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I am submitting herewith a dissertation written by Terry Paul Harrison entitled ''Design and Development of a Decision Support System for Forest Management Planning with Multiple Objectives.'' I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Management Science.

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DESIGN AND DEVELOPMENT OF A DECISION SUPPORT SYSTEM FOR FOREST MANAGEMENT PLANNING WITH MULTIPLE OBJECTIVES

A Dissertation

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Degree

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Terry Paul Harrison

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DEDICATION

The author wishes to dedicate this dissertation to his wife, Debra. She has endured the long hours and emotional drain of this work. Without her support, this research simply would not have been possible.

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ABSTRACT

The purpose of this research was to design, develop, and implement a procedure for forest management planning with multiple objectives. Within the context of this primary objective were the secondary objectives of 1) conducting basic research in mulitobjective optimization, and 2) developing a computer implementation of these results, with a goal of eventual use on microcomputers.

Our research in multiobjective optimization resulted in a new algorithmic approach that links two previously unrelated techniques. We refer to this new technique as an ''implicit/explicit'' approach to multiobjective decision making. It synergistically combines the implicit utility function method with an iterative devlopment of the decision maker's explicit utility function. This results in a robust, interactive algorithm that requires a relatively light information burden of the decision maker. The discussion of the implicit/explicit algorithm concludes with a collection of results supporting its viability.

We have applied the implicit/explicit algorithm in the development of FORMAN (FORest Management plANning system). FORMAN is a computer package designed specifically for forest management planning with multiple objectives, when the decision maker wishes to optimize an unspecified utility function over a linear constraint set.

We conclude with a description of sample FORMAN use; and discuss directions for future research.

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CHAPTER 0

INTRODUCTION

0.1 AN HISTORICAL OVERVIEW OF FOREST MANAGEMENT

Forest management is the formalized application of technical forestry to the operation of a forest property (Meyer [1961]). Implicit in this definition is that the major goal of forest management is to produce most effectively the yield of forest-related benefits, which includes timber production, wildlife, aesthetics, erosion control and others.

Forest management has been practiced since the Middle Ages, when it was initially developed in Western Europe (Smith [1962]). At that time, most land was owned by feudal barons who kept in their employ one or more foresters. The forester's duties included primarily the protection of the forest, along with supervision of harvesting. Over the next few hundred years, the practice of forest management changed from passive to active. That is, the forest manager began to manipulate the forest through harvesting and other activities to achieve a desired forest structure.

The process of manipulating the forest to achieve a particular stand composition has continued to remain more of an art than a science to this day. This is due primarily to the difficulty in predicting the response of the forest to various management practices. For example it is well known that removing a portion of an existing timber stand

(thinning) leads to accelerated diameter growth on the residual stand. However quantifying the amount of diameter growth on individual stems represents a leading edge of research in forest biometrics. Since the early 1930's, studies have been underway to establish quantitative relationships among the various parameters that affect the productivity of forest stands. This capability has continued to evolve, so that in some instances very good quantitative relationships exist that describe a forest's response to various management practices. However this capability does not exist uniformly for all timber benefits in all timber stands.

The types of treatments (also called cultural practices) applied to a forest have also evolved in the recent past. Initially the only way to manipulate the composition of a forest was through harvesting. Today in addition to harvesting, controlled burning, pruning, the use of herbicides, irrigation, planting, and genetic modification are some of the ways in which the structure of a forest is controlled.

While in the past forest management was defined to include only timber production, today forest management deals with achievement of the full spectrum of forest benefits. As a result the considerations facing a forest manager today have greatly expanded over those of managers of only 25 years ago. Forest management has grown from its initial scope of ''how to manipulate the forest to produce timber'' to its current scope of ''how to manage a diverse ecosystem for a multitude of benefits.''

0.2 THE NONINDUSTRIAL PRIVATE FOREST LANDOWNER

In the previous section we noted some of the recent changes in forest management with respect to management objectives and cultural practices. Here the focus is on another aspect of great importance: the nonindustrial private landowner.

When classifying forest land of commercial value in the United States, a typical categorization is 1) government lands, 2) lands held by forest industry, and 3) lands owned by nonindustrial private forest landowners (NIPF's). The bulk of all commercial forest land in the U.S. is held by the NIPF group. Nationally, NIPF's own 58 percent of the forest land, while in the Southeastern States this figure is over 71 percent (Ross [1980]). It is clear that corporately the NIPF's have a dominant influence on the nation's supply of timber and other forest-related benefits.

In the past 40 years over 150 studies have been made on NIPF's to determine their ownerships, their interests, tenure, forestry practices, etc. (A very extensive bibliography of NIPF studies that contains over 190 citations can be found in Kurtz and Crouse [1981].) These studies have often been justified on the basis of a perceived need to motivate NIPF's to practice increased levels of forest management. For the most part this perceived need is well justified. For example, it is estimated that NIPF lands on a national basis produce at only 60 percent of their biological potential (Ross [1980]). A 1969 study by the Tennessee Valley Authority estimated that in the seven state Tennessee Valley Region, NIPF lands produced at only half

of their biological potential. It is important to point out that both industry and government timberlands are also well below biological potential in production. But because of the dominant position of the NIPF's as a group, much effort has been focused on ways to increase the productivity of their lands.

An obvious question is: ''Why is increased productivity needed in any sector?'' In the past the typical response has been to meet the projected future demand for fiber and wood products. According to the United States Forest Service, the U.S. will experience timber shortages and rapid increases in the real price of timber early in the 21st century (Hair [1980]). This sentiment was also reflected by President Carter in his first Environmental Message to the Secretary of Agriculture to '' . . . consider the need for measures to improve the capability of private lands to meet the nation's need for wood.'' On the other hand forest economists do not unanimously agree that projected timber shortages are sufficient justification for attempting to increase timber productivity on private lands. It is argued that the increased demand (if it does appear) will lead to higher price levels which will result in sufficient timber supplies, even when the relatively long lags between stand establishment and harvest are considered. Undoubtedly there are valid points in both arguments. We agree in spirit with the remarks of Duerr [1981], who contends that better forest management in the NIPF sector is a worthwhile objective because <u>increased productivity is a basic axiom of forestry</u>.

Having established that increasing productivity on NIPF lands is a defensible goal, the next question is what can be done to cost-

effectively achieve that goal? Perhaps this question can be best answered by examining what has not worked in the past. Satterfield [1980] echoed what many believe are the primary causes of low productivity on NIPF lands. These are:

1) the long period between investment in forest management and the realization of a return at harvest,
2) a lack of capital to make investments,
3) a lack of information as to what an individual's land can produce, and how to manage to achieve that potential,
4) perceived low rates-of-return on investments,
5) diverse objectives of land ownership beyond just financial criteria.

It is clear that no single effort can overcome all of these obstacles, but that a combination of approaches is needed. For instance the long investment period and lack of capital can be ameliorated through subsidies and preferential tax treatment for forest investments. This in fact has been partially achieved through the Forestry Incentives Program (FIP) which is a federal cost-sharing program for certain forest management cultural treatments.

The purpose of this research is in direct response to points 3 and 5, the lack of information on how to manage forest lands to achieve the best yield of forest benefits. By definition this requires an answer on an individual basis. Here we turn to mathematical optimization to provide a tool to help close this information gap.

Increasingly it is being recognized that NIPF's are interested in obtaining information on specific recommendations for management of their lands. For example, Hickman and Gehlhausen [1981] surveyed over 2000 NIPF's in Texas. These landowners were asked to state their preference on ten very diverse assistance programs. Information on how to effect multiple-use management ranked second only to requiring performance bonds for timber sale contracts. Sedjo and Ostermeier [1977], in their well known report on the NIPF sector, echoed similar sentiments on the importance of specific management assistance for NIPF's.

It is not surprising that numerous studies on NIPF interests find that landowners are very interested in obtaining assistance in forest management. Many NIPF's have professions totally unrelated to forestry and are unaware of where to obtain forest management assistance. Often this lack of information turns to lack of action as the landowner's forest management policy becomes one of benign neglect.

To provide the NIPF with a detailed forest management plan is a time-consuming activity. Considerable time is invested in the field work of measuring the resource. Alternative management strategies must be examined, based on the landowner's objectives. The Tennessee Valley Authority attempted to provide a tool to minimize the time necessary to formulate a management plan. This tool, known as the Woodland Resource Analysis Plan (WRAP) (Harrison and Rosenthal [1981]), relegated to the computer the task of summarizing the resource data and recommending harvesting strategies based on landowner objectives. While the program had some notable deficiencies, it also had some notable successes. In

particular it was very well-received by the NIPF sector in terms of the specifics it presented on management of an individual NIPF property. In a recent study by the U.S. Forest Service [1978] of NIPF's who had used WRAP, 90 percent found the program sufficiently understandable, 76 percent found it sufficiently personalized, and 97 percent believed that WRAP was applicable to their situation. This user acceptance is primarily attributable to the type and method of information presented.

Specifically then, the focus of this reseach effort was to design, develop and implement a forest management decision support tool for the landowner and natural resource manager. This required the leading edge of research in multiobjective optimization to deal with the landowner's diverse management objectives. And it demanded the state-of-the-art capabilities in forest biometrics to determine forest production functions.

0.3 OVERVIEW OF THE REMAINING CHAPTERS

In the remaining chapters we deal with the design and development of our multicriteria optimization model for forest management planning. We refer to this model as FORMAN, FORest Management plANning system.

Over the past ten years there has been a veritable explosion in research to apply optimization methodology to the problems of forest management. In chapter 1 we review some of the recent mathematical programming approaches to forest management planning.

In Chapter 2 we discuss the problems of modeling forest benefits in general, and describe specifically the forest production functions

used in FORMAN.

Chapter 3 contains a review of multicriteria decision making. It too has recently become a field of its own and has already accumulated a rich literature.

Chapter 4 represents the bulk of the original methodological research in this dissertation. It contains a complete development of a new generalized technique for multiobjective optimization. This discussion includes a description of the algorithm, along with results supporting its viability.

In Chapter 5 we develop a formulation for the forest management problem addressed in FORMAN. This formulation is particularly designed for use with nonindustrial private forest landowners, and small computers.

Chapter 6 contains a description of how we adapt the general multiobjective technique of Chapter 4 to solve the forest management formulation of Chapter 5. Also we discuss a heuristic solution procedure to solve problems that contain partitioned cardinality constraints, in addition to outlining the computer implememtation.

Chapter 7 concludes with a discussion of related future research topics. This includes extensions to the model and areas of potential improvement in the solution algorithm.

CHAPTER 1

REVIEW OF OPTIMIZATION APPROACHES TO FOREST MANAGEMENT PLANNING

The use of optimization in forest management planning has a long and rich history. As early as the middle nineteenth century, Faustmann [1849] provided an economic argument that a forest should be harvested at the age where the marginal value of the trees is equal to the marginal cost of investment (timber and land). Forest management also has a long history in terms of applications of operations research/management science. For instance, over twenty years ago Theiler [1959], and Coutu and Ellertsen [1960] provided applications of the use of linear programming to solve for harvest schedules.

In this chapter, we focus on optimization approaches to forest management planning. In all instances, the goal is to maximize the utility of the forest property through time. However the way in which utility is defined leads to a variety of solution techniques. Often utility will be equated with a single financial criterion; while more recent approaches define utility (either implicitly or explicitly) in terms of multiple objectives.

In examining the results of thirty years of research in forest management planning, we believe that it is appropriate to organize the review by application rather than solution technique. We note that there are many studies that deal with the maximization of utility of a single stand, while considering no others (single stand techniques). The results of these single stand approaches also provide insight for

our second class of applications, those dealing simultaneously with multiple stands of timber. We can further dichotomize the multiple stand literature by whether it addresses single or multiple benefits.

Since it is primarily through the temporal and spatial manipulation of the forest that change is introduced, in most situations the decision variables are timber characteristics. Whether dealing with single versus multiple stands or single versus multiple benefits, the forest manager is concerned with the optimal determination of the number, timing, and intensity of harvests, along with regeneration considerations.

This review is intended to selectively highlight the major schools of thought and practice on optimization approaches to forest management planning. More thorough reviews and bibliographies are available in Johnson and Schuermann [1977], Martin and Sendak [1973], and Field [1973].

1.1 SINGLE STAND TECHNIQUES

Here the preponderance of techniques deals with optimizing one criterion over one stand, usually with no restrictions on the decision variables. Frequently this one criterion is a financial measure such as present net worth.

There are many varied approaches used for the management of a single stand. Frequently these differing techniques lead to the same solution. For instance, the dynamic programming approach of Amidon and Akin [1968] can be shown to replicate the economic theory results

of Chappelle and Nelson [1964].

The earliest results in optimal management of a single stand were based purely on economic arguments. Faustmann [1849] investigated the situation of an infinite series of identical even-aged stands, with all management decisions fixed except rotation age. Chappelle and Nelson [1964] used marginal analysis to address this problem. Chang [1981], and Nautiyal [1983] have recently extended the economic analysis approach to deal with the concept of financial maturity in uneven-aged stands.

Pelz [1977] applied inventory theory to determine optimum rotation in a single stand, while Adam and Ek [1974], and Kao and Brodie [1980] used a nonlinear programming technique.

Dynamic programming has also been widely used. Amidon and Akin [1968] formulated the single stand management problem as a dynamic program with a single state variable - total volume. Brodie et al. [1978, 1979], Kao and Brodie [1979a, 1979b, 1980], and Ritters, Brodie and Kao [1982], have extended the use of dynamic programming in forest management to include a multiple state space. Rose, Leary and Chen [1981], and Chen, Rose and Leary [1980] have also used dynamic programming to determine optimal harvesting schedules for a single stand of timber.

Control theory has been used by Anderson [1976], Hellman [1980, 1982], and Naslund [1969]. Anderson's work resulted in a verification of Faustmann's research. Schreuder [1971] used the calculus of variations for this same general problem, while Loucks [1964] employed the use of linear programming.

Buorgiorno and Michie [1980], and Usher [1966, 1969, 1976] used a matrix model for forest management. Hool [1966] was among the first to include elements of uncertainty, which resulted in a finite horizon Markovian decision model. Lembersky and Johnson [1975] extended Hool's work by considering an infinite time horizon.

Giese and Jones [1983] have recently used linear complementarity to determine the optimal rotation of a single stand where the objective is to economically produce wood for energy use.

The single stand techniques do not currently receive as much use as the multiple stand approaches. This is primarily because the single stand techniques ignore important operational aspects, such as the interaction of benefits from various stands. Often the decision maker is quite willing to manage a stand in a suboptimal fashion (that is suboptimal to what could be achieved if no other stands were considered), so as to obtain control of the overall flow of benefits from a collection of stands. Also, the single stand techniques tend to be normative in their approach.

However the single stand models do provide a basis of comparison when used in conjunction with the multiple stand techniques.

1.2 MULTIPLE STAND TECHNIQUES

By definition here we consider only constrained optimization problems; otherwise we can apply the single stand techniques independently to a set of stands.

The constraints arise primarily as one attempts to control the

flow of benefits from a collection of forest stands. In addition to physical limitations on harvesting and regeneration, constraints are usually included to restrict i) woodflow, ii) cashflow, and iii) sustained yield (i.e. guarantee the ability of perpetual, sustained harvests).

The multiple stand approaches have become increasingly popular in the recent past. We believe that this is due, at least in part, to the increased use of optimization to determine prescriptive, individual management plans rather than for the development of normative models of actions.

1.2.1 Optimization of timber benefit

Here the emphasis is on the maximization of the financial utility of the forest (i.e. present net worth) subject to flow constraints. By far the most common solution technique is linear programming (LP), with the decision variables often being the amount of the i'th stand to harvest under the j'th management regime. (Another frequently used formulation is to denote the decision variables as the percentage of the i'th timber age class to be managed under the j'th management regime.)

This approach has proven to be one of the easiest to design and implement. Barros and Weintraub [1982] applied LP to a vertically integrated forest industry in South America. Coutu and Ellertsen [1960] used LP in farm forestry management. Curtis [1962], Kent [1980], Kidd et al. [1966], Leak [1964], Navon and McConnen [1967], Paine [1966], Rorres [1978], Theiler [1959], and

Wardle [1965] applied variations of this basic LP approach to problems in the management of government and industry timberlands.

A few others from this class deserve special attention. Liitschwager and Tcheng [1967] used Dantzig-Wolfe decomposition to solve a forest management problem with 1200 constraints and 28000 decision variables. The subproblems were themselves linear programs. What is noteworthy in this application is that problems of this size were well beyond the current state of technology at that time. Nazareth [1980] used Dantzig-Wolfe decomposition on a similar problem. Here the subproblems possessed a network structure, and were solved with either a network or dynamic programming algorithm.

Kirby, Hager and Wong [1982], and Weintraub and Navon [1976] deal with an interesting variation of multiple stand management. In this situation, the decision to harvest may require that a road be constructed to provide access to the timber. This is often the case in the western United States, where the transportation considerations are major model components. This formulation leads to a problem with an embedded network structure. In both of these examples, however, the formulated problems were solved as general mixed integer linear programs.

Ware and Clutter [1971] provided a fairly standard formulation and solution to the forest management problem when posed as an LP. The noteworthy aspect of their endeavor, MAX MILLION (now called THEO), is that it was (and is) implemented operationally on over 10 million acres of industrial forests.

As a group the single benefit - multiple stand forest management

problem has received widespread attention and application. However in recent years there has been increasing emphasis on multiple objective approaches. We explore these in the following section.

1.2.2 Multicriteria approaches

Part of the recent increase in multiobjective approaches to forest management can be attributed to the fact that multiobjective forest management is in many situations mandated by law. For instance, the National Forest Management Act of 1976 requires that management plans formulated for National Forests address the many benefits that the forest can produce. Another reason for the increased use of multiobjective approaches is that the field of multicriteria optimization has become well establised only in the last ten years.

Within our general framework, the goal of the multiobjective approaches remains the same - maximization of utility; only here utility is derived from more than one benefit. It is not surprising that goal programming is by far the most widely used multiobjective technique. It is relatively easy to conceptualize and implement. (However, as discussed in Chapter 3, it also has some serious theoretical deficiencies.) Some examples of its use are: Bottoms and Bartlett [1975], Dane, Meador and White [1977], Field [1973], Field, Dress and Fortson [1980], Schuler and Meadows [1976], and Schuler, Webster and Meadows [1977]. We note that all of these applications were for government timberlands.

Leuschner et al. [1975] applied the constraint method to a forest management problem in the Jefferson National Forest. In this

technique, minimum acceptable levels are set for all objectives except one (in this case timber income). The remaining objective is optimized subject to the original constraints plus the constraints that guarantee acceptable levels of the other objectives. A difficulty with this approach is determining the minimum acceptable levels.

Steuer and Schuler [1978] reported an application of Steuer's technique for multiobjective linear programming on the Mark Twain National Forest. DeKluyver, Daellenbach and Whyte [1980] described the use of multiobjective dynamic programming in conjunction with multiobjective linear programming to address the forest management problem. Hamner [1975], and Harrison and Rosenthal [1980,1981] discuss the use of a nonlinear utility function to deal with the problem of multiple objectives in forest management planning.

It is remarkable that, with the exception of WRAP, not one of the the applications reviewed in this chapter was designed specifically for use with the NIPF's. We have already discussed in Chapter 1, the dominant influence of the NIPF's in the timber industry. We have also noted their diverse and unique information requirements in forest management planning. In the following chapters we build upon the work reviewed here, to design a multiple stand, multiple benefit approach to forest management planning specifically for (but not limited to) the NIPF group.

CHAPTER 2

FORECASTING OF FOREST BENEFITS

The forest ecosystem, while often giving the impression of complete serenity, is actually in a constant state of change. It is a truly dynamic community, reacting to the stimuli of the environment on a continual basis. The most obvious effect of this change is in the regeneration, growth and death of trees, wildlife, herbaceous groundcover, etc. The growth of individual members of the forest community is the lifeblood of the forest ecosystem. And through this regeneration/growth/death process the forest provides many benefits to man.

Some of the more obvious benefits include timber production, wildlife, watershed protection, erosion control and aesthetics. However there are other benefits provided by the forest that are equally important. For instance there is currently great interest concerning the depletion of the vast tropical rain forests of South America. These forests provide (on a global scale) a large carbon dioxide sink - oxygen source. It is unclear what effect their removal will have on the composition of the earth's atmosphere.

In a system as complex as the forest there are many factors influencing its response. As managers we hope to direct this response in a particular fashion. For example, we often wish to concentrate timber growth on a few large stems rather than on many smaller ones. We prefer to have a small, healthy wildlife population as opposed to a

large, more sickly one. If the forest ecosystem changes in response to stimuli and we wish to control this change, we need to carefully determine what stimuli we should provide. The obvious conclusion is that in some sense we seek to ''optimally'' change the forest. We have no hope of succeeding in this optimal management unless we can accurately forecast the forest's response to management treatments.

The forecasting of a forest's response to management is a topic that has been under investigation for well over a hundred years. Not surprisingly there exist many schools of thought dealing with this question. For the remainder of this chapter we examine some different approaches to the forecasting of forest benefits. And in particular we focus on those techniques used in our forest management planning system, FORMAN.

2.1 TIMBER BENEFITS

Historically, the estimation of timber growth and yield has dealt almost exclusively with the quality and quantity of timber produced. And in comparing quality and quantity, by far the majority of research has dealt with the forecast of timber volumes.

Of the hundreds of research studies dealing with forest growth and yield, all may be classified as to whether they address a pure versus mixed species stand and similarly whether they deal with even-aged stands or stands of many different age classes. Numerous approaches have been used to determine a prediction system for forest stands of each of the preceding types, however all of these studies have at least

one factor in common. Either implicitly or explicitly they must deal with the three components of forest growth and yield - 1) height growth, 2) diameter growth, and 3) mortality/regeneration.

Bruner and Moser [1973] developed a Markov chain approach to solve the question of diameter growth in uneven-aged forest stands. However their model's performance was unimpressive. While it would seem that the Markovian assumption would be valid in the prediction of diameter growth, their results indicated otherwise.

Barnes [1982] has proposed a general growth and yield model based on biochemical arguments. This model considers the distribution of foilage, photosynthetic efficiency and rate, and seeks to predict growth based on photosynthate allocation. While the conceptual model is both intuitively and scientifically appealing, it is far from being an operational approach.

Many other techniques have been used to forecast timber growth and yield. To date, the technique of most widespread acceptance and use is nonlinear regression. This approach attempts to model growth and yield through the development of equations that predict each of the principal components: diameter and height growth, and mortality. These equations are fit to data collected over a period of years from stands of a particular type. This nonlinear regression technique is used to develop models that predict the growth and yield of an individual tree (individual tree models) or on a per acre basis (stand level models). The important point is that the ''right'' model to use is the simplest one that satisfies our criteria for accuracy. For the types of timber stands encountered in the southeast United States, stand level models

are the simplest (in terms of computer resources and methodology); and they are among the most accurate. Therefore we have chosen state-of-the-art stand level forecasting models for use in FORMAN. (Three excellent reviews on stand level growth projection models are Farrar [1978], Moser [1978], and Williston [1975].)

We have selected two stand level models in particular. The first deals with upland oak - hickory stands (Dale [1972]). The other is a model derived from the integration of many different studies of loblolly pine plantations (Hepp [1982]). In the next two sections we give a description of each of these timber growth and yield forecasting approaches.

2.1.1 The Upland Oak Model

The model that we use for predicting the growth and yield of upland oak stands is that developed by Dale [1972] and implemented by Hepp [1982]. It is a stand level model. That is, the inputs (site index, age, and density) and the outputs (stand volumes) are aggregated beyond the individual tree level, and reported on a per acre basis.

We described in the preceding discussion the major components of a growth and yield model. Essentially the model must describe the height growth of the stand, the mortality, and the diameter growth through time. The Dale Upland Oak Model contains two major components to address these requirements; 1) a technique for estimating the yearly basal area growth and 2) given a stand's site index, age, and basal area, the distribution of cordwood versus sawtimber is determined through a set of merchandising equations. Note that since the age of

the stand is one of the independent variables, the model by definition deals only with even-aged stands of upland oaks.

Let

A = the stand age in years

B = the basal area in square feet per acre of all living trees 2.6 inches or larger in diameter at breast height (d.b.h.)

S = site index in feet at age 50, (as described in Hepp [1982] we use the site index curves of Schnur [1937]).

Let

Y_1 = net annual basal growth per acre in square feet for all trees 2.6 inches or larger in d.b.h.,

where

$$Y_1 = 0.011383\ B\ S\ A^{-1.05} - B\ A^{-0.8}\ln(B) + 3.68521\ B\ A^{-0.75}. \quad (1)$$

Therefore to determine the basal area of a stand at some future point, say A_0+t, one repeatedly solves the above equation for ages A_0, A_0+1, ..., A_0+t and sums the result. By adding this quantity to the original basal area, the forecast basal area of the stand is obtained. The other stand parameters can then be determined as a function of site index, age, and predicted basal area.

Let

Y_2 = total cubic foot volume per acre for all trees 2.6 inches in d.b.h. or larger.

Y_3 = quadratic mean stand diameter of trees 2.6 inches d.b.h. or larger.

Y_4 = ratio of merchantable cubic foot volume to total cubic volume. Merchantable volume is for all trees 4.6 inches d.b.h. or larger. $Y_4 = 0.0$ if $D < 2.3$, and $Y_4 = 0.735$ if $D > 16$.

Y_5 = ratio of board foot volume to total cubic foot volume. Board foot volume is based on the International 1/4 inch rule for trees 8.6 inches d.b.h. or larger to a top diameter outside bark of 8.5 inches. $Y_4 = 0.0$ if $D < 4.8$ and $Y_5 = 3.55$ if $D > 16$.

Each of the stand measures is defined mathematically by the relations:

$$Y_2 = \exp[3.09094 + 0.00930176\ S + 1.03909\ \ln(B) - 20.11035\ A^{-1}] \quad (2)$$

$$Y_3 = 1.1341 + 0.0019876\ A\ S \quad (3)$$

$$Y_4 = -0.052676 + 0.7876045\ \exp[-(1.2987 - 0.08117\ Y_3)^{10}] \quad (4)$$

$$Y_5 = -0.088414 + 3.63827\ \exp[-(2.0 - 0.125\ Y_3)^4] \quad (5)$$

Algorithm 2.1 is a condensed description of how the Dale Upland Oak model is implemented in FORMAN.

Algorithm 2.1: Upland oak model

Step 0: Determine the stand's site index from the Schnur [1937] site index curves.

Step 1: For each year between the current year and the age of the current harvest determine the basal area increment by equation (1). From this, calculate the stand's basal area at harvest.

Step 2: Using equations (3)-(5) and the forecast basal area at the time of harvest, determine the forecast cubic volume per acre and board foot volume per acre.

Step 3: Using the specified residual basal area for the stand, along with equations (3)-(5), determine the corresponding cubic and board foot volumes.

Step 4: Define the yields of the current harvest to be the yields determined in step 2 minus the yields determined in step 3.

Step 5: If this is not the final harvest, let the successor harvest become the current and go to step 2; otherwise stop.

2.1.2 The Loblolly Pine Plantation Model

The loblolly pine plantation model in FORMAN, while it addresses a much simpler stand structure than the Upland Oak model, is actually much more complex mathematically. The Dale Upland Oak model aggregates many of the different aspects of growth into the annual basal area increment equation given by (1). In the case of loblolly plantations we do not rely on a single model, but determine our growth forecasts through the use of results from a number of growth and yield studies. As in the case with the Upland Oak Model, the loblolly plantation model is that implemented by Hepp [1982]. For a detailed discussion on the source of the data and the techniques used for deriving the specific functional forms of the growth equations, the reader is referred to the original studies cited both here and in Hepp. The purpose of this discussion is to provide a conceptual description of the loblolly pine plantation model used in FORMAN.

Before proceeding, we briefly review the use of the Weibull probability density function (pdf) in the context of stand table projection. This review will help to illuminate important concepts of the loblolly model.

The distribution of diameters is among the most important single factor characterizing a stand of timber. With a diameter distribution model we seek to predict n_{ij}, the number of trees of the i'th species in the j'th diameter class. Here we make the simplifying assumption that we deal only with stands composed of a single species (e.g. loblolly pine), and therefore we drop the subscript i and seek an estimate of n_j.

The general Weibull pdf is completely described by the location parameter a, the scale parameter b, and the shape parameter c. Mathematically,

$$f(x) = (c/b) \{ (x-a)/b)^{c-1} \exp[-((x-a)/b)^c] \}$$

where f(x) is the Weibull pdf for the random variable x, with $x \geq a$, $b > 0$, and $c > 0$.

The Weibull distribution has been extremely popular in forest biometry, principally because the distribution can assume a great variety of shapes (through the control of the a, b, and c parameters).

