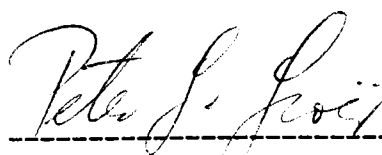


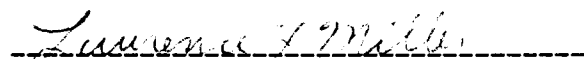
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Dr. Peter G. Groer, Major Professor

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Date 7/13/94

BAYESIAN TECHNIQUES FOR
IMPUTATION OF MISSING OCCUPATIONAL
DOSE/EXPOSURE DATA

A Thesis Presented for
the Master of Science
Degree
The University of Tennessee, Knoxville

Robert A. Cooper III

August 1994

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ABSTRACT

It has been proposed that occupational exposure / dose limits change the distribution of occupational exposures/doses. The often used log normal model represents the distribution of doses quite well for low doses. In dose ranges that approach occupational limits dose distributions deviate from a log normal and tend to be more normal. Kumazawa and Numakunai [7] have developed the hybrid log normal model to account for this shift and applied this new model to several data sets using classical statistics to estimate parameters. One of these parameters is the hybridization parameter which is used to investigate whether a distribution is predominantly normal or log normal. We applied the Bayesian approach to the hybrid log normal model. We obtained predictive densities to impute missing occupational exposures/doses and obtained marginal posterior densities for the parameters, including the hybridization parameter. The marginal posterior density of the hybridization parameter quantifies the remaining uncertainty about the parameter in a quantitative probabilistic fashion and is thus an improvement over published estimation techniques [7]. We also obtained a predictive density using the bivariate normal model which compares quite well to imputation results with a Bayesian linear regression technique used by Dolan [1].

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CHAPTER 1

OVERVIEW

1.1 INTRODUCTION

The log normal model has traditionally been used to represent the distribution of occupational doses. It has been noticed, however, that dose / exposure distributions deviate from a log normal and tend to be more normal in dose ranges that approach occupational dose limits [7]. The hybrid log normal model has been developed by Kumazawa and Numakunai [7] to account for this shift. This shift can be characterized by the hybridization parameter ρ . Classical techniques have been applied by Kumazawa and Numakunai [7] to estimate the parameters, including ρ . These parameter estimates are point estimates and say nothing about the remaining uncertainty about the parameters. As an improvement over this classical estimation, we applied Bayesian parameter estimation techniques. Bayesian parameter estimation results in marginal posterior densities for each parameter. These marginal posterior densities characterize the remaining uncertainty about these parameters in a quantitative probabilistic fashion. We also derived the predictive density for the hybrid log normal model.

As another application of Bayesian technique we obtained the

predictive density of a lung burden conditional on a urinalysis result and a calibration data set using the bivariate normal model.

1.2 RADIOLOGICAL MONITORING DATA

Radiological monitoring data are available for radiation workers from the Fernald Feed Materials Production Facility for the years 1952 through 1983. These data include urinalysis results for total uranium and ^{235}U lung burden measurements for these workers. However, lung burden measurements were not performed for the years 1952 through 1967 since the necessary counting equipment was not available. For epidemiological studies on these workers it is necessary to infer lung burdens for workers for the years in which lung burden measurements were not made. To facilitate the calculation of lung burdens, Dolan [1] used a data set consisting of lung burdens and urinalysis results for workers in plant 8 who were monitored by both methods in the same month during the years 1968 through 1983. This data will be referred to as Data Set 1 and is shown in Appendix A. A subset of this data is shown in Table 1.1.

External radiation exposure data for workers at the Y-12 weapons production facility in Oak Ridge Tennessee was also used. The gamma exposures for workers in department #2038 for the years 1961 and 1962 are shown in tables 1.2 and 1.3, respectively.

TABLE 1.1

A SUBSET OF DATA SET 1:

URINALYSIS AND LUNG BURDEN VALUES

OBTAINED IN THE SAME MONTH

OBSERVATION	URINALYSIS microgram/Litre of U	LUNG BURDEN microgram ²³⁵ U
1	17	27
2	7	19
3	3	0
4	1	97
5	10	5
6	10	0
7	0	28
8	8	129
9	1	112
10	1	70
11	21	125
12	5	79
13	4	95

TABLE 1.2

DATA SET 2: GAMMA EXPOSURES (mR) FOR THE YEAR 1961.

OBS	QRTR	EXPOSURE mR
1	2	39
2	4	44
3	2	70
4	4	110
5	2	64
6	4	79
7	2	47
8	4	37
9	2	40
10	4	48
11	2	146

TABLE 1.3

DATA SET 3: GAMMA EXPOSURES (mR) FOR THE YEAR 1962.

OBS	QRTR	EXPOSURE mR
1	2	43
2	4	28
3	2	76
4	4	45
5	2	50
6	4	25
7	2	43
8	4	53
9	3	46
10	2	32
11	4	72

1.3 A BRIEF REVIEW OF THE BAYESIAN APPROACH

In order to explain the methods for the imputation of missing data used here it is helpful to briefly review the Bayesian approach to parameter estimation. For a given bivariate probability density function $f(\underline{x}, \underline{y} | \underline{\theta})$, where $\underline{\theta} = \{\theta_1, \dots, \theta_i, \dots, \theta_m\}$ is a parameter vector, the posterior density can be constructed using what is known as the prior density of $\underline{\theta}$, $\pi(\underline{\theta})$, and the likelihood $\mathcal{L}(\underline{x}, \underline{y} | \underline{\theta})$. $\pi(\underline{\theta})$ represents the prior knowledge of the parameter $\underline{\theta}$ and $\mathcal{L}(\underline{x}, \underline{y} | \underline{\theta})$ is the probability of the vectors $\underline{x}$ and $\underline{y}$ given $\underline{\theta}$. Here $\underline{x} = \{x_1, \dots, x_i, \dots, x_n\}$ and $\underline{y} = \{y_1, \dots, y_i, \dots, y_n\}$ represent the data. Since all n experiments where each pair $\{x_i, y_i\}$ is measured are assumed to be independent $\mathcal{L}(\underline{x}, \underline{y} | \underline{\theta})$ is obtained by multiplication.

$$\mathcal{L}(\underline{x}, \underline{y} | \underline{\theta}) = f(x_1, y_1 | \underline{\theta}) \dots f(x_i, y_i | \underline{\theta}) \dots f(x_n, y_n | \underline{\theta})$$

The joint posterior density of the parameter $\underline{\theta}$, $p(\underline{\theta} | \underline{x}, \underline{y})$, is proportional to the likelihood times the prior density.

$$p(\underline{\theta} | \underline{x}, \underline{y}) \propto \mathcal{L}(\underline{x}, \underline{y} | \underline{\theta}) \pi(\underline{\theta})$$

To obtain the marginal posterior density of any of the elements of $\underline{\theta}$, for example θ_1 , one has to integrate $p(\underline{\theta} | \underline{x}, \underline{y})$, over all other parameters $\theta_2 \dots \theta_m$.

$$p(\theta_1 | \underline{x}, \underline{y}) = \int \dots \int p(\underline{\theta} | \underline{x}, \underline{y}) d\theta_2 \dots d\theta_m$$

In a similar fashion, the posterior densities can be obtained for any θ_i . The posterior densities of the parameter $\underline{\theta}$ characterize the remaining uncertainty about $\underline{\theta}$.

The predictive density for a new observation $\{x', y'\}$, $p(x', y' | \underline{x}, \underline{y})$, can be obtained by multiplying the probability of the new observation given the parameter, $f(x', y' | \underline{\theta})$, by the joint posterior density of the parameter $p(\underline{\theta} | \underline{x}, \underline{y})$ and integrating over all possible values of $\underline{\theta}$.

$$p(x', y' | \underline{x}, \underline{y}) = \int_{\underline{\theta}} p(\underline{\theta} | \underline{x}, \underline{y}) f(x', y' | \underline{\theta}) d\underline{\theta}$$

Casella and Berger [2] show that the connection between joint and conditional densities can be used to express the predictive density of a new observation x' given a observation y' , $p(x' | y', \underline{x}, \underline{y})$, can be obtained by dividing $p(x', y' | \underline{x}, \underline{y})$ by $p(y' | \underline{x}, \underline{y})$, the marginal predictive density of y' .

$$p(x' | y', \underline{x}, \underline{y}) = \frac{p(x', y' | \underline{x}, \underline{y})}{p(y' | \underline{x}, \underline{y})}$$

where

$$p(y' | \underline{x}, \underline{y}) = \int p(x', y' | \underline{x}, \underline{y}) dx'$$

CHAPTER 2

BAYESIAN IMPUTATION OF MISSING DATA USING THE BIVARIATE NORMAL MODEL

Let L be a measure of lung burden (μg of ^{235}U) and U be a urinalysis result (μg of total uranium/litre). Observations on n persons, where both measurements are performed, yields the following data:

$\underline{D} = \{(U_1, L_1), (U_2, L_2), \dots, (U_n, L_n)\}$. Such data is shown as Data Set 1. in Appendix A. A set of data where only values of U are available is denoted by the vector $\underline{U}$. With the assumption that working conditions stayed the same, $\underline{D}$ can be used to predict a value of L for a worker with urinalysis result U_0 contained in $\underline{U}$. This is represented by the posterior density of L given $\underline{D}$ and U_0 or, $p(L|\underline{D}, U_0)$.

The bivariate normal probability density function is given by Geisser [3].

$$f(\underline{D}|\underline{\mu}, \underline{\Sigma}) \propto |\underline{\Sigma}|^{-1/2} \text{Exp}\left[-\frac{1}{2} \text{tr} \underline{\Sigma}^{-1} (\underline{D} - \underline{\mu})(\underline{D} - \underline{\mu})'\right] \quad (2.1)$$

Here $\underline{\mu}' = \{\mu_1, \mu_2\}$ is the true mean vector and $\underline{\Sigma} = \{\{\Sigma_{11}, \Sigma_{12}\}, \{\Sigma_{21}, \Sigma_{22}\}\}$ is the covariance matrix and $\underline{D}' = \{U, L\}$ is the random variable vector. This is the probability density function for $\underline{D}$ given the parameters $\underline{\mu}$ and $\underline{\Sigma}$.

The noninformative prior for $\underline{\mu}$ and $\underline{\Sigma}$ used here is given by Geisser [3].

$$\pi(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \propto |\boldsymbol{\Sigma}|^{-3/2} \quad (2.2)$$

The likelihood is given by Press [4].

$$\mathcal{L}(\boldsymbol{m}, \boldsymbol{S}) \propto |\boldsymbol{S}|^{1/2(n-4)} |\boldsymbol{\Sigma}^{-1}|^{1/2n} \exp[-1/2 \text{tr} \boldsymbol{\Sigma}^{-1} [(n-1)\boldsymbol{S} + n(\boldsymbol{m}-\boldsymbol{\mu})(\boldsymbol{m}-\boldsymbol{\mu})']] \quad (2.3)$$

Here $\boldsymbol{m}' = \{m_1, m_2\}$ is the sample mean vector and $\boldsymbol{S} = \{\{S_{11}, S_{12}\}, \{S_{21}, S_{22}\}\}$ is the sample covariance matrix. $\boldsymbol{m}$ and $\boldsymbol{S}$ contain the set of observations $\underline{D} = \{(U_1, L_1), \dots, (U_i, L_i), \dots, (U_n, L_n)\}$. Multiplying the prior and the likelihood gives the joint posterior density of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

$$P(\boldsymbol{\mu}, \boldsymbol{\Sigma}^{-1} | \boldsymbol{m}, \boldsymbol{S}) \propto |\boldsymbol{\Sigma}^{-1}|^{1/2(n-3)} \exp[-1/2 \text{tr} \boldsymbol{\Sigma}^{-1} [(n-1)\boldsymbol{S} + n(\boldsymbol{m}-\boldsymbol{\mu})(\boldsymbol{m}-\boldsymbol{\mu})']] \quad (2.4)$$

The predictive density is obtained by multiplying the posterior density by the probability of a new observation given $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and integrating over $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}^{-1}$. This is given by Geisser [3].

$$p(D | \boldsymbol{m}, \boldsymbol{S}) \propto \left[1 + \frac{n}{(n+1)(n-1)} (\boldsymbol{m}-D)' \boldsymbol{S}^{-1} (\boldsymbol{m}-D)\right]^{-1/2n} \quad (2.5)$$

Raiffa and Schlaifer [5] show how equation 2.5 is used to obtain the conditional density of the lung burden for a given urinalysis result, $p(L | U_0, \underline{D})$. This is a Student density and is given by Dunsmore [6]:

$$St(L | m_1 + \frac{S_{12}}{S_{22}}(U_0 - m_2), \frac{n(n-2)}{(n+1)} \frac{S_{22}}{S_{11}S_{22} - S_{12}S_{21}}, n-2) \quad (2.6)$$

A Student density with parameters $\boldsymbol{\theta} = \{\theta_1, \theta_2, \theta_3\}$ has the following form:

$$St(L|\theta_1, \theta_2, \theta_3) \propto (\theta_3 + \theta_2(L - \theta_1)^2)^{-(\theta_3+1)/2} \quad (2.7)$$

A graph of equation 2.6 with the urinalysis result of $U_0=80 \mu\text{g} / \text{liter}$ is shown in figure 2.1. Also shown in figure 2.1 is the result of the Bayesian linear regression performed by Dolan [1]. The computer programs used to obtain figure 2.1 are given in Appendix B.

