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I am submitting herewith a dissertation written by Cheng Liu entitled "Social Welfare Maximization on Capacitated Networks." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Geography.

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SOCIAL WELFARE MAXIMIZATION ON CAPACITATED NETWORKS

A Dissertation
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Degree

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Cheng Liu

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DEDICATION

This work is dedicated:

To my parents in recognition of their constant love and encouragement with which I have been touched and inspired throughout my life.

To my wife, Tai-Lun, whose patience, companionship, concerns, and the sacrifices she made in to support me in the affliction and tribulations of this research, and my entire graduate work in the United States, have made the whole undertaking worthwhile.

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ABSTRACT

Classical location-allocation models consider spatial efficiency or equity as their major objectives, while more recent location-allocation models consider the maximization of social welfare as their primary goal. The purpose of this study are to explore the properties of social welfare functions used in location-allocation modeling, relate those to entropy maximizing models, and use them to determine the optimal location and size of facilities on capacitated networks.

This research introduces a modified version of Wilson's doubly constrained entropy maximizing model, and proves the equivalence between this modified form and other social welfare maximization models; i.e., consumer surplus, users' benefit, and log accessibility. The modified form of entropy maximizing model can also be applied as spatial competition location-allocation model.

The log accessibility maximization model is applied to capacitated networks. The existence and uniqueness of a solution to the log accessibility model on capacitated bipartite and tree networks are proven. A four step solution procedure for the log accessibility maximization model in these cases is then developed. The solution procedure is shown to be both feasible and convenient.

The numerical results for capacitated bipartite and tree-structured networks are discussed. Since many variables are involved in this model--population, facility size, facility location, construction costs, capital budgets, economies of scale, link capacity, transportation cost, flow pattern, and network structure--it is very difficult to obtain generalized results. In the numerical results presented, most variables are hold constant in order to study the effect of congestion on the location-allocation decision process.

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1. Statement of Problem

Location-allocation models have been used in the planning of facility locations for the last thirty years. The type of model that is used depends upon the goal which is sought. Classical location-allocation models have considered spatial efficiency or equity as their major objectives, while more recent location-allocation models consider the maximization of social welfare as their primary goal. The purposes of this study are to explore the properties of social welfare functions used in location-allocation modeling, relate those to entropy maximizing models, and use them to determine the optimal location and size of facilities on capacitated networks. In this research, social welfare is defined in terms of users' benefit, consumer surplus, or log accessibility, as all three of these concepts have been shown to be theoretically equivalent [Coelho, et al., 1976; Leonardi, 1978]. This theoretical equivalence can be further broadened by considering a Modified Entropy Maximization Model (MEMM) that is equivalent to location-allocation models which maximize social welfare.

Before dealing with these issues, however, it is necessary to consider the vast array of location-allocation literature. This literature review is divided into classic location-allocation problems and social welfare location-allocation problems. The classic problems in the 1960s and early 1970s are technique oriented. At this stage, solution techniques were the central focus of research by geographers [Rushton et al., 1979] who adapted the solution techniques developed in other fields, such as operations research, transportation engineering, and applied mathematics, to solve location-allocation problems. In the late 1970s and early 1980s the spatial interaction model and measures of social welfare were integrated into classic location-allocation models.

1.2. Classic Problems in Location-Allocation Modeling

As spatial problems in the economic landscape were articulated, solutions techniques were developed to deal with these problems. From these solutions, a body of theory evolved for each of these separate problems. Three classic location-allocation problems are the p-median, p-center, and covering problems. Each of these problems can be further classified by the type of space in which it is applied and the number of facilities to be located.

Three types of space are usually considered in location-allocation modeling: Euclidean, Manhattan, and network space. [ReVelle, 1970]. In each, $d(X_j, V_i)$ is defined as the travel distance from population center V_i to facility X_j . Travel distance $d(X_j, V_i)$ can be measured in continuous space or discrete space.

Continuous space means that facilities can be located at any point on the plane; i.e., the solution space is infinite. Within this group, two distance measures have been traditionally used. The first, and most common, is the Euclidean metric; i.e.,

$$d(X_j, V_i) = [(x_j - a_i)^2 + (y_j - b_i)^2]^{1/2},$$

where:

(a_i, b_i) : location of source i ,

(x_j, y_j) : location of facility j .

Euclidean metric distances are used in most traditional location problems such as Von Thunen's model, Weber's problem, and central place theory. Another example of the Euclidean metric is the location of facilities involving air and sea transportation, such as helicopter emergency medical stations [Elzinga, 1972].

The other measure of distance in continuous space is the rectilinear, or Manhattan, metric:

$$d(X_j, V_i) = |x_j - a_i| + |y_j - b_i|.$$

Manhattan distance is appropriate in urban areas where travel takes place along an orthogonal set of streets [Handler, 1978].

Discrete space refers to a network where travel occurs only along the links of the network, such as highway or railroad systems. In this class of problems, the distance between any two points is not necessarily straight line but rather the shortest distance along the links between those two points on the network. The facility and demand points also occur on the network either at the nodes or along the links. This distance measure may be formulated as follows:

$$d(X_j, V_i) = \min_{\text{all paths}} \{ d(X_j, V_i) \}.$$

Network space is used by most transport researchers because it is the best representation of the highway and railway system.

