To the Graduate Council:

I am submitting herewith a dissertation written by Suzanne Elizabeth Collier entitled "A Theory on Perturbations of the Dirac Operator." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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A Theory on Perturbations of the Dirac Operator

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Suzanne Elizabeth Collier

August 1997

This dissertation is dedicated to my parents,

Mr. Wayne E. Collier

and

Mrs. Mary C. Collier,

to my brother,

Jonathan,

and to my sister,

Angela,

who have been a constant source of encouragement.

Acknowledgements

I wish to express my sincere thanks to my thesis advisor, Dr. Don B. Hinton, for his patience and guidance during my studies at the University of Tennessee. For his encouragement, respect, and attention to detail, I am truly thankful. I have benefitted from his knowledge and experience in the field of differential equations.

My appreciation and thanks go to Dr. G. Samuel Jordan, Associate Head for Graduate Studies, for the concern, guidance, and support he has given continuously over the last six years.

I would also like to thank Dr. Jordan, Dr. Steven Serbin, and Dr. J. Wayne Waller for reading this manuscript and giving suggestions.

Finally, my appreciation goes to my family and friends who have been a constant source of encouragement and motivation. May God bless each of you with good health physically and mentally.

Abstract

In this work we consider perturbations of a first-order differential operator with matrix coefficients known as the Dirac operator. Our main results involve perturbations B of Dirac operators T and L. These operators have the form

$$Ly = \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p_1 & p \\ p & p_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\};$$

$$Ty = \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}; \text{ and}$$

$$By = \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \begin{pmatrix} q_1 & q_4 \\ q_3 & q_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where the coefficients $p, p_1, p_2, q_1, q_2, q_3$, and q_4 are assumed to be real, locally Lebesgue integrable functions. The operators T and L have one singular point which is allowed to be either zero or infinity. Unitary transformations are used to apply results for an operator with a singularity at infinity to one with a singularity at zero.

After introducing notation and several preliminary results, we give necessary and sufficient conditions for the perturbation B to be relatively bounded or relatively compact with respect to T or L. These conditions involve explicit integral averages of the coefficients of B. Results are given for both limit point and limit circle type operators. One application involves a Dirac operator which arises in the study of the relativistic electron. We then focus on several examples in order to show how the conditions for relative boundedness and relative compactness affect the $\mathcal{L}^2_W$-decay of eigenfunctions.

Preface

In this dissertation we develop a theory on perturbations of a first order ordinary differential operator known as the Dirac operator. The Dirac operator has its origins as a partial differential operator which arises in the study of relativistic quantum mechanics. A detailed explanation is given in Thaller [14]. If we assume a spherically symmetric potential, then we can obtain the ordinary differential Dirac operator via a separation of variables argument. This separation argument can be found in Weidmann [15]. We focus on two general forms of the Dirac operator, L and T, which are given by (1.1) and (1.2), respectively.

As seen in Goldberg [7], Reed and Simon [13], and Kato [10], certain perturbation results involving bounded or compact perturbations still hold if the perturbing operator is only relatively bounded or relatively compact. For example, the essential spectrum is preserved under a relatively compact perturbation. Also, a relatively bounded, symmetric perturbation with relative bound less than one preserves self-adjointness.

We define relevant terms in Chapter 1 as well as introduce notation and theorems which will be used throughout this work. In Chapters 2 through 4 we develop necessary and sufficient conditions for perturbations B of a given Dirac operator to be relatively bounded or relatively compact. These conditions involve explicit integral averages of the coefficients of B. In Chapter 5 we focus on several examples in order to study how the conditions for relatively bounded and relatively compact perturbations, which are given in Chapters 2 and 4, yield results on the decay of eigenfunctions.

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Chapter 1

Preliminaries

The purpose of this chapter is to introduce notation and theorems which will be used throughout this work. We use definitions given by Goldberg [7] and Weidmann [15].

Let X and Y be Banach spaces and let B and T be linear operators, each having domain in X and range in Y. Denote the domains of B and T by D(B) and D(T), respectively.

By definition the *graph norm* of T on D(T), denoted $||\cdot||_T$, is given by

$$||y||_T = ||y|| + ||Ty||.$$

We say that B is *relatively bounded with respect to* T (or T*-bounded*) if $D(T) \subseteq D(B)$ and B is bounded on D(T) with respect to $||\cdot||_T$, i.e., there exists a constant $C > 0$ such that

$$||By|| \leq C(||y|| + ||Ty||) \text{ for all } y \in D(T).$$

A sequence $\{y_n\}_{n=1}^{\infty}$ is T-bounded if there exists a constant $C > 0$ such that

$$||y_n||_T < C \text{ for each n.}$$

We say that B is *relatively compact with respect to* T (or T*-compact*) if $D(T) \subseteq D(B)$ and B is compact on D(T) with respect to $||\cdot||_T$, i.e., if $\{y_n\}_{n=1}^{\infty}$ is a T-bounded sequence, then $\{By_n\}_{n=1}^{\infty}$ contains a convergent subsequence.

We use the separable, weighted Hilbert space $\mathcal{L}^2_w(I)$, I an interval, which is the space of (equivalence classes of) Lebesgue measurable functions $y : I \to \mathbb{C}^2$ such that

$$\int_I (w_1|y_1|^2 + w_2|y_2|^2) < \infty,$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and w_1 and w_2 are positive measurable functions on the interval I.

For $y, z \in \mathcal{L}^2_w(I)$, define

$$\langle y, z\rangle_w = \int_I z^*(t) \begin{pmatrix} w_1(t) & 0 \\ 0 & w_2(t) \end{pmatrix} y(t)\,dt$$

and $||y||^2_w = \langle y, y\rangle_w$. We will omit the subscript w when there is no ambiguity.

Let us consider the differential expressions L, T, and B which satisfy on I

$$(1.1) \qquad Ly = \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p_1 & p \\ p & p_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\};$$

$$(1.2) \qquad Ty = \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}; \text{ and}$$

$$(1.3) \qquad By = \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \begin{pmatrix} q_1 & q_4 \\ q_3 & q_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where the coefficients $p, p_1, p_2, q_1, q_2, q_3$, and q_4 are assumed to be real, locally Lebesgue integrable functions.

The *maximal operator* L_1 is the differential operator defined by L with the largest possible domain in $\mathcal{L}^2_w(I)$ which is mapped into $\mathcal{L}^2_w(I)$, i.e.,

$$D(L_1) = \{y \in \mathcal{L}^2_w(I) : y \in AC_{loc}(I), Ly \in \mathcal{L}^2_w(I)\},$$

where $AC_{loc}(I)$ is the set of functions which are absolutely continuous on compact subsets of I.

We define the *minimal unclosed operator* L_0' to be the restriction of L to the functions with compact support in the interior of I. The *minimal operator* L_0 is defined to be the closure of L_0'.

Similar definitions and statements hold for the maximal operators (T_1 and B_1) and minimal operators (T_0 and B_0) associated with the operators T and B, respectively. Since B is a multiplicative operator, $B_1 = B_0$. If $q_3 \equiv q_4$, then B_1 is self-adjoint.

The operators above are special cases of the more general *formal differential expressions* Γ [15, sections 1,3] of the form

$$\begin{aligned}(1.4) \qquad \Gamma y(t) = W^{-1}(t) \Bigg\{ &\sum_{j=o}^{[\frac{n}{2}]} (-1)^j \left(P_j(t)\, y^{(j)}(t)\right)^{(j)} \\ &+ \sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \left[\left(Q_j(t)\, y^{(j)}(t)\right)^{(j+1)} - \left(Q_j^*(t)\, y^{(j+1)}(t)\right)^{(j)}\right] \Bigg\}\end{aligned}$$

where $[\alpha]$ denotes the largest integer less than or equal to α, the functions $y \in \mathbb{C}^m$ are defined on (a,b), $-\infty \le a < b \le \infty$, the coefficients W, P_j, and Q_j are $m \times m$-matrix valued functions on (a,b), and $W(t)$ is positive definite. Each coefficient is assumed to be locally Lebesgue integrable on I. The natural number n is the *order* of the differential expression Γ. If the $m \times m$-matrix valued functions P_j are hermitian, then these expressions are *formally self-adjoint*, i.e., for "sufficiently smooth" functions $y, z : (a,b) \to \mathbb{C}^m$ with compact support

$$\int_a^b \langle W(t)\, \Gamma y(t), z(t) \rangle \, dt = \int_a^b \langle W(t)\, y(t), \Gamma z(t) \rangle \, dt$$

where $\langle \, . \, , \, . \, \rangle$ denotes the usual inner product in $\mathbb{C}^m$. Hence, Γ generates a hermitian operator which is densely defined (i.e. symmetric) in the separable, weighted Hilbert space $\mathcal{L}^2_W(a,b)$.

Associated with the formal differential expressions Γ are maximal and minimal operators (Γ_1 and Γ_0, respectively) on the Hilbert space $\mathcal{L}^2_W(a,b)$, which are defined in a similar manner as the maximal and minimal operators associated with the operator L. For example, the maximal operator Γ_1 defined by Γ has domain

$$D(\Gamma_1) = \{ y \in \mathcal{L}^2_W(a,b) : y^{\{0\}}, y^{\{1\}}, \ldots, y^{\{n-1\}} \in AC_{loc}(a,b), \Gamma y \in \mathcal{L}^2_W(a,b) \},$$

where each $y^{\{i\}}$, for $i = 0, 1, ..., n-1$, is a quasi-derivative (see [15, pages 26,29]).

We note several well-known facts [15, pages 41,46] for formally self-adjoint Γ :

1. $D(\Gamma_1)$ is dense in $\mathcal{L}^2_W(a,b)$ and Γ_1 is closed.

2. $\Gamma_0^* = \Gamma_1$ and $\Gamma_1^* = \Gamma_0$.

3. Any self-adjoint extension A of Γ_0 satisfies $\Gamma_0 \subset A \subset \Gamma_1$.

Notice that if $n = 1$, then

$$\Gamma y(t) = W^{-1}(t) \left\{ P_0(t)\, y(t) + [Q_0(t)\, y(t)]' - Q_0^*(t)\, y'(t) \right\}. \tag{1.5}$$

Equation (1.5) is a more general case of equation (1.1). In order to see this, we let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $J = -J^*$ and $J\,y'(t) = \frac{1}{2}\{[J\,y(t)]' - J^*\,y'(t)\}$ so that the operator L takes the form of Γ (with $m = 2$ and $n = 1$) when

$$W(t) = \begin{pmatrix} w_1(t) & 0 \\ 0 & w_2(t) \end{pmatrix}, \; P_0(t) = \begin{pmatrix} p_1(t) & p(t) \\ p(t) & p_2(t) \end{pmatrix}, \text{ and } Q_0(t) = \frac{1}{2}\, J.$$

We say that Γ is *regular at a* if $a > -\infty$ and the assumptions on the coefficients are satisfied on $[a, b)$ instead of (a, b). We define *regular at b* similarly. If Γ is regular at a and regular at b, then we say that Γ is *regular*. Otherwise, Γ is *singular*.

Since $\Gamma_0 : D(\Gamma_0) \subset \mathcal{L}^2_W(a,b) \to \mathcal{L}^2_W(a,b)$ is a closed, symmetric operator,

$$D(\Gamma_0^*) = D(\Gamma_0) \oplus N(iE - \Gamma_0^*) \oplus N(-iE - \Gamma_0^*), \tag{1.6}$$

where $N(\pm iE - \Gamma_0^*) = \{y \in D(\Gamma_0^*) : \pm iEy - \Gamma y = 0\}$ and E denotes the identity operator. Equation (1.6) is referred to as the *first formula of Von Neumann* and can be rewritten as

$$D(\Gamma_1) = D(\Gamma_0) \oplus N(iE - \Gamma_1) \oplus N(-iE - \Gamma_1), \tag{1.7}$$

since $\Gamma_0^* = \Gamma_1$.

The *deficiency indices* of Γ_0, denoted by $d_\pm(\Gamma_0)$, are given by

$$d_+(\Gamma_0) = \dim R(iE - \Gamma_0)^\perp = \dim N(-iE - \Gamma_1)$$

and

$$d_-(\Gamma_0) = \dim R(-iE - \Gamma_0)^\perp = \dim N(iE - \Gamma_1).$$

If the coefficients of Γ_0 are real, we have $d_+(\Gamma_0) = d_-(\Gamma_0)$. It is well-known [15, page 52,55] that

$$d_\pm(\Gamma_0) \leq n \times m, \tag{1.8}$$

and if Γ is regular at a and singular at b, then

$$d_+(\Gamma_0) + d_-(\Gamma_0) \geq n \times m. \tag{1.9}$$

If Γ is regular at a and singular at b, we say that Γ is *limit circle at b* if $d_\pm(\Gamma_0) = n\times m$ and Γ is *limit point at b* if $d_+(\Gamma_0) + d_-(\Gamma_0) = n \times m$. This notation stems from the geometric method of Weyl for the second order equation (see [4, Chapter 9]).

As an example, we compute the deficiency indices for the minimal operator Γ_0 corresponding to the differential expression (1.5) with

$$W(t) = \begin{pmatrix} t^\gamma & 0 \\ 0 & t^\gamma \end{pmatrix}, \ P_0(t) = 0, \text{ and } Q_0(t) = \frac{1}{2}\begin{pmatrix} 0 & -t^\alpha \\ t^\alpha & 0 \end{pmatrix}$$

for some constants γ and α. Since the coefficients are real, $d_+(\Gamma_0) = d_-(\Gamma_0)$.

We know (see [2, Theorem 9.11.2]) that if there exists a $\lambda \in \mathbb{C}$ such that each solution of $\Gamma y(t) = \lambda y(t)$, $t \in I$, is square integrable, i.e.,

$$\int_I y^*(t)W(t)y(t)\,dt < \infty ,$$

then for every λ each solution of $\Gamma y(t) = \lambda y(t)$, $t \in I$, is square integrable. Thus, to determine $d_+(\Gamma_0)$ and $d_-(\Gamma_0)$ it is enough to consider $\dim N(\Gamma_1)$. Notice that inequalities (1.8) and (1.9) imply that the deficiency indices are either one or two since $d_+(\Gamma_0) = d_-(\Gamma_0)$. Now, we determine conditions on γ and α for which we have $\mathcal{L}^2_W(I)$-solutions to

$$\Gamma y(t) = 0,$$

i.e.,

$$\frac{1}{2}\begin{pmatrix} t^{-\gamma} & 0 \\ 0 & t^{-\gamma} \end{pmatrix}\left\{\left[\begin{pmatrix} 0 & -t^{\alpha} \\ t^{\alpha} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right]' + \begin{pmatrix} 0 & -t^{\alpha} \\ t^{\alpha} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'\right\} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.10}$$

Via equation (1.10) we need $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in D(\Gamma_1)$ which satisfy the following system:

$$\begin{aligned} -2t^{\alpha}y_2'(t) - \alpha t^{\alpha-1}y_2(t) &= 0 \\ 2t^{\alpha}y_1'(t) + \alpha t^{\alpha-1}y_1(t) &= 0 \end{aligned}$$

By solving each equation, we obtain the general solutions

$$y_1(t) = c_1\, t^{-\frac{\alpha}{2}} \text{ and } y_2(t) = c_2\, t^{-\frac{\alpha}{2}} \text{ for constants } c_1, c_2.$$

Thus, two linearly independent solutions to equation (1.10) are

$$Y_1(t) = \begin{pmatrix} t^{-\frac{\alpha}{2}} \\ 0 \end{pmatrix} \text{ and } Y_2(t) = \begin{pmatrix} 0 \\ t^{-\frac{\alpha}{2}} \end{pmatrix}. \tag{1.11}$$

If we take the interval $I = [a, \infty), a > 0$, then $Y_1 \in D(\Gamma_1)$ iff

$$\int_a^{\infty} Y_1^T(t)\, W(t)\, Y_1(t)\, dt = \int_a^{\infty} t^{\gamma-\alpha}\, dt < \infty,$$

i.e., $\gamma - \alpha < -1$. Similarly, $Y_2 \in D(\Gamma_1)$ iff $\gamma - \alpha < -1$. Therefore, $d_+(\Gamma_0) = d_-(\Gamma_0) = 2$ (implying that Γ is limit circle at ∞) iff $\gamma - \alpha < -1$. We also have, via [2, Theorem 9.11.2], that Γ is limit point at ∞ iff $\gamma - \alpha \geq -1$.

On the other hand, if we take the interval $I = (0, a], a > 0$, then $Y_1 \in D(\Gamma_1)$ iff

$$\int_0^{a} Y_1^T(t)\, W(t)\, Y_1(t)\, dt = \int_0^{a} t^{\gamma-\alpha}\, dt < \infty,$$

i.e., $\gamma-\alpha > -1$. Similarly, $Y_2 \in D(\Gamma_1)$ iff $\gamma-\alpha > -1$. Therefore, $d_+(\Gamma_0) = d_-(\Gamma_0) = 2$ (implying that Γ is limit circle at 0) iff $\gamma - \alpha > -1$. Moreover, Γ is limit point at 0 iff $\gamma - \alpha \leq -1$ (via [2, Theorem 9.11.2]).

Now, we establish some general properties of operators. As above the subscript 0 denotes a minimal operator, and the subscript 1 denotes a maximal operator.

Theorem 1.1 *Let F and G be closed linear operators in a Banach space $\mathcal{B}$ with $D(F) \subseteq D(G)$. Then G is F-bounded.*

Proof:

Let $\hat{G}:$ graph $F \to \mathcal{B}$ be defined by

$$\hat{G}(y, Fy) = Gy.$$

Then $\hat{G}$ is linear since G is linear. Let $\{(y_n, Fy_n)\}_{n=1}^{\infty} \subset D(\hat{G})$ be such that

$$(y_n, Fy_n) \to (y, z) \text{ and } \hat{G}(y_n, Fy_n) \to \zeta \text{ as } n \to \infty.$$

Now, we show that $\hat{G}$ is closed, i.e., $(y, z) \in$ graph F and $\hat{G}(y, z) = \zeta$. Since F is closed, graph F is closed. Thus, $(y, z) \in$ graph F where $z = Fy$; hence,

$$(y_n, Fy_n) \to (y, Fy) \text{ in } D(\hat{G}).$$

Since G is closed, $y_n \to y$ and $Gy_n = \hat{G}(y_n, Fy_n) \to \zeta$ imply that $Gy = \zeta$. So, we have $\zeta = Gy = \hat{G}(y, Fy)$. Thus, $\hat{G}$ is closed. By the Closed Graph Theorem, G is F-bounded. □

Theorem 1.2 *Let F and G be formal differential expressions on an interval I where F is symmetric, the order of G is less than the order of F, and the coefficients of F and G are sufficiently smooth so that $D(F_0') \subseteq D(G_0')$.*

(i)If G_0' is F_0'-bounded, then G_0 is F_0-bounded.

(ii) If G_0' is F_0'-compact, then G_0 is F_0-compact.

Proof:

(i) Let $y \in D(F_0)$. Then there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset D(F_0')$ such that

$$y_n \to y \text{ and } F_0' y_n \to F_0 y \text{ as } n \to \infty.$$

Since G_0' is F_0'-bounded, $\{y_n\}_{n=1}^{\infty} \subset D(G_0')$ and there exists a constant $C_1 > 0$ such that

$$\begin{aligned} (1.12) \qquad \|G_0' y_n - G_0' y_m\| &= \|G_0'(y_n - y_m)\| \\ &\leq C_1 \left(\|y_n - y_m\| + \|F_0'(y_n - y_m)\|\right). \end{aligned}$$

Since $\{y_n\}_{n=1}^{\infty}$ and $\{F_0' y_n\}_{n=1}^{\infty}$ are convergent sequences, they are Cauchy. Hence, by inequality (1.12), $\{G_0' y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete space $\mathcal{L}_W^2(a,b)$ and, therefore, converges. By definition of G_0, $y \in D(G_0)$. Since $y \in D(F_0)$ is arbitrary, we have $D(F_0) \subseteq D(G_0)$. By applying Theorem 1.1 we conclude G_0 is F_0-bounded.

(ii) Since G_0' is F_0'-compact, G_0' is F_0'-bounded. By part (i), G_0 is F_0-bounded. Let $\{y_n\}_{n=1}^{\infty} \subset D(F_0)$ be an F_0-bounded sequence, i.e., there exists a constant $C_2 > 0$ such that for each n

$$(1.13) \qquad \|y_n\| + \|F_0 y_n\| \leq C_2 \,.$$

Since F_0 is the closure of F_0', there exists a $z_n \in D(F_0')$ such that for each n

$$(1.14) \qquad \|y_n - z_n\| + \|F_0 y_n - F_0' z_n\| < \frac{1}{n} \,.$$

Then $\{z_n\}_{n=1}^{\infty} \subset D(F_0')$ is an F_0'-bounded sequence since, via the triangle inequality and inequalities (1.13) and (1.14),

$$\begin{aligned} \|z_n\|_{F_0'} &= \|z_n\| + \|F_0' z_n\| \\ &\leq (\|z_n - y_n\| + \|F_0' z_n - F_0 y_n\|) + (\|y_n\| + \|F_0 y_n\|) \\ &\leq \frac{1}{n} + C_2 \\ &\leq 1 + C_2 \quad \text{for each n.} \end{aligned}$$

Since G_0' is F_0'-compact, there exists a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ such that $\{G_0 z_{n_k}\}_{k=1}^{\infty}$ converges as $k \to \infty$, say to ζ. Thus, via the triangle inequality, the

F_0-boundedness of G_0, and inequality (1.14), we have that $\{y_n\}_{n=1}^{\infty} \subset D(G_0)$ and for some constant $C_3 > 0$

$$\begin{aligned} \|G_0 y_{n_k} - \zeta\| &\leq \|G_0 y_{n_k} - G_0 z_{n_k}\| + \|G_0 z_{n_k} - \zeta\| \\ &= \|G_0(y_{n_k} - z_{n_k})\| + \|G_0 z_{n_k} - \zeta\| \\ &\leq C_3 \left(\|y_{n_k} - z_{n_k}\| + \|F_0 y_{n_k} - F_0 z_{n_k}\|\right) + \|G_0 z_{n_k} - \zeta\| \\ &\leq \frac{C_3}{n_k} + \|G_0 z_{n_k} - \zeta\| \to 0 \text{ as } k \to \infty. \end{aligned}$$

Therefore, $\{G_0 y_n\}_{n=1}^{\infty}$ contains a convergent subsequence. By definition, G_0 is F_0-compact. □

Theorem 1.3 *Let F and G be as in Theorem 1.2 and let $D(F_1) \subseteq D(G_1)$. Then*

(i) G_1 is F_1-bounded.

(ii) G_0 is F_0-bounded.

(iii) If G_1 is F_1-compact, then G_0 is F_0-compact.

Proof:

(i) Apply Theorem 1.1.

(ii) Let $y \in D(F_0')$.

Then via part (i), there exists a constant $C > 0$ such that

$$\begin{aligned} \|G_0' y\| &= \|G_1 y\| \\ &\leq C \left(\|y\| + \|F_1 y\|\right) \\ &= C \left(\|y\| + \|F_0' y\|\right). \end{aligned}$$

Since $y \in D(F_0')$ is arbitrary, we have that G_0' is F_0'-bounded. Apply Theorem 1.2 (i) to complete the proof.

(iii) Let $\{y_n\}_{n=1}^{\infty} \subset D(F_0)$ be an F_0-bounded sequence. Then $\{y_n\}_{n=1}^{\infty} \subset D(F_1)$ and is an F_1-bounded sequence. Since G_1 is F_1-compact, there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that the sequence $\{G_1 y_{n_k}\}_{k=1}^{\infty}$ converges as $k \to \infty$. Now, part (ii) implies that $D(F_0) \subseteq D(G_0)$. Hence, $G_0 y_n = G_1 y_n$ for any n. Therefore, the sequence $\{G_0 y_{n_k}\}_{k=1}^{\infty}$ converges as $k \to \infty$. By definition, G_0 is F_0-compact. $\square$

The following lemma is stated and proved in [1, pages 65-67]:

Lemma 1.4 *Let X and Y be subspaces of a Banach space $\mathcal{B}$, where X is closed, Y is finite dimensional, and $X \cap Y = \{0\}$. Then there exists a constant $K > 0$ such that*

$$\|x + y\| \geq K \ \|y\| \text{ for all } x \in X \text{ and } y \in Y.$$

Theorem 1.5 *Let F and G be as in Theorem 1.2, G_0 be F_0-compact, and $D(F_1) \subseteq D(G_1)$. Then G_1 is F_1-compact.*

Proof:

Let $\{y_n\}_{n=1}^{\infty} \subset D(F_1)$ be an F_1-bounded sequence, i.e., there exists a constant $C_1 > 0$ such that for each n

$$\|y_n\| + \|F_1 y_n\| \leq C_1 \ . \tag{1.15}$$

Since $D(F_1) = D(F_0) \oplus S$ where S is finite dimensional [see equation (1.7) and inequality (1.8)], y_n can be written as

$$y_n = y_{n,0} + y_{n,c}$$

where $y_{n,0} \in D(F_0)$ and $y_{n,c} \in S$ for each n. Thus, we have

$$\begin{aligned} G_1 y_n &= G_1 y_{n,0} + G_1 y_{n,c} \\ &= G_0 y_{n,0} + G_1 y_{n,c} \end{aligned}$$

for each n since $D(F_0) \subseteq D(G_0) \subseteq D(G_1)$. Since $F_0 \subset F_1$ and F_1 is bounded when acting upon a finite dimensional space, there exists a constant $C_2 > 0$ such that for each n

$$\|F_1 y_{n,c}\| \leq C_2 \ \|y_{n,c}\| \ . \tag{1.16}$$

Thus, via the triangle inequality and inequalities (1.15) and (1.16), we have

$$\begin{aligned} \|F_1 y_{n,0}\| &= \|F_1 y_n - F_1 y_{n,c}\| \\ &\le \|F_1 y_n\| + \|F_1 y_{n,c}\| \\ &\le C_1 + C_2 \, \|y_{n,c}\| \text{ for each n.} \end{aligned} \tag{1.17}$$

Via Lemma 1.4,

$$\begin{aligned} \|y_{n,c}\| &\le \|y_{n,c}\|_{F_1} \\ &\le \frac{1}{K} \, \|y_{n,0} + y_{n,c}\|_{F_1} \\ &= \frac{1}{K} \, \|y_n\|_{F_1} \text{ for each n.} \end{aligned} \tag{1.18}$$

Thus, via inequalities (1.15), (1.17), and (1.18), we have that

$$\|F_0 y_{n,0}\| = \|F_1 y_{n,0}\| \le C_1 + \frac{C_2}{K} C_1 \text{ for each n,} \tag{1.19}$$

i.e., $\{F_0 y_{n,0}\}_{n=1}^{\infty}$ is bounded in $\mathcal{L}_w^2(I)$.
Also, via the triangle inequality and inequalities (1.15) and (1.18), we have that for each n

$$\begin{aligned} \|y_{n,0}\| &= \|y_n - y_{n,c}\| \\ &\le \|y_n\| + \|y_{n,c}\| \\ &\le \|y_n\| + \frac{1}{K} \, \|y_n\|_{F_1} \\ &\le C_1 + \frac{1}{K} C_1 \, . \end{aligned} \tag{1.20}$$

Thus, $\{y_{n,0}\}_{n=1}^{\infty}$ is bounded in $\mathcal{L}_w^2(I)$. Via inequalities (1.19) and (1.20), $\{y_{n,0}\}_{n=1}^{\infty}$ is an F_0-bounded sequence. Since G_0 is F_0-compact, $\{G_0 y_{n,0}\}_{n=1}^{\infty}$ contains a convergent subsequence, say $\{G_0 y_{n_k,0}\}_{k=1}^{\infty}$. When acting on a finite dimensional space, G_1 is bounded. Thus, there exists a constant $C_3 > 0$ such that for each k

$$\|G_1 y_{n_k,c}\| \le C_3 \|y_{n_k,c}\| \, .$$

Via the above inequality and inequalities (1.15) and (1.18), we have that

$$||G_1 y_{n_k,c}|| \leq \frac{C_3}{K} C_1 \text{ for each k,}$$

i.e., $\{G_1 y_{n_k,c}\}_{k=1}^{\infty}$ is bounded in a finite dimensional subspace of $\mathcal{L}_w^2(I)$. Hence, $\{G_1 y_{n_k,c}\}_{k=1}^{\infty}$ contains a convergent subsequence. Therefore, $\{G_1 y_n\}_{n=1}^{\infty}$ contains a convergent subsequence since $G_1 y_n = G_0 y_{n,0} + G_1 y_{n,c}$. By definition, G_1 is F_1-compact. □

Theorem 1.6 *Let F and G be as in Theorem 1.2. If F is regular at a and limit point at b, then G_0 is F_0-bounded (G_0 is F_0-compact) if and only if G_1 is F_1-bounded (G_1 is F_1-compact).*

Proof:

Via equation (1.7) and inequality (1.8), we can write $D(F_1) = D(F_0) \oplus S$, where S is finite dimensional. Since F is regular at a and limit point at b, dim $S = n \times m$. We know [15, page 62] that there exists $n \times m$-functions which are in the domain of F_1, have compact support in (a, b), and are linearly independent modulo $D(F_0)$. Let us call this span of functions S_0. WLOG, we can take $S = S_0$. Since the order of G is less than the order of F, we have $S_0 \subseteq D(G_1)$. Thus, $D(F_0) \subseteq D(G_0)$ implies that $D(F_1) \subseteq D(G_1)$. Suppose that G_0 is F_0-bounded. Then $D(F_0) \subseteq D(G_0)$. Hence, G_1 is F_1-bounded via Theorem 1.3 (i). Suppose that G_0 is F_0-compact. Then $D(F_0) \subseteq D(G_0)$. Hence, G_1 is F_1-compact via Theorem 1.5. Moreover, if G_1 is F_1-bounded (G_1 is F_1-compact), then G_0 is F_0-bounded (G_0 is F_0-compact) via Theorem 1.3 (ii) (Theorem 1.3 (iii)). □

The following theorem and lemma [3, pages 570, 575-6] are essential in the proofs of later theorems.

Theorem 1.7 *Let $I = [a, \infty)$ and let N, W, and P be positive measurable functions such that N, W^{-1}, and $P^{-1} \in \mathcal{L}_{loc}(I)$. Suppose there exists an $\varepsilon_0 > 0$ and a positive* *continuous function $f = f(t)$ on I such that*

$$S_1(\varepsilon) := \sup_{t\in I}\{f^2\,[\frac{1}{\varepsilon f}\int_t^{t+\varepsilon f} P^{-1}]\,[\frac{1}{\varepsilon f}\int_t^{t+\varepsilon f} N]\} < \infty$$

and

$$S_2(\varepsilon) := \sup_{t\in I}\{[\frac{1}{\varepsilon f}\int_t^{t+\varepsilon f} W^{-1}]\,[\frac{1}{\varepsilon f}\int_t^{t+\varepsilon f} N]\} < \infty$$

for all $\varepsilon \in (0,\varepsilon_0)$. Then there exists a constant $k > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and $y \in D$,

$$\int_I N|y|^2 \le k\{S_2(\varepsilon)\int_I W|y|^2 + \varepsilon^2 S_1(\varepsilon)\int_I P|y'|^2\}$$

where

$$D = \{y : y \in AC_{loc}(I), \int_I W|y|^2 < \infty, \text{ and } \int_I P|y'|^2 < \infty\}.$$

Note that if $S_1(\varepsilon), S_2(\varepsilon) < \infty$ for $\varepsilon = \varepsilon_1$, then $S_1(\varepsilon), S_2(\varepsilon) < \infty$ for all $\varepsilon \in (0,\varepsilon_1]$.

Lemma 1.8 *Let $f, g \in AC_{loc}(I)$ be positive functions on an interval I satisfying $|f'(t)| \le N_0$ and $|f(t)\ g'(t)| \le M_0\ g(t)$ a.e. on I for some constants N_0 and M_0. Then for fixed $t \in I$, $0 < \varepsilon < \frac{1}{N_0}$, and $t \le \tau \le t + \varepsilon f(t)$, we have that*

$$(1 - \varepsilon N_0) f(t) \le f(\tau) \le (1 + \varepsilon N_0) f(t)$$

and

$$e^{-\frac{M_0}{N_0}} g(t) \le g(\tau) \le e^{\frac{M_0}{N_0}} g(t).$$

This lemma implies that both positive and negative powers of $f(\tau)$ and $g(\tau)$ are essentially constant for $t \le \tau \le t + \varepsilon f(t)$ and fixed t.

In the proofs of several results in this work, we will use unitary transformations. Here, we develop sufficient conditions for a unitary change of dependent and independent variables for a maximal (or minimal or self-adjoint extension of a minimal) operator of the form

$$Mz = \begin{pmatrix} W_1^{-1} & 0 \\ 0 & W_2^{-1} \end{pmatrix} \left\{\frac{1}{2}[(\Theta z)^{\cdot} + \Theta \dot{z}] + \mathcal{N} z\right\}\ ,\quad x \in X, \tag{1.21}$$

where $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$, $\mathcal{N} = \begin{pmatrix} \eta_1 & \eta \\ \eta & \eta_2 \end{pmatrix}$, and $\dot{} = \frac{d}{dx}$.

Let $t = f(x)$ where $f : X \to T$ is a strictly increasing (decreasing) $C^{(1)}$-function, and let

$$(Uz)(t) = \begin{pmatrix} \mu_1(x) & 0 \\ 0 & \mu_2(x) \end{pmatrix} \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \text{ for } z = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \in \mathcal{L}^2_W(X),$$

where $\mu_i \in C(X)$, for $i = 1, 2$, are never zero on X.

If the weight functions w_i satisfy on T

$$w_i(t) = \frac{W_i(x)}{\mu_i^2(x)\,|\dot{f}(x)|} \text{ for } i = 1, 2, \tag{1.22}$$

then we have for $\tilde{z}, z \in \mathcal{L}^2_W(X)$

$$\begin{aligned}
\langle U\tilde{z}, Uz\rangle_{\mathcal{L}^2_w(T)} &= \int_T [(Uz)(t)]^* \begin{pmatrix} w_1^{-1}(t) & 0 \\ 0 & w_2^{-1}(t) \end{pmatrix} (U\tilde{z})(t)\,dt \\
&= \int_T \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}^* \begin{pmatrix} \mu_1^2(x) w_1^{-1}(t) & 0 \\ 0 & \mu_2^2(x) w_2^{-1} \end{pmatrix} \begin{pmatrix} \tilde{z}_1(x) \\ \tilde{z}_2(x) \end{pmatrix} dt \\
&= \int_X \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}^* \begin{pmatrix} \mu_1^2(x) w_1^{-1}(t) & 0 \\ 0 & \mu_2^2(x) w_2^{-1} \end{pmatrix} \begin{pmatrix} \tilde{z}_1(x) \\ \tilde{z}_2(x) \end{pmatrix} |\dot{f}(x)|\,dx \\
&= \int_X \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}^* \begin{pmatrix} W_1^{-1}(x) & 0 \\ 0 & W_2^{-1}(x) \end{pmatrix} \begin{pmatrix} \tilde{z}_1(x) \\ \tilde{z}_2(x) \end{pmatrix} dx \\
&= \langle \tilde{z}, z\rangle_{\mathcal{L}^2_W(X)} ,
\end{aligned}$$

i.e.,

$$\langle U\tilde{z}, Uz\rangle_{\mathcal{L}^2_w(T)} = \langle \tilde{z}, z\rangle_{\mathcal{L}^2_W(X)} \text{ for } \tilde{z}, z \in \mathcal{L}^2_W(X). \tag{1.23}$$

Thus, U is a unitary map from $\mathcal{L}^2_W(X)$ onto $\mathcal{L}^2_w(T)$. Let $L = UMU^{-1}, y = Uz \in D(L)$ for $z \in D(M)$, and $' = \frac{d}{dt}$.

Then

$$\begin{aligned}
(MU^{-1}y)(x) &= W^{-1}\left\{\frac{1}{2}\{(\Theta\mu^{-1}y)'\,\dot{f}(x) + \Theta[(\mu^{-1})^{\cdot}\,y + \mu^{-1}y'\,\dot{f}(x)]\} + \mathcal{N}\mu^{-1}y\right\} \\
&= W^{-1}\left\{\frac{1}{2}\{(\Theta\mu^{-1}y)'\,\dot{f}(x) + \Theta[-\mu^{-1}\dot{\mu}\mu^{-1}y + \mu^{-1}y'\,\dot{f}(x)]\} + \mathcal{N}\mu^{-1}y\right\}
\end{aligned}$$

where $W(x) = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix}$ and $\mu(x) = \begin{pmatrix} \mu_1(x) & 0 \\ 0 & \mu_2(x) \end{pmatrix}$.

Hence, we have, suppressing independent variables,

$$\begin{aligned}
Ly &= UMU^{-1}y \\
&= \mu W^{-1}\left\{\frac{1}{2}\{\Theta\mu^{-1}y'\,\dot{f} + (\Theta\mu^{-1})'y\,\dot{f} - \Theta\mu^{-1}\dot{\mu}\mu^{-1}y + \Theta\mu^{-1}y'\,\dot{f}\}\right. \\
&\quad \left. + \mathcal{N}\mu^{-1}y\right\} \\
&= \mu W^{-1}\mu\,\dot{f}\left\{\frac{1}{2}\{\mu^{-1}\Theta\mu^{-1}y' + \mu^{-1}(\Theta\mu^{-1})'y - \frac{1}{\dot{f}}\,\mu^{-1}\Theta\mu^{-1}\dot{\mu}\mu^{-1}y\right. \\
&\quad \left. + \mu^{-1}\Theta\mu^{-1}y'\} + \frac{1}{\dot{f}}\,\mu^{-1}\mathcal{N}\mu^{-1}y\right\} \\
&= \mu^2\,\dot{f}\,W^{-1}\left\{\frac{1}{2}\{\mu^{-1}\Theta\mu^{-1}y' + (\mu^{-1}\Theta\mu^{-1})'y - (\mu^{-1})'\Theta\mu^{-1}y\right. \\
&\quad \left. - \frac{1}{\dot{f}}\,\mu^{-1}\Theta\mu^{-1}\dot{\mu}\mu^{-1}y + \mu^{-1}\Theta\mu^{-1}y'\} + \frac{1}{\dot{f}}\,\mu^{-1}\mathcal{N}\mu^{-1}y\right\} \\
&= \mu^2\,\dot{f}\,W^{-1}\left\{\frac{1}{2}\{\mu^{-1}\Theta\mu^{-1}y' + (\mu^{-1}\Theta\mu^{-1})'y + \frac{1}{\dot{f}}\,\mu^{-1}\dot{\mu}\mu^{-1}\Theta\mu^{-1}y\right. \\
&\quad \left. - \frac{1}{\dot{f}}\,\mu^{-1}\Theta\mu^{-1}\dot{\mu}\mu^{-1}y + \mu^{-1}\Theta\mu^{-1}y'\} + \frac{1}{\dot{f}}\,\mu^{-1}\mathcal{N}\mu^{-1}y\right\} \\
&= \mu^2\,\dot{f}\,W^{-1}\left\{\frac{1}{2}[\tilde{\Theta}y' + (\tilde{\Theta}y)'] + \tilde{\mathcal{N}}y\right\},
\end{aligned}$$

where

$$\tilde{\Theta}(t) = (\mu^{-1}\Theta\mu^{-1})(x) \tag{1.24}$$

and

$$\tilde{\mathcal{N}}(t) = \frac{1}{\dot{f}(x)}\left\{\frac{1}{2}[\mu^{-1}\dot{\mu}(\mu^{-1}\Theta\mu^{-1}) - (\mu^{-1}\Theta\mu^{-1})\dot{\mu}\mu^{-1}] + \mu^{-1}\mathcal{N}\mu^{-1}\right\}(x). \tag{1.25}$$

Thus, via equation (1.22)

$$(Ly)(t) = (\text{sign } \dot{f}) \begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix} \left\{\frac{1}{2}[(\tilde{\Theta}y)' + \tilde{\Theta}y'] + \tilde{\mathcal{N}}y\right\}, \quad t \in T. \tag{1.26}$$

Also, for $\tilde{y} = U\tilde{z}, \tilde{z} \in D(M)$ we have

$$\langle Ly, L\tilde{y}\rangle_{\mathcal{L}^2_w(T)} = \langle UMU^{-1}y, UMU^{-1}\tilde{y}\rangle_{\mathcal{L}^2_w(T)} \tag{1.27}$$

$$= \langle UMz, UM\tilde{z}\rangle_{\mathcal{L}^2_w(T)}$$
$$= \langle Mz, M\tilde{z}\rangle_{\mathcal{L}^2_W(X)},$$

where the last equality follows by equation (1.23). Therefore, if we have the conditions on the weight functions (1.22) and on $\tilde{\Theta}$ and $\tilde{\mathcal{N}}$, which are given by (1.24) and (1.25), then

$$\|z\|_{\mathcal{L}^2_W(X)} = \|y\|_{\mathcal{L}^2_w(T)} \text{ and } \|Mz\|_{\mathcal{L}^2_W(X)} = \|Ly\|_{\mathcal{L}^2_w(T)}$$

via equations (1.23) and (1.27).