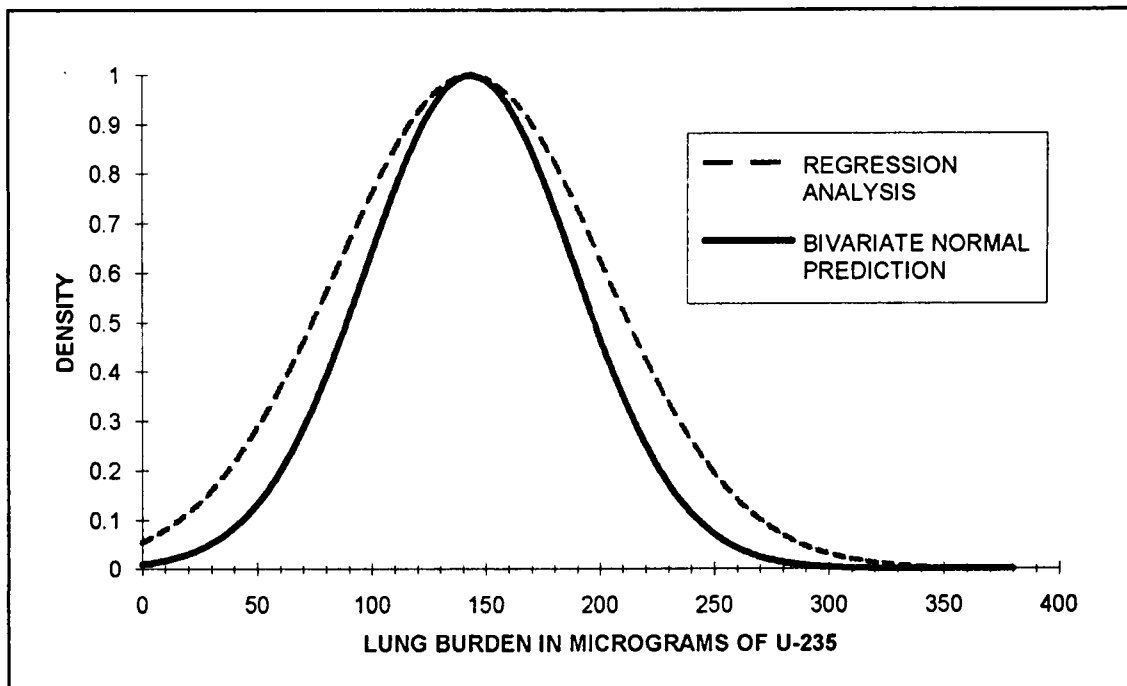


Figure 2.1 The posterior density of lung burden given a urinalysis result of $80 \mu\text{g}/\text{Litre}$ of uranium.

CHAPTER 3

THE HYBRID LOG NORMAL MODEL

3.1 A BRIEF REVIEW OF THE HYBRID LOG NORMAL MODEL

Kumazawa [7] has suggested the following probability density function for occupational doses.

$$f(x|\alpha, \beta, \gamma) \propto \left(\frac{\alpha}{x} + \beta\right) \text{Exp}\left[-(\alpha \ln x + \beta x + \gamma)^2/2\right]$$

$$(0 < x < \infty, \alpha \geq 0, \beta \geq 0, \alpha + \beta > 0) \quad (3.1.1)$$

When $\alpha=0$ equation 3.1.1 is a normal probability density function with mean $-\gamma/\beta$ and variance β^{-2} :

$$f(x|0, \beta, \gamma) \propto \frac{1}{\beta^{-1}} \text{Exp}\left[-(x + \frac{\gamma}{\beta})^2/2\beta^{-2}\right]$$

When $\beta=0$ equation 3.1.1 is a log normal probability density function.

$$f(x|\alpha, 0, \gamma) \propto \frac{1}{x} \frac{1}{\alpha^{-1}} \text{Exp}\left[-(\ln x + \frac{\gamma}{\alpha})^2/2\alpha^{-2}\right]$$

Equation 3.1.1 can be expressed using the parameters ρ , μ and σ where

$$\rho = \frac{\beta}{\alpha}, \mu = \ln\left(\frac{\beta}{\alpha}\right) - \frac{\gamma}{\alpha}, \sigma^2 = \frac{1}{\alpha^2}.$$

Writing equation 3.1.1 in terms of the parameters ρ , μ and σ gives the hybrid lognormal probability density function as equation 3.1.2.

$$f(x|\rho,\mu,\sigma) \propto \frac{1}{\sigma} \left(\rho + \frac{1}{x}\right) \text{Exp}\left[-(\ln \rho x + \rho x - \mu)^2 / 2\sigma^2\right]$$

$$(0 < x < \infty, -\infty < \mu < \infty, 0 < \sigma < \infty, 0 < \rho < \infty) \quad (3.1.2)$$

Since $\rho = \beta / \alpha$, it represents the tendency of a distribution of doses to be lognormal or normal.

Figure 3.1.1 shows a plot of equation 3.1.2 with $\rho=1$, $\mu=0$, $\sigma=1$, $X \sim \text{HLN}(1,0,1)$, compared to the standard normal $X \sim N(0,1)$ and the standard lognormal $X \sim \text{LN}(0,1)$ distributions.

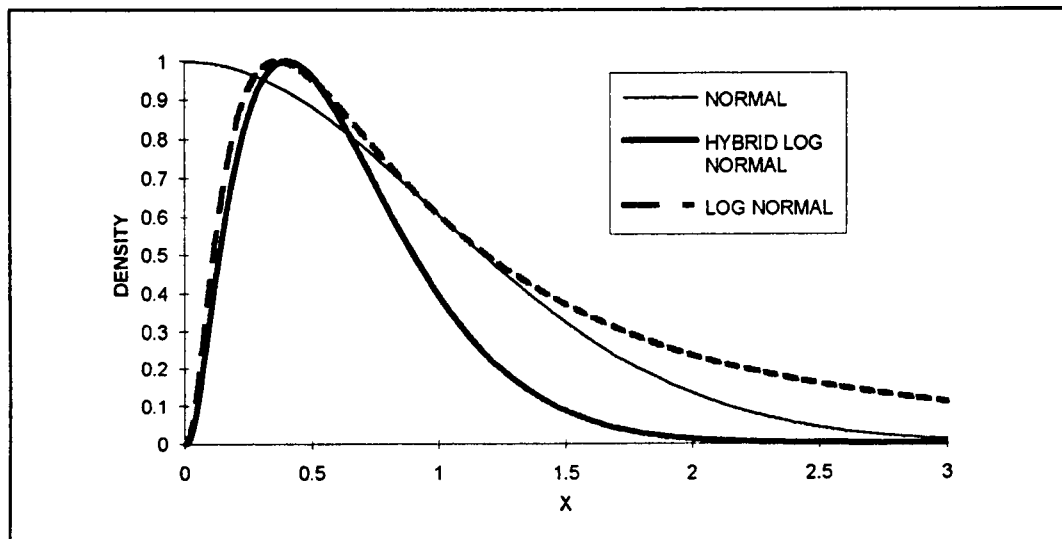


Figure 3.1.1 Examples of the log normal, normal and hybrid log normal densities.

3.2 BAYESIAN PARAMETER ESTIMATION FOR THE HYBRID LOG NORMAL MODEL

The joint posterior density of the parameters ρ , μ and σ given observations represented by the vector $\underline{x} = \{x_1, \dots, x_i, \dots, x_n\}$ is:

$$p(\rho, \mu, \sigma | \underline{x}) \propto \pi(\rho, \mu, \sigma) \mathcal{L}(\underline{x} | \rho, \mu, \sigma). \quad (3.2.1)$$

$\pi(\rho, \mu, \sigma)$ is the prior density of the parameters ρ , μ and σ and $\mathcal{L}(\underline{x} | \rho, \mu, \sigma)$ is the likelihood of data set $\underline{x}$ given the parameters ρ , μ and σ . This posterior density can be integrated over μ and σ to obtain the marginal posterior density of ρ . The tendency of the distribution of doses $\underline{x}$ to be lognormal or normal can be studied with the posterior density of ρ .

For a data vector $\underline{x}$ of size n consisting of independent elements x_i , the likelihood is obtained by applying equation 3.1.2 to each x_i and multiplying these n functions together.

$$\mathcal{L}(\rho, \mu, \sigma | \underline{x}) = f(x_1 | \rho, \mu, \sigma) \dots f(x_i | \rho, \mu, \sigma) \dots f(x_n | \rho, \mu, \sigma) \quad (3.2.2)$$

This Likelihood is given by Groer and Uppuluri [8] as equation 3.2.3.

$$\mathcal{L}(\rho, \mu, \sigma | \underline{x}) \propto \sigma^{-n} \prod_{i=1}^n (\rho + 1/x_i) \text{Exp} \left[-\sum_{i=1}^n (\rho x_i + \ln \rho x_i - \mu)^2 / 2\sigma^2 \right]. \quad (3.2.3)$$

The following prior has been used by Groer and Uppuluri [8]:

$$\pi(\mu, \sigma, \rho) \propto \sigma^{-1} \text{Exp}[-(a/\rho^2 + b\rho^2) + 2\sqrt{ab}]. \quad (3.2.4)$$

The noninformative prior density for μ is an improper uniform distribution and the noninformative prior for σ is taken to be proportional to $1/\sigma$. A prior for the parameter ρ used by Groer and Uppuluri [8] is:

$$\pi(\rho) \propto \text{Exp}[-(a/\rho^2 + b\rho^2) + 2\sqrt{ab}], \quad (3.2.5)$$

where $a=1/100$ and $b=1/100$. This prior density of ρ shown in figure 3.2.1 represents the knowledge about the parameter ρ which indicates whether the distribution of doses follows a hybrid lognormal, lognormal or normal model.

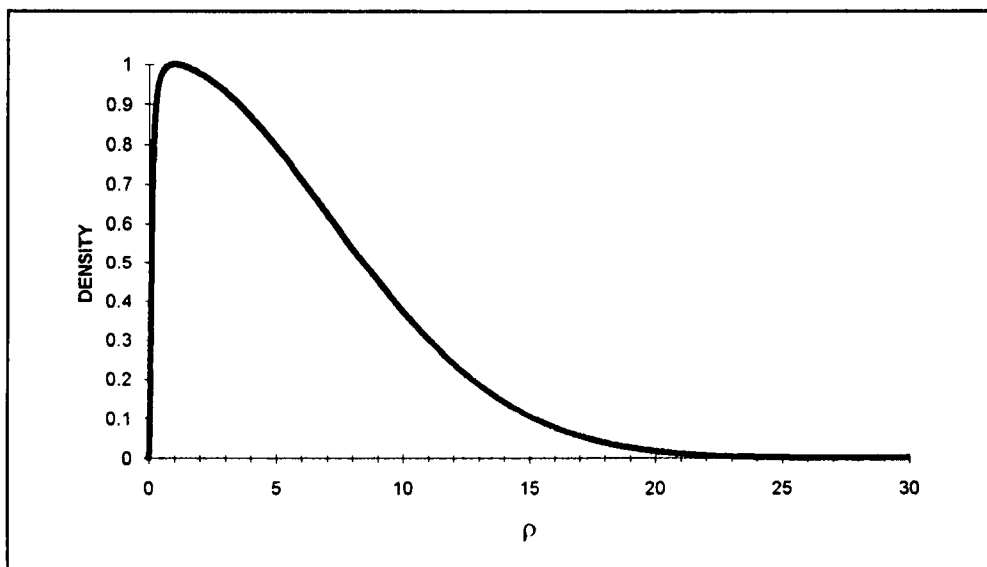


Figure 3.2.1 The prior density of ρ .

