - S(x))\right] dx\right\} - \dot{M} = 0.$$

(III-42)

We substitute for the integral in equation (III-42) the expression obtained for it from equation (III-40), and obtain

$$\dddot{S} + \nu(\ddot{S} - \dot{M}) - \left(\frac{\ddot{S}}{\dot{S}} - \nu\right)\left[\ddot{S} - \nu(\dot{S} - M) - \dot{M} + R\dot{S}(1-\nu^2)\right]$$

$$+ \dot{S}^2(\dot{S} - M) - \dot{M} = 0,$$

(III-43)

and we have succeeded in converting equation (III-38) into a differential equation. Solutions of (III-38) must satisfy (III-43). If $\nu = 0$ and M is a constant, then equation (III-43) becomes

$$\dddot{S} - \frac{\ddot{S}(\ddot{S} + R)}{\dot{S}} + \dot{S}^2(\ddot{S} - M) = 0,$$

(III-44)

Multiplying by $1/\dot{S}^2$ makes equation (III-44) a perfect differential, whose primitive is

$$\frac{(\dddot{S}+R)}{\dot{S}} + \tfrac{1}{2}\dot{S}^2 - MS = C_1 ,$$

or

$$\dddot{S} + \tfrac{1}{2}\dot{S}^3 - MS\dot{S} + R - C_1\dot{S} = 0. \qquad \text{(III-45)}$$

If equation (III-45) is multiplied by $(\ddot{S} - M)$, the result is again a perfect differential, whose primitive is

$$\tfrac{1}{2}\ddot{S}^2 - M\dot{S} + \tfrac{1}{8}\dot{S}^4 - \tfrac{1}{2}MS\dot{S}^2 + \tfrac{1}{2}M^2\dot{S}^2$$

$$+ R\dot{S} - MR\tau + C_1(MS - \tfrac{1}{2}\dot{S}^2) = \tfrac{1}{2}C_2 . \qquad \text{(III-46)}$$

Attempts to find an integrating factor for (III-46) when $M \neq 0$ have not been successful, and for many purposes it appears that (III-46) is no more useful than (III-38). We started with an integro-differential equation, and by differentiating twice and substituting appropriately, we obtained a fourth order differential equation. By assuming $\nu = 0$ and M constant, we succeeded in performing two successive integrations, obtaining (III-38). Thus we have performed an equal number of differentiations and integrations, and, in addition, made restrictive assumptions, so that it appears we have made no net gain, or perhaps have lost on the exchange. Moreover, it appears that (III-46) is more convenient for numerical integration. The great value of (III-46) lies in the fact that for $M = 0$, it can be formally solved with an elliptic integral, and the motion analyzed in considerable detail. This case, in which $M = \nu = 0$, is the conservative case.

Solution of the Conservative Case

We make the substitution $\dot{s} = r$ in equation (III-46), and set $M = 0$:

$$\dot{r}^2 = -\tfrac{1}{4} r^4 + C_1 r^2 - 2R r + C_2 \,. \tag{III-47}$$

The formal solution to equation (III-47) is

$$\tau - \tau_0 = \int_{r_0}^{r} \frac{dx}{\sqrt{-\tfrac{1}{4}x^4 + C_1 x^2 - 2Rx + C_2}} \,, \tag{III-48}$$

which is an elliptic integral of the first kind. It can be put in Legendre's standard form

$$\int_0^{\varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \,, \tag{III-49}$$

where $0 < k < 1$, by a bilinear transformation of the form[7]

$$r = \frac{\alpha t + \beta}{\gamma t + \delta}$$

where the constants α, β, γ and δ depend on the roots of the quartic in r. The integral is fully tabulated.

Actually it is unnecessary to perform the transformation since most of the characteristics of the solution can be

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[7] H. and B. Jefferys, op. cit., p. 650.

discovered simply by investigation of the function

$$f(r) = \dot{r}^2 = -\frac{1}{4} r^4 + C_1 r^2 - 2Rr + C_2 \ .$$

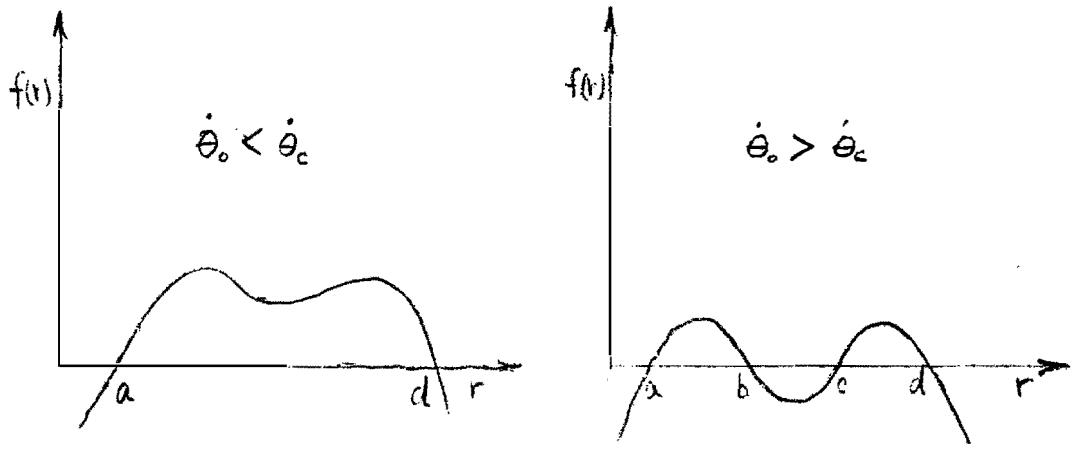
(III-49)

Since $\dot{r}$ is real, the motion is restricted to those values of r for which $f(r)$ is positive. Suppose r_1 and r_2 are two simple roots of $f(r)$, and $f(r) > 0$ between these roots. Suppose further that $\dot{r}$ starts with a given sign, say positive, and $r_1 < r < r_2$. Then $\dot{r}$ will remain positive until r reaches r_2. At this point the motion will reverse, and r will retrace its path toward r_1, with $\dot{r}$ now having a negative sign, since $\dot{r}^2$ cannot be negative. It is also impossible for $\dot{r}$ to remain zero at r_2, since $\ddot{r} = f'(r)$ is not zero at r_2, which is a simple root of $f(r)$. Thus the motion will be periodic between the simple roots r_1 and r_2, with period

$$P = 2 \int_{r_1}^{r_2} \frac{dr}{\sqrt{f(r)}} \ .$$

(III-50)

There are several possibilities for a quartic such as $f(r)$. It can have zero, two, or four real roots. Since $f(\pm\infty) = -\infty$, the case of no real roots is impossible for $f(r)$, since this would imply the motion could not exist anywhere, whereas in any actual case it must at least exist for the initial conditions. Hence $f(r)$ has the form shown in Figure 4 (a) or 4 (b).