Chapter 2

Perturbations of T: Limit Point Case

In this chapter, we consider perturbations B of the higher-ordered differential operator T, where T and B are defined by (1.2) and (1.3), respectively. We develop necessary and sufficient conditions for the perturbations to be relatively bounded or relatively compact with respect to T. These conditions involve explicit integral averages of the coefficients of B. In Theorem 2.1 we consider the maximal operators T_1 and B_1 associated with the differential operators (1.2) and (1.3), respectively. The proof relies heavily upon Theorem 1.7 and Lemma 1.8 and follows an argument similar to the one found in [2, Theorem 1.1].

Corollaries 2.2 and 2.3 apply Theorem 2.1 to perturbations of an operator of the form (1.5) with

$$W(x) = \begin{pmatrix} x^\gamma & 0 \\ 0 & x^\gamma \end{pmatrix}, \; P_0(x) = 0, \text{ and } Q_0(x) = \frac{1}{2}\begin{pmatrix} 0 & -x^\alpha \\ x^\alpha & 0 \end{pmatrix}$$

for some constants γ and α. Each proof makes use of a unitary transformation. Corollary 2.2 considers the operators on the interval $[a,\infty), a > 0$, with $\gamma - \alpha \geq -1$ so that the unperturbed operator is limit point at ∞; whereas, Corollary 2.3 considers the operators on the interval $(0,a], a > 0$, with $\gamma - \alpha \leq -1$ so that the perturbed operator is limit point at 0.

Theorem 2.1 *Let $I = [a, \infty)$ for some $a > 0$ and let $w_1 = w_2 = f^{-1}$ where $f \in AC_{loc}(I)$ is positive and $|f'(t)| \le N_0$ a.e. on I for some constant N_0. Then*
(i) B_1 is T_1-bounded if and only if

$$\text{(2.1)} \qquad \sup_{t \in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\ q_i^2(\tau)\, d\tau < \infty, \quad i = 1, 2, 3, 4,$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$;
(ii) B_1 is T_1-compact if and only if

$$\text{(2.2)} \qquad \lim_{t \to \infty} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\ q_i^2(\tau)\, d\tau = 0, \quad i = 1, 2, 3, 4,$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$.

Proof:
By applying Lemma 1.8 with $g \equiv 1$, we know that positive and negative powers of f are essentially constant on intervals of length εf. Thus, we may replace

$$\frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\ q_i^2(\tau)\, d\tau$$

with either

$$f(t) \int_t^{t+\varepsilon f(t)} q_i^2(\tau)\, d\tau \quad \text{or} \quad \int_t^{t+\varepsilon f(t)} f(\tau)\ q_i^2(\tau)\, d\tau\,.$$

(i) Sufficiency

Let us first consider any $y \in D(T_1)$ whose second component is zero. Then we have

$$\text{(2.3)} \qquad ||y||^2 = \int_I f^{-1}(\tau)|y_1(\tau)|^2\, d\tau,$$

$$\text{(2.4)} \qquad ||Ty||^2 = \int_I f(\tau)|y_1'(\tau)|^2\, d\tau, \text{ and}$$

$$\text{(2.5)} \qquad ||By||^2 = \int_I f(\tau)[q_1^2(\tau) + q_3^2(\tau)]|y_1(\tau)|^2\, d\tau\,.$$

Now, we show that the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2N_0})$ with $N = f(q_1^2 + q_3^2), W = f^{-1}$,and $P = f$. Since positive and negative powers of f are essentially constant on intervals of length εf, we have for some constants $C_1, C_2 > 0$

$$
\begin{aligned}
(2.6)\quad S_1 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} f^{-1}(\tau)\,d\tau\right]\left[\int_t^{t+\varepsilon f(t)} f(\tau)[q_1^2(\tau)+q_3^2(\tau)]\,d\tau\right]\right\}\\
&\le \frac{C_1}{\varepsilon}\sup_{t\in I}\left\{\int_t^{t+\varepsilon f(t)} f(\tau)[q_1^2(\tau)+q_3^2(\tau)]\,d\tau\right\} \text{ and}\\
(2.7)\quad S_2 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} f(\tau)\,d\tau\right]\left[\int_t^{t+\varepsilon f(t)} f(\tau)[q_1^2(\tau)+q_3^2(\tau)]\,d\tau\right]\right\}\\
&\le \frac{C_2}{\varepsilon}\sup_{t\in I}\left\{\int_t^{t+\varepsilon f(t)} f(\tau)[q_1^2(\tau)+q_3^2(\tau)]\,d\tau]\right\}
\end{aligned}
$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$. Inequalities (2.1), (2.6), and (2.7) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2N_0})$. Therefore (via Theorem 1.7), there exists a constant $k_1 > 0$ such that

$$
\begin{aligned}
(2.8)\qquad & \int_I f(\tau)[q_1^2(\tau)+q_3^2(\tau)]|y_1(\tau)|^2\,d\tau\\
&\le k_1\left\{\int_I f^{-1}(\tau)|y_1(\tau)|^2\,d\tau + \varepsilon^2\int_I f(\tau)|y_1'(\tau)|^2\,d\tau\right\}.
\end{aligned}
$$

By substituting (2.3)-(2.5) into (2.8), we obtain for some constant k_1

$$(2.9)\qquad ||By||^2 \le k_1\,(||y||^2 + \varepsilon^2||Ty||^2).$$

Thus, $y \in D(B_1)$. Since y is arbitrary, inequality (2.9) implies that B_1 is T_1-bounded on those $y \in D(T_1)$ with $y_2 \equiv 0$.

Let us now consider any $y \in D(T_1)$ whose first component is zero. Then we have

$$
\begin{aligned}
(2.10)\qquad ||y||^2 &= \int_I f^{-1}(\tau)|y_2(\tau)|^2\,d\tau,\\
(2.11)\qquad ||Ty||^2 &= \int_I f(\tau)|y_2'(\tau)|^2\,d\tau, \text{ and}\\
(2.12)\qquad ||By||^2 &= \int_I f(\tau)[q_2^2(\tau)+q_4^2(\tau)]|y_2(\tau)|^2\,d\tau\ .
\end{aligned}
$$

Following an argument similar to the one above, we show the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2N_0})$ with $N = f(q_2^2+q_4^2), W = f^{-1}$, and $P = f$. For some constants $C_3, C_4 > 0$ we have

$$(2.13)\quad S_1 = \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} f^{-1}(\tau)\,d\tau\right]\left[\int_t^{t+\varepsilon f(t)} f(\tau)[q_2^2(\tau)+q_4^2(\tau)]\,d\tau\right]\right\}$$

$$\leq \frac{C_3}{\varepsilon} \sup_{t\in I} \left\{ \int_t^{t+\varepsilon f(t)} f(\tau)[q_2^2(\tau) + q_4^2(\tau)]\, d\tau \right\} \text{ and}$$

$$(2.14)\quad S_2 = \sup_{t\in I} \left\{ \frac{1}{\varepsilon^2 f^2(t)} \left[\int_t^{t+\varepsilon f(t)} f(\tau)\, d\tau \right] \left[\int_t^{t+\varepsilon f(t)} f(\tau)[q_2^2(\tau) + q_4^2(\tau)]\, d\tau \right] \right\}$$
$$\leq \frac{C_4}{\varepsilon} \sup_{t\in I} \left\{ \int_t^{t+\varepsilon f(t)} f(\tau)[q_2^2(\tau) + q_4^2(\tau)]\, d\tau \right\}$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$. Inequalities (2.1), (2.13), and (2.14) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2N_0})$. Therefore (via Theorem 1.7), there exists a constant $k_2 > 0$ such that

$$(2.15)\quad \int_I f(\tau)[q_2^2(\tau) + q_4^2(\tau)]|y_2(\tau)|^2\, d\tau$$
$$\leq k_2 \left\{ \int_I f^{-1}(\tau)|y_2(\tau)|^2\, d\tau + \varepsilon^2 \int_I f(\tau)|y_2'(\tau)|^2\, d\tau \right\}.$$

By substituting (2.10)-(2.12) into (2.15), we obtain for some constant k_2

$$(2.16)\quad ||By||^2 \leq k_2\, (||y||^2 + \varepsilon^2 ||Ty||^2).$$

Thus, $y \in D(B_1)$. Since y is arbitrary, inequality (2.16) implies that B_1 is T_1-bounded on those $y \in D(T_1)$ with $y_1 \equiv 0$.

Finally, we consider any $y \in D(T_1)$. Note that a shorter argument is possible, but estimates obtained here will be useful later. Then we have, suppressing the independent variable,

$$(2.17)\quad ||y||^2 = \int_I f^{-1}[|y_1|^2 + |y_2|^2]\ ,$$

$$(2.18)\quad ||Ty||^2 = \int_I f[|y_1'|^2 + |y_2'|^2]\ , \text{ and}$$

$$(2.19)\quad ||By||^2 = \int_I f\{[q_1^2 + q_3^2]|y_1|^2 + [q_1q_4 + q_2q_3]y_1\bar{y}_2$$
$$+[q_1q_4 + q_2q_3]\bar{y}_1y_2 + [q_2^2 + q_4^2]|y_2|^2\}$$
$$\leq \int_I f\{[q_1^2 + q_3^2]|y_1|^2 + 2\,|q_1q_4 + q_2q_3|\,|y_1y_2|$$
$$+[q_2^2 + q_4^2]|y_2|^2\}\ .$$

We make use of the inequality $2ab \leq a^2 + b^2$ in inequality (2.19) to obtain

$$(2.20)\quad ||By||^2 \leq \int_I f\{[q_1^2 + q_3^2]|y_1|^2 + |q_1q_4 + q_2q_3|\,[|y_1|^2 + |y_2|^2]$$
$$+[q_2^2 + q_4^2]|y_2|^2\}\ .$$

Claim 2.1.1 *There exist constants $k_3, k_4 > 0$ such that*

$$\int_I f\ |q_1q_4 + q_2q_3|\,|y_1|^2 \le k_3 \left\{ \int_I f^{-1}|y_1|^2 + \varepsilon^2 \int_I f|y_1'|^2 \right\} \tag{2.21}$$

and

$$\int_I f\ |q_1q_4 + q_2q_3|\,|y_2|^2 \le k_4 \left\{ \int_I f^{-1}|y_2|^2 + \varepsilon^2 \int_I f|y_2'|^2 \right\} \tag{2.22}$$

Proof of Claim 2.1.1:

Using the Cauchy-Schwarz inequality and the inequality $(a+b)^2 \le 2(a^2+b^2)$, we have

$$\begin{aligned}
\int_t^{t+\varepsilon f(t)} f\ |q_1q_4 + q_2q_3| &\le \int_t^{t+\varepsilon f(t)} f\ |q_1(q_4 - q_2)| + \int_t^{t+\varepsilon f(t)} f\ |q_2(q_1+q_3)| \\
&\le \left(\int_t^{t+\varepsilon f(t)} f\ q_1^2\right)^{\frac{1}{2}} \left(\int_t^{t+\varepsilon f(t)} f\ [q_4 - q_2]^2\right)^{\frac{1}{2}} \\
&\quad + \left(\int_t^{t+\varepsilon f(t)} f\ q_2^2\right)^{\frac{1}{2}} \left(\int_t^{t+\varepsilon f(t)} f\ [q_1 + q_3]^2\right)^{\frac{1}{2}} \\
&\le \left(\int_t^{t+\varepsilon f(t)} f\ q_1^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} f\ [q_4^2 + q_2^2]\right)^{\frac{1}{2}} \qquad (2.23) \\
&\quad + \left(\int_t^{t+\varepsilon f(t)} f\ q_2^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} f\ [q_1^2 + q_3^2]\right)^{\frac{1}{2}}
\end{aligned}$$

Now, we show that the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2N_0})$ with $N = f|q_1q_4 + q_2q_3|, W = f^{-1}$, and $P = f$. For some constants $C_5, C_6 > 0$ we have

$$\begin{aligned}
S_1 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} f^{-1}\right]\left[\int_t^{t+\varepsilon f(t)} f|q_1q_4 + q_2q_3|\right]\right\} \qquad (2.24) \\
&\le \frac{C_5}{\varepsilon}\sup_{t\in I}\left\{\left(\int_t^{t+\varepsilon f(t)} f\ q_1^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} f\ [q_4^2 + q_2^2]\right)^{\frac{1}{2}}\right. \\
&\quad \left. + \left(\int_t^{t+\varepsilon f(t)} f\ q_2^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} f\ [q_1^2 + q_3^2]\right)^{\frac{1}{2}}\right\} \text{ and}
\end{aligned}$$

$$\begin{aligned}
S_2 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} f\right]\left[\int_t^{t+\varepsilon f(t)} f|q_1q_4 + q_2q_3|\right]\right\} \qquad (2.25) \\
&\le \frac{C_6}{\varepsilon}\sup_{t\in I}\left\{\left(\int_t^{t+\varepsilon f(t)} f\ q_1^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} f\ [q_4^2 + q_2^2]\right)^{\frac{1}{2}}\right. \\
&\quad \left. + \left(\int_t^{t+\varepsilon f(t)} f\ q_2^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} f\ [q_1^2 + q_3^2]\right)^{\frac{1}{2}}\right\}
\end{aligned}$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$. Inequalities (2.1), (2.24), and (2.25) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2N_0})$. An application of Theorem 1.7 gives us inequalities (2.21) and (2.22). #

Therefore, via inequalities (2.8), (2.15), (2.21), and (2.22) we obtain from inequality (2.20) that

$$\begin{aligned} ||By||^2 \ \leq \ & (k_1 + k_3)\left\{\int_I f^{-1}|y_1|^2 + \varepsilon^2 \int_I f\ |y_1'|^2\right\} \\ & +(k_2 + k_4)\left\{\int_I f^{-1}|y_2|^2 + \varepsilon^2 \int_I f\ |y_2'|^2\right\}. \end{aligned} \tag{2.26}$$

Hence, there exists a constant $C > 0$ such that for each $y \in D(T_1)$

$$||By||^2 \leq C\ (||y||^2 + ||Ty||^2).$$

Thus, $y \in D(B_1)$. Since y is arbitrary, we know that B_1 is T_1-bounded.

Necessity

Let ϕ be a function in $C_0^\infty(\mathbb{R})$ such that $\phi \equiv 1$ on $[0,1]$ and supp(ϕ)=$[-2,2]$. Fix $\varepsilon \in (0, \frac{1}{2N_0})$. For each $r \geq a$ we define

$$\phi_r(t) = \phi(\frac{t-r}{\varepsilon f(r)}), \text{ for } t \geq a. \tag{2.27}$$

Then, $\phi_r \equiv 1$ on $[r, r+\varepsilon f(r)]$ and supp(ϕ_r)=$[r - 2\varepsilon f(r), r + 2\varepsilon f(r)]$.

For each $r \geq a$ we define

$$\Phi_r(t) = \begin{pmatrix} \phi_r(t) \\ 0 \end{pmatrix} \text{ and } \Psi_r(t) = \begin{pmatrix} 0 \\ \phi_r(t) \end{pmatrix}, \text{ for } t \geq a. \tag{2.28}$$

Via Lemma 1.8 there exists a constant $C_1 > 0$ such that for each $r \geq a$

$$\begin{aligned} ||\Phi_r||^2 \ &= \ \int_I f^{-1}(t)\phi_r^2(t)\,dt \\ &= \ \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} f^{-1}(t)\phi^2\left(\frac{t-r}{\varepsilon f(r)}\right)\,dt \\ &\leq \ C_1\, f^{-1}(r)\int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \phi^2\left(\frac{t-r}{\varepsilon f(r)}\right)\,dt\,. \end{aligned} \tag{2.29}$$

By a change of variables (letting $u = \dfrac{t-r}{\varepsilon f(r)}$ in (2.29)),

$$\|\Phi_r\|^2 \le C_1\,\varepsilon \int_{-2}^{2} \phi^2(u)\,du\,. \tag{2.30}$$

Since ϕ is continuous, inequality (2.30) implies there exists a constant $C_2 > 0$ such that for $r \ge a$

$$\|\Phi_r\|^2 \le C_2\,. \tag{2.31}$$

Similarly, for $r \ge a$

$$\|\Psi_r\|^2 \le C_2\,. \tag{2.32}$$

Via Lemma 1.8 there exists a constant $C_3 > 0$ such that for each $r \ge a$

$$\begin{aligned} \|T\Phi_r\|^2 &= \int_I f(t)\left[\frac{d}{dt}\phi_r(t)\right]^2 dt \\ &= \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} f(t)\left[\frac{d}{dt}\phi\left(\frac{t-r}{\varepsilon f(r)}\right)\right]^2 dt \\ &\le C_3\, f(r)\int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \left[\frac{d}{dt}\phi\left(\frac{t-r}{\varepsilon f(r)}\right)\right]^2 dt\,. \end{aligned} \tag{2.33}$$

By a change of variables (letting $u = \dfrac{t-r}{\varepsilon f(r)}$ in (2.33)),

$$\|T\Phi_r\|^2 \le C_3\varepsilon^{-1}\int_{-2}^{2} [\phi'(u)]^2\,du\,. \tag{2.34}$$

Since ϕ' is continuous, inequality (2.34) implies there exists a constant $C_4 > 0$ such that for $r \ge a$

$$\|T\Phi_r\|^2 \le C_4\,. \tag{2.35}$$

Similarly, for $r \ge a$

$$\|T\Psi_r\|^2 \le C_4\,. \tag{2.36}$$

Since $\phi_r \equiv 1$ on $[r, r+\varepsilon f(r)]$ and supp(ϕ_r)=$[r-2\varepsilon f(r), r+2\varepsilon f(r)]$,

$$\begin{aligned} \int_r^{r+\varepsilon f(r)} f(t)[q_1^2(t)+q_3^2(t)]\,dt &\le \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} f(t)[q_1^2(t)+q_3^2(t)]\phi_r^2(t)\,dt \\ &= \|B\Phi_r\|^2\,. \end{aligned} \tag{2.37}$$

Thus, via the T_1-boundedness of B_1 and inequalities (2.31), (2.35), and (2.37), there exists a constant $C_5 > 0$ such that

$$\int_r^{r+\varepsilon f(r)} f(t)[q_1^2(t) + q_3^2(t)]\,dt \le C_5\,. \tag{2.38}$$

Since the right-hand side of the above inequality is independent of r, inequality (2.1) holds for $i = 1, 3$.

Moreover,

$$\begin{aligned} \int_r^{r+\varepsilon f(r)} f(t)[q_2^2(t) + q_4^2(t)]\,dt &\le \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} f(t)[q_2^2(t) + q_4^2(t)]\phi_r^2(t)\,dt \\ &= \|B\Psi_r\|^2\,. \end{aligned} \tag{2.39}$$

Via the T_1-boundedness of B_1 and inequalities (2.32), (2.36), and (2.39),

$$\int_r^{r+\varepsilon f(r)} f(t)[q_2^2(t) + q_4^2(t)]\,dt \le C_5\,. \tag{2.40}$$

Since the right-hand side of the above inequality is independent of r, inequality (2.1) holds for $i = 2, 4$.

Note that the proof of necessity shows that (2.1) holds for every $\varepsilon \in (0, \frac{1}{2N_0})$.

(ii) Sufficiency

By the previous argument B_1 is T_1-bounded. Thus, $D(T_1) \subseteq D(B_1)$. For every positive integer $N > a$, define B_N on $D(T_1)$ by

$$B_N y = \begin{cases} By & \text{on } [a, N] \\ 0 & \text{on } (N, \infty) \end{cases}$$

In order to simplify the proof, we break the argument into two Claims.

Claim 2.1.2 *$B_N \to B_1$ in the space of bounded operators on $D(T_1)$ with the T_1-norm.*

Proof of Claim 2.1.2:

By a note in Chapter 1, we know that T_1 is closed since it is a formally self-adjoint maximal operator of the form (1.4). Therefore, $D(T_1)$ is complete under the T_1-norm.

Using the inequality $2ab \le a^2 + b^2$, we have for $y \in D(T_1)$

$$
\begin{aligned}
(2.41) \qquad \|By - B_N y\|^2 &= \int_N^\infty f\{[q_1^2 + q_3^2]|y_1|^2 + [q_1 q_4 + q_2 q_3] y_1 \bar{y}_2 \\
&\quad + [q_1 q_4 + q_2 q_3] \bar{y}_1 y_2 + [q_2^2 + q_4^2]|y_2|^2\} \\
&\le \int_I f\{[q_1^2 + q_3^2]|y_1|^2 + 2\,|q_1 q_4 + q_2 q_3|\,|y_1 y_2| \\
&\quad + [q_2^2 + q_4^2]|y_2|^2\} \\
&\le \int_I f\{[q_1^2 + q_3^2]|y_1|^2 + |q_1 q_4 + q_2 q_3|\,[|y_1|^2 + |y_2|^2] \\
&\quad + [q_2^2 + q_4^2]|y_2|^2\}\ .
\end{aligned}
$$

We apply the sufficiency argument in part (i) with $I_N = [N, \infty)$ to the last inequality in (2.41). Via Theorem 1.7 and inequalities (2.6), (2.7), (2.13), (2.14), (2.24), and (2.25), there exist constants k_1, k_2, k_3, and $k_4 > 0$ such that

$$
\begin{aligned}
(2.42) \qquad & \|By - B_N y\|^2 \\
\le\ & \frac{k_1}{\varepsilon} \sup_{t \in I_N} \left\{ \int_t^{t+\varepsilon f(t)} f[q_1^2 + q_3^2] \right\} \left\{ \int_{I_N} f^{-1}|y_1|^2 \ + \varepsilon^2 \int_{I_N} f|y_1'|^2 \right\} \\
+\ & \frac{k_2}{\varepsilon} \sup_{t \in I_N} \left\{ \left(\int_t^{t+\varepsilon f(t)} f\ q_1^2 \right)^{\frac{1}{2}} \left(2 \int_t^{t+\varepsilon f(t)} f\ [q_4^2 + q_2^2] \right)^{\frac{1}{2}} \right. \\
& \left. + \left(\int_t^{t+\varepsilon f(t)} f\ q_2^2 \right)^{\frac{1}{2}} \left(2 \int_t^{t+\varepsilon f(t)} f\ [q_1^2 + q_3^2] \right)^{\frac{1}{2}} \right\} \\
& \cdot \left\{ \int_{I_N} f^{-1}|y_1|^2 \ + \varepsilon^2 \int_{I_N} f|y_1'|^2 \right\} \\
+\ & \frac{k_3}{\varepsilon} \sup_{t \in I_N} \left\{ \left(\int_t^{t+\varepsilon f(t)} f\ q_1^2 \right)^{\frac{1}{2}} \left(2 \int_t^{t+\varepsilon f(t)} f\ [q_4^2 + q_2^2] \right)^{\frac{1}{2}} \right. \\
& \left. + \left(\int_t^{t+\varepsilon f(t)} f\ q_2^2 \right)^{\frac{1}{2}} \left(2 \int_t^{t+\varepsilon f(t)} f\ [q_1^2 + q_3^2] \right)^{\frac{1}{2}} \right\} \\
& \cdot \left\{ \int_{I_N} f^{-1}|y_2|^2 \ + \varepsilon^2 \int_{I_N} f|y_2'|^2 \right\} \\
+\ & \frac{k_4}{\varepsilon} \sup_{t \in I_N} \left\{ \int_t^{t+\varepsilon f(t)} f[q_2^2 + q_4^2] \right\} \left\{ \int_{I_N} f^{-1}|y_2|^2 \ + \varepsilon^2 \int_{I_N} f|y_2'|^2 \right\}.
\end{aligned}
$$

Adding nonnegative terms to the right-hand side of inequality (2.42) gives

$$
(2.43)\ \|By - B_N y\|^2 \le k_5 \sup_{t \in I_N} F(t) \left\{ \int_{I_N} f^{-1}[|y_1|^2 + |y_2|^2] \ + \varepsilon^2 \int_{I_N} f[|y_1'|^2 + |y_2'|^2] \right\}
$$

for some constant $k_5 > 0$ where

$$\begin{aligned} F(t) \;=\; & \int_t^{t+\varepsilon f(t)} f[q_1^2 + q_3^2] \\ & + \left(\int_t^{t+\varepsilon f(t)} f\, q_1^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} f\, [q_4^2 + q_2^2]\right)^{\frac{1}{2}} \\ & + \left(\int_t^{t+\varepsilon f(t)} f\, q_2^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} f\, [q_1^2 + q_3^2]\right)^{\frac{1}{2}} \\ & + \int_t^{t+\varepsilon f(t)} f[q_2^2 + q_4^2]\,. \end{aligned}$$

Via inequality (2.43) there exists a constant $k_6 > 0$ such that

$$\|By - B_N y\|^2 \le k_6 \sup_{t\in I_N} F(t) \;\cdot \{\|y\| + \|Ty\|\}^2. \tag{2.44}$$

Therefore, inequality (2.44) implies that for $y \neq 0$

$$\frac{\|By - B_N y\|}{\|y\|_{T_1}} \le k_6^{\frac{1}{2}} \left(\sup_{t\in I_N} F(t)\right)^{\frac{1}{2}}. \tag{2.45}$$

Since $I_N = [N, \infty)$, equations (2.2) imply that $\sup_{t\in I_N} F(t) \to 0$ as $N \to \infty$ so that, via inequality (2.45), we have

$$\|B - B_N\| \to 0 \text{ as } N \to \infty\,. \;\#$$

Claim 2.1.3 *Each B_N is T_1-compact.*

Proof of Claim 2.1.3:

Let $\{y_n\}_{n=1}^{\infty} \subset D(T_1)$ be a T_1-bounded sequence where

$$y_n = \begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix} \text{ for each n.}$$

We need to show that for each N the sequence $\{B_N y_n\}_{n=1}^{\infty}$ has a convergent subsequence. We make use of the Arzela-Ascoli Theorem.

Let us consider the sequence of the first component functions $\{y_{n,1}\}_{n=1}^{\infty}$. We show that this sequence is uniformly bounded on $[a, N]$. Partition $[a, N]$ by

$J_i = [t_i, t_{i+1}]$, $1 \le i \le k$ with $t_1 = a$, $t_{i+1} = t_i + \delta f(t_i)$ and $\delta \in (0, \varepsilon)$ chosen so that $N = t_{k+1} = t_k + \delta f(t_k)$. Via equation (2.12) of [3, page 571] in the proof of Theorem 1.7, we know there exists a constant $C_1 > 0$ such that for $t \in J_i$

$$(2.46) \qquad |y_{n,1}(t)|^2 \le C_1 \left\{ \frac{1}{\delta^2 f^2(t_i)} \left[\int_{J_i} f(\tau)\, d\tau \right] \left[\int_{J_i} f^{-1}(\tau) |y_{n,1}(\tau)|^2\, d\tau \right] + \left[\int_{J_i} f^{-1}(\tau)\, d\tau \right] \left[\int_{J_i} f(\tau) |y'_{n,1}(\tau)|^2\, d\tau \right] \right\}.$$

By applying Lemma 1.8, we obtain

$$(2.47) \qquad |y_{n,1}(t)|^2 \le C_1 \left\{ C_2 \delta^{-1} \left[\int_{J_i} f^{-1}(\tau) |y_{n,1}(\tau)|^2\, d\tau \right] + C_3\, \delta \left[\int_{J_i} f(\tau) |y'_{n,1}(\tau)|^2\, d\tau \right] \right\}$$

for some constants $C_2, C_3 > 0$ and for $t \in J_i$. Thus, there exists a constant $C_4 > 0$ such that for $t \in [a, N]$

$$(2.48) \qquad \begin{aligned} |y_{n,1}(t)|^2 &\le C_4 \left\{ \int_a^\infty f^{-1}(\tau) |y_{n,1}(\tau)|^2\, d\tau + \int_a^\infty f(\tau) |y'_{n,1}(\tau)|^2\, d\tau \right\} \\ &\le C_4 \left(\|y_n\|^2 + \|Ty_n\|^2 \right) \\ &\le C_4 \left(\|y_n\| + \|Ty_n\| \right)^2. \end{aligned}$$

Since $\{y_n\}_{n=1}^\infty$ is a T_1-bounded sequence, inequality (2.48) implies that the sequence $\{y_{n,1}\}_{n=1}^\infty$ is uniformly bounded on $[a, N]$.

Now, we show that $\{y_{n,1}\}_{n=1}^\infty$ is equicontinuous on $[a, N]$. For $s, t \in [a, N]$ we have via the Cauchy-Schwarz inequality

$$(2.49) \qquad \begin{aligned} |y_{n,1}(s) - y_{n,1}(t)| &= \left| \int_t^s y'_{n,1}(\tau)\, d\tau \right| \\ &\le \left| \int_t^s f^{-\frac{1}{2}}(\tau) f^{\frac{1}{2}}(\tau) |y'_{n,1}(\tau)|\, d\tau \right| \\ &\le \left| \int_t^s f^{-1}(\tau)\, d\tau \right|^{\frac{1}{2}} \left| \int_t^s f(\tau) |y'_{n,1}(\tau)|^2\, d\tau \right|^{\frac{1}{2}}. \end{aligned}$$

Since f is a positive, continuous function on $[a, \infty)$, f^{-1} is bounded above and below on $[a, N]$. This fact along with inequality (2.49) implies that there exists a constant

$C > 0$ such that for $s, t \in [a, N]$

$$
\begin{aligned}
(2.50) \qquad |y_{n,1}(s) - y_{n,1}(t)| &\leq C\,|s-t|^{\frac{1}{2}} \left| \int_t^s f(\tau) |y'_{n,1}(\tau)|^2 \, d\tau \right|^{\frac{1}{2}} \\
&\leq C\,|s-t|^{\frac{1}{2}} \|Ty_n\| \\
&\leq C\,|s-t|^{\frac{1}{2}} \|y_n\|_{T_1} .
\end{aligned}
$$

Since $\{y_n\}_{n=1}^{\infty}$ is a T_1-bounded sequence, inequality (2.50) implies that the sequence $\{y_{n,1}\}_{n=1}^{\infty}$ is equicontinuous on $[a, N]$.

Therefore, via the Arzela-Ascoli Theorem we conclude that there exists a subsequence $\{y_{n_k,1}\}_{k=1}^{\infty}$ of $\{y_{n,1}\}_{n=1}^{\infty}$ which converges uniformly on $[a, N]$. By repeating the above arguments [inequalities (2.46)-(2.50)] on the subsequence $\{y_{n_k,2}\}_{k=1}^{\infty}$ of $\{y_{n,2}\}_{n=1}^{\infty}$, we conclude that there exists a subsequence of $\{y_{n_k,2}\}_{k=1}^{\infty}$ which converges uniformly on $[a, N]$. Hence, there exists a subsequence of $\{y_n\}_{n=1}^{\infty}$ which converges uniformly on $[a, N]$. WLOG, we assume the sequence $\{y_n\}_{n=1}^{\infty}$ converges uniformly on $[a, N]$.

Now, suppressing the independent variable and using the inequality $2ab \leq a^2 + b^2$ we have for each N

$$
\begin{aligned}
(2.51) \quad \|B_N y_n - B_N y_m\|^2 &= \int_a^N f\{[q_1^2 + q_3^2]|y_{n,1} - y_{m,1}|^2 \\
&\quad + [q_1 q_4 + q_2 q_3](y_{n,1} - y_{m,1})\overline{(y_{n,2} - y_{m,2})} \\
&\quad + [q_1 q_4 + q_2 q_3]\overline{(y_{n,1} - y_{m,1})}(y_{n,2} - y_{m,2}) \\
&\quad + [q_2^2 + q_4^2]|y_{n,2} - y_{m,2}|^2\} \\
&\leq \int_a^N f\{[q_1^2 + q_3^2]|y_{n,1} - y_{m,1}|^2 \\
&\quad + 2\,|q_1 q_4 + q_2 q_3|\,|(y_{n,1} - y_{m,1})(y_{n,2} - y_{m,2})| \\
&\quad + [q_2^2 + q_4^2]|y_{n,2} - y_{m,2}|^2\} \\
&\leq \int_a^N f\{[q_1^2 + q_3^2]|y_{n,1} - y_{m,1}|^2 \\
&\quad + |q_1 q_4 + q_2 q_3|\,(|y_{n,1} - y_{m,1}|^2 + |y_{n,2} - y_{m,2}|^2) \\
&\quad + [q_2^2 + q_4^2]|y_{n,2} - y_{m,2}|^2\} .
\end{aligned}
$$

By adding nonnegative terms to the right-hand side of inequality (2.51), we obtain

$$\begin{aligned}
\|B_N y_n - B_N y_m\|^2 &\leq \int_a^N f\{[q_1^2 + q_3^2]\ \|y_n - y_m\|_2^2 + |q_1 q_4 + q_2 q_3|\ \|y_n - y_m\|_2^2 \\
&\quad +[q_2^2 + q_4^2]\ \|y_n - y_m\|_2^2\} \\
&\leq \sup_{a \leq t \leq N} \|y_n - y_m\|_2^2 \left\{ \int_a^N f\ [q_1^2 + q_3^2] \right. \\
&\quad \left. + \int_a^N f\ |q_1 q_4 + q_2 q_3| + \int_a^N f\ [q_2^2 + q_4^2] \} \right\}
\end{aligned}$$

where

$$\|y_n - y_m\|_2^2 = [y_{n,1}(t) - y_{m,1}(t)]^2 + [y_{n,2}(t) - y_{m,2}(t)]^2 .$$

By applying the argument used in inequality (2.23) to the inequality above, we see that the integrals on the right-hand side are finite on $[a, N]$. Hence, there exists a constant $C > 0$ such that

$$\|B_N y_n - B_N y_m\|^2 \leq C \sup_{a \leq t \leq N} \|y_n - y_m\|_2^2 .$$

Since $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the uniform norm, the above inequality implies that the sequence $\{B_N y_n\}_{n=1}^{\infty}$ is Cauchy in $\mathcal{L}_w^2(I)$ for each N. Therefore, $\{B_N y_n\}_{n=1}^{\infty}$ converges for each N as $n \to \infty$ since $\mathcal{L}_w^2(I)$ is complete. Hence, each B_N is T_1-compact. #

Since B_1 is the uniform limit of T_1-compact operators, B_1 is T_1-compact.

Necessity

We use contradiction arguments to show that each equation in (2.2) must hold.

Suppose that for some $\varepsilon \in (0, \frac{1}{2N_0})$ there exists a $\rho > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers such that $r_n \to \infty$ as $n \to \infty$ and for each n

(2.52) $$\int_{r_n}^{r_n + \varepsilon f(r_n)} f(t) q_i^2(t)\, dt \geq \rho \quad \text{for some } i \in \{1, 2, 3, 4\}.$$

Suppose $i = 1$. Let $\{\Phi_r\}_{r \geq a}$ be defined by (2.28). Then via inequalities (2.31) and (2.35) there exist constants $C_2, C_4 > 0$ such that for each n

(2.53) $$\begin{aligned}
\|\Phi_{r_n}\|_{T_1}^2 &= (\|\Phi_{r_n}\| + \|T\Phi_{r_n}\|)^2 \\
&\leq (C_2^{\frac{1}{2}} + C_4^{\frac{1}{2}})^2.
\end{aligned}$$

Thus, $\{\Phi_{r_n}\}_{n=1}^{\infty}$ is a T_1-bounded sequence. Since B_1 is T_1-compact, $\{B\Phi_{r_n}\}_{n=1}^{\infty}$ has a convergent subsequence. WLOG, we assume $\{B\Phi_{r_n}\}$ converges, say to some y_0. Now, since $\phi_{r_n} \equiv 1$ on $[r_n, r_n + \varepsilon f(r_n)]$ and supp(ϕ_{r_n})=$[r_n - 2\varepsilon f(r_n), r_n + 2\varepsilon f(r_n)]$, we have for each n

$$
\begin{aligned}
(2.54) \qquad \rho &\leq \int_{r_n}^{r_n+\varepsilon f(r_n)} f(t) q_1^2(t)\, dt \\
&\leq \int_{r_n}^{r_n+\varepsilon f(r_n)} f(t)[q_1^2(t) + q_3^2(t)]\, dt \\
&\leq \int_{r_n-2\varepsilon f(r_n)}^{r_n+2\varepsilon f(r_n)} f(t)[q_1^2(t) + q_3^2(t)]\phi_{r_n}^2(t)\, dt \\
&= \|B\Phi_{r_n}\|^2.
\end{aligned}
$$

Notice that a contradiction is reached if we show that $y_0 = 0$ a.e. in $[a, \infty)$. Let J_0 be a finite subinterval of $[a, \infty)$. Since $r_n \to \infty$ as $n \to \infty$ and supp(ϕ_{r_n})= $[r_n - 2\varepsilon f(r_n), r_n + 2\varepsilon f(r_n)]$, we conclude that $\phi_{r_n} \equiv 0$ on J_0 for sufficiently large n. Hence, $\Phi_{r_n} \equiv 0$ and $B\Phi_{r_n} \equiv 0$ on J_0 for sufficiently large n. For such n

$$
\begin{aligned}
\|y_0\|_{J_0} &= \|y_0 - B\Phi_{r_n}\|_{J_0} \\
&\leq \|y_0 - B\Phi_{r_n}\|
\end{aligned}
$$

Since $B\Phi_{r_n} \to y_0$ as $n \to \infty$, and the left-hand side of the above inequality is independent of n, we have $\|y_0\|_{J_0} = 0$. Thus, $y_0 = 0$ a.e. in $[a, \infty)$ since the interval J_0 is arbitrary. This contradiction implies that equation (2.2) holds for $i = 1$. If we begin with q_3 instead of q_1 in inequality (2.54), then the above argument shows that equation (2.2) holds for $i = 3$.

Moreover, by repeating the above argument with q_1 replaced by either q_2 or q_4 and Φ_r replaced by Ψ_r (as defined by (2.28)), we conclude that equation (2.2) holds for $i = 2$ and $i = 4$.