Figure 3.2.1 shows a mode at $\rho=1$ and how the prior density tends toward zero as ρ approaches zero. Point estimates of ρ less than 1 and greater than zero represent a data set that tends to be lognormally distributed. For example, a value of $\rho=1/30$ would indicate this tendency. Figure 3.2.1 also shows how the prior density approaches zero for values of ρ greater than 30. Similarly a mode of, for example, $\rho=30$ would indicate that a data set is more normally distributed than lognormaly.

The joint posterior density for ρ , μ and σ is obtained by multiplying the prior and the likelihood and is shown in equation 3.2.6.

$$p(\rho, \mu, \sigma | \mathbf{x}) \propto \sigma^{-(n+1)} \prod_{i=1}^n \left(\rho + \frac{1}{x_i}\right) \text{Exp}\left[-\sum_{i=1}^n (\rho x_i + \ln \rho x_i - \mu)^2 / 2\sigma^2\right] \text{Exp}\left[\left(\frac{-a}{\rho^2} - b\rho^2\right)\right] \quad (3.2.6)$$

The marginal posterior density of ρ is obtained by integrating equation 3.2.6 over μ and σ .

$$p(\rho | \mathbf{x}) \propto \int_{-\infty}^{\infty} \int_0^{\infty} p(\rho, \mu, \sigma | \mathbf{x}) d\mu d\sigma \quad (3.2.7)$$

Integrating equation 3.2.6 over μ gives the joint posterior density of ρ and σ . This is accomplished using

$$y_i = \rho x_i + \ln \rho x_i$$

$$\bar{y} = \sum_{i=1}^n \frac{y_i}{n},$$

and

$$\sum_{i=1}^n (y_i - \mu)^2 = \sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2.$$

Inserting these into equation 3.2.6 gives

$$p(\rho, \mu, \sigma | \mathcal{X}) \propto \sigma^{-(n+1)} \prod_{i=1}^n \left(\rho + \frac{1}{x_i} \right) \text{Exp}\left[\left(\frac{-a}{\rho^2} - b\rho^2\right)\right] \text{Exp}\left[-\frac{n}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2\right], \quad (3.2.8)$$

or

$$p(\rho, \mu, \sigma | \mathcal{X}) \propto \sigma^{-(n+1)} \prod_{i=1}^n \left(\rho + \frac{1}{x_i} \right) \text{Exp}\left[\left(\frac{-a}{\rho^2} - b\rho^2\right)\right] \text{Exp}\left[\frac{-1}{2\sigma^2} \left(\sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2 \right)\right].$$

Integration of equation 3.2.8 over μ involves only the integral given below since all terms not including μ are considered "constant" in the integration over μ .

$$\int_{-\infty}^{\infty} \text{Exp}\left[\frac{-1}{2\sigma^2} n(\bar{y} - \mu)^2\right] d\mu \propto \sigma$$

Inserting the value of this integral gives the following joint posterior density of ρ and σ .

$$\pi(\sigma, \rho | \mathcal{X}) \propto \prod_{i=1}^n \left(\rho + \frac{1}{x_i} \right) * \text{Exp}\left[-(a/\rho^2 + b\rho^2)\right] * \sigma^{-n} \text{Exp}\left[-\frac{n}{2\sigma^2} \sum_{i=1}^n (y_i - \bar{y})^2\right] \quad (3.2.9)$$

Integrating (3.2.9) over σ gives the marginal posterior density of ρ .

$$\pi(\rho | \mathbf{X}) \propto \prod_{i=1}^n (\rho + 1/x_i) \cdot \text{Exp}[-(a/\rho^2 + b\rho^2)] \cdot [\sum_{i=1}^n (y_i - \bar{y})^2]^{-(n-1)/2} \quad (3.2.10)$$

The posterior density of σ can be obtained by integrating equation 3.2.9 over ρ .

$$\pi(\sigma | \mathbf{X}) \propto \int_0^\infty \prod_{i=1}^n (\rho + 1/x_i) \text{Exp}[-(a/\rho^2 + b\rho^2)] \sigma^{-n} \text{Exp}[-\sum_{i=1}^n (y_i - \bar{y})^2 / 2\sigma^2] d\rho \quad (3.2.11)$$

It is not possible to perform this integration analytically. It is therefore necessary to use numerical methods.

The joint posterior density of μ and ρ can be obtained by integrating equation 3.2.6 over σ .

$$p(\rho, \mu | \mathbf{X}) \propto \prod_{i=1}^n (\rho + \frac{1}{x_i}) \text{Exp}[-\frac{a}{\rho^2} - b\rho^2] (\text{Exp}[-\sum_{i=1}^n (\rho x_i + \ln \rho x_i - \mu)^2])^{-n} \quad (3.2.12)$$

The marginal posterior density of μ can be obtained by integrating equation 3.2.12 over ρ . It is again impossible to perform this integration analytically. Again numerical methods have to be used.

The predictive density is used to "predict" a future or past missing observation x' based on data already acquired $\underline{x}$. This density is obtained when the joint posterior density of ρ , μ and σ times the probability of x' given ρ , μ and σ and is integrated over all parameters. The predictive density of a new observation x' is then:

$$p(x'|\underline{x}) \propto \int_0^\infty \int_{-\infty}^\infty \int_0^\infty p(\rho, \mu, \sigma | \underline{x}) * f(x' | \rho, \mu, \sigma) d\rho d\mu d\sigma.$$

Inserting the expressions for $p(\rho, \mu, \sigma | \underline{x})$ and $p(x' | \rho, \mu, \sigma)$ gives:

$$p(x'|\underline{x}) \propto \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \sigma^{-(n+1)} \prod_{i=1}^n \left(\rho + \frac{1}{X_i}\right) \text{Exp}\left[-\sum_{i=1}^n (\rho X_i + \ln \rho X_i - \mu)^2 / 2\sigma^2\right] \\ \text{Exp}\left[\left(\frac{-a}{\rho^2} - b\rho^2\right)\right] \frac{1}{\sigma} \left(\rho + \frac{1}{X'}\right) \text{Exp}\left[-(\ln \rho X' + \rho X' - \mu)^2 / 2\sigma^2\right] d\rho d\mu d\sigma.$$

If we define X to include x' and $\underline{x}$ such that $X = \{x_1 \dots x_i \dots x_n, x'\}$ with $n+1$ elements, $p(x' | \underline{x})$ can be written:

$$p(x'|\underline{x}) \propto \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \sigma^{-(n+2)} \prod_{i=1}^{n+1} \left(\rho + \frac{1}{X_i}\right) \text{Exp}\left[-\sum_{i=1}^{n+1} (\rho X_i + \ln \rho X_i - \mu)^2 / 2\sigma^2\right] \\ \text{Exp}\left[\left(\frac{-a}{\rho^2} - b\rho^2\right)\right] d\rho d\mu d\sigma. \quad (3.2.13)$$

Equation 3.2.13 is of the same form as equation 3.2.7, therefore

$$\pi(x'|\underline{x}) \propto \int_0^\infty \prod_{i=1}^{n+1} \left(\rho + \frac{1}{X_i}\right) * \text{Exp}\left[-(a/\rho^2 + b\rho^2)\right] * \left[\sum_{i=1}^{n+1} (Y_i - \bar{Y})^2\right]^{-n/2} d\rho \quad (3.2.14)$$

Here

$$Y_i = \rho X_i + \ln \rho X_i$$

$$\bar{Y} = \sum_{i=1}^{n+1} \frac{Y_i}{n+1}$$

$\pi(x' | \underline{x})$ can now be obtained by integrating (3.2.14) numerically over ρ .

3.3 APPLYING THE HYBRID LOG NORMAL MODEL TO OCCUPATIONAL EXPOSURE / DOSE DATA

We have obtained marginal posterior densities for the parameters ρ , μ and σ along with predictive densities of new observations with the use of computer programs. All computer programming was done in "Mathematica" [9]. Examples of these programs are shown in Appendix B. To aid in performing the numerical integrations contour plots of joint posterior densities were obtained. These contour plots indicate the ranges of both parameters over which the mass of the joint posterior density is concentrated. Numerical integrations were done with two independent methods; the Gauss - Kronrod method used in "NIntegrate" [9] and the Gauss - Legendre method. Identical results were obtained with both methods.

Figure 3.3.1 through Figure 3.3.3 shows the marginal posterior density of ρ , the marginal posterior density of μ , and the marginal posterior density of σ , respectively, using lung burden data from Data Set 1. Figure 3.3.4 shows the predictive density of a future or missing lung burden measurement. The marginal posterior densities and the predictive densities for the other data are shown in Appendix C. Zero doses were eliminated from all data sets due to the $1/x$ term in the marginal posterior density equations.

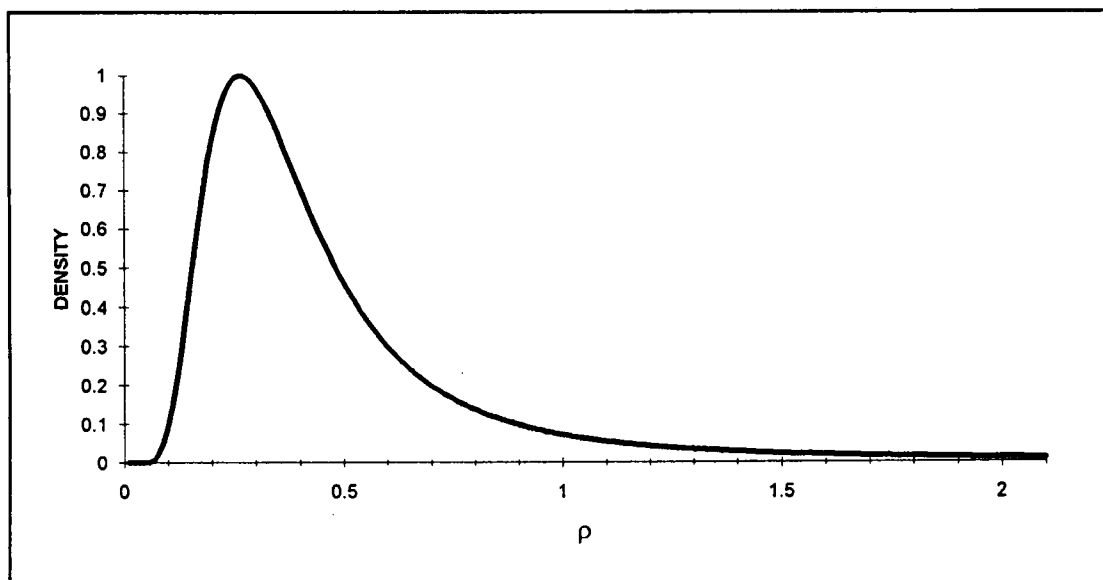


Figure 3.3.1 The marginal posterior density of ρ using lung burden data.

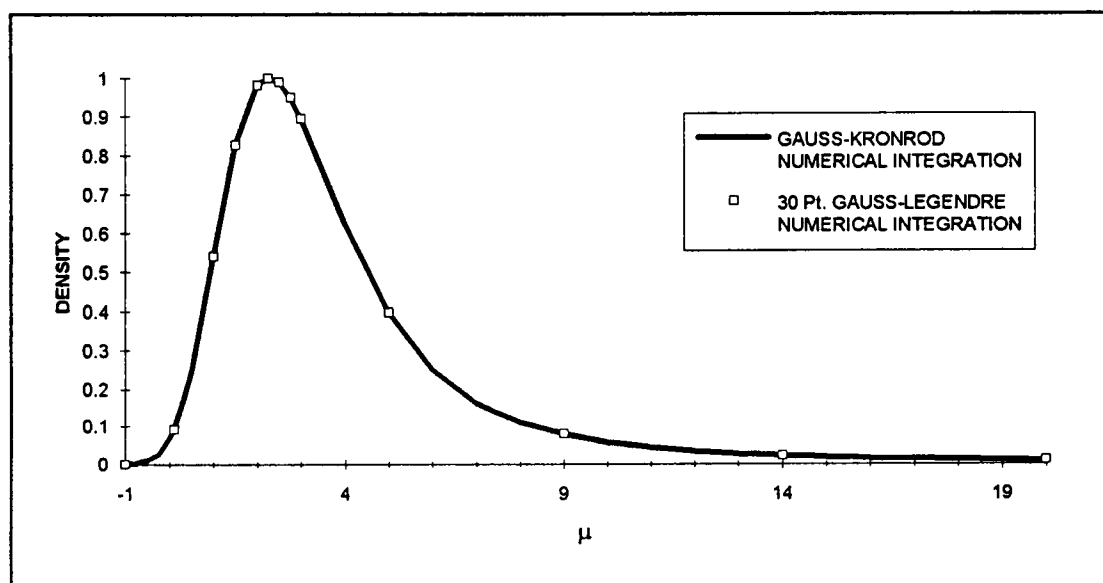


Figure 3.3.2 The marginal posterior density of μ using lung burden data.