(a) Case 1. Two real roots (b) Case 2. Four real roots

Figure 4. Possible forms of $f(r)$

If $f(r)$ has two real roots, the motion will be periodic between a and d in Figure 4(a), and if $f(r)$ has four real roots, the motion could exist either in the region $a \le r \le b$, or $c \le r \le d$ in Figure 4(b), but could not move from one region to the other. There exists the interesting possibility that changing the initial conditions and other parameters in the proper fashion may cause $f(r)$ to change from case 1 to case 2, with an abrupt change in period. This indeed does occur, as we now proceed to demonstrate by examining the positions of the roots of $f(r)$ as a function of the initial conditions and R, the coupling constant.

We assume the lateral vibrations to vanish initially, and the rotor to have an initial angular velocity $\dot{\theta}_0 = k_0 + 1$. This situation can be obtained by the sudden release of constraints at certain points on a rotating shaft. With these initial conditions, it is obvious from physical considerations

that r_0 must be the largest root of $f(r)$, since the rotor certainly will not increase its speed as the lateral vibrations begin to grow.

We now evaluate C_1 and C_2 in terms of the initial conditions, using equations (III-45) and (III-46). Thus

$$C_1 = \tfrac{1}{2}\dot{S}_0^2 + \frac{(\dddot{S}_0 + R)}{\dot{S}_0} ,$$

and

$$C_2 = \ddot{S}_0^2 + \tfrac{1}{4}\dot{S}_0^4 + 2R\dot{S}_0 - C_1\dot{S}_0^2 .$$

(III-51)

We want C_1 and C_2 in terms of $r_0 = \dot{S}_0$ and R. Thus $\dddot{S}_0$ and $\ddot{S}_0$ are needed, and can be obtained from (III-40) and (III-41)

$$\dddot{S}_0 = -R - R\,r_0(r_0+2) ;$$

$$\ddot{S}_0 = 0 \qquad \text{(III-52)}$$

Substituting (III-52) in (III-51), we obtain

$$C_1 = \tfrac{1}{2}r_0^2 - R(r_0+2)$$

$$C_2 = -\tfrac{1}{4}r_0^4 + R\,r_0^2(r_0+2) + 2Rr_0 .$$

(III-53)

Let $r_0\,C_3 = 4C_2$, then

$$C_3 = -r_0^3 + 4R\, r_0(r_0+2) + 8R\,. \tag{III-54}$$

Thus

$$f(r) = \tfrac{1}{4}r^4 + (\tfrac{1}{2}r_0^2 - R(r_0+2))r^2 - 2Rr + \tfrac{1}{4}C_3 r_0\,. \tag{III-55}$$

One root of (III-55) is r_0. Note that, if $R=0$, $f(r) = \tfrac{1}{4}[(r-r_0)(r+r_0)]^2$, and there is a double root at $r=r_0$. This means the rotation remains constant, which is not surprising if the coupling vanishes.

Consider the equation

$$f_1(r) = \frac{-4fr}{r-r_0} = r^3 + r_0 r^2 - (r_0^2 - 4R(r_0+2))r + C_3 = 0. \tag{III-56}$$

If $C_3 = 0$, then $r=0$ is a root, and

$$\frac{f_1(r)}{r} = r^2 - r r_0 - (r_0^2 - 4R(r_0+2)) = 0\,. \tag{III-57}$$

From $C_3=0$ we obtain $R = \frac{r_0^3}{4[r_0(r_0+2)+2]}$, which, when substituted in (III-57), yields an equation with roots approximately at $-\tfrac{3}{2}r_0$ and $\tfrac{1}{2}r_0$. Thus, for $C_3=0$, the roots are at $-\tfrac{3}{2}r_0, 0, \tfrac{1}{2}r_0$ and r_0; and this is an example of case 2, Figure 4(b).

The case when the two inner roots are equal is termed the critical case. We denote the critical initial angular velocity by $\dot\theta_c = r_c + 1$. This is not to be confused with the

critical speed, which will always be lower. For $r_0 < r_c$, the motion will be between the roots c and d in Figure 4(b), and, for $r_0 < r_c$, the motion will be between a and d in Figure 4(a) and will have a longer period. Also, the lateral vibrations will have a much larger maximum amplitude because of the greater swing in r in Figure 4(a).

We denote the double root of the critical case by γ, and it is easy to prove that it is always positive if r_c is. The proof consists first in observing that $f(r)$ has no cubic term, and therefore the sum of the roots of $f(r)$ is zero. We put $-r_c$ for r in $f(r)$, and check the resulting sign of $f(r)$:

$$f(-r_c) = -\tfrac{1}{4} r_c^4 + (\tfrac{1}{2} r_c^2 - R(r_c+2)) r_c^2 + 2Rr_c + R r_c^2 (r_c+2) + 2R r_c$$

$$= 4 R r_c . \qquad \text{(III-58)}$$

Thus, $f(-r_c) \geq 0$ if $r_c \geq 0$, since R is never negative. But $f(-\infty) = +\infty$, hence the one remaining root, a, satisfies $a \leq -r_c$. Because of the absence of the cubic term in $f(r)$, we have $a + 2\gamma + r_c = 0$, hence γ must be ≥ 0 if $a \leq -r_c$. It will be seen shortly that $r_c > 0$ for all $R > 0$, which is certainly expected physically. The fact that $\gamma > 0$ means that the rotational speed of the rotor never reaches the critical speed $r = 0$ in any case in which $r_0 > r_c$.

It is desirable to know the relation which exists

between R and r_0 for the critical case, i.e., when $f(r)$ has the double root γ. Using the theory of cubic equations, we show in the appendix that

$$R = \frac{4r_c^3}{27 + 36r_c + 14r_c^2 - 4r_c^3 - r_c^4} \tag{III-59}$$

is the desired relation, where r_c is the value of r_0 which causes $f(r)$ to have a double root.

The Period as a Function of the Initial Condition

The question arises as to whether the period (III-50) approaches infinity as $r_0 \to r_c$ from either direction. This question can be answered by considering the contribution to the integral in the neighborhood of γ as the minimum approaches the axis from above or below.

When the minimum approaches the axis from above, i.e., $r_0 \to r_c$ from the right, we integrate over r between γ and $\gamma+\varepsilon$, where γ is the value of r_0 at which the minimum eventually touches the axis. In this case $f(r)$ will have two real roots at a and d as in Figure 4(a), and two conjugate complex roots, which we denote by $\gamma+i\beta$ and $\gamma-i\beta$. Then

$$\begin{aligned} f(r) &= (r-a)(r-b)(r-(\gamma+i\beta))(r-(\gamma-i\beta)) \\ &= (r-a)(r-b)(r^2-2\gamma r+\gamma^2+\beta^2). \end{aligned} \tag{III-60}$$

From (III-50), we consider

$$P_c = \int_{\gamma}^{\gamma+\varepsilon} \frac{dx}{\sqrt{X}\sqrt{x^2-2\gamma x+\gamma^2+\beta^2}} \tag{III-61}$$

as $\beta \to 0$. P_c denotes the contribution to the period over the interval $\gamma < r < \gamma+\varepsilon$. We chose ε small enough that $X = (x-a)(x-b)$ has no roots in the interval, hence $\frac{1}{\sqrt{X}}$ has a lower bound M in the interval. Thus,

$$P_c \geq M \int_{\gamma}^{\gamma+\varepsilon} \frac{dx}{\sqrt{x^2-2\gamma x+\gamma^2+\beta^2}} . \tag{III-62}$$

The integral (III-62) can be evaluated if $\beta \neq 0$ and has the value

$$P_c \geq M \ln\left(\frac{+\sqrt{\varepsilon^2+\beta^2}+\varepsilon}{\beta}\right). \tag{III-63}$$

As $\beta \to 0$, $P_c \to \infty$. Thus the period of the motion becomes infinite as $r_0 \to r_c$ from the right.