The cumulative distribution function (cdf) of the Weibull pdf is given by

$$F(x) = 1 - \exp\{ -((x-a)/b)^c \} \quad x \geq a, b > 0, c > 0.$$

Our interest here is to use the Weibull as a model for predicting the number of trees in a particular diameter class. With N stems per acre, the number of trees in a diameter class having midpoint x and

width 2w is given by:

$$N_x = N \{ \exp[-((x-w)/b)^c] - \exp[-((x+w)/b)^c] \}.$$

This equation provides a convenient closed form solution. Therefore to determine the answer to our original question, that of determining n_j, we have a computationally useful, closed form result.

A common way to use the Weibull distribution is to make the a, b, and c parameters vary as functions of stand characteristics. Therefore the Weibull pdf will respond dynamically to the change in stand structure through time. Regression analysis is the most common technique for deriving functional forms for a, b, and c. (An excellent discussion on the use of the Weibull pdf for modeling diameter distributions may be found in Bailey and Dell [1973]).

We now return to a discussion of the loblolly plantation model. The fundamental model used here is that of Smalley and Bailey [1974]. This work provides for the basic diameter distribution model (a Weibull approach) along with a mortality function and a model for distributing height growth among the diameter classes. It is important to realize that this study was based on the use of unthinned stand data, therefore a difficulty arises when one attempts to use the Smalley-Bailey model to simulate the effects of thinning. The problem is that the equations to determine the a, b, and c parameters cannot be ''regrown'' to show the effects of thinning due to the nature of the functional relations. Therefore we use the technique outlined in Clutter and Jones [1980] to update the stand table after thinning.

The site index and cubic foot volume equations are from Smalley

and Bower [1968, 1971]. Finally, the board foot and cubic foot/cord conversion results are from work by Burkhart et al. [1972].

Algorithm 2.2 presents an overview of the loblolly plantation model used in FORMAN.

Algorithm 2.2: Loblolly pine plantation model

Step 0: Determine site index from Smalley and Bower [1971].

Step 1: Calculate the updated stand table at the time of the current harvest. If there has been a thinning prior to this harvest, update the stand table by the procedure of Clutter and Jones [1980], otherwise use the results of Smalley and Bailey [1974].

Step 2: Remove trees, starting with the smallest diameter class and continuing in increasing diameter class order, until the stand table reflects the required residual basal area. (Obviously if this is a final harvest, the residual basal area is zero.) Merchandise the removals from step 3 using Smalley and Bower [1968] to determine cubic foot volume and Burkhart et al. [1972] to obtain board foot volumes.

Step 3: If this is a final harvest, stop; otherwise let the successor harvest become the current harvest and go to step 1.

2.2 WILDLIFE BENEFITS

There exist two fundamental approaches to forecasting wildife benefits. The first approach is to predict the population levels of a given species. The other approach is to forecast the habitat potential

of a forest stand to support wildlife of a certain species.

The problems encountered in forecasting timber benefits are amplified with attempts to deal with wildlife. Timber stands are stationary communities compared to wildlife. Also it is usually not possible to accurately inventory an existing wildlife population without a considerably greater expenditure of time and money. And it is difficult to develop good forecasting models that predict population levels without knowing an initial population distribution. Primarily for these reasons, the population level approach to forecasting wildlife is decreasing in popularity; and habitat suitability models are becoming more of the accepted norm. Note that this is the opposite philosophy of the timber forecasting models. There we actually do predict the timber population.

Habitat suitability models use as inputs various factors describing the forest ecosystem such as physiographic region, soil characteristics, timber characteristics and others. The output is a measure of how suitable the site is to support a wildlife population of a given species. This habitat measure, called a habitat suitability index, is a dimensionless number ranging from 0 (worst possible habitat) to 1 (optimum habitat). An obvious assumption that is required to use a habitat suitability index is that better habitat is perfectly correlated with better populations. While in isolated cases this can be totally false (i.e. it is possible to have a perfect habitat with no population), in general and in the aggregate this is a reasonable assumption.

Habitat suitability models have become increasingly popular over the past ten years. For example, the U.S. Fish and Wildlife Service uses a habitat suitability model (HEP) as an aid in decsion making on a national basis (Schenberger and Farmer [1978]). Lines and Perry [1978] have developed a habitat suitability model for wildlife in the Palouse River Basin of Washington and Idaho. And Whitaker and McCuen [1976] described a model used for habitat suitability evaluation in Maryland. The development of habitat suitability models have been published in numerous proceedings. One particularly significant proceedings is ''The Use of Multivariate Statistics in Studies of Wildlife Habitat'', USDA Forest Service [1978]. In that study Shugart [1981] provides a comprehensive overview of efforts to model wildlife population through the use of habitat suitablity measures.

We have implemented the habitat suitability models of Williamson [1983], which were developed specifically for FORMAN. These models deal with gray squirrel (Sciurus carolinensis), white-tailed deer (Odocoileus virginianus), and bobwhite quail (Colinus virginianus) in the Southeast United States. The remainder of this chapter is an overview of the Williamson models. We refer the reader to Williamson's original work for the specific details of model coefficients and supporting studies.

2.2.1 The Gray Squirrel Model

The gray squirrel model is a linear additive approach based on two primary components, hard mast production (i.e. acorns and nuts) and den site availability. Each of these components is scaled to range in

value from 0 (not available) to 1 (optimally abundant).

The specific functional form is:

$$squirrel_i = 0.555\ dens_i + 0.445\ mast_i$$

where

$squirrel_i$ - habitat suitability index for gray squirrel in a given stand in year i.

$dens_i$ - den site availability for gray squirrel in a given stand in year i, $0 \leq dens_i \leq 1$.

$mast_i$ - hard mast production in a given stand in year i, $0 \leq mast_i \leq 1$.

The weights 0.555 and 0.445 represent subjective valuations of the relative importance of the two criteria, dens and hard mast, and reflect the opinions of wildlife biologists at the Tennessee Valley Authority (Williamson [1983]).

The den site availability index is a function of stand type (i.e. principal tree species represented) and average stand diameter at breast height (d.b.h). The den site availability index (in those stands capable of den site production) increases linearly from a value of 0 at average stand diameters of less than 4 inches to a maximum of 1 for average stand diameters greater than 16 inches.

The hard mast index is a function of stand type, physiographic region (i.e. major river bottoms, coastal plain, ridge and valley (cf. Hunt [1974])), landform (i.e. uplands or bottomlands), and stand basal area expressed on a per acre basis. The use of stand type, physiographic region and landform permits the estimation of stand

composition with respect to mast producing species. Stand basal area acts as a correction factor in determining the estimated basal area in mast producing species. Mast production is calculated in terms of pounds of hard mast produced per square foot of basal area in mast producing species.

Note that hard mast production is a factor in both the gray squirrel and white-tailed deer habitat suitability indices. Obviously a resident squirrel population requires a different level of mast production than a corresponding deer population. This aspect was incorporated into the model by including an estimate of optimal mast requirements for deer and squirrel. Any level of mast production that met or exceeded this level generated a mast index of 1.

2.2.2 The White-Tailed Deer Model

The white-tailed deer model is based on two habitat factors, mast availability and browse availability. Three other important factors, cover availability, proximity to water, and vegetative interspersion were not included.

Cover availability was not included since it is believed that it is rarely a limiting factor in the southeast United States. Likewise, distance to water was omitted since it also is seldom a critical habitat factor for white-tailed deer in the Southeast. Vegetative interspersion and juxtaposition, while recognized as important factors, were omitted to permit stand independence in the estimation of habitat suitability and therefore provide a more mathematically tractable model.

Deer habitat suitability is defined as:

$$deer_i = 0.5\ mast_i + 0.5\ browse_i$$

where

$deer_i$ - habitat suitability index for white-tailed deer in a given stand in year i

$mast_i$ - hard mast production in a given stand in year i, $0 \leq mast_i \leq 1$

$browse_i$ - browse production for a given stand in year i, $0 \leq browse_i \leq 1$.

The hard mast model is the same as that used for gray squirrel. The browse production model is a function of stand type, stand age, current stand basal area (expressed on a per acre basis), number of years since the stand was last thinned, and residual stand basal area since the last thinning.

The browse function is a complex model in that it has more than a dozen different functional forms, based on the combination of stand type and stand age (i.e. a young pine plantation has a different functional form than does a mature hardwood stand).

For the most part the browse function was derived from published results. However wherever gaps in the literature existed, the browse function was based on the subjective opinion of a group of wildlife biologists rather than empirical results.

2.2.3 The Bobwhite Quail Model

The bobwhite quail model differs from the two preceeding models in modeling philosophy. The gray squirrel and white-tailed deer models were additive in nature. That is the habitat suitability index is a weighted average of the individual components. This approach is vaild when the individual components that contribute to habitat suitability are in some sense exchangeable. For instance a browse index of 1 and a mast index of 0 will result in a habitat suitability index of 0.5 for deer, indicating that loosely speaking neither browse or mast is so critical that the absence of one precludes the possibility of reasonable habitat.

The bobwhite quail model does not embody that approach. Instead of being a linear additive model, it is multiplicative in nature. Specifically the functional form is:

$$quail_i = base_i * thin_i$$

where

$quail_i$ - habitat suitability index for bobwhite quail in a given stand for year i

$base_i$ - a component determined principally on stand age, and ranges in value from 0 to 1

$thin_i$ - a component based primarily on the effects of recent (i.e. 3-5 years) thinning activity, and ranges in value from 0 to 1.

The important deviation from the two previous models is that a low value of $base_i$ or $thin_i$ precludes a reasonable bobwhite quail

habitat suitability index. These factors must occur simultaneously.

The $base_i$ component is a function of stand type and stand age. Type and age are important factors in determining if there is sufficient cover. The $thin_i$ component is a function of stand basal area (expressed on a per acre basis), number of years since the last thinning operation, and per cent of the stand removed in the last thin. This thin component is critical in that forest stands need to be ''opened'' by thinning every 3 to 5 years to stimulate the production of weed species, which are the principal source of food for quail.

CHAPTER 3

A DISCUSSION OF MULTICRITERIA DECISION MAKING

Consider the following problem. A decision maker (DM) wishes to choose the best course of action among many possible alternatives. Let X denote the set of all possible alternatives, with $\underline{x}$ being any member of X. Also, there exist I criteria or objectives of interest to the DM. For any $\underline{x} \in X$, let $f_i(\underline{x})$, i=1,...,I, measure the attainment of the i'th objective. Additionally, define $F(\underline{x})$ as the vector function $(f_1(\underline{x}),\ldots,f_I(\underline{x}))$.

The Multiobjective Problem (MOP) can be defined as: ''Find the most preferred $\underline{x} \in X$.'' However, this problem is ill-defined. The reason for this lack of definition is that there is not universal agreement as to what constitutes the ''most preferred'' or ''best'' solution. For instance, it is likely that what is the most preferred $\underline{x}$ to one DM may not be the most preferred alternative to another. In addition, what is most preferred to a DM at one point in time may not be a most preferred solution for the same DM, at another point in time.

Clearly, if I = 1, there is no disagreement as to what defines a best solution. Yet with I > 1, there does not exist a universal, well-defined statement of what characterizes the most preferred alternative. In the case of I = 1, we have a single objective optimization problem. No difficulty arises in this situation, since R^1 possesses a natural ordering. Given any two numbers a, b $\in R^1$, we can always determine whether i) a < b, ii) a > b, or iii) a = b. However,

given $\underline{u}$, $\underline{v} \in R^k$, $k > 1$, this natural ordering no longer exists.

It is precisely this lack of a natural ordering of R^n, $n > 1$, that leads to the difficulty and ill-definition of the MOP. Given $F(\underline{x})$ and $F(\underline{y})$, $\underline{x}$, $\underline{y} \in X$, in general it is not possible to determine if $F(\underline{x})$ is preferred to $F(\underline{y})$ or vice versa. Obviously if it is not possible to choose a preferred solution between two alternatives, it is not possible to choose the most preferred $\underline{x} \in X$, when $|X|$ is large. One way to overcome this difficulty is to define a function $U[F(\underline{x})]$ with the property that $U[F(\underline{x})] > U[F(\underline{y})]$ if and only if $F(\underline{x})$ is preferred to $F(\underline{y})$. As we shall see in the following sections, many multiobjective techniques are founded on this principal.

Much of the research in multiobjective optimization deals with i) finding suitable definitions for ''most preferred'', and ii) developing solution techniques to determine the most preferred solution, once a definition is given. Frequently the resulting multiobjective techniques make use of the wealth of existing single objective optimization methodologies.

Often a distinction is made between the terms multiobjective optimization and multicriteria decision making. For this discussion, it is not useful to make that distinction. Either of these two terms will apply equally to problem MOP. In addition, it is assumed that there exists a single DM. When dealing with more than one DM, problems arise not only with the difficulty of suitably defining the best alternative for each DM or even how to measure each objective, but also with combining these individually preferred solutions into a corporate,

most preferred alternative.

One additional point of definition. A large body of literature exists dealing with ''multiattribute utility theory'' (c.f. Keeney and Raiffa [1976]). The word utility, when used precisely in this context, implies that elements of uncertainty are modeled in the problem. Complementary to multiattribute utility theory is ''multiattribute value theory'' (c.f. Dyer and Sarin [1979]); where multiattribute value theory implies that no element of uncertainty is directly modeled. However, in the literature the term utility function is often construed to mean either a utility or value function. Since in this review of multiobjective optimization, (and indeed in all succeding chapters), we assume that no uncertainty exists in any of the models, we use the term utility function (for readability) when in, fact the term value function would actually be more accurate.

3.1 NO UTILITY FUNCTION ASSUMED

One major class of algorithms for multiobjective optimization does not require the definition of a utility function. Instead, this class of algorithms works directly with the partial ordering of R^n. Since in some sense, these techniques do not have available as much information as thoses approaches that have a well-defined utility function, it is to be expected that these techniques do not have as great an ability to discriminate between various points in X. However, even without the use of a utility function, much can be done to assist the decision maker in identifying a most preferred solution.

A key assumption in these approaches is that the decision maker's

satisfaction increases monotonically as the level of any benefit increases, while the level of all other benefits remains constant. This situation holds quite frequently in practice, as it assumes that the decision maker's desire for all benefits is insatiable.

There are examples where this monotonicity assumption may not be satisfied. Harrison and Rosenthal [1981] discuss a forest mangement example where the monotonicity assumption may not hold for certain forest benefits. For instance, there may be a level of deer population, beyond which the DM would experience decreasing satisfaction due to disease in the herd, over-browsing of the understory, etc. However, in general, this monotonicity property is a reasonable assumption.

Given that monotonicity holds, there are often many points in X that can be discarded from further consideration. The removal of certain points in X is based on the concept of <u>efficiency</u> (also known as Pareto optimality, nondominance, or noninferiority).

3.1.1 Efficiency techniques

An efficient solution is one in which no one objective function can be improved without a simultaneous detriment to at least one of the other objectives. More formally, $\underline{x}$ is an <u>efficient solution</u> if and only if there does not exist any $\underline{y} \in X$ such that $f_i(\underline{x}) \leq f_i(\underline{y})$ for all i and $f_j(\underline{x}) < f_j(\underline{y})$ for at least one j. A <u>dominated solution</u> is any solution that is not efficient.

Given the monotonicty assumption, a rational decision maker would never choose a dominated point. In addition, we can make some stronger

statements about efficiency, in that some efficient points are more desireable than others. Specifically, a DM would likely choose a properly efficient point over an improperly efficient one. An efficient solution $\underline{x}$, is properly efficient if there exists a scalar, $M > 0$ such that for each i, i=1,...,I,

$$\frac{f_i(\underline{y}) - f_i(\underline{x})}{f_j(\underline{x}) - f_j(\underline{y})} \leq M$$

for some j such that $f_j(\underline{y}) < f_j(\underline{x})$ whenever $\underline{y} \in X$ and $f_j(\underline{y}) > f_i(\underline{x})$ (Geoffrion [1968]).

Essentially a properly efficient point guarantees that an efficient point is not chosen where an arbitrarily large gain in one criterion can be made at the expense of only an infinitesimal loss of another. Since this would always be an acceptable tradeoff to a rational decision maker, one would always wish to choose properly efficient points. In general there does not exist a guarantee of whether a given efficient point is properly efficient. However, for the case where each f_i is linear and all constraints are linear, Benson and Morin [1977] have demonstrated that all efficient points are properly efficient.

Given that identifying efficient solutions is one goal of the multiobjective analysis, what techniques exist to achieve this? Soland [1979] provided a general and flexible characterization of the efficient set. Let g be any function mapping R^I into R which is strictly increasing in each component. Let $\underline{b} \in R^I$ and P(g,b) be the problem :

$$\max\ g[F(\underline{x})]$$

$$\text{s.t. } \underline{x} \in X$$

$$F(\underline{x}) \geq \underline{b}$$

Soland shows that $\underline{x}^*$ is an optimal solution to this problem if and only if it is efficient.

The actual solution technique to the MOP, based on efficiency, varies from author to author. Some researchers advocate providing the set of efficient solutions to the DM, and permitting the decision maker to choose the most preferred solution. Other authors recommend generation of only a portion of the efficient set. For instance, Hultz et al. [1981] described an example where the analyst provided the necessary tools for the DM to probe the efficient set. No mechanism was included to lead the decision maker to a most preferred solution.

An opposite approach is that of Ecker and Schoemaker [1980], who define special classes of efficient points that are more desireable than others. The goal of analysis, in this case, is to provide the decision maker with efficient points from this class.

A middle ground approach is reported in Steuer and Schuler [1978]. Here the DM is presented with a collection of efficient points at each step of their algorithm. Based on the DM's previous answers, the size of the efficient set that contains interesting solutions is decreased. Eventually only a few points remain. Note that this procedure only applies to the all linear case. However some interesting variations on this technique are found in Steuer and Choo [1981], and Steuer [1981]. These are also applicable in the case where the f_i's may be nonlinear, and/or X may be nonconvex.

3.2 AN EXPLICITLY ASSUMED UTILITY FUNCTION

A very different approach to the MOP is when an explicit utility function is assumed. Recall that a major difficulty in determining the most preferred solution is simply characterizing what is meant by most preferred. In the cases where an explicit utility function is assumed, no ambiguity exists. For all techniques in this class, in one form or another, a function U is defined with the property that for any $\underline{x}$, $\underline{y} \in X$, $U[F(\underline{x})] > U[F(\underline{y})]$ if and only if $F(\underline{x})$ is preferred to $F(\underline{y})$. Given a precise definition of U, the solution procedes as a single objective optimization problem.

There exist many techniques and variants that use an explicitly defined utility function to solve the MOP. Three important classes are i) multiple objective linear programming, ii) goal programming, and iii) multiattribute utility theory. As will be discussed in the following three sections, the underlying assumptions on the form and nature of U lead to very different solution techniques for the MOP.

3.2.1 Multiple objective linear programming

For the special case of the MOP where all of the f_i's and constraints are linear, there exists some convenient results characterizing efficient solutions. Obviously, even if an explicit utility function technique is used, we would still desire that the recommended solution is efficient. Any efficient solution in X can be generated by a suitable collection of weights, $w_i > 0$, where U is defined as the weighted sum of the f_i's (c.f. Cohon [1978]).

The question becomes, what are suitable values for the w_i's? It is often very difficult to obtain representative values of the w_i's from the decision maker a priori. Therefore a number of interactive techniques have been proposed to develop an estimate of the w_i's iteratively. Ho [1979] describes an approach where the w_i's are determined through a sequence of parametric linear programs. Steuer and Schuler's [1978] method also iteratively refines an estimate of these weights based on the DM's preference for various efficient points.

Note that in these two approaches the functional form is known a priori, however in each case, the exact functional representation of the explicit utility function is determined only after completion of the iterative algorithm. This is a different situation from the other explicit techniques described in this chapter, where the exact representation of the explicit utility function is known before the solution procedure is initiated.

3.2.2 Goal programming

Perhaps more than any other technique, goal programming is often used to solve the MOP. Indeed, to some persons, goal programming is multiobejctive optimization. While goal programming has been successfully used in a number of situations (Ignizio [1977]), it also has some serious drawbacks that must be considered (Dyer [1977], Rosenthal [1983]).

There are a number of variations of goal programming. However in all cases, the DM must specify a target or aspiration level, b_i, for

each of the I objectives. Deviational variables (n_i, and p_i) are then defined as the amount that f_i is over or under the specified level b_i (i.e. $f_i(\underline{x}) + n_i - p_i = b_i$, with $n_i \cdot p_i = 0$, $n_i, p_i \geq 0$). This results in two broad classes of goal programming: i) minimizing the weighted sum of the deviational variables, or ii) preemptive goal programming.

In the first case, weights are specified to penalize the over or under achievement of the aspiration levels. Denote these weights as o_i and u_i. The objective function is then:

$$\min \sum_i o_i p_i + u_i n_i$$

$$\text{s.t. } \underline{x} \in X$$

$$\underline{n} \cdot \underline{p} = \underline{0}.$$

Note that this objective function provides a well-defined ordering of all $F(\underline{x})$, $\underline{x} \in X$. The major criticism of this approach is that it assumes that the instantaneous tradeoff rate between objectives i and j, (in economic terms, the marginal rate of substitution), can take on only four distinct values. This tradeoff rate is given mathematically as:

$$\left. \frac{\partial U/\partial f_i}{\partial U/\partial f_j} \right| F(\underline{x}).$$

Note that this leads to only the following four values for the marginal rate of substitution, o_i/o_j, $-o_i/u_j$, $-u_i/o_j$, and u_i/u_j. This is clearly inconsistent with many years of economic thought and observation of human behaviour. Normally we would expect that these rates should vary as the level of benefits i and j change.

The second approach is preemptive goal programming. Here the objectives are ordered by increasing importance so that $o_1 \gg o_2 \gg o_3 \ldots o_I$. The relation $o_1 \gg o_2$ means that $o_1 > o_2$ S, for any finite S.

With preemptive goal programming, the optimum is found for the first objective, then if any ties exist (dual degeneracy), the second objective is optimized subject to the constraint that no degredation of the first objective is allowed. Of course, the objection raised in the prior case of goal programming, (limited values for the marginal rates of substitution), is even more pronounced here. The reason is that it may be possible to gain a very large increased in one objective for only a very small decrease in another, higher priority objective. This also contradicts our rationality of properly efficient solution.

3.2.3 Multiattribute utility theory

In this technique, the idea is to construct a function U, that is based on several axioms of rational choice (c.f. Keeney and Raiffa [1976] or Dyer and Sarin [1979]). Given these axioms, various assumptions are made (or verified) regarding the preferences of the decision maker. These axioms, in conjuction with the observation of the preferences of the DM, leads to a particular functional form of U.

Two of the most popular forms of these functions are the additive and multiplicative forms. The use of either of these requires the analyst to assess a series of one dimensional utility functions (oduf's) which map the level of the f_i's into [0,1]. The least desireable level of f_i, say $\underline{f}_i$, results in a one dimensional utility of

zero (i.e. $u_i(\underline{f}_i) = 0$). In a similar fashion, the most desireable level of f_i, say $\bar{f}_i$, results in a one dimensional utility of one (i.e. $u_i(\bar{f}_i) = 1$). Some functional forms of u_i are discussed in Chapter 4.

If using either the additive or multiplicative form, the analyst must obtain the decision maker's response to a series of questions designed to determine a set of scale factors, k_i, $i=1,\ldots,I$.
The additive form is given by:

$$U[F(\underline{x})] = \sum_i k_i \, u_i(f_i(\underline{x}))$$

where

$$\sum_i k_i = 1.$$

The multiplicative form is given by:

$$U[F(\underline{x})] = k^{-1} \left[\prod_i \right] (1 + k \, k_i \, u_i(f_i(\underline{x}))) - 1$$

where

$$\left[\prod_i \right] (1 + k \, k_i) = 1 + k, \text{ and}$$

$$0 < k_i < 1.$$

A major drawback to the use of explicit utility functions is that they are often difficult to assess. Another criticism is that the decision maker's preference structure may change as he/she gains more insight into the MOP. In Chapter 4, we discuss a technique that overcomes some of these objections. It is based on the use of both an explicit and implicit utility function.

3.3 AN IMPLICITLY ASSUMED UTILITY FUNCTION

A very different approach to the MOP is the use of an implicit utility function. Here the assumption is made that the decision maker is operating under the guidance of an unknown utility function. Often other assumptions are made on the nature of this unknown function, such as concavity, smoothness, etc. What is particularly clever is that this utility function is maximized without ever knowing its representation. The primary requirement of any of these implicit techniques is that one can obtain the DM's local tradeoff rates at any point. These rates are then used to determine increasingly better solutions, until a most preferred solution is found.

Geoffrion, Dyer, and Feinberg [1972] presented one of first formal application of this technique. Their method (discussed in greater detail in Chapter 4), is an iterative ascent technique employing the Frank-Wolfe method. Musselman and Talavage [1980], and Carlson and Shukla [1979a], used a cutting plane approach. In these algorithms the local tradeoff rates are used to determine cuts that successively remove uninteresting portions of X. Eventually only a small subset of X remains (possibly singleton), so that the decision maker can choose a most preferred point.

A criticism of these implicit methods is that they require too great an information burden of the decision maker. We demonstrate in Chapter 4 that to a large extent, this criticism can be overcome by combining an implicit utility function approach with an explicit one.

CHAPTER 4

AN IMPLICIT/EXPLICIT UTILITY FUNCTION APPROACH TO MULTIOBJECTIVE OPTIMIZATION

4.1 INTRODUCTION

A representation of the multiobjective optimization problem is:

$$\text{maximize } U[f_1(\underline{x}), f_2(\underline{x}), \ldots, f_r(\underline{x})] \qquad \text{(P1)}$$

$$\text{subject to } \underline{x} \in X,$$

where $f_1, \ldots, f_r$ ($f_i: R^n \rightarrow R^1$) are functions that describe the acheivement of each of the various objectives with respect to the decision vector $\underline{x}$, X is the set of all feasible decisions, and $U: R^r \rightarrow R^1$ is a utility function which provides an ordering of all elements in X. If U, $\underline{f}$, and X are explicitly defined then P1 is a single-objective mathematical program. Thus, one common approach to multiobjective optimization is to determine an explicit representation for U and then solve P1 by a single-objective optimization technique. Sometimes the explicit a priori definition of U may be difficult. Thus, a second common approach to multiobjective optimization is to relax the requirement of an explicit utility function and maximize U implicitly (assuming that it exists with appropriate properties but is unknown). We refer to the first approach as the <u>explicit utility function</u> approach, since U is determined before P1 is solved. We refer

to the second approach as the implicit utility function approach since no specific representation for U is assumed prior to solving to P1.

As noted in chapter 3, the assumption of whether or not U is known a priori leads to very different techniques for solving P1. For instance, in the case of an explicit utility function, much effort is spent determining an appropriate form and representation for U. Various independence properties are examined, and the decision maker is required to respond to a collection of hypothetical scenarios. The end result of this process is a specific form for the decision maker's utility function. The solution of P1, once the process of determining U is complete, is often trivial. Rosenthal [1982] has observed that explicit utility function approaches have been traditionally applied in cases where the number of feasible alternatives is small, say 50 or less. In this case, enumeration and evaluation of the entire feasible set is frequently used to solve P1 after U is determined.

Implicit approaches have typically been used where the number of feasible alternatives is large (very likely infinite). With an implicit approach the elicitation of preference information often occurs concurrently with the actual solution procedure. This implies that many of the implicit techniques are interactive.

To contrast the explicit and implicit approaches then, the explicit techniques require a large effort and information burden on the decision maker to determine a global representation of U, however the solution of P1 then reduces to an ordinary (single-objective) optimization problem. The implicit techniques often require only local preference information from the decision maker but may lead to an

increased number of iterations, and more importantly, may never converge to a most preferred solution. Also they often require the assumption that the underlying utility function is concave, which may be difficult to defend when U is unknown.

The remainder of this chapter describes a technique designed to overcome some of the deficiencies of both explicit and implicit approaches by combining the strengths of each into an implicit/explicit algorithm. It is somewhat surprising that few examples have appeared in the literature that combine an explicit and an implicit technique. We can cite only the work of Oppenheimer [1978] and Zionts and Wallenius [1983] as being similar in spirit to the approach described here.

4.2 STRUCTURE OF THE IMPLICIT/EXPLICIT TECHNIQUE

4.2.1 Review of the Geoffrion-Dyer-Feinberg algorithm

Here we illustrate the general class of implicit utility function algorithms in the context of the work of Geoffrion, Dyer, and Feinberg [1972]. In this seminal work, Geoffrion et al. outlined a procedure for solving problem P1 when X is a compact and convex set in R^n, and U is differentiable and concave on X but unknown. The solution procedure occurs iteratively, with two major steps.

1) Given a feasible point, $\underline{x}^k$, determine a direction of improvement, $\underline{d}^k$.

2) From the current point find the best feasible point along the improvement direction, which yields $\underline{x}^{k+1}$. Therefore $\underline{x}^{k+1} = \underline{x}^k + t^k\underline{d}^k$, with t^k a step size parameter along $\underline{d}^k$.