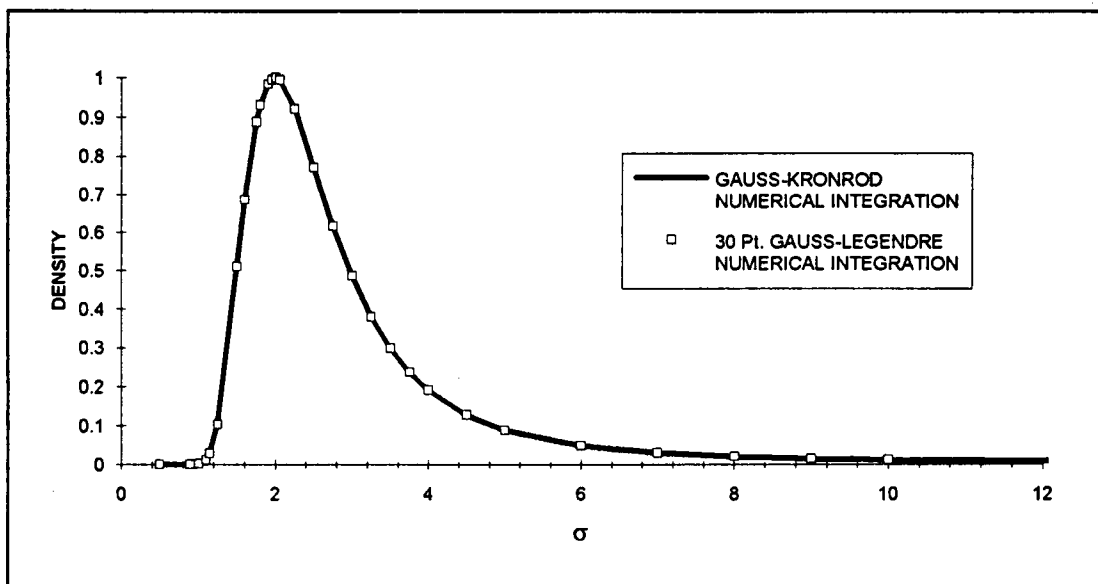


Figure 3.3.3 The marginal posterior density of σ using lung burden data.

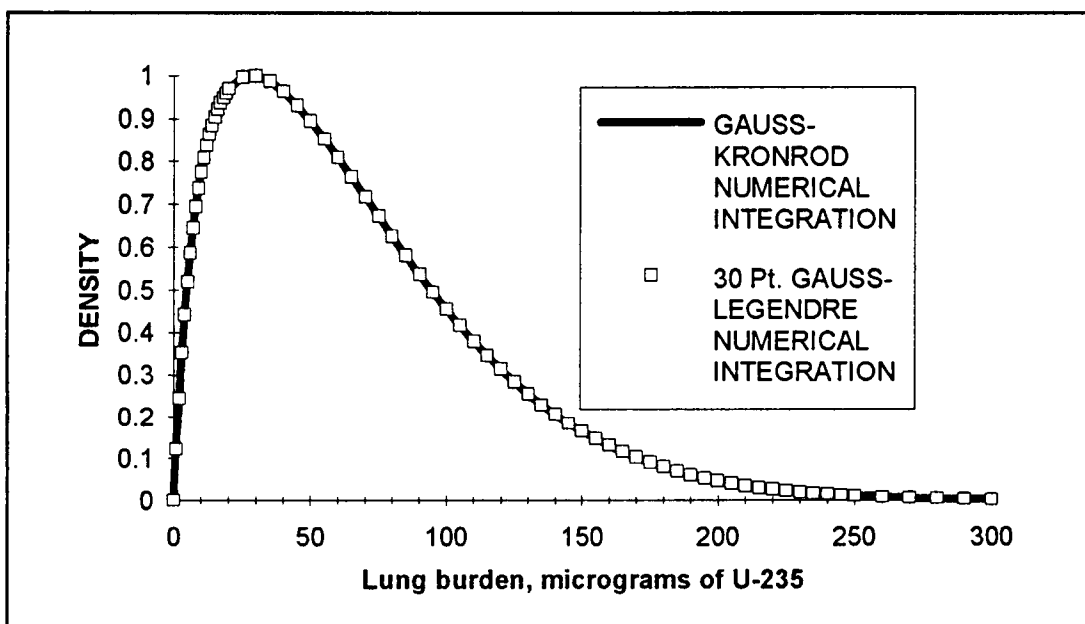


Figure 3.3.4 The predictive density of a new lung burden observation.

CHAPTER 4

DISCUSSION

We have presented a method to estimate missing ^{235}U lung burden measurements given a calibration data set that consists of concurrent urinalysis (total amounts of uranium in $\mu\text{g}/\text{litre}$) and ^{235}U lung burden measurements (micrograms of ^{235}U). Bayesian methods have been applied to the bivariate normal model to obtain a predictive density of the lung burden conditional on a given urinalysis result and the calibration data. Some epidemiological studies investigate the biological effects of occupational radiation exposures where doses are never precisely known. The predictive density derived here describes the remaining uncertainty about missing lung burden measurements in a quantitative fashion and should be used in a rigorous analysis of epidemiological data with uncertain exposures. We have also compared this predictive density to the Bayesian linear regression technique performed by Dolan [1]. The predictive density using the bivariate normal model is more realistic since it considers a distribution for the urinalysis data.

We also used Bayesian procedures for the hybrid log normal model to obtain the marginal posterior densities of the model parameters ρ , μ , and σ . Again, these marginal posterior densities represent the

remaining uncertainty about these parameters quantitatively and are an improvement over published parameter estimation techniques for this distribution [7].

According to Groer and Uppuluri [8], if $p(\rho | \underline{x})$ concentrates its mass at values of $\rho < 1$ or $\rho > 1$, this indicates that the distribution of dose is predominantly log normal or normal, respectively. According to Kumazawa [7], occupational dose distributions from NRC licensed facilities are predominantly normal in the range approaching the occupational dose limit. This "limit effect" indicates that individual doses in this range do not deviate much from the anticipated values due to the fact that workers likely to receive larger doses are monitored more carefully and at more frequent intervals. It is therefore reasonable to suggest that the posterior density of ρ can be used to indicate the degree of this limit effect for exposure situations similar to those analyzed by Kumazawa [7]. It is also reasonable to suggest that the posterior density of ρ indicates compliance with the principle of ALARA for a dose distribution that: 1) fits the hybrid log normal model and 2) concentrates its mass at low values of dose and 3) has negligible mass as the dose approaches the limit from below.

We also derived for the hybrid log normal model the predictive density of a new observation. This density is an improvement over the classical approach for inferring missing doses since the remaining

uncertainty about the inferred dose is characterized in a quantitative probabilistic fashion.

LIST OF REFERENCES

LIST OF REFERENCES

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APPENDIXES

APPENDIX A

**Data set 1: Urinalysis and lung burden values
obtained in the same month.**

TABLE A. DATA SET 1: URINALYSIS AND LUNG BURDEN VALUES OBTAINED
IN THE SAME MONTH

OBSERVATION	URINALYSES microgram/Liter	LUNG BURDEN microgram U-235
1	17	27
2	7	19
3	3	0
4	1	97
5	10	5
6	10	0
7	0	28
8	8	129
9	1	112
10	1	70
11	21	125
12	5	79
13	4	95
14	20	98
15	7	102
16	7	102
17	10	114
18	10	33
19	6	40
20	1	14
21	9	58
22	16	101
23	8	0
24	18	116
25	55	116
26	4	2.8
27	7	30
28	11	40
29	18	222
30	11	78
31	39	154
32	4	89
33	10	32
34	7	33
35	3	33
36	6	33
37	3	92
38	28	33
39	18	0
40	7	0
41	11	57
42	16	93
43	8	28
44	11	0
45	10	29
46	15	17
47	7	25
48	1	0
49	29	59
50	8	91
51	18	89
52	3	67.2
53	5	38

TABLE A (CONTINUED). DATA SET 1: URINALYSIS AND LUNG BURDEN
VALUES OBTAINED IN THE SAME MONTH

OBSERVATION	URINALYSES microgram/Liter	LUNG BURDEN microgram U-235
54	18	11
55	11	11
56	3	161
57	1	152
58	5	0
59	12	71
60	18	42
61	3	110
62	3	164
63	9	95
64	6	97
65	29	73
66	14	112
67	26	52
68	9	32.65
69	7	44.1
70	11	17
71	4	17
72	5	44
73	17	24
74	9	73
75	8	73
76	2	27
77	26	49
78	19	49
79	6	49
80	17	18
81	7	46
82	7	72
83	16	157
84	16	85
85	19	102
86	17	124
87	23	65
88	4	23
89	10	116
90	7	38
91	6	2
92	5	83
93	2	24
94	11	24
95	36	123
96	7	0
97	16	0
98	12	0
99	7	43
100	5	58.34
101	3	28
102	5	0

APPENDIX B
EXAMPLES OF COMPUTER PROGRAMS

APPENDIX B.1

A PROGRAM USED TO OBTAIN THE POSTERIOR DENSITY OF LUNG BURDEN GIVEN A URINALYSIS RESULT OF 80 MICROGRAMS/ LITRE USING BAYESIAN LINEAR REGRESSION ANALYSIS.

```
x={27.,19.,97.,5.,0,28.,129.,112.,70.,125.,79.,95.,98.,102.,102.,
114.,33.,40.,14.,58.,101.,0,116.,116.,2.8,30.,40.,222.,78.,154.,
89.,32.,33.,33,33.,92.,33.,0,0,57.,93.,28.,0,29.,17.,25.,0,59.,91.,
89.,67.2,38.,11.,11.,161.,152.,0,71.,42.,110.,164.,95.,97.,73.,
112.,52.,32.65,44.1,17.,17.,44.,24.,73.,73.,27.,49.,49.,49.,18.,
46.,72.,157.,85.,102.,124.,65.,23.,116.,38.,2.,83.,24.,24.,123.,
0,0,0,43.,58.34,28.,0};
```

```
y={17.,7.,3.,1.,10.,10.,8.,1.,1.,21.,5.,4.,20.,7.,7.,10.,10.,6.,1.,
9.,16.,8.,18.,5.,4.,7.,11.,18.,11.,39.,4.,10.,7.,3.,6.,3.,28.,18.,7.,
11.,16.,8.,11.,10.,15.,7.,1.,29.,8.,18.,3.,5.,18.,11.,3.,1.,5.,12.,18.,
3.,3.,9.,6.,29.,14.,26.,9.,7.,11.,4.,5.,17.,9.,8.,2.,26.,19.,6.,17.,7.,7.,
16.,16.,19.,17.,23.,4.,10.,7.,6.,5.,211.,36.,7.,16.,12.,7.,5.,3.,5.};
```

```
Uo=80;
```

```
b1=(Length[x]*Apply[Plus,x*y]-Apply[Plus,x]*Apply[Plus,y])/
(Length[x]*Apply[Plus,y^2]-(Apply[Plus,y])^2);
```

```
bo=(Apply[Plus,x]-b1*Apply[Plus,y])/Length[x];
```

```
c=Sqrt[1+1/Length[x]+((Uo-Apply[Plus,y]/Length[y])^2/
Apply[Plus,(y-(Apply[Plus,y]/Length[y]))^2]);
```

```
s=Sqrt[Apply[Plus,(x-(Apply[Plus,x]/Length[x]))^2]/Length[x];
```

```
Do[Print[FortranForm[N[1/(1+((L-bo-Uo*b1)/(s*c))^2/(Length[x]-1))^((Le
ngth[x]+1)/2)]]],{L,1,400,10}]
```