We now examine the opposite case, where the minimum approaches the axis from below, or $r_0 \to r_c$ from the left. If we let $\beta = i\beta'$, then (III-62) becomes

$$P_c \geq M \int_{\gamma+\beta'}^{\gamma+\beta'+\varepsilon} \frac{dx}{\sqrt{x^2-2\gamma x+\gamma^2-\beta'^2}} , \tag{III-64}$$

where we integrate from the larger real root $\gamma+\beta'$ toward r_0. Evaluating (III-64), we have

$$P_c \geq M \ln\left(\frac{+\sqrt{\mathcal{E}^2+2\mathcal{E}\beta'}+\beta'+\mathcal{E}}{\beta'}\right) ,$$

(III-65)

and again $P_c \to \infty$ as $\beta' \to 0$, or $r_0 \to r_c$ from the left.

Thus we see that $P \to \infty$ as $r_0 \to r_c$ from either direction. However, the period falls off more slowly as r_0 moves to the left away from r_c than when moving to the right away from r_c . This occurs for two reasons: the representative point has farther to go between the roots a and d in Figure 4(a) than between c and d in Figure 4(b), and it must traverse a longer region of low velocity $\dot{r}$. If we plot the period against r_0 for a given R , we obtain a graph of the general form indicated in Figure 5.

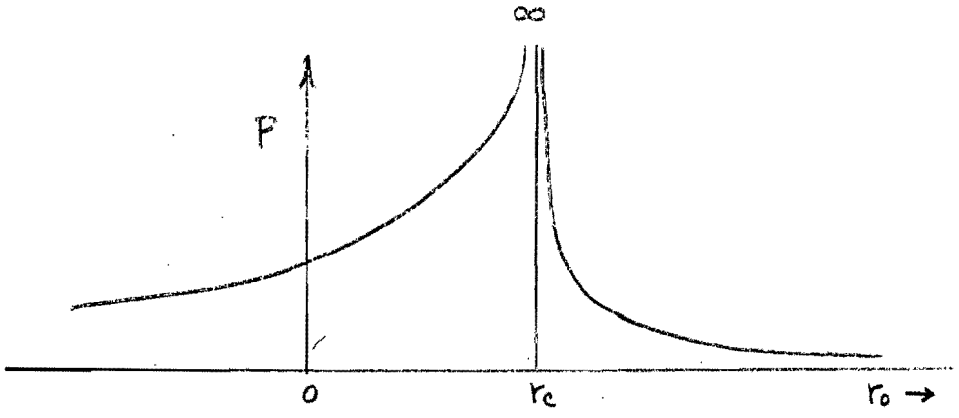


Figure 5. Qualitative graph of the period as a function of r_0 for fixed R .

The period rises very steeply near r_c , since it goes to infinity logarithmically, and hence does not attain high values until β is very small, or r_0 is very close to r_c . It is doubtful whether the sharp rise in period could even be

detected experimentally, since the slightest disturbance would tend to alter the system to a state which could be attained from a different, shorter period, set of initial conditions. Even the effect on the motion from other resonances would probably prevent very long periods.

Amplitude as a Function of Initial Conditions

If we plot amplitude of the rotational motion against r_0 for a given R, we obtain a graph like that displayed in Figure 6.

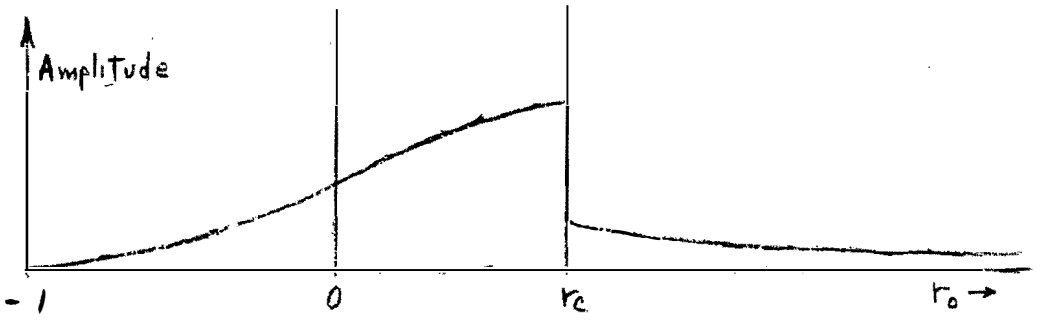


Figure 6. Qualitative graph of the amplitude as a function of r_0 for fixed R.

There is a finite discontinuity, or "jump," in the amplitude at $r_0 = r_c$ because the motion changes its turning points from a and d in Figure 4(a) to c and d in Figure 4(b). The motion must vanish at $r_0 = -1$ since there is no rotation at this point, and above r_c the amplitude approaches $4R$ as r_0 becomes very large. This can be seen by examining the function $f_1(r)$, (III-56), after substituting for C_3 from (III-54):

$$f_1(r) = r^3 + r_0 r^2 - r_0^2 r - r_0^3 + 8R(r_0+2)(r_0+r) + 8R. \qquad \text{(III-66)}$$

We make the substitution $r = r_0 - \alpha$, and obtain the cubic

$$\varphi(\alpha) = (16R - 4\alpha)r_0^2 + 4r_0\alpha^2 - \alpha^3 - 8R(r_0+2)\alpha + 8R .$$

As $r_0 \to \infty$, the coefficient of r_0^2 approaches zero for that root of $\varphi(\alpha)=0$ corresponding to the root c in Figure 4(b). Hence $\alpha \to 4R$ as $r_0 \to \infty$.

Typical values of R. In order to have a better idea of the magnitude of the coupling constant R, it is desirable to calculate its value for several typical cases. Before doing this however, we point out a relationship between the initial energies of rotation and vibration and R which may be of use in finding R in some cases. It was shown in equation (III-27) that, for $T(0)=\dot{T}(0)=0$, there is still a residual lateral motion which occurs because of the rotation of the equilibrium configuration. From (III-27), the energy of this initial motion, for the mode of vibration being considered, is

$$E_{Vi} = \tfrac{1}{2}\int_0^1 (\dot{\xi}^2 + \dot{\eta}^2)\rho_0 D(u) U^2(u)\,du \cdot \omega^2$$

$$= \frac{l}{2}\,\dot{\bar{T}} * \dot{\bar{T}}\, l^2\omega^2 \cdot \rho_0$$

$$= \frac{l}{2}\,\Delta_c^2\, l^2 \rho_0\, \omega^2\, \theta_0^2 .$$

(III-67)

The initial energy of the rotation is

$$E_{Ri} = \frac{l}{2}\, I_{aa}\, \dot{\theta}_0^2\, \omega^2 .$$

(III-68)

The ratio of the quantities E_{V_i} and E_{R_i} is

$$\frac{E_{V_i}}{E_{R_i}} = \frac{\Delta_c^2 \ell^2 f_o}{I_{ave}} \quad . \tag{III-69}$$

If we compare (III-69) with (III-39) we find

$$\frac{E_{V_i}}{E_{R_i}} = 2R \tag{III-70}$$

if f_o , the lateral motion damping force, vanishes.