Note that step 1 is incompletely specified. Geoffrion et al. overcome this by taking a first order approximation to U. Then through the use of the well known Frank-Wolfe (Wolfe [1970]) algorithm they determine a direction of improvement through a solution to the problem:

$$\begin{aligned} &\max \ \nabla_{\underline{x}} U[f_1(\underline{x}^k), \ \ldots, \ f_r(\underline{x}^k)] \cdot \underline{z} \\ &\text{subject to } \underline{z} \in X. \end{aligned} \qquad \text{(P2)}$$

The solution of P2 determines $\underline{z}^k$, which yields the improving direction $\underline{d}^k = \underline{z}^k - \underline{x}^k$. To solve P2 we need the ability to evaluate the gradient of U at $\underline{f}(\underline{x}^k)$. However the gradient of U with respect to $\underline{x}$ is unavailable since U is unknown. We can rewrite $\nabla_{\underline{x}} U$ as

$$\nabla_{\underline{x}} U[f_1(\underline{x}^k), \ \ldots, \ f_r(\underline{x}^k)] = \sum_{i=1}^{r} \frac{(\partial U)^k}{(\partial f_i)} \nabla_{\underline{x}} f_i(\underline{x}^k),$$

where $(\partial U/\partial f_i)^k$ is the i'th partial derivative of U at the point $[f_1(\underline{x}^k), \ \ldots, \ f_r(\underline{x}^k)]$ and $\nabla_{\underline{x}} f_i(\underline{x}^k)$ is the gradient of f_i at $\underline{x}^k$.

The f_i's (and by assumption their first partial derivatives) are explicitly known, therefore if $(\partial U/\partial f_i)^k$ were known problem P2 would be well defined. Note that the optimal solution to problem P2 is not changed by multiplying the objective function by a positive coefficient. Without loss of generality, rearrange the indices of f_i so that $(\partial U/\partial f_1)^k$ is positive and divide the objective function of P2 by this term. This results in the following problem:

$$\max \sum_{i=1}^{r} y_i^k \nabla f_i(\underline{x}^k) \cdot \underline{z} \qquad \text{(P3)}$$

$$\text{subject to } \underline{z} \in X$$

where

$$y_i^k = \frac{(\partial U/\partial f_i)^k}{(\partial U/\partial f_1)^k} \qquad i = 1, \ldots, r.$$

The y_i^k are often referred to as the marginal rate of substitution between objective i and objective 1. They represent the infinitesimal amount of change between objectives i and 1 where the decision maker is indifferent, when all other objectives are held constant. Since more of any objective is assumed to be prefered by the decision maker, the marginal rate of substitution is the infintesimal degradation of one objective relative to the infinitesimal increase in another. Therefore the marginal rate of substitution can be approximated by the relation

$$y_i^k \cong \frac{-\Delta f_1(\underline{x}^k)}{\Delta f_i(\underline{x}^k)} .$$

This equation suggests that the y_i^k may be calculated through a procedure to determine the values of $\Delta f_1(\underline{x}^k)$ and $\Delta f_i(\underline{x}^k)$ at the current point $\underline{x}^k$. Assuming this capability, we can restate the Geoffrion-Dyer-Feinberg algorithm as follows.

Algorithm 4.1: Geoffrion-Dyer-Feinberg

Step 0: Choose $\underline{x}^1 \in X$, set $k = 1$.

Step 1: For objectives 2,...,r, determine the decision maker's marginal rate of substitution between objective i and objective 1 (MRS_{i1}) at the current point $\underline{x}^k$.

Step 2: Compute an optimal solution, $\underline{z}^k$, to problem P3.

Step 3: Set $\underline{d}^k = \underline{z}^k - \underline{x}^k$.

Step 4: Display values of $\underline{f}(\underline{x}^k + t\underline{d}^k)$ for several $t\varepsilon[0,1]$. Have the decision maker designate the most preferred value of t as t^k. Set $\underline{x}^{k+1} = \underline{x}^k + t^k\underline{d}^k$. Go to step 1.

The algorithm is now well specified. In the next two sections, we describe techniques to elicit $\underline{y}^k$, the marginal rate of substitution estimates, and t^k, the step size parameter.

4.2.2 Assessment of the decision maker's marginal rate of substitution

The assessment of the decision maker's marginal rate of substitution requires the largest part of the information burden in the implicit approaches to multiobjective optimization. Accurate results from this step help accelerate the solution process, possibly leading to a decreased number of iterations. However obtaining this information directly is generally difficult (and frequently impossible). For example, the decision maker often feels at a loss to respond to ''if your current criterion levels are $f_1(\underline{x}^k)$, ..., $f_r(\underline{x}^k)$, how many units of criterion 1 are you willing to sacrifice to gain one additional unit of criterion i.''

Dyer [1973] has proposed a procedure to avoid some of the difficulty in providing this tradeoff information. Using the relation $y_i^k \cong (-\Delta f_1(\underline{x}^k))/(\Delta f_i(\underline{x}^k))$, the procedure iteratively determines an

approximate marginal rate of substitution between objectives i and 1 through a series of binary comparisions. To obtain an estimate of $MRS_i(\underline{f}(\underline{x}^k))$, one first specifies a pertubation quantity, $\Delta f_i(\underline{x}^k)$, for objectives i and 1. The decision maker is presented with two vectors of attainments, $A = (f_1(\underline{x}^k),\ldots,f_r(\underline{x}^k))$ and $B = (f_1(\underline{x}^k)+\Delta f_1(\underline{x}^k), f_2(\underline{x}^k),\ldots,f_i(\underline{x}^k)-\Delta f_i(\underline{x}^k),\ldots,f_r(\underline{x}^k))$. If the decision maker prefers the second vector to the first, $\Delta f_i(\underline{x}^k)$ is increased and the question is repeated until the first vector of attainments is considered to be more attractive. A variant of the bisection algorithm is then applied to $\Delta f_i(\underline{x}^k)$ to iteratively determine a value at which the decision maker is indifferent between A and B. Denote this value as $\Delta f_i(\underline{x}^k)^*$. At this point y_i^k is approximated by $y_i^k \cong (-\Delta f_1(\underline{x}^k))/(\Delta f_i(\underline{x}^k)^*)$. The entire process is repeated to obtain the remaining y_i^k's.

Note that this procedure avoids the difficulties of questioning the decision maker for the value of the y_i^k's directly. It does lead to an increased number of iterations at each step, but the required information is easier to provide. This procedure also results (hopefully) in a more accurate estimate of y_i^k, since the decision maker defines it iteratively rather than with a single estimate. Also the decision maker is required at any time only to compare two collections of benefit attainments.

A detailed description of Dyer's procedure is contained in Algorithm 4.2.

Algorithm 4.2: Dyer's procedure for determining the marginal rates of substitution

Step 0: Let $A = \underline{f}(\underline{x}^k)$, and $i = 2$.

Step 1: Select Δf_1^k.

Step 2: Let $j = 0$, and select Δf_i^j.

Step 3: Let $B = (f_1(\underline{x}^k)+\Delta f_1^k,\ldots,f_i(\underline{x}^k)-\Delta f_i^j,\ldots)$. Compare A and B.

Step 4: If B is preferred to A, let $\Delta f_i^{j+1} = 2(\Delta f_i^j)$, set $j=j+1$, and go to step 3. If A is preferred to B, let $\delta^j = \Delta f_i^j$, and go to step 6. If the decision maker is indifferent between A and B, go to step 8.

Step 5: Compare A and B. If A is preferred, go to step 6. If B is preferred, go to step 7. If the decision maker is indifferent between A and B, go to step 8.

Step 6: Increase the desirability of B by letting $\Delta f_i^{j+1} = \Delta f_i^j-\delta^j/2$. Let $\delta^{j+1} = \delta^j/2$ and $j=j+1$. Go to step 5.

Step 7: Decrease the desirability of B by letting $\Delta f_i^{j+1} = \Delta f_i^j+\delta^j/2$. Let $\delta^{j+1} = \delta^j/2$ and $j=j+1$. Go to step 5.

Step 8: Compute $y_i^k = \Delta_i^k/\Delta f_i^j$. If $i\neq r$, set $i=i+1$ and go to step 2. Otherwise stop.

4.2.3 Determination of the step size parameter

Here we describe an algorithm to determine t^k, the step size parameter. Recall that as part of the implicit algorithm we determine $\underline{d}^k$, a direction of improvement from the current point $\underline{x}^k$. Problem P3 is based on a first order approximation to U. If the decision maker's true utility function were linear, one would set $t^k = 1$ and traverse $\underline{d}^k$

to the point $\underline{z}^k$. However, frequently the decision maker exhibits behaviour that does not support an underlying linear utility function. In these cases we must find the point in decision space that leads to the most preferred point on the line segment $\underline{f}(\underline{x}^k+t^k\underline{d}^k)$, $t^k\varepsilon[0,1]$. Without any additional assumptions on the nature of U, the step size is usually calculated through a grid search over the parameter t^k. Algorithm 4.3 presents a straightforward implementation of this concept.

Algorithm 4.3: Step size determination in the GDF algorithm

Step 0: Fix N, the number of mesh points to be examined. Set i = 1, $F^* = \underline{f}(\underline{x}^k)$, and $i^* = 0$.

Step 1: Display F^* and $F^i = \underline{f}(\underline{x}^k+(i/N)\underline{d}^k)$. If F^i is preferred to F^*, set $F^* = F^i$, and $i^* = i$.

Step 2: Let i = i+1. If $i \leq N$ go to step 1. Otherwise set $t^k = i^*/N$ and stop.

4.2.4 Overview of the implicit/explicit approach

In the previous three sections we described a procedure for solving P1 when the utility function is assumed to exist but is unknown. We have already discussed the differing assumptions and computational procedures needed depending on whether we try to explicitly build a global representation of U, or solve P1 implicitly. We have also noted some of the tradeoffs. The explicit methods require a large information burden initially but result in a (relatively) easy problem to solve. The implicit approaches require more information from the decision maker, but the information is local in nature and

somewhat easier to provide. Here we present an overview of a technique that is described in detail in the following sections. It is based on synergistically combining the strengths of both implicit and explicit techniques.

Assuming that one started with an implicit approach (such as the one described in 4.2.1-4.2.3), the question arises: ''What areas of the algorithm lend themselves to improvement?''. It is clear that the largest volume of interaction with the decision maker results from the determination of the marginal rates of substitution. Also, if there existed a functional representation of $MRS_i(\underline{f}(\underline{x}))$, $i = 2,,,r$, we could completely specify $U(\cdot)$ under some very reasonable assumptions (as will be demonstrated in subsequent sections). Therefore we seek a way to improve the implicit procedure by a technique of approximating the decision maker's $MRS_i(\underline{f}(\underline{x}))$. Note that at each iteration we obtain point estimates of these functions in the form of $\underline{y}^k$. The pure implicit approaches discard this information after each iteration. Here we propose to introduce the explicit flavor into the algorithm by using the value of the $\underline{y}^k$'s to interactively derive a representation of $MRS_i(\underline{f}(\underline{x}))$ through the use of nonlinear least squares analysis. Once the $MRS_i(\underline{f}(\underline{x}))$ are estimated, we can temporarily use the functions rather than the decision maker, to drive the implicit algorithm. Then we can revert to questioning the decision maker to obtain another collection of observations of $\underline{y}^k$.

This general approach is attractive for a number of reasons. Specifically, 1) we maintain the interactive nature and robust

convergence characteristics of the implicit technique, 2) we improve convergence in the case where we approximate the $MRS_i(\underline{f}(\underline{x}))$ well, and 3) we place more of the work on the machine segment of the man-machine interaction, while still providing the decision maker all of the flexibility of the pure implicit techniques.

4.3 APPROXIMATING A DECISION MAKER'S MARGINAL RATE OF SUBSTITUTION

In the previous section we identified a potential improvement in the implicit approaches to multiobjective decision making. Specifically, we seek the ability to predict the $MRS_{ij}(\underline{f})$ based on the decision maker's past responses to local tradeoff questions. If a ''sufficiently good'' model could be constructed to predict $MRS_{ij}(\underline{f})$ then some of the information burden required of the decision maker could be relegated to the computer, while still maintaining the ascent characteristics of the implicit algorithm.

To implement this modification to the basic implicit approach, we must first define the functional form of $MRS_{ij}(\underline{f})$. This is where the implicit/explicit algorithm takes on some of the characteristics of an explicit utility approach. In an implicit approach, no specific functional form is assumed for U. We keep this assumption since the overall approach is still an implicit one, but we do assume that the decision maker is providing marginal rate of substitution information according to a particular underlying utility function.

It is important to note that we have not introduced any additional assumptions beyond those of the implicit approach. We assume a

particular form of an underlying utility function only to provide a means of estimating a decision maker's marginal rate of substitution at a particular point. If these estimates prove to be unacceptably inaccurate, then the implicit/explicit algorithm reverts to a pure implicit approach (in essence, we simply discard the predicted marginal rate of substitution information). However, in the case where the assumed underlying utility function is representative of the decision maker's utility function, the predicted marginal rate of substitution provides a means for rapid convergence to a most preferred solution. This also leads to a corresponding decrease in the information burden on the decision maker.

In the next section we deal with the case where the assumed underlying utility function is additive, and following that we describe the case where we assume a multiplicative form. The multiplicative form is applicable when the benefits of one objective may substitute for the benefits of another. The additive form is appropriate when the benefit derived from each objective is independent of all others. Both the additive and multiplicative utility functions are composed of a collection of one dimensional utility functions (odufs). These odufs (also known in the literature as single-attribute or conditional utility (value) functions) represent the strength of preference of the decision maker for various levels of a particular objective while considering no others. For instance, one oduf might describe the level of satisfaction of the decision maker for various amounts of dollar income.

For the remainder of this chapter we assume that all odufs have the following form. Let $u_i(f_i)$ measure the utility to the decision maker for having f_i units of the i'th objective, irrespective of other objectives. For values of f_i, $0 \leq f_i \leq 1$, let

$$u_i(f_i) = s_i[1 - \exp(b_i f_i)]$$

with $b_i < 0$, and $s_i = [1 - \exp(b_i)]^{-1}$.

This form for the oduf has been used successfully in the past as reported by Keeney [1979], and has a number of desirable properties. Most importantly, it is strictly concave and increasing. This reflects the common perception that more of a desirable objective is good, but one enjoys each additional increment at a decreasing rate. (The canonical example is that one's first million dollars has more value to an individual than one's second million dollars.) Because of the restriction on the b_i's and f_i's, $s_i > 1$ and $u_i(0) = 0$, $u_i(1) = 1$. Also note that once b_i is specified, u_i is completely defined.

We now prove the strictly concave, increasing assertion for u_i.

Lemma 4.4

Let $u_i(f_i) = s_i[1 - \exp(b_i f_i)]$, (1)

with $b_i < 0$, and $s_i = [1 - \exp(b_i)]^{-1}$.

Then for $0 \leq f_i \leq 1$, $u_i(f_i)$ is a strictly concave and increasing function.

Proof:

It suffices to show $\dfrac{\partial u_i}{\partial f_i} > 0$ and $\dfrac{\partial^2 u_i}{\partial f_i^2} < 0$ for all $f_i \in [0,1]$.

Now, $\frac{\partial u_i}{\partial f_i} = s_i\ (-b_i)\ \exp(b_i f_i)$

$$= \frac{(-b_i)\ \exp(b_i f_i)}{(1 - \exp(b_i))}$$

> 0 for all $f_i \in [0,1]$ since $b_i < 0$ and $0 < \exp(b_i) < 1$.

Similarly, $\frac{\partial^2 u_i}{\partial f_i^2} = (-s_i)\ b_i^2\ \exp(b_i f_i)$

$$= \frac{-(b_i^2)\ \exp(b_i f_i)}{(1 - \exp(b_i))}$$

< 0 for all $f_i \in [0,1]$

Q.E.D.

4.3.1 Assuming an additive form of the utility function

In this case we use the well-known additive utility function (Dyer and Sarin [1979], Keeney and Raiffa [1976], Keelin [1981], Kirkwood and Sarin [1980]) to provide a basis for determining the form of $MRS_{ij}(\underline{f})$. Again, the purpose of this section is to describe a technique for predicting the $MRS_{ij}(\underline{f})$, which is then used to find an improving direction as part of the iterative process of an implicit approach to multiobjective decisionmaking.

The general additive form is:

$$U(\underline{f}) = \sum_i k_i\ u_i(f_i) \qquad (2)$$

where: $u_i(f_i)$ is an increasing one dimensional utility function

k_i is a positive scale factor for the i'th objective, $0 < k_i < 1$.

f_i is the level of acheivement of objective i. For this discussion, we assume without loss of generality that f_i has been scaled so that $0 \leq f_i \leq 1$.

Also, $u_i(0) = 0$, $u_i(1) = 1$, and $\sum k_i = 1$. Therefore $U(\underline{0}) = 0$, and $U(\underline{1}) = 1$.

4.3.1.1 Form of the MRS

We assume that the underlying utility function is (2). We also assume that the form of the oduf's is exponential and given by (1). Therefore an expression for the $MRS_{ij}(\underline{f})$ is:

$$MRS_{ij}(\underline{f}) = \frac{\partial U/\partial f_j}{\partial U/\partial f_i}$$

$$= \frac{\dfrac{\partial}{\partial f_j}\left\{ k_j \dfrac{(1 - \exp(b_j f_j))}{(1 - \exp(b_j))} \right\}}{\dfrac{\partial}{\partial f_i}\left\{ k_i \dfrac{(1 - \exp(b_i f_i))}{(1 - \exp(b_i))} \right\}}$$

$$= \frac{k_j\, b_j \exp(b_j f_j)\,(1 - \exp(b_i))}{k_i\, b_i \exp(b_i f_i)\,(1 - \exp(b_j))} \qquad (3)$$

Once we know the values for k_i, k_j, b_i and b_j, then $MRS_{ij}(\underline{f})$ is completely defined. Conversely, once the functional form of $MRS_{ij}(\underline{f})$ is known for all (i,j) pairs, then the overall utility function (2) is completely known.

Now, suppose that we have observed the decision maker's responses to the marginal rate of substitution questions. Specifically, suppose that we have observed a collection of ordered triples:

$$T_m = \{(y_j^q, f_1^q, f_j^q) : q = 1, \ldots, m\},$$

with y_j^q = empirically observed marginal rate of substitution between objective 1 and objective j recorded at the q'th observation

f_1^q = level of the first objective at the q'th observation

f_j^q = level of the j'th objective at the q'th observation

At each major iteration of the implicit procedure, we obtain estimates of the marginal rate of substitution between the reference objective and each of the other objectives. Therefore at each iteration we observe estimates of $\{MRS_2(\underline{f}), MRS_3(\underline{f}), \ldots, MRS_r(\underline{f})\}$. We can now rewrite (3) as:

$$MRS_j(\underline{f}) = \frac{k_j\, b_j\, \exp(b_j f_j)\, (1 - \exp(b_1))}{k_1\, b_1\, \exp(b_1 f_1)\, (1 - \exp(b_j))} \qquad (4)$$

Let $a_{i1} = k_i \quad i = 1, \ldots, r$

$a_{i2} = -b_i \quad i = 1, \ldots, r$.

Taking into account the index for the q'th observation, (4) can be written as:

$$MRS_j^q(\underline{f}) = \frac{a_{j1}\, a_{j2}\, \exp(a_{12} f_1^q)\, (1 - \exp(-a_{12}))}{a_{11}\, a_{12}\, \exp(a_{j2} f_j^q)\, (1 - \exp(-a_{j2}))} \qquad (5)$$

4.3.1.2 Form of the error function

The value of $MRS_j^q(\underline{f})$ given by (5) is an estimate of the observed value of y_j^q. We seek a method of determining an appropriate value for the parameter $\underline{a} = (a_{11}, a_{12}, \ldots, a_{r1}, a_{r2})$ so that this estimate is as accurate as possible. Let $Y = (y_2^1, y_3^1, \ldots, y_r^m)$. Consider MRS_j^q as a function of $\underline{a}$. Define $\underline{MRS}(\underline{a}) = (MRS(\underline{a})_2^1, MRS(\underline{a})_3^1, \ldots, MRS(\underline{a})_r^m)$. Given T_m, we seek a value for $\underline{a}$ so as to minimize the following <u>error function</u>:

$$E(\underline{a}) = || \underline{MRS}(\underline{a}) - Y ||_p \qquad (6)$$

where $||\cdot||_p$ denotes the 1_p norm. Zeleny [1982] discusses various choices for p in the context of using regression analysis for estimating the explicit functional form of U directly. He notes that p = 1 is appropriate when the errors are independent and follow a Cauchy probability distribution. A value of p = 2 is correct when the errors are normally distributed with mean zero and common (possibly unknown) variance. Setting p to infinity is proper when the errors are uniformly distributed. We use the value of p = 2.

To approximate the decision maker's marginal rate of substitution

as well as possible, we wish to choose a value for $\underline{a}$ such that we

$$\text{minimize} \quad || \, MRS'(\underline{a}) - Y \, ||_2 \tag{7}$$

$$\text{subject to} \quad 0 < a_{j1} < 1 \qquad j = 1, \ldots, r$$

$$0 < a_{j2} \qquad j = 1, \ldots, r$$

$$\sum_j a_{j1} = 1$$

The objective function in (7) is the square root of the sum of squared error between the observed and predicted marginal rates of substitution. The bounds on the a_{jk} are inherited from restrictions on the original underlying utility function. Since $0 < k_j < 1$ in (2), $0 < a_{j1} < 1$ in (7). Likewise, since $b_j < 0$ in (2), and $b_j = -a_{j2}$, $0 < a_{j2}$ in (7). Also note that the optimal solution to (7) is the same as the optimal solution to a related problem where we square the objective function while defining everything else the same as in (7). We will solve this related problem because of the convenience of avoiding the square root.

Observe that the bounds in (7) are strict inequalities. We avoid that difficulty by fixing an appropriate $\varepsilon > 0$ and then replacing the inequalities in (7) by $\varepsilon \leq a_{j1} \leq 1-\varepsilon$, and $\varepsilon \leq a_{j2}$.

Therefore to find the best approximation to the decision maker's marginal rate of substitution under the assumptions of this section, we seek a value for $\underline{a}$ such that we

$$\min E(\underline{a}) = \sum_j \sum_q \{MRS_j^q(\underline{a}) - y_j^q\}^2 \qquad (8)$$

subject to $\quad \varepsilon \leq a_{j1} \leq 1-\varepsilon \qquad j = 1, \ldots, r$

$$\varepsilon \leq a_{j2} \qquad j = 1, \ldots, r$$

$$\sum_j a_{j1} = 1$$

where

$$MRS_j^q(\underline{a}) = \frac{(a_{j1}\, a_{j2} \exp(a_{12} f_1^q)\ (1 - \exp(-a_{12}))}{(a_{11}\, a_{12} \exp(a_{j2} f_j^q)\ (1 - \exp(-a_{j2}))}$$

4.3.1.3 Near quasiconvexity of the error function

Clearly (8) describes a nonlinear program in $\underline{a}$ over a convex, compact feasible region. To obtain a ''best'' approximation to the decision maker's marginal rate of substitution we would like to find an optimal value of $\underline{a}$ in (8). A particularly desirable property for $E(\underline{a})$ would be strict quasiconvexity. (Recall that a function f defined on a convex set S is strictly quasiconvex if for each $\underline{x}_1$, $\underline{x}_2 \in S$ with $f(\underline{x}_1) \neq f(\underline{x}_2)$, $f[\alpha\underline{x}_1+(1-\alpha)\underline{x}_2] < \max\{f(\underline{x}_1), f(\underline{x}_2)\}$ for each $\alpha \in (0,1)$.) The significance of strict quasiconvexity is that when minimizing a function f with this property any local minimum is a global minimum (e.g. Bazaraa and Shetty [1979]). Hence if we could show that the objective function of (8) is strictly quasiconvex over the feasible region, we can use a nonlinear programming algorithm to find a locally optimal solution, knowing that this solution will also be globally optimal. Unfortunately, we can show that the objective function in (8) is not strictly quasiconvex. Strict quasiconvexity is the weakest

condition under which we can guarantee a global optimum. Therefore we have no way of theoretically guaranteeing through convexity arguments that any solution obtained by a nonlinear programming algorithm is a global minimum to (8).

However we did test (4) for strict quasiconvexity empirically, and the results are reported in Table 4.1. Points $\underline{a}^1$ and $\underline{a}^2$, and values of f_1, f_2 were systematically chosen. At each $\underline{a}^1$, $\underline{a}^2$, f_1, f_2 quadruple, (4) was tested for strict quasiconvexity. This test was replicated for 2.0×10^7 points as described in Table 4.1. The percentage of these test points at which (4) satisfied strict quasiconvexity was 98.69. Since the objective function of (8) is the sum of squared differences between (4) and a constant, the percentage of points satisfying strict quasiconvexity for (4) is representative also for (8). Essentially these results imply that although the objective function to (8) is not strictly quasiconvex, it is nearly so. While we have no guarantee of convergence to a globally optimal solution to (8), the results of Table 4.1 imply that this may occur often in practice.

This was in fact found to be true. A test problem was constructed with a zero value of the error function attained in the feasible region. Starting from eight different points spread throughout the feasible region, all eight converged to a zero valued solution to (8). Each of these eight optimal solutions were different (note that strict quasiconvexity does not guarantee uniqueness). In fact it is demonstrated in Section 4.3.1.5 that there exists an infinite number of locally optimal solutions to (8) with the same objective function

TABLE 4.1

TEST OF STRICT QUASICONVEXITY IN THE CASE
OF AN ASSUMED ADDITIVE UTILITY FUNCTION

	Minimum Value	Maximum Value	Mesh Size
alpha	0.1	0.9	0.2
f_I^q	0.1	0.9	0.2
f_I^q	0.1	0.9	0.2
a_{I1}^1	0.1	0.9	0.2
a_{j1}^1	0.1	0.9	0.2
a_{I2}^1	1.1	2.6	0.5
a_{j2}^1	1.1	2.6	0.5
a_{I1}^2	0.1	0.9	0.2
a_{j1}^2	0.1	0.9	0.2
a_{I2}^2	1.1	2.6	0.5
a_{j2}^2	1.1	2.6	0.5

Total number of grid points examined = 2.0×10^7

Percentage of points failing strict quasiconvexity = 1.31

value. These solutions are closely related however.

To summarize the results of this section, we cannot be guaranteed that a locally optimal solution to (8) is globally optimal. However, empirically it appears that in practice this will be true with high probability.

4.3.1.4 Gradient of the error function

Many nonlinear programming algorithms require evaluation of the gradient of the objective function throughout the solution procedure. Even in those cases where the gradient can be approximated, having an exact representation may lead to improved convergence. The purpose of this section is to develop an analytic representation of the gradient of the error function in the case of additive U and exponential u_i's. This is required by the nonlinear programming algorithm that was selected for minimizing $E(\underline{a})$. Since the error function is:

$$E(\underline{a}) = \sum_j \sum_q \{ MRS_j^q(\underline{a}) - y_j^q \}^2,$$

$$\frac{\partial E}{\partial a_{ik}} = 2 \sum_j \sum_q \{MRS_j^q(\underline{a}) - y_j^q\} \frac{\partial MRS_j^q(\underline{a})}{\partial a_{ik}}$$

where

$$\frac{\partial MRS_j^q(\underline{a})}{\partial a_{11}} = \frac{(a_{j1}\ a_{j2})\ (1 - \exp(-a_{12}))\ (\exp(a_{12}f_I^q))}{(a_{12})\ (1 - \exp(-a_{j2}))\ (\exp(a_{j2}f_j^q))}\ \frac{-1}{(a_{11})^2}$$

$$= \frac{-MRS_j^q(\underline{a})}{a_{11}}$$

$$\frac{\partial MRS_j^q(\underline{a})}{\partial a_{j1}} = \frac{(a_{j2})\ (1 - \exp(-a_{12}))\ (\exp(a_{12}f_I^q))}{(a_{11}a_{12})\ (1 - \exp(-a_{j2}))\ (\exp(a_{j2}f_j^q))}$$

$$= \frac{MRS_j^q(\underline{a})}{a_{j1}}$$

$$\frac{\partial MRS_j^q(\underline{a})}{\partial a_{12}} = \frac{(1 - \exp(-a_{12}))}{(1 - \exp(-a_{j2}))}\ \frac{(\exp(a_{12}f_I^q))}{(\exp(a_{j2}f_j^q))}\ \frac{(-a_{j1}\ a_{j2})}{(a_{11}\ a_{12}^2)} +$$

$$\frac{(a_{j1}\ a_{j2})}{(a_{11}\ a_{12})}\ \frac{\partial}{\partial a_{12}}\left\{\frac{(1-\exp(-a_{12}))\ \exp(a_{12}\ f_I^q)}{(1-\exp(-a_{j2}))\ \exp(a_{j2}\ f_j^q)}\right\}$$

$$= \frac{(1 - \exp(-a_{12}))}{(1 - \exp(-a_{j2}))}\ \frac{(\exp(a_{12}f_I^q))}{(\exp(a_{j2}f_j^q))}\ \frac{(-a_{j1}\ a_{j2})}{(a_{11}\ a_{12}^2)} +$$

$$\frac{(a_{j1}\ a_{j2})}{(a_{11}\ a_{12})}\ \frac{(1-\exp(-a_{12}))(f_I^q\ \exp(a_{12}\ f_I^q))+(\exp(a_{12}\ f_I^q+a_{12}))}{(1 - \exp(-a_{j2}))\ (\exp(a_{j2}\ f_j^q))}$$

$$= MRS_j^q(\underline{a})\ \left\{\ f_I^q - \frac{1}{a_{12}} + \frac{\exp(a_{12})}{(1 - \exp(-a_{12}))}\ \right\}$$

$$\frac{\partial MRS_j^q(\underline{a})}{\partial a_{j2}} = \frac{a_{j1}}{(a_{11}\ a_{12})}(1-\exp(-a_{12}))(\exp(a_{12}\ f_1^q)) *$$

$$\frac{\partial}{\partial a_{j2}} \left\{ \frac{a_{j2}}{(1 - \exp(-a_{j2}))\ (\exp(a_{j2}\ f_j^q))} \right\}$$

$$= MRS_j^q(\underline{a}) \left\{ f_j^q + \frac{1}{a_{j2}} + \frac{\exp(a_{j2})}{(1 - \exp(-a_{j2}))} \right\}$$

4.3.1.5 Solution of the error minimization problem (additive form)

This section integrates the results of 4.3.1.1 - 4.3.1.4 to form an algorithm for estimating the decision maker's marginal rate of substitution when the underlying utility function is assumed to be additive. We seek a solution procedure for the nonlinear program:

$$\min E(\underline{a}) = \sum_j \sum_q \{MRS_j^q(\underline{a}) - y_j^q\}^2 \qquad (9)$$

subject to $\varepsilon \leq a_{j1} \leq 1-\varepsilon \qquad j = 1, \ldots, r$

$\varepsilon \leq a_{j2} \qquad j = 1, \ldots, r$

$\sum_j a_{j1} = 1$

where

$$MRS_j^q(\underline{a}) = \frac{(a_{j1}\ a_{j2}\ \exp(a_{12}f_1^q)\ (1 - \exp(-a_{12}))}{(a_{11}\ a_{12}\ \exp(a_{j2}f_j^q)\ (1 - \exp(-a_{j2}))}$$

First notice that this problem has only one constraint, ignoring the bounds on the variables. Define problem (10) as the same problem as (9) except omitting the single constraint. The following theorem demonstrates that once we have an optimal solution to (10), we can

determine another solution with the same objective function value that will also satisfy the constraint of (9). This transformation can be performed in time that is a linear function of the number of objectives considered.