Note that the proof of necessity shows that (2.2) holds for every $\varepsilon \in (0, \frac{1}{2N_0})$. □

Next, we prove two corollaries of Theorem 2.1. We consider the maximal operators associated with the following differential expressions on the interval I:

$$(\widehat{T}z)(x) = \frac{1}{2}\begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix}\left\{\left[\begin{pmatrix} 0 & -x^{\alpha} \\ x^{\alpha} & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right]^{\bullet} + \begin{pmatrix} 0 & -x^{\alpha} \\ x^{\alpha} & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^{\bullet}\right\}$$

(2.55)

and

$$(\widehat{B}z)(x) = \begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix}\begin{pmatrix} b_1 & b_4 \\ b_3 & b_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{2.56}$$

where the coefficients b_1, b_2, b_3, and b_4 are assumed to be real, locally Lebesgue integrable functions and $^{\bullet} = \frac{d}{dx}$.

Corollary 2.2 *Let $I = [a, \infty)$ for some $a > 0$ and let $\gamma - \alpha \geq -1$ and $N_0 = |\alpha - \gamma| a^{\alpha-\gamma-1}$. Then*

(i) $\widehat{B}_1$ is $\widehat{T}_1$-bounded if and only if

$$\sup_{x \in I}\; x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{-2\gamma}\, b_i^2(u)\, du < \infty, \quad i = 1, 2, 3, 4, \tag{2.57}$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$;

(ii) $\widehat{B}_1$ is $\widehat{T}_1$-compact if and only if

$$\lim_{x \to \infty} x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{-2\gamma}\, b_i^2(u)\, du = 0, \quad i = 1, 2, 3, 4, \tag{2.58}$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$.

Proof:

We begin by applying an argument in Chapter 1 to transform the differential expressions unitarily. Notice that $\widehat{T}$ is of the form (1.21) with $W_1(x) = W_2(x) = x^{\gamma}$, $\Theta = \begin{pmatrix} 0 & -x^{\alpha} \\ x^{\alpha} & 0 \end{pmatrix}$, and $\mathcal{N} = 0$. Let $\mu_1(x) = \mu_2(x) = x^{\frac{\alpha}{2}}$ so that $y(x) = x^{\frac{\alpha}{2}} z(x)$ and, via equations (1.22), (1.24), (1.25), and (1.26),

$$(Ty)(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} \tag{2.59}$$

and

$$(2.60) \qquad (By)(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix} \begin{pmatrix} x^{-\alpha}b_1 & x^{-\alpha}b_4 \\ x^{-\alpha}b_3 & x^{-\alpha}b_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Now, we apply Theorem 2.1 (with $f(x) = x^{\alpha-\gamma}$ and $q_i(x) = x^{-\alpha}b_i(x)$, for $i = 1, 2, 3$, and 4) to the maximal operators T_1 and B_1 associated with the transformed differential expressions (2.59) and (2.60), respectively. Notice that for $x \in I$

$$\begin{aligned} |\dot{f}(x)| &= |\alpha - \gamma| x^{\alpha-\gamma-1} \\ &\leq |\alpha - \gamma| a^{\alpha-\gamma-1} \end{aligned}$$

since $\alpha - \gamma - 1 \leq 0$. Thus, the sharpest constant N_0 in Theorem 2.1 is given by

$$N_0 = |\alpha - \gamma| a^{\alpha-\gamma-1}.$$

Hence, B_1 is T_1-bounded if and only if

$$\sup_{x \in I} \; x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{2(\alpha-\gamma)} [u^{-\alpha} \, b_i(u)]^2 \, du < \infty, \quad i = 1, 2, 3, 4,$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$, i.e., B_1 is T_1-bounded if and only if

$$\sup_{x \in I} \; x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{-2\gamma} \, b_i^2(u) \, du < \infty, \quad i = 1, 2, 3, 4,$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$. Since the transformation is unitary, inequalities (2.57) hold if and only if $\widehat{B}_1$ is $\widehat{T}_1$-bounded.

Similarly, B_1 is T_1-compact if and only if

$$\lim_{x \to \infty} \; x^{\gamma-\alpha} \int_x^{x+\varepsilon x^{\alpha-\gamma}} u^{-2\gamma} \, b_i^2(u) \, du = 0, \quad i = 1, 2, 3, 4,$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$. Thus, via the unitary transformation, equations (2.58) hold if and only in $\widehat{B}_1$ is $\widehat{T}_1$-compact. □

Corollary 2.3 *Let* $I = (0, \frac{1}{a}]$ *for some* $a > 0$ *and let* $\gamma - \alpha \leq -1$ *and* $N_0 = |\gamma - \alpha + 2| a^{\gamma-\alpha+1}$. *Then*

(i) $\widehat{B}_1$ *is* $\widehat{T}_1$*-bounded if and only if*

$$(2.61) \qquad \sup_{x \in I} \; x^{\gamma-\alpha+2} \int_{x-\varepsilon' x^{\alpha-\gamma}}^{x} u^{-2\gamma-2} \, b_i^2(u) \, du < \infty, \quad i = 1, 2, 3, 4,$$

for some sufficiently small ε';

(ii) $\widehat{B}_1$ is $\widehat{T}_1$-compact if and only if

$$(2.62) \qquad \lim_{x\to\infty} x^{\gamma-\alpha+2} \int_{x-\varepsilon' x^{\alpha-\gamma}}^{x} u^{-2\gamma-2}\, b_i^2(u)\, du = 0, \quad i = 1,2,3,4,$$

for some sufficiently small ε'.

Proof:

We prove this result by using a unitary transformation to transform the singularity at 0 to a singularity at ∞ and then applying the Theorem 2.1 to the new operators.

Again, we use the argument in Chapter 1 to transform the operators T_1 and B_1 unitarily. Let $\mu_1(x) = \mu_2(x) = x^{\frac{\alpha}{2}}$ so that $y(t) = x^{\frac{\alpha}{2}} z(x), t = \frac{1}{x}$ and, via equations (1.22), (1.24), (1.25), and (1.26),

$$(2.63) \qquad (Ty)(t) = -\begin{pmatrix} t^{\gamma-\alpha+2} & 0 \\ 0 & t^{\gamma-\alpha+2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$$

and

$$(2.64) \qquad (By)(t) = \begin{pmatrix} t^{\gamma-\alpha+2} & 0 \\ 0 & t^{\gamma-\alpha+2} \end{pmatrix} \begin{pmatrix} t^{\alpha-2}b_1 & t^{\alpha-2}b_4 \\ t^{\alpha-2}b_3 & t^{\alpha-2}b_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Note that if $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is replaced with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the results of Theorem 2.1 hold. Let us call this version Theorem 2.1*.

Now, we apply Theorem 2.1* (with $f(t) = t^{\gamma-\alpha+2}$ and $q_i(t) = x^{2-\alpha} b_i(x), t = \frac{1}{x}$, for $i = 1, 2, 3,$ and 4) to the maximal operators T_1 and B_1 associated with the transformed differential expressions (2.63) and (2.64), respectively. Notice that for $t \in [a, \infty)$

$$\begin{aligned} |f'(t)| &= |\gamma - \alpha + 2| t^{\gamma-\alpha+1} \\ &\leq |\gamma - \alpha + 2| a^{\gamma-\alpha+1} \end{aligned}$$

since $\gamma - \alpha + 1 \leq 0$. Thus, the sharpest constant N_0 in Theorem 2.1 is given by

$$N_0 = |\gamma - \alpha + 2| a^{\gamma-\alpha+1}.$$

Hence, B_1 is T_1-bounded if and only if

$$\sup_{a\le t\le\infty} \quad t^{\alpha-\gamma-2} \int_t^{t+\varepsilon t^{\gamma-\alpha+2}} \tau^{2(\gamma-\alpha+2)}[\tau^{\alpha-2}\, b_i(\frac{1}{\tau})]^2\, d\tau < \infty, \quad i=1,2,3,4,$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$, i.e., B_1 is T_1-bounded if and only if

$$\sup_{a\le t\le\infty} \quad t^{\alpha-\gamma-2} \int_t^{t+\varepsilon t^{\gamma-\alpha+2}} \tau^{2\gamma}\, b_i^2(\frac{1}{\tau})\, d\tau < \infty, \quad i=1,2,3,4,$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$. By a change of variables (letting $u = \frac{1}{\tau}$),

$$\int_t^{t+\varepsilon t^{\gamma-\alpha+2}} \tau^{2\gamma}\, b_i^2(\frac{1}{\tau})\, d\tau = \int_{\frac{1}{t+\varepsilon t^{\gamma-\alpha+2}}}^{\frac{1}{t}} u^{-2\gamma-2}\, b_i^2(u)\, du\,.$$

Again, we apply a change of variables (letting $x = \frac{1}{t}$) to obtain

$$t^{\alpha-\gamma-2} \int_{\frac{1}{t+\varepsilon t^{\gamma-\alpha+2}}}^{\frac{1}{t}} u^{-2\gamma-2}\, b_i^2(u)\, du = x^{\gamma-\alpha+2} \int_{\frac{x}{1+\varepsilon x^{\alpha-\gamma-1}}}^{x} u^{-2\gamma-2}\, b_i^2(u)\, du\,.$$

Thus, B_1 is T_1-bounded if and only if

$$\sup_{x\in I} \quad x^{\gamma-\alpha+2} \int_{\frac{x}{1+\varepsilon x^{\alpha-\gamma-1}}}^{x} u^{-2\gamma-2}\, b_i^2(u)\, du < \infty, \quad i=1,2,3,4, \tag{2.65}$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$. Now, we show that (2.65) is equivalent to (2.61). Suppose (2.65) holds. We seek an $\varepsilon' > 0$ such that

$$\frac{x}{1+\varepsilon x^{\alpha-\gamma-1}} < x - \varepsilon' x^{\alpha-\gamma}\,,$$

i..e.,

$$\varepsilon' < \frac{\varepsilon}{1+\varepsilon x^{\alpha-\gamma-1}}\,.$$

Since $0 < x \le a$ and $\alpha-\gamma-1 \ge 0$, the above inequality holds if we choose

$$\varepsilon' < \frac{\varepsilon}{1+\varepsilon a^{\alpha-\gamma-1}}\,.$$

Thus, for $i = 1,2,3,$ and 4

$$\int_{\frac{x}{1+\varepsilon x^{\alpha-\gamma-1}}}^{x} u^{-2\gamma-2}\, b_i^2(u)\, du > \int_{x-\varepsilon' x^{\alpha-\gamma}}^{x} u^{-2\gamma-2}\, b_i^2(u)\, du$$

so that (2.65) implies (2.61) holds for all $\varepsilon' \in (0, \frac{\varepsilon}{1+\varepsilon a^{\alpha-\gamma-1}})$. Now, suppose (2.61) holds for some $\varepsilon' \in (0, \frac{\varepsilon}{1+\varepsilon a^{\alpha-\gamma-1}})$, $\varepsilon \in (0, \frac{1}{2N_0})$. We seek an $\varepsilon'' > 0$ such that

$$\frac{x}{1+\varepsilon'' x^{\alpha-\gamma-1}} > x - \varepsilon' x^{\alpha-\gamma},$$

i..e.,

$$\varepsilon'' < \frac{\varepsilon'}{1-\varepsilon' x^{\alpha-\gamma-1}}.$$

If we choose $\varepsilon'' < \varepsilon'$, then the above inequality holds. Hence, for $i = 1, 2, 3$, and 4

$$\int_{\frac{x}{1+\varepsilon'' x^{\alpha-\gamma-1}}}^{x} u^{-2\gamma-2}\, b_i^2(u)\, du < \int_{x-\varepsilon' x^{\alpha-\gamma}}^{x} u^{-2\gamma-2}\, b_i^2(u)\, du$$

so that (2.61) implies (2.65) holds for all $\varepsilon'' \in (0, \varepsilon')$. Therefore, (2.65) is equivalent to (2.61). Since the transformation is unitary, inequalities (2.61) hold if and only if $\widehat{B}_1$ is $\widehat{T}_1$-bounded. □

We conclude this chapter with an argument to show that the operator T of Theorem 2.1 is limit point at ∞. Since T is a formally self-adjoint operator of the form (1.5), T is limit point at ∞ iff $d_+(T_0) + d_-(T_0) = 2$. Hence, T is limit point at ∞ iff $d_+(T_0) = d_-(T_0) = 1$ since the coefficients of T are real. In order to determine $d_+(T_0)$ and $d_-(T_0)$, it is enough to consider dim $N(T_1)$ (see [2, Theorem 9.11.2]). Note that inequalities (1.8) and (1.9) imply that the deficiency indices are either one or two since $d_+(T_0) = d_-(T_0)$. Suppose that $d_+(T_0) = d_-(T_0) = 2$. Then every solution to $Ty = 0$ is in $\mathcal{L}_w^2(I)$. Hence, we consider $y \in D(T_1)$ which satisfy the following system:

$$\begin{aligned} y_1'(t) &= 0 \\ y_2'(t) &= 0 \end{aligned}$$

By solving each equation we obtain the general solutions $y_1(t) = c_1$ and $y_2(t) = c_2$ for constants c_1, c_2. Thus, each solution of $Ty = 0$ is a linear combination of

$$Y_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } Y_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let Y_3 be a nontrivial solution of $Ty = 0$. Then for some constant $C > 0$

$$\begin{aligned} \|Y_3\|^2 &= \int_a^\infty Y_3^*(t) \begin{pmatrix} f^{-1}(t) & 0 \\ 0 & f^{-1}(t) \end{pmatrix} Y_3(t)\, dt \\ &= C \int_a^\infty f^{-1}(t)\, dt\,. \end{aligned}$$

Via the hypothesis on f, we conclude that $f(t) \le N_0\, t + b$ for some constant b. Hence,

$$\int_a^\infty f^{-1}(t)\, dt \ge \int_a^\infty (N_0\, t + b)^{-1}\, dt = \infty$$

so that $Y_3 \notin D(T_1)$. Since Y_3 is arbitrary, there exist no nontrivial $\mathcal{L}_w^2(I)$-solutions to $Ty = 0$, i.e., $d_+(T_0) = d_-(T_0) \ne 2$. This contradiction implies that the deficiency indicies are one. Therefore, T is limit point at ∞.

Chapter 3

Perturbations of T: Limit Circle Case

In this chapter, we consider perturbations $\widehat{B}$ of the higher-ordered differential operator $\widehat{T}$, where $\widehat{T}$ and $\widehat{B}$ are defined by (2.55) and (2.56), respectively. In Theorem 3.1 we consider the minimal operators $\widehat{T}_0$ and $\widehat{B}_0$ associated with the differential expressions (2.55) and (2.56), respectively, on the interval $[a,\infty), a > 0$, with $\gamma - \alpha < -1$. The proof follows a similar argument as that of Theorem 2.1 and applies the Hardy inequality:

$$\int_a^b t^\beta |y(t)|^2 dt \;\leq\; \frac{4}{(\beta+1)^2} \int_a^b t^{\beta+2}|y'(t)|^2 dt\,, \tag{3.1}$$

where $-\infty < a < b < \infty, \beta \neq -1$, and $y(a) = 0 = y(b)$.

As an application of Theorem 3.1, Corollary 3.2 deals with the maximal operators $\widehat{T}_1$ and $\widehat{B}_1$ associated with the differential expressions (2.55) and (2.56), respectively, on the interval $[a,\infty)$, $a \geq 1$, with $\gamma - \alpha < -1$. We make use of several results of Chapter 1 in the proof. In this chapter we let $' = \frac{d}{dx}$.

Theorem 3.1 *Let* $I = [a,\infty)$ *for some* $a > 0$ *and let* $\gamma - \alpha < -1$. *Then*
(i) $\widehat{B}_0$ *is* $\widehat{T}_0$*-bounded if and only if*

$$\sup_{x\in I} \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha}\, b_i^2(u)\, du < \infty, \quad i = 1,2,3,4, \tag{3.2}$$

for some $\varepsilon \in (0, \frac{1}{2})$ *;*

(ii) $\widehat{B}_0$ *is* $\widehat{T}_0$*-compact if and only if*

$$(3.3) \qquad \lim_{x\to\infty} \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha}\, b_i^2(u)\, du = 0, \quad i = 1,2,3,4,$$

for some $\varepsilon \in (0, \frac{1}{2})$.

Proof:

(i) Sufficiency

As in Corollary 2.2, we make a unitary transformation. Thus, we can consider $\widehat{T}$ and $\widehat{B}$ to be of the form (2.59) and (2.60), respectively. We begin by showing that $\widehat{B}_0'$ is $\widehat{T}_0'$-bounded if inequalities (3.2) hold. Let us first consider $z \in D(\widehat{T}_0')$ whose second component is zero. Since z has compact support in the interior of I, there exists a $b < \infty$ such that the support of z_1 is contained in $[a, b]$. Then we have

$$(3.4) \qquad \|z\|^2 = \int_a^b u^{\gamma-\alpha}|z_1(u)|^2\, du,$$

$$(3.5) \qquad \|\widehat{T}z\|^2 = \int_a^b u^{\alpha-\gamma}|z_1'(u)|^2\, du, \text{ and}$$

$$(3.6) \qquad \|\widehat{B}z\|^2 = \int_a^b u^{-\gamma-\alpha}[b_1(u)^2 + b_3(u)^2]|z_1(u)|^2\, du\ .$$

Now, we show that the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2})$ with $N = x^{-\gamma-\alpha}(b_1^2 + b_3^2)$, $W = x^{\alpha-\gamma-2}$, $P = x^{\alpha-\gamma}$, and $f = x$. By applying Lemma 1.8 with $g \equiv 1$ and $f = x$, we know that positive and negative powers of x are essentially constant on intervals of length εx. Thus, we have for some constants $C_1, C_2, \hat{C}_1$, and $\hat{C}_2 > 0$

$$\begin{aligned}
(3.7) \quad S_1 &= \sup_{x\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_x^{x+\varepsilon x} u^{\gamma-\alpha}\, du\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}[b_1(u)^2 + b_3(u)^2]\, du\right]\right\} \\
&\le \frac{C_1}{\varepsilon}\sup_{x\in I}\left\{x^{\gamma-\alpha+1}\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}[b_1(u)^2 + b_3(u)^2]\, du\right\} \\
&\le \frac{\hat{C}_1}{\varepsilon}\sup_{x\in I}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_1(u)^2 + b_3(u)^2]\, du\right\} \text{ and} \\
(3.8) \quad S_2 &= \sup_{x\in I}\left\{\frac{1}{\varepsilon^2 x^2}\left[\int_x^{x+\varepsilon x} u^{\gamma-\alpha+2}\, du\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}[b_1(u)^2 + b_3(u)^2]\, du\right]\right\}
\end{aligned}$$

$$\begin{aligned} &\leq \frac{C_2}{\varepsilon} \sup_{x\in I} \left\{ x^{\gamma-\alpha+1} \int_x^{x+\varepsilon x} u^{-\gamma-\alpha}[b_1(u)^2 + b_3(u)^2]\, du] \right\} \\ &\leq \frac{\hat{C}_2}{\varepsilon} \sup_{x\in I} \left\{ \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha}[b_1(u)^2 + b_3(u)^2]\, du] \right\} \end{aligned}$$

for some $\varepsilon \in (0, \frac{1}{2})$. Inequalities (3.2), (3.7), and (3.8) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2})$. Therefore (via Theorem 1.7), there exists a constant $k_1 > 0$ such that

$$\begin{aligned} &\int_a^b u^{-\gamma-\alpha}[b_1(u)^2 + b_3(u)^2]|z_1(u)|^2\, du \\ &\qquad \leq k_1 \left\{ \int_a^b u^{\alpha-\gamma-2}|z_1(u)|^2\, du + \varepsilon^2 \int_a^b u^{\alpha-\gamma}|z_1'(u)|^2\, du \right\}. \end{aligned}$$

Applying the Hardy inequality (3.1) to the first integral on the right-hand side of the above inequality gives

$$\begin{aligned} &\int_a^b u^{-\gamma-\alpha}[b_1(u)^2 + b_3(u)^2]|z_1(u)|^2\, du \\ &\qquad \leq k_1 \left\{ \frac{4}{(\alpha-\gamma-1)} \int_a^b u^{\alpha-\gamma}|z_1'(u)|^2\, du + \varepsilon^2 \int_a^b u^{\alpha-\gamma}|z_1'(u)|^2\, du \right\}. \end{aligned} \tag{3.9}$$

By substituting (3.5) and (3.6) into inequality (3.9), we obtain for some constant $\hat{k}_1 > 0$

$$\begin{aligned} ||\widehat{B}z||^2 &\leq \hat{k}_1\, ||\widehat{T}z||^2 \\ &\leq \hat{k}_1\, (||z|| + ||\widehat{T}z||)^2 . \end{aligned}$$

Thus, $z \in D(\widehat{B}_0')$. Since z is arbitrary, the inequality above implies that $\widehat{B}_0'$ is $\widehat{T}_0'$-bounded on those $z \in D(\widehat{T}_0')$ with $z_2 \equiv 0$.

Let us now consider any $z \in D(\widehat{T}_0')$ whose first component is zero. Since z has compact support in the interior of I, there exists a $b < \infty$ such that the support of z_2 is contained in $[a, b]$. Then we have

$$||z||^2 = \int_a^b u^{\gamma-\alpha}|z_2(u)|^2\, du, \tag{3.10}$$

$$||\widehat{T}z||^2 = \int_a^b u^{\alpha-\gamma}|z_2'(u)|^2\, du, \text{ and} \tag{3.11}$$

$$||\widehat{B}z||^2 = \int_a^b u^{-\gamma-\alpha}[b_2(u)^2 + b_4(u)^2]|z_2(u)|^2\, du\ . \tag{3.12}$$

Following an argument similar to the one above, we show the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2})$ with $N = x^{-\gamma-\alpha}(b_2^2 + b_4^2)$, $W = x^{\alpha-\gamma-2}$, $P = x^{\alpha-\gamma}$, and $f = x$. For some constants $C_3, C_4, \hat{C}_3$, and $\hat{C}_4 > 0$, we have

$$\begin{aligned}
(3.13)\; S_1 &= \sup_{x\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_x^{x+\varepsilon x} u^{\gamma-\alpha}\,du\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}[b_2(u)^2 + b_4(u)^2]\,du\right]\right\} \\
&\leq \frac{C_3}{\varepsilon}\sup_{x\in I}\left\{x^{\gamma-\alpha+1}\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}[b_2(u)^2 + b_4(u)^2]\,du\right\} \\
&\leq \frac{\hat{C}_3}{\varepsilon}\sup_{x\in I}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_2(u)^2 + b_4(u)^2]\,du\right\} \text{ and} \\
(3.14)\; S_2 &= \sup_{x\in I}\left\{\frac{1}{\varepsilon^2 x^2}\left[\int_x^{x+\varepsilon x} u^{\gamma-\alpha+2}\,du\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}[b_2(u)^2 + b_4(u)^2]\,du\right]\right\} \\
&\leq \frac{C_4}{\varepsilon}\sup_{x\in I}\left\{x^{\gamma-\alpha+1}\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}[b_3(u)^2 + b_4(u)^2]\,du]\right\} \\
&\leq \frac{\hat{C}_4}{\varepsilon}\sup_{x\in I}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_2(u)^2 + b_4(u)^2]\,du]\right\}
\end{aligned}$$

for some $\varepsilon \in (0, \frac{1}{2})$. Inequalities (3.2), (3.13), and (3.14) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2})$. Therefore (via Theorem 1.7), there exists a constant $k_2 > 0$ such that

$$\begin{aligned}
&\int_a^b u^{-\gamma-\alpha}[b_2(u)^2 + b_4(u)^2]|z_2(u)|^2\,du \\
&\quad\leq k_2\left\{\int_a^b u^{\alpha-\gamma-2}|z_2(u)|^2\,du + \varepsilon^2\int_a^b u^{\alpha-\gamma}|z_2'(u)|^2\,du\right\}.
\end{aligned}$$

Applying the Hardy inequality (3.1) to the first integral on the right-hand side of the above inequality gives

$$\begin{aligned}
(3.15)\quad &\int_a^b u^{-\gamma-\alpha}[b_2(u)^2 + b_4(u)^2]|z_2(u)|^2\,du \\
&\quad\leq k_2\left\{\frac{4}{(\alpha-\gamma-1)}\int_a^b u^{\alpha-\gamma}|z_2'(u)|^2\,du + \varepsilon^2\int_a^b u^{\alpha-\gamma}|z_2'(u)|^2\,du\right\}.
\end{aligned}$$

By substituting (3.11) and (3.12) into inequality (3.15), we obtain for some constant $\hat{k}_2 > 0$

$$\begin{aligned}
||\widehat{B}z||^2 &\leq \hat{k}_2\,||\widehat{T}z||^2 \\
&\leq \hat{k}_2\,(||z|| + ||\widehat{T}z||)^2.
\end{aligned}$$

Thus, $z \in D(\widehat{B}_0')$. Since z is arbitrary, the inequality above implies that $\widehat{B}_0'$ is $\widehat{T}_0'$-bounded on those $z \in D(\widehat{T}_0')$ with $z_1 \equiv 0$.

Finally, we consider any $z \in D(\widehat{T}_0')$. As before there exists a $b < \infty$ such that the support of z_1 and the support of z_2 are contained in $[a, b]$. Then we have, suppressing the independent variable,

$$\|z\|^2 = \int_a^b u^{\gamma-\alpha}[|z_1|^2 + |z_2|^2] , \tag{3.16}$$

$$\|\widehat{T}z\|^2 = \int_a^b u^{\alpha-\gamma}[|z_1'|^2 + |z_2'|^2] , \text{ and} \tag{3.17}$$

$$\begin{aligned} \|\widehat{B}z\|^2 &= \int_a^b u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|z_1|^2 + [b_1b_4 + b_2b_3]z_1\bar{z}_2 \\ &\quad +[b_1b_4 + b_2b_3]\bar{z}_1 z_2 + [b_2^2 + b_4^2]|z_2|^2\} \\ &\leq \int_a^b u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|z_1|^2 + 2\,|b_1b_4 + b_2b_3|\,|z_1z_2| \\ &\quad +[b_2^2 + b_4^2]|z_2|^2\} . \end{aligned} \tag{3.18}$$

We make use of the inequality $2ab \leq a^2 + b^2$ in inequality (3.18) to obtain

$$\begin{aligned} \|\widehat{B}z\|^2 &\leq \int_a^b u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|z_1|^2 + |b_1b_4 + b_2b_3|\,[|z_1|^2 + |z_2|^2] \\ &\quad +[b_2^2 + b_4^2]|z_2|^2\} . \end{aligned} \tag{3.19}$$

Claim 3.1.1 *There exist constants $k_3, k_4 > 0$ such that*

$$\begin{aligned} &\int_a^b u^{-\gamma-\alpha}\ |b_1b_4 + b_2b_3|\,|z_1|^2 \\ &\quad \leq k_3 \left\{\frac{4}{(\alpha-\gamma-1)}\int_a^b u^{\alpha-\gamma}|z_1'|^2 + \varepsilon^2 \int_a^b u^{\alpha-\gamma}|z_1'|^2\right\} \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\int_a^b u^{-\gamma-\alpha}\ |b_1b_4 + b_2b_3|\,|z_2|^2 \\ &\quad \leq k_4 \left\{\frac{4}{(\alpha-\gamma-1)}\int_a^b u^{\alpha-\gamma}|z_2'|^2 + \varepsilon^2 \int_a^b u^{\alpha-\gamma}|z_2'|^2\right\} \end{aligned} \tag{3.21}$$

Proof of Claim 3.1.1:

By following the argument in Claim 2.1.1, we know that

$$\int_x^{x+\varepsilon x} u^{2-2\alpha}|b_1b_4 + b_2b_3|\,du \tag{3.22}$$

$$\leq \left(\int_x^{x+\varepsilon x} u^{2-2\alpha} b_1^2\, du\right)^{\frac{1}{2}} \left(2\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_4^2 + b_2^2]\, du\right)^{\frac{1}{2}}$$
$$+ \left(\int_x^{x+\varepsilon x} u^{2-2\alpha} b_2^2\, du\right)^{\frac{1}{2}} \left(2\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_1^2 + b_3^2]\, du\right)^{\frac{1}{2}}$$

Now, we show that the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2})$ with $N = x^{-\gamma-\alpha}|b_1 b_4 + b_2 b_3|$, $W = x^{\alpha-\gamma-2}$, $P = x^{\alpha-\gamma}$, and $f = x$. Via Lemma 1.8 and inequality (3.22) we have for some constants $C_5, C_6, \hat{C}_5$, and $\hat{C}_6 > 0$

$$
\begin{aligned}
(3.23)\quad S_1 &= \sup_{x\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_x^{x+\varepsilon x} u^{\gamma-\alpha}\, du\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}|b_1 b_4 + b_2 b_3|\, du\right]\right\} \\
&\leq \frac{C_5}{\varepsilon}\sup_{x\in I}\left\{x^{\gamma-\alpha+1}\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}|b_1 b_4 + b_2 b_3|\, du\right\} \\
&\leq \frac{\hat{C}_5}{\varepsilon}\sup_{x\in I}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}|b_1 b_4 + b_2 b_3|\, du\right\} \\
&\leq \frac{\hat{C}_5}{\varepsilon}\sup_{x\in I}\left\{\left(\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha} b_1^2\, du\right)^{\frac{1}{2}}\left(\frac{2}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_4^2 + b_2^2]\, du\right)^{\frac{1}{2}}\right. \\
&\qquad \left. + \left(\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha} b_2^2\, du\right)^{\frac{1}{2}}\left(\frac{2}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_1^2 + b_3^2]\, du\right)^{\frac{1}{2}}\right\} \text{ and} \\
(3.24)\quad S_2 &= \sup_{x\in I}\left\{\frac{1}{\varepsilon^2 x^2}\left[\int_x^{x+\varepsilon x} u^{\gamma-\alpha+2}\, du\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}|b_1 b_4 + b_2 b_3|\, du\right]\right\} \\
&\leq \frac{C_6}{\varepsilon}\sup_{x\in I}\left\{x^{\gamma-\alpha+1}\int_x^{x+\varepsilon x} u^{-\gamma-\alpha}|b_1 b_4 + b_2 b_3|\, du]\right\} \\
&\leq \frac{\hat{C}_6}{\varepsilon}\sup_{x\in I}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}|b_1 b_4 + b_2 b_3|\, du]\right\} \\
&\leq \frac{\hat{C}_6}{\varepsilon}\sup_{x\in I}\left\{\left(\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha} b_1^2\, du\right)^{\frac{1}{2}}\left(\frac{2}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_4^2 + b_2^2]\, du\right)^{\frac{1}{2}}\right. \\
&\qquad \left. + \left(\frac{1}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha} b_2^2\, du\right)^{\frac{1}{2}}\left(\frac{2}{x}\int_x^{x+\varepsilon x} u^{2-2\alpha}[b_1^2 + b_3^2]\, du\right)^{\frac{1}{2}}\right\}
\end{aligned}
$$

for some $\varepsilon \in (0, \frac{1}{2})$. Inequalities (3.2), (3.23), and (3.24) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2})$. An application of Theorem 1.7 gives us

$$\int_a^b u^{-\gamma-\alpha}\, |b_1 b_4 + b_2 b_3|\, |z_1|^2$$
$$\leq k_3\, \left\{\int_a^b u^{\alpha-\gamma-2}|z_1|^2 \; + \varepsilon^2 \int_a^b u^{\alpha-\gamma}|z_1'|^2\right\}$$

and

$$\int_a^b u^{-\gamma-\alpha}\, |b_1 b_4 + b_2 b_3|\, |z_2|^2$$
$$\leq k_4 \left\{ \int_a^b u^{\alpha-\gamma-2} |z_2|^2 + \varepsilon^2 \int_I u^{\alpha-\gamma} |z_2'|^2 \right\}$$

for some constants $k_3, k_4 > 0$. Inequalities (3.20) and (3.21) follow by applying the Hardy inequality (3.1) to the first integral on the right-hand side of each inequality above. #

Therefore, via inequalities (3.9), (3.15), (3.20), and (3.21) we obtain from inequality (3.19) that

$$\begin{aligned} \|\widehat{B}z\|^2 \leq\ & (k_1 + k_3) \left\{ \frac{4}{(\alpha-\gamma-1)} \int_a^b u^{\alpha-\gamma} |z_1'|^2 + \varepsilon^2 \int_a^b u^{\alpha-\gamma} |z_1'|^2 \right\} \\ & + (k_2 + k_4) \left\{ \frac{4}{(\alpha-\gamma-1)} \int_a^b u^{\alpha-\gamma} |z_2'|^2 + \varepsilon^2 \int_I u^{\alpha-\gamma} |z_2'|^2 \right\}, \end{aligned}$$

i.e., there exists a constant $C > 0$ such that

$$\begin{aligned} \|\widehat{B}z\|^2 &\leq C\, \|\widehat{T}z\|^2 \\ &\leq C\, (\|z\| + \|\widehat{T}z\|)^2 . \end{aligned}$$

Thus, $z \in D(\widehat{B}_0')$. Since z is arbitrary, the inequality above implies that $\widehat{B}_0'$ is $\widehat{T}_0'$-bounded. Via Theorem 1.2, $\widehat{B}_0$ is $\widehat{T}_0$-bounded.

<u>Necessity</u>

Fix $\varepsilon \in (0, \frac{1}{2})$. For each $r \geq a$ define Φ_r and Ψ_r to be the vector-valued functions, with compact support, given by (2.28) where $f(r) = r$. Then via Lemma 1.8 there exists a constant $C_1 > 0$ such that for each $r \geq a$

$$\begin{aligned} \|\Phi_r\|^2 &= \int_I x^{\gamma-\alpha} \phi_r^2(x)\, dx \\ &= \int_{r-2\varepsilon r}^{r+2\varepsilon r} x^{\gamma-\alpha} \phi^2 \left(\frac{x-r}{\varepsilon r} \right) dx \\ &\leq C_1 r^{\gamma-\alpha} \int_{r-2\varepsilon r}^{r+2\varepsilon r} \phi^2 \left(\frac{x-r}{\varepsilon r} \right) dx . \end{aligned} \tag{3.25}$$

By a change of variables (letting $u = \dfrac{x-r}{\varepsilon r}$ in (3.25)),

$$\|\Phi_r\|^2 \le C_1\, \varepsilon r^{\gamma-\alpha+1} \int_{-2}^{2} \phi^2(u)\, du\,. \tag{3.26}$$

Since ϕ is continuous, inequality (3.26) implies there exists a constant $C_2 > 0$ such that for $r \ge a$

$$\|\Phi_r\|^2 \le C_2\, r^{\gamma-\alpha+1}. \tag{3.27}$$

Similarly, for $r \ge a$

$$\|\Psi_r\|^2 \le C_2\, r^{\gamma-\alpha+1}. \tag{3.28}$$

Via Lemma 1.8 there exists a constant $C_3 > 0$ such that for each $r \ge a$

$$\begin{aligned} \|\widehat{T}\Phi_r\|^2 &= \int_I x^{\alpha-\gamma} \left[\frac{d}{dx}\phi_r(x)\right]^2 dx \\ &= \int_{r-2\varepsilon r}^{r+2\varepsilon r} x^{\alpha-\gamma} \left[\frac{d}{dx}\phi\left(\frac{x-r}{\varepsilon r}\right)\right]^2 dx \\ &\le C_3\, r^{\alpha-\gamma} \int_{r-2\varepsilon r}^{r+2\varepsilon r} \left[\frac{d}{dx}\phi\left(\frac{x-r}{\varepsilon r}\right)\right]^2 dx\,. \end{aligned} \tag{3.29}$$

By a change of variables (letting $u = \dfrac{x-r}{\varepsilon r}$ in (3.29)),

$$\|\widehat{T}\Phi_r\|^2 \le C_3 \varepsilon^{-1} r^{\alpha-\gamma-1} \int_{-2}^{2} [\phi'(u)]^2\, du\,. \tag{3.30}$$

Since ϕ' is continuous, inequality (3.30) implies there exists a constant $C_4 > 0$ such that for $r \ge a$

$$\|\widehat{T}\Phi_r\|^2 \le C_4\, r^{\alpha-\gamma-1}. \tag{3.31}$$

Similarly, for $r \ge a$

$$\|\widehat{T}\Psi_r\|^2 \le C_4\, r^{\alpha-\gamma-1}. \tag{3.32}$$

Since $\phi_r \equiv 1$ on $[r, r+\varepsilon r]$ and $\operatorname{supp}(\phi_r)=[r-2\varepsilon r, r+2\varepsilon r]$,

$$\begin{aligned} \int_{r}^{r+\varepsilon r} x^{-\gamma-\alpha}[b_1^2(x) + b_3^2(x)]\, dx &\le \int_{r-2\varepsilon r}^{r+2\varepsilon r} x^{-\gamma-\alpha}[b_1^2(x) + b_3^2(x)]\phi_r^2(x)\, dx \\ &= \|\widehat{B}\Phi_r\|^2\,. \end{aligned} \tag{3.33}$$

Thus, via the $\widehat{T}_0$-boundedness of $\widehat{B}_0$ and inequalities (3.27), (3.31), and (3.33), there exists a constant $C_5 > 0$ such that

$$\int_r^{r+\varepsilon r} x^{-\gamma-\alpha}[b_1^2(x) + b_3^2(x)]\,dx \le C_5\,(r^{\gamma-\alpha+1} + r^{\alpha-\gamma-1})\,.$$

After multiplying the above inequality by $r^{\gamma-\alpha+1}$, we apply Lemma 1.8 to the left-hand side and obtain a constant $C_6 > 0$ such that

$$\frac{1}{r}\int_r^{r+\varepsilon r} x^{2-2\alpha}[b_1^2(x) + b_3^2(x)]\,dx \le C_6\,(r^{2(\gamma-\alpha+1)} + 1) \tag{3.34}$$

Since $\gamma - \alpha + 1 < 0$, the right-hand side of the above inequality is bounded on I. Hence, inequality (3.2) holds for $i = 1, 3$.

Moreover,

$$\begin{aligned} \int_r^{r+\varepsilon r} x^{-\gamma-\alpha}[b_2^2(x) + b_4^2(x)]\,dx &\le \int_{r-2\varepsilon r}^{r+2\varepsilon r} x^{-\gamma-\alpha}[b_2^2(x) + b_4^2(x)]\phi_r^2(t)\,dx \\ &= \|\widehat{B}\Psi_r\|^2\,. \end{aligned} \tag{3.35}$$

Via the $\widehat{T}_0$-boundedness of $\widehat{B}_0$ and inequalities (3.28), (3.32), and (3.35),

$$\int_r^{r+\varepsilon r} x^{-\gamma-\alpha}[b_2^2(x) + b_4^2(x)]\,dx \le C_5\,(r^{\gamma-\alpha+1} + r^{\alpha-\gamma-1})\,.$$

After multiplying the above inequality by $r^{\gamma-\alpha+1}$, we apply Lemma 1.8 to the left-hand side to obtain a constant $C_7 > 0$ such that

$$\frac{1}{r}\int_r^{r+\varepsilon r} x^{2-2\alpha}[b_2^2(x) + b_4^2(x)]\,dx \le C_7\,(r^{2(\gamma-\alpha+1)} + 1) \tag{3.36}$$

Since $\gamma - \alpha + 1 < 0$, the right-hand side of the above inequality is bounded on I. Hence, inequality (3.2) holds for $i = 2, 4$.

Note that the proof of necessity shows that (3.2) holds for every $\varepsilon \in (0, \frac{1}{2})$.