Some amplification of the exact meaning of the quantities E_V and E_R may be helpful. We expand the function giving the central line of the rotor in terms of the lateral motion eigenfunctions and consider the eigenfunction of the mode near whose frequency the rotor is turning. Consider all of the mass of the rotor to be condensed on the line of this eigenfunction and the line to be rotated rigidly. The kinetic energy of this motion, plus the potential energy (which vanishes in E_{V_i}) involved in bending the rotor to this shape, is E_V . Now consider the entire central line (not the eigenfunction component) to be fixed rigidly in space, and the elemental wafers which compose the rotor to rotate about this line. The total energy of this motion is E_R , which is entirely kinetic energy.

Unfortunately, it is not easy to determine E_{V_i} for a single mode. Sometimes the value of $\sum_n E_{V_i n}$, where the summation is over all the modes, can be determined, and thus an upper limit to R can be established. R , of course, may be

different for each mode, and the estimate $\sum_n E_{Vin}/E_{Ri}$ is an estimate of $\sum_n 2R_n$. If one R is very much larger than the rest, then $\sum_n E_{Vin}/E_{Ri} \cong 2R$.

As an example we calculate the value of R for a naturally bent circular shaft. For simplicity, assume the shaft to be bent in the shape of its first eigenfunction for vibration with both ends free. Referring to Rayleigh[8] we find the eigenfunction to be, for the first mode,

$$U = (\sin 4.73 - \sinh 4.73)\{\cos 4.73u + \cosh 4.73u\} - (\cos 4.73 - \cosh 4.73)\{\sin 4.73u + \sinh 4.73u\}. \tag{III-71}$$

Normalization of (III-71) gives us

$$U = (\cos 4.73u + \cosh 4.73u) - 1.02366(\sin 4.73u + \sinh 4.73u). \tag{III-72}$$

We take as a measure for the deviation from straightness the quantity Δ_{max}, defined by

$$\Delta_{max} = \Delta_c |U_{max} - U_{min}|. \tag{III-73}$$

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[8]Lord Rayleigh, op. cit., p. 219.

The quantity Δ_{max} is thus the ratio of the maximum departure of the eigenfunction from a straight line coinciding with it at the ends to the length l. Let us consider a shaft which is one hundred inches long, and is one tenth of an inch out of line. Then $\Delta_{max}=.001$ for this case. From (III-72), we find $|U_{max} - U_{min}|=2.7711$. Thus

$$\Delta_c^2 = \left(\frac{\Delta_{max}}{2.7711}\right)^2 = .1302\,\Delta_{max}^2 . \qquad \text{(III-74)}$$

For $\Delta_{max}=.001$,

$$\Delta_c^2 = .1302\times 10^{-6} .$$

Assume the shaft has a diameter of three inches and, thus, has a radius of gyration $3/\sqrt{2}$ units. Then $I_{ave}=\rho_0\, 9/2$ units, and the coupling constant R has the value

$$R = \frac{\rho_0(100)^2 \cdot .1302\times 10^{-6}}{\rho_0\, 9/2}$$
$$= .000289 .$$

If the deviation from straightness is doubled, R is quadrupled. Hence $R\sim.001$ may be taken as a typical value.

As a second example, consider the same shaft with a small mass m attached at the midpoint two inches from the central line. Suppose $m<<M$, where M is the mass of the shaft. If we compute R by (III-70), we obtain

$$R = \frac{m}{M} \cdot \frac{4}{9} 10^4 ,$$

This is actually $\sum_n R_n$, since we have not attempted to find E_{Vi} for the first mode alone. We now find the correct value of R_1 for the first mode. Suppose

$$\Delta(u) = 2 \qquad \text{for} \quad \frac{\ell-\delta}{2} < u < \frac{\ell+\delta}{2}$$

and $$\Delta(u) = 0 \qquad \text{for} \quad 0 \le u \le \frac{\ell-\delta}{2}, \quad \ell \ge u \ge \frac{\ell+\delta}{2} .$$

Also, let $m = \rho_m \delta$, where ρ_m is constant.

Then

$$\Delta_c = \rho_m \int_0^1 \Delta(u)\, U(u)\, du .$$

Now let $\delta \to 0, \rho_m \to \infty$ in such a fashion that m remains constant. Then Δ_c becomes

$$\Delta_c = \frac{m}{\delta} \cdot 2 \cdot \delta \cdot U(1/2) = 2m\, U(1/2) .$$

From (III-72), $\qquad U(1/2) = .7711 .$

Thus, from (III-39),

$$R_1 = \frac{m}{M} \cdot \frac{4}{9} \cdot 10^4 \cdot (.7711)^2 ,$$

or $$R_1 = .59 \sum_n R_n .$$

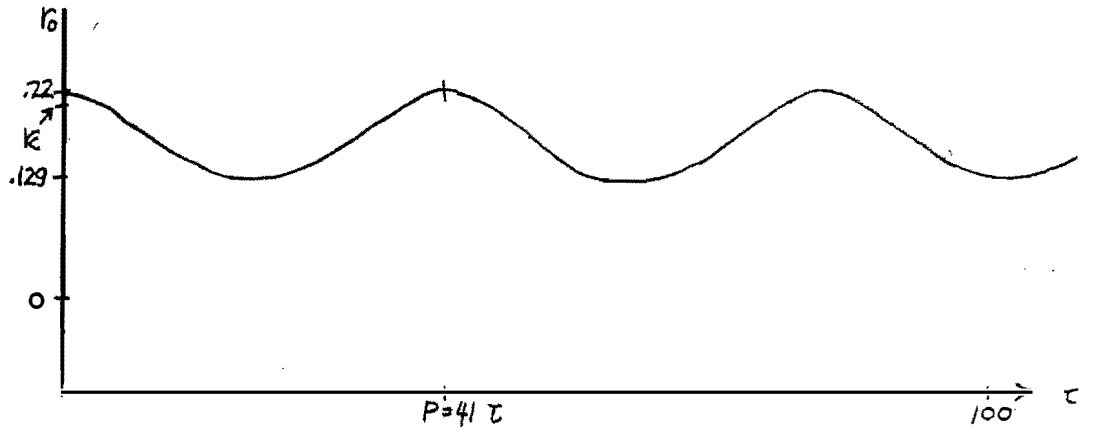
We see that the value of R from (III-70) is too large by 1/0.59, or nearly 70 percent. However, it does give a good upper bound on R without knowing the eigenfunctions. If the small mass m is at some other point besides the middle of

the rotor, R will be less, and higher modes will have more coupling.