Another way to classify problems is to specify the number of facilities to be located, i.e., either a single

facility or multiple facilities. Theoretically, the solution methods for each case could be the same, but single facility problems are usually much easier to solve. In most cases, there are particular efficient and optimal solution methods for the single facility problem, while multifacility problems often must be addressed with heuristics.

1.2.1. The Family of p-Median Problems

The p-median problem is to find facility locations for which the sum of total weighted distance from the users to the facilities is a minimum. This emphasis on minimizing the distance traveled by all of the users tends to shift the location of the facilities to the most populated areas.

Formally, the p-median problem is to choose X^* to minimize:

$$Z = g(X^*) = \{ \min_X g(X) \text{ s.t. } |X| = p, X \in N \},$$

where:

- N : network or continuous space,
- X : vector of facility locations,
- V : vector of sources,
- p : number of facilities,
- $g(X)$: weighted total travel time and
 $g(X) = \{W d(X,V) \mid X \in N\},$
- W : vector of weight functions,
- $|X|$: the cardinality of X.

Since the distance measurements $d(X,V)$ vary in different spaces, the formulations and solution techniques for p-median problems are also different.

1.2.1.1. The p-Median Problem in Euclidean Space

Assuming p equals 1, this problem becomes a single facility median problem whose objective is:

$$\begin{aligned} Z &= \min_{x,y} f(x,y), \\ &= \min_{x,y} \sum_i W_i [(x-a_i)^2 + (y-b_i)^2]^{\frac{1}{2}}. \end{aligned} \quad (1.1)$$

The approach to solving the problem is to compute the partial derivatives of (1.1) and set them equal to zero:

$$\frac{\partial f(x,y)}{\partial x} = \sum_{i=1}^m \frac{W_i (x - a_i)}{d(X,V)} = 0, \quad (1.2)$$

$$\frac{\partial f(x,y)}{\partial y} = \sum_{i=1}^m \frac{W_i (y - b_i)}{d(X,V)} = 0. \quad (1.3)$$

Solving for x and y , yields:

$$x = \frac{\sum_{i=1}^m \frac{w_i a_i}{d(X,V)}}{\sum_{i=1}^m \frac{w_i}{d(X,V)}}, \quad (1.4)$$

$$y = \frac{\sum_{i=1}^m \frac{w_i b_i}{d(X,V)}}{\sum_{i=1}^m \frac{w_i}{d(X,V)}}. \quad (1.5)$$

An iteration algorithm starts by choosing any arbitrary point, $x^{(1)}$, $y^{(1)}$ as an initial guess point.

Substitute $x^{(1)}$, $y^{(1)}$ into the right hand side of (1.4) and (1.5). A new point $x^{(2)}$, $y^{(2)}$ will be obtained.

Substitute $x^{(2)}$, $y^{(2)}$ into (1.4) and (1.5) again and repeat

this process until $x^{(m)}$, $y^{(m)}$ converges to a new set of x^* ,

y^* . The point x^* , y^* may be an optimal solution. If the

estimates of x^* , y^* never coincide with a source, a populated node, then the above algorithm will yield the optimal solution [Kuhn, 1973]. If, however, it leads to a source node, a modified version of this algorithm,

developed by Ostresh [1978], must be used. Ostresh's method guarantees optimality.

If p is greater than one, this problem is called the multi-facility median problem. The solution procedure is to choose a set of p points in the space for an initial guess point. The space is then partitioned into p subsets according to the rule that every source goes to the nearest facility. Within each subset, the iteration method of single facility median problem (1.4) and (1.5) is adopted to find a new facility location. After processing every subset, demand points are reallocated to the facilities, and the p -single facility problem is solved again. This process is repeated until stability is achieved. This approach, developed by Cooper [1967], is a heuristic method and the optimal solution is not guaranteed.

1.2.1.2. The p -Median Problem in Rectangular Space

The single facility problem in a rectangular space was analyzed by Francis and White [1974]. The task is to choose a location (x,y) which minimizes

$$\begin{aligned}
 Z &= \min_{x,y} f(x,y), \\
 &= \min_{x,y} \sum_{i=1}^m W_i (|x - a_i| + |y - b_i|).
 \end{aligned}$$

The above equation can be decomposed into two individual optimization problems:

$$\min_x f_1(x) = \sum_{i=1}^m W_i |x - a_i|,$$

$$\min_y f_2(y) = \sum_{i=1}^m W_i |y - b_i|.$$

The problem thus reduces to two one dimensional location problems which can be solved easily.