APPENDIX B.2

A PROGRAM USED TO OBTAIN THE POSTERIOR DENSITY OF LUNG BURDEN GIVEN A URINALYSIS RESULT OF 80 MICROGRAMS/ LITRE USING THE BIVARIATE NORMAL MODEL.

```
x={27.,19.,97.,5.,0,28.,129.,112.,70.,125.,79.,95.,98.,102.,102.,
114.,33.,40.,14.,58.,101.,0,116.,116.,2.8,30.,40.,222.,78.,154.,89.,
32.,33.,33,33.,92.,33.,0,0,57.,93.,28.,0,29.,17.,25.,0,59.,91.,89.,
67.2,38.,11.,11.,161.,152.,0,71.,42.,110.,164.,95.,97.,73.,112.,
52.,32.65,44.1,17.,17.,44.,24.,73.,73.,27.,49.,49.,49.,18.,46.,72.,
157.,85.,102.,124.,65.,23.,116.,38.,2.,83.,24.,24.,123.,0,0,0,43.,
58.34,28.,0};
```

```
y={17.,7.,3.,1.,10.,10.,8.,1.,1.,21.,5.,4.,20.,7.,7.,10.,10.,6.,1.,
9.,16.,8.,18.,5.,4.,7.,11.,18.,11.,39.,4.,10.,7.,3.,6.,3.,28.,18.,7.,
11.,16.,8.,11.,10.,15.,7.,1.,29.,8.,18.,3.,5.,18.,11.,3.,1.,5.,12.,18.,
3.,3.,9.,6.,29.,14.,26.,9.,7.,11.,4.,5.,17.,9.,8.,2.,26.,19.,6.,17.,7.,7.,
16.,16.,19.,17.,23.,4.,10.,7.,6.,5.,211.,36.,7.,16.,12.,7.,5.,3.,5.};
```

```
n=Length[x];
m1=Apply[Plus,x]/Length[x];
m2=Apply[Plus,y]/Length[y];
s11=Apply[Plus,(x-m1)^2];
s22=Apply[Plus,(y-m2)^2];
s12=Apply[Plus,(x-m1)*(y-m2)];
s21=Apply[Plus,(x-m1)*(y-m2)];
a=m1+(s12/s22)*(U-m2);
b=n*(n-2)*s22/((n+1)*(s11*s22-s12*s21));
c=n-2;
U=80;
```

```
Do[Print[FortranForm[N[(c+b*(L-a)^2)^(-.5*(c+1))]],
{L,-80,380,20}]
```

APPENDIX B.3

A PROGRAM USED TO OBTAIN THE POSTERIOR DENSITY OF RHO
USING LUNG BURDEN DATA.

```
x={27.,19.,97.,5.,28.,129.,112.,70.,125.,79.,95.,98.,102.,102.,114.,3  
3.,40.,14.,58.,101.,116.,116.,2.8,30.,40.,222.,78.,154.,89.,32.,33.,33,  
33.,92.,33.,57.,93.,28.,29.,17.,25.,59.,91.,89.,67.2,38.,11.,11.,1  
61.,152.,71.,42.,110.,164.,95.,97.,73.,112.,52.,32.65,44.1,17.,17.,44.  
,24.,73.,73.,27.,49.,49.,49.,18.,46.,72.,157.,85.,102.,124.,65.,23.,116  
,38.,2.,83.,24.,24.,123.,43.,58.34,28.};
```

```
a=1/100;
```

```
b=1/100;
```

```
f[r_]=Apply[Times,r+10/x]*Exp[-a/r^2-b*r^2]*  
  (Apply[Plus,(r*x+Log[r*x/10]-Apply[Plus,  
    (r*x/10+Log[r*x/10])/Length[x]])^2])^(-Length[x]/2 +1/2);
```

```
Do[Print[FortraForm[N[[f[r]]]],{r,.01,30,.1}]
```

APPENDIX B.4

A PROGRAM USED TO OBTAIN THE POSTERIOR DENSITY OF MU USING LUNG BURDEN DATA.

THE JOINT DENSITY OF ρ AND μ .

```
x={27.,19.,97.,5.,28.,129.,112.,70.,125.,79.,95.,98.,102.,102.,114.,3
3.,40.,14.,58.,101.,116.,116.,2.8,30.,40.,222.,78.,154.,89.,32.,33.,33,
33.,92.,33.,57.,93.,28.,29.,17.,25.,59.,91.,89.,67.2,38.,11.,11.,1
61.,152.,71.,42.,110.,164.,95.,97.,73.,112.,52.,32.65,44.1,17.,17.,44.
,24.,73.,73.,27.,49.,49.,49.,18.,46.,72.,157.,85.,102.,124.,65.,23.,116
.,38.,2.,83.,24.,24.,123.,43.,58.34,28.};
```

```
a=1/100;
```

```
b=1/100;
```

```
f[r_,u_]=Exp[-a/r^2-b*r^2]*10^148*
Apply[Times,r+10/x]*
(Apply[Plus,((r*x/10)+Log[r*x/10]-u)^2])^(-Length[x]/2);
ContourPlot[f[r,u],{r,.01,34},{u,-1,200},PlotRange->All,
ContourShading->False]
```

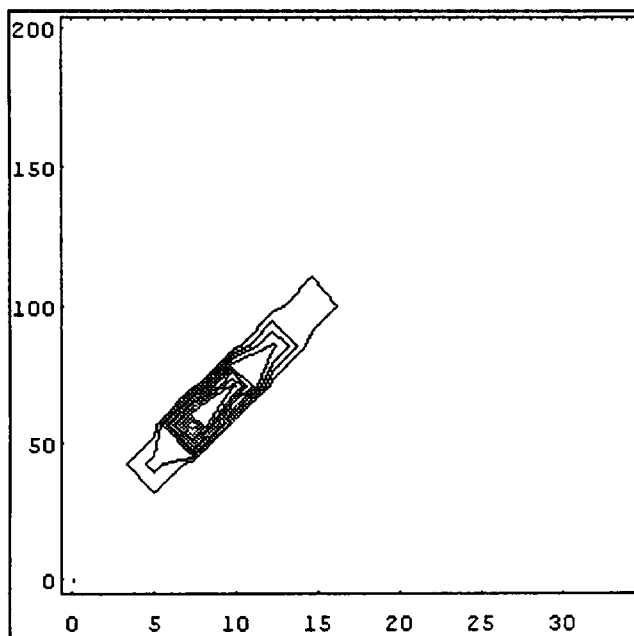


Figure B.1 Constant density contours of the joint density of ρ (horizontal axis) and μ (vertical axis) using lung burden data.

THE NUMERICAL INTEGRATION OF THE JOINT DENSITY OVER RHO

GAUSS - KRONROD METHOD

```
x={27.,19.,97.,5.,28.,129.,112.,70.,125.,79.,95.,98.,102.,102.,114.,3
3.,40.,14.,58.,101.,116.,116.,2.8,30.,40.,222.,78.,154.,89.,32.,33.,33,
33.,92.,33.,57.,93.,28.,29.,17.,25.,59.,91.,89.,67.2,38.,11.,11.,1
61.,152.,71.,42.,110.,164.,95.,97.,73.,112.,52.,32.65,44.1,17.,17.,44.
,24.,73.,73.,27.,49.,49.,49.,18.,46.,72.,157.,85.,102.,124.,65.,23.,116
.,38.,2.,83.,24.,24.,123.,43.,58.34,28.};
```

```
a=1/100;
```

```
b=1/100;
```

```
f[r_,u_]=Exp[-a/r^2-b*r^2]*Apply[Times,r+10/x]*
      (Apply[Plus,(r*x/10+Log[r*x/10]-u)^2])^(-Length[x]/2);
```

```
Do[Print[FortranForm[N[NIntegrate[f[r,u],{r,.1,34}]]], {u,-1,200,2}]
```

30 POINT GAUSS - LEGENDRE METHOD

```
x={27.,19.,97.,5.,28.,129.,112.,70.,125.,79.,95.,98.,102.,102.,114.,3
3.,40.,14.,58.,101.,116.,116.,2.8,30.,40.,222.,78.,154.,89.,32.,33.,33,
33.,92.,33.,57.,93.,28.,29.,17.,25.,59.,91.,89.,67.2,38.,11.,11.,1
61.,152.,71.,42.,110.,164.,95.,97.,73.,112.,52.,32.65,44.1,17.,17.,44.
,24.,73.,73.,27.,49.,49.,49.,18.,46.,72.,157.,85.,102.,124.,65.,23.,116
.,38.,2.,83.,24.,24.,123.,43.,58.34,28.};
```

```
a=1/100;
```

```
b=1/100;
```

```
f[r_]=Exp[-a/r^2-b*r^2]* Apply[Times,r+10/x]*
      (Apply[Plus,(r*x/10+Log[r*x/10]-u)^2])^(-Length[x]/2);
```

```
m=GaussianQuadratureWeights[30,1,34]
```

```
t=Transpose[m];
```

```
n=t[[1]];
```

```
w=t[[2]];
```

```
Do[Print[FortranForm[N[Apply[Plus,f[n]*w ] ]],{u,-1,200,10}]
```

APPENDIX B.5

A PROGRAM USED TO OBTAIN THE POSTERIOR DENSITY OF SIGMA
USING LUNG BURDEN DATA.

THE JOINT DENSITY OF RHO AND SIGMA

```
x={27.,19.,97.,5.,28.,129.,112.,70.,125.,79.,95.,98.,102.,102.,114.,3  
3.,40.,14.,58.,101.,116.,116.,2.8,30.,40.,222.,78.,154.,89.,32.,33.,33,  
33.,92.,33.,57.,93.,28.,29.,17.,25.,59.,91.,89.,67.2,38.,11.,11.,1  
61.,152.,71.,42.,110.,164.,95.,97.,73.,112.,52.,32.65,44.1,17.,17.,44.  
,24.,73.,73.,27.,49.,49.,49.,18.,46.,72.,157.,85.,102.,124.,65.,23.,116  
,38.,2.,83.,24.,24.,123.,43.,58.34,28.};
```

```
a=1/100;  
b=1/100;  
f[r_,s_]=s^(-Length[x])*  
Apply[Times,r+10/x]*  
Exp[-a/r^2-b*r^2]*  
Exp[(-1/(2*s^2))*Apply[Plus,(r*x/10+Log[r*x/10]-  
Apply[Plus,(r*x+Log[r*x/10])/Length[x]])^2]];  
ContourPlot[10^74*f[r,s],{r,.05,1.25},{s,.5,5},ContourShading->False,  
PlotRange->All]
```

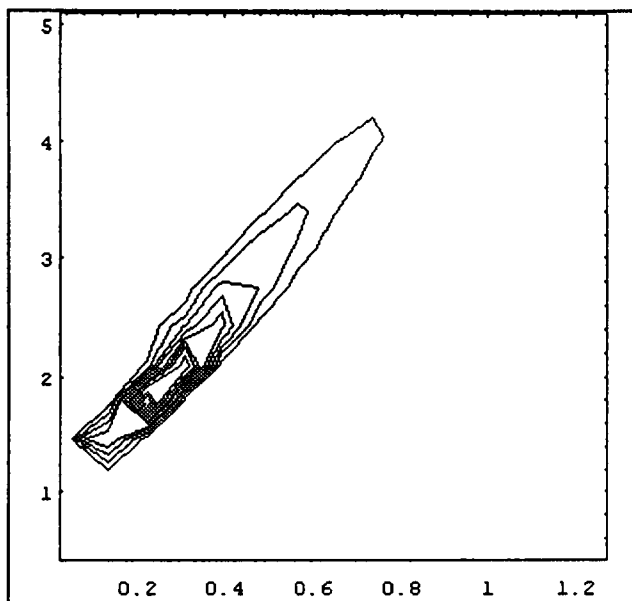


Figure B.2 Constant density contours of the joint density of rho (horizontal axis) and sigma (vertical axis) using lung burden data.