Theorem 4.5: Consider problems (9) and (10). Assume that $\underline{a}$ is an optimal solution to (10).

Let

$$a'_{j1} = c\, a_{j1}$$

$$a'_{j2} = a_{j2}$$

with

$$c = \frac{1}{\sum_j a_{j1}}$$

then $\underline{a}'$ is an optimal solution to (9) with equal objective function value as $E(\underline{a})$ in (10).

Proof:

$$\sum_j a'_{j1} = \sum_j c\, a_{j1}$$

$$= \frac{1}{\sum_j a_{j1}} \sum_j a_{j1}$$

$$= 1$$

Therefore the equality constraint is satisfied. Also since $a'_{j2} = a_{j2}$ and a_{j2} satifies the bounds in (10) by assumption, a'_{j2} also satisfies the bounds in (9).

Now $\sum a'_{j1} = 1$, and $a_{j1}, c > 0$ implies $0 < a'_{j1} < 1$ for all j.

Finally, we show that $\underline{a}'$ is an optimal solution for (9). Note that $E(\underline{a}) = E(\underline{a}')$ since the c's in (9) cancel. The objective function of (10) provides a lower bound for (9), therefore $\underline{a}'$ is optimal for (9).

Q.E.D.

Theorem 4.5 leads to the following algorithm to solve problem (9).

Algorithm 4.6: Determining an explicit utility function (additive form)

Step 0: Set p = 1, set $\underline{a}^0$ to infinity, set $\underline{a}^1$ to a point that satisfies the bounds on $\underline{a}$, and fix $\varepsilon > 0$, a convergence parameter.

Step 1: Use an unconstrained nonlinear optimization technique to solve (10), starting from $\underline{a}^p$.

Step 2: Define

$$b^p_{j1} = \frac{a^p_{j1}}{\sum_j a^p_{j1}}$$

$$b^p_{j2} = a^p_{j2}$$

Step 3: Compute $\varepsilon^p = || \underline{b}^p - \underline{b}^{p-1} ||_2$

If $\varepsilon^p \leq \varepsilon$ stop

otherwise for each a_i set a_i^{p+1} to $a_i^p + 0.5$, set p = p+1 and go to Step 1.

There are two primary attractions to Algorithm 4.6. First, as a result of Theorem 4.5, we can use an unconstrained nonlinear optimization technique to solve the subproblem in Step 2. For a

technique such as a modified Newton approach (e.g. Bazarra and Shetty [1979], Luenberger [1973]), the addition of ranges on the variables is a trivial extension. Second, by requiring two successive iterations to be within an ε distance of each other, based on the empirical results of section 4.3.1.3, we can be reasonably confident of finding a globally optimal solution to (9). Convergence is based on a heuristic argument. Note that the algorithm requires two successive iterations to be within an ε tolerance of each other. Again, invoking the ''almost strictly quasiconvex'' property, it appears that in practice relatively few iterations will be required to converge.

4.3.2 Assuming a multiplicative form of the utility function

Here we mirror the discussion of the previous sections (4.3.1.1 - 4.3.1.5) to derive a representation of $MRS_{ij}(\underline{f})$ when the underlying utility function is assumed to be multiplicative. This form, like the additive version, has been used successfully in a variety of applications (Golabi, Kirkwood, and Sicherman [1981], Gros [1975], and Keeney [1979]).

The general multiplicative form is:

$$U(\underline{f}) = k^{-1} \{ \prod_i (1 + k\, k_i\, u_i(f_i)) - 1 \} \qquad (11)$$

where $u_i(f_i)$ is an increasing one dimensional utility function.

k_i is a positive scale factor for the i'th objective, $0 < k_i < 1$.

k is another scale factor to ensure that $U(\underline{1}) = 1$. Therefore k must satisfy the relation $1+k=\overline{|}_i|(1+kk_i)$, with $k > -1$ and $k \neq 0$.

f_i is the level of acheivement of objective i. For this discussion we assume without loss of generality that $0 \leq f_i \leq 1$.

4.3.2.1 Form of the MRS

As in the addtive case, we assume an underlying utility function to permit the derivation of an exact functional form for $MRS_{ij}(\underline{f})$. In this case we assume that this underlying utility function is (11), and that the form of the odufs is (1). Then

$$\frac{\partial U}{\partial f_i} = \frac{\partial}{\partial f_i} \left\{ k^{-1} \left[\overline{|}_j| \, (1 + k\, k_j\, u_j(f_j)) - 1 \right] \right\}$$

$$= \frac{\partial}{\partial f_i} \left\{ k^{-1} \left[\overline{|}_{j \neq i} | \, (1 + k\, k_j\, u_j(f_j))(1 + k\, k_i\, u_i(f_i)) - 1 \right] \right\}$$

$$= k_i \frac{\partial u_i}{\partial f_i} \left\{ \overline{|}_{j \neq i} | \, (1 + k\, k_j\, u_j(f_j)) \right\}$$

Therefore

$$MRS_{ij}(\underline{f}) = \frac{\partial U / \partial f_j}{\partial U / \partial f_i}$$

$$= \frac{k_j \dfrac{\partial u_j}{\partial f_j} \left\{ \prod_{1 \neq j} (1 + k\, k_1\, u_1(f_1)) \right\}}{k_i \dfrac{\partial u_i}{\partial f_i} \left\{ \prod_{1 \neq i} (1 + k\, k_1\, u_1(f_1)) \right\}}$$

$$= \frac{(k_j \dfrac{\partial u_j}{\partial f_j})\ (1 + k\, k_i\, u_i(f_i))}{(k_i \dfrac{\partial u_i}{\partial f_i})\ (1 + k\, k_j\, u_j(f_j))} \qquad (12)$$

Now

$$\frac{\partial u_i}{\partial f_i} = \frac{-b_i \exp(b_i f_i)}{(1 - \exp(b_i))} \qquad (13)$$

Therefore substituting (13) into (12) yields:

$$MRS_{ij}(\underline{f}) = \frac{(k_j\, b_j)\exp(b_j f_j)(1-\exp(b_i)) \left\{1 + k\, k_i \dfrac{(1-\exp(b_i f_i))}{(1-\exp(b_i))}\right\}}{(k_i\, b_i)\exp(b_i f_i)(1-\exp(b_j)) \left\{1 + k\, k_j \dfrac{(1-\exp(b_j f_j))}{(1-\exp(b_j))}\right\}} \qquad (14)$$

As in the additive case suppose that we have observed the decision maker's responses to the marginal rate of substitution questions. This results in the collection T_m, where

$$T_m = \{(y_j^q,\ f_I^q,\ f_j^q) : q = 1,\ \ldots,\ m\},$$

with y_j^q = marginal rate of substitution between objective 1 and objective j recorded at the q'th observation

f_1^q = level of the first objective at the q'th observation

f_j^q = level of the j'th objective at the q'th observation.

Again for simplicity of notation define $MRS_{1j}(\underline{f}) = MRS_j(\underline{f})$. Then (14) becomes

$$MRS_j(\underline{f}) = \frac{(k_j\ b_j)\exp(b_j f_j)(1-\exp(b_1))\left\{1 + k\ k_1 \frac{(1-\exp(b_1 f_1))}{(1-\exp(b_1))}\right\}}{(k_1\ b_1)\exp(b_1 f_1)(1-\exp(b_j))\left\{1 + k\ k_j \frac{(1-\exp(b_j f_j))}{(1-\exp(b_j))}\right\}} \qquad (15)$$

Let $\underline{a} = (a_0, a_{11}, a_{12}, \ldots, a_{r1}, a_{r2})$, with

$a_0 = k$

$a_{i1} = k_i \quad i = 1, \ldots, r$

$a_{i2} = -b_i \quad i = 1, \ldots, r$.

Taking into account the index for the q'th observation, (15) can be rewritten as:

$$MRS_j^q(\underline{f}) = \frac{(a_{j1}a_{j2})\exp(a_{12}f_1^q)(1-\exp(-a_{12}))\left\{1+a_0a_{11}\frac{(1-\exp(-a_{12}f_1^q))}{(1-\exp(-a_{12}))}\right\}}{(a_{11}a_{12})\exp(a_{j2}f_j^q)(1-\exp(-a_{j2}))\left\{1+a_0a_{j1}\frac{(1-\exp(-a_{j2}f_j^q))}{(1-\exp(-a_{j2}))}\right\}} \qquad (16)$$

4.3.2.2 Form of the error function

Consider MRS as a function of $\underline{a}$ instead of $\underline{f}$. Here we wish to choose the ''best'' value for $\underline{a}$, for a given set of values for $\underline{f}$. Again we wish to minimize the error function, $E(\underline{a})$, subject to the restrictions on $\underline{a}$. From (16) we can express this as the nonlinear program:

$$\text{minimize} \quad E(\underline{a}) = \sum_j \sum_q \{MRS_j^q(\underline{a}) - y_j^q\}^2 \tag{17}$$

$$\text{subject to} \quad \varepsilon \leq a_{j1} \leq 1-\varepsilon \qquad j = 1, \ldots, r$$

$$\varepsilon \leq a_{j2} \qquad j = 1, \ldots, r$$

$$-1 < a_0, \quad a_0 \neq 0$$

$$1 + a_0 = \overline{|_j|} \, (1 + a_0 \, a_{j1})$$

where $MRS_j^q(\underline{a})$ is defined by (16). As in the additive case, we will show that the constraint which is not a bound can be ignored in the solution procedure.

4.3.2.3 Near quasiconvexity of the error function

Like its additive counterpart, the objective function of (17) for minimizing $E(\underline{a})$ in the multiplicative case is not strictly quasiconvex. However it does exhibit the same empirical behavior as the additive case when tested at a number of mesh points throughout its feasible region. Notice that (17) is the sum of squared differences of (16) and a constant term. Therefore we empirically tested (16), whose behavior is used as a surrogate for (17), for strict quasiconvexity. We found that when testing (16) at the more than 5 million mesh points listed in

Table 4.2, strict quasiconvexity failed less than 2.5 percent of the time. Also when solving (17) from a number of different starting points, all convergered to an equivalent-valued solution. Therefore while we have no theoretical guarantee of global convergence, it appears that it will also occur frequently in practice for the multiplicative case.

4.3.2.4 Gradient of the error function

In this section we develop an analytic representation of the gradient of the error function in (17) for the multiplicative case.

Let

$$M1_j^q(\underline{a}) = \frac{a_{j1}\, a_{j2}\, \exp(a_{12} f_1^q)\, (1 - \exp(-a_{12}))}{a_{11}\, a_{12}\, \exp(a_{j2} f_j^q)\, (1 - \exp(-a_{j2}))}$$

$$M2_j^q(\underline{f}) = \frac{\left\{ 1 + a_0\, a_{11} \dfrac{(1 - \exp(-a_{12} f_1^q))}{(1 - \exp(-a_{12}))} \right\}}{\left\{ 1 + a_0\, a_{j1} \dfrac{(1 - \exp(-a_{j2} f_j^q))}{(1 - \exp(-a_{j2}))} \right\}}$$

This implies $MRS_j^q(\underline{a}) = M1_j^q(\underline{a})\ M2_j^q(\underline{a})$ in the multiplicative case.

The error function is:

$$E(\underline{a}) = \sum_j \sum_q \{ MRS_j^q(\underline{a}) - y_j^q \}^2$$

Therefore,

$$\frac{\partial E}{\partial a_{ik}} = 2 \sum_j \sum_q \{MRS_j^q(\underline{a}) - y_j^q\} \quad \frac{\partial MRS_j^q(\underline{a})}{\partial a_{ik}}$$

TABLE 4.2

TEST OF STRICT QUASICONVEXITY IN THE CASE OF AN ASSUMED MULTIPLICATIVE UTILITY FUNCTION

	Minimum Value	Maximum Value	Mesh Size
alpha	0.1	0.9	0.4
f_1^q	0.1	0.9	0.4
f_1^q	0.1	0.9	0.4
a_0^1	0.1	0.9	0.4
a_{11}^1	0.1	0.9	0.4
a_{j1}^1	0.1	0.9	0.4
a_{12}^1	1.1	2.6	0.5
a_{j2}^1	1.1	2.6	0.5
a_0^2	0.1	0.9	0.4
a_{11}^2	0.1	0.9	0.4
a_{j1}^2	0.1	0.9	0.4
a_{12}^2	1.1	2.6	0.5
a_{j2}^2	1.1	2.6	0.5

Total number of grid points examined = 5,038,848

Percentage of points failing strict quasiconvexity = 2.47

$$= 2 \sum_j \sum_q \{MRS_j^q(\underline{a}) - y_j^q\} \left\{ M1_j^q(\underline{a}) \frac{\partial M2_j^q(\underline{a})}{\partial a_{ik}} + M2_j^q(\underline{a}) \frac{\partial M1_j^q(\underline{a})}{\partial a_{ik}} \right\}$$

Now $\frac{\partial M1_j^q(\underline{a})}{\partial a_0} = 0$. Since $MRS_j^q(\underline{a}) = M1_j^q(\underline{a})$ when the underlying form is additive, we have already determined $\frac{\partial M1_j^q(\underline{a})}{\partial a_{ik}}$ for i=1,...,r and k=1,2 (see section 4.3.1.4). All that remains is to determine $\frac{\partial M2_j^q(\underline{a})}{\partial a_{ik}}$. We simply state the results to avoid the detail of algebraic manipulation.

$$\frac{\partial M2_j^q(\underline{a})}{\partial a_0} = \frac{a_{11} \dfrac{1-\exp(-a_{12}f_1^q)}{1-\exp(-a_{12})}}{1+a_0 a_{j1} \dfrac{1-\exp(-a_{j2}f_j^q)}{1-\exp(-a_{j2})}} - M2_j^q(\underline{a}) \frac{a_{j1} \dfrac{1-\exp(-a_{j2}f_j^q)}{1-\exp(-a_{j2})}}{1+a_0 a_{j1} \dfrac{1-\exp(-a_{j2}f_j^q)}{1-\exp(-a_{j2})}}$$

$$\frac{\partial M2_j^q(\underline{a})}{\partial a_{11}} = \frac{a_0 \dfrac{1-\exp(-a_{12}f_1^q)}{1-\exp(-a_{12})}}{1+a_0 a_{j1} \dfrac{1-\exp(-a_{j2}f_j^q)}{1-\exp(-a_{j2})}}$$

$$\frac{\partial M2_j^q(\underline{a})}{\partial a_{j1}} = \frac{a_0 \dfrac{1-\exp(-a_{j2}f_j^q)}{1-\exp(-a_{j2})}}{1+a_0 a_{j1} \dfrac{1-\exp(-a_{j2}f_j^q)}{1-\exp(-a_{j2})}} \{-M2_j^q(\underline{a})\}$$

$$\frac{\partial M2_j^q(\underline{a})}{\partial a_{12}} = \frac{a_0\, a_{11}}{1+a_0 a_{j1}\,\dfrac{1-\exp(-a_{j2}f_j^q)}{1-\exp(-a_{j2})}} \left\{ \frac{\exp(-a_{12}(1+f_1^q))-\exp(-a_{12})}{(1-\exp(-a_{12}))^2} + \frac{f_1^q \exp(-a_{12}\, f_1^q)}{1-\exp(-a_{12})} \right\}$$

$$\frac{\partial M2_j^q(\underline{a})}{\partial a_{j2}} = \frac{1}{1+a_0 a_{j1}\,\dfrac{1-\exp(-a_{j2}f_j^q)}{1-\exp(-a_{j2})}} \left\{ \frac{\exp(-a_{j2}(1+f_j^q))-\exp(-a_{j2})}{(1-\exp(-a_{j2}))^2} + \frac{f_j^q \exp(-a_{j2}\, f_j^q)}{1-\exp(-a_{j2})} \right\} \{-M2_j^q(\underline{a})\}$$

4.3.2.5 Solution of the error minimization problem (multiplicative form)

Here we develop an efficient technique for solving (17) which results in a best approximation of $MRS_j(\underline{f})$. A key component in this technique results from the following theorem.

Theorem 4.7: Consider the nonlinear program:

$$\text{minimize} \quad E(\underline{a}) = \sum_j \sum_q \{MRS_j^q(\underline{a}) - y_j^q\}^2 \qquad (18)$$

$$\text{subject to} \quad \varepsilon \le a_{j1} \le 1-\varepsilon \qquad j = 1, \ldots, r$$

$$\varepsilon \le a_{j2} \qquad j = 1, \ldots, r$$

$$-1 < a_0, \quad a_0 \neq 0$$

$$1 + a_0 = \overline{|_j|}\ (1 + a_0\, a_{j1})$$

where

$$MRS_j^q(\underline{f}) = \frac{(a_{j1}a_{j2})\exp(a_{12}f_I^q)(1-\exp(-a_{12}))\left\{1+a_0a_{11}\frac{(1-\exp(-a_{12}f_I^q))}{(1-\exp(-a_{12}))}\right\}}{(a_{11}a_{12})\exp(a_{j2}f_j^q)(1-\exp(-a_{j2}))\left\{1+a_0a_{j1}\frac{(1-\exp(-a_{j2}f_j^q))}{(1-\exp(-a_{j2}))}\right\}}$$

Define a similar problem, (19), the same as (18) except omit the single nonlinear constraint. Assume that $\underline{a}$ is an optimal solution to (19). Let

$$a_0' = a_0/c$$

$$a_{j1}' = c\, a_{j1}$$

$$a_{j2}' = a_{j2}$$

with

$$c = \frac{a_0}{\overline{|_j|}\,(1 + a_0\, a_{j1}) - 1}$$

Then $\underline{a}'$ is an optimal solution to (18).

Proof:

$$1 + a_0' = 1 + a_0/c$$

$$= 1 + \frac{a_0\,[\,\overline{|_j|}\,(1 + a_0\, a_{j1}) - 1]}{a_0}$$

$$= \overline{|_j|}\,(1 + a_0\, a_{j1})$$

$$= \overline{|_j|}\ (1 + (a_0/c)\ (c\ a_{j1}))$$

$$= \overline{|_j|}\ (1 + a_0'\ a_{j1}')$$

Therefore the nonlinear constraint is satisfied. Since $a_{j2} \geq \varepsilon$, $a_{j2}' \geq \varepsilon$ for all j.

Now we show that $a_0' > -1$ and $\neq 0$. First, $c > 0$ and $a_0 \neq 0$ implies $a_0' \neq 0$.

$$a_0' = \frac{a_0}{c}$$

$$= \overline{|_j|}\ (1 + a_0\ a_{j1}) - 1$$

Case 1: $-1 < a_0 < 0 \quad ===> \quad 0 < \overline{|_j|}\ (1 + a_0\ a_{j1}) < 1.$

Therefore $-1 < a_0' < 0$.

Case 2: $a_0 > 0 \quad ===> \quad \overline{|_j|}\ (1 + a_0\ a_{j1}) > 1$

Therefore $a_0' > 0$.

To show that a_{j1}' satisfies the bounds for all j, since $c > 0$ and $a_{j1} > \varepsilon$ for all j, $a_{j1}' > \varepsilon$, for ε chosen suitably small. To show that $a_{j1}' < 1 - \varepsilon$ for all j, we examine two cases.

Case 1: $-1 < a_0 < 0$

Therefore, $1 + a_0\ a_{j1} > \overline{|_i|}\ (1 + a_0\ a_{i1})$

or,

$$\frac{a_0\, a_{j1}}{\prod_i (1 + a_0\, a_{i1}) - 1} < 1$$

which implies $a'_{j1} < 1$.

Case 2: $a_0 > 0$

Therefore, $1 + a_0\, a_{j1} < \prod_i (1 + a_0\, a_{i1})$

or,

$$\frac{a_0\, a_{j1}}{\prod_i (1 + a_0\, a_{i1}) - 1} < 1$$

which implies $a'_{j1} < 1$.

We have just demonstrated that $\underline{a}'$ is feasible for (18). Now we demonstrate optimality. Since (19) is a relaxation of (18), $E(\underline{a}) \leq E(\underline{a}')$. Note that the c's in (18) cancel in the objective function, so $E(\underline{a}) = E(\underline{a}')$. Hence $\underline{a}'$ is optimal for (18).

Q.E.D

This theorem provides an extremely useful result. We can solve the unconstrained version of (18) (while implicitly handling the bounds). Then in time that is a linear function of the number of objectives we can adjust the unconstrained solution to satisfy the nonlinear constraint while still maintaining optimality.

This result leads to the following algorithm for determining a

representation of $MRS_j(\underline{f})$ when the underlying utility function is assumed to be multiplicative.

Algorithm 4.8: Determining an explicit utility function (multiplicative form)

Step 0: Set $p = 1$, set $\underline{a}^0$ to infinity, set $\underline{a}^1$ to a point that satisfies the bounds on $\underline{a}$, and fix $\varepsilon > 0$, a convergence parameter.

Step 1: Use an unconstrained nonlinear optimization technique to solve (19), starting from $\underline{a}^p$.

Step 2: Define

$$c = \frac{a_0^p}{\overline{|_j|}\ (1 + a_0^p\ a_{j1}^p) - 1}$$

$$b_0^p = a_0^p / c$$

$$b_{j1}^p = c\ a_{j1}^p$$

$$b_{j2}^p = a_{j2}^p$$

Step 3: Compute $\varepsilon^p = || \underline{b}^p - \underline{b}^{p-1} ||_2$

If $\varepsilon^p \leq \varepsilon$ stop

otherwise for each a_i^{p+1} set $a_i^{p+1} = a_i^p + 0.5$, set $p = p+1$ and go to Step 1.

4.4 SOLUTION OF THE UTILITY MAXIMIZATION PROBLEM

In section 4.3.2, we noted some mathematical programming applications of explicit utility theory. Our purpose in this section is to develop a collection of results that are useful when the explicit utility maximization problem is solved as a mathematical program. Specifically, we show that if the additive form of an explicit utility function is assumed, under reasonable assumptions any locally optimal solution is also globally optimal. Also, while the multiplicative version of an explicit utility function does not permit us to universally guarantee global optimality, our results indicate that we may acheive it frequently in practice.

We first deal with the additive case.

Theorem 4.9: (Concavity of the additive form) Let

$$U(\underline{f}(\underline{x})) = \sum k_i \, u_i(f_i(\underline{x}))$$

where: $u_i(f_i)$ is a strictly concave and increasing one dimensional utility function.

$\underline{x} = (x_1, x_2, \ldots, x_n)$ is the value of the decision vector, with X being the convex set of all feasible $\underline{x}$.

k_i is a positive scale factor for the i'th objective, $0 < k_i < 1$.

$f_i(x_i)$ is a concave function of the decision variables and represents the level of acheivement of objective i. For this discussion, we assume without loss of generality that f_i has been scaled so that $0 \leq f_i \leq 1$.

Also, assume $u_i(0) = 0$, $u_i(1) = 1$, and $\sum_i k_i = 1$. Then $U(\underline{f}(\underline{x}))$ is a concave function over $\underline{x} \in X$.

Proof: Fix $\underline{x}$, $\underline{y} \in X$.

$$U(\underline{f}(\alpha\underline{x} + (1-\alpha)\underline{y})) = \sum_i k_i\, u_i(f_i(\alpha\underline{x} + (1-\alpha)\underline{y}))$$

$$\geq \sum_i k_i\, u_i(\alpha f_i(\underline{x}) + (1-\alpha) f_i(\underline{y}))$$

$$\geq \sum_i k_i\, (\alpha u_i(f_i(\underline{x})) + (1-\alpha) u_i(f_i(\underline{y})))$$

$$= \alpha\, U(\underline{f}(\underline{x})) + (1-\alpha)\, U(\underline{f}(\underline{y}))$$

Q.E.D.

A common special case for this theorem is when the f_i's are linear. It is then possible to use the explicit utility function as an objective function in a mathematical programming algorithm to obtain globally optimal solutions.

We now deal with the multiplicative case. Here we do not derive a global result as in the additive case. However we report some empirical behaviour that appears to be useful in practice. Our analysis of the multiplicative form deals with two cases: 1) $k>0$, and 2) $k<0$. We first show that when $k>0$, under some reasonable assumptions, the multiplicative form is strictly quasiconcave, and therefore yields globally optimal solutions when used as an objective function in a mathematical programming algorithm. Unfortunately, we show a counterexample to this situation when $k<0$. However, we report some test results indicating that often the multiplicative form (with $k<0$) will behave in practice as if it were strictly quasiconcave.

First for the multiplicative case with $k>0$, we develop some necessary preliminary results, and then prove strict quasiconcavity.

Proposition 4.10:

Let $a_i, b_i \geq 0$, $i=1,\ldots,n$, such that $\overline{\prod_{i=1}^{n}} a_i \leq \overline{\prod_{i=1}^{n}} b_i$. Then,

$$\overline{\prod_{i=1}^{n}} 2a_i \leq \overline{\prod_{i=1}^{n}} (a_i+b_i).$$

Proof: We first prove the result for n=2. Let n=2 and without loss of generality assume $a_1<a_2$. Given $a_1a_2<b_1b_2$, we must show that $4a_1a_2\leq(a_1+b_1)(a_2+b_2)$.

We note

$$\begin{aligned}(a_1+b_1)(a_2+b_2)-(a_1+b_2)(a_2+b_1) &= (a_1b_2+b_1a_2)-(a_1b_1+a_2b_2)\\ &= (a_2-a_1)(b_1-b_2).\end{aligned}$$

Therefore,

$$\min[(a_1+b_1)(a_2+b_2),(a_1+b_2)(a_2+b_1)] = \begin{cases}(a_1+b_1)(a_2+b_2) & \text{if } b_1\leq b_2\\ (a_1+b_2)(a_2+b_1) & \text{if } b_2\leq b_1.\end{cases}$$

It follows that if the claim is true for $b_1\leq b_2$ then it also must be true for $b_1>b_2$. Therefore assume $b_1\leq b_2$.

With $0\leq a_1\leq a_2$, $0\leq b_1\leq b_2$, and $a_1a_2\leq b_1b_2$, there are four possible cases:

i) $a_1 \leq a_2 \leq b_1 \leq b_2$

ii) $a_1 \leq b_1 \leq a_2 \leq b_2$

iii) $a_1 \leq b_1 \leq b_2 \leq a_2$

iv) $b_1 \leq a_1 \leq a_2 \leq b_2$.

In cases (i) and (ii), $2a_i \leq a_i+b_i$ for i=1,2 so $\prod_i 2a_i \leq \prod_i (a_i+b_i)$.

In cases (iii) and (iv) we have:

$(a_1-b_1)(a_2-b_2) \leq 0$, so

$$a_1b_2+a_2b_1 \geq a_1a_2+b_1b_2$$
$$\geq 2a_1a_2.$$

Therefore

$$a_1b_2+a_2b_1+b_1b_2 \geq 2a_1a_2+b_1b_2 \geq 3a_1a_2$$

Q.E.D. for n=2

The proof for $n > 2$ requires the following lemma.