The multi-facility median problem in rectangular space was examined by Cabot [1970]. The objective function is:

$$Z = \min_j \sum_i \sum W_{ji} (|x_j - a_i| + |y_j - b_i|). \quad (1.6)$$

Equation (1.6) can be decomposed into x-axis and y-axis problems; i.e.,

$$\min_{x_1, \dots, x_n} f(x_1, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^p W_{ji} |x_j - a_i|, \quad (1.7)$$

and

$$\min_{y_1, \dots, y_n} f(y_1, \dots, y_n) = \sum_{i=1}^n \sum_{j=1}^p W_{ji} |y_j - b_i|.$$

To convert (1.7) to a linear program, define the following auxiliary variables [Rosenthal, 1981]:

Let:

$$r_{ji} = (x_j - a_i)^+ = \max(x_j - a_i, 0), \quad (1.8)$$

$$s_{ji} = (x_j - a_i)^- = -\min(x_j - a_i, 0). \quad (1.9)$$

Substituting (1.8) and (1.9) into (1.7) yields

$$Z = \min \sum_{j=1}^m \sum_{i=1}^n W_i (r_{ji} + s_{ji}), \quad (1.10)$$

s.t.:

$$x_j - r_{ji} + s_{ji} = a_i, \quad \text{for all } i, j, \quad (1.11)$$

$$x_j, r_{ji}, s_{ji} \geq 0, \quad \text{for all } i, j. \quad (1.12)$$

Equations (1.10)-(1.12) can be solved by linear programming. There are $m+2(m*n)$ decision variables in this problem. If the dual of (1.10)-(1.12) is taken, the decision variables will be reduced to $m*n$. Therefore, the dual is computationally easier to solve than the primal.

1.2.1.3. The p-Median Problem on a Network

In a network the flows occur only along the links and the facilities can only be located at nodes or on links. In the case of a single facility, Hakimi [1964] proved that there must exist an optimal solution which is a node. This result greatly simplifies the solution procedure. Instead of examining all nodes and links, Hakimi suggested only

examination of the nodes is necessary. The node with the minimal objective function is the optimal solution. Hakimi [1965] also found similar results for the multi-facility problem, i.e.; that there is at least one set of nodes in a multi-facility median problem which is an optimal solution.

Many approaches for solving multi-facility problems have been developed. This section reviews three of the most common approaches: node substitution, mathematical programming, and dual-based mathematical programming.

Node substitution was first suggested by Kuhn and Hamburger [1963]. Teitz and Bart [1968] modified this method and presented a successful heuristic node substitution technique. The node substitution approach starts by selecting any p nodes as an initial solution set. It then examines the nodes which are not in the current solution set. If the objective function can be improved by substituting any node not in the solution set with any node in the current solution set, then these two nodes should be exchanged. This substitution process continues until no further improvement in the objective function occurs.

The problem can also be formulated as a mixed integer program [Balinski, 1965].

Let :

$$y_i = \begin{cases} 1 & \text{if facility is established at } V_i, \\ 0 & \text{otherwise,} \end{cases}$$

f_i : the fixed cost at location i ,

$$x_{ij} = \begin{cases} 1 & \text{if source } i \text{ use the facility } j, \\ 0 & \text{otherwise,} \end{cases}$$

d_{ij} : shortest travel time between i and j .

The p -median problem can be formulated as:

$$\min \sum_{i=1}^n \sum_{j=1}^m d_{ij} x_{ij} + \sum_{i=1}^n f_i y_i, \quad (1.13)$$

s.t.:

$$\sum_{i=1}^m x_{ij} = 1, \quad \text{for all } j, \quad (1.14)$$

$$y_i \geq x_{ij}, \quad \text{for all } i, j, \quad (1.15)$$

$$y_i \in \{0, 1\}, \quad \text{for all } i, \quad (1.16)$$

$$x_{ij} \geq 0, \quad \text{for all } i, j. \quad (1.17)$$

In the p -median problem, the fixed cost at any place often is assumed to be uniform. Therefore, the $f_i y_i$ term

can be dropped from (1.13)-(1.17). ReVelle and Swain [1970] presented an equivalent integer program for this case:

$$\min \sum_i \sum_j d_{ij} x_{ij}, \quad (1.18)$$

s.t.:

$$\sum_j x_{jj} = 1,$$

$$x_{jj} \geq x_{ij}, \quad \text{for all } i, \quad (1.19)$$

$$\sum_i x_{ii} = p, \quad (1.20)$$

$$x_{ij} \in \{0,1\}, \quad \text{for all } i,j. \quad (1.21)$$

Both forms (1.13)-(1.17) and (1.18)-(1.21) can be solved by integer programming in which branch and bound or relaxation techniques are applied.

Erlenkotter [1978] suggested an approach developed from dual based mathematical programming. First, replace constraint (1.16) by its relaxation:

$$y_i \geq 0. \quad (1.22)$$

The dual of (1.13) - (1.15), (1.17) and (1.22) is:

$$\max \sum_j z_j,$$

s.t.:

$$\sum_j w_{ij} \leq f_i, \quad \text{for all } i, \quad (1.23)$$

$$z_j - w_{ij} \leq d_{ij}, \quad \text{for all } i,j,$$

$$w_{ij} \geq 0, \quad \text{for all } i, j,$$

where:

z_j, w_{ij} : the dual variables associated with (1.14) and (1.15), respectively.

Erlenkotter suggested a simple solution procedure called the dual ascent-and-adjustment method. Instead of finding the primal solution, this procedure solves the dual problem very efficiently. The solution for various values of p can be found by varying the construction cost, f_i , in (1.23). The branch and bound technique may be necessary, however, because the relaxation of constraint (1.16) does not guarantee an integer solution.