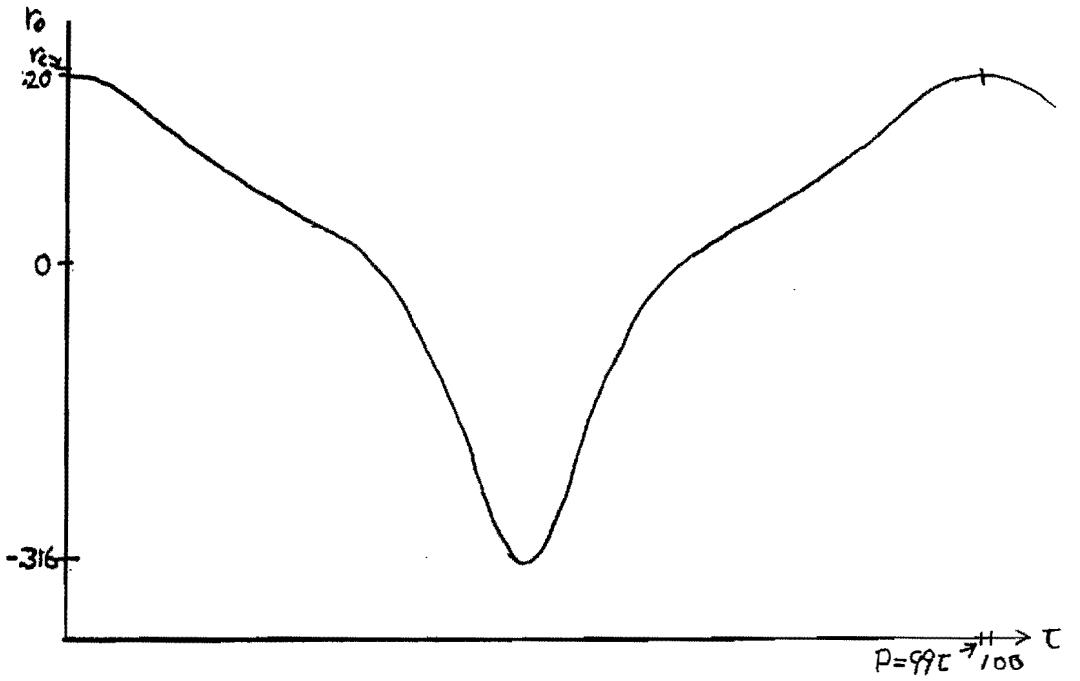
R has been considered here for the conservative case, but it does not change when external forces are applied, except very slightly, as indicated by (III-39), when there is damping of the lateral motion. This change can readily be seen to arise from the slight shift in natural frequency caused by damping.

Numerical integration for a typical case. Equation (III-38) has been integrated numerically for $R=.001$ and $R=.000333$ for several values of r_0, all with $M=N=0$. These integrations were performed before the solution was discovered. For $R=.001$, it was found that $.2<r_c<.22$, and, for $R=.000333$, r_c was slightly less than .14. From (III-39), the exact values of r_c are calculated to be: $R=.001$, $r_c=.20668$, and $R=.000333$, $r_c=.13905$. The cases for $R=.001$, $r_0=.20$ and .22 are displayed in Figure 7(a) and (b). From Figure 7(a) we note that the period for $r_0=.22$ is about $41.0\,\tau$, and for $r_0=.20$ is about $99\,\tau$, or 2.42 times longer. Also, the amplitude of the rotational motion for $r_0=.22$ is about .091; while for $r_0=.20$, the amplitude is .517, or 5.68 times as great. The amplitudes can also be calculated by solving (III-56) for the appropriate root, and the results agree almost exactly, indicating that the numerical integrations are fairly accurate.

The lateral vibrations. In order to obtain the lateral motion, we could take the solution for θ from equation



(a) $R = .001$, $r_0 = .22$, $r_c = .20603$.



(b) $R = .001$, $r_0 = .20$, $r_c = .20603$.

Figure 7. Numerical integration of equation (III-38) .

(III-38), invert to obtain elliptic functions, substitute it in T in equation (III-26), and solve the resulting equation. In view of the fact that elliptic functions are hard to manage and, moreover, would appear in the equation for T as arguments of trigonometric functions, this procedure could probably be followed only with numerical methods. There is, however, a much easier method which yields the envelope of the lateral motion and is based on conservation of energy.

Let E be the total energy. Then

$$E = E_V + E_R \,, \tag{III-75}$$

where E_V and E_R are the vibrational and rotational energies which we have previously defined. Also, $E_V/E_R = 2R$ from equation (III-70).

We want to find the amplitude of the vibration in a particular mode as a multiple of the equilibrium displacement of the mode, Δ_c. Because of the axial symmetry of the end conditions, the vibration may be considered as two equal perpendicular harmonic vibrations ninety degrees out of phase, which produce a circular motion of the central line. For the case in which the equilibrium configuration is a straight line, and the vibrations are of constant amplitude, the kinetic and potential energies are equal and constant since the kinetic energy does not change for circular motion and the average kinetic and potential energies are equal for simple harmonic motion. If the radius of the circle varies

slowly compared to the rotation speed, the kinetic and potential energies are slowly varying, but still approximately equal. The total energy of vibration, for a straight equilibrium control line, is proportional to the square of the amplitude. For a curved equilibrium central line, this is also very nearly true if the amplitude is large compared to the equilibrium central line amplitude. Since we wish to obtain the ratio of vibrational amplitude to the equilibrium amplitude, we must compare the energies of vibration of a rotor with a straight central line (or one whose eigenfunction component for the particular resonance examined is zero) which has the same amplitude as the rotor with the curved equilibrium central line. A straight rotor vibrating with amplitude Δ_c has energy $2E_{V_i}$; a rotor with the equilibrium central line amplitude Δ_c has energy E_{V_i}, since it has no potential energy. Thus

$$V^2 = \frac{E_V}{2E_{V_i}} \tag{III-76}$$

is the expression we seek, where V is the ratio of the amplitude to the equilibrium amplitude. It is not correct for $E_V \sim 2E_{V_i}$, but is correct for $E_V \gg 2E_{V_i}$. Generally the displacements will be sufficiently large to make (III-76) a good approximation.