NUMERICAL INTEGRATION OF THE JOINT DENSITY OVER RHO

GAUSS - KRONROD METHOD

```
x={27.,19.,97.,5.,28.,129.,112.,70.,125.,79.,95.,98.,102.,102.,114.,3
3.,40.,14.,58.,101.,116.,116.,2.8,30.,40.,222.,78.,154.,89.,32.,33.,33,
33.,92.,33.,57.,93.,28.,29.,17.,25.,59.,91.,89.,67.2,38.,11.,11.,1
61.,152.,71.,42.,110.,164.,95.,97.,73.,112.,52.,32.65,44.1,17.,17.,44.
,24.,73.,73.,27.,49.,49.,49.,18.,46.,72.,157.,85.,102.,124.,65.,23.,116
.,38.,2.,83.,24.,24.,123.,43.,58.34,28.};
```

```
a=1/100;
```

```
b=1/100;
```

```
f[r_,s_]=s^(-Length[x])*10^74*
Apply[Times,r+10/x]*
Exp[-a/r^2-b*r^2]*
Exp[(-1/(2*s^2))*Apply[Plus,(r*x/10+Log[r*x/10]-
Apply[Plus,(r*x/10+Log[r*x/10])/Length[x]])^2]];
```

```
Do[Print[FortranForm[N[NIntegrate[f[r,s],{r,.05,1.25}] ]],{s,.5,5,.1}]
```

30 POINT GAUSS - LEGENDRE METHOD

```
x={27.,19.,97.,5.,28.,129.,112.,70.,125.,79.,95.,98.,102.,102.,114.,3
3.,40.,14.,58.,101.,116.,116.,2.8,30.,40.,222.,78.,154.,89.,32.,33.,33,
33.,92.,33.,57.,93.,28.,29.,17.,25.,59.,91.,89.,67.2,38.,11.,11.,1
61.,152.,71.,42.,110.,164.,95.,97.,73.,112.,52.,32.65,44.1,17.,17.,44.
,24.,73.,73.,27.,49.,49.,49.,18.,46.,72.,157.,85.,102.,124.,65.,23.,116
.,38.,2.,83.,24.,24.,123.,43.,58.34,28.};
```

```
a=1/100;
```

```
b=1/100;
```

```
f[r_]=s^(-Length[x])*10^74*
Apply[Times,r+10/x]*
Exp[-a/r^2-b*r^2]*
Exp[(-1/(2*s^2))*Apply[Plus,(r*x/10+Log[r*x/10]-
Apply[Plus,(r*x/10+Log[r*x/10])/Length[x]])^2]];
m=GaussianQuadratureWeights[30,.05,1.25];
```

```
t=Transpose[m];
```

```
n=t[[1]];
```

```
w=t[[2]];
```

```
Do[Print[FortranForm[N[Apply[Plus,f[n]*w] ]],{s,.5,5,.1}]
```

APPENDIX B.6

A PROGRAM USED TO OBTAIN THE PREDICTIVE DENSITY OF A NEW OBSERVATION USING LUNG BURDEN DATA

THE JOINT DENSITY OF RHO AND Y

```
x={.027,.019,.097,.005,.028,.129,.112,.070,.125,.079,.095,
.098,.102,.102,.114,.033,.040,.014,.058,.101,.116,.116,.0028,
.030,.040,.222,.078,.154,.089,.032,.033,.033,.033,.092,.033,
.057,.093,.028,.029,.017,.025,.059,.091,.089,.0672,.038,.011,
.011,.161,.152,.071,.042,.110,.164,.095,.097,.073,.112,.052,
.03265,.0441,.017,.017,.044,.024,.073,.073,.027,.049,.049,
.049,.018,.046,.072,.157,.085,.102,.124,.065,.023,.116,.038,
.002,.083,.024,.024,.123,.043,.05834,.028,y};
a=1/100;
b=1/100;
ContourPlot[(N[10^(-118)*(Apply[Times,r+1/x]])*
N[(10^85*Exp[-a/r^2-b*r^2] *(Apply[Plus,((r*x)+Log[r*x]-
Apply[Plus,((r*x)+Log[r*x])/Length[x]])^2])^(-Length[x]/2+1/2))],
{r,.1,30},{y,.001,.25},PlotRange->All,ContourShading->False]
```

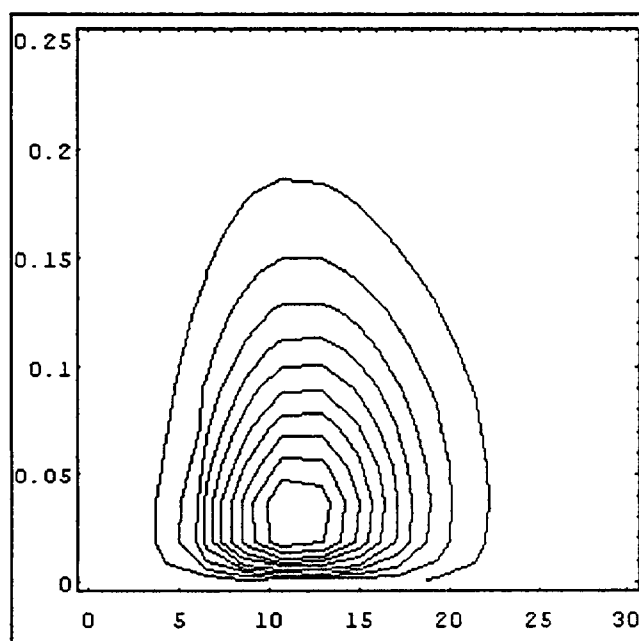


Figure B.3 Constant density contours of the joint density of rho (horizontal axis) and y (vertical axis), a new observation, using lung burden data.

THE NUMERICAL INTEGRATION OF THE JOINT DENSITY OVER RHO

GAUSS - KRONROD METHOD

```
x={.027,.019,.097,.005,.028,.129,.112,.070,.125,.079,.095,.098,
.102,.102,.114,.033,.040,.014,.058,.101,.116,.116,.0028,.030,
.040,.222,.078,.154,.089,.032,.033,.033,.033,.092,.033,.057,
.093,.028,.029,.017,.025,.059,.091,.089,.0672,.038,.011,.011,
.161,.152,.071,.042,.110,.164,.095,.097,.073,.112,.052,.03265,
.0441,.017,.017,.044,.024,.073,.073,.027,.049,.049,.049,.018,
.046,.072,.157,.085,.102,.124,.065,.023,.116,.038,.002,.083,.024,
.024,.123,.043,.05834,.028,y};
```

```
a=1/100;
b=1/100;
```

```
f[r_,y_]=Apply[Times,r+1/x]*
Exp[-a/r^2-b*r^2] *
(Apply[Plus,(r*x+Log[r*x]-
Apply[Plus,(r*x+Log[r*x])/Length[x]])^2])^(-Length[x]/2 +1/2);
```

```
Do[Print[FortranForm[N[NIntegrate[f[r,y],{r,.01,30}]
]],{y,.001,.25,.001}]
```

30 POINT GAUSS - LEGENDRE METHOD

```
x={.027,.019,.097,.005,.028,.129,.112,.070,.125,.079,.095,.098,
.102,.102,.114,.033,.040,.014,.058,.101,.116,.116,.0028,.030,
.040,.222,.078,.154,.089,.032,.033,.033,.033,.092,.033,.057,.093
,.028,.029,.017,.025,.059,.091,.089,.0672,.038,.011,.011,.161,.152,
.071,.042,.110,.164,.095,.097,.073,.112,.052,.03265,.0441,.017,
.017,.044,.024,.073,.073,.027,.049,.049,.049,.018,.046,.072,.157
,.085,.102,.124,.065,.023,.116,.038,.002,.083,.024,.024,.123,.043,
.05834,.028,y};
```

```
a=1/100;
b=1/100;
```

```
f[r_,y_]=10^(-7)*(N[10^(-118)*(Apply[Times,r+1/x]])*
N[(10^85*Exp[-a/r^2-b*r^2] *
(Apply[Plus,((r*x)+Log[r*x]-
Apply[Plus,((r*x)+Log[r*x])/Length[x]])^2])^(-Length[x]/2+1/2))];
```

```
m=GaussianQuadratureWeights[30,.1,30];
```

```
t=Transpose[m];
```

```
n=t[[1]];
w=t[[2]];
```

```
Do[Print[FortranForm[N[Apply[Plus,f[n,y]*w] ]]] ,{y,.001,.3,.005}]
```

APPENDIX C

**THE POSTERIOR AND PREDICTIVE DENSITIES FOR
URINALYSIS RESULTS AND GAMMA EXPOSURES.**

APPENDIX C.1
THE POSTERIOR DAND PREDICTIVE DENSITIES FOR URINALYSIS RESULTS

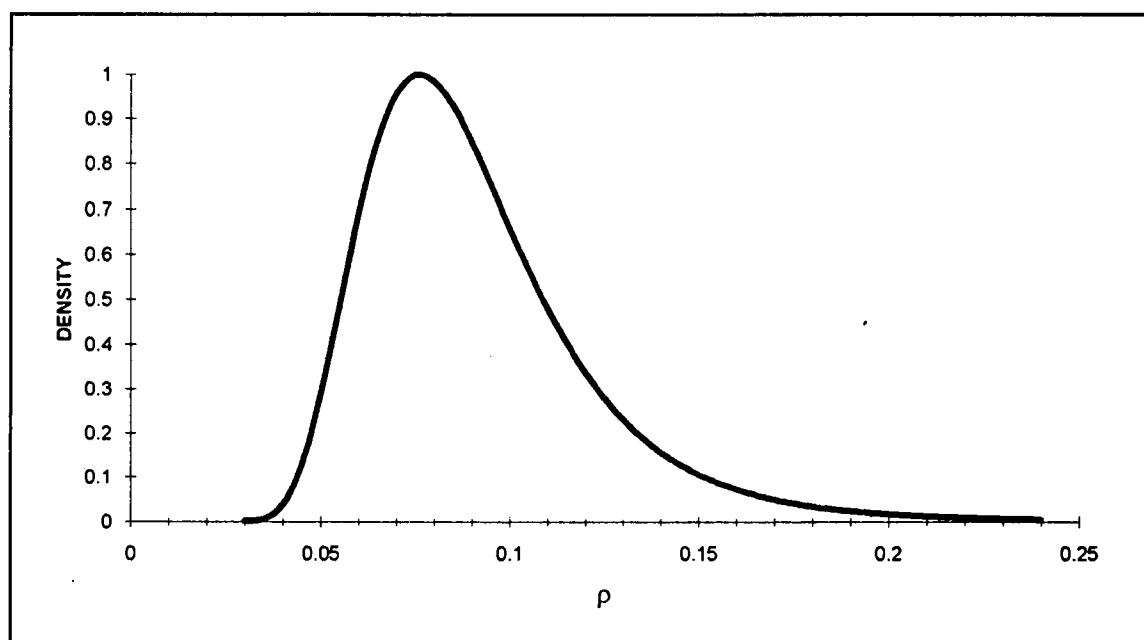


Figure C.1.1 The marginal posterior density of ρ using urinalysis data.

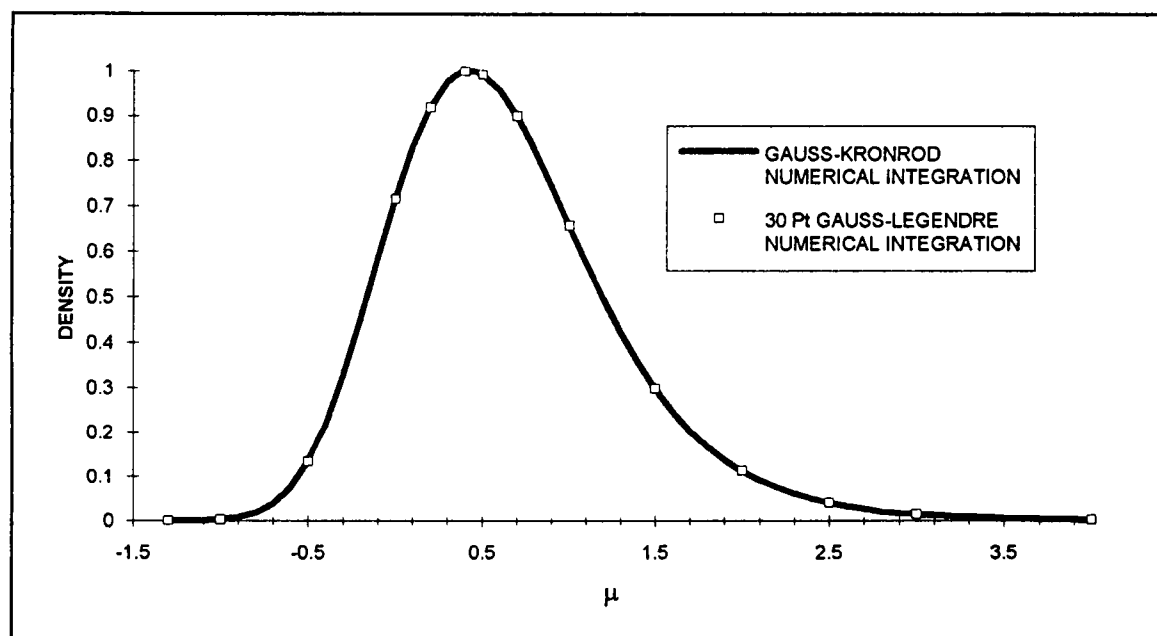


Figure C.1.2 The marginal posterior density of μ using urinalysis data.