1.2.2. The Family of p-Center Problems

The p-center problem is also known as the minmax location problem because it minimizes the maximum distance to be travelled from a user's location to a facility. If small outlying population centers exist, the facility locations will be pulled toward these outlying nodes. To discuss p-center problems, we define the following:

Let:

$$g_{ji}(X_j) = w_{ji} d(X_j, V_i)$$

be the weighted travel time between facility X_j and source V_i . The p center problem is to choose x^* such that:

$$Z = g(x^*) = \min_X \max_{j,i} g_{ji}(X).$$

1.2.2.1. The p -Center Problem in Euclidean Space

Elzinga and Hearn [1972] examined the problem of locating a single facility to serve equally weighted sources. The objective function is:

$$\min_{x,y} \max_i [(x - a_i)^2 + (y - b_i)^2]^{\frac{1}{2}}. \quad (1.24)$$

Equation (1.24) can be transformed to a mathematical programming problem with a linear objective function and non-linear constraints; i.e.,

$$\min_{x,y,r} r, \quad (1.25)$$

s.t.:

$$[(x - a_i)^2 + (y - b_i)^2]^{\frac{1}{2}} \leq r. \quad (1.26)$$

This problem, (1.25)-(1.26), is equivalent to finding the smallest circle with radius r which covers all n

sources. As soon as the circle is defined, the center of the circle is the optimal solution. Elzinga and Hearn [1972] suggest a simple, efficient algorithm to solve this problem.

When $p > 1$, the objective function is:

$$\min_{x_1 \dots x_n} \max_{j=1 \dots n} \{ W_{ji} [(x_j - a_i)^2 + (y_j - b_i)^2]^{\frac{1}{2}} \}. \quad (1.27)$$

Equation (1.27) can be transformed into the following mathematical programming problem:

$$\min Z_j, \quad (1.28)$$

s.t.:

$$W_{ji} [(x_j - a_i)^2 + (y_j - b_i)^2]^{\frac{1}{2}} \leq Z_j. \quad (1.29)$$

The optimal solution is:

$$Z^* = \max_j Z_j. \quad (1.30)$$

Since the constraint (1.29) is non-linear, linear programming technique can not be used. However, the dual of (1.28) - (1.29) is:

$$\max_{\mathbf{q}} \frac{\sum_{i=1}^n (a_i^t a_i^t) q_i - \mathbf{q}^t \mathbf{A} \mathbf{A} \mathbf{q}}{r \mathbf{q}^t},$$

s.t.:

$$\sum_{i=1}^n q_i = 1,$$

$$q_i \geq 0,$$

where:

$$q_i = \frac{u_i}{\sum_{k=1}^n u_k},$$

u_i : dual variables associated with constraint (1.29),

$\mathbf{A}_i$: the matrix with a_i as i th column

This is a non-linear objective function with linear constraints. It can be solved by the steepest descent method or parametric programming [Elzinga, et al., 1976]. Since the objective function is concave, there is a unique local optimal solution.

1.2.2.2. The p-Center Problem in Rectangular Space

The single facility problem without a weight function is formulated as:

$$\min_{x,y} \max_i (|x - a_i| + |y - b_i|).$$

This objective function is different from the p - median problem in rectangular space, which can be separated into two subprograms. However, the concept of Chebyshev distance is useful in the p -center problem. Denote the Chebyshev distance between two points X_j, V_i as:

$$t(X_j, V_i) = \max (|x_j - a_i|, |y_j - b_i|).$$

It can be proven that if the plane is rotated 45 degree counterclockwise, the Chebyshev distance in the rotated plane equals to the rectangular distance in the original plane [Dearing and Francis, 1974]. This can be presented as:

let:

X_j, V_i : the points on the original plane,

X'_j, V'_i : the points on the rotated plane,

$$d(X'_j, V'_i) = t(X_j, V_i), \quad (1.31)$$

Equation (1.31) can be transformed as:

$$\min_{x', y'} \max_i (|x' - a'_i|, |y' - b'_i|). \quad (1.32)$$

Equation (1.32) can be decomposed as two subprograms:

$$Z_1 = \min_{x'} \max_i (|x' - a'_i|),$$

$$Z_2 = \min_{y'} \max_i (|y' - b'_i|).$$

The optimal solution is:

$$Z = \max(Z_1, Z_2).$$

The algorithm for this problem is first to rotate the plane 45 degrees counterclockwise. This converts all the Manhattan distances to straight line distance. Then find the outermost four points (max a' , min a' , max b' , and min b'). The center of these four points is the optimal solution. Elzinga and Hearn [1972] developed a single equation to solve this problem.

Dearing and Francis [1974] use the same technique to solve the p-center problem with a weighting function. Let X' , V' present the points on the rotated plane. The objective of weighted p-center problem on a rectangular space is:

$$\min_{x'_1 \dots x'_p} \max_{j=1 \dots p} W'_j t(X'_j, V'_j), \quad (1.33)$$

where:

$t(X', V')$ is Chebyshev distance defined as (1.31).