Lemma 4.11:

If $a_i, b_i \geq 0$, $i=1,\ldots,n$, then

$$2^{n-1}(\prod_{i=1}^{n} a_i + \prod_{i=1}^{n} b_i) \leq \prod_{i=1}^{n} (a_i + b_i).$$

Proof of Lemma 4.11:

By the Kuhn-Tucker theorem, it can be shown that the nonlinear program

$$\min_{\underline{a},\underline{b}} \prod_{i=1}^{n} (a_i + b_i)$$

$$\text{s.t.} \quad \prod_{i=1}^{n} a_i + \prod_{i=1}^{n} b_i = C$$

$$a_i, b_i \geq 0 \quad i=1,\ldots,n$$

for any feasible value of C, is solved at $a_i = b_i$, and has minimum objective function value $\prod_{i=1}^{n} 2a_i$, in which case,

$$\frac{\prod_{i=1}^{n} (a_i + b_i)}{\prod_{i=1}^{n} a_i + \prod_{i=1}^{n} b_i} = 2^{n-1}$$

which completes the proof of the lemma.

Returning to the proof of proposition 4.10, let n>2 and let:

$a_1' = \prod_{i=1}^{n-1} a_i$, $a_2' = a_n$, $b_1' = \prod_{i=1}^{n-1} b_i$, $b_2' = b_n$. Then,

$$\prod_{i=1}^{n} 2a_i = 2^n \prod_{i=1}^{n} a_i$$

$$= 2^n \prod_{i=1}^{2} a_i'$$

$$= 2^{n-2} \prod_{i=1}^{2} 2a'_i$$

$$\leq 2^{n-2} \prod_{i=1}^{2} (a_i' + b_i') \quad \text{by proposition for n=2}$$

$$= 2^{n-2} \left(\prod_{i=1}^{n-1} a_i + \prod_{i=1}^{n-1} b_i\right)(a_n+b_n)$$

$$\leq 2^{n-2} \left(2^{2-n} \prod_{i=1}^{n-1} (a_i+b_i) \right)(a_n+b_n) \text{ by lemma 4.11}$$

$$= \prod_{i=1}^{n} (a_i + b_i).$$

Q.E.D

With proposition 4.10, we can now prove the following theorem.

Theorem 4.12: Let

$$U(\underline{f}(\underline{x})) = k^{-1} \{ \prod_i (1 + k\, k_i\, u_i(f_i(\underline{x}))) - 1 \}$$

where $u_i(f_i)$ is an increasing one dimensional utility function.

$\underline{x} = (x_1, x_2, \ldots, x_n)$ is the value of the decision vector, with X being the convex set of all feasible $\underline{x}$.

k_i is a positive scale factor for the i'th objective, $0 < k_i < 1$.

k is another scale factor to ensure that $U(\underline{1}) = 1$. Therefore k must satisfy the relation $1+k=\prod_i(1+kk_i)$, with $k > -1$ and $k \neq 0$.

$f_i(\underline{x})$ is the level of acheivement of objective i. For this discussion we assume without loss of generality that $0 \leq f_i(\underline{x}) \leq 1$.

Also, assume $u_i(0) = 0$, and $u_i(1) = 1$. Then $U(\underline{f}(\underline{x}))$ is a strictly quasiconcave function over $\underline{x} \in X$ for $k > 0$.

Proof: Fix $\underline{x}$, $\underline{y} \in X$, and assume $k > 0$. Also assume without loss of generality that $U(\underline{f}(\underline{x})) \preceq U(\underline{f}(\underline{y}))$. Therefore,

$$U(0.5\underline{x} + 0.5\underline{y}) = k^{-1} \{ \prod_i (1 + k\, k_i\, u_i(f_i(0.5\underline{x}+0.5\underline{y}))) - 1 \}$$

By concavity of f_i, monotonicity of u_i, positivity of k, and nonnegativity of $1+kk_iu_i(f_i(\underline{x}))$, we have the previous right hand side:

$$\geq k^{-1} \{ \prod_i (1 + k\, k_i\, u_i(0.5f_i(\underline{x})+0.5f_i(\underline{y}))) - 1 \}$$

By strict concavity of u_i, positivity of k, and nonnegativity of $1+kk_iu_i(0.5f_i(\underline{x})+0.5f_i(\underline{y}))$, we have the previous right hand side:

$$> k^{-1} \{ \prod_i (1 + k\, k_i\, 0.5u_i(f_i(\underline{x}))+k\, k_i\, 0.5u_i(f_i(\underline{y})))-1\}$$

By proposition 4.10, with $a_i=0.5+0.5kk_iu_i(f_i(\underline{x}))$, and $b_i=0.5+0.5kk_iu_i(f_i(\underline{y}))$, along with $U(\underline{f}(\underline{x})) \preceq U(\underline{f}(\underline{y}))$, we have:

$$\geq k^{-1} \{ \prod_i (1 + k\, k_i\, 0.5u_i(f_i(\underline{x}))+k\, k_i\, 0.5u_i(f_i(\underline{x})))-1\}$$

$$= U(\underline{f}(\underline{x}))$$

Q.E.D

For the case $k<0$, we tested the multiplicative form of an explicit utility function for the case of three criteria. First note that in this situation there does exist cases where the multiplicative form is

not strictly quasiconcave. We originally tested our three variable case with 1 million mesh points (Table 4.3) and found a strict quasiconcavity failure rate of 1.58 per cent. However after numerous computer runs we began to acquire the insight that the strict quasiconcavity failure was occurring as the original overall scaling constant, k, approached its lower bound of -1.

TABLE 4.3

TEST OF STRICT QUASICONCAVITY OF THE MULTIPLICATIVE UTILITY FUNCTION - CASE 1

	Minimum Value	Maximum Value	Mesh Size
x_i	0.0	1.0	0.25
y_i	0.0	1.0	0.25
k_i	0.05	0.95	0.30

Number of criteria = 3, alpha = 0.5, $b_1 = -1.0$, $b_2 = -2.0$, $b_3 = -3.0$, and $f(x_i) = x_i$

Total number of grid points examined = 1.0×10^6

Percentage of points failing strict quasiconcavity = 1.58

We conjectured that if we chose a fixed value of k, say k', close to -1, and examined the utility function with values of k in the range

(-1,k'], we would observe a large increase in the percentage of strict quasiconcavity failure. This was found to be true. For example, we set k'= -0.95 and tested our hypothesis on nearly 2 million grid points (Table 4.4). We observed a failure rate of nearly 15 per cent, a ten-fold increase over the failure rate reported in Table 4.3. Even more importantly when we reexamined the failures occurring in Table 4.3, not one failure was found when k was greater than our arbitrary cutoff point of -0.95.

TABLE 4.4

TEST OF STRICT QUASICONCAVITY OF THE MULTIPLICATIVE UTILITY FUNCTION - CASE 2

	Minimum Value	Maximum Value	Mesh Size
x_i	0.0	1.0	0.25
y_i	0.0	1.0	0.25
k_i	0.95	0.99	0.01

Number of criteria = 3, alpha = 0.5, $b_1 = -1.0$, $b_2 = -2.0$, $b_3 = -3.0$, and $f(x_i) = x_i$

Total number of grid points examined = 1.95×10^6

Percentage of points failing strict quasiconcavity = 14.86

In fact in all the tests that were made (totaling over 10 million grid points) for our three criteria problem, we never found a single case of strict quasiconcavity failure of the multiplicative utility function when k was greater than -0.95.

Of course these results guarantee nothing but they do provide a strong indication that the multiplicative utility function (for $k<0$) may often behave in practice as if it were strictly quasiconcave (and therefore lead to globally optimal solutions to the utility maximization problem). Also, based on our results it appears that we have an easy-to-determine criterion to alert us when the utility function may behave undesireably

To summarize the results of this section, under appropriate assumptions the additive form of a utility function is strictly concave and leads to globally optimal solutions when solved by the use of mathematical programming. The multiplicative form is strictly quasiconcave only for $k>0$, however our empirical investigation indicates that it may lead to globally optimal solutions even when $k<0$.

4.5 AN ALGORITHM FOR AN IMPLICIT/EXPLICIT APPROACH TO MULTIOBJECTIVE DECISION MAKING

Here we combine the results of Sections 4.1-4.4 to derive an implicit/explicit algorithm for multiobjective decisionmaking. As in the previous sections, we assume that the underlying utility function is either additive or multiplicative, the odufs are concave and increasing, the f_i's are concave, and X describes a compact and convex set.

Algorithm 4.13: An implicit/explicit approach to mulitobjective decision making

Step 0: Choose $\underline{x}^1 \in X$, set $k = 1$, set $j = 1$, and fix j_max, the number of observations used in approximating the decision maker's explicit utility function.

Step 1: Assess the decision maker's tradeoffs at the current point, $\underline{x}^k$, using Algorithm 4.2. This results in $y_2^k, \ldots, y_r^k$.

Step 2: If $j < j_max$ go to Step 3, otherwise set $j = 0$. If the underlying form of the utility function is additive use Algorithm 4.6 to determine an explicit representation of the decision maker's utility function, based on the past j_max observations. If the underlying form is multiplicative, use Algorithm 4.8. Solve to optimality the explicit utility problem (P1), resulting in the solution $\underline{z}^k$. If $\underline{f}(\underline{z}^k)$ is preferred to $\underline{f}(\underline{x}^k)$ go to Step 4, otherwise go to Step 3.

Step 3: Solve $\sum_{i=2}^{r} y_i^k \nabla_{\underline{x}} f_i(\underline{x}^k) \cdot \underline{z}$, $\underline{z} \in X$, resulting in $\underline{z}^k$.

Step 4: Set $\underline{d}^k = \underline{z}^k - \underline{x}^k$.

Step 5: Determine the step size parameter, t^k, using Algorithm 4.3. Set $\underline{x}^{k+1} = \underline{x}^k + t^k \underline{d}^k$, $k = k + 1$, $j = j + 1$. If $|\underline{x}^k - \underline{x}^{k-1}| <$ convergence tolerance stop. Otherwise go to Step 1.

This algorithm posesses a number of desirable features.

1) At any time, it requires the decision maker only to compare two feasible points in criterion space.
2) It generates efficient solutions.
3) It is ''adaptable''. That is, it permits the decision maker to develop his/her preference structure as the procedure progresses.
4) It maintains (or exceeds) the same robust convergence characteristics of the implicit utility function approaches.

The first property, that of only requiring comparisons in criterion space, is particularly useful. Typically, the number of criteria is small (say three to eight), while the number of decision variables may be in the thousands. Therefore by dealing only with criterion space the decision maker can operate in a more meaningful and reduced region. Also the algorithm requires only binary comparisons. Anytime the decision maker is required to respond, he/she must indicate only that one collection of benefits is preferred, not preferred, or equal to another. This leads to a relatively light information burden.

Consider the claim of efficient solutions. Candidate solutions are generated in Steps 2 and 3. In Step 3, the solutions result from a problem whose objective function is the positively-weighted sum of all objectives, therefore an efficient solution results (c.f. Cohon [1978]). In Step 2, where the explicit utility function is maximized, the solutions are also efficient due to the strictly increasing nature of the odufs and the convexity properties of the

objective functions. (For example, suppose that $\underline{x}^*$ is an optimal solution to the utility maximization problem. Also suppose that $\underline{x}^*$ is not efficient. Then there must exist another feasible solution $\underline{x}'$ such that $f_i(x_i') \geq f_i(x_i^*)$ for all i and $f_i(x_i') > f_i(x_i^*)$ for at least one i. But then $U[\underline{f}(\underline{x}')] > U[\underline{f}(\underline{x}^*)]$, contradicting the optimality of $\underline{x}^*$. Therefore $\underline{x}^*$ is efficient.) Hence, the algorithm generates efficient solutions.

The implicit/explicit algorithm permits the decision maker to learn during the man-machine interaction. Often a decision maker gains new insight into the dynamics of a problem through the solution procedure. Therefore the decision maker's preference structure may change throughout the course of an interactive dialog. A recent paper by Dickmeyer [1983] reinforces the value of this man-machine interaction. The implicit/explicit procedure fosters this learning process.

Finally, the algorithm has the robust convergence characteristics of an implicit utility approach, which are described in Geoffrion, Dyer, and Feinberg [1972]. The implicit/explicit algorithm contains an acceleration step every j_max iterations. When the decision maker operates under a preference structure that is well-approximated by the assumed utility function, convergence is quickened by this acceleration step. However if the acceleration step leads to a less-preferred solution, the implicit/explicit algorithm temporarily reverts to a pure implicit approach, which maintains the ascent characteristics noted above in Geoffrion et al.

An optimal solution is found when two successive iterations yield the same solution. In practice however, the procedure may be terminated any time the decision maker wishes (due to the primal feasible, efficient characteristics of the algorithm).

CHAPTER 5

FORMULATION OF THE FOREST MANAGEMENT PROBLEM

Probably no other aspect of problem analysis has a greater impact on the value of a solution effort than problem formulation. The same problem under different formulations can vary from useful and useable to uninformative and intractable. The value of a good formulation, and the time spent to derive it, is of obvious importance.

In chapter 1 we discussed at length various approaches used in modeling the forest management problem. We noted that these differing formulations often are a result of the diverse orientations and objectives of the user. This chapter deals both descriptively and mathematically with a formulation of the forest management problem as especially concerns the nonindustrial private forest landowner.

One of the most fundamental uses of mathematical optimization is to construct an abstract model of a real-world system. The model is solved and the results are inferred to the original problem. If the mathematical model is a sufficiently good reflection of reality, the results of the optimization may provide useful insight for management of the real-world system. In this process the system designer is often faced with the tradeoff of including sufficient detail in the model versus obtaining a solvable problem. This points to two key elements of a useful formulation: 1) it must sufficiently represent the system under study, and 2) it must be reasonably solvable.

In the following sections of this chapter we provide, first a

descriptive discussion of our formulation for forest management planning, then a mathematical representation. This discussion focuses on important factors relating to usefulness and tractability, such as necessary simplifying assumptions. We orient the discussion from a functional point of view. That is, we describe the desirable aspects of our forest management formulation and then express them mathematically rather than assuming a particular mathematical form for the model a priori.

5.1 MODEL CONSIDERATIONS

5.1.1 Assumptions

We assume that the forest is divided into one or more homogeneous components, referred to as stands. A stand represents a contiguous area of land containing trees of the same species, age, site index and density. The process of delineating the stand boundaries is critical to the solution procedure and often very difficult in practice, requiring the judgement of a professional forester. In our current implementation, no provision is made for mixed species stands. However this is not a limitation of the model itself, but reflects the lack (in our opinion) of ''sufficiently good'' general techniques for simulating the growth and yield of mixed species stands. (For instance, in our current implementation the decision variables are x_{ij}, the percentage of stand i managed under harvesting regime j. During the process of describing the various ways of harvesting the forest, one could stratify additionally by species.)

All cultural treatments that are designated for a portion of a stand are assumed to be applied uniformly. For instance, if the model recommends cutting a portion of a given stand to a residual basal area of 70 square feet per acre, it is assumed that all acres in that portion are treated likewise.

We also assume that for each stand there exists a finite number of harvesting regimes. These regimes are enumerated prior to the solution procedure, under the direction of the decision maker. Each regime describes completely a specific management practice along with the corresponding benefits.

The planning horizon length is fixed by the decision maker. We assume that the overall objective is to maximize benefits only within the planning period.

Probably the most restrictive assumption we make is that of stand independence with respect to response to cultural treatments. This means that the amount of benefit k on stand i is independent of the management activites on any other stands. This is not a completely accurate representation of reality in the case of highly mobile wildlife populations, but it is assumed here to provide a more tractable problem. We do feel that it is acceptable when judged in light of the design requirements of this analysis.

We implicitly assume that the models used to predict the response of the forest ecosystem to the prescribed cultural treatments are acceptably accurate. In addition, for the wildlife criteria we attempt to optimize habitat suitability. This requires the assumption that wildlife populations are perfectly correlated with habitat. (These

models are described in detail in chapter 2).

One final caveat regarding assumptions. A purpose of this model is to define ''optimal'' courses of action, but more importantly another purpose is to provide insight for the decision maker. In the case of some optimization problems, the mathematical solution often can be used directly as a management prescription. Not so here. In this case the data are soft, the management goals are often imprecisely stated, and the decision maker is frequently unsure as to just what is a good solution. In addition there are freqently long time lags between analysis, implementation, and harvest. Geoffrion's [1976] remarks are very appropriate for this context when he stated that the purpose of optimization is not numbers but insight.

5.1.2 Objective functions

Clearly the forest produces a multitude of benefits. These benefits are a particularly diverse group. For instance, wildlife, recreation, aesthetics, erosion control and fiber are all very tangible results of careful forest management. We noted in the introduction that the nonindustrial private forest landowner (NIPF) is the target group of this analysis effort. And they as a group are often interested in all of these forest benefits in addition to many others.

We have included in FORMAN, five forest benefits as objectives. These are 1) the amount of wood harvested, 2) the expected present financial value of the sale of harvested timber products, 3) production of quail, 4) production of deer and 5) production of squirrel.

Obviously there are many other forest benefits that are of

interest to forest landowners. The group above was chosen for two primary reasons. First, our past experience indicates that NIPF's in the southeastern U.S. are predominantly interested in timber and wildlife production. Second, for the benefits we have chosen, quantitative relationships exist that facilitate their inclusion in an optimization process. This is not to discount the importance of other, perhaps less quantifiable, forest benefits. But in order to impose a mathematical structure, we were forced to limit ourselves to those forest benefits for which reasonable functional relationships exist (see Chapter 2 for a thorough discussion of the models used for the simulation of forest benefits).

Also we wished to limit the number of objectives to a manageable number, especially for an interactive procedure. Interest in timber production is a frequent consideration of most landowners. The three wildlife objectives are associated with the various stages of forest succession. High quail production most often occurs in open fields or young stands. Deer habitat requirements are well satisfied by stands (of the proper type) of medium age. And squirrel habitat is best met by mature, mast-producing stands. (These statements represent generalities and are not meant to be interpreted as absolutes). The point is that these wildlife species tend to be conflicting in their habitat requirements, which has useful theoretical implications for multiobjective optimization. Keeney and Raiffa [1976] point out the usefulness of removing correlations between objectives to avoid the problem of redundancy in the analysis of impacts.

Therefore these five criteria were chosen as appropriate objective functions because 1) they are of importance to NIPF's and 2) functional relationships exist that relate them to timber stand characteristics.

5.1.3 Constraints

We assume three classes of constraints in our formulation. The first class represents the typical constraints on wood and cash flow. These constraints require that for each year of the planning horizon, the amount of timber and cash produced across the aggregation of all stands must be between a user-specified upper and lower bound. For example, a landowner might wish to avoid a large cashflow in any one year because of the corresponding tax liability. It may be preferred to spread this return over a few years. Conversely a landowner might wish to ensure a minimum level of cash to meet an anticipated need, such as financing a college education. The timber flow constraints are likewise grounded in reality. For example, we have worked with landowners who have in their employ a crew that performs timber harvesting on a year-round basis. In this case the landowner would like to ensure that the optimization model recommends harvesting schedules that would keep this harvesting crew reasonably employed throughout the year. In addition, many other examples exist that recommend the inclusion of these smoothing constraints.

The second class of constraints simply reflects physical limitations imposed by the formulation. These constraints require that each acre of land can be assigned to at most one management strategy. Each management regime describes completely all cultural treatments

that will take place throughout the planning horizon, which includes harvesting and regeneration. This implies that the same acreage cannot be simultaneously under the prescription of two or more management regimes.

The third class of constraints is new to the forestry problem per se, and also lightly researched in the area of optimization theory in general. These we have denoted as partitioned cardinality constraints. In the optimization context, given I mutually exclusive and collectively exhaustive partitions of the decision variables, the partitioned cardinality constraints require that at most p_i decision variables can be at positive levels in the i'th partition. This is a generalization of the cardinality constraint concept discussed in Tanahashi and Luenberger [1972] and Rubin [1975]. In the context of the forest management problem these constraints require that at most p_i different management regimes may be used for the i'th stand. (Note that because of the nature of the benefits simulation and since the mapping of management regimes to stand acreage is 1 to 1, new management regimes cannot be created by merely taking a convex combination of existing regimes.)

5.2 MATHEMATICAL FORMULATION

Let

i - index stand number

j - index harvesting regime

k - index objectives

t - index year of the planning horizon

and

a_t = discount factor for monetary benefits in year t

b_{kt} = lower bound for objective k in year t

I = number of stands

J_i = set of harvesting regimes considered for stand i

K = number of objectives

$m_i = |J_i|$

p_i = maximum number of regimes that can be used for stand i

T = number of years in the planning horizon.

u_{kt} = upper bound for objective k in year t

w_i = percent of total acreage contained in stands i

Define the decision variables as x_{ij} = percentage of stand i managed under regime j, with $\underline{x} = (x_{ij})$. Also let $\delta_{ij} = 1$ if $x_{ij} > 0$, 0 otherwise.

Define the measures of the objective as:

c_{ij1t} - volume of timber removed (cubic feet) in year t on tract i under harvesting regime j,

c_{ij2t} - worth in dollars of management activities in year t on tract i under harvesting regime j,

c_{ij3t} - habitat suitability index per acre for deer in year t on tract i under regime j,

c_{ij4t} - habitat suitability index per acre for squirrel in year t on tract i under regime j,

c_{ij5t} - habitat suitability index per acre for quail in year t on tract i under regime j.

Define the objective functions

$$f_1(\underline{x}) = \sum_i \sum_j \sum_t c_{ij1t}\, x_{ij}$$

$$f_2(\underline{x}) = \sum_i \sum_j \sum_t a_t\, c_{ij2t}\, x_{ij}$$

$$f_3(\underline{x}) = \sum_i \sum_j \sum_t w_i\, c_{ij3t}\, x_{ij}$$

$$f_4(\underline{x}) = \sum_i \sum_j \sum_t w_i\, c_{ij4t}\, x_{ij}$$

$$f_5(\underline{x}) = \sum_i \sum_j \sum_t w_i\, c_{ij5t}\, x_{ij} .$$

Let $\underline{f}(\underline{x}) = \{f_1(\underline{x}),\ldots,f_K(\underline{x})\}$, and let $U[\underline{f}(\underline{x})]$ be the (unknown) utility function of the decision maker. The forest management problem (FMP) that we investigate can be expressed as:

(FMP) $\max\ U[\underline{f}(\underline{x})]$

s.t.

$$b_{kt} \le \sum_i \sum_j c_{ijkt}\, x_{ij} \le u_{kt} \qquad k=1,2 \text{ and } t=1,\ldots,T$$

$$\sum_{j \in J_i} x_{ij} \le 1 \qquad i=1,\ldots,I$$

$$\sum_{j \in J_i} \delta_{ij} \le p_i \qquad i=1,\ldots,I$$

$$0 \le x_{ij} \le 1 \qquad i=1,\ldots,I \text{ and } j=1,\ldots,m_i$$

$$x_{ij} \le \delta_{ij} \qquad i=1,\ldots,I \text{ and } j=1,\ldots,m_i.$$

CHAPTER 6

IMPLEMENTATION OF THE IMPLICIT/EXPLICIT TECHNIQUE IN THE FOREST MANAGEMENT PROBLEM

In Chapter 4, we described a new algorithm for multiobjective optimization. In Chapter 5, we developed a mathematical description of the forest management problem that we wish to solve. However there still remain some issues of implementation that have not been addressed. In this chapter we focus on those issues and provide a description of their resolution in FORMAN. Specifically, we deal with:

1) determination of the exact form of the explicit utility function,
2) a heuristic solution of partitioned cardinality constrained linear programming subproblems, and
3) solution of the explicit utility maximization problem.

6.1 SOLVING FOR THE EXPLICIT UTILITY FUNCTION

We demonstrated in Chapter 4 that solving for a representation of an explicit utility function, based on the decision maker's marginal rates of substitution, requires the solution of a nonlinear program. This nonlinear program is a least squares problem subject to simple bounds on the variables and either i) a single linear constraint in the additive case, or ii) a single nonlinear constraint in the multiplicative case. Recall that by Theorems 4.5 and 4.7, we can solve either of these problems (additive or multiplicative form) without

regard to the linear or nonlinear constraint. Then in time that is a linear function of the number of benefits, this solution can be modified to provide an optimal solution to the original problem with the linear or nonlinear constraint included. Therefore the question of solving for an explicit utility function becomes in this case, a question of solving a nonlinear least squares problem subject to simple bounds on the variables.

6.1.1 A quasi-Newton approach to unconstrained minimization

The nonlinear least squares problems subject to simple bounds were given in Chapter 4 by (10) for the additive case, and (19) for the multiplicative case. We repeat these formulations here as problems (1) and (2), respectively.

Problem 1:

$$\min E(\underline{a}) = \sum_j \sum_q \{MRS_j^q(\underline{a}) - y_j^q\}^2 \qquad (1)$$

subject to $\quad \varepsilon \leq a_{j1} \leq 1-\varepsilon \qquad j = 1, \ldots, r$

$$\varepsilon \leq a_{j2} \qquad j = 1, \ldots, r$$

where

$$MRS_j^q(\underline{a}) = \frac{(a_{j1}\, a_{j2} \exp(a_{12} f_1^q)\, (1 - \exp(-a_{12}))}{(a_{11}\, a_{12} \exp(a_{j2} f_j^q)\, (1 - \exp(-a_{j2}))} .$$

Problem 2:

$$\text{minimize} \quad E(\underline{a}) = \sum_j \sum_q \{MRS_j^q(\underline{a}) - y_j^q\}^2 \qquad (2)$$

$$\text{subject to} \qquad \varepsilon \leq a_{j1} \leq 1-\varepsilon \qquad j = 1, \ldots, r$$

$$\varepsilon \leq a_{j2} \qquad j = 1, \ldots, r$$

$$-1 < a_0, \quad a_0 \neq 0$$

where

$$MRS_j^q(\underline{a}) = \frac{(a_{j1}a_{j2})\exp(a_{12}f_1^q)(1-\exp(-a_{12}))\left\{1+a_0a_{11}\dfrac{(1-\exp(-a_{12}f_1^q))}{(1-\exp(-a_{12}))}\right\}}{(a_{11}a_{12})\exp(a_{j2}f_j^q)(1-\exp(-a_{j2}))\left\{1+a_0a_{j1}\dfrac{(1-\exp(-a_{j2}f_j^q))}{(1-\exp(-a_{j2}))}\right\}}.$$

As a solution technique, we decided upon the use of the BFGS (Broyden-Fletcher-Goldfarb-Shanno) algorithm (c.f. Fletcher [1980] or Dennis and More [1977]), which was modified to handle the simple bounds. BFGS has been successful on a wide variety of problems, and currently ranks as one the best overall algorithms for unconstrained optimization. Before describing our modified version of BFGS, it is useful to review the Newton and quasi-Newton approaches to unconstrained, nonlinear optimization.

We seek a minimum of $f(\underline{x})$, $\underline{x} \in R^n$. Suppose that f is continuous and has continuous second partial derivatives on R^n. Let the gradient of f at $\underline{x}$ be denoted by $\underline{g}(\underline{x})$. Let $F(\underline{x})$ represent the Hessian matrix of second partial derivatives at $\underline{x}$. The basic idea of Newton's method is that f is approximated locally by a quadratic function, and we minimize this approximated quadratic exactly. In the neighborhood of $\underline{x}_k$, (k is an iteration count), we can approximate f by the truncated Taylor series:

$$f(\underline{x}) \cong f_k + \underline{g}_k(\underline{x} - \underline{x}_k) + (1/2)(\underline{x} - \underline{x}_k)^T F_k (\underline{x} - \underline{x}_k),$$

where:

$$f_k = f(\underline{x}_k),$$

$$\underline{g}_k = \underline{g}(\underline{x}_k), \text{ and}$$

$$F_k = F(\underline{x}_k).$$

If F_k is positive definite, we can minimize the approximation of f (i.e. the right hand side of the above relation) at:

$$\underline{x}_{k+1} = \underline{x}_k - F_k^{-1} \underline{g}_k^T.$$

The pure form of Newton's method then uses $\underline{x}_{k+1}$ to define a quadratic approximation of f and repeats the process. The solution technique is to iteratively solve for $\underline{x}_{k+1}$ until $|\underline{x}_{k+1} - \underline{x}_k| <$ some convergence tolerance. This results in an iterative descent approach. It is iterative since each successive step is generated based on information from the previous step. It provides descent because it can be shown that f decreases at each iteration, if the starting solution is sufficiently close to a local minimum. In addition, the pure Newton's method possesses some desireable convergence properties. To characterize the convergence behaviour of Newton's method more precisely, we define the following terms.