(ii) Sufficiency

By the previous argument $\widehat{B}_0$ is $\widehat{T}_0$-bounded. Thus, $D(\widehat{T}_0) \subseteq D(\widehat{B}_0)$. For every positive integer $N > a$, define $\widehat{B}_N$ on $D(\widehat{T}_0)$by

$$\widehat{B}_N z = \begin{cases} \widehat{B}z & \text{on } [a, N] \\ 0 & \text{on } (N, \infty) \end{cases}$$

Notice that each $\widehat{B}_N$ is $\widehat{T}_0$-bounded with the same norm as $\widehat{B}_0$ since $||\widehat{B}_N z|| \leq ||Bz||$. In order to simplify the proof, we break the argument into two Claims.

Claim 3.1.2 *$\widehat{B}_N \to \widehat{B}_0$ in the space of bounded operators on $D(\widehat{T}_0)$ with the $\widehat{T}_0$-norm.*

Proof of Claim 3.1.2:

By definition $\widehat{T}_0$ is closed. Therefore, $D(\widehat{T}_0)$ is complete under the $\widehat{T}_0$-norm.

Let $z \in D(\widehat{T}_0)$. Since $\widehat{T}_0$ is the closure of $\widehat{T}_0'$, for each positive integer $n \geq 1$ there exists a $z_n \in D(\widehat{T}_0')$ such that

$$||z - z_n|| + ||\widehat{T}z - \widehat{T}z_n|| < \frac{1}{n}, \tag{3.37}$$

where $z_n = \begin{pmatrix} z_{n,1} \\ z_{n,2} \end{pmatrix}$ for each n.

Using the inequality $2ab \leq a^2 + b^2$, we have for each $z_n \in \widehat{T}_0'$

$$\begin{aligned} ||\widehat{B}z_n - \widehat{B}_N z_n||^2 &= \int_N^\infty u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|z_{n,1}|^2 + [b_1 b_4 + b_2 b_3] z_{n,1}\overline{z_{n,2}} \\ &\quad + [b_1 b_4 + b_2 b_3]\overline{z_{n,1}} z_{n,2} + [b_2^2 + b_4^2]|z_{n,2}|^2\} \\ &\leq \int_N^\infty u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|z_{n,1}|^2 + 2\,|b_1 b_4 + b_2 b_3|\,|z_{n,1} z_{n,2}| \\ &\quad + [b_2^2 + b_4^2]|z_{n,2}|^2\} \\ &\leq \int_N^\infty u^{-\gamma-\alpha}\{[b_1^2 + b_3^2]|z_{n,1}|^2 \\ &\quad + |b_1 b_4 + b_2 b_3|\,[|z_{n,1}|^2 + |z_{n,2}|^2] + [b_2^2 + b_4^2]|z_{n,2}|^2\}\ . \end{aligned} \tag{3.38}$$

Since each z_n has compact support in the interior of I, there exists a $b < \infty$ such that the support of $z_{n,1}$ and the support of $z_{n,2}$ are contained in $[a, b]$. Thus, we can apply the sufficiency argument in part (i) with $I_N = [N, \infty)$ to the last inequality in (3.38).

Via Theorem 1.7 and inequalities (3.7), (3.8), (3.13), (3.14), (3.23), and (3.24), there exists a constants K_1, K_2, K_3, and $K_4 > 0$ such that

$$||\widehat{B}z_n - \widehat{B}_N z_n||^2 \ \leq\ K_1 \sup_{x \in I_N} \left\{\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha}[b_1^2 + b_3^2]\, du\right\}$$

$$
\begin{aligned}
& \cdot \left\{ \int_a^b \; u^{\alpha-\gamma-2} |z_{n,1}|^2 \; du + \varepsilon^2 \int_a^b \; u^{\alpha-\gamma} |z_{n,1}'|^2 \; du \right\} \\
+ \;\; & K_2 \sup_{x \in I_N} \left\{ \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_1^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_4^2 + b_2^2] \, du \right)^{\frac{1}{2}} \right. \\
& \left. + \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_2^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_1^2 + b_3^2] \, du \right)^{\frac{1}{2}} \right\} \\
& \cdot \left\{ \int_a^b \; u^{\alpha-\gamma-2} |z_{n,1}|^2 \; + \varepsilon^2 \int_a^b \; u^{\alpha-\gamma} |z_{n,1}'|^2 \right\} \\
+ \;\; & K_3 \sup_{x \in I_N} \left\{ \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_1^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_4^2 + b_2^2] \, du \right)^{\frac{1}{2}} \right. \\
& \left. + \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_2^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_1^2 + b_3^2] \, du \right)^{\frac{1}{2}} \right\} \\
& \cdot \left\{ \int_a^b \; u^{\alpha-\gamma-2} |z_{n,2}|^2 \; + \varepsilon^2 \int_|^b \; u^{\alpha-\gamma} |z_{n,2}'|^2 \right\} \\
+ \;\; & K_4 \sup_{x \in I_N} \left\{ \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_2^2 + b_4^2] \, du \right\} \\
& \cdot \left\{ \int_a^b \; u^{\alpha-\gamma-2} |z_{n,2}|^2 \; du + \varepsilon^2 \int_a^b \; u^{\alpha-\gamma} |z_{n,2}'|^2 \; du \right\} .
\end{aligned}
$$

We apply the Hardy inequality (3.1), as before, to obtain constants $\hat{K}_1, \hat{K}_2, \hat{K}_3$, and $\hat{K}_4 > 0$ such that

$$
\begin{aligned}
\|\widehat{B} z_n - \widehat{B}_N z_n\|^2 \;\; \leq \;\; & \hat{K}_1 \sup_{x \in I_N} \left\{ \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_1^2 + b_3^2] \, du \right\} \cdot \left\{ \int_a^b \; u^{\alpha-\gamma} |z_{n,1}'|^2 \; du \right\} \\
+ \;\; & K_2 \sup_{x \in I_N} \left\{ \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_1^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_4^2 + b_2^2] \, du \right)^{\frac{1}{2}} \right. \\
& \left. + \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_2^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_1^2 + b_3^2] \, du \right)^{\frac{1}{2}} \right\} \\
& \cdot \left\{ \int_a^b \; u^{\alpha-\gamma} |z_{n,1}'|^2 \right\} \\
+ \;\; & K_3 \sup_{x \in I_N} \left\{ \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_1^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_4^2 + b_2^2] \, du \right)^{\frac{1}{2}} \right. \\
& \left. + \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_2^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_1^2 + b_3^2] \, du \right)^{\frac{1}{2}} \right\}
\end{aligned}
$$

$$\cdot \left\{ \int_a^b \ u^{\alpha-\gamma} |z'_{n,2}|^2 \right\}$$

$$+ \ K_4 \sup_{x \in I_N} \left\{ \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_2^2 + b_4^2] \, du \right\} \cdot \left\{ \int_a^b \ u^{\alpha-\gamma} |z'_{n,2}|^2 \ du \right\} .$$

Adding nonnegative terms to the right-hand side of the above inequality gives

$$\begin{aligned} (3.39) \qquad \|\widehat{B} z_n - \widehat{B}_N z_n\|^2 &\leq K_5 \sup_{x \in I_N} F(x) \int_a^b u^{\alpha-\gamma} [|z'_{n,1}|^2 + |z'_{n,2}|^2] \, du \\ &\leq K_5 \sup_{x \in I_N} F(x) \ \|\widehat{T} z_n\|^2 \\ &\leq K_5 \sup_{x \in I_N} F(x) \ \|z_n\|^2_{\widehat{T}'_0} \end{aligned}$$

for some constant $K_5 > 0$, where

$$\begin{aligned} F(x) \ = \ & \sup_{x \in I_N} \left\{ \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_1^2 + b_3^2] \, du \right. \\ & + \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_1^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_4^2 + b_2^2] \, du \right)^{\frac{1}{2}} \\ & + \left(\frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} b_2^2 \, du \right)^{\frac{1}{2}} \left(\frac{2}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_1^2 + b_3^2] \, du \right)^{\frac{1}{2}} \\ & \left. + \frac{1}{x} \int_x^{x+\varepsilon x} u^{2-2\alpha} [b_2^2 + b_4^2] \, du \right\} . \end{aligned}$$

Therefore, via the the triangle inequality, the $\widehat{T}_0$-boundedness of $\widehat{B}_0$, and inequalities (3.37) and (3.39), we have for each n

$$\begin{aligned} \|\widehat{B} z - \widehat{B}_N z\| &\leq \|\widehat{B} z - \widehat{B} z_n\| + \|\widehat{B} z_n - \widehat{B}_N z_n\| + \|\widehat{B}_N z_n - \widehat{B}_N z\| \\ &\leq 2K_6 \left(\|z - z_n\| + \|\widehat{T} z - \widehat{T} z_n\| \right) + \|\widehat{B} z_n - \widehat{B}_N z_n\| \\ &< \frac{2K_6}{n} + K_5^{\frac{1}{2}} \left(\sup_{x \in I_N} F(x) \right)^{\frac{1}{2}} \|z_n\|_{\widehat{T}'_0} \ \text{ for some constant } K_6 > 0 \, . \end{aligned}$$

By applying the triangle inequality and inequality (3.37) again, we obtain for each n

$$\begin{aligned} \|\widehat{B} z - \widehat{B}_N z\| &\leq \frac{2K_6}{n} + K_5^{\frac{1}{2}} \left(\sup_{x \in I_N} F(x) \right)^{\frac{1}{2}} \left(\|z_n - z\|_{\widehat{T}_0} + \|z\|_{\widehat{T}_0} \right) \\ &< \frac{2K_6}{n} + K_5^{\frac{1}{2}} \left(\sup_{x \in I_N} F(x) \right)^{\frac{1}{2}} \left(\frac{1}{n} + \|z\|_{\widehat{T}_0} \right) . \end{aligned}$$

We let $n \to \infty$ to obtain for each N

$$\|\widehat{B}z - \widehat{B}_N z\| \ \leq \ K_5^{\frac{1}{2}} \left(\sup_{x \in I_N} F(x) \right)^{\frac{1}{2}} \|z\|_{\widehat{T}_0} .$$

Since $I_N = [N, \infty)$, equations (3.3) imply that $\sup\limits_{x \in I_N} F(x) \to 0$ as $N \to \infty$ so that, via the above inequality, we have for $z \neq 0$

$$\|\widehat{B} - \widehat{B}_N\| \to 0 \ \text{ as } N \to \infty . \quad \#$$

Claim 3.1.3 *Each $\widehat{B}_N$ is $\widehat{T}_0$-compact.*

Proof of Claim 3.1.3:

Let $\{z_n\}_{n=1}^{\infty} \subset D(\widehat{T}_0)$ be a $\widehat{T}_0$-bounded sequence where $z_n = \begin{pmatrix} z_{n,1} \\ z_{n,2} \end{pmatrix}$ for each n. We need to show that for each N the sequence $\{\widehat{B}_N z_n\}_{n=1}^{\infty}$ has a convergent subsequence. We make use of the Arzela-Ascoli Theorem.

Let us consider the sequence of the first component functions $\{z_{n,1}\}_{n=1}^{\infty}$. We show that this sequence is uniformly bounded on $[a, N]$. Partition $[a, N]$ by $J_i = [x_i, x_{i+1}]$, $1 \leq i \leq k$ with $x_1 = a$, $x_{i+1} = x_i + \hat{\varepsilon} x_i$ and $\hat{\varepsilon} \in (0, \varepsilon)$ chosen so that $N = x_{k+1} = x_k + \hat{\varepsilon} x_k$. Via equation (2.12) of [3, page 571] in the proof of Theorem 1.7, we know there exists a constant $C_1 > 0$ such that for $x \in J_i$

$$\begin{aligned} (3.40) \qquad |z_{n,1}(x)|^2 \ &\leq \ C_1 \left\{ \frac{1}{\hat{\varepsilon}^2 x_i^2} \left[\int_{J_i} u^{\alpha-\gamma}\, du \right] \left[\int_{J_i} u^{\gamma-\alpha} |z_{n,1}(u)|^2\, du \right] \right. \\ &\quad \left. + \left[\int_{J_i} u^{\gamma-\alpha}\, du \right] \left[\int_{J_i} u^{\alpha-\gamma} |z'_{n,1}(u)|^2\, du \right] \right\} . \end{aligned}$$

Via Lemma 1.8 and inequality (3.40), there exists a constant $C_2 > 0$ such that for $x \in J_i$

$$|z_{n,1}(x)|^2 \leq C_2 \left\{ \frac{1}{\hat{\varepsilon}}\, x_i^{\alpha-\gamma-1} \int_{J_i} u^{\gamma-\alpha} |z_{n,1}(u)|^2\, du + \hat{\varepsilon}\, x_i^{\gamma-\alpha+1} \int_{J_i} u^{\alpha-\gamma} |z'_{n,1}(u)|^2\, du \right\} .$$

Since $x_i \in [a, N]$, the inequality above implies that for $x \in J_i$

$$(3.41) \qquad |z_{n,1}(x)|^2 \leq C_3 \left(\|z_n\|^2 + \|\widehat{T} z_n\|^2 \right)$$

for some constant $C_3 > 0$. Since $\{z_n\}_{n=1}^{\infty}$ is a $\widehat{T}_0$-bounded sequence, inequality (3.41) implies that the sequence $\{z_{n,1}\}_{n=1}^{\infty}$ is uniformly bounded on $[a, N]$.

Now, we show that $\{z_{n,1}\}_{n=1}^{\infty}$ is equicontinuous on $[a, N]$. For $s, t \in [a, N]$ we have via the Cauchy-Schwarz inequality

$$
\begin{aligned}
(3.42) \qquad |z_{n,1}(s) - z_{n,1}(t)| &= \left| \int_t^s z'_{n,1}(u)\, du \right| \\
&\leq \left| \int_t^s |z'_{n,1}(u)|\, du \right| \\
&= \left| \int_t^s u^{\frac{1}{2}(\gamma-\alpha)}\, u^{\frac{1}{2}(\alpha-\gamma)} |z'_{n,1}(u)|\, du \right| \\
&\leq \left| \int_t^s u^{\gamma-\alpha}\, du \right|^{\frac{1}{2}} \left| \int_t^s u^{\alpha-\gamma} |z'_{n,1}(u)|^2\, du \right|^{\frac{1}{2}} .
\end{aligned}
$$

Since $u^{\gamma-\alpha}$ is a positive, continuous function on $[a, N]$, there exists a constant $C > 0$ such that for $s, t \in [a, N]$ we have via (3.42)

$$
\begin{aligned}
(3.43) \qquad |z_{n,1}(s) - z_{n,1}(t)| &\leq C\, |s - t|^{\frac{1}{2}}\, \|\widehat{T} z_n\| \\
&\leq C\, |s - t|^{\frac{1}{2}}\, \|z_n\|_{\widehat{T}_0}
\end{aligned}
$$

for each n. Since $\{z_n\}_{n=1}^{\infty}$ is a $\widehat{T}_0$-bounded sequence, (3.43) implies that the sequence $\{z_{n,1}\}_{n=1}^{\infty}$ is equicontinuous on $[a, N]$.

Therefore, via the Arzela-Ascoli Theorem we conclude that there exists a subsequence $\{z_{n_k,1}\}_{k=1}^{\infty}$ of $\{z_{n,1}\}_{n=1}^{\infty}$ which converges uniformly on $[a, N]$. By repeating the above arguments (inequalities (3.40)-(3.43)) on the subsequence $\{z_{n_k,2}\}_{k=1}^{\infty}$ of $\{z_{n,2}\}_{n=1}^{\infty}$, we conclude that there exists a subsequence of $\{z_{n_k,2}\}_{k=1}^{\infty}$ which converges uniformly on $[a, N]$. Hence, there exists a subsequence of $\{z_n\}_{n=1}^{\infty}$ which converges uniformly on $[a, N]$. WLOG, we assume the sequence $\{z_n\}_{n=1}^{\infty}$ converges uniformly on $[a, N]$.

Now, suppressing the independent variable and using the inequality $2ab \leq a^2 + b^2$ we have for each N

$$
\begin{aligned}
(3.44) \qquad \|\widehat{B}_N z_n - \widehat{B}_N z_m\|^2 &= \int_a^N u^{-\gamma-\alpha} \{ [b_1^2 + b_3^2] |z_{n,1} - z_{m,1}|^2 \\
&\quad + [b_1 b_4 + b_2 b_3](z_{n,1} - z_{m,1}) \overline{(z_{n,2} - z_{m,2})}
\end{aligned}
$$

$$
\begin{aligned}
&+[b_1b_4+b_2b_3]\overline{(z_{n,1}-z_{m,1})}(z_{n,2}-z_{m,2})\\
&+[b_2^2+b_4^2]|z_{n,2}-z_{m,2}|^2\}\,du\\
\le\; &\int_a^N u^{-\gamma-\alpha}\{[b_1^2+b_3^2]|z_{n,1}-z_{m,1}|^2\\
&+2\,|b_1b_4+b_2b_3|\,|(z_{n,1}-z_{m,1})(z_{n,2}-z_{m,2})|\\
&+[b_2^2+b_4^2]|z_{n,2}-z_{m,2}|^2\}\,du\\
\le\; &\int_a^N u^{-\gamma-\alpha}\{[b_1^2+b_3^2]|z_{n,1}-z_{m,1}|^2\\
&+|b_1b_4+b_2b_3|\,(|z_{n,1}-z_{m,1}|^2+|z_{n,2}-z_{m,2}|^2)\\
&+[b_2^2+b_4^2]|z_{n,2}-z_{m,2}|^2\}\,du\,.
\end{aligned}
$$

By adding nonnegative terms to the right-hand side of inequality (3.44), we obtain

$$
\begin{aligned}
||\widehat{B}_Nz_n-\widehat{B}_Nz_m||^2 \;\le\; &\int_a^N u^{-\gamma-\alpha}\{[b_1^2+b_3^2]\;||z_n-z_m||_2^2+|b_1b_4+b_2b_3|\;||z_n-z_m||_2^2\\
&+[b_2^2+b_4^2]\;||z_n-z_m||_2^2\}\,du\\
\le\; &\sup_{a\le x\le N}\;||z_n-z_m||_2^2\left\{\int_a^N u^{-\gamma-\alpha}[b_1^2+b_3^2]\,du\right.\\
&\left.+\int_a^N u^{-\gamma-\alpha}|b_1b_4+b_2b_3|\,du+\int_a^N u^{-\gamma-\alpha}[b_2^2+b_4^2]\,du\right\}
\end{aligned}
$$

where

$$
||z_n-z_m||_2^2=[z_{n,1}(x)-z_{m,1}(x)]^2+[z_{n,2}(x)-z_{m,2}(x)]^2\,.
$$

By applying the argument used in inequality (3.22) to the inequality above, we see that the integrals on the right-hand side are finite on $[a,N]$. Hence, there exists a constant $C>0$ such that

$$
||\widehat{B}_Nz_n-\widehat{B}_Nz_m||^2\le C\sup_{a\le x\le N}\;||z_n-z_m||_2^2\,.
$$

Since $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the uniform norm, the above inequality implies that the sequence $\{\widehat{B}_Nz_n\}_{n=1}^{\infty}$ is Cauchy in $\mathcal{L}_w^2(I)$ for each N. Therefore, $\{\widehat{B}_Nz_n\}_{n=1}^{\infty}$ converges for each N as $n\to\infty$ since $\mathcal{L}_w^2(I)$ is complete. Hence, each $\widehat{B}_N$ is $\widehat{T}_0$-compact. #

Since $\widehat{B}_0$ is the uniform limit of $\widehat{T}_0$-compact operators, $\widehat{B}_0$ is $\widehat{T}_0$-compact.

Necessity

We use contradiction arguments to show that each equation in (3.3) must hold.

Suppose that for some $\varepsilon \in (0, \frac{1}{2})$ there exists a $\rho > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers such that $r_n \to \infty$ as $n \to \infty$ and for each n

$$\frac{1}{r_n}\int_{r_n}^{r_n+\varepsilon r_n} x^{2-2\alpha} b_i^2(x)\,dx \geq \rho \text{ for some } i \in \{1,2,3,4\}. \tag{3.45}$$

Suppose $i = 1$. Let $\{\Phi_r\}_{r\geq a}$ be defined as in (2.28) with $f(r) = r$. Then via inequalities (3.27) and (3.31) there exist constants $C_2, C_4 > 0$ such that for each n

$$\begin{aligned} \|\Phi_{r_n}\|^2_{\widehat{T}_0} &= (\|\Phi_{r_n}\| + \|\widehat{T}\Phi_{r_n}\|)^2 \\ &\leq 2\,(\|\Phi_{r_n}\|^2 + \|\widehat{T}\Phi_{r_n}\|^2) \\ &\leq 2C_2\, r_n^{\gamma-\alpha+1} + 2C_4\, r_n^{\alpha-\gamma-1}. \end{aligned} \tag{3.46}$$

For each $r \geq a$ define

$$\widehat{\Phi}_r(x) = r^{\frac{1}{2}(\gamma-\alpha+1)}\Phi_r(x) \text{ for } x \geq a. \tag{3.47}$$

Via inequality (3.46) we have for each n

$$\begin{aligned} \|\widehat{\Phi}_{r_n}\|^2_{\widehat{T}_0} &= r_n^{\gamma-\alpha+1}\, \|\Phi_{r_n}\|^2_{\widehat{T}_0} \\ &\leq 2C_2\, r_n^{2(\gamma-\alpha+1)} + 2C_4\,. \end{aligned} \tag{3.48}$$

Since $\gamma - \alpha + 1 < 0$, inequality (3.48) implies that $\{\widehat{\Phi}_{r_n}\}_{n=1}^{\infty}$ is a $\widehat{T}_0$-bounded sequence. Via the $\widehat{T}_0$-compactness of $\widehat{B}_0$, $\{\widehat{B}\widehat{\Phi}_{r_n}\}_{n=1}^{\infty}$ has a convergent subsequence. WLOG, we assume $\{\widehat{B}\widehat{\Phi}_{r_n}\}_{n=1}^{\infty}$ converges, say to some z_0. Now, since $\phi_{r_n} \equiv 1$ on $[r_n, r_n + \varepsilon r_n]$ and supp(ϕ_{r_n})=$[r_n - 2\varepsilon r_n, r_n + 2\varepsilon r_n]$, we have for each n

$$\begin{aligned} \rho &\leq \frac{1}{r_n}\int_{r_n}^{r_n+\varepsilon r_n} x^{2-2\alpha} b_1^2(x)\,dx \\ &\leq \frac{1}{r_n}\int_{r_n}^{r_n+\varepsilon r_n} x^{2-2\alpha}[b_1^2(x) + b_3^2(x)]\,dx \\ &\leq \frac{1}{r_n}\int_{r_n-2\varepsilon r_n}^{r_n+2\varepsilon r_n} x^{2-2\alpha}[b_1^2(x) + b_3^2(x)]\phi_{r_n}^2(x)\,dx\,. \end{aligned} \tag{3.49}$$

We apply Lemma 1.8 to the above inequality to obtain for some constant $C > 0$ and for each n

$$(3.50) \qquad \begin{aligned} & \frac{1}{r_n}\int_{r_n-2\varepsilon r_n}^{r_n+2\varepsilon r_n} x^{2-2\alpha}[b_1^2(x)+b_3^2(x)]\phi_{r_n}^2(x)\,dx \\ \leq \;& C\, r_n^{\gamma-\alpha+1}\int_{r_n-2\varepsilon r_n}^{r_n+2\varepsilon r_n} x^{-\gamma-\alpha}[b_1^2(x)+b_3^2(x)]\phi_{r_n}^2(x)\,dx \\ = \;& C\,\|\widehat{B}\widehat{\Phi}_{r_n}\|^2, \end{aligned}$$

i.e., $\|\widehat{B}\widehat{\Phi}_{r_n}\| \geq \left(\frac{\rho}{C}\right)^{\frac{1}{2}} > 0$ for each n.

By an argument similar to the one used in the proof of Theorem 2.1 (ii), we conclude that $z_0 = 0$ a.e. in $[a, \infty)$. This contradiction implies that equation (3.3) holds for $i = 1$. If we begin with b_3 instead of b_1 in inequality (3.49), then the above argument shows that equation (3.3) holds for $i = 3$.

Now, for each $r \geq a$ define

$$\widehat{\Psi}_r(x) = r^{\frac{1}{2}(\gamma-\alpha+1)}\Psi_r(x) \quad \text{for } x \geq a$$

where $\{\Psi_r\}_{r\geq a}$ is given in (2.28) with $f(r) = r$. Via inequalities (3.28) and (3.32) we have for each n

$$\begin{aligned} \|\widehat{\Psi}_{r_n}\|^2_{\widehat{T}_0} &= r_n^{\gamma-\alpha+1}\,\|\Psi_{r_n}\|^2_{\widehat{T}_0} \\ &\leq 2C_2\, r_n^{2(\gamma-\alpha+1)} + 2C_4\,. \end{aligned}$$

Thus, $\{\widehat{\Psi}_{r_n}\}_{n=1}^{\infty}$ is a $\widehat{T}_0$-bounded sequence.

By repeating the above argument with b_1 replaced by either b_2 or b_4 and $\widehat{\Phi}_r$ replaced by $\widehat{\Psi}_r$, we conclude that equation (3.3) holds for $i = 2$ and $i = 4$.

Note that the proof of necessity shows that (3.3) holds for every $\varepsilon \in (0, \frac{1}{2})$. □

Corollary 3.2 *Let $I = [a, \infty), a \geq 1$, and $\gamma - \alpha < -1$. Then the following three statements are equivalent.*

(i) $\int_I x^{-\gamma-\alpha}\, b_i^2(x)dx < \infty$, *for $i = 1, 2, 3$, and 4.*

(ii) $\widehat{B}_1$ is $\widehat{T}_1$-bounded.

(iii) $\widehat{B}_1$ is $\widehat{T}_1$-compact.

Proof:

We break the proof into two cases.

Case I: $\alpha = 0$

We consider the maximal operators, $\widehat{T}_1$ and $\widehat{B}_1$ and the minimal operators, $\widehat{T}_0$ and $\widehat{B}_0$, associated with the following differential expressions on the interval I:

$$(\widehat{T}z)(x) = \begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} \tag{3.51}$$

and

$$(\widehat{B}z)(x) = \begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix} \begin{pmatrix} b_1 & b_4 \\ b_3 & b_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{3.52}$$

(i) $\Rightarrow$ (iii): Since $\gamma < -1$ and $x \geq a \geq 1$, we have

$$\int_I x\, b_i^2(x)dx \leq \int_I x^{-\gamma}\, b_i^2(x)dx\,.$$

Hence, (i) implies that $\widehat{B}_0$ is $\widehat{T}_0$-compact via Theorem 3.1. Therefore, $D(\widehat{T}_0) \subseteq D(\widehat{B}_0)$.

Via equation (1.7) we can write

$$D(\widehat{T}_1) = D(\widehat{T}_0) \oplus S, \tag{3.53}$$

where S has dimension four since $\widehat{T}$ is regular at a and limit circle at ∞.

Claim 3.2.1 $S \subseteq D(\widehat{B}_1)$.

Proof of Claim 3.2.1:

Let z_1 and z_2 be functions in $C^{(1)}(\mathbb{R})$ such that $z_1(a) = 1$, $\operatorname{supp}(z_1) = [a-1, a+1]$, $z_2(x) = 0$ for $x \leq a$, and $z_2(x) = 1$ for $x \geq a+1$.

For $x \geq a$ we define $Z_1(x) = \begin{pmatrix} z_1(x) \\ 0 \end{pmatrix}$, $Z_2(x) = \begin{pmatrix} 0 \\ z_1(x) \end{pmatrix}$, $Z_3(x) = \begin{pmatrix} z_2(x) \\ 0 \end{pmatrix}$, and $Z_4(x) = \begin{pmatrix} 0 \\ z_2(x) \end{pmatrix}$. Notice that these four vector-valued functions are linearly independent. Since z_1 has compact support,

$$\|Z_1\|^2 = \|Z_2\|^2 = \int_a^{a+1} x^{\gamma}\, |z_1(x)|^2\, dx\,,$$

which is finite since z_1 is continuous. Moreover,

$$\begin{aligned}\|Z_3\|^2 = \|Z_4\|^2 &= \int_a^\infty x^\gamma\,|z_2(x)|^2\,dx \\ &= \int_a^{a+1} x^\gamma\,|z_2(x)|^2\,dx + \int_{a+1}^\infty x^\gamma\,dx\,,\end{aligned}$$

which is finite since z_2 is continuous and $\gamma < -1$. Thus, each $Z_i \in \mathcal{L}^2_w(I)$, with weights x^γ.

We show that each $\widehat{T}Z_i \in \mathcal{L}^2_w(I)$, with weights x^γ, so that each $Z_i \in D(\widehat{T}_1)$. Since z_1 has compact support,

$$\|\widehat{T}Z_1\| = \|\widehat{T}Z_2\| < \infty\,.$$

Moreover, since $z_2'(x) = 0$ on $[a+1, \infty)$,

$$\|\widehat{T}Z_3\|^2 = \|\widehat{T}Z_4\|^2 = \int_a^{a+1} x^{-\gamma}\,|z_2'(x)|^2\,dx\,,$$

which is finite since z_2' is continuous.

Define $S = \operatorname{span}\{Z_1, Z_2, Z_3, Z_4\}$. We now prove that

$$D(\widehat{T}_1) = D(\widehat{T}_0) \oplus S\,,$$

i.e., we show that no linear combination of the Z_i is in $D(\widehat{T}_0)$. Suppose to the contrary that there exist constants c_1, c_2, c_3, c_4 (not all zero) such that

$$Z := c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4 \in D(\widehat{T}_0)\,. \tag{3.54}$$

Since $\widehat{T}$ is regular at a and $Z \in D(\widehat{T}_1)$, we have that $Z \in D(\widehat{T}_0)$ if and only if $Z(a) = 0$ and $[Z, Y](x) \to 0$ as $x \to \infty$ for every $Y \in D(\widehat{T}_1)$, where the *Lagrange identity* is defined by

$$[Y, \hat{Y}](x) = (y_1\hat{y}_2 - y_2\hat{y}_1)(x)$$

for real vector-valued functions Y and $\hat{Y}$ (see [15, Theorem 3.12]). Since $z_1(a) \neq 0$, we must take $c_1 = c_2 = 0$. Therefore, $Z = c_3 Z_3 + c_4 Z_4$. In order to satisfy the second condition of [15, Theorem 3.12], we must have $[Z, Z_3](x) \to 0$ as $x \to \infty$ and $[Z, Z_4](x) \to 0$ as $x \to \infty$, where

$$[Z, Z_3](x) = c_3[Z_3, Z_3](x) + c_4[Z_4, Z_3](x) = c_4[Z_4, Z_3](x)$$

and

$$[Z, Z_4](x) = c_3[Z_3, Z_4](x) + c_4[Z_4, Z_4](x) = c_3[Z_3, Z_4](x)\,.$$

Since

$$\lim_{x\to\infty}[Z_3, Z_4](x) = \lim_{x\to\infty}\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right] = 1\,,$$

we must have $c_3 = 0$. Moreover, since

$$\lim_{x\to\infty}[Z_4, Z_3](x) = \lim_{x\to\infty}\left[\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right] = 1\,,$$

we must have $c_4 = 0$. Hence, no linear combination of the Z_i is in $D(\widehat{T}_0)$.

Since z_1 has compact support,

$$\|\widehat{B}Z_1\| = \|\widehat{B}Z_2\| < \infty\,.$$

Moreover,

$$\begin{aligned}
\|\widehat{B}Z_3\|^2 &= \int_a^\infty x^{-\gamma}\,[b_1^2(x) + b_3^2(x)]\,|z_2(x)|^2\,dx \\
&= \int_a^{a+1} x^{-\gamma}\,[b_1^2(x) + b_3^2(x)]\,|z_2(x)|^2\,dx + \int_{a+1}^\infty x^{-\gamma}\,[b_1^2(x) + b_3^2(x)]\,dx
\end{aligned}$$

and

$$\begin{aligned}
\|\widehat{B}Z_4\|^2 &= \int_a^\infty x^{-\gamma}\,[b_2^2(x) + b_4^2(x)]\,|z_2(x)|^2\,dx \\
&= \int_a^{a+1} x^{-\gamma}\,[b_2^2(x) + b_4^2(x)]\,|z_2(x)|^2\,dx + \int_{a+1}^\infty x^{-\gamma}\,[b_2^2(x) + b_4^2(x)]\,dx\,,
\end{aligned}$$

which are finite by (i). Therefore, each $Z_i \in D(\widehat{B}_1)$. #

Equation (3.53) and Claim 3.2.1 imply that $D(\widehat{T}_1) \subseteq D(\widehat{B}_1)$. Via Theorem 1.5 $\widehat{B}_1$ is $\widehat{T}_1$-compact.

(iii) $\Rightarrow$ (ii): Since $\widehat{B}_1$ is $\widehat{T}_1$-compact, $D(\widehat{T}_1) \subseteq D(\widehat{B}_1)$. Via Theorem 1.3 $\widehat{B}_1$ is $\widehat{T}_1$-bounded.

(ii) $\Rightarrow$ (i): Since $\widehat{B}_1$ is $\widehat{T}_1$-bounded, $D(\widehat{T}_1) \subseteq D(\widehat{B}_1)$. Via equation (3.53) $S \subseteq D(\widehat{B}_1)$. Therefore, $\|BZ_i\| < \infty$ for each i, i.e.,

$$\int_I x^{-\gamma}\, b_i^2(x)\,dx < \infty\,,\ i = 1, 2, 3, 4\,.$$

Case II: $\alpha \neq 0$

As in the proof of Corollary 2.2, we transform the differential expressions $\widehat{T}$ and $\widehat{B}$ unitarily into T and B, respectively, where

$$(Ty)(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$$

and

$$(By)(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix} \begin{pmatrix} x^{-\alpha}b_1 & x^{-\alpha}b_4 \\ x^{-\alpha}b_3 & x^{-\alpha}b_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Replacing γ with $\gamma - \alpha$, b_i with $x^{-\alpha}b_i$, and z with y in Case I, we have that the following three statements are equivalent:

(i′) $\int_I x^{-\gamma-\alpha}\, b_i^2(x)\, dx < \infty$ for $i = 1, 2, 3$, and 4.

(ii′) B_1 is T_1-bounded.

(iii′) B_1 is T_1-compact.

Since the transformation is unitary, (ii′) holds iff $\widehat{B}_1$ is $\widehat{T}_1$-bounded, and (iii′) holds iff $\widehat{B}_1$ is $\widehat{T}_1$-compact. □

Chapter 4

Perturbations of L

In this chapter, we consider perturbations B of the higher-ordered differential operator L, where L and B are defined by (1.1) and (1.3), respectively. We assume that the functions $w_1, w_2, p, p_1, p_2 \in AC_{loc}(I)$. Again, we develop necessary and sufficient conditions for the perturbations to be relatively bounded or relatively compact with respect to L. As in Chapter 2 these conditions involve explicit integral averages of the coefficients of B. In Theorems 4.2 and 4.4 we consider the maximal operators L_1 and B_1 associated with the differential operators (1.1) and (1.3), respectively. Each of the proofs follows the general pattern of Theorem 2.1 but requires additional estimates on $||Ly||$.

Corollary 4.5 applies Theorem 4.4 to perturbations of an operator of the form (1.5) with

$$W(x) = \begin{pmatrix} x^\gamma & 0 \\ 0 & x^\gamma \end{pmatrix}, \; P_0(x) = \begin{pmatrix} r_1 & r \\ r & r_2 \end{pmatrix}, \text{ and } Q_0(x) = \frac{1}{2}\begin{pmatrix} 0 & -x^\alpha \\ x^\alpha & 0 \end{pmatrix},$$

where γ and α are constants which satisfy $\gamma - \alpha \leq -1$; whereas, Corollary 4.6 applies Theorem 4.4 to perturbations of an operator of the form (1.5) with

$$W(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; P_0(x) = \begin{pmatrix} \frac{a}{x} + d & \frac{b}{x} + \frac{c}{x^2} \\ \frac{b}{x} + \frac{c}{x^2} & \frac{a}{x} - d \end{pmatrix}, \text{ and } Q_0(x) = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where a, b, c and d are constants and $c \neq 0$. This operator arises from the study of the relativistic electron (see [14, page 129]). The proof of each corollary makes use of a unitary transformation.

We need the following lemma in the proof of Theorem 4.2. In this lemma, we consider the minimal unclosed operators, L_0' and B_0', associated with the differential expressions (1.1) and (1.3), respectively, where the weight functions $w_1 = w_2 \equiv 1$ and the coefficient functions p_1 and p_2 satisfy the relationship $p_2 = -p_1$ on an interval I. We establish a lower bound on the $\mathcal{L}_w^2(I)$-norm of L on $D(L_0')$.

Lemma 4.1 *Let $0 < \delta < \frac{1}{2}$ and suppose $w_1 = w_2 \equiv 1$ and $p_2 = -p_1$. If $|p_1'(t)| \leq \delta\,[p_1^2(t) + p^2(t)]$ and $|p'(t)| \leq \delta\,[p_1^2(t) + p^2(t)]$ on an interval I, then there exists a positive definite matrix Δ_1 such that for $y \in D(L_0')$*

$$||Ly||^2 \geq \int_I \{[y'(t)]^* y'(t) + y^*(t)\Delta_1 y(t)\}\, dt \tag{4.1}$$

where $\Delta_1 = (1 - 2\delta)[p_1^2(t) + p^2(t)] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof:

Let $y \in D(L_0')$. Then after some simplification we have

$$\begin{aligned} ||Ly||^2 &= \int_I [(Ly)(t)]^*[(Ly)(t)]\, dt \\ &= \int_I \{|y_1'(t)|^2 + |y_2'(t)|^2 + p(t)\,[|y_1(t)|^2 - |y_2(t)|^2]' \\ &\quad -2p_1(t)\,[\Re\mathrm{e}(y_1(t)\bar{y}_2(t))]' \\ &\quad +[p_1^2(t) + p^2(t)]\,[|y_1(t)|^2 + |y_2(t)|^2]\}\, dt\,. \end{aligned} \tag{4.2}$$

Now, by integration by parts

$$\begin{aligned} &\int_I \{p(t)\,[|y_1(t)|^2 - |y_2(t)|^2]' - 2p_1(t)\,[\Re\mathrm{e}(y_1(t)\bar{y}_2(t))]'\}\, dt \\ &= \int_I \{-p'(t)\,[|y_1(t)|^2 - |y_2(t)|^2] + 2p_1'(t)\,\Re\mathrm{e}(y_1(t)\bar{y}_2(t))\}\, dt. \end{aligned} \tag{4.3}$$

Therefore, if we suppress the independent variable, we have

$$\begin{aligned}
(4.4) \qquad \|Ly\|^2 &= \int_I \{|y_1'|^2 + |y_2'|^2 - p'\,[|y_1|^2 - |y_2|^2] \\
&\quad +2p_1'\,\Re\mathrm{e}(y_1\bar{y}_2) + [p_1^2 + p^2]\,[|y_1|^2 + |y_2|^2]\}\,dt \\
&\geq \int_I \{|y_1'|^2 + |y_2'|^2 - |p'|\,[|y_1|^2 + |y_2|^2] \\
&\quad -2\,|p_1'|\,|\Re\mathrm{e}(y_1\bar{y}_2)| + [p_1^2 + p^2]\,[|y_1|^2 + |y_2|^2]\}\,dt \\
&\geq \int_I \{|y_1'|^2 + |y_2'|^2 - |p'|\,[|y_1|^2 + |y_2|^2] \\
&\quad -|p_1'|\,[|y_1|^2 + |y_2|^2] + [p_1^2 + p^2]\,[|y_1|^2 + |y_2|^2]\}\,dt
\end{aligned}$$

where the last inequality in (4.4) follows from the inequality

$$2\,|\Re\mathrm{e}(y_1\bar{y}_2)| \leq 2\,|y_1\,y_2| \leq |y_1|^2 + |y_2|^2\,.$$

Use of the hypotheses in (4.4) implies that

$$\|Ly\|^2 \geq \int_I \{[y'(t)]^* y'(t) + (1-2\delta)[p_1^2(t) + p^2(t)]y^*(t)y(t)\}\,dt. \quad \square$$

Theorem 4.2 *Let $I = [a,\infty)$ for some $a > 0$, $w_1 = w_2 \equiv 1$, $p_2 = -p_1$, and $f(t) = [p_1^2(t) + p^2(t) + 1]^{-\frac{1}{2}}$ on I, where p_1 and p satisfy the conditions of Lemma 4.1. Suppose L is limit point at ∞. Then*

(i) B_1 is L_1-bounded if and only if

$$(4.5) \qquad \sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\; q_i^2(\tau)\,d\tau < \infty, \quad i = 1,2,3,4,$$

for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$;

(ii) B_1 is L_1-compact if and only if

$$(4.6) \qquad \lim_{t\to\infty} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\; q_i^2(\tau)\,d\tau = 0, \quad i = 1,2,3,4,$$

for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$.