Equation (1.33) can be decomposed into two subprograms which are solved independently. These are:

$$\min_{1} Z', \quad (1.34)$$

s.t.:

$$W_{ji} |x'_j - a'_i| \leq Z'_1, \quad (1.35)$$

and

$$\min_{2} Z',$$

s.t.:

$$W_{ji} |y'_j - b'_i| \leq Z'_2. \quad (1.35)$$

The optimal solution is then:

$$Z' = \max (Z'_1, Z'_2).$$

The absolute values in (1.34) can be separated into two constraints:

$$W_{ji} (x'_j - a'_i) \leq Z'_1, \quad (1.36)$$

$$- W_{ji} (x'_j - a'_i) \leq Z'_1. \quad (1.37)$$

Equations (1.34) and (1.36) - (1.37) represent linear problems and can be solved. An interesting interpretation can be given to the dual of this problem. Dearing and

Francis [1974] determined that the dual of (1.34) and (1.36) - (1.37) is a network maximal flow problem which has one starting node, one sink, and n intermediate nodes with p arcs between each node.

1.2.3. The Family of Covering Problems

Covering problems are special cases of p - center problems on the network. In the p -center problems the facilities can be located at any place on the network; either on the nodes or on the links. Covering problems restrict the solutions to the nodes. Three covering problems are chosen for discussion; set covering problems, maximal covering problems, and maximal covering problems with mandatory closeness constraints.

1.2.3.1. The Location Set Covering Problem

If the goal of locating facilities is to assure that every user will be served within a minimum service level, S , the location set covering problem finds the minimal number of facilities and their locations to achieve this goal [Toregas, et al., 1971]. This problem can be represented as:

$$\min Z = \sum_{j \in N_i} x_j, \quad (1.38)$$

s.t.:

$$\sum_{j \in N_i} x_j \geq 1, \quad \text{for all } i, \quad (1.39)$$

$$x_j = (0,1), \quad \text{for all } j, \quad (1.40)$$

where:

$$N_i = \{ j \mid d_{ji} \leq S \},$$

S : minimal service level.

The solution algorithm consists of two steps. The first step determines the set N_i for a given value of S and the shortest distance matrix D , where:

$$D = [d_{ij}]. \quad (1.41)$$

The second step formulates a linear program according to N_i , and replaces (1.40) by:

$$x_j \geq 0. \quad (1.42)$$

In a later paper, Toregas and ReVelle [1972] developed a reduction method to preprocess the data matrix, D . This reduction method greatly decreases the size of this

problem. This relaxation problem, (1.38), (1.39), and (1.42), can be solved by standard linear programming, but a zero-one solution is not guaranteed. Toregas et al. [1971] suggested that the cutting plane method be used if fractional solutions are obtained.

1.2.3.2. The Maximal Covering Problem

The maximal covering problem is a special case of the set covering problem. If the budget is too small to support a sufficient number of facilities to cover every user within a minimum service level, S , the maximal covering problem locates facilities to cover as many users as possible. The model is formulated as:

$$\max Z = \sum_{i=1}^n a_i y_i,$$

s.t.:

$$\sum_{j \in N_i} x_j \geq y_i, \quad \text{for all } i,$$

$$\sum_{j \in N_i} x_j = p, \quad \text{for all } i,$$

$$x_j, y_i = (0,1), \quad \text{for all } i,j,$$

where:

$$x_j = \begin{cases} 1 : \text{if } j \text{ is assigned to } i, \\ 0 : \text{otherwise,} \end{cases}$$

$$y_i = \begin{cases} 1 : \text{if } i \text{ is chosen as a facility location,} \\ 0 : \text{otherwise.} \end{cases}$$

For computational reasons, Church and ReVelle [1974] transformed the maximal covering model to minimize the uncovered population.

$$\min Z = \sum_{i=1}^m a_i \bar{y}_i, \quad (1.43)$$

s.t.:

$$\sum_{j \in N_i} x_j + \bar{y}_i \geq 1, \quad \text{for all } i, \quad (1.44)$$

$$\sum_{j \in N_i} x_j = p, \quad \text{for all } i, \quad (1.45)$$

$$x_j, \bar{y}_i = (0,1), \quad \text{for all } i,j, \quad (1.46)$$

where:

$$\bar{y}_i = 1 - y_i, \quad \text{for all } i.$$

Replace (1.46) by (1.47),

$$x_j, \bar{y}_i \geq 0, \quad \text{for all } i, j. \quad (1.47)$$

This problem, (1.43), (1.44), (1.45), and (1.47), is solved by linear programming, with branch and bound suggested if fractional solutions are obtained [Church and ReVelle, 1972].

1.2.3.3. The Maximal Covering with Mandatory Closeness Constraints

One weakness of maximal covering problem is that some users will be too far away from facilities. Mandatory closeness constraints can be introduced into the maximal covering problem to assure that no users are beyond a given service level, T . A new constraint is added to the formulation (1.43) - (1.47), which is:

$$\sum_{j \in M} x_j \geq 1,$$

where:

$$M = \{ j \mid d_{ij} \leq T \}.$$

The solution technique is the same as in the previous section [Church and ReVelle, 1972].