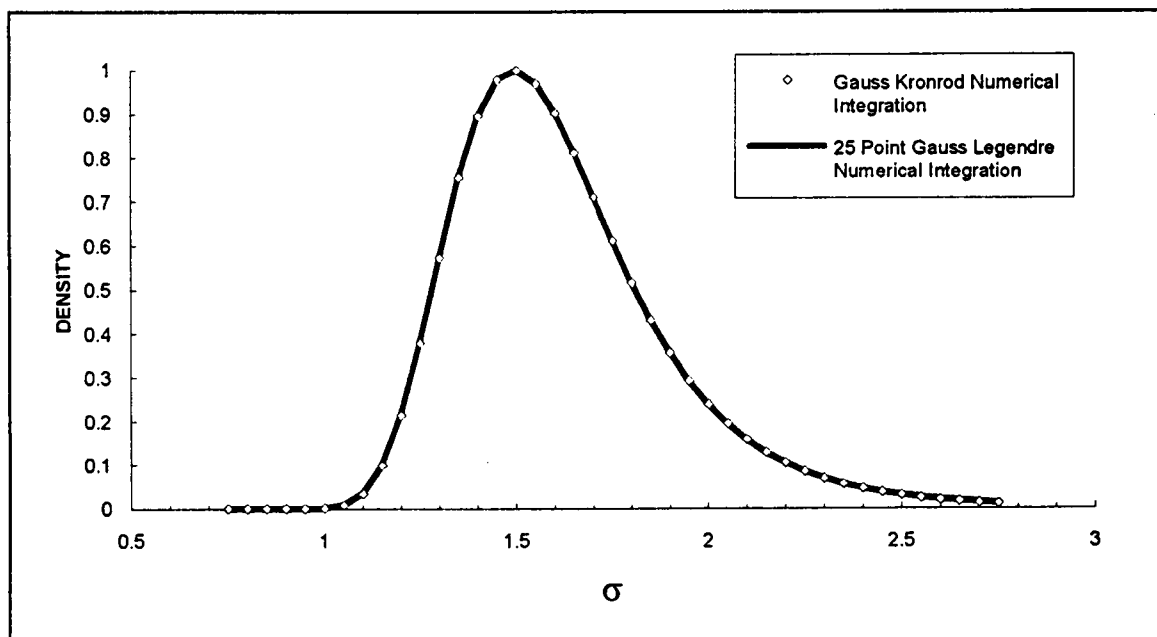


Figure C.1.3 The marginal posterior density of σ using urinalysis data.

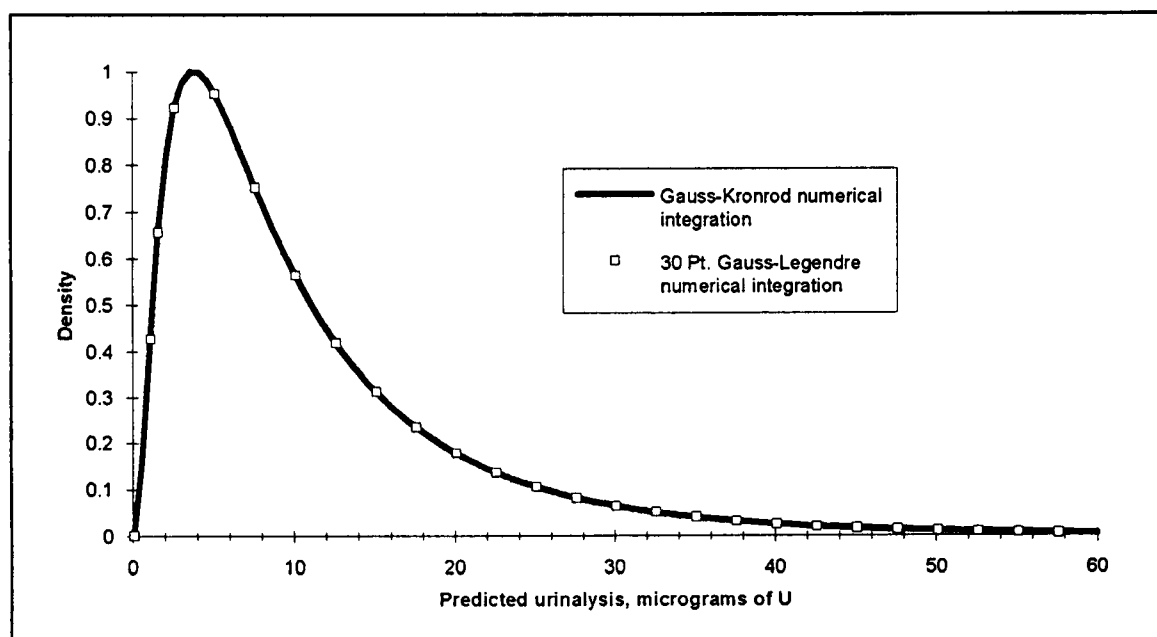


Figure C.1.4 The predictive density of a new urinalysis observation.

APPENDIX C.2

THE POSTERIOR AND PREDICTIVE DENSITIES FOR 1961 GAMMA EXPOSURE DATA.

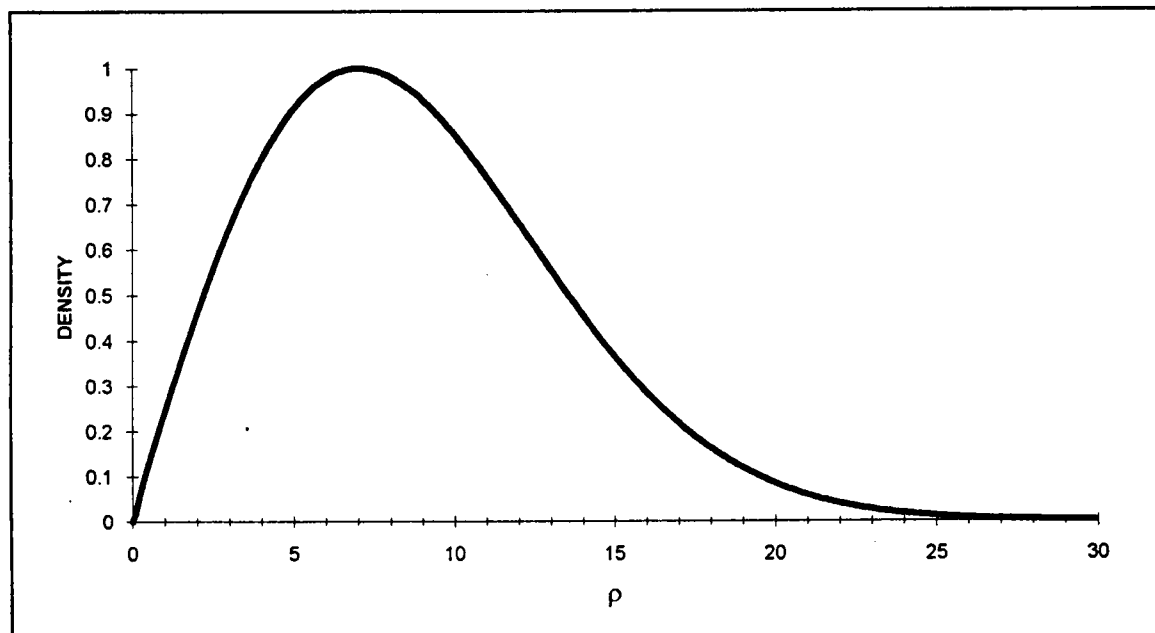


Figure C.2.1 The marginal posterior density of ρ using 1961 gamma exposure data.

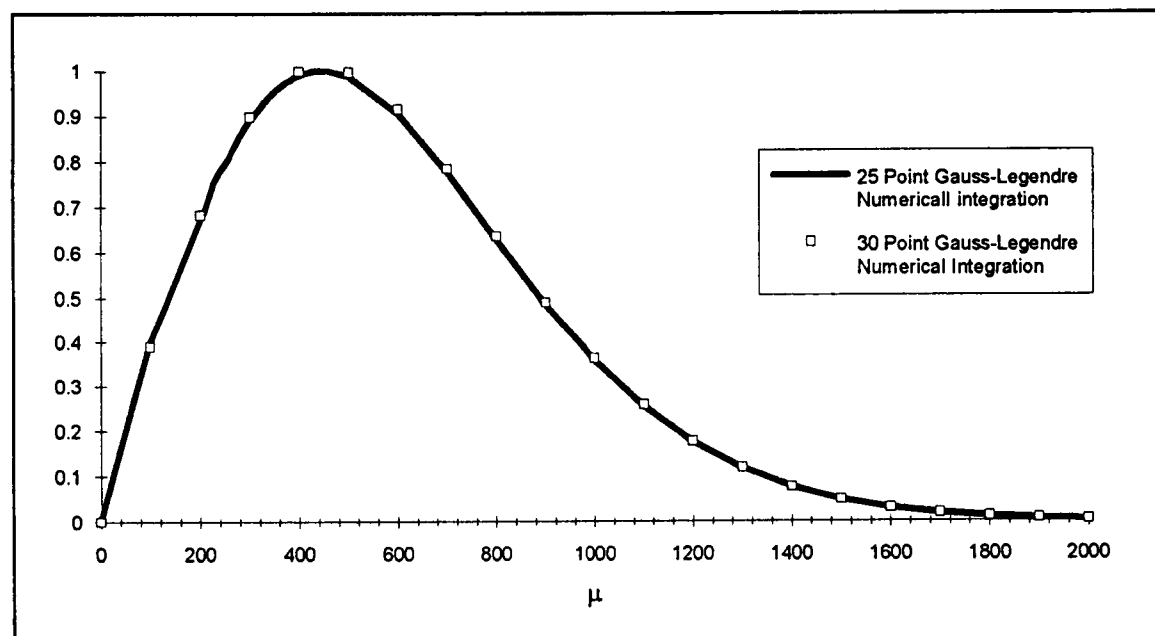


Figure C.2.2 The marginal posterior density of μ using 1961 gamma exposure data.

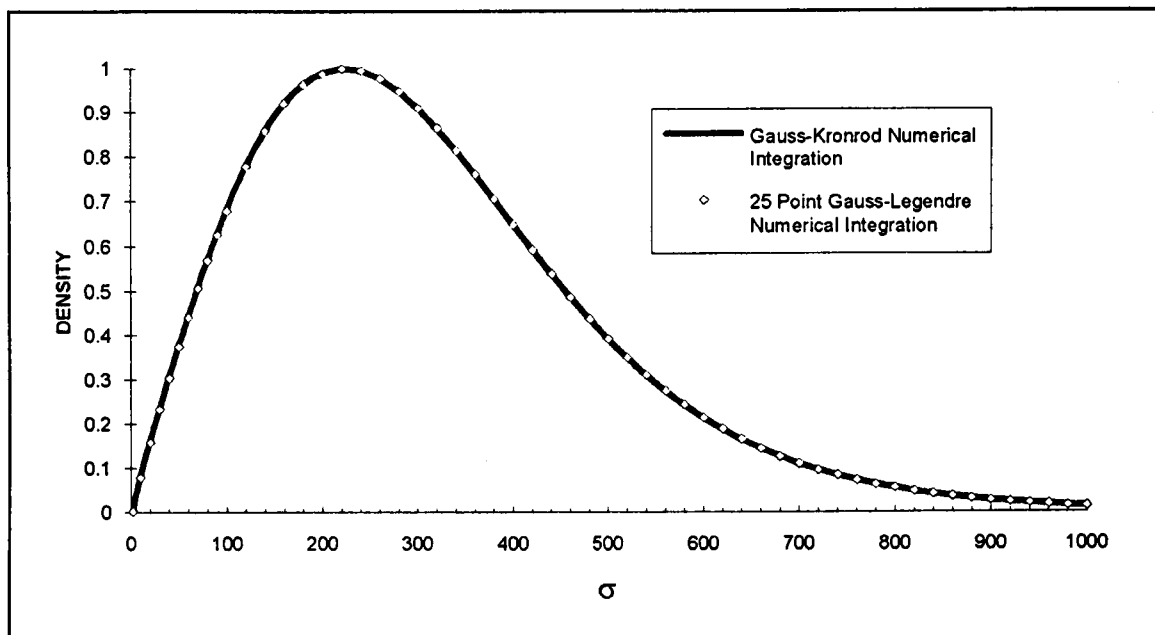


Figure C.2.3 The marginal posterior density of σ using 1961 gamma exposure data.