Proof:

Via the triangle inequality and the bounds on $|p_1'|$ and $|p'|$ on I, we have

$$|f'(t)| \;=\; \left|\frac{p_1(t)p_1'(t) + p(t)p'(t)}{[p_1^2(t) + p^2(t) + 1]^{\frac{3}{2}}}\right|$$

$$\leq \frac{\delta[p_1^2(t)+p^2(t)]\,[|p_1(t)|+|p(t)|]}{[p_1^2(t)+p^2(t)+1]^{\frac{3}{2}}}$$
$$\leq \frac{\delta[|p_1(t)|+|p(t)|]}{[p_1^2(t)+p^2(t)+1]^{\frac{1}{2}}}$$

Since $[|p_1(t)|+|p(t)|]^2 \leq 2[p_1^2(t)+p^2(t)]$, we obtain the inequality

$$|f'(t)| \leq \sqrt{2}\,\delta.$$

Hence, the hypotheses of Lemma 1.8 hold with $N_0 = \sqrt{2}\,\delta$.

Now, we show that B_0 is L_0-bounded if and only if inequalities (4.5) hold. Note that $D(L_0') \subseteq D(B_0')$ since the order of B is less than the order of L and the coefficients are sufficiently smooth.

(i) Sufficiency

Let us first consider any $y \in D(L_0')$ whose second component is zero. Then we have

$$(4.7)\qquad ||y||^2 = \int_I |y_1(\tau)|^2\,d\tau,$$

$$(4.8)\qquad ||By||^2 = \int_I [q_1(\tau)^2+q_3(\tau)^2]|y_1(\tau)|^2\,d\tau, \quad \text{and via Lemma 4.1}$$

$$(4.9)\qquad \begin{aligned} ||Ly||^2 &\geq \int_I \{|y_1'(\tau)|^2+(1-2\delta)[p_1^2(\tau)+p^2(\tau)]|y_1(\tau)|^2\}d\tau \\ &= \int_I \{|y_1'(\tau)|^2+(1-2\delta)[f^{-2}(\tau)-1]|y_1(\tau)|^2\}d\tau\,. \end{aligned}$$

Now, we show that the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$ with $N = q_1^2+q_3^2, W = (1-2\delta)f^{-2}+2\delta$ and $P=1$. By applying Lemma 1.8 with $g \equiv 1$, we know that positive and negative powers of f are essentially constant on intervals of length εf. Thus, we have for some constant $C_1 > 0$

$$(4.10)\qquad \begin{aligned} S_1 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} d\tau\right]\left[\int_t^{t+\varepsilon f(t)}[q_1(\tau)^2+q_3(\tau)^2]\,d\tau\right]\right\} \\ &= \frac{1}{\varepsilon}\sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_1(\tau)^2+q_3(\tau)^2]\,d\tau\right\} \quad \text{and} \end{aligned}$$

$$(4.11)\qquad \begin{aligned} S_2 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)}\frac{d\tau}{(1-2\delta)f^{-2}(\tau)+2\delta}\right]\right. \\ &\qquad \left.\cdot\left[\int_t^{t+\varepsilon f(t)}[q_1(\tau)^2+q_3(\tau)^2]\,d\tau\right]\right\} \\ &\leq \frac{C_1}{\varepsilon(1-2\delta)}\sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_1(\tau)^2+q_3(\tau)^2]\,d\tau]\right\} \end{aligned}$$

for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$.

Inequalities (4.5), (4.10), and (4.11) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$. Therefore (via Theorem 1.7), there exists a constant $k_1 > 0$ such that

$$\begin{aligned}(4.12)\qquad &\int_I [q_1(\tau)^2 + q_3(\tau)^2]\ |y_1(\tau)|^2\ d\tau \\ &\quad \le k_1 \left\{ \int_I [(1-2\delta)f^{-2}(\tau) + 2\delta]\ |y_1(\tau)|^2\ d\tau + \varepsilon^2 \int_I\ |y_1'(\tau)|^2\ d\tau \right\}.\end{aligned}$$

By substituting (4.7)-(4.9) into (4.12) and applying Lemma 4.1, we obtain for some constant $\hat{k}_1 > 0$

$$(4.13)\qquad ||By||^2 \le \hat{k}_1\,(||y||^2 + ||Ly||^2).$$

Since y is arbitrary and $D(L_0') \subseteq D(B_0')$, inequality (4.13) implies that B_0' is L_0'-bounded on those $y \in D(L_0')$ with $y_2 \equiv 0$.

Let us now consider any $y \in D(L_0')$ whose first component is zero. Then we have

$$\begin{aligned}(4.14)\qquad ||y||^2 &= \int_I\ |y_2(\tau)|^2\ d\tau,\\ (4.15)\qquad ||By||^2 &= \int_I [q_2(\tau)^2 + q_4(\tau)^2]|y_2(\tau)|^2\ d\tau, \quad \text{and via Lemma 4.1}\\ (4.16)\qquad ||Ly||^2 &\ge \int_I \{|y_2'(\tau)|^2 + (1-2\delta)[p_1^2(\tau) + p^2(\tau)]|y_2(\tau)|^2\}\ d\tau\\ &= \int_I \{|y_2'(\tau)|^2 + (1-2\delta)[f^{-2}(\tau) - 1]|y_2(\tau)|^2\}\ d\tau\,.\end{aligned}$$

Following an argument similar to the one above, we show the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$ with $N = q_2^2 + q_4^2, W = (1-2\delta)f^{-2} + 2\delta$, and $P = 1$. For some constant $C_2 > 0$ we have

$$\begin{aligned}(4.17)\qquad S_1 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} d\tau\right]\left[\int_t^{t+\varepsilon f(t)} [q_2(\tau)^2 + q_4(\tau)^2]\,d\tau\right]\right\}\\ &= \frac{1}{\varepsilon}\ \sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)} [q_2(\tau)^2 + q_4(\tau)^2]\,d\tau\right\} \ \text{and}\\ (4.18)\qquad S_2 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} \frac{d\tau}{(1-2\delta)f^{-2} + 2\delta}\right]\right.\\ &\qquad \left.\cdot\left[\int_t^{t+\varepsilon f(t)} [q_2(\tau)^2 + q_4(\tau)^2]\,d\tau\right]\right\}\\ &\le \frac{C_2}{\varepsilon(1-2\delta)}\ \sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)} [q_2(\tau)^2 + q_4(\tau)^2]\,d\tau\right\}\end{aligned}$$

for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\delta})$. Inequalities (4.5), (4.17), and (4.18) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\delta})$. Therefore (via Theorem 1.7), there exists a constant $k_2 > 0$ such that

$$
\begin{aligned}
(4.19) \qquad & \int_I [q_2(\tau)^2 + q_4(\tau)^2] \, |y_2(\tau)|^2 \, d\tau \\
& \leq k_2 \left\{ \int_I [(1 - 2\delta f^{-2}(\tau) + 2\delta] \, |y_2(\tau)|^2 \, d\tau + \varepsilon^2 \int_I |y_2'(\tau)|^2 \, d\tau \right\}.
\end{aligned}
$$

By substituting (4.14)-(4.16) into (4.19) and applying Lemma 4.1, we obtain for some constant $\hat{k}_2 > 0$

$$
(4.20) \qquad ||By||^2 \leq \hat{k}_2 \, (||y||^2 + ||Ly||^2).
$$

Since y is arbitrary and $D(L_0') \subseteq D(B_0')$, inequality (4.20) implies that B_0' is L_0'-bounded on those $y \in D(L_0')$ with $y_1 \equiv 0$.

Finally, we consider any $y \in D(L_0')$. Then we have, suppressing the independent variable,

$$
\begin{aligned}
(4.21) \qquad ||y||^2 &= \int_I [|y_1|^2 + |y_2|^2] \,, \\
(4.22) \qquad ||By||^2 &= \int_I \{[q_1^2 + q_3^2]|y_1|^2 + [q_1q_4 + q_2q_3]y_1\bar{y_2} \\
& \quad + [q_1q_4 + q_2q_3]\bar{y_1}y_2 + [q_2^2 + q_4^2]|y_2|^2\} \\
&\leq \int_I \{[q_1^2 + q_3^2]|y_1|^2 + 2\,|q_1q_4 + q_2q_3|\,|y_1y_2| \\
& \quad + [q_2^2 + q_4^2]|y_2|^2\}, \quad \text{and via Lemma 4.1} \\
(4.23) \qquad ||Ly||^2 &\geq \int_I \{|y_1'|^2 + |y_2'|^2 + (1 - 2\delta)(p_1^2 + p^2)[|y_1|^2 + |y_2|^2]\} \\
&= \int_I \{|y_1'|^2 + |y_2'|^2 + (1 - 2\delta)(f^{-2} - 1)[|y_1|^2 + |y_2|^2]\}\,.
\end{aligned}
$$

We make use of the inequality $2ab \leq a^2 + b^2$ in inequality (4.22) to obtain

$$
\begin{aligned}
(4.24) \qquad ||By||^2 \leq & \int_I \{[q_1^2 + q_3^2]|y_1|^2 + |q_1q_4 + q_2q_3|\,[|y_1|^2 + |y_2|^2] \\
& + [q_2^2 + q_4^2]|y_2|^2\}\,.
\end{aligned}
$$

Claim 4.2.1 *There exist constants $k_3, k_4 > 0$ such that*

$$
(4.25) \qquad \int_I |q_1q_4 + q_2q_3|\,|y_1|^2 \leq k_3 \, \left\{ \int_I [(1 - 2\delta)f^{-2} + 2\delta] \, |y_1|^2 + \varepsilon^2 \int_I |y_1'|^2 \right\}
$$

and

$$(4.26) \quad \int_I |q_1q_4 + q_2q_3|\,|y_2|^2 \le k_4 \left\{\int_I [(1-2\delta)f^{-2} + 2\delta]\,|y_2|^2 + \varepsilon^2 \int_I |y_2'|^2\right\}$$

Proof of Claim 4.2.1:

By following the argument in Claim 2.1.1, we know that

$$(4.27) \quad \int_t^{t+\varepsilon f(t)} |q_1q_4 + q_2q_3| \le \left(\int_t^{t+\varepsilon f(t)} q_1^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} [q_4^2 + q_2^2]\right)^{\frac{1}{2}} + \left(\int_t^{t+\varepsilon f(t)} q_2^2\right)^{\frac{1}{2}} \left(2\int_t^{t+\varepsilon f(t)} [q_1^2 + q_3^2]\right)^{\frac{1}{2}}$$

Now, we show that the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$ with $N = |q_1q_4 + q_2q_3|, W = (1-2\delta)f^{-2} + 2\delta$, and $P = 1$. Via inequality (4.27) and the definitions of S_1 and S_2, we have for some constant $C_3 > 0$

$$\begin{aligned}(4.28)\ S_1 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} 1\right]\left[\int_t^{t+\varepsilon f(t)} |q_1q_4 + q_2q_3|\right]\right\} \\ &\le \frac{1}{\varepsilon}\sup_{t\in I}\left\{\left(f(t)\int_t^{t+\varepsilon f(t)} q_1^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2 + q_2^2]\right)^{\frac{1}{2}}\right. \\ &\quad \left. + \left(f(t)\int_t^{t+\varepsilon f(t)} q_2^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2 + q_3^2]\right)^{\frac{1}{2}}\right\} \text{ and}\end{aligned}$$

$$\begin{aligned}(4.29)\ S_2 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} \frac{1}{(1-2\delta)f^{-2} + 2\delta}\right]\left[\int_t^{t+\varepsilon f(t)} |q_1q_4 + q_2q_3|\right]\right\} \\ &\le \frac{C_3}{\varepsilon(1-2\delta)}\sup_{t\in I}\left\{\left(f(t)\int_t^{t+\varepsilon f(t)} q_1^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2 + q_2^2]\right)^{\frac{1}{2}}\right. \\ &\quad \left. + \left(f(t)\int_t^{t+\varepsilon f(t)} q_2^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2 + q_3^2]\right)^{\frac{1}{2}}\right\}\end{aligned}$$

for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$. Inequalities (4.5), (4.28), and (4.29) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$. An application of Theorem 1.7 gives us inequalities (4.25) and (4.26). #

Therefore, via inequalities (4.12), (4.19), (4.25), and (4.26) we obtain from inequality (4.24) that

$$(4.30) \quad ||By||^2 \le (k_1 + k_3)\left\{\int_I [(1-2\delta)f^{-2} + 2\delta]\,|y_1|^2 + \varepsilon^2 \int_I f\,|y_1'|^2\right\}$$

$$+(k_2+k_4)\left\{\int_I [(1-2\delta)f^{-2}+2\delta]\ |y_2|^2+\varepsilon^2\int_I f\ |y_2'|^2\right\}.$$

Hence, via Lemma 4.1 there exists a constant $C > 0$ such that for each $y \in D(L_0')$

$$||By||^2 \le C\ (||y||^2 + ||Ly||^2).$$

Since y is arbitrary and $D(L_0') \subseteq D(B_0')$, we know that B_0' is L_0'-bounded. Via Theorem 1.2 we have that B_0 is L_0-bounded.

Necessity

Fix $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$. For each $r \ge a$ define Φ_r and Ψ_r to be the vector-valued functions, with compact support, given by (2.28). Then for each $r \ge a$

$$\begin{aligned} ||\Phi_r||^2 &= \int_I \phi_r^2(t)\,dt \\ &= \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \phi^2\left(\frac{t-r}{\varepsilon f(r)}\right)\,dt\,. \end{aligned} \tag{4.31}$$

By a change of variables (letting $u = \dfrac{t-r}{\varepsilon f(r)}$ in (4.31)),

$$||\Phi_r||^2 = \varepsilon f(r)\int_{-2}^{2}\phi^2(u)\,du\,. \tag{4.32}$$

Since ϕ is continuous, inequality (4.32) implies there exists a constant $C_1 > 0$ such that for $r \ge a$

$$||\Phi_r||^2 \le C_1\, f(r)\,. \tag{4.33}$$

Similarly, for $r \ge a$

$$||\Psi_r||^2 \le C_1\, f(r)\,. \tag{4.34}$$

Via the hypothesis on p' and Lemma 1.8, there exists a constant $C_2 > 0$ such that for each $r \ge a$

$$\begin{aligned} ||L\Phi_r||^2 &= \int_I \left\{\left[\frac{d}{dt}\phi_r(t)\right]^2 + [p_1^2(t)+p^2(t)-p'(t)]\,\phi_r^2(t)\right\}\,dt \\ &= \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \left\{\left[\frac{d}{dt}\phi\left(\frac{t-r}{\varepsilon f(r)}\right)\right]^2 + [p_1^2(t)+p^2(t)-p'(t)]\,\phi^2\left(\frac{t-r}{\varepsilon f(r)}\right)\right\}\,dt \end{aligned}$$

$$\begin{aligned}
&\le \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \left\{ \left[\frac{d}{dt}\phi\left(\frac{t-r}{\varepsilon f(r)}\right)\right]^2 + [p_1^2(t)+p^2(t)+|p'(t)|]\,\phi^2\left(\frac{t-r}{\varepsilon f(r)}\right)\right\} dt \\
&\le \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \left\{ \left[\frac{d}{dt}\phi\left(\frac{t-r}{\varepsilon f(r)}\right)\right]^2 + (1+\delta)f^{-2}(t)\,\phi^2\left(\frac{t-r}{\varepsilon f(r)}\right)\right\} dt \\
&\le \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \left[\frac{d}{dt}\phi\left(\frac{t-r}{\varepsilon f(r)}\right)\right]^2 dt \\
&\quad + C_2\,(1+\delta)f^{-2}(r)\int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \phi^2\left(\frac{t-r}{\varepsilon f(r)}\right) dt\,.
\end{aligned}$$

By a change of variables (letting $u=\dfrac{t-r}{\varepsilon f(r)}$ in the above inequality),

$$\|L\Phi_r\|^2 \le \frac{1}{\varepsilon f(r)}\int_{-2}^{2}[\phi'(u)]^2\,du + C_2\,\frac{\varepsilon(1+\delta)}{f(r)}\int_{-2}^{2}\phi^2(u)\,du\,. \tag{4.35}$$

Since ϕ and ϕ' are continuous, there exists a constant $C_3>0$ such that for $r\ge a$

$$\|L\Phi_r\|^2 \le C_3\,f^{-1}(r)\,. \tag{4.36}$$

Similarly, for $r\ge a$

$$\|L\Psi_r\|^2 \le C_3\,f^{-1}(r)\,. \tag{4.37}$$

Since $\phi_r\equiv 1$ on $[r, r+\varepsilon f(r)]$ and supp(ϕ_r)=$[r-2\varepsilon f(r), r+2\varepsilon f(r)]$,

$$\begin{aligned}
\int_r^{r+\varepsilon f(r)}[q_1^2(t)+q_3^2(t)]\,dt &\le \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)}[q_1^2(t)+q_3^2(t)]\phi_r^2(t)\,dt \\
&= \|B\Phi_r\|^2\,.
\end{aligned} \tag{4.38}$$

Thus, via the L_0-boundedness of B_0 and inequalities (4.33), (4.36), and (4.38), there exists a constant $C_4>0$ such that

$$f(r)\int_r^{r+\varepsilon f(r)}[q_1^2(t)+q_3^2(t)]\,dt \le C_4\,[C_2\,f^2(r)+C_3]\,. \tag{4.39}$$

Since $f(r)=[p_1^2(r)+p^2(r)+1]^{-\frac{1}{2}}\le 1$ on I, equation (4.39) implies that inequality (4.5) holds for $i=1,3$.

Moreover,

$$\begin{aligned}
\int_r^{r+\varepsilon f(r)}[q_2^2(t)+q_4^2(t)]\,dt &\le \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)}[q_2^2(t)+q_4^2(t)]\phi_r^2(t)\,dt \\
&= \|B\Psi_r\|^2\,.
\end{aligned} \tag{4.40}$$

Via the L_0-boundedness of B_0 and inequalities (4.34), (4.37), and (4.40),

$$f(r)\int_r^{r+\varepsilon f(r)}[q_2^2(t)+q_4^2(t)]\,dt \le C_4\left[C_2 f^2(r)+C_3\right]. \tag{4.41}$$

is bounded on I, inequality (4.41) implies that inequality (4.5) holds for $i = 2, 4$. (Note that the proof of necessity shows that (4.5) holds for every $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$.)

Therefore, we have shown that B_0 is L_0-bounded if and only if inequalities (4.5) hold. We apply Theorem 1.6 to conclude that B_1 is L_1-bounded if and only if inequalities (4.5) hold.

Now, we show that B_0 is L_0-compact if and only if equations (4.6) hold.

(ii) Sufficiency

By the previous argument B_0 is L_0-bounded. Thus, $D(L_0) \subseteq D(B_0)$. For every positive integer $N > a$, define B_N on $D(L_0)$ by

$$B_N y = \begin{cases} By & \text{on } [a, N] \\ 0 & \text{on } (N, \infty) \end{cases}$$

Notice that each B_N is L_0-bounded with the same norm as B_0 since $||B_N y|| \le ||By||$. In order to simplify the proof, we break the argument into two Claims.

Claim 4.2.2 *$B_N \to B_0$ in the space of bounded operators on $D(L_0)$ with the L_0-norm.*

Proof of Claim 4.2.2:

By definition L_0 is closed. Therefore, $D(L_0)$ is complete under the L_0-norm.

Let $y \in D(L_0)$. Since L_0 is the closure of L_0', for each positive integer $n \ge 1$ there exists a $y_n \in D(L_0')$ such that

$$||y - y_n|| + ||Ly - Ly_n|| < \frac{1}{n}, \tag{4.42}$$

where $y_n = \begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix}$ for each n.

Using the inequality $2ab \leq a^2 + b^2$, we have for each $y_n \in D(L_0')$

$$
\begin{aligned}
(4.43)\quad \|By_n - B_N y_n\|^2 &= \int_N^\infty \{[q_1^2+q_3^2]|y_{n,1}|^2 + [q_1q_4+q_2q_3]y_{n,1}\overline{y_{n,2}} \\
&\quad +[q_1q_4+q_2q_3]\overline{y_{n,1}}y_{n,2} + [q_2^2+q_4^2]|y_2|^2\} \\
&\leq \int_I \{[q_1^2+q_3^2]|y_{n,1}|^2 + 2\,|q_1q_4+q_2q_3|\,|y_{n,1}y_{n,2}| \\
&\quad +[q_2^2+q_4^2]|y_{n,2}|^2\} \\
&\leq \int_I \{[q_1^2+q_3^2]|y_{n,1}|^2 + |q_1q_4+q_2q_3|\,[|y_{n,1}|^2+|y_{n,2}|^2] \\
&\quad +[q_2^2+q_4^2]|y_{n,2}|^2\}\ .
\end{aligned}
$$

We apply the sufficiency argument in part (i) with $I_N = [N,\infty)$ to the last inequality in (4.43). Via Theorem 1.7 and inequalities (4.10), (4.11), (4.17), (4.18), (4.28), and (4.29), there exist constants K_1, K_2, K_3, and $K_4 > 0$ such that for each n

$$
\begin{aligned}
(4.44)\ \|By_n - B_N y_n\|^2 &\leq \frac{K_1}{\varepsilon}\sup_{t\in I_N}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2]\right\} \\
&\quad \cdot\left\{\int_{I_N}[(1-2\delta)f^{-2}+2\delta]|y_{n,1}|^2 \ +\varepsilon^2\int_{I_N}|y_{n,1}'|^2\right\} \\
&+\ \frac{K_2}{\varepsilon}\sup_{t\in I_N}\left\{\left(f(t)\int_t^{t+\varepsilon f(t)}q_1^2\right)^{\frac12}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2+q_2^2]\right)^{\frac12}\right. \\
&\quad \left.+\left(f(t)\int_t^{t+\varepsilon f(t)}q_2^2\right)^{\frac12}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2]\right)^{\frac12}\right\} \\
&\quad \cdot\left\{\int_{I_N}[(1-2\delta)f^{-2}+2\delta]|y_{n,1}|^2 \ +\varepsilon^2\int_{I_N}|y_{n,1}'|^2\right\} \\
&+\ \frac{K_3}{\varepsilon}\sup_{t\in I_N}\left\{\left(f(t)\int_t^{t+\varepsilon f(t)}q_1^2\right)^{\frac12}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2+q_2^2]\right)^{\frac12}\right. \\
&\quad \left.+\left(f(t)\int_t^{t+\varepsilon f(t)}q_2^2\right)^{\frac12}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2]\right)^{\frac12}\right\} \\
&\quad \cdot\left\{\int_{I_N}[(1-2\delta)f^{-2}+2\delta]|y_{n,2}|^2 \ +\varepsilon^2\int_{I_N}|y_{n,2}'|^2\right\} \\
&+\ \frac{K_4}{\varepsilon}\sup_{t\in I_N}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_2^2+q_4^2]\right\} \\
&\quad \cdot\left\{\int_{I_N}[(1-2\delta)f^{-2}+2\delta]|y_{n,2}|^2 \ +\varepsilon^2\int_{I_N}|y_{n,2}'|^2\right\}.
\end{aligned}
$$

Adding nonnegative terms to the right-hand side of inequality (4.44) gives for each n

$$(4.45)\ \|By_n - B_N y_n\|^2 \le K_5 \sup_{t\in I_N} F(t) \left\{ \int_{I_N} [(1-2\delta)f^{-2} + 2\delta][|y_{n,1}|^2 + |y_{n,2}|^2] + \varepsilon^2 \int_{I_N} [|y'_{n,1}|^2 + |y'_{n,2}|^2] \right\}$$

for some constant $K_5 > 0$ where

$$\begin{aligned} F(t) &= f(t)\int_t^{t+\varepsilon f(t)} [q_1^2 + q_3^2] \\ &\quad + \left(f(t)\int_t^{t+\varepsilon f(t)} q_1^2\right)^{\frac{1}{2}} \left(2f(t)\int_t^{t+\varepsilon f(t)} [q_4^2 + q_2^2]\right)^{\frac{1}{2}} \\ &\quad + \left(f(t)\int_t^{t+\varepsilon f(t)} q_2^2\right)^{\frac{1}{2}} \left(2f(t)\int_t^{t+\varepsilon f(t)} [q_1^2 + q_3^2]\right)^{\frac{1}{2}} \\ &\quad + f(t)\int_t^{t+\varepsilon f(t)} [q_2^2 + q_4^2]\,. \end{aligned}$$

Via Lemma 4.1 and inequality (4.45) there exists a constant $K_6 > 0$ such that for each n

$$(4.46)\qquad \|By_n - B_N y_n\|^2 \le K_6 \sup_{t\in I_N} F(t)\ (\|y_n\| + \|Ly_n\|)^2.$$

Therefore, via the the triangle inequality, the L_0-boundedness of B_0, and inequalities (4.42) and (4.46), we have for each n

$$\begin{aligned} \|By - B_N y\| &\le \|By - By_n\| + \|By_n - B_N y_n\| + \|B_N y_n - B_N y\| \\ &\le 2K_7(\|y - y_n\| + \|Ly - Ly_n\|) + \|By_n - B_N y_n\| \\ &< \frac{2K_7}{n} + K_6^{\frac{1}{2}} \left(\sup_{t\in I_N} F(t)\right)^{\frac{1}{2}} \|y_n\|_{L_0'} \quad \text{for some constant } K_7 > 0. \end{aligned}$$

By applying the triangle inequality and inequality (4.42) again, we obtain for each n

$$\begin{aligned} \|By - B_N y\| &\le \frac{2K_7}{n} + K_6^{\frac{1}{2}} \left(\sup_{t\in I_N} F(t)\right)^{\frac{1}{2}} (\|y_n - y\|_{L_0} + \|y\|_{L_0}) \\ &< \frac{2K_7}{n} + K_6^{\frac{1}{2}} \left(\sup_{t\in I_N} F(t)\right)^{\frac{1}{2}} \left(\frac{1}{n} + \|y\|_{L_0}\right)\,. \end{aligned}$$

We let $n \to \infty$ to obtain for each N

$$||By - B_N y|| \leq K_6^{\frac{1}{2}} \left(\sup_{t \in I_N} F(t) \right)^{\frac{1}{2}} ||y||_{L_0}.$$

Hence, the above inequality implies that for $y \neq 0$

$$\frac{||By - B_N y||}{||y||_{L_0}} \leq K_6^{\frac{1}{2}} \left(\sup_{t \in I_N} F(t) \right)^{\frac{1}{2}}. \tag{4.47}$$

Since $I_N = [N, \infty)$, equations (4.6) imply that $\sup_{t \in I_N} F(t) \to 0$ as $N \to \infty$ so that, via inequality (4.47), we have

$$||B - B_N|| \to 0 \text{ as } N \to \infty. \quad \#$$

Claim 4.2.3 *Each B_N is L_0-compact.*

Proof of Claim 4.2.3:

Let $\{y_n\}_{n=1}^{\infty} \subset D(L_0)$ be an L_0-bounded sequence where

$$y_n = \begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix} \text{ for each n.}$$

Since L_0 is the closure of L_0', for each n there exists a $\hat{y}_n \in D(L_0')$ such that

$$||y_n - \hat{y}_n|| + ||Ly_n - L\hat{y}_n|| < \frac{1}{n}. \tag{4.48}$$

Notice that $\{\hat{y}_n\}_{n=1}^{\infty}$ is an L_0-bounded sequence since there exists a constant $C > 0$ such that for each n

$$\begin{aligned} ||\hat{y}_n|| + ||L\hat{y}_n|| &\leq (||\hat{y}_n - y_n|| + ||L\hat{y}_n - Ly_n||) + (||y_n|| + ||Ly_n||) \\ &< \frac{1}{n} + C \\ &\leq 1 + C \end{aligned}$$

via the triangle inequality, inequality (4.48), and the L_0-boundedness of the sequence $\{y_n\}_{n=1}^{\infty}$. We need to show that for each N the sequence $\{B_N y_n\}_{n=1}^{\infty}$ has a convergent subsequence. First, we make use of the Arzela-Ascoli Theorem to show that the sequence $\{B_N \hat{y}_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Let us consider the sequence of the first component functions $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$. We show that this sequence is uniformly bounded on $[a, N]$. Partition $[a, N]$ by $J_i = [t_i, t_{i+1}]$, $1 \le i \le k$ with $t_1 = a$, $t_{i+1} = t_i + \hat{\varepsilon} f(t_i)$ and $\hat{\varepsilon} \in (0, \varepsilon)$ chosen so that $N = t_{k+1} = t_k + \hat{\varepsilon} f(t_k)$. Via equation (2.12) of [3, page 571] in the proof of Theorem 1.7, we know there exists a constant $C_1 > 0$ such that for $t \in J_i$

$$
\begin{aligned}
(4.49) \qquad |\hat{y}_{n,1}(t)|^2 \;&\le\; C_1 \left\{ \frac{1}{\hat{\varepsilon}^2 f^2(t_i)} \left[\int_{J_i} \frac{d\tau}{(1-2\delta) f^{-2}(\tau) + 2\delta} \right] \right. \\
&\quad \cdot \left[\int_{J_i} [(1-2\delta) f^{-2}(\tau) + 2\delta] |\hat{y}_{n,1}(\tau)|^2 \, d\tau \right] \\
&\quad \left. + \left[\int_{J_i} d\tau \right] \left[\int_{J_i} |\hat{y}'_{n,1}(\tau)|^2 \, d\tau \right] \right\} . \\
&=\; C_1 \left\{ \frac{1}{\hat{\varepsilon}^2 f^2(t_i)} \left[\int_{J_i} \frac{d\tau}{(1-2\delta) f^{-2}(\tau) + 2\delta} \right] \right. \\
&\quad \cdot \left[\int_{J_i} [(1-2\delta) f^{-2}(\tau) + 2\delta] |\hat{y}_{n,1}(\tau)|^2 \, d\tau \right] \\
&\quad \left. + \hat{\varepsilon} f(t_i) \int_{J_i} |\hat{y}'_{n,1}(\tau)|^2 \, d\tau \right\} .
\end{aligned}
$$

By applying Lemma 1.8, we obtain

$$
\begin{aligned}
(4.50) \qquad |\hat{y}_{n,1}(t)|^2 \;&\le\; C_1 \left\{ \frac{C_2 f(t_i)}{\hat{\varepsilon}} \int_{J_i} [(1-2\delta) f^{-2}(\tau) + 2\delta] |\hat{y}_{n,1}(\tau)|^2 \, d\tau \right. \\
&\quad \left. + \hat{\varepsilon} f(t_i) \int_{J_i} |\hat{y}'_{n,1}(\tau)|^2 \, d\tau \right\}
\end{aligned}
$$

for some constant $C_2 > 0$ and for $t \in J_i$. Since f is continuous on [a,N], there exists a constant $C_3 > 0$ such that for $t \in [a, N]$

$$
(4.51) \; |\hat{y}_{n,1}(t)|^2 \;\le\; C_3 \left\{ \int_a^N [(1-2\delta) f^{-2}(\tau) + 2\delta] |\hat{y}_{n,1}(\tau)|^2 \, d\tau + \int_a^N |\hat{y}'_{n,1}(\tau)|^2 \, d\tau \right\}
$$

so that via Lemma 4.1 there exists a constant $C_4 > 0$

$$
\begin{aligned}
(4.52) \qquad |\hat{y}_{n,1}(t)|^2 \;&\le\; C_4 \left(\|\hat{y}_n\|^2 + \|L\hat{y}_n\|^2 \right) \\
&\le\; C_4 \left(\|\hat{y}_n\| + \|L\hat{y}_n\| \right)^2 .
\end{aligned}
$$

Since $\{\hat{y}_n\}_{n=1}^{\infty}$ is a L_0-bounded sequence, inequality (4.52) implies that the sequence $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$ is uniformly bounded on $[a, N]$.

Now, we show that $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$ is equicontinuous on $[a, N]$. For $s, t \in [a, N]$ we have via the Cauchy-Schwarz inequality

$$
(4.53) \qquad \begin{aligned}
|\hat{y}_{n,1}(s) - \hat{y}_{n,1}(t)| &= \left| \int_t^s \hat{y}'_{n,1}(\tau)\, d\tau \right| \\
&\leq \left| \int_t^s |\hat{y}'_{n,1}(\tau)|\, d\tau \right| \\
&\leq \left| \int_t^s d\tau \right|^{\frac{1}{2}} \left| \int_t^s |\hat{y}'_{n,1}(\tau)|^2\, d\tau \right|^{\frac{1}{2}} \\
&\leq |s-t|^{\frac{1}{2}} \| L\hat{y}_n \| \\
&\leq |s-t|^{\frac{1}{2}} \| \hat{y}_n \|_{L_0} .
\end{aligned}
$$

Since $\{\hat{y}_n\}_{n=1}^{\infty}$ is a L_0-bounded sequence, inequality (4.53) implies that the sequence $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$ is equicontinuous on $[a, N]$.

Therefore, via the Arzela-Ascoli Theorem we conclude that there exists a subsequence $\{\hat{y}_{n_k,1}\}_{k=1}^{\infty}$ of $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$ which converges uniformly on $[a, N]$. By repeating the above arguments (inequalities (4.49)-(4.53)) on the subsequence $\{\hat{y}_{n_k,2}\}_{k=1}^{\infty}$ of $\{\hat{y}_{n,2}\}_{n=1}^{\infty}$, we conclude that there exists a subsequence of $\{\hat{y}_{n_k,2}\}_{k=1}^{\infty}$ which converges uniformly on $[a, N]$. Hence, there exists a subsequence of $\{\hat{y}_n\}_{n=1}^{\infty}$ which converges uniformly on $[a, N]$. WLOG, we assume the sequence $\{\hat{y}_n\}_{n=1}^{\infty}$ converges uniformly on $[a, N]$.

Now, suppressing the independent variable and using the inequality $2ab \leq a^2 + b^2$ we have for each N

$$
(4.54) \quad \begin{aligned}
\| B_N\hat{y}_n - B_N\hat{y}_m \|^2 &= \int_a^N \{ [q_1^2 + q_3^2] |\hat{y}_{n,1} - \hat{y}_{m,1}|^2 \\
&\quad + [q_1q_4 + q_2q_3](\hat{y}_{n,1} - \hat{y}_{m,1})\overline{(\hat{y}_{n,2} - \hat{y}_{m,2})} \\
&\quad + [q_1q_4 + q_2q_3]\overline{(\hat{y}_{n,1} - \hat{y}_{m,1})}(\hat{y}_{n,2} - \hat{y}_{m,2}) \\
&\quad + [q_2^2 + q_4^2] |\hat{y}_{n,2} - \hat{y}_{m,2}|^2 \} \\
&\leq \int_a^N \{ [q_1^2 + q_3^2] |\hat{y}_{n,1} - \hat{y}_{m,1}|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\,|q_1q_4+q_2q_3|\,|(\hat{y}_{n,1}-\hat{y}_{m,1})(\hat{y}_{n,2}-\hat{y}_{m,2})| \\
&+[q_2^2+q_4^2]|\hat{y}_{n,2}-\hat{y}_{m,2}|^2\} \\
\le\ & \int_a^N \{[q_1^2+q_3^2]|\hat{y}_{n,1}-\hat{y}_{m,1}|^2 \\
&+|q_1q_4+q_2q_3|\,(|\hat{y}_{n,1}-\hat{y}_{m,1}|^2+|\hat{y}_{n,2}-\hat{y}_{m,2}|^2) \\
&+[q_2^2+q_4^2]|\hat{y}_{n,2}-\hat{y}_{m,2}|^2\}\,.
\end{aligned}
$$

By adding nonnegative terms to the right-hand side of inequality (4.54), we obtain

$$
\begin{aligned}
\|B_N\hat{y}_n-B_N\hat{y}_m\|^2 \ \le\ & \int_a^N \{[q_1^2+q_3^2]\ \|\hat{y}_n-\hat{y}_m\|_2^2+|q_1q_4+q_2q_3|\ \|\hat{y}_n-\hat{y}_m\|_2^2 \\
&+[q_2^2+q_4^2]\ \|\hat{y}_n-\hat{y}_m\|_2^2\} \\
\le\ & \sup_{a\le t\le N}\|\hat{y}_n(t)-\hat{y}_m(t)\|_2^2\left\{\int_a^N\ [q_1^2+q_3^2]\right. \\
&\left.+\int_a^N\ |q_1q_4+q_2q_3|+\int_a^N\ [q_2^2+q_4^2]\}\right\}
\end{aligned}
$$

where

$$
\|\hat{y}_n(t)-\hat{y}_m(t)\|_2^2=|\hat{y}_{n,1}(t)-\hat{y}_{m,1}(t)|^2+|\hat{y}_{n,2}(t)-\hat{y}_{m,2}(t)|^2\ .
$$

By applying the argument used in inequality (4.27) to the inequality above, we see that the integrals on the right-hand side are finite on $[a,N]$. Hence, there exists a constant $C>0$ such that

$$
\|B_N\hat{y}_n-B_N\hat{y}_m\|^2\le C\ \sup_{a\le t\le N}\ \|\hat{y}_n(t)-\hat{y}_m(t)\|_2^2\,.
$$

Since $\{\hat{y}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the uniform norm, the above inequality implies that the sequence $\{B_N\hat{y}_n\}_{n=1}^{\infty}$ is Cauchy in $\mathcal{L}_w^2(I)$ for each N. Therefore, $\{B_N\hat{y}_n\}_{n=1}^{\infty}$ converges for each N as $n\to\infty$ since $\mathcal{L}_w^2(I)$ is complete. Let $\xi=\lim_{n\to\infty}B_N\hat{y}_n$.

Via the triangle inequality, the L_0-boundedness of B_0, and inequality (4.48), there exists a constant $C>0$ such that for each n and for each N

$$
\begin{aligned}
\|B_Ny_n-\xi\| \ \le\ & \|B_Ny_n-B_N\hat{y}_n\|+\|B_N\hat{y}_n-\xi\| \\
\le\ & C\,(\|y_n-\hat{y}_n\|+\|Ly_n-L\hat{y}_n\|)+\|B_N\hat{y}_n-\xi\| \\
<\ & \frac{C}{n}+\|B_N\hat{y}_n-\xi\|\,.
\end{aligned}
$$

Since the right-hand side of the above inequality converges for each N as $n \to \infty$, we conclude that $\{B_N y_n\}_{n=1}^{\infty}$ converges for each N as $n \to \infty$. Hence, each B_N is L_0-compact. #

Since B_0 is the uniform limit of L_0-compact operators, B_0 is L_0-compact.