1.3. Issues in the Classic Location-Allocation Problems

This section discusses three issues arising in location-allocation problems: the difference between equity and efficiency; the relationship between covering problems and p-median problems; and, the allocation of users to facilities.

1.3.1. Equity and Efficiency

Private and public sector location models have been developed with several goals. Private sector location models usually stress cost efficiency, while public sector location models normally emphasize equity. Equity is measured by the degree of equality in the distribution of services among the population [McAllister, 1976].

The typical example of efficiency in private sector location is the p-median problem. (It is also well known in the literature as the uncapacitated facilities location problem with p facilities [Erlenkotter, 1977]). The p-median problem, with its objective of minimizing the sum of total travel distance or travel time, is normally used in the location of plants or warehouses which serve other plants, warehouses, markets, or consumers.

The typical example of equity in public sector location is the p-center problem, whose objective is to locate

facilities for which the worst case is minimized. This is a measure of equity. The p-center problem can be applied to locating emergency services such as fire, police, and ambulance stations.

It is worth noting that efficiency is not associated only with the private sector; it is also considered in public facility location modeling. The public sector must be efficient when budgets are very tight. Khumawala [1973] presented such a model, called the p-median problem with maximum distance constraints. It consists of the same initial constraints as the p-median problem, (1.14)-(1.17), with an additional constraint which sets a limit on the maximum distance any user can travel. That is, efficiency is maximized subject to a minimal level of equity. Church and ReVelle's [1975] covering problem with mandatory closeness constraints is another example of combining equity and efficiency in public sector location analysis.

1.3.2. The Structural Relationship between Median and Covering Problems

Church and ReVelle [1976] proved that the maximal covering problem is equivalent to the p-median problem by modifying the shortest distance matrix, D , to D' where:

$$d'_{ij} = \begin{cases} 0 & : \text{if } d_{ij} \leq S, \\ 1 & : \text{otherwise.} \end{cases}$$

The maximal covering problem with mandatory closeness constraint was shown to be equivalent to a p median problem by modifying matrix D to be D'' where:

$$d''_{ij} = \begin{cases} 0 & : \text{if } d_{ij} \leq S, \\ 1 & : \text{if } S \leq d_{ij} \leq T, \\ M & : \text{if } d_{ij} > T, \end{cases}$$

where:

M : a large number.

Although the relationship between set covering and p-median problems has not been considered in the literature, it is obvious that the set covering problem is related to the maximal covering problem. By using sensitivity analysis and parametric programming to adjust the right hand side of (1.45), the optimal solution, the smallest value of p which covers all the populations, can be found.

The benefit of considering the linkage between these models is that the solution techniques of the p-median problem are well developed and more efficient than covering

models. Furthermore, all covering problems are special cases of the p-median problem [Church and ReVelle, 1976]. Therefore, if these two models can be shown to be equivalent, the better developed and more efficient solution techniques of the p-median problem can be used to solve the covering problem.

1.3.3. The Allocation Rule

The all or nothing shortest trip assignment method is unrealistic when dealing with facilities dependent on public patronage. That is, people do not always travel to the closest facility. Spatial interaction modeling, on the other hand, assumes that there is a trade off between facility size and the distance users are willing to travel [Coelho and Wilson, 1976]. If consumers consider distance and facility size, their behavior can be represented by:

$$s_{ij} = O_i \frac{\sum_j^{\alpha} w_j \exp(-\beta c_{ij})}{\sum_j^{\alpha} w_j \exp(-\beta c_{ij})}, \quad (1.48)$$

where:

s_{ij} : number of trips from i to j ,

O_i : population at location i ,

- w_j : facility size at location j ,
- β : distance discount,
- c_{ij} : travel cost from i to j ,
- α : economies of scale parameter.

The spatial interaction model is a descriptive model derived from empirical data. It predicts the number of trips between origins and destinations based on the attractiveness of facilities and resistance of travel distance, or cost, to travel between them. The incorporation of the spatial interaction model into the location allocation model has become an important research topic [Bigman and ReVelle, 1978].

One of the first efforts to integrate the spatial interaction model into the location allocation model was made by Hodgson [1978]. He introduced spatial interaction into a p -median problem by substituting constraints. Instead of shortest distance trip assignment, spatial interaction constraints were used. Assuming that the facility sizes are all equal, this model finds the locations of p facilities such that the total travel time is minimized, subject to a spatial interaction model, s_{ij} .

That is:

$$\min Z = \sum_{i=1}^n \sum_{j \in G} s_{ij} c_{ij},$$

s.t.:

$$s_{ij} = A_i B_j O_{ij} D_{ij} f(c_{ij}), \quad (1.49)$$

$$A_i = \sum_{j=1}^n [B_j O_{ij} D_{ij} f(c_{ij})]^{-1},$$

$$B_j = \sum_{i=1}^n [A_i O_{ij} D_{ij} f(c_{ij})]^{-1},$$

where:

$$|G| = p,$$

$$D_{ij} = \begin{cases} D & \text{if } j \in G, \\ 0 & \text{otherwise,} \end{cases}$$

A_i, B_j : balance factors.