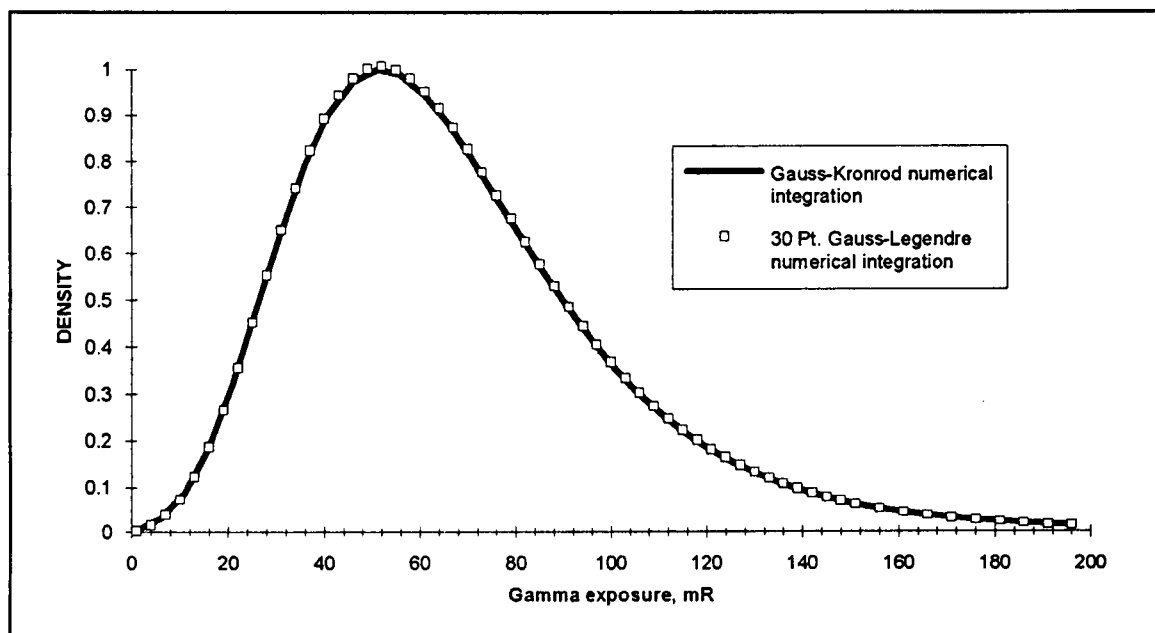


Figure C.2.4 The predictive density of a new observation using 1961 gamma exposure data.

APPENDIX C.3

THE POSTERIOR AND PREDICTIVE DENSITIES FOR 1962 GAMMA EXPOSURE DATA

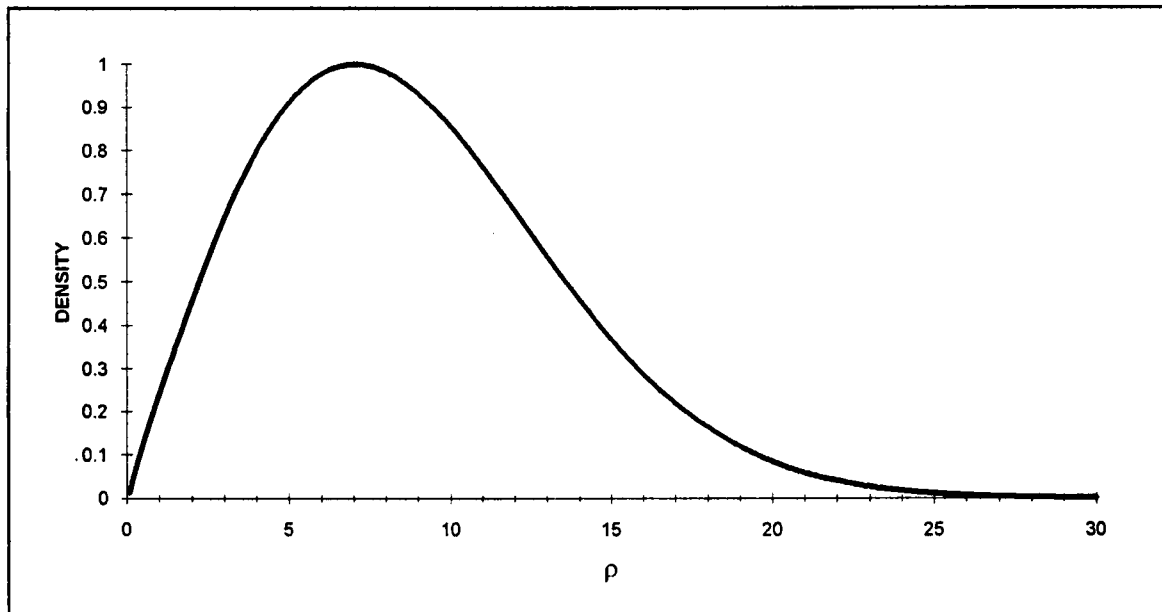


Figure C.3.1 The marginal posterior density of ρ using 1962 gamma exposure data

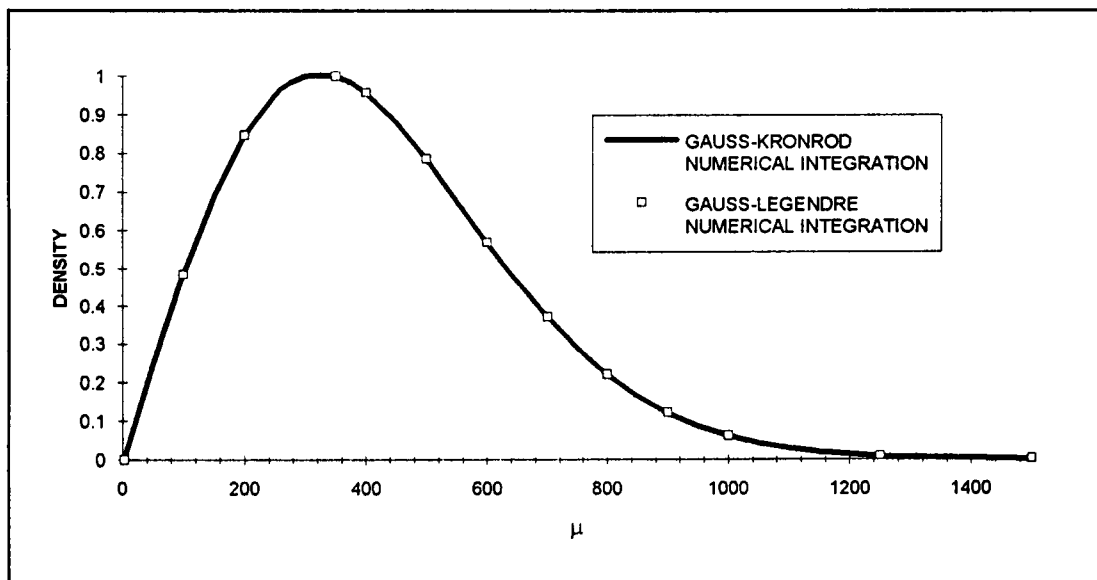


Figure C.3.2 The marginal posterior density of μ using 1962 gamma exposure data

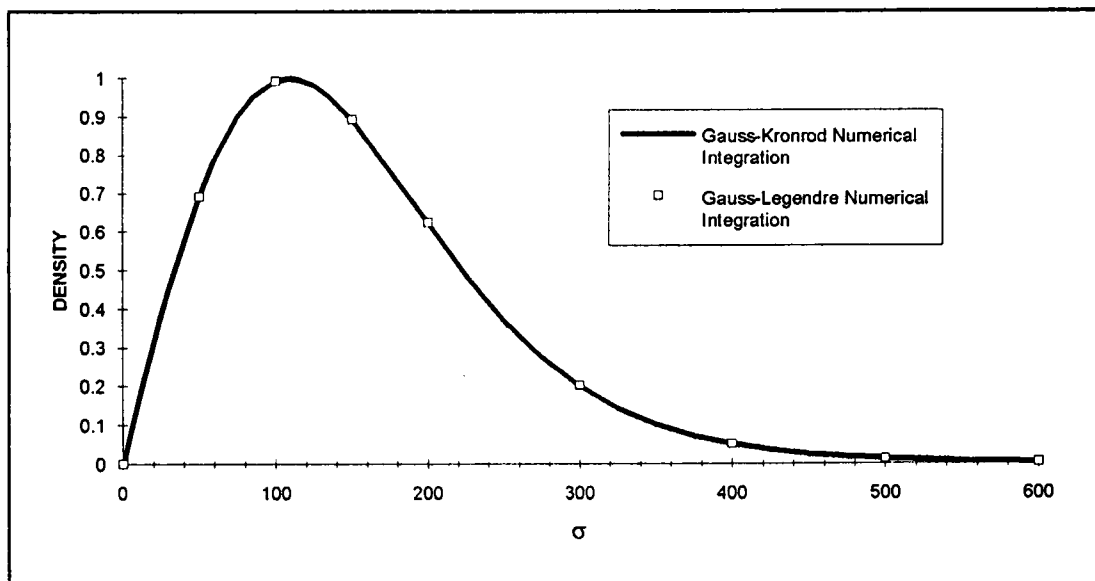


Figure C.3.3 The marginal posterior density of σ using 1962 gamma exposure data.

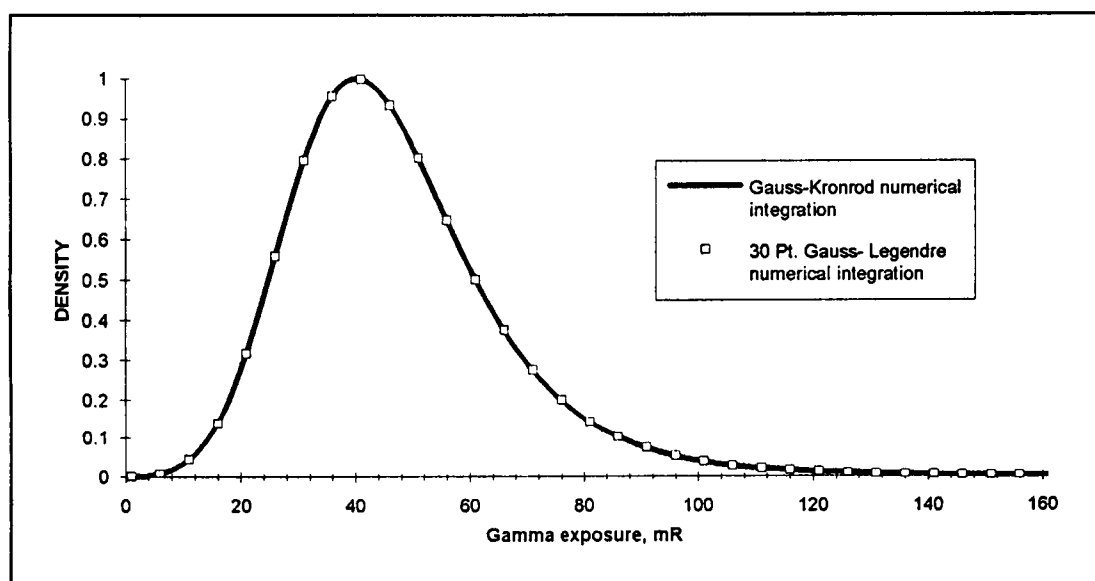


Figure C.3.4 The predictive density of a new observation using 1962 gamma exposure data.

VITA

Robert A. Cooper III was born in Hudson, Wisconsin on January 12, 1967. He attended elementary and high school in Hudson, Wisconsin and was graduated from Hudson High School in May of 1985. The following September he entered the University of Wisconsin at River Falls and in December, 1991 received the degree of Bachelor of Science in Physics. During the following eight months he worked at the Northern States Power- Prairie Island Nuclear Generation Plant as a health physicist as an employee of Bartlett Nuclear Inc. In August 1992, he entered the University of Tennessee, Knoxville and in August, 1994 received a Master of Science degree in Nuclear Engineering.

The author is a member of the Health Physics Society.