If an algorithm is guaranteed to generate a sequence of points converging to a local minimizer, starting from an arbitrary point, then the algorithm is <u>globally convergent</u>. Let the sequence of solutions, $\{r_k\}$, converge to the optimal solution r^*. The <u>order of convergence</u> of $\{r_k\}$ is defined as the supremum of the nonnegative numbers p satisfying

$$0 \leq \overline{\lim_{k \to \text{infinity}}} \frac{| r_{k+1} - r^* |}{| r_k - r^* |^p} \leq \text{infinity}.$$

Given that the initial solution, x_0, is sufficiently close to the

optimal solution x^*, Newton's method converges with order two. However if the initial solution is not sufficiently close, Newton's method may fail to even satisfy global convergence.

It can be shown that a sufficient condition for descent is that F^{-1} is positive definite at each step. However for Newton's method, F^{-1} may not even be defined away from the optimal solution, $\underline{x}^*$. Also, determining F (and/or F^{-1}) may prove to be expensive computationally. To overcome some of these difficulties inherent in a pure Newton approach, the quasi-Newton techniques have been developed.

The basic idea of the quasi-Newton approaches is to iteratively develop an increasingly better approximation to the inverse Hessian, while overcoming some of the deficiencies of the pure Newton technique.

Let:

H_k be the approximation to F_k^{-1} at $\underline{x}_k$,

$H_0 = I$ (identity matrix)

$\underline{y}_k = \underline{g}_{k+1} - \underline{g}_k$

$\underline{s}_k = \underline{x}_{k+1} - \underline{x}_k$.

The general quasi-Newton iteration scheme is given by Algorithm 6.1.

Algorithm 6.1 Quasi-Newton Iteration Scheme

a) Set $\underline{s}_k = -H_k \underline{g}_k$.

b) Determine α_k, a line search parameter by searching along $\underline{s}_k$ giving $\underline{x}_{k+1} = \underline{x}_k + \alpha_k \underline{s}_k$.

c) Update the approximation to the inverse Hessian.

The major differences among variants of the quasi-Newton approach lie

in step (c); updating the Hessian inverse. We use the BFGS update, which is given by:

$$H_{k+1}^{bfgs} = \left(I - \frac{s_k y_k^T}{y_k^T s_k} \right) H_k^{bfgs} \left(I - \frac{y_k s_k^T}{y_k^T s_k} \right) + \frac{s_k s_k^T}{y_k^T s_k} .$$

Some of the important considerations of this technique are:

i) Only first derivatives are used versus the determination of the Hessian inverse in the pure Newton method. (In fact, this method proves robust even if differencing is used to approximate the first derivatives (Fletcher [1980]). We used a differencing technique in FORMAN.)

ii) In the absence of roundoff error, H_k is always positive definite and therefore s_k is a descent direction.

iii) It has proven to be extremely robust over a wide collection of problems (Fletcher [1980]).

iv) H_k is positive definite if and only if $y_k^T s_k > 0$. Therefore there exists an easy-to-verify condition to determine when numerical error may cause the descent property not to hold.

6.1.2 Determination of the step size parameter

Before completing the description of the algorithm for the nonlinear least squares problem, we describe our approach to determining the stepsize parameter, α_k.

Recall that in step (b) of Algorithm 6.1, α_k, the stepsize paramter, is determined. Note also that evaluations of the error function of Chapter 4 are computationally very expensive. Therefore we

seek a solution to the stepsize problem that gives sufficient accuracy for the quasi-Newton algorithm to proceed, but requires as few functional evaluations as possible.

Since there are bounds on the variables, before the line search can be performed, we must perform an operation similar to the simplex ratio test. Given a currently feasible value of the decision variables, we must determine the maximum distance along the descent direction that can be traveled without violating primal feasibility. Denote this as α_k^{max}.

Given α_k^{max} the stepsize determination proceeds by evaluating the function at three points, and then minimizing exactly the quadratic interpolation of the function based on the three data points. To ensure that the quadratic interpolation actually results in a decrease in the function, we enforce the condition that the middle interpolation point has a function value less than the left endpoint. Algorithm 6.2 has a more precise description of the stepsize solution procedure.

Algorithm 6.2 Determination of the step size parameter in a quasi-Newton iteration scheme

Step 0: Fix max_count, MC:= 0.

Set ITERATION_COUNT:=0.

Set left bound, LB:= 0.

Set right bound, RB:= α_k^{max}.

Fix $\rho \in (0,0.5)$.

Compute the function value at the left bound, LV:= $f(\underline{x}_k)$.

Step 1: ITERATION_COUNT:= ITERATION_COUNT + 1.

If ITERATION_COUNT > MAX_COUNT

then α_k:= 0, stop.

Compute the function value at the right bound, RV:=f($\underline{x}_k$+RB*$\underline{s}_k$).

Step 2: Compute the function value at the midpoint,

MB:= (RB+LB)/2.0,

MV:= f($\underline{x}_k$+MB*$\underline{s}_k$).

Step 3: If MV > LV

then RB:= (RB+LB)/2.0, go to 1.

Step 4: $$\alpha_k := \frac{1}{2}\left\{\frac{(MB^2-RB^2)\,LV + (RB^2-LB^2)\,MV + (LB^2-MB^2)\,RV}{(MB-RB)\,LV + (RB-LB)\,MV + (LB-MB)\,RV}\right\}$$

Step 5: If $f(\underline{x}_k+\alpha_k\underline{s}_k) \leq f_k + \rho\,\alpha_k\,\underline{g}_k^T\underline{s}_k$

then go to 6

else RB:= α_k,

RV:= f($\underline{x}_k$+RB*$\underline{s}_k$),

MB:= (RB+LB)/2.0,

MV:= f($\underline{x}_k$+MB*$\underline{s}_k$),

go to 4.

Step 6: If $f(\underline{x}_k+\alpha_k\underline{s}_k) \geq f_k + (1-\rho)\,\alpha_k\,\underline{g}_k^T\underline{s}_k$

then stop

else LB:= α_k,

LV:= f($\underline{x}_k$+LB*$\underline{s}_k$),

MB:= (RB+LB)/2.0,

MV:= f($\underline{x}_k$+MB*$\underline{s}_k$),

go to 4.

At step 0 we evaluate the function at the extremes of the line search parameter. Then we examine the midpoint of the two bounds to determine if it has a value less than the left bound. Since $\underline{s}_k$ is a descent direction, unless we are at an optimum there must exist a midpoint that satisfies this condition. In step 4 we minimize exactly a quadratic interpolation of the function, $f(\alpha)$, based on LV, MV, and RV. Then in steps 5 and 6, we test to determine if the inexact line search is exact enough for sufficient descent. The tests in 5 and 6 are often referred to as the Goldstein-Armijo conditions (c.f. Fletcher [1980]). Once a value for α_k is found that satisfies these conditions, the line search is terminated.

Note that the line search algorithm of 6.2, used in conjunction with BFGS, does not guarantee global convergence for general functions (Fletcher [1980]). However, if the sequence of matrices $\{H_k\}$ has a uniformly bounded condition number, then a line search that satisfies sufficient descent will guarantee that $\{\underline{g}_k\}$ will converge to zero (Dembo and Steihaug [1983]). While we have not verified in FORMAN that all possible H_k satisfy this condition, in practice, we have always observed that the gradient vanishes at the optimal solution.

6.1.3 A quasi-Newton algorithm with treatment of simple bounds

We now complete the discussion of finding a representation of the explicit utility function. All that remains is to describe the modification of the BFGS approach to handle simple bounds. We make use of the following result discussed in Fletcher [1980], p. 61.

Proposition 6.3

Consider the BFGS method. If the i'th row and column of H_j is zeroed, then this property is preserved in all matrices H_k, k=j+1, j+2,..., and $x_k^i = x_j^i$ for all $k > j$.

It follows from proposition 6.3 that if a variable moves to a bound during a BFGS update, one can merely zero out the corresponding row and column of H and continue. To permit a variable, say x^p, to move from a bound, we check for a gradient element of proper sign that will maintain primal feasibility (i.e. positive if x^p is at an upper bound, and negative if x^p is at a lower bound). If this condition is satisfied, place a one in the (p,p) element of H.

Infrequently after a number of iterations, the descent direction actually becomes an ascent direction due to accumulation of numerical error. When the line search routine detects this, we reset $H_k = I$, resulting in an attempted steepest descent step. If this step fails, a minimum has been found. If this step succeeds (i.e. results in a decrease in the objective function), we restart BFGS with $H_k = I$. This spacer step of steepest descent theoretically guarantees convergence (Goldstein and Price [1967]).

We summarize the solution procedure for determining an explicit representation of the decision maker's utility function in Algorithm 6.4.

Algorithm 6.4 A quasi-Newton algorithm including treatment of bounds on variables

Step 0: Let $k = 0$, $H_0 = I$, $\underline{x}_0 =$ a feasible interior solution. Calculate $\underline{g}_0$.

Step 1: $\underline{s}_k = -H_k \underline{g}_k$

Step 2: Determine α_k^{max}.

Step 3: Using Algorithm 6.2, determine α_k.

Step 4: If $\alpha_k = 0$

then $H_k = I$,

$\underline{s}_k = -H_k \underline{g}_k$,

check for variables at bounds,

free any variables ready to leave bounds,

determine α_k^{max},

using Algorithm 6.2, determine α_k,

if $\alpha_k > 0$

then go to 5

else stop.

Step 5: $\underline{x}_{k+1} = \underline{x}_k + \alpha_k \underline{s}_k$, $k = k + 1$

Step 6: Check for variables at bounds.

Step 7: Determine $\underline{g}_k$.

Step 8: Free any variables ready to leave bounds.

Step 9: Perform a BFGS update.

Step 10: Test for convergence

if converged

then stop

else go to 1.

For a small set of test problems where the explict utility function has had eleven variables, starting from the center of the feasible region, Algorithm 6.4 has averaged approximately 6 CPU seconds on an IBM 4341 using the Pascal VS compiler, Release 2.1. For this collection of test problems, there existed an optimal solution with objective function value of zero. The convergence tolerance was a sum of squared error less than 1.0E-10.

6.2 A HEURISTIC PROCEDURE FOR PARTITIONED CARDINALITY CONSTRAINED LINEAR PROGRAMS

Examining the formulation presented in Chapter 5, it is clear that the partitioned cardinality constrained linear programming (PCCLP) problem is combinatorial in nature. This is supported by the fact that the PCCLP is a generalization of the cardinality constrained LP, which can be formulated as an integer program. NP-completeness theory provides the insight that exact solutions may require considerable computational effort. Therefore the design of a solution procedure may be best directed toward a heuristic approach. A heuristic approach also seems to be a reasonable compromise in the forestry problem because:

1) the multiobjective procedure requires the solution of a number of partitioned cardinality constrained LP'S in an interactive setting,
2) we wish to eventually implement FORMAN on microcomputers, and

3) the partitioned cardinality constraints are among the most ''soft'' problem data (i.e. a decision maker often can provide only rough estimates of suitable values).

In light of these considerations, we have derived a heuristic technique to deal with the partitioned cardinality constraints.

6.2.1 Design aspects of the LP procedure

The PCCLP code contains a number of features collectively designed to exploit the structure of the subproblems encountered during the solution of the forest management problem.

First, the code uses a sparse data format. That is, only non-zeroes are stored. The sparse format is implemented through the creation of a data structure in Pascal that contains the following items: 1) the variable number, 2) the stand number associated with the variable, 3) a link field to the next variable in the same stand, 4) the upper bound on the variable, 5) an array of the objective function coefficients for each benefit, 6) an array to store the nonzero column elements of the constraint matrix, and 7) an array to store the row subscripts of the column elements. The sparse format is implemented both in-core and out-of-core to save storage and execution time.

The only user input required (other than interaction at the terminal) is a relatively free format enumeration of the harvesting regimes and linking constraints. Again, only non-zeroes are required. (An exact description of the record formats, along with a sample dataset, is contained in the Appendix.)

The problem generator section of the PCCLP code converts the user

input into an internal representation of the problem. It performs a number of setup tasks such as converting each of the range constraints into a single constraint with an upper bounded slack variable; adding logical variables; determining a starting basis; and initializing a direct access file that permits relative record access to each set of column data.

The PCCLP code employs partial pricing and suboptimization. It uses the revised simplex algorithm with explicit form of the basis inverse and implicit treatment of upper bounds. The partial pricing and suboptimization are implemented through an in-core/out-of-core ''bucket'' mechanism. The bucket (which is an in-core storage area) contains at any time, only a small subset of the problem data. These data are organized in columns corresponding to the problem variables. Whenever it is necessary to obtain a variable to enter the basis, the variables in the bucket are examined. If none of the variables in the bucket are attractive, the bucket is refilled with variables from the direct access dataset containing the problem data. In addition, the bucket-filling mechanism employs the use of an opening gambit along with a middle, and ending game (Bradley, Brown and Graves [1977]). This technique dynamically changes the size of the bucket during the solution process. During the early pivots, when it is easy to find variables that price out attractively, the bucket is expanded to accomodate as many variables as possible. During the middle game, attractive variables become harder to find, and subsequently the bucket size is decreased. In the final pivots, any variable that prices out attractive is a valuable find, and therefore the bucket is again

decreased so that not too much time is spent attempting to fill it.

Since the constraint matrix tends to be very long and narrow (usually at least ten times as many columns as rows), we employ an explicit representation of the basis inverse. This can be accomodated even on the smallest of machines for which FORMAN is targeted. The PCCLP code itself requires 14K bytes of storage when compiled with the IBM Pascal/VS compiler, Release 2.1. For a problem containing 50 rows and 500 columns, an additional 25k bytes of data storage are needed when a bucket of maximum size of 15 variables is employed.

Finally, the code employs elastic optimization on the linking constraints. Elastic optimization permits a constraint to be violated at the expense of incurring a cost which is linear in the amount of the violation. This is implemented by adding two additional logical variables to each constraint.

Together these techniques, along with the heuristics for the partitioned cardinality constraints, have resulted in an LP package that is well suited to exploit the structure of the subproblems encountered during the solution of the implicit/explicit algorithm used in FORMAN.

6.2.2 The partitioned cardinality constraints heuristic

Our heuristic for solving the partitioned cardinality constrained linear programs is based on the solution of a collection of LP relaxations. We first demonstrate that if an optimal solution exists to the partitioned cardinality constrained LP, then a basic optimal solution exists.

Theorem 6.5

Consider the problem:

$$\begin{aligned} &\max \ \underline{cx} \\ &\text{s.t.} \quad A\underline{x} = \underline{b} \qquad\qquad (PCCLP) \\ &\qquad | \ \underline{x}_+^i \ | \leq p_i \qquad i = 1, \ldots, t \\ &\qquad \underline{x} \geq \underline{0}, \end{aligned}$$

where the variables are partitioned into t mutually exclusive and collectively exhaustive sets, and $| \ \underline{x}_+^i \ |$ denotes the number of positive variables contained in the i'th set.

If there exists an optimal solution to PCCLP, then there exists a basic optimal solution.

Proof:

Denote the columns of A by $a_1, a_2, \ldots, a_n$. Suppose that $(x_1, x_2, \ldots, x_n)$ is an optimal solution to PCCLP. Therefore,

$x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = \underline{b}$, and $| \ \underline{x}_+^i \ | \leq p_i$ for all i.

Assume that exactly p of the variables x_j are greater that zero, and without loss of generality, assume that they are the first s variables. Therefore,

$x_1 a_1 + x_2 a_2 + \ldots + x_s a_s = \underline{b}$, and $| \ \underline{x}_+^i \ | \leq p_i$ for all i.

Case 1: Assume that $a_1, \ldots, a_s$ are linearly independent. Clearly $s \leq m$ where m is the rank of A.

If $m = s$, the proof is complete.

If $s < m$, then since A has rank m, m-s columns can be found from the remaining n-s columns so that the resulting set of m columns is linearly independent. Assign the value of zero to $x_{s+1}, \ldots, x_n$, which

yields a basic solution having the same objective function value as the original optimal solution, and still satisfies the cardinality constraints, hence it is optimal.

Case 2: Assume that $a_1, \ldots, a_s$ are linearly dependent. Thus there are constants $y_1, y_2, \ldots, y_s$, at least one of which is positive, such that:

$$y_1 a_1 + y_2 a_2 + \ldots + y_s a_s = \underline{0}. \tag{3}$$

Now,

$$x_1 a_1 + x_2 a_2 + \ldots + x_s a_s = \underline{b}. \tag{4}$$

Multiply (3) by a positive ε and add to (4). This yields:

$$(x_1 - \varepsilon y_1) a_1 + \ldots + (x_s - \varepsilon y_s) a_s = \underline{b}. \tag{5}$$

Let $\varepsilon = \min \{x_i/y_i;\ y_i > 0\}$. Therefore this solution to (5) is feasible and has s-1 positive variables. Repeat this process until the solution is linearly independent, so that case 1 applies. (Note that by eliminating variables, one can never violate a cardinality constraint that is already satisfied.)

To complete the proof, we must demonstrate that (5) is optimal. The objective function value of (5) is:

$$\underline{c}\ \underline{x} - \varepsilon\ \underline{c}\ \underline{y}.$$

For ε sufficiently small, $\underline{x} - \varepsilon\underline{y}$ is a feasible solution for positive or negative values of ε. Therefore $\underline{cy} = 0$. Otherwise if $\underline{cy} \neq 0$, one could deterimine a new feasible solution by setting ε to the proper sign and sufficient magnitude so as to have a feasible solution with a better objective function value than our original solution. This violates the assumption that the original solution was optimal. Q. E. D.

This theorem demonstrates that a vertex enumeration solution technique for the PCCLP will eventually find an optimal solution. Returning now to our PCCLP heuristic, it is composed of two rules: 1) when considering variables to price out, price out first those that will not lead to a violation of cardinality (if any), and 2) rescale the reduced cost of each variable as it is priced out to reward/penalize a variable with respect to its current contribution to the cardinality constraints.

The first idea is easily accomodated in our implementation during pricing, when the PCCLP procedure refills the in-core bucket from auxilliary (disk) storage.

All variables of a given stand are linked together into a circular queue through the use of a link field. Just prior to replenishing the bucket, the timber stands are sorted into decreasing order. The sort criterion is the amount of ''cardinality slack'', where cardinality slack for stand i:= maximum allowable cardinality for stand i - current cardinality for stand i. When the bucket is filled, the stand with the most cardinality slack is examined first. If the bucket is not filled with variables from the first stand, then the next stand with the most remaining cardinality slack is examined. The process continues until the bucket is replenished, or there are no more variables to price out. This ensures that if there are some attractive variables that do not lead to cardinality violations, they are examined first.

The second technique in this heuristic is to rescale the reduced cost of a variable as a function of the impact that making the variable

basic would have on the cardinality constraints. Specifically, each time a variable associated with the i'th stand is priced out, its reduced cost is modified by the relation:

$$RC_{new} := RC_{old} * scale_factor,$$

where:

$$scale_factor := 2^{(cardinality_slack_i)}.$$

However to avoid numerical difficulties, we restrict the scale_factor to the range [-4096, 4096]. Note that the use of this scale_factor causes does not affect the ability of the simplex LP technique to determine an optimal solution. It merely provides a second criterion to discriminate which attractive variable to pivot into the basis.

These heuristics do not prevent the cardinality constraints from being violated, but they do tend to disperse the violation throughout all stands, which is generally what the decision maker desires. In section 6.4, we describe the results of these heuristics on a sample problem.

6.3 MAXIMIZATION OF THE EXPLICIT UTILITY FUNCTION

The final aspect of implementation is the maximization of the explicit utility function. This problem is given in Chapter 5 as the Forest Management Problem (FMP), where $U[\underline{f}(\underline{x})]$ is defined by either (2) or (11) of Chapter 4. Note that when the partitioned cardinality constraints are handled implicitly by our heuristic procedure, the resulting problem is the maximization of a nonlinear objective function subject to a linear set of constraints. To solve this problem we use the well-known Frank-Wolfe algorithm (Wolfe [1970]). The general

Frank-Wolfe procedure is given in Algorithm 6.6. Here we assume a problem of the form:

$$\max f(\underline{x})$$

$$\text{s.t.} \quad A\,\underline{x} = \underline{b}$$

$$\underline{l} \leq \underline{x} \leq \underline{u} \,.$$

Algorithm 6.6 The Frank-Wolfe method for nonlinear programs

Step 0: Choose an initial point, $\underline{x}_1 \in X$. Let $k = 1$.

Step 1: Determine a solution to the direction finding problem, (which is based on a linear approximation of f), given by:

$$\max \nabla f(\underline{x}_k)\ \underline{y}_k$$

$$\text{s.t.} \quad A\,\underline{y}_k = \underline{b}$$

$$\underline{l} \leq \underline{y}_k \leq \underline{u}$$

Step 2: Let $\underline{d}_k := \underline{y}_k^* - \underline{x}_k$.

Solve the stepsize problem for t_k:

$$\max \quad f(\underline{x}_k + t_k\,\underline{d}_k)$$

$$\text{s.t.} \quad 0 \leq t_k \leq 1 \,.$$

Step 3: Let $\underline{x}_{k+1} := \underline{x}_k + t_k\,\underline{d}_k$.

If $|\underline{x}_{k+1} - \underline{x}_k| <$ convergence tolerance

then stop

else $k := k + 1$, go to step 1.

This algorithm was chosen because of its robust convergence characteristics (Hogan [1971], Wolfe [1970]), and also because it was compatible with existing software built for use in FORMAN. Given that a computer procedure already existed for solving the Frank-Wolfe steps

in the Geoffrion-Dyer-Feinberg algorithm, it was relatively easy to implement this approach.

The primary requirement for its use was to determine the gradient of U with respect to $\underline{x}$ for use in step 1. We develop this representation in the next section.

6.3.1 Gradient of the explicit utility function

Let i index stands, i=1,...,I, j index harvesting regimes, j=1,...,J, and r index benefits, r=1,...,R. Denote the set of feasible solutions of the FMP of Chapter 5 as X. Given $U[\underline{f}(\underline{x})]$, where $U[\cdot]$ is either an additive or multiplicative utility function as described in Chapter 4, we seek $\partial U/\partial x_{ij}$.

By the chain rule,

$$\frac{\partial U}{\partial x_{ij}} = \sum_r \left\{ \frac{\partial U}{\partial f_r} \right\}_{|\underline{f}(\underline{x})} \left\{ \frac{\partial f_r}{\partial x_{ij}} \right\}_{|\underline{x}}$$

$$= \sum_r \left\{ \frac{\partial U}{\partial f_r} \right\}_{|\underline{f}(\underline{x})} c_{ijr}$$

To completely specify $\partial U/\partial x_{ij}$, we seek $\partial U/\partial f_r$, r=1,...,R.

Recall that the one dimensional utility functions of Chapter 4 were given by:

$$u_i(f_i) = s_i[1 - \exp(-a_{i2}f_i)]$$

with $a_{i2} > 0$, $s_i = [1 - \exp(-a_{i2})]^{-1}$, and $0 \leq f_i \leq 1$.

For the additive case, using the oduf's of Chapter 4,

$$\frac{\partial U}{\partial f_r} = \frac{\partial}{\partial f_r} \left\{ \sum_{p=1}^{R} a_{p1} \frac{1 - \exp(-a_{p2} f_p)}{1 - \exp(-a_{p2})} \right\}$$

$$= \frac{a_{r1}\, a_{r2}\, \exp(-a_{r2} f_r)}{1 - \exp(-a_{r2})}$$

For the multiplicative case, again using the oduf's of Chapter 4:

$$\frac{\partial U}{\partial f_r} = \frac{\partial}{\partial f_r} \left\{ a_0^{-1} \left(\prod_{p=1}^{R} (1 + a_0\, a_{p1} \frac{1 - \exp(-a_{p2} f_p)}{1 - \exp(-a_{p2})}) - 1 \right) \right\}$$

$$= \frac{a_{r1}\, a_{r2}\, \exp(-a_{r2} f_r)}{1 - \exp(-a_{r2})} \left(\prod_{\substack{p=1 \\ p \neq r}}^{R} (1 + a_0\, a_{p1} \frac{1 - \exp(-a_{p2} f_p)}{1 - \exp(-a_{p2})}) \right)$$

An important computational observation is that as $\underline{x}$ changes, only those terms containing $\underline{f}$ must be recomputed. Hence, much of the expensive exponentiation can be performed once during problem setup.

We give a description of the solution procedure for the explicit utility maximization problem in Algorithm 6.7.

Algorithm 6.7 Maximization of the explicit utility function

Step 0: Calculate $M_r = (a_{r1} a_{r2})/(1 - \exp(-a_{r2}))$ for r=1,...,R.

If the utility function is multiplicative

then calculate $S_r = (a_0 a_{r2})/(1 - \exp(-a_{r2}))$ for r=1,...,R.

Let k = 0.

Given an initial starting solution, $\underline{x}_0$, determine $\underline{f}(\underline{x}_0)$.

Step 1: Calculate $Q_r = \exp(-a_{r2} f_r)$, r=1,...,R.

If the utility function is additive

then $\partial U/\partial f_r = M_r\ Q_r$, r=1,...,R.

If the utility function is multiplicative

$$\text{then } \partial U/\partial f_r = (M_r\ Q_r) \left(\prod_{\substack{p=1 \\ p \neq r}}^{R} 1 + (S_p)(1-Q_p) \right), \quad r=1,\ldots,R.$$

Step 2: Solve the direction-finding problem at the current point $\underline{x}_k$.

$$\max \sum_r \left\{ \frac{\partial U}{\partial f_r} \right\} \Big|_{\underline{f}(\underline{x})} \sum_i \sum_j c_{ijr}\ y_{ij(k)}$$

$$\text{s.t. } \underline{y}_k \in X$$

Step 3: Solve the step size problem.

$$\max\ U[\underline{f}(\underline{x}_k + t_k\ (\underline{y}_k - \underline{x}_k)]$$

$$\text{s.t. } 0 \leq t_k \leq 1$$

Step 4: Let $\underline{x}_{k+1} := \underline{x}_k + t_k\ (\underline{y}_k - \underline{x}_k)$

Step 5: If $|\underline{x}_{k+1} - \underline{x}_k| <$ convergence tolerance

then stop

else go to step 1.

6.4 A SAMPLE FORMAN SESSION

Here we present the results of a sample session of the FORMAN procedure. The management regimes, timber characteristics, cardinality constraints, and woodflow/cashflow constraints were provided by the Tennessee Valley Authority, Division of Land and Forest Resources. The

hypothetical forest property contained four stands, totaling 350 acres. The stands were respectively, a loblolly pine plantation, an upland oaks stand, a yellow-poplar stand, and a stand of natural shortleaf pine. The problem characteristics of these four stands are summarized in Table 6.1.

TABLE 6.1

PROBLEM CHARACTERISTICS OF THE SAMPLE FORMAN SESSION

Stand No.	Stand Type	Number of Management Regimes	Cardinality Constraint
1	Loblolly pine	11	3
2	Upland oaks	9	1
3	Yellow-poplar	9	3
4	Shortleaf pine	7	4

In this session, the author acted as the decision maker. A rough ranking of the benefits prior to the session resulted in i) present net worth of timber harvesting, ii) deer, iii) squirrel, iv) cords of timber (perhaps for firewood), and v) quail.

The necessary timber and wildlife data were generated as described in Chapter 2. The data is listed in the Appendix in a format useable by FORMAN.

The results were most encouraging. The session required a total of 29 minutes, much of which was time spent considering collections of benefits. The decision maker was required to respond to 28 binary comparisions of the type listed in Figure 6-1. (Note that these binary comparisons are the only information burden required of the decision maker during a session.) The final solution resulted in the attainment of 99.5% of the maximum present net worth, 91.5% of the maximum deer habitat suitability, 87.8% of the maximum attainable habitat suitability for squirrel, 84.6% for cords of timber harvested, and 77.4% for quail. The results of the final solution are summarized in Table 6.2.

The final solution required one management regime for stand 1, three for stands 2, 3, and 4. All of these were within the cardinality limits, except stand 2, which had a cardinality limit of one.

The procedure produced an acceptable solution, and the information burden required to complete the procedure was reasonable. It remains to be determined if these successful results will be consistently achieved, however the initial results are encouraging. The Tennessee Valley Authority plans to field implement FORMAN in the near future. We will be closely monitoring these results, and adapting the procedure based on these real-world experiments

TABLE 6.2

RESULTS OF THE SAMPLE FORMAN SESSION

Benefit	Maximum Value	Value Attained
Present Net Worth	$34,441	$34,261
Cords Harvested	18,358	15,531
Deer Habitat Suitability	0.55	0.50
Squirrel Habitat Suitability	0.56	0.49
Quail Habitat Suitability	0.52	0.40

OBJECTIVE	STRATEGY 1	STRATEGY 2
PNW OF TIMBER	34040.35	37794.55
CORDS OF WOOD	17943.09	16148.78
DEER	0.49	0.49
SQUIRREL	0.55	0.55
BOBWHITE QUAIL	0.40	0.40

```
ENTER 'L' IF YOU PREFER STRATEGY 1,
      'R' IF YOU PREFER STRATEGY 2,
      'I' IF YOU ARE INDIFFERENT, OR
      'Q' IF YOU WISH TO QUIT          ?
```

FIGURE 6.1

SAMPLE QUESTION FOR ELICITING PREFERENCE INFORMATION

CHAPTER 7

FUTURE RESEARCH AND EXTENSIONS TO THE MODEL

In Chapters 4 and 5, we provided the development of a new algorithm for multiobjective optimization, along with a formulation for the use of this algorithm in forest management planning. In Chapter 6, we discussed the implementation of our forest management package, FORMAN, and described a sample of its use. We conclude with a discussion of related future research topics. In particular we focus on research questions to extend 1) the implicit/explicit algorithm, and 2) FORMAN's capabilities.