The solution technique is more complex because of the non-linear constraint (1.49). Hodgson [1978] adopted the heuristic node substitution method developed by Teitz and Bart to solve this problem. As mentioned before, heuristic algorithms offer the advantage of computational efficiency, but the global optimum is not guaranteed.

1.4. Measures of Social Welfare

The measurement of efficiency or equity in p-median problems or p-center problems is not suitable to evaluate the location of some public facilities. While it is understandable and reasonable to use total distance or total profit to measure the spatial efficiency of a private facility, development of a measure of efficiency in the public facility location setting is much more complex and difficult [Lea, 1979]. Some measures of efficiency have been adapted from economic welfare theory. These measures include users' benefits [Neuburger, 1971], consumer surplus [Coelho, 1977], and log accessibility [Leonardi, 1978]. The following sections review these major measures of social objectives as they have been developed in conjunction with the spatial interaction hypothesis. All three can be shown to be equivalent and are related to the maximum entropy approach popularized by Wilson [1967].

1.4.1. Users' Benefit, Consumer Surplus, and their Equivalence

The spatial interaction function can be used to develop a users' benefit equation for each individual in society. Consider the following expression:

$$w_j^{\alpha} \exp(-\beta c_{ij}),$$

where:

w_j : facility size,

α : an economy of scale parameter,

β : distance decay factor,

c_{ij} : travel time from location i to location j .

This can be transformed to:

$$\exp \beta \left(\frac{\alpha}{\beta} \ln w_j - c_{ij} \right).$$

The term,

$$\frac{\alpha}{\beta} \ln w_j - c_{ij},$$

measures the "size benefits" at facility j , "which are set against the disutility of travel cost" [Coelho and Wilson, 1976, p. 414]. Users try to maximize the difference between size benefits and travel cost. Therefore, the total users' benefits are the summation of the difference between size benefit and travel cost for all users:

$$U = \sum_i \sum_j s_{ij} \left(\frac{\alpha}{\beta} \ln w_j - c_{ij} \right). \quad (1.50)$$

The location-allocation problem is to maximize U when the number of trips, s_{ij} , are determined by the spatial interaction model (1.48).

Marshall's measure of consumer surplus is the area under the demand curve [Coelho et al., 1976]. In this location-allocation problem, demand is a function of facility size and transportation cost. The number of users generated by the system, the demand function, depends on the facility size w_j and transportation cost c_{ij} . Neuburger [1971] and Williams [1976] have shown that the consumers' surplus, CS , is given by:

$$CS = - \sum_i \sum_j \int s_{ij} (w_j) dc_{ij}, \quad (1.51)$$

where:

s_{ij} , c_{ij} , and w_j are defined as above.

Consumer surplus can be rewritten as:

$$\begin{aligned} CS &= - \sum_i \sum_j O_i \int \frac{w_j^\alpha \exp(-\beta c_{ij})}{\sum_i \sum_j w_j^\alpha \exp(-\beta c_{ij})} dc_{ij}, \\ &= - \frac{1}{\beta} \sum_i O_i \ln \left(\sum_k w_k^\alpha \exp(-\beta c_{ik}) \right), \end{aligned}$$

$$= \frac{1}{\beta} \sum_i \sum_j s_{ij} \ln \left(\frac{s_{ij}}{\alpha_{w_j}^0} \right) - \sum_i \sum_j s_{ij} c_{ij},$$

$$= \frac{1}{\beta} \sum_i \sum_j s_{ij} \ln \left(\frac{s_{ij}}{\alpha_{w_j}} \right) - \sum_i 0 \ln 0 - \sum_i \sum_j s_{ij} c_{ij}. \quad (1.52)$$

1.4.2. Log Accessibility and its' Equivalence with Consumer Surplus

A significant change in the goal of location-allocation modeling occurred when Leonardi [1979] introduced the log accessibility measurement into the objective function of location-allocation models. Leonardi is a pioneer in introducing spatial interaction into the location-allocation process. In his work, consumer allocation is not confined to the nearest facility, nor are facility sizes fixed, as in the p-median and covering problems. Leonardi's approach determines both the size and the location of facilities simultaneously and allocates users to facilities based on the spatial interaction model. The objective function maximizes log accessibility, subject to a budget constraint. Economies of scale in both facility attractiveness and construction cost are also introduced into his approach.

Consider the following measure of accessibility:

$$\phi_i = \sum_j w_j^\alpha \exp(-\beta c_{ij}) , \quad (1.55)$$

where:

α , β , w_j and c_{ij} are defined as above,

ϕ_i : total accessibility of all users at location i.

Equation (1.55) is an index which expresses the total accessibility of a user located at i to all facility locations j . The measure of total accessibility for all users and locations has the following form:

$$\Phi' = \prod_i \phi_i^{O_i}, \quad (1.56)$$

where:

O_i : population at location i .

The log accessibility maximization model (LAMM) maximizes the logarithm of (1.56) subject to a budget constraint. That is:

$$\max \ln \Phi' = \max \sum_i O_i \ln \left(\sum_j w_j^\alpha \exp(-\beta c_{ij}) \right), \quad (1.57)$$

s.t.:

$$\sum_j w_j^\tau \Gamma_j \leq B,$$

where:

Γ_j : construction cost at location j ,

τ : a construction cost economy of scale parameter,

B : total budget.