Necessity

We use contradiction arguments to show that each equation in (4.6) must hold.

Suppose that for some $\varepsilon \in (0, \frac{1}{2\sqrt{2}\,\delta})$ there exists a $\rho > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers such that $r_n \to \infty$ as $n \to \infty$ and for each n

$$(4.55) \qquad f(r_n)\int_{r_n}^{r_n+\varepsilon f(r_n)} q_i^2(t)\,dt \geq \rho \quad \text{for some } i \in \{1,2,3,4\}.$$

Suppose $i = 1$. Let $\{\Phi_r\}_{r\geq a}$ be defined as in (2.28). Then via inequalities (4.33) and (4.36) there exist constants $C_1, C_3 > 0$ such that for each n

$$(4.56) \qquad \begin{aligned} \|\Phi_{r_n}\|_{L_0}^2 &= (\|\Phi_{r_n}\| + \|L\Phi_{r_n}\|)^2 \\ &\leq 2\,(\|\Phi_{r_n}\|^2 + \|L\Phi_{r_n}\|^2) \\ &\leq 2C_1\, f(r_n) + 2C_3\, f^{-1}(r_n). \end{aligned}$$

For each $r \geq a$ define

$$(4.57) \qquad \tilde{\Phi}_r(t) = f^{\frac{1}{2}}(r)\Phi_r(t) \quad \text{for } t \geq a.$$

Via inequality (4.56) we have for each n

$$(4.58) \qquad \begin{aligned} \|\tilde{\Phi}_{r_n}\|_{L_0}^2 &= f(r_n)\,\|\Phi_{r_n}\|_{L_0}^2 \\ &\leq 2C_1\, f^2(r_n) + 2C_3\,. \end{aligned}$$

Since f is bounded on I, inequality (4.58) implies that $\{\tilde{\Phi}_{r_n}\}_{n=1}^{\infty}$ is an L_0-bounded sequence. Via the L_0-compactness of B_0, $\{B\tilde{\Phi}_{r_n}\}_{n=1}^{\infty}$ has a convergent subsequence. WLOG, we assume $\{B\tilde{\Phi}_{r_n}\}_{n=1}^{\infty}$ converges, say to some y_0. Now, since $\phi_{r_n} \equiv 1$ on $[r_n, r_n + \varepsilon f(r_n)]$ and supp(ϕ_{r_n})=$[r_n - 2\varepsilon f(r_n), r_n + 2\varepsilon f(r_n)]$, we have for each n

$$(4.59) \qquad \rho \;\leq\; f(r_n)\int_{r_n}^{r_n+\varepsilon f(r_n)} q_1^2(t)\,dt$$

$$
\begin{aligned}
&\leq f(r_n)\int_{r_n}^{r_n+\varepsilon f(r_n)}[q_1^2(t)+q_3^2(t)]\,dt\\
&\leq f(r_n)\int_{r_n-2\varepsilon f(r_n)}^{r_n+2\varepsilon f(r_n)}[q_1^2(t)+q_3^2(t)]\phi_{r_n}^2(t)\,dt\\
&= \|B\tilde{\Phi}_{r_n}\|^2 .
\end{aligned}
$$

By an argument similar to the one used in the proof of Theorem 2.1 (ii), we conclude that $y_0 = 0$ a.e. in $[a,\infty)$. This contradiction implies that equation (4.6) holds for $i = 1$. If we begin with q_3 instead of q_1 in inequality (4.59), then the above argument shows that equation (4.6) holds for $i = 3$.

Now, for each $r \geq a$ define

$$
\tilde{\Psi}_r(t) = f^{\frac{1}{2}}(r)\Psi_r(t) \quad \text{for } t \geq a \tag{4.60}
$$

where $\{\Psi_r\}_{r\geq a}$ is given in (2.28). Via inequalities (4.34) and (4.37) we have for each n

$$
\begin{aligned}
\|\tilde{\Psi}_{r_n}\|_{L_0}^2 &= f(r_n)\,\|\Psi_{r_n}\|_{L_0}^2\\
&\leq 2C_1\,f^2(r_n) + 2C_3 .
\end{aligned} \tag{4.61}
$$

Thus, $\{\tilde{\Psi}_{r_n}\}_{n=1}^{\infty}$ is an L_0-bounded sequence.

By repeating the above argument with q_1 replaced by either q_2 or q_4 and $\tilde{\Phi}_r$ replaced by $\tilde{\Psi}_r$, we conclude that equation (4.6) holds for $i = 2$ and $i = 4$. (Note that the proof of the necessity shows that (4.6) holds for every $\varepsilon \in (0, \frac{1}{2\sqrt{2}\delta})$.)

Therefore, we have shown that B_0 is L_0-compact if and only if equations (4.6) hold. We apply Theorem 1.6 to conclude that B_1 is L_1-compact if and only if equations (4.6) hold. □

We need the following lemma in the proof of Theorem 4.4. In this lemma, we consider the minimal unclosed operators, L_0' and B_0', associated with the differential expressions (1.1) and (1.3), respectively, where the weight functions $w_1 = w_2$. Let $\omega = w_1 = w_2$. We use a completing-the-square technique, as in [8, pages 371-2], to establish a lower bound on the $\mathcal{L}_w^2(I)$-norm of L.

Lemma 4.3 *Let* $0 < \delta < 1$ *and* $\omega = w_1 = w_2$. *There exists an* $\hat{\varepsilon} > 0$ *and positive definite matrices* Δ_1, Δ_2 *such that if* $|p_i(t)| \leq \hat{\varepsilon}|p(t)|$, *for* $i = 1, 2$, *and* $|[\omega^{-1}p]'(t)| \leq \hat{\varepsilon}^2\omega^{-1}(t)p^2(t)$ *on an interval I, then for* $y \in D(L_0')$

$$\|Ly\|^2 \geq \int_I \{[y'(t)]^*\Delta_1 y'(t) + y^*(t)\Delta_2 y(t)\}\, dt \tag{4.62}$$

where $\Delta_2 \geq \delta\omega^{-1}(t)p^2(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* Δ_1, Δ_2 *are defined by (4.64) and (4.65), respectively.*

Note: By the matrix inequality $A \geq B$, we mean that the matrix $A - B$ is positive semi-definite.

PROOF:

Let $y \in D(L_0')$. Then after some simplification we have

$$\begin{aligned}
\|Ly\|^2 &= \int_I [(Ly)(t)]^* \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} [(Ly)(t)]\, dt \\
&= \int_I \{\omega^{-1}(t)|y_1'(t)|^2 + \omega^{-1}(t)|y_2'(t)|^2 + \omega^{-1}(t)p(t)\,[|y_1(t)|^2 - |y_2(t)|^2]' \\
&\quad -\omega^{-1}(t)p_1(t)\,[y_1(t)\bar{y}_2'(t) + \bar{y}_1(t)y_2'(t)] \\
&\quad +\omega^{-1}(t)p_2(t)\,[\bar{y}_1'(t)y_2(t) + y_1'(t)\bar{y}_2(t)] + \omega^{-1}(t)[p_1^2(t) + p^2(t)]\,|y_1(t)|^2 \\
&\quad +2\omega^{-1}(t)p(t)[p_1(t) + p_2(t)]\,[\Re\mathrm{e}(y_1(t)\bar{y}_2(t))] \\
&\quad +\omega^{-1}(t)[p_2^2(t) + p^2(t)]\,|y_2(t)|^2\}\, dt.
\end{aligned}$$

After integrating the third term of the integrand by parts, we obtain

$$\begin{aligned}
\|Ly\|^2 &= \int_I \{\omega^{-1}(t)|y_1'(t)|^2 + \omega^{-1}(t)|y_2'(t)|^2 - (\omega^{-1}p)'(t)\,[|y_1(t)|^2 - |y_2(t)|^2] \\
&\quad -\omega^{-1}(t)p_1(t)\,[y_1(t)\bar{y}_2'(t) + \bar{y}_1(t)y_2'(t)] \\
&\quad +\omega^{-1}(t)p_2(t)\,[\bar{y}_1'(t)y_2(t) + y_1'(t)\bar{y}_2(t)] + \omega^{-1}(t)[p_1^2(t) + p^2(t)]\,|y_1(t)|^2 \\
&\quad +2\omega^{-1}(t)p(t)[p_1(t) + p_2(t)]\,[\Re\mathrm{e}(y_1(t)\bar{y}_2(t))] \\
&\quad +\omega^{-1}(t)[p_2^2(t) + p^2(t)]\,|y_2(t)|^2\}\, dt.
\end{aligned} \tag{4.63}$$

Now, define

$$\Delta_1 = \omega^{-1} \begin{pmatrix} 1-\beta_1^2 & 0 \\ 0 & 1-\beta_2^2 \end{pmatrix}, \tag{4.64}$$

where $0 < \beta_i < 1$, for $i = 1, 2$, and

$$\Delta_2 = \begin{pmatrix} \omega^{-1}(1-\beta_2^{-2})p_1^2 + \omega^{-1}p^2 - (\omega^{-1}p)' & \omega^{-1}p[p_1+p_2] \\ \omega^{-1}p[p_1+p_2] & \omega^{-1}(1-\beta_1^{-2})p_2^2 + \omega^{-1}p^2 - (\omega^{-1}p)' \end{pmatrix}. \tag{4.65}$$

Notice that Δ_1 is positive definite. Now, we show that for sufficiently small $\hat{\varepsilon} > 0$, $\Delta_2 \geq \delta\omega^{-1}(t)p^2(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Via the hypotheses on p_1 and $(\omega^{-1}p)'$ we have, suppressing the independent variable,

$$\begin{aligned} \omega^{-1}(1-\beta_2^{-2})p_1^2 + \omega^{-1}p^2 - (\omega^{-1}p)' &\geq \omega^{-1}(1-\beta_2^{-2})\hat{\varepsilon}^2p^2 + \omega^{-1}p^2 - \omega^{-1}\hat{\varepsilon}^2p^2 \\ &= \omega^{-1}p^2(1-\beta_2^{-2}\hat{\varepsilon}^2) \\ &\geq \delta\omega^{-1}p^2 \ \text{ iff } \ \hat{\varepsilon} \leq (1-\delta)^{\frac{1}{2}}\beta_2 . \end{aligned} \tag{4.66}$$

Also, via the hypotheses on p_2 and $(\omega^{-1}p)'$ we have

$$\begin{aligned} \omega^{-1}(1-\beta_1^{-2})p_2^2 + \omega^{-1}p^2 - (\omega^{-1}p)' &\geq \omega^{-1}(1-\beta_1^{-2})\hat{\varepsilon}^2p^2 + \omega^{-1}p^2 - \omega^{-1}\hat{\varepsilon}^2p^2 \\ &= \omega^{-1}p^2(1-\beta_1^{-2}\hat{\varepsilon}^2) \\ &\geq \delta\omega^{-1}p^2 \ \text{ iff } \ \hat{\varepsilon} \leq (1-\delta)^{\frac{1}{2}}\beta_1 . \end{aligned} \tag{4.67}$$

Hence, via (4.66), (4.67), and the hypotheses on p_1 and p_2,

$$\begin{aligned} \det\Delta_2 &= [\omega^{-1}(1-\beta_2^{-2})p_1^2 + \omega^{-1}p^2 - (\omega^{-1}p)'][\omega^{-1}(1-\beta_1^{-2})p_2^2 + \omega^{-1}p^2 - (\omega^{-1}p)'] \\ &\quad -\omega^{-2}p^2(p_1+p_2)^2 \\ &\geq [\omega^{-1}p^2(1-\beta_2^{-2}\hat{\varepsilon}^2)][\omega^{-1}p^2(1-\beta_1^{-2}\hat{\varepsilon}^2)] - \omega^{-2}p^2(4\hat{\varepsilon}^2p^2) \\ &\geq \omega^{-2}p^4(1-\beta_1^{-2}\hat{\varepsilon}^2 - \beta_2^{-2}\hat{\varepsilon}^2 + \hat{\varepsilon}^4\beta_1^{-2}\beta_2^{-2} - 4\hat{\varepsilon}^2) \\ &\geq \delta^2\omega^{-2}p^4 \ \text{ for sufficiently small } \hat{\varepsilon} . \end{aligned}$$

Therefore, Δ_2 is a positive definite matrix which satisfies $\Delta_2 \geq \delta\omega^{-1}(t)p^2(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for sufficiently small $\hat{\varepsilon} > 0$.

Via equation (4.63)

$$\begin{aligned}
\|Ly\|^2 &= \int_I \{\omega^{-1}(1-\beta_1^2)|y_1'|^2 + \omega^{-1}(1-\beta_2^2)|y_2'|^2 + \omega^{-1}\beta_1^2|y_1'|^2 + \omega^{-1}\beta_2^2|y_2'|^2 \\
&\quad -[(\omega^{-1}p)' + \omega^{-1}\beta_2^{-2}p_1^2]|y_1|^2 + \omega^{-1}\beta_2^{-2}p_1^2|y_1|^2 \\
&\quad +[(\omega^{-1}p)' - \omega^{-1}\beta_1^{-2}p_2^2]|y_2|^2 + \omega^{-1}\beta_1^{-2}p_2^2|y_2|^2] \\
&\quad -\omega^{-1}p_1\,(y_1\bar{y}_2' + \bar{y}_1 y_2') + \omega^{-1}p_2\,(\bar{y}_1' y_2 + y_1'\bar{y}_2) + \omega^{-1}(p_1^2+p^2)\,|y_1|^2 \\
&\quad +2\omega^{-1}p(p_1+p_2)\,[\Re\mathrm{e}(y_1\bar{y}_2)] + \omega^{-1}(p_2^2+p^2)\,|y_2|^2\}\,dt \\
&= \int_I \{(y')^*\Delta_1 y' + [\omega^{-1}\beta_1^2|y_1'|^2 + \omega^{-1}p_2\,(\bar{y}_1' y_2 + y_1'\bar{y}_2) + \omega^{-1}\beta_1^{-2}p_2^2|y_2|^2] \\
&\quad +[\omega^{-1}\beta_2^2|y_2'|^2 - \omega^{-1}p_1\,(y_1\bar{y}_2' + \bar{y}_1 y_2') + \omega^{-1}\beta_2^{-2}p_1^2|y_1|^2] + y^*\Delta_2 y\}\,dt \\
&= \int_I \{(y')^*\Delta_1 y' + \omega^{-1}|\beta_1 y_1' + \beta_1^{-1}p_2 y_2|^2 + \omega^{-1}|\beta_2 y_2' + \beta_2^{-1}p_1 y_1|^2 \\
&\quad +y^*\Delta_2 y\}\,dt \\
&\geq \int_I \{(y')^*\Delta_1 y' + y^*\Delta_2 y\}\,dt\,. \quad \square
\end{aligned}$$

Theorem 4.4 *Let $I=[a,\infty)$ for some $a>0$ and let $f(t)=[\omega^2(t)+\delta p^2(t)]^{-\frac{1}{2}}$. In addition to the hypotheses of Lemma 4.3, suppose L is limit point at ∞ and suppose there exists a constant $N_1>0$ such that $|[p^{-1}(t)]'| \leq N_1$ a.e. on I. Then*

(i) B_1 is L_1-bounded if and only if

$$\sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\; q_i^2(\tau)\,d\tau < \infty, \quad i=1,2,3,4, \tag{4.68}$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$;

(ii) B_1 is L_1-compact if and only if

$$\lim_{t\to\infty} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\; q_i^2(\tau)\,d\tau = 0, \quad i=1,2,3,4, \tag{4.69}$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$, where N_0 is defined below.

Proof:

Remark: There exists a constant $N_2>0$ such that $|\omega'(t)| \leq N_2\,\omega(t)|p(t)|$ on I since via the triangle inequality and the bounds on $|(\omega^{-1}p)'|$ and $|(p^{-1})'|$

$$\left|\frac{\omega'(t)}{\omega(t)p(t)}\right| \;\leq\; \left|\frac{\omega'(t)}{\omega(t)p(t)} - [p^{-1}(t)]'\right| + |[p^{-1}(t)]'|$$

$$
\begin{aligned}
&\leq \frac{\omega(t)}{p^2(t)} \left| \frac{\omega'(t)p(t)}{\omega^2(t)} - \frac{p'(t)}{\omega(t)} \right| + N_1 \\
&= \frac{\omega(t)}{p^2(t)} \left| [\omega^{-1}p]'(t) \right| + N_1 \\
&\leq \hat{\varepsilon}^2 + N_1 =: N_2,
\end{aligned}
$$

where $\hat{\varepsilon}$ is as in Lemma 4.3.

Let $N_0 = \max \left\{ N_2, \frac{N_2}{\sqrt{2\delta}} + \sqrt{\frac{2}{\delta}} N_1 \right\}$. Now, we state and prove two claims.

Claim 4.4.1 *Positive and negative powers of f are essentially constant on intervals of length εf, where $\varepsilon \in (0, \frac{1}{2N_0})$.*

Proof of Claim 4.4.1:

We show that the hypotheses of Lemma 1.8 hold with $f(t) = [\omega^2(t) + \delta p^2(t)]^{-\frac{1}{2}}$ and $g \equiv 1$. Notice that the hypotheses on ω and p imply that $f \in AC_{loc}(I)$. Via the triangle inequality and the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we have

$$
\begin{aligned}
|f'(t)| &= \left| \frac{\omega(t)\omega'(t) + \delta p(t)p'(t)}{[\omega^2(t) + \delta p^2(t)]^{\frac{3}{2}}} \right| \\
&\leq \frac{\omega(t)|\omega'(t)| + \delta |p(t)|\,|p'(t)|}{[\omega^2(t) + \delta p^2(t)]^{\frac{3}{2}}} \\
&\leq \frac{(|\omega'(t)| + \delta^{\frac{1}{2}}|p'(t)|)(\omega(t) + \delta^{\frac{1}{2}}|p(t)|)}{[\omega^2(t) + \delta p^2(t)]^{\frac{3}{2}}} \\
&\leq \frac{(|\omega'(t)| + \delta^{\frac{1}{2}}|p'(t)|)[2\,(\omega^2(t) + \delta p^2(t))]^{\frac{1}{2}}}{[\omega^2(t) + \delta p^2(t)]^{\frac{3}{2}}} \\
&\leq \frac{\sqrt{2}\,(|\omega'(t)| + \delta^{\frac{1}{2}}|p'(t)|)}{[\omega^2(t) + \delta p^2(t)]}
\end{aligned}
$$

Applying the inequality $\frac{1}{a^2+b^2} \leq \frac{1}{2\,|ab|}$, the remark above, and the bound on $|(p^{-1})'|$, we conclude from the above inequality that

$$
\begin{aligned}
|f'(t)| &\leq \frac{\sqrt{2}\,|\omega'(t)|}{[\omega^2(t) + \delta p^2(t)]} + \sqrt{\frac{2}{\delta}} \frac{|p'(t)|}{p^2(t)} \\
&\leq \frac{|\omega'(t)|}{\sqrt{2\delta}|\omega(t)p(t)|} + \sqrt{\frac{2}{\delta}}\, |[p^{-1}(t)]'| \\
&\leq \frac{N_2}{\sqrt{2\delta}} + \sqrt{\frac{2}{\delta}} N_1 \leq N_0 .
\end{aligned}
$$

Hence, Lemma 1.8 implies that positive and negative powers of f are essentially constant on intervals of length εf. #

Claim 4.4.2 *Positive and negative powers of ω are essentially constant on intervals of length εf, where $\varepsilon \in (0, \frac{1}{2N_0})$.*

Proof of Claim 4.4.2:

Note that since $(p^{-1})'$ exists on I, either $p > 0$ or $p < 0$ on I. Suppose $p > 0$. Now, we show that the hypotheses of Lemma 1.8 hold with f of the lemma replaced by $f_1 = \delta^{-\frac{1}{2}}p^{-1}$ and $g = \omega$. (If $p < 0$, then we take $f_1 = -\delta^{-\frac{1}{2}}p^{-1}$.) Notice that the hypotheses on ω and p imply that $f_1, \omega \in AC_{loc}(I)$. Via the bound on $|(p^{-1})'|$, $|f_1'(t)| = \delta^{-\frac{1}{2}}|[p^{-1}(t)]'| \leq \delta^{-\frac{1}{2}} N_1 \leq N_0$; and via the above remark

$$\left|\frac{f_1(t)g'(t)}{g(t)}\right| = \delta^{-\frac{1}{2}} \left|\frac{\omega'(t)}{\omega(t)p(t)}\right| \leq \delta^{-\frac{1}{2}} N_2 .$$

Hence, Lemma 1.8 implies that positive and negative powers of ω are essentially constant on intervals of length εf_1. Since $\omega^2(t) + \delta p^2(t) > \delta p^2(t)$, we have that $[\omega^2(t) + \delta p^2(t)]^{-\frac{1}{2}} < \delta^{-\frac{1}{2}}p^{-1}$, i.e., $f < f_1$. Thus, positive and negative powers of ω are essentially constant on intervals of length εf. #

Now, we show that B_0 is L_0-bounded if and only if inequalities (4.68) hold. Note that $D(L_0') \subseteq D(B_0')$ since the order of B is less than the order of L and the coefficients are sufficiently smooth.

(i) Sufficiency

Let us first consider any $y \in D(L_0')$ whose second component is zero. Then we have

$$(4.70) \quad ||y||^2 = \int_I \omega(\tau)|y_1(\tau)|^2 \, d\tau,$$

$$(4.71) \quad ||By||^2 = \int_I \omega^{-1}(\tau)[q_1(\tau)^2 + q_3(\tau)^2]|y_1(\tau)|^2 \, d\tau, \quad \text{and via Lemma 4.3}$$

$$(4.72) \quad ||Ly||^2 \geq \int_I \omega^{-1}(\tau)\{(1-\beta_1^2)|y_1'(\tau)|^2 + \delta p^2(\tau)]|y_1(\tau)|^2\}d\tau$$

Now, we show that the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2N_0})$ with $N = \omega^{-1}(q_1^2 + q_3^2), W = \omega + \delta\omega^{-1}p^2$ and $P = \omega^{-1}(1 - \beta_1^2)$.

Via Claims 4.4.1 and 4.4.2 we have for some constants $C_1, C_2 > 0$

$$
\begin{aligned}
(4.73)\quad S_1 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} \frac{\omega(\tau)}{(1-\beta_1^2)}\,d\tau\right]\left[\int_t^{t+\varepsilon f(t)} \omega^{-1}(\tau)[q_1(\tau)^2+q_3(\tau)^2]\,d\tau\right]\right\} \\
&\leq \frac{C_1}{\varepsilon(1-\beta_1^2)}\sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_1(\tau)^2+q_3(\tau)^2]\,d\tau\right\} \text{ and} \\
(4.74)\quad S_2 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} \frac{d\tau}{\omega(\tau)+\delta\omega^{-1}(\tau)p^2(\tau)}\right]\right. \\
&\quad \left.\cdot\left[\int_t^{t+\varepsilon f(t)} \omega^{-1}(\tau)[q_1(\tau)^2+q_3(\tau)^2]\,d\tau\right]\right\} \\
&= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} \omega(\tau)f^2(\tau)\,d\tau\right]\right. \\
&\quad \left.\cdot\left[\int_t^{t+\varepsilon f(t)} \omega^{-1}(\tau)[q_1(\tau)^2+q_3(\tau)^2]\,d\tau\right]\right\} \\
&\leq \frac{C_2}{\varepsilon}\sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_1(\tau)^2+q_3(\tau)^2]\,d\tau\right\}
\end{aligned}
$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$.

Inequalities (4.68), (4.73), and (4.74) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2N_0})$. Therefore (via Theorem 1.7), there exists a constant $k_1 > 0$ such that

$$
\begin{aligned}
(4.75)\qquad & \int_I \omega^{-1}(\tau)[q_1(\tau)^2+q_3(\tau)^2]\,|y_1(\tau)|^2\,d\tau \\
&\leq k_1\left\{\int_I [\omega(\tau)+\delta\omega^{-1}(\tau)p^2(\tau)]\,|y_1(\tau)|^2\,d\tau \right. \\
&\quad \left. +\varepsilon^2\int_I \omega^{-1}(\tau)(1-\beta_1^2)|y_1'(\tau)|^2\,d\tau\right\}.
\end{aligned}
$$

By substituting (4.70)-(4.72) into (4.75) and applying Lemma 4.3, we obtain for some constant $\hat{k}_1 > 0$

$$(4.76)\qquad ||By||^2 \leq \hat{k}_1\,(||y||^2 + ||Ly||^2).$$

Since y is arbitrary and $D(L_0') \subseteq D(B_0')$, inequality (4.76) implies that B_0' is L_0'-bounded on those $y \in D(L_0')$ with $y_2 \equiv 0$.

Let us now consider any $y \in D(L_0')$ whose first component is zero. Then we have

$$
\begin{aligned}
(4.77)\quad ||y||^2 &= \int_I \omega(\tau)|y_2(\tau)|^2\,d\tau, \\
(4.78)\quad ||By||^2 &= \int_I \omega^{-1}(\tau)[q_2(\tau)^2+q_4(\tau)^2]|y_2(\tau)|^2\,d\tau, \text{ and via Lemma 4.1}
\end{aligned}
$$

$$(4.79)\quad ||Ly||^2 \;\geq\; \int_I \{\omega^{-1}(\tau)(1-\beta_2^2)|y_2'(\tau)|^2 + \delta\omega^{-1}(\tau)p^2(\tau)|y_2(\tau)|^2\}\, d\tau\,.$$

Following an argument similar to the one above, we show the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2N_0})$ with $N = \omega^{-1}(q_2^2+q_4^2), W = \omega + \delta\omega^{-1}p^2$, and $P = \omega^{-1}(1-\beta_2^2)$. Via Claims 4.4.1 and 4.4.2 we have for some constants $C_3, C_4 > 0$

$$\begin{aligned}
(4.80)\quad S_1 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} \frac{\omega(\tau)}{(1-\beta_2^2)}\, d\tau\right]\left[\int_t^{t+\varepsilon f(t)} \omega^{-1}(\tau)[q_2(\tau)^2+q_4(\tau)^2]\, d\tau\right]\right\}\\
&\leq \frac{C_3}{\varepsilon(1-\beta_2^2)}\sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_2(\tau)^2+q_4(\tau)^2]\, d\tau\right\} \text{ and}\\
(4.81)\quad S_2 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} \frac{d\tau}{\omega(\tau)+\delta\omega^{-1}(\tau)p^2(\tau)}\right]\right.\\
&\qquad \left.\cdot\left[\int_t^{t+\varepsilon f(t)} \omega^{-1}(\tau)[q_2(\tau)^2+q_4(\tau)^2]\, d\tau\right]\right\}\\
&= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} \omega(\tau)f^2(\tau)\, d\tau\right]\right.\\
&\qquad \left.\cdot\left[\int_t^{t+\varepsilon f(t)} \omega^{-1}(\tau)[q_2(\tau)^2+q_4(\tau)^2]\, d\tau\right]\right\}\\
&\leq \frac{C_4}{\varepsilon}\sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_2(\tau)^2+q_4(\tau)^2]\, d\tau\right\}
\end{aligned}$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$. Inequalities (4.68), (4.80), and (4.81) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2N_0})$. Therefore (via Theorem 1.7), there exists a constant $k_2 > 0$ such that

$$\begin{aligned}
(4.82)\qquad &\int_I \omega^{-1}(\tau)[q_2(\tau)^2+q_4(\tau)^2]\; |y_2(\tau)|^2\, d\tau\\
&\quad\leq\; k_2\left\{\int_I [\omega(\tau)+\delta\omega^{-1}(\tau)p^2(\tau)]\; |y_2(\tau)|^2\, d\tau\right.\\
&\qquad\left. +\varepsilon^2\int_I \omega^{-1}(\tau)(1-\beta_2^2)|y_2'(\tau)|^2\, d\tau\right\}.
\end{aligned}$$

By substituting (4.77)-(4.79) into (4.82) and applying Lemma 4.3, we obtain for some constant $\hat{k}_2 > 0$

$$(4.83)\qquad ||By||^2 \leq \hat{k}_2\,(||y||^2 + ||Ly||^2).$$

Since y is arbitrary and $D(L_0') \subseteq D(B_0')$, inequality (4.83) implies that B_0' is L_0'-bounded on those $y \in D(L_0')$ with $y_1 \equiv 0$.

Finally, we consider any $y \in D(L_0')$. Then we have, suppressing the independent variable,

$$(4.84) \qquad \|y\|^2 = \int_I \omega[|y_1|^2 + |y_2|^2],$$

$$(4.85) \qquad \begin{aligned} \|By\|^2 &= \int_I \omega^{-1}\{[q_1^2 + q_3^2]|y_1|^2 + [q_1q_4 + q_2q_3]y_1\bar{y_2} \\ &\quad +[q_1q_4 + q_2q_3]\bar{y_1}y_2 + [q_2^2 + q_4^2]|y_2|^2\} \\ &\leq \int_I \omega^{-1}\{[q_1^2 + q_3^2]|y_1|^2 + 2\,|q_1q_4 + q_2q_3|\,|y_1y_2| \\ &\quad +[q_2^2 + q_4^2]|y_2|^2\}, \text{ and via Lemma 4.3} \end{aligned}$$

$$(4.86) \qquad \|Ly\|^2 \geq \int_I \{(y')^*\Delta_1 y' + y^*\Delta_2 y\}.$$

We make use of the inequality $2ab \leq a^2 + b^2$ in inequality (4.85) to obtain

$$(4.87) \qquad \begin{aligned} \|By\|^2 &\leq \int_I \omega^{-1}\{[q_1^2 + q_3^2]|y_1|^2 + |q_1q_4 + q_2q_3|\,[|y_1|^2 + |y_2|^2] \\ &\quad +[q_2^2 + q_4^2]|y_2|^2\}. \end{aligned}$$

Claim 4.4.3 *There exist constants $k_3, k_4 > 0$ such that*

$$(4.88) \qquad \begin{aligned} &\int_I \omega^{-1}|q_1q_4 + q_2q_3|\,|y_1|^2 \\ &\quad \leq k_3 \left\{\int_I [\omega + \delta\omega^{-1}p^2]\,|y_1|^2 + \varepsilon^2 \int_I \omega^{-1}(1 - \beta_1^2)|y_1'|^2\right\} \end{aligned}$$

and

$$(4.89) \qquad \begin{aligned} &\int_I \omega^{-1}|q_1q_4 + q_2q_3|\,|y_2|^2 \\ &\quad \leq k_4 \left\{\int_I [\omega + \delta\omega^{-1}p^2]\,|y_2|^2 + \varepsilon^2 \int_I \omega^{-1}(1 - \beta_2^2)|y_2'|^2\right\} \end{aligned}$$

Proof of Claim 4.4.3:

By following the argument in Claim 2.1.1, we know that

$$(4.90) \qquad \begin{aligned} f(t)\int_t^{t+\varepsilon f(t)} |q_1q_4 + q_2q_3| &\leq \left(f(t)\int_t^{t+\varepsilon f(t)} q_1^2\right)^{\frac{1}{2}} \left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2 + q_2^2]\right)^{\frac{1}{2}} \\ &\quad + \left(f(t)\int_t^{t+\varepsilon f(t)} q_2^2\right)^{\frac{1}{2}} \left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2 + q_3^2]\right)^{\frac{1}{2}} \end{aligned}$$

Now, we show that the hypotheses of Theorem 1.7 hold for some $\varepsilon \in (0, \frac{1}{2N_0})$ with $N = \omega^{-1}|q_1q_4 + q_2q_3|$, $W = \omega + \delta\omega^{-1}p^2$, and $P = \omega^{-1}(1-\beta_1^2)$. Via inequality (4.90) and Claims 4.4.1 and 4.4.2 we have for some constants $C_5, C_6 > 0$

$$\begin{aligned}
(4.91)\quad S_1 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_t^{t+\varepsilon f(t)} \frac{\omega}{(1-\beta_1^2)}\right]\left[\int_t^{t+\varepsilon f(t)} \omega^{-1}|q_1q_4+q_2q_3|\right]\right\}\\
&\le \frac{C_5}{\varepsilon(1-\beta_1^2)}\sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)}|q_1q_4+q_2q_3|\right\}\\
&\le \frac{C_5}{\varepsilon}\sup_{t\in I}\left\{\left(f(t)\int_t^{t+\varepsilon f(t)} q_1^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2+q_2^2]\right)^{\frac{1}{2}}\right.\\
&\quad\left.+\left(f(t)\int_t^{t+\varepsilon f(t)} q_2^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2]\right)^{\frac{1}{2}}\right\} \text{ and}
\end{aligned}$$

$$\begin{aligned}
(4.92)\quad S_2 &= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} \frac{1}{\omega+\delta\omega^{-1}p^2}\right]\left[\int_t^{t+\varepsilon f(t)} \omega^{-1}|q_1q_4+q_2q_3|\right]\right\}\\
&= \sup_{t\in I}\left\{\frac{1}{\varepsilon^2 f^2(t)}\left[\int_t^{t+\varepsilon f(t)} \omega f^2\right]\left[\int_t^{t+\varepsilon f(t)} \omega^{-1}|q_1q_4+q_2q_3|\right]\right\}\\
&\le \frac{C_6}{\varepsilon}\sup_{t\in I}\left\{f(t)\int_t^{t+\varepsilon f(t)}|q_1q_4+q_2q_3|\right\}\\
&\le \frac{C_6}{\varepsilon}\sup_{t\in I}\left\{\left(f(t)\int_t^{t+\varepsilon f(t)} q_1^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2+q_2^2]\right)^{\frac{1}{2}}\right.\\
&\quad\left.+\left(f(t)\int_t^{t+\varepsilon f(t)} q_2^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2]\right)^{\frac{1}{2}}\right\}
\end{aligned}$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$. Inequalities (4.68), (4.91), and (4.92) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2N_0})$. An application of Theorem 1.7 gives us inequality (4.88). Replacing β_1 with β_2 in the above argument and applying Theorem 1.7, we obtain (4.89). #

Therefore, via inequalities (4.75), (4.82), (4.88), and (4.89) we obtain from inequality (4.87) that

$$\begin{aligned}
(4.93)\quad ||By||^2 &\le (k_1+k_3)\left\{\int_I[\omega+\delta\omega^{-1}p^2]\,|y_1|^2+\varepsilon^2\int_I\omega^{-1}(1-\beta_1^2)|y_1'|^2\right\}\\
&\quad+(k_2+k_4)\left\{\int_I[\omega+\delta\omega^{-1}p^2]\,|y_2|^2+\varepsilon^2\int_I\omega^{-1}(1-\beta_2^2)|y_2'|^2\right\}.
\end{aligned}$$

Hence, via Lemma 4.3 there exists a constant $C > 0$ such that for each $y \in D(L_0')$

$$||By||^2 \le C\ (||y||^2 + ||Ly||^2).$$

Since y is arbitrary and $D(L_0') \subseteq D(B_0')$, we know that B_0' is L_0'-bounded. Via Theorem 1.2 we have that B_0 is L_0-bounded.

<u>Necessity</u>

Fix $\varepsilon \in (0, \frac{1}{2N_0})$. For each $r \geq a$ define Φ_r and Ψ_r to be the vector-valued functions, with compact support, given by (2.28). Then via Claim 4.4.1 we have for each $r \geq a$

$$\begin{aligned} (4.94) \qquad \|\Phi_r\|^2 &= \int_I \omega(t)\phi_r^2(t)\,dt \\ &= \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \omega(t)\phi^2\left(\frac{t-r}{\varepsilon f(r)}\right) dt \\ &\leq C_1\omega(r)\int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \phi^2\left(\frac{t-r}{\varepsilon f(r)}\right) dt. \end{aligned}$$

By a change of variables (letting $u = \dfrac{t-r}{\varepsilon f(r)}$ in (4.94)),

$$\|\Phi_r\|^2 \leq \varepsilon C_1\omega(r)f(r)\int_{-2}^{2} \phi^2(u)\,du\,.$$

Since $\omega(r)f(r) \leq 1$, we conclude from the above inequality that for $r \geq a$

$$(4.95) \qquad \|\Phi_r\|^2 \leq \varepsilon C_1 \int_{-2}^{2} \phi^2(u)\,du\,.$$

Since ϕ is continuous, inequality (4.95) implies there exists a constant $C_2 > 0$ such that for $r \geq a$

$$(4.96) \qquad \|\Phi_r\|^2 \leq C_2\,.$$

Similarly, for $r \geq a$

$$(4.97) \qquad \|\Psi_r\|^2 \leq C_2\,.$$

Via integration by parts and the bounds on $|(\omega^{-1}p)'|$ and $|p_1|$, there exists a constant $C_3 > 0$ such that for each $r \geq a$

$$\begin{aligned} (4.98)\ \|L\Phi_r\|^2 & \\ &= \int_I \left\{\omega^{-1}(t)\left[\frac{d}{dt}\phi_r(t)\right]^2 - (\omega^{-1}p)'(t)\,\phi_r^2(t) + \omega^{-1}(t)[p_1^2(t)+p^2(t)]\phi_r^2(t)\right\} dt \\ &\leq \int_I \left\{\omega^{-1}(t)\left[\frac{d}{dt}\phi_r(t)\right]^2 + (2\hat{\varepsilon}+1)\omega^{-1}(t)p^2(t)\phi_r^2(t)\right\} dt \end{aligned}$$

Now, via Claims 4.4.1 and 4.4.2 there exists a constant $C_4 > 0$ such that for each $r \geq a$

$$\begin{aligned}\int_I \omega^{-1}(t)\left[\frac{d}{dt}\phi_r(t)\right]^2 dt &= \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \omega^{-1}(t)\left[\frac{d}{dt}\phi_r(t)\right]^2 dt \\ &\leq C_4\omega^{-1}(r)\int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)}\left[\frac{d}{dt}\phi\left(\frac{t-r}{\varepsilon f(r)}\right)\right]^2 dt\,.\end{aligned}$$

By a change of variables (letting $u = \dfrac{t-r}{\varepsilon f(r)}$ in the above inequality),

$$\int_I \omega^{-1}(t)\left[\frac{d}{dt}\phi_r(t)\right]^2 dt \leq \frac{C_4}{\varepsilon}\,\omega^{-1}(r)f^{-1}(r)\int_{-2}^{2}[\phi'(u)]^2\,du\,. \tag{4.99}$$

Moreover, via the inequality $\delta p^2 \leq f^{-2}$ and Claims 4.4.1 and 4.4.2, there exists a constant $C_5 > 0$ such that for each $r \geq a$

$$\begin{aligned}\int_I \omega^{-1}(t)p^2(t)\phi_r^2(t)\,dt &= \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \omega^{-1}(t)p^2(t)\phi_r^2(t)\,dt \\ &\leq \frac{C_5}{\delta}\,\omega^{-1}(r)f^{-2}(r)\int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)}\phi^2\left(\frac{t-r}{\varepsilon f(r)}\right) dt\,.\end{aligned}$$

By a change of variables (letting $u = \dfrac{t-r}{\varepsilon f(r)}$ in the above inequality),

$$\int_I \omega^{-1}(t)p^2(t)\phi_r^2(t)\,dt \leq \frac{\varepsilon C_5}{\delta}\,\omega^{-1}(r)f^{-1}(r)\int_{-2}^{2}\phi^2(u)\,du\,. \tag{4.100}$$

Substituting (4.99) and (4.100) into inequality (4.98) gives

$$\begin{aligned}\|L\Phi_r\|^2 \;\leq\; & \frac{C_4}{\varepsilon}\,\omega^{-1}(r)f^{-1}(r)\int_{-2}^{2}[\phi'(u)]^2\,du \\ & +(2\hat{\varepsilon}+1)\frac{\varepsilon C_5}{\delta}\,\omega^{-1}(r)f^{-1}(r)\int_{-2}^{2}\phi^2(u)\,du\,.\end{aligned} \tag{4.101}$$

Since ϕ and ϕ' are continuous, inequality (4.101) implies there exists a constant $C_6 > 0$ such that for $r \geq a$

$$\|L\Phi_r\|^2 \leq C_6\,\omega^{-1}(r)f^{-1}(r)\,. \tag{4.102}$$

Similarly, for $r \geq a$

$$\|L\Psi_r\|^2 \leq C_6\,\omega^{-1}(r)f^{-1}(r)\,. \tag{4.103}$$

Since $\phi_r \equiv 1$ on $[r, r+\varepsilon f(r)]$ and $\text{supp}(\phi_r)=[r-2\varepsilon f(r), r+2\varepsilon f(r)]$,

$$\begin{aligned}(4.104)\quad \int_r^{r+\varepsilon f(r)} \omega^{-1}(t)[q_1^2(t)+q_3^2(t)]\,dt &\leq \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \omega^{-1}(t)[q_1^2(t)+q_3^2(t)]\phi_r^2(t)\,dt \\ &= \|B\Phi_r\|^2 .\end{aligned}$$

Via Claim 4.4.2 and inequality (4.104) there exists constant $C_7 > 0$ such that for each $r \geq a$

$$\begin{aligned} f(r)\int_r^{r+\varepsilon f(r)} [q_1^2(t)+q_3^2(t)]\,dt &\leq C_7\,\omega(r)f(r)\int_r^{r+\varepsilon f(r)} \omega^{-1}(t)[q_1^2(t)+q_3^2(t)]\,dt \\ &\leq C_7\,\omega(r)f(r)\,\|B\Phi_r\|^2 .\end{aligned}$$

Thus, via the L_0-boundedness of B_0, (4.96), (4.102), and the inequality $\omega f \leq 1$, there exists a constant $C_8 > 0$ such that for each $r \geq a$

$$\begin{aligned}(4.105)\quad f(r)\int_r^{r+\varepsilon f(r)} [q_1^2(t)+q_3^2(t)]\,dt &\leq C_8\,\omega(r)f(r)[C_2 + C_6\omega^{-1}(r)f^{-1}(r)] \\ &\leq C_8\,[C_2\omega(r)f(r) + C_6] \\ &\leq C_8\,[C_2 + C_6] .\end{aligned}$$

Since the right-hand side of inequality (4.105) is independent of r, we conclude that inequality (4.68) holds for $i = 1, 3$.