In Chapter 2 when presenting the timber/wildlife production functions, we made the assumption that these functions were separable with respect to each timber stand. That is, the amount of the j'th benefit from the i'th stand depended only on how one managed the i'th stand. Management activities on adjacent stands were not considered. For timber and squirrel, this is a reasonable assumption. However for deer and quail, this is somewhat of an abstraction from reality. We assumed separability in FORMAN so as to limit the number of decision variables and results in a tractable problem. Future work on the nonseparable version of FORMAN would permit a more realistic representation of the forest management problem.

In Chapter 4, we noted the empirical behaviour of the multiplicative utility function as it was investigated for strict quasiconcavity. Recall that our results were that as the scaling

constant, k, approached its lower bound of -1, the percentage of strict quasiconcavity failure increased. Also in our tests, no example of strict quasiconcavity failure was found above the arbitrary cutoff of $k > -0.95$. Some immediate questions come to mind. Is there a point for k, above which the multiplicative utility function is strictly quasiconcave? Also, if it exists, can it be determined? As k approaches -1, it implies in an economic sense that the benefits are becoming substitutes for each other. This poses the question: ''What is the economic implication of the phenomenon of strict quasiconcavity violation as k approaches -1?''

This research did not deal with any of the behavioural aspects of eliciting information from the decision maker. Shilling [1980] and Shilling et al. [1982] have some interesting results on this subject. As FORMAN is implemented by the Tennessee Valley Authority, our follow-up study on its use will focus more on these user-related aspects.

Finally, there are some interesting questions that remain relating to the partitioned cardinality constrained LP's. While our heuristic has performed well on the sample problems, other techniques may also prove fruitful, and merit further investigation.

There are additional research questions that are more fundamental in nature. For instance, improving convergence of the maximization of the explicit utility function. However in a large part, our future research relating to FORMAN will be directed by the results of its use in the field.

REFERENCES

REFERENCES

Adams, D. M. and A. R. Ek. 1974. Optimizing the Management of Uneven-aged Forest Stands. Canadian Journal of Forest Research 4:274-287.

Al-Ani, F. S. 1979. Multi-Product Versus Single-Product Timber Rotation: Comparative by Regression and Computer Simulation. Unpublished Master's Thesis, The University of Tennessee.

Amidon, E. L. and G. S. Akin. 1968. Dynamic Programming to Determine Optimum Levels of Growing Stock. Forest Science 14:287-291.

Anderson, F. J. 1976. Control Theory and the Optimum Timber Rotation. Forest Science 22:242-246.

Anderson, M. D. 1981. Resolving Tradeoffs Among Conflicting Objectives Using Mathematical Programming and Interactive Graphics. Report 81-3, School of Engineering and Applied Science, Princeton University.

Arthur, J. L. and D. P. Dykstra. 1980. Timebr Harvest Planning - A Combined Optimization/Simulation Model. Journal of Environmental Management 4:491-498.

Bailey, R. L. and T. R. Dell. 1973. Quantifying Diameter Distributions with the Weibull Function. Forest Science 19:97-104.

Bare, B. B. and B. F. Anholt. 1976. Selecting Forest Residue Treatment Alternatives Using Goal Programming. USDA Forest Service General Technical Report PNW-43.

Bare, B. B. and G. F. Schreuder. 1976. Applications of Systems Analysis in Wildland Management: An Overview. Presented paper at the ORSA/TIMS Joint National Meeting, March 31 - April 2, 1976, Philadelphia, PA.

Barnes, R. B. 1979. Wildlife Habitat From a Forest Inventory: Is It Possible? Transactions of the Northeast Fish and Wildlife Conference 36:151-159.

Barnes, R. L. 1982. A New Look at Growth and Yield. FOREM, Spring/Summer:4-13.

Barros, O. and A. Weintraub. 1982. Planning for a Vertically Integrated Forest Industry. Operations Research 30:1168-1182.

Bazaraa, M. S. and C. M. Shetty. 1979. Nonlinear Programming Theory and Algorithms. John Wiley and Sons.

Beck, D. E. 1977. Twelve-Year Acorn Yield in Southern Appalachian Oaks. USDA Forest Service Research Note SE-244.

Bell, E. F. 1976. Goal Programming for Land Use Planning. USDA Forest Service General Technical Report PNW-53.

Bell, E. F. 1977. Mathematical Programming in Forestry. Journal of Forestry 75:317-319.

Benayoun, R., J. de Montgolfier and J. Tergny. 1971. Linear Programming with Multiple Objective Functions: Step Method (STEM). Mathematical Programming 1:366-375.

Benson, H. P. 1978. Existence of Efficient Solutions for Vector Maximization Problems. Journal of Optimization Theory and Applications 26:569-580.

Benson, H. P. and T. L. Morin. 1977. The Vector Maximization Problem: Proper Efficiency and Stability. SIAM Journal on Applied Mathematics 32:64-72.

Bertier, P. and J. de Montgolfier. 1973. On Multicriteria Analysis: An Application to a Forest Management Problem. Presented paper at the NATO conference on Mathematical Analysis of Decision Problems in Ecology, July 9-13, 1973, Istambul, Turkey.

Blair, R. M. 1969. Timber Stand Density Influences Food and Cover. In proceedings of the symposium on White-Tailed Deer in the Southern Forest. Southern Forest Experiment Station, Nacogdoches, TX.

Blair, R. M. and L. E. Brunett. 1976. Phytosocialogical Changes After Timber Harvest in a Southern Pine Ecosystem. Ecology 57:18-32.

Blair, R. M. and L. E. Brunett. 1980. Seasonal Browse Selection by Deer in a Southern Pine-Hardwood Habitat. Journal of Wildlife Management 44:79-88.

Blair, R. M. and H. G. Enghart. 1976. Deer Forage and Overstory Dynamics in a Loblolly Pine Plantation. Journal of Range Management 29:104-108.

Bottoms, K. E. and E. T. Bartlett. 1975. Resource Allocation Through Goal Programming. Journal of Range Management 28:442-447.

Boyce, S. G. 1977. Management of Eastern Hardwood Forests for Multiple Benefits (DYNAST-MB). USDA Forest Service Research Paper SE-168.

Boyce, S. G. 1978. Management of Forests for Timber and Related Benefits (DYNAST-TM). USDA Forest Service Research Paper SE-184.

Bradley, G. H., G. G. Brown and G. W. Graves. 1977. Design and Implementation of Large Scale Primal Transshipment Algorithms. Management Science 24:1-34.

Brill, E. D., S. Chang and L. D. Hopkins. 1982. Modeling to Generate Alternatives: The HSJ Approach and an Illustration. Management Science 28:221-235.

Brodie, J. D., D. M. Adams and C. Kao. 1978. Analysis of Economic Impacts on Thinning and Rotation for Douglas-fir, Using Dynamic Programming. Forest Science 24:513-522.

Brodie, J. D. and C. Kao. 1979. Optimizing Thinning in Douglas-fir with Three-Descriptor Dynamic Programming to Account for Accelrated Diameter Growth. Forest Science 25:665-672.

Broido, A., R. J. McConnen and W. G. O'Regan. 1965. Some Operations Research Application in the Conservation of Wildland Resources. Management Science 11:802-814.

Bruner, H. D. and J. W. Moser. 1973. A Markov Chain Approach to the Prediction of Diameter Distributions in Uneven-Aged Forest Stands. Canadian Journal of Forest Research 3:409-417.

Brunswig, N. L. and A. S. Johnson. 1972. Bobwhite Quail Foods and Populations on Pine Plantations in the Georgia Piedmont During the First Seven Years Following Site Preparation. In Proceedings of the Southeast Association of Game and Fish Commissions 26:96-107.

Buongiorno, J. and B. R. Michie. 1980. A Matrix Model of Uneven-Aged Forest Management. Forest Science 26:609-625.

Burkhart, H. E., R. C. Parker, M. R. Strub and R. G. Oderwald. 1972. Yields of Old Field Loblolly Pine Plantations. Division of Forestry and Wildlife Resources. VPI Publication FWS-2-72.

Burroughs, J. N. and W. B. Kurtz. 1982. Linear Programming in the Classroom: A Simplified Approach. Journal of Forestry 80:222-223.

Carlson, R. C. and P. R. Shukla. 1979a. Interactive Solution of Discrete Multicriterion Problems: An Algorithm and Computational Results for the Binary Case. Technical Report 79-2, Industrial Engineering and Engineering Management, Stanford University.

Carlson, R. C. and P. R. Shukla. 1979b. Robustness of the Interactive Algorithm for Multicriterion Zero-One Variable Problems Under Probabilistic Tradeoffs. Technical Report 79-3, Industrial Engineering and Engineering Management, Stanford University.

Chang, S. J. 1981. Determination of the Optimal Growing Stock and Cutting Cycle for an Uneven-Aged Stand. Forest Science 27:739-744.

Chang, S. J. and J. Buongiorno. 1981. A Programming Model for Multiple Use Forestry. Journal of Environmental Management 13:45-58.

Chappelle, D. E., M. Mang and R. C. Miley. 1976. Evaluation of Timber RAM as a Forest Management Planning Model. Journal of Forestry 74:288-293.

Chappelle, D. E. and T. C. Nelson. 1964. Estimation of Optimal Stocking Levels and Rotation Ages of Loblolly Pine. Forest Science 10:471-502.

Charnes, A. and W. W. Cooper. 1977. Goal Programming and Multiple Objective Optimization Part 1. European Journal of Operations Research 1:39-54.

Chen, C. M., D. W. Rose and R. A. Leary. 1980. Derivation of Optimal Stand Density Over Time - A Discrete Stage, Continuous State Dynamic Programming Solution. Forest Science 26:217-227.

Choo, E. U. and D. R. Atkins. 1980. An Interactive Algorithm for Multicriteria Programming. Computers and Operations Research 7:81-87.

Clawson, M. 1979. The Economics of U. S. Nonindustrial Private Forests. Resources for the Future Research Paper 14, Washington, D.C.

Clawson, M. 1981. An Economic Classification of U. S. ''Commercial'' Forests. Journal of Forestry 79:727-730.

Clutter, J. L. and E. P. Jones. 1980. Prediction of Growth after Thinning in Old-Field Alash Pine Plantations. USDA Forest Service Research Paper SE-217.

Cochrane, J. L. and M. Zeleny (eds.). 1973. Multiple Criteria Decision Making. University of South Carolina Press.

Cohon, J. L. 1978. Multiobjective Programming and Planning. Academic Press.

Cohon, J. L. and D. H. Marks. 1975. A Review and Evaluation of Multiobjective Programming Techniques. Water Resources Research 11:208-220.

Coutu, A. J. and B. W. Ellertsen. 1960. Farm Forestry Planning Through Linear Programming. Tennessee Valley Authority Technical Report 236-60.

Curtis, F. H. 1962. Linear Programming the Management of a Forest Property. Journal of Forestry 60:611-616.

Daellenbach, H. G. and C. A. de Kluyver. 1980. Note on Multiple Objective Dynamic Programming. Journal of the Operational Research Society 31:591-594.

Dale, M. E. 1972. Growth and Yield Predictions for Upland Oak Stands. USDA Forest Service Research Paper NE-241.

Dane, C. W., N. C. Meador and J. B. White. 1977. Goal Programming in Land-Use Planning. Journal of Forestry 75:325-329.

Daniel, C. and R. Lamaire. 1974. Evaluating Effects of Water Resource Development on Wildlife Habitat. Wildlife Society Bulletin 2:114-118.

de Kluyver, C. A., H. G. Daellenbach and A. G. D. Whyte. 1980. A Two-Stage, Multiple Objective Mathematical Programming Approach to Optimal Thinning and Harvesting. Forest Science 26:674-686.

Dembo, R. S. and T. Steihaug. 1983. Truncated-Newton Algorithms for Large-Scale Optimization. Presented at the ORSA/TIMS Joint National Meeting, Chicago, IL, April, 1983. (Also to appear in Mathematical Programming.)

Dennis, J. E. and J. J. More. 1977. Quasi-Newton Methods, Motivation and Theory. SIAM Review 19:46-79.

Dickmeyer, N. 1983. Measuring Effects of a University Planning Decision Aid. Management Science 29:673-685.

Duerr, W. A. 1981. Productivity As a Forestry Theme. Journal of Forestry 79:520-522.

Dyer, A. A., J. G. Hof, J. W. Kelly, S. A. Crim and G. S. Alward. 1979. Implications of Goal Programming in Forest Resource Allocation. Forest Science 25:535-543.

Dyer, J. S. 1972. Interactive Goal Programming. Management Science 19:62-70.

Dyer, J. S. 1973. An Empirical Investigation of a Man-Machine Interactive Approach to the Solution of the Multiple Criteria Problem. in Cochrane and Zeleny op. cit.:202-216.

Dyer, J. S. 1977. On the Relationship Between Goal Programming and Multiattribute Utility Theory. Discussion paper No. 69, Management Study Center, Graduate School of Management, U.C.L.A.

Dyer, J. S. and R. K. Sarin. 1979. Measurable Multiattribute Value Functions. Operations Research 27: 810-822.

Ecker, J. G., N. S. Hegner and I. A. Kouada. 1980. Generating All Maximal Efficient Faces for Multiobjective Linear Programs. Journal of Optimization Theory and Applications 30:353-381.

Ecker, J. G. and I. A. Kouada. 1975. Finding Efficient Points for Multiple Objective Linear Programs. Mathematical Programming 8:375-377.

Ecker, J. G. and I. A. Kouada. 1978. Finding All Efficient Points for Multiple Objective Linear Programs. Mathematical Programming 14:249-261.

Ecker, J. G. and N. E. Shoemaker. 1980. Selecting Subsets from the Set of Nondominated Vectors in Multiple Objective Linear Programming. Department of Mathematical Sciences, Rensselaer Polytechnic Institute.

Ellefson, P. V., S. L. Palm and D. C. Lothner. 1982. From Public Land to Nonindustrial Private Forest: A Minnesota Case Study. Journal of Forestry 80:219-222.

Farrar, R. M. 1979. Status of Growth and Yield Information in the South. Southern Journal of Applied Forestry 3:132-137.

Faustmann, M. 1849. Berechnung des Werthes, welchen Waldboden Sowie nach nicht haubare Holzbestande fur die Weldwirtschaft besitzen. Allegemeine Forst und Jagd Zeitung 25:441.

Field, D. B. 1973. Goal Programming for Forest Management. Forest Science 19:125-135.

Field, D. B. 1977. Linear Programming: Out of the Classroom and into the Woods. Journal of Forestry 75:330-334.

Field, R. C. 1980. Timber Harvest Scheduling with Multiple Decision Criteria. in Case Studies in Forest Resource Information Analysis, R. Rajagopal (ed.), USDA Forest Service, Atlanta, GA.

Field, R. C., P. E. Dress and J. C. Fortson. 1980. Complementary Linear and Goal Programming Procedures for Timber Harvest Scheduling. Forest Science 26:121-133.

Fletcher, R. 1980. Practical Methods of Optimization. John Wiley and Sons.

Gal, T. 1977. A General Method for Determining the Set of All Efficient Solutions to a Linear Vector Maximum Problem. European Journal of Operations Research:307-322.

Garey, M. R. and D. S. Johnson. 1979. Computers and Intractability - A Guide to the Theory of NP-Completeness. W. H. Freeman and Co.

Geoffrion, A. M. 1968. Proper Efficiency and the Theory of Vector Maximization. Journal of Mathematical Analysis and Applications 22:618-630.

Geoffrion, A. M. 1976. The Purpose of Mathematical Programming Is Insight, Not Numbers. Interfaces 7:81-92.

Geoffrion, A. M., J. S. Dyer and A. Feinberg. 1972. An Interactive Approach for Multicriterion Optimization with an Application to the Operation of an Academic Department. Management Science 19:357-386.

Giese, R. F. and P. C. Jones. 1983. An Economic Model of Short-Rotation Forestry. To appear in Mathematical Programming.

Golabi, K., C. W. Kirkwood and A. Sicherman. 1981. Selecting a Portfolio of Solar Energy Projects Using Multiattribute Preference Theory. Management Science 27:174-189.

Goldstein, A. A. and J. F. Price. 1967. An Effective Algorithm for Minimization. Numerische Mathematik:184-189.

Goodrum, P. D., V. H. Reid and C. E. Boyd. 1971. Acorn Yields, Characteristics, and Management Criteria of Oaks for Wildlife. Journal of Wildlife Management 35:520-532.

Gros, J. 1975. Power Plant Siting: A Paretian Environmental Approach. Nuclear Engineering and Design 34:281-292.

Hair, D. 1980. Timber Situation in the United States - 1952-2030. Journal of Forestry 78:683-686.

Halla, L. K. 1973. Managing Deer Habitat in Loblolly-Shortleaf Pine Forests. Journal of Forestry 71:752-757.

Hamner, L. N. 1975. WRAP - A Multiple Resource Model for Allocation of Wildland Resources. Tennessee Valley Authority, Division of Forestry, Fisheries and Wildlife Development, Technical Note B12.

Harrison, T. P. and R. E. Rosenthal. 1980. Quantifying Landowner Objectives in a Forest Management Optimization Model. Working Paper No. 102, Management Science Program, The University of Tennessee.

Harrison, T. P. and R. E. Rosenthal. 1981. A Multiobjective Optimization Program for Scheduling Timber Harvests on Nonindustrial Private Forest Lands. Working Paper No. 119, Management Science Program, The University of Tennessee. (To appear in TIMS Study on Systems Analysis in Forestry and Forest Industries).

Hazen, G. B. and T. L. Morin. 1982. Optimality Conditions in Nonconical Multiple Objective Programming. Presented paper at the ORSA/TIMS Joint National Meeting, October 25-27, San Diego, CA.

Hellman, O. 1980. A Special Problem of Large Scale Forest Management. European Journal of Operational Research 4:16-18.

Hellman, O. 1982. Mathematical Modeling for an Energy Forest. Management Science 28:1247-1257.

Hepp, T. E. 1982. YIELD: Timber Yield Forecasting and Planning Tool. Southern Journal of Applied Forestry 6:135-140.

Hickman, C. A. and R. J. Gehlhausen. 1981. Landowner Interest in Forestry Assistance Programs in East Texas. Journal of Forestry 79:211-213.

Ho, J. K. 1979. Multiple Criteria Optimization: A Unified Framework. Presented paper at the Conference on Human Aided Optimization, August 6-8, the Wharton School, Philadelphia, PA.

Hogan, W. W. 1971. Convergence Results for Some Extensions of the Frank-Wolfe Method. Working Paper No. 169, Western Management Science Institute, University of California, Los Angeles.

Holmes, D. D. 1982. A Multiobjective Optimization Model for Land Allocation. Unpublished Master's Thesis, The University of Tennessee.

Hool, J. N. 1966. A Dynamic Programming - Markov Chain Approach to Forest Production Control. Forest Science Monograph 12.

Hotvedt, J. E., W. A. Leuschner and G. J. Buhyoff. 1982. A Heuristic Weight Determination Procedure for Goal Programs Used for Harvest Scheduling. Canadian Journal of Forest Research 12:292-298.

Hultz, J. W., D. D. Klingman, G. T. Ross and R. M. Soland. 1981. An Interactive Computer System for Multicriteria Facility Location. Computers and Operations Research 8:249-261.

Hunt, C. B. 1974. Natural Regions of the United States and Canada. W. H. Freeman and Co., San Francisco, CA.

Hwang, C. L., A. S. M. Masud, S. R. Paidy and K. Yoon. 1979. Multiple Objective Decision Making -- Methods and Applications. Springer-Verlag.

Hwang, C. L., S. R. Paidy, K. Yoon and A. S. M. Masud. 1980. Mathematical Programming with Multiple Objectives: A Tutorial. Computers and Operations Research 7:5-31.

Ignizio, J. P. 1976. Goal Programming and Extensions. D. C. Heath Publishing Co.

Ignizio, J. 1977. A Review of Goal Programming: A Tool for Multiobjective Analysis. Journal of the Operational Research Society 29:1109-1119.

Isermann, H. 1974. Proper Efficiency and the Linear Vector Maximum Problem. Operations Research 22:189-191.

Jahn, L. R. 1977. Habitat Data Needs for Managing Wildlife. In proceedings of Classification, Inventory, and Analysis of Fish and Wildlife Habitat. January 24-27, Phoenix, AZ.

Johnson, K. N. and H. L. Schuerman. 1977. Techniques for Prescribing Optimal Timber Harvest and Investment Under Different Objectives - Discussion and Synthesis. Forest Science Monograph 18.

Johnson, K. N. and D. B. Jones. 1980. Timber Harvest Scheduling Model User's Guide and Operations Manual. U. S. Forest Service Working Draft.

Johnson, L. E. and D. P. Loucks. 1980. Interactive Multiobjective Planning Using Computer Graphics. Computers and Operations Research 7:89-97.

Kao, C. 1982. Optimal Stocking Levels and Rotation Under Risk. Forest Science 28:711-719.

Kao, C. and J. D. Brodie. 1979a. Determination of Optimal Thinning Entry Interval Using Dynamic Programming. Forest Science 25:672-674.

Kao, C. and J. D. Brodie. 1979b. Goal Programming for Reconciling Economic, Even-Flow, and Regulation Objectives in Forest Harvesting. Canadian Journal of Forest Research 9:525-531.

Kao, C. and J. D. Brodie. 1980. Simultaneous Optimization of Thinnings and Rotation with Continuous Stocking and Entry Intervals. Forest Science 26:338-346.

Keefer, D. L. 1978. Applying Multiobjective Decision Analysis to Resource Allocation Planning Problems. in Zionts op. cit.:299-320.

Keefer, D. L. and S. M. Pollock. 1980. Approximations and Sensitivity in Multiobjective Resource Allocation. Operations Research 28:114-128.

Keelin, T. W. 1981. A Parametric Representation of Additive Value Functions. Management Science 27: 1200-1208.

Keeney, R. L. 1979. Evaluation of Proposed Pumped Storage Sites. Operations Research 27:1-36.

Keeney, R. L. 1982. Decision Analysis: An Overview. Operations Research 30:803-838.

Keeney, R. L. and H. Raiffa. 1976. Decisions with Multiple Objectives: Preferences and Value Tradeoffs. John Wiley and Sons.

Kent, B. M. 1980. Linear Programming in Land-Management Planning on National Forests. Journal of Forestry 78:469-471.

Kidd, W. E., E. F. Thompson and P. H. Hoepner. 1966. Forest Regulation by Linear Programming - A Case Study. Journal of Forestry 64:611-613.

Kingsley, N. P. 1975. The Forest Landowners of New Jersey. USDA Forest Service, Research Bulletin NE-39.

Kingsley, N. P. 1980. The Northeastern Forest Landownership Survey. in Royer and Convery op. cit., 83-96.

Kirby, M. W., W. A. Hager and P. Wong. 1982. Simultnaeous Planning of Wildland Management and Transportation Alternatives. Presented paper at the ORSA/TIMS Joint National Meeting, April 18-21, 1982, Detroit, MI.

Kirkwood, C. W. and R. K. Sarin. 1980. Preference Conditions for Multiattribute Value Functions. Operations Research 28:225-232.

Khairullah, Z. and S. Zionts. 1979. An Experiment with Some Algorithms for Multiple Criteria Decision Making. Working Paper 422, School of Management, State University of New York at Buffalo.

Knierim, H. A., K. L. Carvell and J. D. Gill. 1971. Browse in Thinned Oak and Cove Hardwood Stands. Journal of Widllife Management 35:163-168.

Knoll, A. L. and A. Engelberg. 1978. Weighting Multiple Objectives - The Churchman-Ackoff Technique Revisited. Computers and Operations Research 5:165-177.

Korhonen, P. and M. Soismaa. 1981. An Interactive Multiple Criteria Approach to Ranking Alternatives. Journal of the Operational Research Society 32:577-585.

Kornbluth, J. S. H. 1973. A survey of Goal Programming. Omega 1:193-205.

Kornbluth, J. S. H. and R. E. Steuer. 1981. Multiple Objective Linear Fractional Programming. Management Science 27:1024-1039.

Kuhn, H. W. and A. W. Tucker. 1950. Nonlinear Programming. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (J. Neyman, ed.). 481-492.

Kurtz, W. B. and C. K. Crouse. 1981. Nonindustrial Private Forest Ownership Studies: A Bibliography. Council of Planning Librarians Bibliographies, 1313 East 60th Street, Chicago, Illinois.

Kurtz, W. B. and B. J. Lewis. 1981. Decision-making Framework for Nonindustrial Private Forest Owners: An Application in the Missouri Ozarks. Journal of Forestry 79:285-288.

Leak, W. B. 1964. Estimating Maximum Allowable Timber Yields By Linear Programming. USDA Forest Service, Research Paper NE-17.

Lee, S. M., M. Gen and C. Snyder. 1982. The Microcomputer: Experience and Implications for the Future of Multiple Criteria Decision Making. Presetned paper at the ORSA/TIMS Joint National Meeting, October 25-27, San Diego, CA.

Lee, S. M., G. I. Green and C. S Kim. 1981. A Multiple Criteria Model for the Location-Allocation Problem. Computers and Operations Research 8:1-8.

Lembersky, M. R. and K. N. Johnson. 1975. Optimal Policies for Managed Stands: An Infinite Horizon Markov Decision Process Approach. Forest Science 21:109-122.

Leuschner, W. A., J. R. Porter, M. R. Reynolds and H. E. Burkhart. 1975. A Linear Programming Model for Multiple-Use Planning. Canadian Journal of Forest Research 5:485-491.

Liittschwager, J. M. and T. H. Tcheng. 1967. Solution of a Large-Scale Forest Scheduling Problem by Linear Programming Decompostion. Journal of Forestry 65:644-646.

Lines, I. L. and C. J. Perry. 1978. A Numerical Wildlife Habitat Evaluation Procedure. Proceedings of the Forty-Third North American Wildlife Conference:284-301.

Loucks, D. P. 1964. The Development of an Optimal Program for Sustained-Yield Management. Journal of Forestry 62:485-490.

Luenberger, D. G. 1973. Introduction to Linear and Nonlinear Programming. Addison-Wesley.

Marcotte, O. and R. M. Soland. 1981. An Interactive Branch-and-Bound Algorithm for Multiple Criteria Optimization. Serial T-442, School of Engineering and Applied Science, The George Washington University.

Martin, A. J. and P. E. Sendak. 1973. Operations Research in Forestry: A Bibliography. USDA Forest Service General Technical Report NE-8.

Martinson, F. K. 1982. A Linear Programming Model for Vegetation Allocation. Presented paper at the ORSA/TIMS Joint National Meeting, October 25-27, 1982, San Diego, CA.

Matney, T. G. and A. D. Sullivan. 1982. Compatible Stand and Stock Tables for Thinned and Unthinned Loblolly Pine Stands. Forest Science 28:161-171.

Mehrez, A. and Zilla Sinuany-Stern. 1983. Resource allocation to Interrelated Rsiky Projects Using a Multiattribute Utility Function. Management Science 29:430-439.

Meyer, H. A., A. B. Recknagel, D. D. Stevenson and R. A. Bartoo. 1961. Forest Management. Ronald Press.

Miller, H. A. 1957. Capabilities and Problems in the Management of Hardwood Forests for Timber and Game. In Special Problems in Southern Forest Management. Sixth Annual Forest Symposium. Louisiana State University, Baton Rouge.

Morse, J. N. 1980. Reducing the Size of the Nondominated Set: Pruning by Clustering. Computers and Operations Research 7:55-66.

Moser, J. W. 1978. Stand-Level Growth Projection Models. In: State of the Art Workshop, Biometrics Working Group, Society of American Foresters and School of Forest and Wildlife Resources, Virginia Polytechnic Institute and State University, Roanoke, VA.

Mullaney, G. E. and V. L. Robinson. 1980. Forest Investment by Nonindustrial, Private Landowners in Georgia. Georgia Forest Research Paper 12, Georgia Forestry Commission.

Murphy, D. A. and J. H. Ehrenreich. 1965. Effects of Timber Harvest and Stand Improvement on Forage Production. Journal of Wildlife Management 29:8734-739.

Murtagh, B. A. and M. A. Saunders. 1978. Large-Scale Linearly Constrained Optimization. Mathematical Programming 14:41-72.

Musselman, K. and J. Talavage. 1980. A Tradeoff Cut Approach to Multiple Objective Optimization. Operations Research 28:1424-1435.

Naslund, B. 1969. Optimal Rotation and Thinning. Forest Science 15:446-451.