Using (III-75) and (III-70) we obtain

$$V^2 = \frac{E - E_R}{2E_{Vi}} = \frac{E - E_R}{4RE_{Ri}} \quad . \qquad \text{(III-77)}$$

Also, $E = E_{V_i} + E_{Ri} \cong E_{Ri}$. Hence (III-77) becomes

$$V^2 = \frac{E_{Ri} - E_R}{4E_{Ri}} \quad ,$$

or

$$V^2 = \frac{\dot{\theta}_o^2 - \dot{\theta}^2}{4R\dot{\theta}_o^2} \quad . \qquad \text{(III-78)}$$

Using $\dot{\theta} = r+1$, we obtain

$$V = \sqrt{\frac{(r_o+1)^2 - (r+1)^2}{4R(r_o+1)^2}} \qquad \text{(III-79)}$$

for the lateral motion amplitude in terms of the initial amplitude. The maximum amplitude is attained when r reaches the root of $f(r)$ which is next to r_o . Thus, if we call this root γ ,

$$V_{max} = \sqrt{\frac{(r_o+1)^2 - (\gamma+1)^2}{4R(r_o+1)^2}} \quad . \qquad \text{(III-80)}$$

The greatest value of V_{max} occurs when γ changes from the double root of the critical case to the root a in Figure 4(a). We denote V_{max} in this case by V_{max_c}, which is the largest possible amplitude for a given R . For $R = .001$, $r_c = .20605$, then $\gamma = -.52$, approximately. Hence

$$V_{max_c} = \sqrt{170.5} = 13.07 \quad ,$$

which means, in the critical case, that the time-dependent coefficient of the eigenfunction increases periodically to a value 13.07 times its initial value. Since Δ_c is the initial value, $13.07\,\Delta_c$ is the maximum value of the coefficient for $R = .001$. Lower maximum values of the coefficient will be obtained for initial rotation speeds different from r_c.

Constant Torque Acceleration

The Θ equation, (III-38) apparently cannot be treated analytically when $M \neq 0$ or $\nu \neq 0$. The case when M is a constant, representing a constant torque applied to a rotor, is fairly important since it approximates the actual conditions under which many types of rotors are brought to their operating speed. Therefore the behavior of equation (III-38) for several values of M and R has been examined with an analogue computer.

The quantity M is a dimensionless torque, and it is desirable to examine the definition in order to determine the size of the units of M. From equation (III-39), we have

$$M = \frac{\rho_o \ell^2}{\omega^2 b^4 I_{ave}} G(\tau).$$

We recall that $G(\tau)$ in equation (III-1) was obtained by integrating $g(u,\tau)$, which was defined in (II-38), and adding to it the terminal torques arising from the evaluation of certain small terms at the endpoints. Since $g(u,\tau) = \frac{\ell^2}{S_o} Q_\theta(u,\tau)$ from equation (II-38), we have

$$M = \frac{\rho_0 l^2}{\omega^2 b^4 I_{ave}} \cdot \frac{l^2}{S_0} \cdot m_{ave} ,$$

(III-81)

where m_{ave} is the average external torque per unit length. We use $b^4 = \frac{\rho_0}{S_0} l^4$ from equation (II-36), and (III-81) becomes

$$M = \frac{m_{ave}}{I_{ave} \omega^2} .$$

(III-82)

Since ω^2 has been chosen to be the resonant frequency, we see that M is just the ratio of the total external torque to twice the kinetic energy of rotation at the critical speed.

A constant torque acting on the rotor means that there is an energy input proportional to the speed. Since there is a continuous energy input, either the rotational motion or the vibrational motion is unbounded if there is no friction. Because the rotational speed must remain near a critical speed if the vibrational motion is unbounded, rotational and vibrational motion cannot both be unbounded.

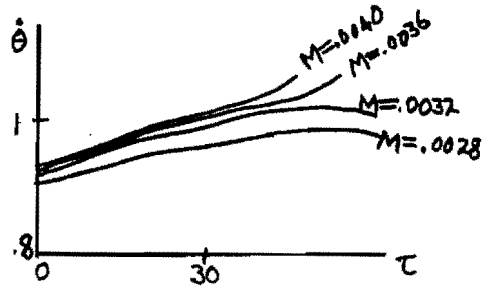
The computer analysis shows that, for a given value of R and for the system starting from rest or a speed well below the critical speed, there exists a critical positive value of M, denoted by M_c, such that, if $0 < M < M_c$, the vibrational motion is unbounded, and, if $M > M_c$, the rotational motion is unbounded.

This phenomenon can be qualitatively understood from physical reasoning. The amplitude of an undamped harmonic oscillator increases linearly with time when it is driven

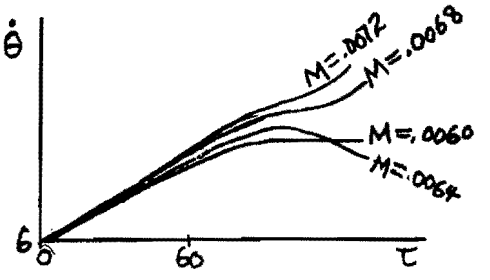
at its natural frequency, and the amplitude becomes large, but is bounded, at speeds near the natural frequency. Since the total energy is proportional to the square of the amplitude, this means that the rate of energy absorption at resonance is proportional to the amplitude. Thus, if the torque is sufficiently large or the coupling is sufficiently small, the rotor accelerates through the critical speed without exciting the lateral vibrations to the point where their power absorption is greater than the input power, and the rotor continues to accelerate monotonically. If, on the other hand, the torque is sufficiently small or the coupling sufficiently large, then the vibrations grow to the point where their power absorption is greater than the input power. When this happens, the acceleration becomes negative, and the lateral vibrations grow at the expense of the rotational energy as well as the input power, until the rotational speed is sufficiently different from the critical speed to stop the growth of the vibrations. The computer solutions show that if $0 < M < M_c$, the rotational speed becomes "trapped" in the neighborhood of the critical speed and appears to oscillate about the critical speed with a varying amplitude.

Figure 8 illustrates the solution with their values of M and R, as taken from the computer.

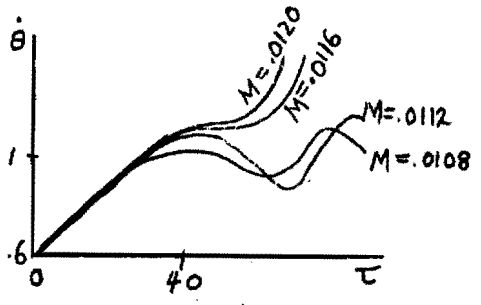
Figure 9 is a graph of M_c vs. R, plotted from the approximate values of M_c taken from Figure 8.



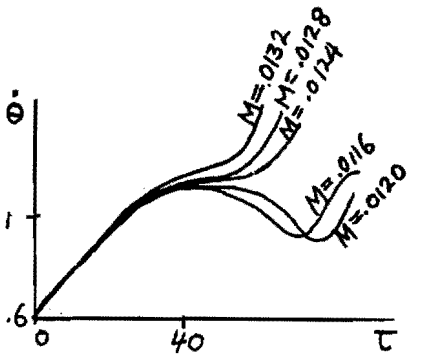
(a) $M_c = .0034$
$R = .0001$



(b) $M_c = .0066$
$R = .00033$



(c) $M_c = .0114$
$R = .001$



(d) $M_c = .0122$
$R = .002$

Figure 8. Computer solution of equation (III-38) for constant torque M near the critical torque M_c.