Moreover,

$$\begin{aligned}(4.106)\quad \int_r^{r+\varepsilon f(r)} \omega^{-1}(t)[q_2^2(t)+q_4^2(t)]\,dt &\leq \int_{r-2\varepsilon f(r)}^{r+2\varepsilon f(r)} \omega^{-1}(t)[q_2^2(t)+q_4^2(t)]\phi_r^2(t)\,dt \\ &= \|B\Psi_r\|^2 .\end{aligned}$$

Via the L_0-boundedness of B_0 and inequalities (4.97), (4.103), and (4.106),

$$(4.107)\qquad \int_r^{r+\varepsilon f(r)} \omega^{-1}(t)[q_2^2(t)+q_4^2(t)]\,dt \leq C_7\,[C_2 + C_6\omega^{-1}(r)f^{-1}(r)] .$$

Using an argument similar to the one above, we conclude from (4.107) that inequality (4.68) holds for $i = 2, 4$. (Note that the proof of necessity shows that (4.68) holds for every $\varepsilon \in (0, \frac{1}{2N_0})$.)

Therefore, we have shown that B_0 is L_0-bounded if and only if inequalities (4.68) hold. We apply Theorem 1.6 to conclude that B_1 is L_1-bounded if and only if inequalities (4.68) hold.

Now, we show that B_0 is L_0-compact if and only if equations (4.69) hold.

(ii) Sufficiency

By the previous argument B_0 is L_0-bounded. Thus, $D(L_0) \subseteq D(B_0)$. For every positive integer $N > a$, define B_N on $D(L_0)$ by

$$B_N y = \begin{cases} By & \text{on } [a, N] \\ 0 & \text{on } (N, \infty) \end{cases}$$

Notice that each B_N is L_0-bounded with the same norm as B_0 since $||B_N y|| \le ||By||$. In order to simplify the proof, we break the argument into two Claims.

Claim 4.4.4 *$B_N \to B_0$ in the space of bounded operators on $D(L_0)$ with the L_0-norm.*

Proof of Claim 4.4.4:

By definition L_0 is closed. Therefore, $D(L_0)$ is complete under the L_0-norm.

Let $y \in D(L_0)$. Since L_0 is the closure of L_0', for each positive integer $n \ge 1$ there exists a $y_n \in D(L_0')$ such that

$$(4.108) \qquad ||y - y_n|| + ||Ly - Ly_n|| < \frac{1}{n},$$

where $y_n = \begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix}$ for each n.

Using the inequality $2ab \le a^2 + b^2$, we have for each $y_n \in D(L_0')$

$$\begin{aligned} (4.109) \quad ||By_n - B_N y_n||^2 &= \int_N^\infty \omega^{-1} \{[q_1^2 + q_3^2]|y_{n,1}|^2 + [q_1q_4 + q_2q_3]y_{n,1}\overline{y_{n,2}} \\ &\quad + [q_1q_4 + q_2q_3]\overline{y_{n,1}}y_{n,2} + [q_2^2 + q_4^2]|y_2|^2\} \\ &\le \int_I \omega^{-1} \{[q_1^2 + q_3^2]|y_{n,1}|^2 + 2\,|q_1q_4 + q_2q_3|\,|y_{n,1}y_{n,2}| \\ &\quad + [q_2^2 + q_4^2]|y_{n,2}|^2\} \end{aligned}$$

$$\leq \int_I \omega^{-1}\{[q_1^2+q_3^2]|y_{n,1}|^2 + |q_1q_4+q_2q_3|\,[|y_{n,1}|^2+|y_{n,2}|^2] \\ +[q_2^2+q_4^2]|y_{n,2}|^2\}\ .$$

We apply the sufficiency argument in part (i) with $I_N = [N,\infty)$ to the last inequality in (4.109). Via Theorem 1.7 and inequalities (4.73), (4.74), (4.80), (4.81), (4.91), and (4.92), there exist constants K_1, K_2, K_3, and $K_4 > 0$ such that for each n

$$\begin{aligned}
(4.110)\ \|By_n - B_Ny_n\|^2 \ \leq\ & \frac{K_1}{\varepsilon}\sup_{t\in I_N}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2]\right\} \\
&\cdot\left\{\int_{I_N}(\omega+\delta\omega^{-1}p^2)|y_{n,1}|^2 \ +\varepsilon^2\int_{I_N}\omega^{-1}(1-\beta_1^2)|y'_{n,1}|^2\right\} \\
+\ & \frac{K_2}{\varepsilon}\sup_{t\in I_N}\left\{\left(f(t)\int_t^{t+\varepsilon f(t)}q_1^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2+q_2^2]\right)^{\frac{1}{2}}\right. \\
&\left.+\left(f(t)\int_t^{t+\varepsilon f(t)}q_2^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2]\right)^{\frac{1}{2}}\right\} \\
&\cdot\left\{\int_{I_N}(\omega+\delta\omega^{-1}p^2)|y_{n,1}|^2 \ +\varepsilon^2\int_{I_N}\omega^{-1}(1-\beta_1^2)|y'_{n,1}|^2\right\} \\
+\ & \frac{K_3}{\varepsilon}\sup_{t\in I_N}\left\{\left(f(t)\int_t^{t+\varepsilon f(t)}q_1^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2+q_2^2]\right)^{\frac{1}{2}}\right. \\
&\left.+\left(f(t)\int_t^{t+\varepsilon f(t)}q_2^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2]\right)^{\frac{1}{2}}\right\} \\
&\cdot\left\{\int_{I_N}(\omega+\delta\omega^{-1}p^2)|y_{n,2}|^2 \ +\varepsilon^2\int_{I_N}\omega^{-1}(1-\beta_2^2)|y'_{n,2}|^2\right\} \\
+\ & \frac{K_4}{\varepsilon}\sup_{t\in I_N}\left\{f(t)\int_t^{t+\varepsilon f(t)}[q_2^2+q_4^2]\right\} \\
&\cdot\left\{\int_{I_N}(\omega+\delta\omega^{-1}p^2)|y_{n,2}|^2 \ +\varepsilon^2\int_{I_N}\omega^{-1}(1-\beta_2^2)|y'_{n,2}|^2\right\}.
\end{aligned}$$

Adding nonnegative terms to the right-hand side of inequality (4.110) gives for each n

$$\begin{aligned}
(4.111)\quad \|By_n - B_Ny_n\|^2 \ \leq\ & K_5\sup_{t\in I_N}F(t)\left\{\int_{I_N}(\omega+\delta\omega^{-1}p^2)[|y_{n,1}|^2+|y_{n,2}|^2]\right. \\
&\left.+\varepsilon^2\int_{I_N}\omega^{-1}[(1-\beta_1^2)|y'_{n,1}|^2+(1-\beta_2^2)|y'_{n,2}|^2]\right\}
\end{aligned}$$

for some constant $K_5 > 0$ where

$$\begin{aligned} F(t) &= f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2] \\ &\quad + \left(f(t)\int_t^{t+\varepsilon f(t)} q_1^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_4^2+q_2^2]\right)^{\frac{1}{2}} \\ &\quad + \left(f(t)\int_t^{t+\varepsilon f(t)} q_2^2\right)^{\frac{1}{2}}\left(2f(t)\int_t^{t+\varepsilon f(t)}[q_1^2+q_3^2]\right)^{\frac{1}{2}} \\ &\quad + f(t)\int_t^{t+\varepsilon f(t)}[q_2^2+q_4^2]. \end{aligned}$$

Via Lemma 4.3 and inequality (4.111) there exists a constant $K_6 > 0$ such that for each n

$$\|By_n - B_N y_n\|^2 \le K_6 \sup_{t\in I_N} F(t)\ (\|y_n\| + \|Ly_n\|)^2. \tag{4.112}$$

Therefore, via the the triangle inequality, the L_0-boundedness of B_0, and inequalities (4.108) and (4.112), we have for each n

$$\begin{aligned} \|By - B_N y\| &\le \|By - By_n\| + \|By_n - B_N y_n\| + \|B_N y_n - B_N y\| \\ &\le 2K_7(\|y-y_n\| + \|Ly - Ly_n\|) + \|By_n - B_N y_n\| \\ &< \frac{2K_7}{n} + K_6^{\frac{1}{2}}\left(\sup_{t\in I_N} F(t)\right)^{\frac{1}{2}} \|y_n\|_{L_0'} \quad \text{for some constant } K_7 > 0. \end{aligned}$$

By applying the triangle inequality and inequality (4.108) again, we obtain for each n

$$\begin{aligned} \|By - B_N y\| &\le \frac{2K_7}{n} + K_6^{\frac{1}{2}}\left(\sup_{t\in I_N} F(t)\right)^{\frac{1}{2}} (\|y_n - y\|_{L_0} + \|y\|_{L_0}) \\ &< \frac{2K_7}{n} + K_6^{\frac{1}{2}}\left(\sup_{t\in I_N} F(t)\right)^{\frac{1}{2}} \left(\frac{1}{n} + \|y\|_{L_0}\right). \end{aligned}$$

We let $n \to \infty$ to obtain for each N

$$\|By - B_N y\| \le K_6^{\frac{1}{2}}\left(\sup_{t\in I_N} F(t)\right)^{\frac{1}{2}} \|y\|_{L_0}.$$

Hence, the above inequality implies that for $y \neq 0$

$$\frac{\|By - B_N y\|}{\|y\|_{L_0}} \le K_6^{\frac{1}{2}}\left(\sup_{t\in I_N} F(t)\right)^{\frac{1}{2}}. \tag{4.113}$$

Since $I_N = [N, \infty)$, equations (4.69) imply that $\sup_{t \in I_N} F(t) \to 0$ as $N \to \infty$ so that, via inequality (4.113), we have

$$||B - B_N|| \to 0 \text{ as } N \to \infty. \quad \#$$

Claim 4.4.5 *Each B_N is L_0-compact.*

Proof of Claim 4.4.5:

Let $\{y_n\}_{n=1}^{\infty} \subset D(L_0)$ be an L_0-bounded sequence where

$$y_n = \begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix} \text{ for each n.}$$

Since L_0 is the closure of L_0', for each n there exists a $\hat{y}_n \in D(L_0')$ such that

$$||y_n - \hat{y}_n|| + ||Ly_n - L\hat{y}_n|| < \frac{1}{n}. \tag{4.114}$$

Notice that $\{\hat{y}_n\}_{n=1}^{\infty}$ is an L_0-bounded sequence since there exists a constant $C > 0$ such that for each n

$$\begin{aligned} ||\hat{y}_n|| + ||L\hat{y}_n|| &\leq (||\hat{y}_n - y_n|| + ||L\hat{y}_n - Ly_n||) + (||y_n|| + ||Ly_n||) \\ &< \frac{1}{n} + C \\ &\leq 1 + C \end{aligned}$$

via the triangle inequality, inequality (4.114), and the L_0-boundedness of the sequence $\{y_n\}_{n=1}^{\infty}$. We need to show that for each N the sequence $\{B_N y_n\}_{n=1}^{\infty}$ has a convergent subsequence. First, we make use of the Arzela-Ascoli Theorem to show that the sequence $\{B_N \hat{y}_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Let us consider the sequence of the first component functions $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$. We show that this sequence is uniformly bounded on $[a, N]$. Partition $[a, N]$ by $J_i = [t_i, t_{i+1}]$, $1 \leq i \leq k$ with $t_1 = a$, $t_{i+1} = t_i + \tilde{\varepsilon} f(t_i)$ and $\tilde{\varepsilon} \in (0, \varepsilon)$ chosen so that $N = t_{k+1} = t_k + \tilde{\varepsilon} f(t_k)$. Via equation (2.12) of [3, page 571] in the proof of

Theorem 1.7, we know there exists a constant $C_1 > 0$ such that for $t \in J_i$

$$
\begin{aligned}
(4.115)\quad |\hat{y}_{n,1}(t)|^2 &\le C_1 \left\{ \frac{1}{\tilde{\varepsilon}^2 f^2(t_i)} \left[\int_{J_i} \frac{d\tau}{\omega(\tau) + \delta\omega^{-1}(\tau)p^2(\tau)} \right] \right. \\
&\quad \cdot \left[\int_{J_i} [\omega(\tau) + \delta\omega^{-1}(\tau)p^2(\tau)] |\hat{y}_{n,1}(\tau)|^2 \, d\tau \right] \\
&\quad \left. + \left[\int_{J_i} \frac{\omega(\tau)}{(1-\beta_1^2)} \, d\tau \right] \left[\int_{J_i} (1-\beta_1^2)\omega^{-1}(\tau) |\hat{y}'_{n,1}(\tau)|^2 \, d\tau \right] \right\} . \\
&= C_1 \left\{ \frac{1}{\tilde{\varepsilon}^2 f^2(t_i)} \left[\int_{J_i} \omega(\tau) f^2(\tau) \, d\tau \right] \right. \\
&\quad \cdot \left[\int_{J_i} [\omega(\tau) + \delta\omega^{-1}(\tau)p^2(\tau)] |\hat{y}_{n,1}(\tau)|^2 \, d\tau \right] \\
&\quad \left. + \left[\int_{J_i} \frac{\omega(\tau)}{(1-\beta_1^2)} \, d\tau \right] \left[\int_{J_i} (1-\beta_1^2)\omega^{-1}(\tau) |\hat{y}'_{n,1}(\tau)|^2 \, d\tau \right] \right\} .
\end{aligned}
$$

By applying Claims 4.4.1 and 4.4.2, we obtain

$$
\begin{aligned}
(4.116)\quad |\hat{y}_{n,1}(t)|^2 &\le C_2 \left\{ \frac{\omega(t_i) f(t_i)}{\tilde{\varepsilon}} \int_{J_i} [\omega(\tau) + \delta\omega^{-1}(\tau)p^2(\tau)] |\hat{y}_{n,1}(\tau)|^2 \, d\tau \right. \\
&\quad \left. + \frac{\tilde{\varepsilon}\omega(t_i) f(t_i)}{(1-\beta_1^2)} \int_{J_i} (1-\beta_1^2)\omega^{-1}(\tau) |\hat{y}'_{n,1}(\tau)|^2 \, d\tau \right\}
\end{aligned}
$$

for some constant $C_2 > 0$ and for $t \in J_i$. Since $\omega f \le 1$, there exists a constant $C_3 > 0$ such that for $t \in [a, N]$

$$
\begin{aligned}
(4.117)\quad |\hat{y}_{n,1}(t)|^2 &\le C_3 \left\{ \int_a^N [\omega(\tau) + \delta\omega^{-1}(\tau)p^2(\tau)] |\hat{y}_{n,1}(\tau)|^2 \, d\tau \right. \\
&\quad \left. + \int_a^N (1-\beta_1^2)\omega^{-1}(\tau) |\hat{y}'_{n,1}(\tau)|^2 \, d\tau \right\}
\end{aligned}
$$

so that via Lemma 4.3 there exists a constant $C_4 > 0$ such that for $t \in [a, N]$

$$
\begin{aligned}
(4.118)\quad |\hat{y}_{n,1}(t)|^2 &\le C_4 \left(\|\hat{y}_n\|^2 + \|L\hat{y}_n\|^2 \right) \\
&\le C_4 \left(\|\hat{y}_n\| + \|L\hat{y}_n\| \right)^2 .
\end{aligned}
$$

Since $\{\hat{y}_n\}_{n=1}^{\infty}$ is an L_0-bounded sequence, inequality (4.118) implies that the sequence $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$ is uniformly bounded on $[a, N]$.

Now, we show that $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$ is equicontinuous on $[a, N]$. For $s, t \in [a, N]$ we have via the Cauchy-Schwarz inequality

$$
(4.119)\quad |\hat{y}_{n,1}(s) - \hat{y}_{n,1}(t)| = \left| \int_t^s \hat{y}'_{n,1}(\tau) \, d\tau \right|
$$

$$
\begin{aligned}
&\leq \left|\int_t^s |\hat{y}'_{n,1}(\tau)|\,d\tau\right| \\
&= \left|\int_t^s [(1-\beta_1^2)^{-\frac{1}{2}}\omega^{\frac{1}{2}}(\tau)][(1-\beta_1^2)^{\frac{1}{2}}\omega^{-\frac{1}{2}}(\tau)]|\hat{y}'_{n,1}(\tau)|\,d\tau\right| \\
&\leq \left|\int_t^s \frac{\omega(\tau)}{(1-\beta_1^2)}\,d\tau\right|^{\frac{1}{2}} \left|\int_t^s (1-\beta_1^2)\omega^{-1}(\tau)|\hat{y}'_{n,1}(\tau)|^2\,d\tau\right|^{\frac{1}{2}}.
\end{aligned}
$$

Since $\omega \in AC_{loc}(I)$, ω is bounded above and below on $[a,N]$. This fact along with (4.119) and Lemma 4.3 implies that there exists a constant $C > 0$ such that for $s,t \in [a,N]$

$$
\begin{aligned}
(4.120) \qquad |\hat{y}_{n,1}(s) - \hat{y}_{n,1}(t)| &\leq C\,|s-t|^{\frac{1}{2}} \left|\int_t^s (1-\beta_1^2)\omega^{-1}(\tau)|\hat{y}'_{n,1}(\tau)|^2\,d\tau\right|^{\frac{1}{2}} \\
&\leq C\,|s-t|^{\frac{1}{2}}\|L\hat{y}_n\| \\
&\leq C\,|s-t|^{\frac{1}{2}}\|\hat{y}_n\|_{L_0}.
\end{aligned}
$$

Since $\{\hat{y}_n\}_{n=1}^{\infty}$ is a L_0-bounded sequence, inequality (4.120) implies that the sequence $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$ is equicontinuous on $[a,N]$.

Therefore, via the Arzela-Ascoli Theorem we conclude that there exists a subsequence $\{\hat{y}_{n_k,1}\}_{k=1}^{\infty}$ of $\{\hat{y}_{n,1}\}_{n=1}^{\infty}$ which converges uniformly on $[a,N]$. By repeating the above arguments [inequalities (4.115)-(4.120)] on the subsequence $\{\hat{y}_{n_k,2}\}_{k=1}^{\infty}$ of $\{\hat{y}_{n,2}\}_{n=1}^{\infty}$, we conclude that there exists a subsequence of $\{\hat{y}_{n_k,2}\}_{k=1}^{\infty}$ which converges uniformly on $[a,N]$. Hence, there exists a subsequence of $\{\hat{y}_n\}_{n=1}^{\infty}$ which converges uniformly on $[a,N]$. WLOG, we assume the sequence $\{\hat{y}_n\}_{n=1}^{\infty}$ converges uniformly on $[a,N]$.

Now, suppressing the independent variable and using the inequality $2ab \leq a^2 + b^2$ we have for each N

$$
\begin{aligned}
(4.121) \quad \|B_N\hat{y}_n - B_N\hat{y}_m\|^2 &= \int_a^N \omega^{-1}\{[q_1^2+q_3^2]|\hat{y}_{n,1}-\hat{y}_{m,1}|^2 \\
&\quad +[q_1q_4+q_2q_3](\hat{y}_{n,1}-\hat{y}_{m,1})\overline{(\hat{y}_{n,2}-\hat{y}_{m,2})} \\
&\quad +[q_1q_4+q_2q_3]\overline{(\hat{y}_{n,1}-\hat{y}_{m,1})}(\hat{y}_{n,2}-\hat{y}_{m,2}) \\
&\quad +[q_2^2+q_4^2]|\hat{y}_{n,2}-\hat{y}_{m,2}|^2\} \\
&\leq \int_a^N \omega^{-1}\{[q_1^2+q_3^2]|\hat{y}_{n,1}-\hat{y}_{m,1}|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\,|q_1q_4+q_2q_3|\,|(\hat{y}_{n,1}-\hat{y}_{m,1})(\hat{y}_{n,2}-\hat{y}_{m,2})| \\
&+[q_2^2+q_4^2]|\hat{y}_{n,2}-\hat{y}_{m,2}|^2\} \\
\le\ & \int_a^N \omega^{-1}\{[q_1^2+q_3^2]|\hat{y}_{n,1}-\hat{y}_{m,1}|^2 \\
&+|q_1q_4+q_2q_3|\,(|\hat{y}_{n,1}-\hat{y}_{m,1}|^2+|\hat{y}_{n,2}-\hat{y}_{m,2}|^2) \\
&+[q_2^2+q_4^2]|\hat{y}_{n,2}-\hat{y}_{m,2}|^2\}\,.
\end{aligned}
$$

By adding nonnegative terms to the right-hand side of inequality (4.121), we obtain

$$
\begin{aligned}
\|B_N\hat{y}_n-B_N\hat{y}_m\|^2 \ \le\ & \int_a^N \omega^{-1}\{[q_1^2+q_3^2]\ \|\hat{y}_n-\hat{y}_m\|_2^2+|q_1q_4+q_2q_3|\ \|\hat{y}_n-\hat{y}_m\|_2^2 \\
&+[q_2^2+q_4^2]\ \|\hat{y}_n-\hat{y}_m\|_2^2\} \\
\le\ & \sup_{a\le t\le N}\ \|\hat{y}_n(t)-\hat{y}_m(t)\|_2^2\left\{\int_a^N \omega^{-1}[q_1^2+q_3^2]\right. \\
&\left.+\int_a^N \omega^{-1}|q_1q_4+q_2q_3|+\int_a^N \omega^{-1}[q_2^2+q_4^2]\}\right\}
\end{aligned}
$$

where

$$
\|\hat{y}_n(t)-\hat{y}_m(t)\|_2^2=|\hat{y}_{n,1}(t)-\hat{y}_{m,1}(t)|^2+|\hat{y}_{n,2}(t)-\hat{y}_{m,2}(t)|^2\,.
$$

By applying the argument used in inequality (4.90) to the inequality above, we see that the integrals on the right-hand side are finite on $[a, N]$. Hence, there exists a constant $C > 0$ such that

$$
\|B_N\hat{y}_n-B_N\hat{y}_m\|^2\le C\sup_{a\le t\le N}\ \|\hat{y}_n(t)-\hat{y}_m(t)\|_2^2\,.
$$

Since $\{\hat{y}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the uniform norm, the above inequality implies that the sequence $\{B_N\hat{y}_n\}_{n=1}^{\infty}$ is Cauchy in $\mathcal{L}_w^2(I)$ for each N. Therefore, $\{B_N\hat{y}_n\}_{n=1}^{\infty}$ converges for each N as $n\to\infty$ since $\mathcal{L}_w^2(I)$ is complete. Let $\xi=\lim_{n\to\infty} B_N\hat{y}_n$.

Via the triangle inequality, the L_0-boundedness of B_0, and inequality (4.114), there exists a constant $C > 0$ such that for each n and for each N

$$
\begin{aligned}
\|B_Ny_n-\xi\| \ \le\ & \|B_Ny_n-B_N\hat{y}_n\|+\|B_N\hat{y}_n-\xi\| \\
\le\ & C\,(\|y_n-\hat{y}_n\|+\|Ly_n-L\hat{y}_n\|)+\|B_N\hat{y}_n-\xi\| \\
<\ & \frac{C}{n}+\|B_N\hat{y}_n-\xi\|\,.
\end{aligned}
$$

Since the right-hand side of the above inequality converges for each N as $n \to \infty$, we conclude that $\{B_N y_n\}_{n=1}^{\infty}$ converges for each N as $n \to \infty$. Hence, each B_N is L_0-compact. #

Since B_0 is the uniform limit of L_0-compact operators, B_0 is L_0-compact.

Necessity

We use contradiction arguments to show that each equation in (4.69) must hold.

Suppose that for some $\varepsilon \in (0, \frac{1}{2N_0})$ there exists a $\rho > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers such that $r_n \to \infty$ as $n \to \infty$ and for each n

$$(4.122) \qquad f(r_n) \int_{r_n}^{r_n+\varepsilon f(r_n)} q_i^2(t)\, dt \geq \rho \quad \text{for some } i \in \{1, 2, 3, 4\}.$$

Suppose $i = 1$. Let $\{\Phi_r\}_{r \geq a}$ be defined as in (2.28). Then via inequalities (4.96) and (4.102) there exist constants $C_1, C_3 > 0$ such that for each n

$$(4.123) \qquad \begin{aligned} \|\Phi_{r_n}\|_{L_0}^2 &= (\|\Phi_{r_n}\| + \|L\Phi_{r_n}\|)^2 \\ &\leq 2\left(\|\Phi_{r_n}\|^2 + \|L\Phi_{r_n}\|^2\right) \\ &\leq 2C_2 + 2C_6\, \omega^{-1}(r_n) f^{-1}(r_n). \end{aligned}$$

For each $r \geq a$ define

$$(4.124) \qquad \tilde{\Phi}_r(t) = \omega^{\frac{1}{2}}(r) f^{\frac{1}{2}}(r) \Phi_r(t) \quad \text{for } t \geq a.$$

Via inequality (4.124) we have for each n

$$(4.125) \qquad \begin{aligned} \|\tilde{\Phi}_{r_n}\|_{L_0}^2 &= \omega(r_n) f(r_n)\, \|\Phi_{r_n}\|_{L_0}^2 \\ &\leq 2C_2\, \omega(r_n) f(r_n) + 2C_6\,. \end{aligned}$$

Since $\omega f \leq 1$, inequality (4.125) implies that $\{\tilde{\Phi}_{r_n}\}_{n=1}^{\infty}$ is an L_0-bounded sequence. Via the L_0-compactness of B_0, $\{B\tilde{\Phi}_{r_n}\}_{n=1}^{\infty}$ has a convergent subsequence. WLOG, we assume $\{B\tilde{\Phi}_{r_n}\}_{n=1}^{\infty}$ converges, say to some y_0. Now, since $\phi_{r_n} \equiv 1$ on $[r_n, r_n + \varepsilon f(r_n)]$ and supp(ϕ_{r_n})=$[r_n - 2\varepsilon f(r_n), r_n + 2\varepsilon f(r_n)]$, we have for each n

$$\rho \;\leq\; f(r_n) \int_{r_n}^{r_n+\varepsilon f(r_n)} q_1^2(t)\, dt$$

$$
\begin{aligned}
&\le f(r_n)\int_{r_n}^{r_n+\varepsilon f(r_n)}[q_1^2(t)+q_3^2(t)]\,dt \\
&\le f(r_n)\int_{r_n-2\varepsilon f(r_n)}^{r_n+2\varepsilon f(r_n)}[q_1^2(t)+q_3^2(t)]\phi_{r_n}^2(t)\,dt\,.
\end{aligned}
$$

Via Claim 4.4.2 and the inequality $\omega f \le 1$, there exists a constant $C > 0$ such that for each n

$$
\begin{aligned}
(4.126)\qquad & f(r_n)\int_{r_n-2\varepsilon f(r_n)}^{r_n+2\varepsilon f(r_n)}[q_1^2(t)+q_3^2(t)]\phi_{r_n}^2(t)\,dt \\
&\le C\,\omega(r_n)f(r_n)\int_{r_n-2\varepsilon f(r_n)}^{r_n+2\varepsilon f(r_n)}\omega^{-1}(t)[q_1^2(t)+q_3^2(t)]\phi_{r_n}^2(t)\,dt \\
&\le C\,\omega(r_n)f(r_n)\|B\tilde{\Phi}_{r_n}\|^2 \\
&\le C\,\|B\tilde{\Phi}_{r_n}\|^2,
\end{aligned}
$$

i.e., $\|B\tilde{\Phi}_{r_n}\| \ge (\frac{\varrho}{C})^{\frac{1}{2}} > 0$ for each n.

By a similar argument to the one used in the proof of Theorem 2.1 (ii), we conclude that $y_0 = 0$ a.e. in $[a,\infty)$. This contradiction implies that equation (4.69) holds for $i = 1$. If we begin with q_3 instead of q_1, then the above argument shows that equation (4.69) holds for $i = 3$.

Now, for each $r \ge a$ define

$$
(4.127)\qquad \tilde{\Psi}_r(t) = \omega^{\frac{1}{2}}(r)f^{\frac{1}{2}}(r)\Psi_r(t) \quad \text{for } t \ge a
$$

where $\{\Psi_r\}_{r\ge a}$ is given in (2.28). Via inequalities (4.97), (4.103), and $\omega f \le 1$ we have for each n

$$
\begin{aligned}
(4.128)\qquad \|\tilde{\Psi}_{r_n}\|_{L_0}^2 &= \omega(r_n)f(r_n)\,\|\Psi_{r_n}\|_{L_0}^2 \\
&\le 2C_2\,\omega(r_n)f(r_n) + 2C_6 \\
&\le 2C_2 + 2C_6\,.
\end{aligned}
$$

Thus, $\{\tilde{\Psi}_{r_n}\}_{n=1}^{\infty}$ is an L_0-bounded sequence.

By repeating the above argument with q_1 replaced by either q_2 or q_4 and $\tilde{\Phi}_r$ replaced by $\tilde{\Psi}_r$, we conclude that equation (4.69) holds for $i = 2$ and $i = 4$. (Note that the proof of the necessity shows that (4.69) holds for every $\varepsilon \in (0, \frac{1}{2N_0})$.)

Therefore, we have shown that B_0 is L_0-compact if and only if equations (4.69) hold. We apply Theorem 1.6 to conclude that B_1 is L_1-compact if and only if equations (4.69) hold. □

Remark: If we drop the hypothesis that L is limit point at ∞, Theorem 4.4 holds for the minimal operators, L_0 and B_0.

Next, we prove a corollary of Theorem 4.4 which concerns the minimal operators associated with the following differential expressions on the interval $I = [a, \infty), a > 0$:

$$(4.129) \quad (\widehat{L}z)(x) = \begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix} \left\{ \frac{1}{2} \left[\begin{pmatrix} 0 & -x^{\alpha} \\ x^{\alpha} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right]^{\bullet} + \frac{1}{2} \begin{pmatrix} 0 & -x^{\alpha} \\ x^{\alpha} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^{\bullet} + \begin{pmatrix} r_1 & r \\ r & r_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}$$

and

$$(4.130) \quad (\widehat{B}z)(x) = \begin{pmatrix} x^{-\gamma} & 0 \\ 0 & x^{-\gamma} \end{pmatrix} \begin{pmatrix} b_1 & b_4 \\ b_3 & b_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

where the coefficients $r, r_1, r_2, b_1, b_2, b_3$, and b_4 are assumed to be real, locally Lebesgue integrable functions and $^{\bullet} = \frac{d}{dx}$.

Corollary 4.5 *Let $0 < \delta < 1$ and $f(x) = x^{\alpha}[x^{2\gamma} + \delta r^2(x)]^{-\frac{1}{2}}$. Suppose there exist a sufficiently small $\hat{\varepsilon} = \hat{\varepsilon}(\delta) > 0$ and a constant $N > 0$ such that*

(1) $|r_i(x)| \le \hat{\varepsilon}|r(x)|$,

(2) $|[x^{-\gamma}r(x)]^{\bullet}| \le \hat{\varepsilon}^2 x^{-\gamma-\alpha} r^2(x)$, and

(3) $|[x^{-\alpha}r(x)]^{\bullet}| \le N x^{-2\alpha} r^2(x)$. Then

(i) $\widehat{B}_0$ is $\widehat{L}_0$-bounded if and only if

$$(4.131) \quad \sup_{x \in I} \frac{1}{f(x)} \int_x^{x+\varepsilon f(x)} f^2(u) u^{-2\alpha}\, b_i^2(u)\, du < \infty, \quad i = 1, 2, 3, 4,$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$;

(ii) $\widehat{B}_0$ is $\widehat{L}_0$-compact if and only if

$$(4.132) \quad \lim_{x \to \infty} \frac{1}{f(x)} \int_x^{x+\varepsilon f(x)} f^2(u) u^{-2\alpha}\, b_i^2(u)\, du = 0, \quad i = 1, 2, 3, 4,$$

for some $\varepsilon \in (0, \frac{1}{2N_0})$, *where* $N_0 = \dfrac{N_1}{\sqrt{2\delta}} + \sqrt{\dfrac{2}{\delta}}N$ *and* $N_1 = \sup_{x\in I} \left|\dfrac{x^{\alpha-1}}{(\gamma-\alpha)r(x)}\right|$. *Note that (2) and (3) imply* N_1 *exists.*

Proof:

We begin by applying an argument in Chapter 1 to transform the differential expressions unitarily. Notice that $\widehat{T}$ is of the form (1.21) with $W_1(x) = W_2(x) = x^\gamma$, $\Theta = \begin{pmatrix} 0 & -x^\alpha \\ x^\alpha & 0 \end{pmatrix}$, and $\mathcal{N} = \begin{pmatrix} r_1 & r \\ r & r_2 \end{pmatrix}$. Let $\mu_1(x) = \mu_2(x) = x^{\frac{\alpha}{2}}$ so that $y(x) = x^{\frac{\alpha}{2}}z(x)$ and, via equations (1.22), (1.24), (1.25), and (1.26),

$$(Ly)(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix} \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} + \begin{pmatrix} x^{-\alpha}r_1 & x^{-\alpha}r \\ x^{-\alpha}r & x^{-\alpha}r_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\} \tag{4.133}$$

and

$$(By)(x) = \begin{pmatrix} x^{\alpha-\gamma} & 0 \\ 0 & x^{\alpha-\gamma} \end{pmatrix} \begin{pmatrix} x^{-\alpha}b_1 & x^{-\alpha}b_4 \\ x^{-\alpha}b_3 & x^{-\alpha}b_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{4.134}$$

Notice that conditions (1)-(3) imply that the hypotheses of Theorem 4.4 hold with $\omega(x) = x^{\gamma-\alpha}$, $p(x) = x^{-\alpha}r(x)$, $p_i(x) = x^{-\alpha}r_1(x)$ for $i = 1, 2$, and $q_i(x) = x^{-\alpha}b_i(x)$ for $i = 1, 2, 3, 4$. The result follows. □

In Corollary 4.6, we consider the maximal operators associated with the following differential expressions on the interval I_0:

$$(\tilde{L}z)(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} + \begin{pmatrix} \frac{a}{x} + d & \frac{b}{x} + \frac{c}{x^2} \\ \frac{b}{x} + \frac{c}{x^2} & \frac{a}{x} - d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{4.135}$$

and

$$(\tilde{B}z)(x) = \begin{pmatrix} s_1 & s_4 \\ s_3 & s_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{4.136}$$

where a, b, c and d are constants, $c \neq 0$, $I_0 = (0, x_0]$ for some $x_0 > 0$ such that $\dfrac{b}{x} + \dfrac{c}{x^2} \neq 0$ on I_0, the coefficients s, s_1, and s_2 are assumed to be real, locally Lebesgue integrable functions on I_0, and $\dot{} = \frac{d}{dx}$.

Corollary 4.6 *(i)* $\tilde{B}_1$ *is* $\tilde{L}_1$*-bounded if and only if*

$$\sup_{x\in I_0} \frac{1}{x^2}\int_{x-\varepsilon''x^2}^{x} u^4 s_i^2(u)\,du < \infty,\ i=1,2, \tag{4.137}$$

for some sufficiently small ε''.

(ii) $\tilde{B}_1$ *is* $\tilde{L}_1$*-compact if and only if*

$$\lim_{x\to 0} \frac{1}{x^2}\int_{x-\varepsilon''x^2}^{x} u^4 s_i^2(u)\,du = 0,\ i=1,2, \tag{4.138}$$

for some sufficiently small ε''.

Proof:

We prove this result by using a unitary transformation to transform the singularity at 0 to a singularity at ∞ and then applying Theorem 4.4 to the new operators.

Again, we use the argument in Chapter 1 to transform the operators $\tilde{L}_1$ and $\tilde{B}_1$ unitarily. Let $\mu_1(x) = \mu_2(x) = 1$ so that $y(t) = z(x)$, $t = \frac{1}{x}$ and, via equations (1.22), (1.24), (1.25), and (1.26),

$$\begin{aligned}(Ly)(t) \;=\; &-\begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix}\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}\right. \\ &\left.+\begin{pmatrix} -\frac{a}{t}-\frac{d}{t^2} & -\frac{b}{t}-c \\ -\frac{b}{t}-c & -\frac{a}{t}+\frac{d}{t^2} \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}\right\}\end{aligned} \tag{4.139}$$

and

$$(By)(t) = \begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix}\begin{pmatrix} t^{-2}s_1 & t^{-2}s_4 \\ t^{-2}s_3 & t^{-2}s_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{4.140}$$

Note that if $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is replaced with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the results of Theorem 4.4 hold. Let us call this version Theorem 4.4*.