Nautiyal, J. C. 1983. Towards a Method of Uneven-Aged Forest Management Based on a Theory of Financial Maturity. Forest Science 29:47-58.

Nautiyal, J. C. and P. H. Pearse. 1967. Optimizing the Conversion to Sustained Yield - A Programming Solution. Forest Science 13:131-139.

Navon, D. I. 1971. Timber RAM ... A Long Range Planning Method for Commercial Timber Lands Under Multiple-Use Management. USDA Forest Service Research Paper PSW-70.

Navon, D. I. and R. J. McConnen. 1967. Evaluating Forest Management Policies by Parametric Linear Programming. USDA Forest Service Research Paper PSW-42.

Navon, D. I. and A. Weintraub. 1982. Operations Research in Forestry - State of the Art. Presented paper at the ORSA/TIMS Joint National Meeting, April 18-21, 1982, Detroit, MI.

Nazareth, L. 1980. A Land Management Model Using Dantzig-Wolfe Decomposition. Management Science 26:510-523.

Omi, P. N., J. L. Murphy and L. C. Wensel. 1981. A Linear Programming Model for Wildland Fuel Management Planning. Forest Science 27:81-94.

Oppenheimer, K. R. 1978. A Proxy Approach to Multi-Attribute Decision Making. Management Science 24:675-689.

Osteen, C. and D. E. Chappelle. 1980. Forest Management Options and Policy Implications in the Kallamazoo River Basin of Michigan. Journal of Forestry 80:287-290.

Paine, D. W. M. 1966. Analysis of a Forest Management Situation by Linear Programming. Australian Forestry:293-303.

Pelz, D. R. 1977. Determination of Optimal Growing Stock Levels by Inventory Theory. Forest Science 23:183-189.

Philip, J. 1972. Algorithms for the Vector Maximization Problem. Mathematical Programming 2:207-229.

ReVelle, C., J. L. Cohon and D. Shobrys. 1981. Multiple Objectives in Facility Location: A Review. Operations Research Group Report Series 81-01, The Johns Hopkins University, Baltimore, Maryland.

Riitters, K., J. D. Brodie and D. W. Hann. 1982. Dynamic Programming for Optimization of Timber Production and Grazing in Ponderosa Pine. Forest Science 28:517-526.

Riitters, K., J. D. Brodie and C. Kao. 1982. Volume versus Value Maximization Illustrated for Douglas-fir with Thinning. Journal of Forestry 80:86-89.

Rorres, C. 1978. A Linear Programming Approach to the Optimal Sustainable Harvesting of a Forest. Journal of Environmental Management 6:245-254.

Rose, D. W., R. A. Leary and C. M. Chen. 1981. Maximum Cubic Volume Production Through Easily Derived Optimum Thinning Schedules. Journal of Forestry 79:32-35.

Rosene, W. 1969. The Bobwhite Quail: Its Life and Management. Rutgers University Press, New Brunswick, New Jersey.

Rosenthal, R. E. 1979. Critique of Goal Programming. Working Paper No. 81, Management Science Program, The University of Tennessee.

Rosenthal, R. E. 1982. Principles of Multiobjective Optimization. Working Paper No. 154, Management Science Program, The University of Tennessee.

Rosenthal, R. E. 1983. Goal Programming - A Critque. New Zealand Journal of Operations Research 11:1-7.

Ross, E. W. 1980. Nonindustrial Forest-Land Owners in the Southeast. in Proceedings of the Eighth Annual Hardwood Symposium, Hardwood Research Council, Asheville, North Carolina.

Roy, B. and P. Vincke. 1981. Multicriteria Analysis: Survey and New Directions. European Journal of Operational Research 8:207-218.

Royer, J. P. and F. J. Convery (eds.). 1980. Nonindustrial Private Forests: Data and Informations Needs Conference Proceedings. Duke University, Durham, North Carolina.

Rubin, D. S. 1975. Vertex Generation and Cardinality Constrained Linear Programs. Operations Research 23:555-565.

Sadagopan, S. and A. Ravindran. 1980. Multicriteria Mathematical Programming - A Unified Interactive Approach. Research Memorandum 80-3, School of Industrial Engineering, Purdue University.

Sanderson, H. R., W. M. Healy, J. C. Pack, J. D. Gill and J. W. Thomas. 1975. Gray Squirrel Habitat and Nest-Tree Preference. In Proceedings of the Southeast Association of Game and Fish Commissions 29:609-617.

Satterfield, S. 1980. A Perspective From the Office of Management and Budget. in Royer and Convery, op. cit.:11-14.

Schamberger, M. and A. Farmer. 1978. The Habitat Evaluation Procedures: Their Application in Project Planning and Impact Evaluation. In Proccedings of the Forty-Third North American Wildlife Conference:274-283.

Schilling, D. A. 1980. An Approach to the Display and Analysis of Multiobjective Problems. Working Paper 80-61, Department of Management Science, Ohio State University.

Schilling, D. A., A. McGarity and C. ReVelle. 1982. Display of Information in Multiobjective Analysis. Management Science 28:236-242.

Schnur, G. L. 1937. Yield, Stand and Volume Tables for Even-AgedUpland Oak Forests. USDA Technical Bulletin No. 560.

Schmidt, J. S. and P. L. Tedder. 1981. A Comprehensive Examination of Economic Harvest Optimization Simulation Methods. Forest Science 27:523-536.

Schoemaker, P. J. H. and C. C. Waid. 1982. An Experimental Comparison of Different Approaches to Determining Weights in Additive Utility Models. Management Science 28:182-196.

Schreuder, G. F. 1971. The Simultaneous Determination of Optimal Thinning Schedule and Rotation for an Even-Aged Forest. Forest Science 17:333-339.

Schreuder, G. F., K. P. Rustagi and B. B. Bare. A Computerized System for Wild Land Use and Planning and Environmental Impact Assesment. Computers and Operations Research 3:217-228.

Schuler, A. and J. C. Meadows. 1976. Planning Resource Use on National Forests to Acheive Multiple Objectives. Journal of Environmental Management 3:351-366.

Schuler, A. T., H. H. Webster and J. C. Meadows. 1977. Goal Programming in Forest Management. Journal of Forestry 75:320-324.

Schuster, J. L. and L. K. Halls. 1962. Timber Overstory Determines Deer Forage in Shortleaf-Loblolly Pine-Hardwood Forests. In Proceedings of the Society of American Foresters:165-167.

Sedjo, R. A. and D. M. Ostermeier. 1977. Policy Alternatives for Nonindustrial Private Forests. Report of the Workshop on Policy Alternatives for Nonindustrial Private Forests, Sponsored by the Society of American Foresters and Resources for the Future.

Shaw, S. P. 1971. Wildlife and Oak Management. In Proceedings of the Oak Symposium, USDA Forest Service.

Shugart, H. H. 1981. An Overview of Multivariate Methods and Their Applications to Studies of Wildlife Habitat. In The Use of Multivariate Statistics in Studies of Wildlife Habitat, USDA Forest Service General Technical Report RM-87.

Smalley, G. W. and R. L. Bailey. 1974. Yield Tables and Stand Structure for Loblolly Pine Plantations in Tennessee, Alabama, and Georgia Highlands. USDA Forest Service Research Paper SO-96.

Smalley, G. W. and D. R. Bower. 1968. Volume Tables and Point Sampling Factors for Loblolly Pine Plantations on Abandoned Fields in Tennessee, Alabama, and Georgia Highlands. USDA Forest Service Research Paper SO-32.

Smalley, G. W. and D. R. Bower. 1971. Site Index Curves for Loblolly and Shortleaf Pine Plantations on Abandoned Fields in Tennessee, Alabama, and Georgia Highlands. USDA Forest Service Research Note SO-126.

Smith, D. M. 1962. The Practice of Silviculture. John Wiley and Sons.

Soland, R. M. 1979. Multicriteria Optimization: A General Characterization of Efficient Solutions. Decision Sciences 10:26-38.

Steuer, R. E. 1976. Multiple Objective Linear Programming with Interval Criterion Weights. Management Science 23:305-317.

Steuer, R. E. 1981. On Sampling the Efficient Set Using Weighted Tchebycheff Metrics. Working Paper 81-113, College of Business Adminstration, University of Georgia.

Steuer, R. E. 1983. Multiple Criterion Function Goal Programming Applied to Managerial Compensation Planning. Working Paper 82-119, College of Business Adminstration, University of Georgia.

Steuer, R. E. and E. U. Choo. 1981. An Interactive Weighted Tchebycheff Procedure for Multiple Objective Programming. Working Paper 81-109, College of Business Administration, University of Georgia.

Steuer, R. E. and F. W. Harris. 1980. Intra-Set Point Generation and Filtering in Decision and Criterion Space. Computers and Operations Research 7:41-53.

Steuer, R. E. and A. T. Schuler. 1978. An Interactive Multiple-Objective Linear Programming Approach to a Problem in Forest Management. Operations Research 26:254-269.

Tanahashi, T. and D. G. Luenberger. 1972. Cardinality-Constrained Linear Programming. Report SE-79R-5, Department of Engineering-Economic Systems, Stanford University.

Tedder, P. L., J. S. Schmidt and J. Gourley. 1980. TREES: Timber Resource Economic Estimation System - Volume I-IV. Research Bulletin 31, Forest Research Lab, Oregon State University.

Tennessee Valley Authority. 1969. Private Forest Management in the Tennessee Valley. Division of Land and Forest Resources, Norris, Tennessee.

Theiler, T. C. 1959. Linear Programming and Optimal Cutting Policies. The Paper Industry:384-391.

Thomas, C. E. and M. S. Hedlund. 1980. Thinning Opportunites in the Midsouth During the 1980's. in Proceedings of the Southern Forest Economic Workshop, Lester Holley, ed.

Thompson, E. F. 1968. The Theory of Decision Under Uncertainty and Possible Applications in Forest Management. Forest Science 14:156-163.

Torn, A. A. 1980. A Sampling-Search-Clustering Approach for Exploring the Feasible/Efficient Solutions of MCDM Problems. Computers and Operations Research 7:67-79.

USDA Forest Service. 1971. Wildlife Management Handbook, Southern Region. FSH 2609.

USDA Forest Service. 1974. The Outlook for Timber in the United States. Forest Resources Report 20.

USDA Forest Service. 1978. Evaluation of the Alabama Pilot Test of the Woodland Resource Analysis Program (WRAP). Nathan Byrd, USFS, SA - State and Private Forestry, Chairman.

USDA Forest Service. 1979. The Use of Multivariate Statistics in Studies of Wildlife Habitat. General Technical Report RM-87.

Usher, M. B. 1966. A Matrix Approach to the Management of Renewable Resources, with Special Reference to Selection Forests. Journal of Applied Ecology 3:355-367.

Usher, M. B. 1969. A Matrix Model for Forest Management. Biometrics 25:309-315.

Usher, M. B. 1976. Extensions to Models Used in Renewable Resources Management Which Incorporate an Arbitrary Strucutre. Journal of Environmental Management 4:123-140.

Wallenius, J. 1975. Comparative Evaluation of Some Interactive Approaches to Multicriterion Optimization. Management Science 21:1387-1396.

Wallmo, O. C., L. H. Carpenter, W. L. Regelin, R. B. Gill and D. L. Baker. 1977. Evaluation of Deer Habitat on a Nutritional Basis. Journal of Range Management 30:122-127.

Wardle, P. A. 1965. Forest Management and Operational Research: A Linear Programming Study. Management Science 11:B260-270.

Ware, G. O. and J. L. CLutter. 1971. A Mathematical Programming System for the Management of Industrial Forests. Forest Science 17:428-445.

Weintraub, A. and D. I. Navon. 1976. A Forest Management Planning Model Integrating Silvicultural and Transpotation Activities. Management Science 22:1299-1309.

Wehrung, D. A. 1978. Interactive Identification and Optimization Using a Binary Preference Realtion. Operations Research 26:322-331.

Whitaker, G. A. and R. H. McCuen. 1976. A Proposed Methodology for Assessing the Quality of Wildlife Habitat. Ecological Modelling 2:251-272.

White, D. J. 1980. Multiobjective Interactive Programming. Journal of the Operational Research Society 31:517-523.

Williams, D. L. and W. C. Hopkins. 1968. Converting Factors for Southern Pine Products. USDA Forest Service Bulletin No. 626.

Williamson, J. F. 1983. Wildlife Benefit Simulation in the FORMAN Program. Tennessee Valley Authority Technical Note B-48. Norris, Tennessee.

Williston, H. L. 1975. Selected Bibliography on Growth and Yield of the Four Major Southern Pines. USDA Forest Service, Southeast Area, State and Private Forestry.

Wolfe, P. 1970. Convergence Theory in Nonlinear Programming. in J. Abadie (ed.), Integer and Nonlinear Programming, North-Holland.

Worrell, A. C. and L. C. Irland. 1975. Alternative Means of Motivating Investment in Private Forestry. Journal of Forestry 73:206-209.

Yu, P. L. and M. Zeleny. 1975. The Set of All Non-Dominated Solutions in Linear Cases and a Multicriteria Simplex Method. Journal of Mathematical Analysis and Applications 49:430-468.

Zeedyk, W. D. 1980. Critical Factors. in Habitat Appraisal. Widllife Management Techniques Manual, 4th. Ed. The Wildlife Society, Washington, D. C.

Zeleny, M. 1982. Multiple Criterion Decision Making. Prentice-Hall.

Zionts, S. (ed.). 1978. Multiple Criteria Problem Solving:Proceedings, Buffalo, N.Y. (U.S.A.), 1977. Springer-Verlag.

Zionts, S. and Jyrki Wallenius. 1976. An Interactive Programming Method for Solving the Multiple Criteria Problem. Management Science 22:652-663.

Zionts, S. and Jyrki Wallenius. 1980. Identifying Efficient Vectors: Some Theory and Computational Results. Operations Research 28:788-793.

Zionts, S. and Jyrki Wallenius. 1983. Interactive Linear Programming. Management Science 29:519-529.

APPENDICES

APPENDIX A. GLOSSARY OF FORESTRY TERMS

basal area - the cross sectional area of a tree at 4.5 feet above ground level. It is usually determined by assuming that the trunk is circular and using the diameter to calculate the area of the resulting circle. This measure may also be expressed on a per acre basis by summing the indiviual basal areas of all trees contained on that acre.

cultural practice - refers to any type of activity that is performed on the stand for the purpose of improved forest management.

d.b.h. (diameter at breast height) - the diameter of the bole at 4.5 feet above ground level.

even-aged stand - a stand composed of trees which are all of the same age (or nearly so).

habitat suitability index - a dimensionless number ranging from 0 to 1. Zero represents the worst possible habitat, while a one indicates optimum habitat.

management regime - a complete specification of harvesting and cultural practices for a given stand.

mixed species stand - a stand composed of trees of many (usually more then three) species, where no species represents more than 50 percent of the stand.

pure stand - a stand composed of trees with greater than 50 percent from one species.

site index - a measure of overall site productivity. It is expressed as the height of the dominant portion of the stand at some reference age (e.g. 25 or 50 years). Since the height growth of a forest stand is directly related to site productivity, site index integrates the productivity factors such as soil characteristics, rainfall, temperature, etc. over an extended period of time.

APPENDIX B. A SAMPLE FORMAN DATASET

Following is the sample FORMAN dataset which was described in Chapter 6. The format is primarily free form with the following exceptions.

1) The dataset starts with a section describing the linking constraints. Column 1 in this section is always blank, except for the last record, which contains '*'. Each record contains the name of a linking constraint (8 characters), followed by (in free form) the lower bound, upper bound, lower elastic cost, and upper elastic cost.

2) The remaining records describe the individual stands and enumerated management regimes. The first record for a stand contains a '%' in column 1, followed by (in free form) the percentage of the total acreage that this stand occupies, and the partitioned cardinality limit for the stand. Stand data is terminated by a record with a '!' in column 1.

3) Each management regime starts with the benefits section. These records have a '=' in column 1 along with the benefit (on a stand basis) for the given management regime.

4) Following the benefits section, the elements of the constraint matrix are listed (nonzeroes only). These records have a blank in column 1, followed by the eight character row (constraint) name, with the constraint element last.

5) The dataset is terminated by a record with a '?' in column 1.

```
|--------|---------|---------|---------|---------|

WOOD1983    0.0   6000.0    0.0    15.0
WOOD1984    0.0   6000.0    0.0    15.0
WOOD1985    0.0   6000.0    0.0    15.0
WOOD1986    0.0   6000.0    0.0    15.0
WOOD1987    0.0   6000.0    0.0    15.0
WOOD1988    0.0   6000.0    0.0    15.0
WOOD1989    0.0   6000.0    0.0    15.0
WOOD1990    0.0   6000.0    0.0    15.0
WOOD1991    0.0   6000.0    0.0    15.0
WOOD1992    0.0   6000.0    0.0    15.0
WOOD1993    0.0   6000.0    0.0    15.0
WOOD1994    0.0   6000.0    0.0    15.0
WOOD1995    0.0   6000.0    0.0    15.0
WOOD1996    0.0   6000.0    0.0    15.0
WOOD1997    0.0   6000.0    0.0    15.0
WOOD1998    0.0   6000.0    0.0    15.0
WOOD1999    0.0   6000.0    0.0    15.0
WOOD2000    0.0   6000.0    0.0    15.0
WOOD2001    0.0   6000.0    0.0    15.0
WOOD2002    0.0   6000.0    0.0    15.0
WOOD2003    0.0   6000.0    0.0    15.0
CASH1983   -40000.0    300000.0    1.0    0.0
CASH1984   -40000.0    300000.0    1.0    0.0
CASH1985      500.0    300000.0    1.0    0.0
CASH1986      500.0    300000.0    1.0    0.0
CASH1987      500.0    300000.0    1.0    0.0
CASH1988      500.0    300000.0    1.0    0.0
CASH1989      500.0    300000.0    1.0    0.0
CASH1990      500.0    300000.0    1.0    0.0
CASH1991      500.0    300000.0    1.0    0.0
CASH1992      500.0    300000.0    1.0    0.0
CASH1993      500.0    300000.0    1.0    0.0
CASH1994      500.0    300000.0    1.0    0.0
CASH1995      500.0    300000.0    1.0    0.0
CASH1996      500.0    300000.0    1.0    0.0
CASH1997      500.0    300000.0    1.0    0.0
CASH1998      500.0    300000.0    1.0    0.0
CASH1999      500.0    300000.0    1.0    0.0
CASH2000      500.0    300000.0    1.0    0.0
CASH2001      500.0    300000.0    1.0    0.0
CASH2002      500.0    300000.0    1.0    0.0
CASH2003      500.0    300000.0    1.0    0.0
*
```

```
|--------|---------|---------|---------|---------|

% 0.285    3  {loblolly plantation, 0 years, 100 acres, SI=60}
= 12400.0
= 3370.0
= 0.360
= 0.000
= 0.531
 WOOD1996     1020.0
 WOOD2001      540.0
 WOOD2002     1810.0
 CASH1983   -12000.0
 CASH1996     9208.0
 CASH2001     4652.0
 CASH2002    20598.0
!
= 5400.0
= 3730.0
= 0.45
= 0.0
= 0.6
 WOOD1988    15700.0
 WOOD2003    21600.0
 CASH1983   -12000.0
 CASH1998    14131.0
 CASH2000     -200.0
 CASH2003    19468.0
!
= 4200.0
= 2690.0
= 0.225
= 0.0
= 0.39995
 WOOD1993      250.0
 WOOD1998      570.0
 WOOD2003     1870.0
 CASH1983   -12000.0
 CASH1993     2270.0
 CASH1998     5128.0
 CASH2000     -200.0
 CASH2003    24084.0
!
= 6500.0
= 4080.0
= 0.09
= 0.0
= 0.2
 WOOD2002     4080.0
 CASH1983   -12000.0
 CASH2000     -200.0
```

```
|--------|---------|---------|---------|---------|

 CASH2002    37422.0
!
= 2800.0
= 2550.0
= 0.3
= 0.0
= 0.4
 WOOD1993      250.0
 WOOD2000     2300.0
 CASH1983   -12000.0
 CASH1993     2227.0
 CASH2000    20665.0
!
= 5000.0
= 3070.0
= 0.3
= 0.0
= 0.4
 WOOD1997     1020.0
 WOOD2002     2050.0
 CASH1984   -12000.0
 CASH1997     9208.0
 CASH2000     -200.0
 CASH2002    18436.0
!
= 4100.0
= 2790.0
= 0.3
= 0.0
= 0.4
 WOOD1996      760.0
 WOOD2001     2030.0
 CASH1984   -12000.0
 CASH1996     6821.0
 CASH1999     -200.0
 CASH2001    18275.0
!
= 7100.0
= 3790.0
= 0.01
= 0.0
= 0.1
 WOOD2003     3790.0
 CASH1985   -12000.0
 CASH2001     -200.0
 CASH2003    34120.0
!
```

```
|--------|---------|---------|---------|---------|

= 6400.0
= 3480.0
= 0.01
= 0.0
= 0.1
 WOOD2000     3480.0
 CASH1984   -12000.0
 CASH1998     -200.0
 CASH2000    31347.0
!
= 5900.0
= 3700.0
= 0.01
= 0.0
= 0.1
 WOOD2000     3700.0
 CASH1983   -14000.0
 CASH1998     -200.0
 CASH2000    33304.0
!
= 7100.0
= 4280.0
= 0.01
= 0.0
= 0.1
 WOOD2003     4280.0
 CASH1984   -14000.0
 CASH2000     -200.0
 CASH2003    38537.00
!
% 0.145  1  {upland oak-hickory, 40 years, 50 acres, SI=60, BA=80}
= 5650.0
= 875.0
= 0.92
= 0.93
= 0.50
 WOOD1983      875.0
 CASH1983     5351.0
!
= 13050.0
= 1535.0
= 0.85
= 0.96
= 0.30
 WOOD1983      225.0
 WOOD2003     1310.0
 CASH1983    -3529.0
 CASH2003    35690.5
!
```

```
|--------|---------|---------|---------|---------|

= 11150.0
= 1440.0
= 0.85
= 0.96
= 0.50
 WOOD1983     225.0
 WOOD1991     180.0
 WOOD1999    1035.0
 CASH1983   -3529.0
 CASH1991    2994.5
 CASH1999   24225.0
!
= 11100.0
= 1435.0
= 0.85
= 0.96
= 0.30
 WOOD1989     355.0
 WOOD1999    1080.0
 CASH1983   -6220.0
 CASH1989    5402.0
 CASH1999   25231.5
!
= 8000.0
= 1415.0
= 0.7
= 0.95
= 0.01
 WOOD2000    1415.0
 CASH1983   -6220.0
 CASH2000   28481.5
!
= 7700.0
= 1365.0
= 0.75
= 0.95
= 0.01
 WOOD1998    1365.0
 CASH1983   -6220.0
 CASH1998   23460.0
!
= 8200.0
= 1460.0
= 0.70
= 0.95
= 0.01
```

```
|--------|---------|---------|---------|---------|

 WOOD2002    1460.0
 CASH1983   -6220.0
 CASH2002  -28481.5
!
= 8650.0
= 1500.0
= 0.8
= 0.95
= 0.40
 WOOD1994     310.0
 WOOD2002    1190.0
 CASH1983   -6220.0
 CASH1994    4694.5
 CASH2002   23183.5
!
= 8550.0
= 1465.0
= 0.80
= 0.95
= 0.40
 WOOD1989     355.0
 WOOD2000    1110.0
 CASH1983   -6220.0
 CASH1989    4540.5
 CASH2000   20327.5
!
% 0.285  3  {Yellow-poplar, 30 years, 100 acres, SI=90, BA=100}
= 22800.0
= 3320.0
= 0.75
= 0.953
= 0.50
 WOOD1983    3320.0
 CASH1983   23919.0
!
= 36300.0
= 3770.0
= 0.8
= 0.953
= 0.56
 WOOD1986    3770.0
 CASH1983  -12360.0
 CASH1986   58384.0
!
= 66400.0
= 55870.0
= 0.75
= 0.97
```

```
|--------|---------|---------|---------|---------|

= 0.40
 WOOD1983     320.0
 WOOD1991     640.0
 WOOD1997     430.0
 WOOD2002    4180.0
 CASH1983   -9455.0
 CASH1991   13486.0
 CASH1997   11936.0
 CASH2002  134272.0
!
= 66500.0
= 5680.0
= 0.70
= 0.97
= 0.30
 WOOD1986     580.0
 WOOD1996     750.0
 WOOD2003    4350.0
 CASH1983  -12360.0
 CASH1986    8113.0
 CASH1996   19774.0
 CASH2003  140766.0
!
= 42700.0
= 5660.0
= 0.55
= 0.97
= 0.10
 WOOD1988     740.0
 WOOD1993     370.0
 WOOD1998     350.0
 WOOD2003    4200.0
 CASH1983  -12000.0
 CASH1988    9643.0
 CASH1993    6046.0
 CASH1998    6554.0
 CASH2003   87604.0
!
= 41000.0
= 4880.0
= 0.66
= 0.67
= 0.23
 WOOD1989     820.0
 WOOD1995    4060.0
 CASH1983  -12000.0
 CASH1989   12726.0
 CASH1995   86169.0
!
```

```
|--------|---------|---------|---------|---------|

= 41000.0
= 5410.0
= 0.7
= 0.9
= 0.45
 WOOD1990     900.0
 WOOD2000    4510.0
 CASH1983  -12000.0
 CASH1990   12726.0
 CASH2000   86169.0
!
= 40400.0
= 3850.0
= 0.63
= 0.95
= 0.35
 WOOD1997    1050.0
 WOOD2002    2800.0
 CASH1983  -12000.0
 CASH1997   24723.0
 CASH2002   81773.0
!
= 40900.0
= 5520.0
= 0.63
= 0.95
= 0.40
 WOOD1994    1210.0
 WOOD2001    4310.0
 CASH1983  -12000.0
 CASH1994   19482.0
 CASH2001   83280.0
!
% 0.285  4  {Natural shortleaf, 20 years, 100 acres, SI=80, BA=)}
= 41400.0
= 3040.0
= 0.20
= 0.45
= 0.30
 WOOD1988    3606.0
 WOOD1998   96089.0
 CASH1983  -12360.0
 CASH1988    3606.0
 CASH1990    -200.0
 CASH1998   96089.0
!
= 41500.0
= 3500.0
```

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|--------|---------|---------|---------|---------|

= 0.20
= 0.50
= 0.40
 WOOD1988      400.0
 WOOD1998       60.0
 WOOD2003     2500.0
 CASH1983   -12360.0
 CASH1988     3606.0
 CASH1990     -200.0
 CASH1998     5408.0
 CASH2002     -200.0
 CASH2003   110245.0
!
= 43100.0
= 3500.0
= 0.20
= 0.50
= 0.35
 WOOD1988      400.0
 WOOD1996      480.0
 WOOD2003     2620.0
 CASH1983   -12360.0
 CASH1988    -3606.0
 CASH1990     -200.0
 CASH1996     4300.0
 CASH2002     -200.0
 CASH2003   111347.0
!
= 43700.0
= 3340.0
= 0.01
= 0.45
= 0.1
 WOOD2000     3340.0
 CASH1983   -12360.0
 CASH2000   110978.0
!
= 43700.0
= 3500.0
= 0.10
= 0.45
= 0.35
 WOOD1998     1230.0
 WOOD2002     2270.0
 CASH1983   -12000.0
 CASH1998   -13597.0
 CASH1999     -200.0
 CASH2002   103445.0
!
```

```
|--------|---------|---------|---------|---------|

= 43800.0
= 3411.0
= 0.2
= 0.45
= 0.3
 WOOD1997     1030.0
 WOOD2001     2380.0
 CASH1983   -12000.0
 CASH1997     9227.0
 CASH1999     -200.0
 CASH2001   103829.0
!
= 43600.0
= 3370.0
= 0.20
= 0.50
= 0.35
 WOOD1995      890.0
 WOOD1998      160.0
 WOOD2001     2320.0
 CASH1983   -12000.0
 CASH1995     8000.0
 CASH1996     -200.0
 CASH1998     1478.0
 CASH1999     -200.0
 CASH2001   102772.0
!
?
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VITA

Terry Paul Harrison was born in Braddock, Pennsylvania on July 20, 1955. He attended elementary schools in Irwin, Pennsylvania and was graduated from Norwin High School in June 1973. The following September he entered The Pennsylvania State University, and in May 1977 he received a Bachelor of Science (high honors) in Forest Science. In June of 1977, he began employment with the Forest Products Division of Koppers Co., Inc.

In September 1978, under the support of Koppers Co., Inc. and The Tennessee Valley Authority, Mr. Harrison began graduate studies at The University of Tennessee. He was awarded a Master of Science degree in Management Science in December 1981. He received the Doctor of Philosophy in August 1983.

The author is a member of Xi Sigma Phi, Phi Kappa Phi, The Institute of Management Sciences, The Operations Research Society of America, and The Society of American Foresters. Mr. Harrison is currently employed in the Management Science Department, College of Business Administration, at The Pennsylvania State University.

He is married to the former Debra Marie Slye of Oil City, Pennsylvania.