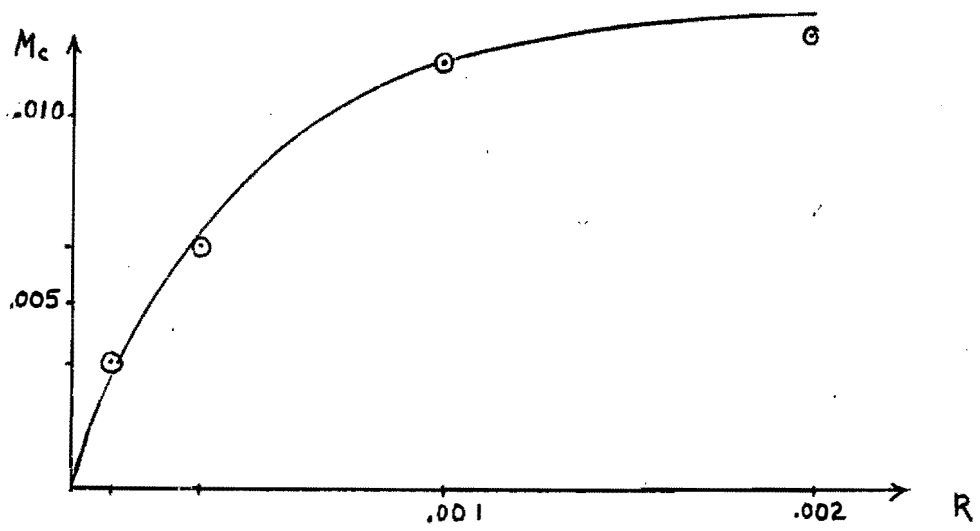


Figure 9. The critical torque M_c vs. the coupling constant R.

There is still a question as to whether or not the rotational speed, after becoming trapped, might not subsequently break out of the resonance neighborhood and continue to increase. Several solutions appeared to do this when $M \lesssim M_c$. Unfortunately, the limitations of the computer prevented an analysis of this point. The several times that it did happen might well have been due to an accumulation of errors by the computer, which was not particularly accurate. The value of M_c was taken to be the value at which the speed failed to increase monotonically through the resonance, since this was usually as far as the computer's limited range of variables would allow. For practical purposes this is probably a good choice, since the lateral vibrations become quite large if the speed has a local maximum in the neighborhood of the critical speed, and it is generally the smallest value of the torque which will avoid large vibrations that it is useful

to know. Whether or not this value of M_c is the value for which the speed is permanently trapped in the neighborhood of the critical speed has not been answered.

CHAPTER IV

SUMMARY

Review of Aims and Results

We set out at the beginning of Chapter II to derive the proper equations of motion for an unbalanced rotor being accelerated through a critical speed. The approach employed differed from others on the subject in that the equations were desired for the case in which the torque, rather than the acceleration, is specified. In order to obtain these equations, Lagrange's equations for a continuous medium were employed, after first being extended slightly to include a system in which the potential energy density depends on the second spatial derivative. The potential and kinetic energy densities were developed in a fairly general form and substituted in the extended Lagrange's equations to obtain three partial differential equations of motion: two which govern the lateral motion and one which governs the rotational motion. The rotational motion equation was reduced to an ordinary differential equation by assuming the torsional motion negligible and averaging over the length. This averaging permitted the separation of the lateral motion equations into time- and space-dependent ordinary differential equations. The spatial equation was the same as the well known one representing the lateral vibrations of a non-rotating bar.

The time-dependent equations for the displacement

coordinates were then examined in some detail. The equations for the lateral motion were obtained in a slightly more general form than previously reported since they included the effect of skew stiffness. The third equation, which governs the rotational motion, is new. It allows the external torque to be specified. It was found that in the general case each equation involved all three displacement coordinates, but that if the rotor did not possess skew stiffness, the rotational motion equation could be obtained, by substitution from the lateral motion equations, as an integro-differential equation in the rotation coordinate, θ, alone. This equation was examined and numerically integrated for several sets of initial conditions in the conservative case, in which there is no friction or external torque. An analogue computer was employed to examine the behavior in the case of a constant external torque with no friction.

It was discovered the the non-linear integro-differential equation in θ could be reduced to a fourth order differential equation, which could be integrated twice if the external torque were constant and there were no damping of the lateral vibrations. It was found that in this (conservative) case, the differential equation could be completely solved in terms of an elliptic integral. This case was examined in some detail, without evaluating the integral, since every property of the motion except the period can be deduced by inspection of the differential equation whose formal solution is the elliptic integral.

The motion in the conservative case consists of a periodic variation in the rotation speed coinciding with the growth and decay of the lateral vibrations. The solution, considered as a function of the initial conditions, exhibits a remarkable behavior. It was found that there exists a critical initial angular velocity, above which the rotational speed variations have a small amplitude and below which they have a large amplitude, with a finite discontinuity in the amplitude at the critical value. Also, the period of the motion approaches infinity as the critical initial angular velocity is approached.

In the case of a constant external torque, the analysis with an analogue computer shows that there exists a critical torque for a rotor with given unbalance to accelerate monotonically through a critical speed. For torques less than the critical torque, the rotational acceleration is at least temporarily "trapped" in the neighborhood of the critical speed with the consequent growth of large lateral vibrations. For torques greater than the critical torque, the rotor increases rotational speed monotonically through the critical speed and, therefore, has much less time to build up lateral vibrations. Knowledge of the critical torque is of value in designing rotating machinery.

Further Research

Further investigation is indicated in several directions. In the conservative case, an examination of the motion

should be made for a rotor in which the lateral vibrations are not assumed to vanish initially, as they were in the present work. This extension should be fairly easy and should result in a different dependence of the critical initial rotation speed on the initial conditions.

In the non-conservative case, torque functions other than constants may be examined and are directly applicable to actual mechanical systems. Also, the effect of damping of the lateral vibrations should be included. The effect of skew stiffness should be examined, although the complications which result from its inclusion appear formidable. In addition, the case of axially asymmetric end conditions might be examined in both the conservative and non-conservative cases.

Experimental Verification

It would be of interest to know if the motion predicted for the conservative case could be physically realized in an experiment. Something approaching this has already been done. Robertson[1] reported an experiment which, although it does not meet the conditions of the conservative case exactly, nevertheless closely approximates the type of motion predicted. His apparatus consisted of a small D. C. motor driving a long shaft supported in bearings at each end. The motor speed could be regulated with a rheostat. It was found that, for certain settings of the rheostat for which the speed of the motor was

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[1]D. Robertson, Phil. Mag. 21 (1936), p. 492 ff.

in the neighborhood of the critical speed, the speed varied periodically. As the rheostat resistance was reduced, the period first increased, then became shorter as the average speed approached the critical speed.

The justification for considering the experiment to approximate the conservative case consists in observing that when the motor and shaft are running at a steady speed, the power from the motor just equals the power lost in friction. If the speed varies periodically, and the motor torque remains approximately constant over the period, then the system is approximately conservative in the sense that, if the torque nearly balances the frictional forces at all speeds, the energy of motion remains nearly constant over the period. The degree to which the motor torque balances the frictional forces throughout the period is the degree to which the motion should approximate the conservative case. If the average speed remains constant, then the torque averaged over the period must balance the frictional forces averaged over the period, but this of course does not mean the system is conservative in the sense that the torque just balances friction at all times. In nearly all D. C. motors, the torque does vary with the speed, therefore the approximation of a conservative system is fairly crude.