Now, we apply Theorem 4.4* (with $t = \frac{1}{x}$, $f(t) = [t^{-4} + \delta t^{-2}(b+ct)^2]^{-\frac{1}{2}}$, $p(t) = -bx-c$, $p_1(t) = -ax-dx^2$, $p_2(t) = -ax+dx^2$, and $q_i(t) = x^2 s_i(x)$, for $i = 1,2,3$, and 4) to the maximal operators L_1 and B_1 associated with the transformed differential expressions (4.139) and (4.140), respectively. Notice that $f(t) > 0$ on $[t_0, \infty)$, $t_0 = \frac{1}{x_0}$, and

$$\lim_{t \to \infty} f(t) = (\delta c^2)^{-\frac{1}{2}} > 0 .$$

Hence, there exists constants $k_1, k_2 > 0$ such that $k_1 \leq f(t) \leq k_2$ for $t_0 \leq t < \infty$. Since $c \neq 0$, we have that $\tilde{L}$ is limit point at zero via [9, page 116]. Thus, L is limit point at ∞. In order to satisfy the hypotheses of Theorem 4.4*, we need

$$\left|\frac{at+d}{bt+ct^2}\right| \text{ and } \left|\frac{at-d}{bt+ct^2}\right| \text{ to be sufficiently small,} \tag{4.141}$$

$$\frac{|b+2ct|}{(b+ct)^2} \text{ to be sufficiently small, and} \tag{4.142}$$

$$\frac{2}{|b+ct|} \text{ to be bounded.} \tag{4.143}$$

Notice that (4.143) holds since $b + ct \neq 0$ on $[t_0, \infty)$, by the choice of x_0. Clearly, we can choose a $t_1 \geq t_0$ large enough so that the quantities in (4.141) and (4.142) are sufficiently small on $[t_1, \infty)$. An application of Theorem 4.4* gives that B_1 is L_1-bounded if and only if

$$\sup_{t_1 \leq t < \infty} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\, \tau^{-4} s_i^2(\tau^{-1})\, d\tau < \infty, \; i = 1, 2, 3, 4, \tag{4.144}$$

for some $\varepsilon \in (0, \frac{1}{2k_2})$.

Claim 4.6.1 *Inequality (4.144) is equivalent to*

$$\sup_{t_1 \leq t < \infty} \int_t^{t+\varepsilon'} \tau^{-4} s_i^2(\tau^{-1})\, d\tau < \infty, \; i = 1, 2, 3, 4, \tag{4.145}$$

for some sufficiently small ε'.

Proof of Claim 4.6.1:

Via the bounds on f

$$\frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\, \tau^{-4} s_i^2(\tau^{-1})\, d\tau \;\geq\; \frac{k_1^2}{k_2} \int_t^{t+\varepsilon k_1} \tau^{-4} s_i^2(\tau^{-1})\, d\tau .$$

Thus, if (4.144) holds for some $\varepsilon \in (0, \frac{1}{2k_2})$, then (4.145) holds for all $\varepsilon' \in (0, k_1\varepsilon)$. Moreover,

$$\frac{1}{f(t)} \int_t^{t+\tilde{\varepsilon} f(t)} f^2(\tau)\, \tau^{-4} s_i^2(\tau^{-1})\, d\tau \;\leq\; \frac{k_2^2}{k_1} \int_t^{t+\tilde{\varepsilon} k_2} \tau^{-4} s_i^2(\tau^{-1})\, d\tau\,.$$

Thus, if (4.145) holds for some $\varepsilon' \in (0, k_1\varepsilon)$, $\varepsilon \in (0, \frac{1}{2k_2})$, then (4.144) for all $\tilde{\varepsilon} \in (0, \frac{\varepsilon'}{k_2})$. Therefore, (4.144) is equivalent to (4.145).

By a change of variables (letting $u = \frac{1}{\tau}$), we have that

$$\int_t^{t+\varepsilon'} \tau^{-4} s_i^2(\tau^{-1})\, d\tau = \int_{\frac{1}{t+\varepsilon'}}^{\frac{1}{t}} u^2 s_i^2(u)\, du.$$

Thus, (4.145) is equivalent to

$$\sup_{0<x\leq x_1} \int_{\frac{x}{1+\varepsilon' x}}^{x} u^2 s_i^2(u)\, du < \infty,\ i = 1, 2, 3, 4, \tag{4.146}$$

for some sufficiently small ε', where $x_1 = \frac{1}{t_1}$. #

Claim 4.6.2 *Inequality (4.146) is equivalent to*

$$\sup_{0<x\leq x_1} \int_{x-\varepsilon'' x^2}^{x} u^2 s_i^2(u)\, du < \infty,\ i = 1, 2, 3, 4, \tag{4.147}$$

for some sufficiently small ε''.

Proof of Claim 4.6.2:

Suppose (4.146) holds. We seek an $\varepsilon'' > 0$ such that

$$\frac{x}{1+\varepsilon' x} < x - \varepsilon'' x^2\,,$$

i..e.,

$$\varepsilon'' < \frac{\varepsilon'}{1+\varepsilon' x}\,.$$

Since $0 < x \leq x_1$, the above inequality holds if we choose

$$\varepsilon'' < \frac{\varepsilon'}{1+\varepsilon' x_1}\,.$$

Thus, for $i = 1, 2, 3$, and 4

$$\int_{\frac{x}{1+\varepsilon' x}}^{x} u^2\, s_i^2(u)\, du > \int_{x-\varepsilon'' x^2}^{x} u^2\, s_i^2(u)\, du$$

so that (4.146) implies (4.147) holds for all $\varepsilon'' \in (0, \frac{\varepsilon'}{1+\varepsilon' x_1})$. Now, suppose (4.147) holds for some sufficiently small ε''. We seek an $\tilde{\varepsilon} > 0$ such that

$$\frac{x}{1+\tilde{\varepsilon} x} > x - \varepsilon'' x^2\,,$$

i.e.,

$$\tilde{\varepsilon} < \frac{\varepsilon''}{1-\varepsilon'' x}\,.$$

If we choose $\tilde{\varepsilon} < \varepsilon''$, then the above inequality holds. Hence, for $i = 1, 2, 3$, and 4

$$\int_{\frac{x}{1+\tilde{\varepsilon} x}}^{x} u^2\, s_i^2(u)\, du < \int_{x-\varepsilon'' x^2}^{x} u^2\, s_i^2(u)\, du$$

so that (4.147) implies (4.146) holds for all $\varepsilon' \in (0, \varepsilon'')$. Therefore, (4.146) is equivalent to (4.147). #

Since the transformation is unitary, inequalities (4.147) hold if and only if $\tilde{B}_1$ is $\tilde{L}_1$-bounded.

Now, we show that $\tilde{B}_1$ is $\tilde{L}_1$-bounded on $(0, x_0]$. We consider the transformed operators B_1 and L_1 on $[t_0, \infty)$. Let $y \in D(L_1)$. Then y can be written as

$$y = y \cdot \chi_{[t_0,t_1)} + y \cdot \chi_{[t_1,\infty)},$$

where the *characteristic function*

$$\chi_{[a,b)}(x) = \begin{cases} 1 & \text{for } x \in [a,b) \\ 0 & \text{for } x \notin [a,b). \end{cases}$$

The triangle inequality yields

$$\begin{aligned} (4.148) \quad \|By\|^2 &\leq \|B(y \cdot \chi_{[t_0,t_1)})\|^2 + \|B(y \cdot \chi_{[t_1,\infty)})\|^2 \\ &= \int_{t_0}^{t_1} t^{-2}[(By)(t)]^*(By)(t)\, dt + \int_{t_1}^{\infty} t^{-2}[(By)(t)]^*(By)(t)\, dt\,, \end{aligned}$$

where the second integral is finite since $y \cdot \chi_{[t_1,\infty)} \in D(B_1)$ by the preceding argument. Since $y \in AC_{loc}[t_0,\infty)$, there exists a constant $C > 0$ such that

$$
\begin{aligned}
\int_{t_0}^{t_1} t^{-2}[(By)(t)]^*(By)(t)\,dt \;\le\; & \int_{t_0}^{t_1} t^{-2}\{[s_1^2(t^{-1}) + s_3^2(t^{-1})]|y_1(t)|^2 \\
& + \left|s_1(t^{-1})s_4(t^{-1}) + s_2(t^{-1})s_3(t^{-1})\right| [|y_1(t)|^2 + |y_2(t)|^2] \\
& + [s_2^2(t^{-1}) + s_4^2(t^{-1})]|y_2(t)|^2\}\,dt \\
\le\; & C\left\{\int_{t_0}^{t_1} t^{-2}[s_1^2(t^{-1}) + s_3^2(t^{-1})]\,dt\right. \\
& + \int_{t_0}^{t_1} t^{-2}\left|s_1(t^{-1})s_4(t^{-1}) + s_2(t^{-1})s_3(t^{-1})\right|\,dt \\
& \left. + \int_{t_0}^{t_1} t^{-2}[s_2^2(t^{-1}) + s_4^2(t^{-1})]\,dt\right\}.
\end{aligned}
$$

By applying the argument used in inequality (4.90) to the inequality above, we see that the integrals on the right-hand side are finite on $[t_0, t_1]$. Hence, (4.148) implies that $\|By\| < \infty$ for each $y \in D(L_1)$, i.e., $D(L_1) \subseteq D(B_1)$. We apply Theorem 1.3 to conclude that B_1 is L_1-bounded on $[t_0,\infty)$. Since the transformation is unitary, $\tilde{B}_1$ is $\tilde{L}_1$-bounded on $(0, x_0]$. Therefore, (i) holds.

(ii) Let $t_1 \ge t_0$ be large enough so that the quantities in (4.141) and (4.142) are sufficiently small on $[t_1,\infty)$. For $y \in D(L_1)$ we write

$$
By = B(y \cdot \chi_{[t_0,t_1)}) + B(y \cdot \chi_{[t_1,\infty)})\,. \tag{4.149}
$$

An application of Theorem 4.4* gives that B_1 is L_1-compact on $[t_1,\infty)$ if and only if

$$
\lim_{t\to\infty} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f^2(\tau)\,\tau^{-4} s_i^2(\tau^{-1})\,d\tau = 0,\;\; i = 1,2,3,4,
$$

for some $\varepsilon \in (0, \frac{1}{2k_2})$. Via an argument similar to the one in (i), we conclude that $\tilde{B}_1$ is $\tilde{L}_1$-compact on $(0, x_1]$ if and only if (4.138) holds for some sufficiently small ε''.

Moreover, via Claim 4.4.5 and Theorem 1.6 we conclude that B_1 is L_1-compact on $[t_0, t_1]$. Hence, via (4.149) and a unitary transformation, we conclude that (ii) holds. □

Chapter 5

Exponential Decay of Eigenfunctions

In this chapter we consider $\mathcal{L}^2_W(I)$-solutions of the differential equation

$$\Gamma y = \lambda y \,, \; \lambda \in \mathbb{C} \,, \tag{5.1}$$

where Γ is a formally self-adjoint differential expression of the form (1.5) and $I = [a, \infty)$, $a > 0$. By note 3 in Chapter 1, we know that any self-adjoint extension A of Γ_0 satisfies $\Gamma_0 \subset A \subset \Gamma_1$. Hence, if λ is an eigenvalue of A, then λ is an eigenvalue of Γ_1 and the corresponding eigenfunction y is in the domain of the maximal operator Γ_1. Since $y \in \mathcal{L}^2_W(I)$ and the weights are positive, $\|y(t)\|_2$ has some average decay, where

$$\|y(t)\|_2^2 = |y_1(t)|^2 + |y_2(t)|^2 \,, \; t \in I \,.$$

We focus on several examples which show how the conditions for relatively bounded and relatively compact perturbations of Γ affect this decay of eigenfunctions. First, we introduce notation and definitions taken from Goldberg [7].

The *kernel index* of a linear operator Λ, denoted $\alpha(\Lambda)$, is given by

$$\alpha(\Lambda) = \dim N(\Lambda) \,,$$

where $N(\Lambda)$ is the null space of Λ. The *deficiency index* of Λ, denoted $\beta(\Lambda)$, is given by

$$\beta(\Lambda) = \dim R(\Lambda)^{\perp},$$

where $R(\Lambda)$ is the range of Λ. [See further explanation of deficiency indices $d_{\pm}$ in Chapter 1.] If not both $\alpha(\Lambda)$ and $\beta(\Lambda)$ are infinite, then Λ has a *Fredholm index* defined as

$$\kappa(\Lambda) := \alpha(\Lambda) - \beta(\Lambda).$$

Note that for any real number r we take

$$\infty - r = \infty \quad \text{and} \quad r - \infty = -\infty.$$

A closed linear operator with a finite index is called a *Fredholm operator.*

We note several well-known facts :

1. $R(\Gamma)$ is closed iff $0 \notin \sigma_{ess}(\Gamma)$, where the *essential spectrum* of Γ

   $$\sigma_{ess}(\Gamma) := \{\lambda : R(\lambda E - \Gamma) \ \text{ is not closed}\}.$$

2. The essential spectrum is conserved under a relatively compact perturbation [10, Theorem IV-5.35].

3. Since the deficiency indices of Γ_0 are equal and finite, all self-adjoint extensions of Γ_0 have the same essential spectrum, which is the same as the essential spectrum of Γ_1 and the essential spectrum of Γ_0. [See [15, page162]and [6, page 10].]

4. For a minimal operator Γ_0 with a regular endpoint a, $\alpha(\Gamma_0) = 0$. [See [15, Theorem 3.12] and [7, Lemma VI.2.4].]

We need the following two lemmas in the proof of Theorem 5.3. Lemma 5.1 is taken in part from [11, Lemma 2.19] as well as the method of Lemma 5.2.

Lemma 5.1 *Let F and G be formal differential expressions on the interval I where F is regular at a, the order of G is less than the order of F, the coefficients of F and G are sufficiently smooth so that $D(F_0') \subseteq D(G_0')$, and F_0 and G_0 satisfy the following two conditions:*

(1) $R(F_0)$ is closed.

(2) Either G_0 is F_0-compact or there exist constants $b, c > 0$ such that

$$||Gy|| \leq b\,||y|| + c\,||Fy|| \quad \textit{for every } y \in D(F_0)$$

and

$$b + c\gamma(F_0) < \gamma(F_0)\,,$$

where $\gamma(F_0) = ||F_0^{-1}||$. Then

(i) $D((F+G)_0) = D(F_0)$ and $(F+G)_0 = F_0 + G_0|_{D(F_0)}$.

(ii) $R((F+G)_0)$ is closed.

(iii) $\alpha((F^+ + G^+)_1) = \alpha((F^+)_1)$, where F^+, G^+ denote the formal adjoints of F, G, respectively, and $(F+G)^+ = F^+ + G^+$. The infimum of such constants c is called the relative bound *of G.*

Proof:

Note that F_0 is one-to-one since $\alpha(F_0) = 0$. Since F_0 is closed and condition (1) holds, we conclude F_0^{-1} is bounded [7, page 98].

(i) Let $\mathcal{G} = G_0|_{D(F_0)}$. Then Lemma V.3.5 or Corollary V.3.8 of [7, page 122-3] implies that $F_0 + \mathcal{G}$ is closed. Since $C_0^\infty \subset D(F_0 + \mathcal{G})$ and $(F+G)_0$ is the minimal closed extension of $(F+G)_0'$, we have

$$(F+G)_0 \subset F_0 + \mathcal{G}\,.$$

Hence,

$$D((F+G)_0) \subseteq D(F_0 + \mathcal{G}) = D(F_0)\,. \tag{5.2}$$

Now, let $y \in D(F_0)$. Then there exists a sequence $\{y_n\}_{n=1}^\infty \subset D(F_0')$ such that

$$y_n \to y \quad \text{and} \quad Fy_n \to Fy \quad \text{as} \quad n \to \infty\,. \tag{5.3}$$

The F_0-boundedness of G_0 implies that $D(F_0) \subseteq D(G_0)$. Hence, there exists a constant $C > 0$ such that

$$\|Gy_n - Gy\| \leq C(\|y_n - y\| + \|Fy_n - Fy\|) .$$

Via (5.3) and the above inequality, we have

$$Gy_n \to Gy \quad \text{as} \quad n \to \infty .$$

Therefore the sequence $\{(F+G)y_n\}_{n=1}^{\infty}$ converges to $(F+G)y$. By definition, $y \in D((F+G)_0)$. Thus,

$$D(F_0) \subseteq D((F+G)_0) . \tag{5.4}$$

Via (5.2) and (5.4) we have that (i) holds.

(ii) By note 4 , we know that $\alpha(F_0) = 0$ since F_0 is a minimal operator with a regular endpoint a. Thus, F_0 has an index. Theorem V.3.6 (i) or Corollary V.3.8 of [7, page 122-3] implies that $R((F+G)_0)$ is closed.

(iii) Since $R(F_0)^{\perp} = N(F_0^*)$ (see [10, IV-(5.16)]) and $F_0^* = (F^+)_1$ (see [5, page 139]),

$$R(F_0)^{\perp} = N((F^+)_1) .$$

Hence,

$$\beta(F_0) = \alpha((F^+)_1) . \tag{5.5}$$

Similarly,

$$\beta((F+G)_0) = \alpha((F+G)_1^+) . \tag{5.6}$$

Theorem V.3.6 (ii) or Corollary V.3.8 of [7, page 122-3] implies that

$$\kappa(F_0) = \kappa((F+G)_0) .$$

Via note (4) and the above equation, we have that

$$\beta(F_0) = \beta((F+G)_0) . \tag{5.7}$$

Equations (5.5) - (5.7) imply (iii) holds. □

Lemma 5.2 *Let F be a formal differential expression on the interval I with smooth coefficients satisfying $F = F^+$. Let*

$$G = gFg^{-1} - F\,, \tag{5.8}$$

where g is a smooth positive function on I and $g^{-1} = \dfrac{1}{g}$. Then the order of G is less than the order of F and $(gFg^{-1})^+ = g^{-1}Fg$.

Proof:

For simplicity, we show that if F is of the form (1.5), then the order of G is less than the order of F. The argument for F of the form (1.4) is similar. Let $y \in \mathcal{L}^2_W(I)$. Then

$$\begin{aligned}
(Gy)(t) &= (gFg^{-1}y)(t) - (Fy)(t) \\
&= g(t)W^{-1}(t)\left\{P_0(t)\,(g^{-1}y)(t) + [Q_0(t)\,(g^{-1}y)(t)]' - Q_0^*(t)\,(g^{-1}y)'(t)\right\} \\
&\quad - W^{-1}(t)\,\{P_0(t)\,y(t) + [Q_0(t)\,y(t)]' - Q_0^*(t)\,y'(t)\} \\
&= W^{-1}(t)\left\{[Q_0(t)\,y(t)]' - \frac{g'(t)}{g(t)}[Q_0(t)\,y(t)] - Q_0^*(t)\,y'(t) + \frac{g'(t)}{g(t)}Q_0^*(t)\,y(t)\right\} \\
&\quad - W^{-1}(t)\,\{[Q_0(t)\,y(t)]' - Q_0^*(t)\,y'(t)\} \\
&= \frac{g'(t)}{g(t)}\,W^{-1}(t)\,\{Q_0^*(t)\,y(t) - [Q_0(t)\,y(t)]\}\,.
\end{aligned}$$

For $y, z \in \mathcal{L}^2_W(I)$ of class $C^{(1)}$ with compact support in the interior of I, we have

$$\begin{aligned}
\langle gFg^{-1}y, z\rangle &= \int_I z^*(t)\; W(t)\; (gFg^{-1}y)(t)\,dt \\
&= \int_I [(gz)(t)]^*\; W(t)\; (Fg^{-1}y)(t)\,dt \\
&= \int_I [(F^+gz)(t)]^*\; W(t)\; (g^{-1}y)(t)\,dt \\
&= \int_I [(g^{-1}F^+gz)(t)]^*\; W(t)\; y(t)\,dt \\
&= \langle y, g^{-1}F^+gz\rangle\,.
\end{aligned}$$

Since $F = F^+$,

$$\langle gFg^{-1}y, z\rangle = \langle y, g^{-1}Fgz\rangle\,,$$

i.e., $(gFg^{-1})^+ = g^{-1}Fg$. $\square$

Theorem 5.3 *Let E be the identity operator and let F and G be as in Lemma 5.2. Suppose $(F-\bar{\lambda}E)_0$ and G_0 satisfy conditions (1) and (2) of Lemma 5.1. If g is bounded above on I, then the map $y \mapsto gy$ is one-to-one from $N((g^{-1}(F-\lambda E)g)_1)$ onto $N((F-\lambda E)_1)$.*

Proof:

Since g is bounded above,

$$\int_I W(t)[|y_1(t)|^2 + |y_2(t)|^2]\,dt \;<\; \infty$$

implies that

$$\int_I W(t)g^2(t)[|y_1(t)|^2 + |y_2(t)|^2]\,dt \;<\; \infty\,.$$

Moreover, if we let $g^{-1}(F-\lambda E)gy = 0$ for $y \in \mathcal{L}^2_W(I)$, then $(F-\lambda E)gy = 0$. Therefore, $gy \in N((F-\lambda E)_1)$. Hence, the map $y \mapsto gy$ maps $N((g^{-1}(F-\lambda E)g)_1)$ into $N((F-\lambda E)_1)$ and is one-to-one since g is positive. By Lemma 5.2

$$G^+ = (gFg^{-1} - F)^+ = (gFg^{-1})^+ - F = g^{-1}Fg - F\,.$$

Via Lemma 5.1 (iii) we have

$$\begin{aligned}\alpha((F-\lambda E)_1) &= \alpha((F-\lambda E + G^+)_1)\\ &= \alpha((g^{-1}(F-\lambda E)g)_1)\end{aligned}$$

since $(F-\bar{\lambda}E)^+ = F-\lambda E$. Since the null spaces have the same finite dimension and the map $y \mapsto gy$ is one-to-one, the map is onto. □

Note that if A is a self-adjoint operator and $\Im m\ \lambda \neq 0$, then $\bar{\lambda} \notin \sigma_{ess}(A)$ [13, page 136].

Remark: We conclude from Theorem 5.3 that the inverse map $z \mapsto g^{-1}z$ is one-to-one from $N((F-\lambda E)_1)$ onto $N((g^{-1}(F-\lambda E)g)_1)$. Therefore, $z \in N((F-\lambda E)_1)$ implies that $g^{-1}z \in N((g^{-1}(F-\lambda E)g)_1)$, i.e.,

$$\int_I W(t)g^{-2}(t)[|z_1(t)|^2 + |z_2(t)|^2]\,dt \;<\; \infty\,. \tag{5.9}$$

Now, we examine several examples to obtain upper bounds on the rate of eigenfunction decay.

Example 1

Consider the differential operator T defined by (1.2) with $w_1(t) = w_2(t) = t^\beta$, $\beta \geq -1$. Let $F = T + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$, $c \geq 0$. Suppose $\bar{\lambda} \notin \sigma_{ess}(F_0)$ so that $R(F_0 - \bar{\lambda}E)$ is closed. Notice that $\bar{\lambda} \notin \sigma_{ess}(F_1)$ via note 3.

Let $B = gTg^{-1} - T$, where $g(t) = e^{-\sigma t^{\beta+1}}$ for some sufficiently small $\sigma > 0$. Then after some computation, we have for $y \in D(T_1)$

$$\begin{aligned}(By)(t) &= (gTg^{-1}y)(t) - (Ty)(t) \\ &= \begin{pmatrix} t^{-\beta} & 0 \\ 0 & t^{-\beta} \end{pmatrix} \begin{pmatrix} 0 & \frac{g'(t)}{g(t)} \\ -\frac{g'(t)}{g(t)} & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.\end{aligned}$$

The proof of Theorem 2.1 shows that the quantities $S_1(\varepsilon)$ and $S_2(\varepsilon)$ are proportional to

$$\sup_{t\in I} t^\beta \int_t^{t+\varepsilon t^{-\beta}} \tau^{-2\beta} \left[\frac{g'(\tau)}{g(\tau)}\right]^2 d\tau\,,$$

where

$$t^{-2\beta}\left[\frac{g'(t)}{g(t)}\right]^2 = \sigma^2(\beta+1)^2\,.$$

Hence, $S_1(\varepsilon)$ and $S_2(\varepsilon)$ are proportional to $\varepsilon\sigma^2(\beta+1)^2$. Therefore, via Theorem 1.7 we can choose $\sigma > 0$ sufficiently small so that there exist constants $b_0, c_0 > 0$ such that

$$||By|| \leq b_0\, ||y|| + c_0\, ||Ty|| \quad \text{for every } y \in D(T_1)$$

and

$$[b_0 + c_0\,(|\bar{\lambda}| + c)] + c_0\,\gamma(F_0 - \bar{\lambda}E) \;<\; \gamma(F_0 - \bar{\lambda}E)\,. \tag{5.10}$$

Notice that $B = gFg^{-1} - F$ and

$$||By|| \leq [b_0 + c_0\,(|\bar{\lambda}| + c)]\, ||y|| + c_0\, ||(F - \bar{\lambda}E)y|| \quad \text{for every } y \in D(T_1)\,,$$

where constants b_0, c_0 satisfy (5.10). Therefore, F and B satisfy the hypotheses of Theorem 5.3.

Now, we apply the remark following Theorem 5.3 to conclude that if $y \in N((F - \lambda E)_1)$, then

$$\int_I t^{\beta} e^{2\sigma t^{\beta+1}} \|y(t)\|_2^2 \, dt \; < \; \infty . \tag{5.11}$$

Hence, if A is any self-adjoint realization of T, then every eigenfunction y of A associated with an eigenvalue $\lambda \notin \sigma_{ess}(A)$ decays exponentially as in (5.11).

Let us consider the special case of $\beta = 0$. In this case we have, via [15, pages 248-9],

$$\sigma_{ess}(F) = (-\infty, -c] \; \cup \; [c, \infty) .$$

We determine the $\mathcal{L}^2_w(I)$-solutions to $Fy = \lambda y$, where $\lambda \in (-c, c)$, i.e., we seek $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in D(T_1)$ which satisfy the following system:

$$\begin{aligned} -y_2'(t) + c\, y_1(t) &= \lambda y_1(t) \\ y_1'(t) - c\, y_2(t) &= \lambda y_2(t) . \end{aligned}$$

By solving the system of equations, we obtain the $\mathcal{L}^2_w(I)$-solution

$$y_1(t) = c_1\, e^{-\sqrt{c^2-\lambda^2}\; t} \;\text{ and }\; y_2(t) = c_2\, e^{-\sqrt{c^2-\lambda^2}\; t} \;\text{ for constants } c_1, c_2.$$

Hence, the eigenfunction y associated with the eigenvalue λ satisfies for $t \in I$

$$\begin{aligned} \|y(t)\|_2^2 &= |y_1(t)|^2 + |y_2(t)|^2 \\ &= (c_1^2 + c_2^2)\, e^{-2\sqrt{c^2-\lambda^2}\; t} , \end{aligned}$$

which is consistent with (5.11), $\beta = 0$. We can use a similar argument for the case $\Im m\, \lambda \neq 0$.

Example 2

Consider the differential operator L defined by (1.1) with $w_1(t) = w_2(t) = 1$ and $p_2(t) = -p_1(t)$. Let $p_1(t) = b\, t^{\beta}$ and $p(t) = c\, t^{\beta}$, where $\beta \geq -1$ and the constants b

and c are such that the hypotheses of Lemma 4.1 are satisfied. Suppose $\bar{\lambda} \notin \sigma_{ess}(L_0)$ so that $R(L_0 - \bar{\lambda}E)$ is closed. Notice that $\bar{\lambda} \notin \sigma_{ess}(L_1)$ via note 3.

Let $B = gLg^{-1} - L$, where $g(t) = e^{-\sigma t^{\beta+1}}$ for some sufficiently small $\sigma > 0$. Then after some computation, we have for $y \in D(L_1)$

$$\begin{aligned}(By)(t) &= (gLg^{-1}y)(t) - (Ly)(t) \\ &= \begin{pmatrix} 0 & \frac{g'(t)}{g(t)} \\ -\frac{g'(t)}{g(t)} & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.\end{aligned}$$

The proof of Theorem 4.2 shows that the quantities $S_1(\varepsilon)$ and $S_2(\varepsilon)$ are proportional to

$$\sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f(\tau)^2 \left[\frac{g'(\tau)}{g(\tau)}\right]^2 d\tau\,,$$

where

$$f(t) = [(b^2+c^2)t^{2\beta}+1]^{-\frac{1}{2}}\,.$$

Now, if $\beta \geq 0$,

$$\begin{aligned}\sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f(\tau)^2 \left[\frac{g'(\tau)}{g(\tau)}\right]^2 d\tau &= \sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} \frac{\sigma^2(\beta+1)^2\tau^{2\beta}}{(b^2+c^2)\tau^{2\beta}+1}\, d\tau \\ &\leq \varepsilon\, \frac{\sigma^2(\beta+1)^2}{(b^2+c^2)}\,.\end{aligned}$$

Hence, via Theorem 1.7 we can choose $\sigma > 0$ sufficiently small so that there exist constants $b_0, c_0 > 0$ such that

$$\|By\| \leq b_0\,\|y\| + c_0\,\|Ly\| \;\text{ for every } y \in D(L_1)$$

and

$$(b_0 + c_0\,|\bar{\lambda}|) + c_0\,\gamma(L_0 - \bar{\lambda}E) \;<\; \gamma(L_0 - \bar{\lambda}E)\,. \tag{5.12}$$

Notice that

$$\|By\| \leq (b_0 + c_0\,|\bar{\lambda}|)\,\|y\| + c_0\,\|(L-\bar{\lambda}E)y\| \;\text{ for every } y \in D(L_1)\,,$$

where constants b_0, c_0 satisfy (5.12). Moreover, if $-1 \le \beta < 0$, then

$$\lim_{t\to\infty} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f(\tau)^2 \left[\frac{g'(\tau)}{g(\tau)}\right]^2 d\tau \;=\; 0$$

which implies that B_0 is L_0-compact; and hence, B_0 is $(L_0 - \bar{\lambda}E)$-compact. Therefore, L and B satisfy the hypotheses of Theorem 5.3 for $\beta \ge -1$.

Now, we apply the remark following Theorem 5.3 to conclude that if $y \in N((L-\lambda E)_1)$, then

$$\int_I e^{2\sigma t^{\beta+1}} \|y(t)\|_2^2 \, dt \;<\; \infty\,. \tag{5.13}$$

Hence, if A is any self-adjoint realization of L, then every eigenfunction y of A associated with an eigenvalue $\lambda \notin \sigma_{ess}(A)$ decays exponentially as in (5.13).

Let us consider the special case of $\beta > 0$ and $b = 0$. Note that $0 \notin \sigma_{ess}(L_1)$ [9, Theorem 2]. We determine the $\mathcal{L}^2_w(I)$-solutions to $Ly = 0$, i.e., we seek $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in D(L_1)$ which satisfy the following system:

$$\begin{aligned} -y_2'(t) + c\,t^{\beta} y_2(t) \;&=\; 0 \\ y_1'(t) + c\,t^{\beta} y_1(t) \;&=\; 0\,. \end{aligned}$$

By solving each equation, we obtain the $\mathcal{L}^2_w(I)$-solution

$$y_1(t) = c_1\, e^{-\frac{c}{\beta+1}\, t^{\beta+1}}, \;\; \text{for constant } c_1, \;\; \text{and} \;\; y_2(t) = 0\,.$$

Hence, the eigenfunction y associated with the eigenvalue zero satisfies for $t \in I$

$$\begin{aligned} \|y(t)\|_2^2 \;&=\; |y_1(t)|^2 + |y_2(t)|^2 \\ &=\; c_1^2\, e^{-\frac{2c}{\beta+1}\, t^{\beta+1}}\,, \end{aligned}$$

which is consistent with (5.13).

Example 3

Consider the differential operator L defined by (1.1) with $w_1(t) = w_2(t) = t^{\alpha}$, for $\alpha \in \mathbb{R}$, $p(t) = c\,t^{\beta}$, for $\beta \ge -1$, and the positive constant c and the functions p_1, p_2

are such that the conditions of Lemma 4.3 are satisfied. Suppose $\bar{\lambda} \notin \sigma_{ess}(L_0)$ so that $R(L_0 - \bar{\lambda}E)$ is closed. Notice that $\bar{\lambda} \notin \sigma_{ess}(L_1)$ via note 3.

Case I: $\alpha \leq \beta$

Let $B = gLg^{-1} - L$, where $g(t) = e^{-\sigma t^{\beta+1}}$ for some sufficiently small $\sigma > 0$. Then after some computation, we have for $y \in D(L_1)$

$$\begin{aligned}(By)(t) &= (gLg^{-1}y)(t) - (Ly)(t) \\ &= \begin{pmatrix} t^{-\alpha} & 0 \\ 0 & t^{-\alpha} \end{pmatrix} \begin{pmatrix} 0 & \frac{g'(t)}{g(t)} \\ -\frac{g'(t)}{g(t)} & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.\end{aligned}$$

The proof of Theorem 4.4 shows that the quantities $S_1(\varepsilon)$ and $S_2(\varepsilon)$ are proportional to

$$\sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f(\tau)^2 \left[\frac{g'(\tau)}{g(\tau)}\right]^2 d\tau\,,$$

where

$$f(t) = [t^{2\alpha} + \delta c\, t^{2\beta}]^{-\frac{1}{2}}.$$

Now,

$$\begin{aligned}\sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f(\tau)^2 \left[\frac{g'(\tau)}{g(\tau)}\right]^2 d\tau &= \sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} \frac{\sigma^2(\beta+1)^2\tau^{2\beta}}{\tau^{2\alpha} + \delta c\,\tau^{2\beta}}\, d\tau \\ &= \sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} \frac{\sigma^2(\beta+1)^2\tau^{2\beta-2\alpha}}{1 + \delta c\,\tau^{2\beta-2\alpha}}\, d\tau \\ &\leq \varepsilon\, \frac{\sigma^2(\beta+1)^2}{\delta c}.\end{aligned}$$

Hence, via Theorem 1.7 we can choose $\sigma > 0$ sufficiently small so that there exist constants $b_0, c_0 > 0$ such that

$$\|By\| \leq b_0\,\|y\| + c_0\,\|Ly\| \quad \text{for every } y \in D(L_1)$$

and

$$(b_0 + c_0\,|\bar{\lambda}|) + c_0\,\gamma(L_0 - \bar{\lambda}E) \;<\; \gamma(L_0 - \bar{\lambda}E)\,. \tag{5.14}$$

Notice that

$$\|By\| \le (b_0 + c_0\,|\bar{\lambda}|)\,\|y\| + c_0\,\|(L - \bar{\lambda}E)y\| \quad \text{for every } y \in D(L_1)\,,$$

where constants b_0, c_0 satisfy (5.14). Therefore, L and B satisfy the hypotheses of Theorem 5.3.

Now, we apply the remark following Theorem 5.3 to conclude that if $y \in N((L - \lambda E)_1)$, then

$$\int_I t^{\alpha}\, e^{2\sigma t^{\beta+1}}\, \|y(t)\|_2^2\, dt \;<\; \infty\,. \tag{5.15}$$

Hence, if A is any self-adjoint realization of L, then every eigenfunction y of A associated with an eigenvalue $\lambda \notin \sigma_{ess}(A)$ decays exponentially as in (5.15).

Case II: $\alpha > \beta$

Let $B = gLg^{-1} - L$, where $g(t) = e^{-\sigma t^{\alpha+1}}$ for some sufficiently small $\sigma > 0$. Then after some computation, we have for $y \in D(L_1)$

$$\begin{aligned}
(By)(t) \;&=\; (gLg^{-1}y)(t) - (Ly)(t) \\
&=\; \begin{pmatrix} t^{-\alpha} & 0 \\ 0 & t^{-\alpha} \end{pmatrix} \begin{pmatrix} 0 & \frac{g'(t)}{g(t)} \\ -\frac{g'(t)}{g(t)} & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.
\end{aligned}$$

The proof of Theorem 4.4 shows that the quantities $S_1(\varepsilon)$ and $S_2(\varepsilon)$ are proportional to

$$\sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f(\tau)^2 \left[\frac{g'(\tau)}{g(\tau)}\right]^2 d\tau\,,$$

where

$$f(t) = [t^{2\alpha} + \delta c\, t^{2\beta}]^{-\frac{1}{2}}\,.$$

Now,

$$\begin{aligned}
\sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} f(\tau)^2 \left[\frac{g'(\tau)}{g(\tau)}\right]^2 d\tau \;&=\; \sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} \frac{\sigma^2(\alpha+1)^2\tau^{2\alpha}}{\tau^{2\alpha} + \delta c\,\tau^{2\beta}}\, d\tau \\
&=\; \sup_{t\in I} \frac{1}{f(t)} \int_t^{t+\varepsilon f(t)} \frac{\sigma^2(\alpha+1)^2\tau^{2\alpha-2\beta}}{\tau^{2\alpha-2\beta} + \delta c}\, d\tau \\
&\le\; \varepsilon\, \sigma^2(\alpha+1)^2\,.
\end{aligned}$$

Hence, via Theorem 1.7 we can choose $\sigma > 0$ sufficiently small so that there exist constants $b_0, c_0 > 0$ such that

$$||By|| \le b_0\,||y|| + c_0\,||Ly|| \quad \text{for every } y \in D(L_1)$$

and

$$(b_0 + c_0\,|\bar{\lambda}|) + c_0\,\gamma(L_0 - \bar{\lambda}E) \;<\; \gamma(L_0 - \bar{\lambda}E)\,. \tag{5.16}$$

Notice that

$$||By|| \le (b_0 + c_0\,|\bar{\lambda}|)\,||y|| + c_0\,||(L - \bar{\lambda}E)y|| \quad \text{for every } y \in D(L_1)\,,$$

where constants b_0, c_0 satisfy (5.16). Therefore, L and B satisfy the hypotheses of Theorem 5.3.

Now, we apply the remark following Theorem 5.3 to conclude that if $y \in N((L - \lambda E)_1)$, then

$$\int_I t^{\alpha}\, e^{2\sigma t^{\alpha+1}}\, ||y(t)||_2^2\, dt \;<\; \infty\,. \tag{5.17}$$

Hence, if A is any self-adjoint realization of L, then every eigenfunction y of A associated with an eigenvalue $\lambda \notin \sigma_{ess}(A)$ decays exponentially as in (5.17).

Let us consider the special case of $\beta > \alpha$ and $p_1 = p_2 = \frac{1}{2}\,p$. Note that $0 \notin \sigma_{ess}(L_1)$ [9, Theorem 2]. We seek $\mathcal{L}_w^2(I)$-solutions to $Ly = 0$, i.e., we seek $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in D(L_1)$ which satisfy the following system:

$$\begin{aligned} -y_2'(t) + \tfrac{c}{2}\, t^{\beta} y_1(t) + c\, t^{\beta} y_2(t) &= 0 \\ y_1'(t) + c\, t^{\beta} y_1(t) + \tfrac{c}{2}\, t^{\beta} y_2(t) &= 0\,. \end{aligned}$$

Although we can not obtain an explicit $\mathcal{L}_w^2(I)$-solution to the above system, we know via [9, equation (3.6)] that this solution is asymptotic to

$$K\, e^{-kt^{\beta+1}}, \quad \text{constant } k > 0, \text{ constant vector } K,$$

as $t \to \infty$. Hence, the behavior of the eigenfunction y associated with the eigenvalue zero is consistent with that implied by (5.15).

In this chapter we have presented three examples to show how the necessary and sufficient conditions for relatively bounded and relatively compact perturbations, which are given in Chapters 2 through 4, yield results on the $\mathcal{L}^2_W$-decay of eigenfunctions. Further work is needed to obtain pointwise estimates on eigenfunction decay.

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Vita

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