Robertson reports that the lower limit of the speed oscillations must always be below the critical speed, while the upper limit may lie either above or below the critical speed. This is in accord with the theory, if large amplitude

oscillations occur. But the theory also predicts speed oscillations with both limits above the critical speed. These oscillations would have small amplitude in both the lateral and rotational motion, which may account for the fact that they were not observed, or it may be that the system differs sufficiently from the conservative case to cause this feature to vanish entirely.

Robertson offers an explanation which he considers unsatisfactory, since it implies the speed oscillations should die out, whereas they apparently persist indefinitely. There seems little doubt that the motion is quite similar to that predicted in the conservative case.

Another example of apparently the same type of motion has been reported by Orr.[2] A shaft running in three bearings and electrically driven was observed to have lateral vibrations whose amplitude varied with a period about twenty times the natural period of the shaft. No explanation is offered, and apparently the concomitant variations in rotation speed escaped notice since they were not mentioned.

In the case of constant torque acceleration, the phenomenon referred to as "trapping," in which the rotational speed remains in the neighborhood of the critical speed if the torque is below the critical value, should be experimentally verified. Of course, in any actual system the lateral vibrations cannot be permitted to increase indefinitely, but it should be possible to establish the existence of the phenomenon.

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[2]W. McF. Orr, Phil. Mag. (6) 45 (1923), p. 712.

BIBLIOGRAPHY

BIBLIOGRAPHY

Baker, J. G. "Mathematical Machine Determination of Vibration of an Accelerated Unbalanced Rotor," Trans. ASME, APM 61, 1939, p. A 145.

Goldstein, H. Classical Mechanics. Cambridge, Mass.: Addison-Wesley Press, Inc., 1951.

Jeffreys, H. and B. Methods of Mathematical Physics. Cambridge: Cambridge University Press, 1950.

Lees, S. "The Whirling of an Overhung Eccentrically Loaded Shaft," Phil Mag. (6) 45, 1923, pp. 708-714.

Lewis, F. M. "Vibration During Acceleration through a Critical Speed," Trans ASME, APM 54, 1932, p. 24 ff.

Love, A. E. H. The Mathematical Theory of Elasticity, 4th ed. New York: Dover Publications, 1944.

Orr, W. McF. "Note on Mr. Lees' Paper on the Whirling of an Overhung Shaft," Phil. Mag. (6) 45, 1923, pp. 708-714.

Rankine, W. J. M'Q. "The Centrifugal Whirling of Shafts," The Engineer, xxvii, 1869, pp. 219-36.

Rayleigh, J. W., Strutt, Baron. The Theory of Sound, 2nd ed. London: Macmillan and Co., 1937, vol. 1.

Robertson, D. "Subsidiary Whirling of Rotor due to Speed Oscillation," Phil. Mag. (7) 20, 1936, pp. 474-501.

_____. "Transient Whirling of a Rotor," Phil. Mag. (7) 20, 1935, pp. 793-825.

_____. "The Whirling of Shafts," The Engineer, clviii, 1934, pp. 216-17, 228-9, 231.

_____. "The Whirling of a Shaft with Skew Stiffness," The Engineer, clvi, 1933, pp. 152-3, 179-81, 213-4.

Tamarkin, J. D. and Feller, W. Partial Differential Equations. Providence, R. I.: Brown University, 1941.

Timoshenko, S. Vibration Problems in Engineering. New York: D. Van Nostrand & Co., 1928.

White, W. T. "An Integral Equation Approach to the Problem of Vibrating Beams," J. Franklin Inst. 245, 1948, pp. 25-36, 117-34.

APPENDIX

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We now proceed to develop the relation between R and r_0 which causes $f(r)$ to have the double root γ, which is the critical case. The cubic $f_1(r)$ has the same roots as $f(r)$, except r_0. From the theory of cubic equations, the condition that a cubic $x^3 + px^2 + qx + r = 0$ have a double root is that the discriminant D should vanish:

$$D = \tfrac{1}{4} m^2 + \tfrac{1}{2} n^3 = 0 , \tag{A-1}$$

where

$$m = \tfrac{1}{3}(3q - p^2)$$

$$n = \tfrac{1}{27}(2p^3 - 9pq + 27r),$$

Thus for $f_1(r)$ in (III-56),

$$m = \tfrac{1}{3}(3C_4 - r_c)$$

$$n = \tfrac{1}{27}(2r_c^3 - 9r_c C_4 + 27C_3), \tag{A-2}$$

where we have defined

$$C_4 = -r_c^2 + 4R(r_c + 2). \tag{A-3}$$

Thus

$$n^3 = C_4^3 - C_4^2 r_c^2 + \tfrac{1}{3} C_4 r_c^4 - \tfrac{1}{27} r_c^6 ,$$

and

$$27 m^2 = \tfrac{4}{27} r_c^6 + 3 r_c^2 C_4 + 27 C_3^2 + 4 C_3 r_c^6 - 18 C_3 C_4 r_c - \tfrac{4}{3} r_c^4 C_4 .$$

(A-4)

The condition (A-1) for a double root is, therefore:

$$4 C_4^3 - C_4^2 r_c^2 + 2 r_c C_3 (2 r_c^2 - 9 C_4) + 27 C_3^2 = 0 .$$

(A-5)

From (III-54) and (A-3) we obtain

$$C_4 = \frac{C_3 - 8R}{r_c} ,$$

(A-6)

and substituting this in (A-5) yields

$$C_3^3 + 2 C_3^2 (r_c^3 - 12R) + C_3 (192 R^2 + 40 R r_c^3 + r_c^6) - 512 R^3 - 16 R^2 r_c^3 = 0 .$$

(A-7)

We note that C_3 is of the form

$$C_3 = AR - r_c^3 ,$$

(A-8)

and, substituting this into (A-7), we obtain

$$(A^3-24A^2+192A-512)R^3+(88A-A^2-208)R^2r_c^3-64Rr_c^6=0\,.$$

(A-9)

Fortunately, $R=0$ is one root of (A-9), and we have only to solve a quadratic to obtain the desired relation between R and r_c. The solution is

$$R=\frac{-Q_2\pm\sqrt{Q_2^2+(16)^2Q_1}}{2Q_1}\,,$$

(A-10)

where Q_1 and Q_2 denote the coefficients of R^3 and $R^2r_c^3$, respectively, in (A-9). We must take the positive sign in (A-10) because R is a positive quantity. Also, it turns out that $Q_2^2\sim 1000(16)^2Q_1$, hence

$$\sqrt{Q_2^2+(16)^2Q_1}=Q_2\left(1+\frac{(16)^2Q_1}{2Q_2^2}\right)$$

(A-11)

with very high precision. Thus,

$$R=\frac{64}{Q_2}r_c^3=\frac{4r_c^3}{27+36r_c+14r_c^2-4r_c^3-r_c^4}$$

(A-12)

when the expression Q_2 is evaluated.

Equation (A-12) expresses the relation between the critical initial angular velocity $\theta_c=r_c+1$ and the coupling constant R.