Leonardi [1978] proved that the optimal solution of this model allocates users via the spatial interaction model. Another benefit of this model is that it not only decides the locations of facilities, but also determines the size of the facilities.

Leonardi also showed the equivalence between log accessibility and consumer surplus [1978, p. 1291]. Take the logarithms of spatial interaction model on both sides,

$$\ln s_{ij} = \ln O_i + \alpha \ln w_j - \beta c_{ij} - \ln \Phi, \quad (1.58)$$

where:

Φ is defined as (1.57).

Substituting (1.58) into (1.57) yields:

$$CS = \frac{1}{\beta} \Phi - \frac{1}{\beta} \sum_i O_i \ln O_i.$$

This means that the consumer surplus can be obtained simply by multiplying log-accessibility by $1/\beta$ and subtracting a constant term.

1.4.3. The Spatial Interaction Model and the Fixed Facility Size Entropy Model

The spatial interaction model with equal facility sizes can be derived from the maximum entropy model with only production constraints [Wilson, 1970]. The maximum entropy model with production constraints can be written as:

$$\max Z = \ln \frac{S!}{\prod_{ij} s_{ij}!}, \quad (1.59)$$

s.t.:

$$\sum_j s_{ij} = O_i, \quad (1.60)$$

$$\sum_i \sum_j c_{ij} s_{ij} = C, \quad (1.61)$$

where:

S : total trips,

C : constant, C implies that the total amount spent on trips in the region at this point of time is a fixed number.

The objective of the maximum entropy model, (1.59)-(1.61), is to find the set of trips, s_{ij} , which has the highest probability of occurring, given the total costs C and source populations O_i .

Using Lagrangian multipliers, Wilson [1969] presented the problem as:

$$\begin{aligned} \max L(s_{ij}, \tau_i, \beta) \\ = \ln\left(\frac{S!}{\prod_{ij} s_{ij}}\right) + \sum_i \tau_i (O_i - \sum_j s_{ij}) + \beta (C - \sum_{ij} c_{ij} s_{ij}). \end{aligned}$$

This requires that:

$$s_{ij} = \frac{O_i \exp(-\beta c_{ij})}{\sum_j \exp(-\beta c_{ij})}. \quad (1.62)$$

It is worth noting that the entropy model with production constraint does not consider facility attractiveness, i.e.; there is no w_j in (1.62). In other words, the attractiveness of all facilities is assumed equal. That is:

$$w_j = W', \quad \text{for all } j.$$

Since w_j is a constant for all j , it is possible to introduce any constant α to this constraint. Therefore,

$$w_j = \alpha W, \quad \text{for all } j. \quad (1.63)$$

Adding this constraint (1.63) to (1.62) yields:

$$s_{ij} = \frac{\alpha w_j \exp(-\beta c_{ij})}{\sum_j \alpha w_j \exp(-\beta c_{ij})}$$

Where α can be interpreted as the parameter which measures economies of scale in attractiveness.

This is the same equation as the spatial interaction model, (1.48), if we assume all the w_j 's are fixed. In the next chapter, it is shown that the spatial interaction model with varying facility sizes can be derived from an augmented entropy maximizing model.

1.4.4. The Equivalence of the Fixed Facility Size Maximum Entropy Model and the Users' Benefit Model

The maximum entropy model (1.59)-(1.61) can be modified by integrating the cost constraint (1.61) into objective function, as follows:

$$\max L = \frac{1}{\beta} \left(\sum_i \sum_j s_{ij} \ln s_{ij} - \sum_i \sum_j c_{ij} s_{ij} \right), \quad (1.64)$$

s.t.:

$$\sum_j s_{ij} = O_i$$

The maximum users' benefit model (1.54) can be presented as:

$$\max U = \frac{1}{\beta} \sum_i \sum_j s_{ij} \ln \frac{s_{ij}}{\alpha_j} - \sum_i \sum_j c_{ij} s_{ij}, \quad (1.65)$$

s.t.:

$$\sum_i \sum_j s_{ij} = O_i.$$

If it is assumed that $w_j = W$ for all j , equation (1.64) is equivalent to (1.65). Therefore, the users' benefits maximizing model is equivalent to the entropy maximizing model if all facility sizes are the same.

1.5. Concluding Remarks

This chapter has reviewed the formulation of various location-allocation models, different concepts between private and public facility location modeling, and the measurement of objective functions in the location-allocation planning process. It also explored the properties of the various equivalent social welfare functions. The equivalence between maximizing users' benefits and maximizing entropy when facility sizes are equal and constant was also shown.

The same equivalence relationship will be shown to hold when facility sizes vary. This is done in chapter 2. In chapter 3, the log accessibility model is applied to

capacitated networks. The theoretical issues of the existence and uniqueness of a system equilibrium solution of the log accessibility model are explored. A mathematical program for finding the solution and a numerical example are developed in chapter 4. The fifth chapter uses these results in a simple tree network to explore the pattern of flows and the changes in facility sizes brought about by the log accessibility model applied to a capacitated network. The final chapter discusses some extended research topics which are an outgrowth of this research.

CHAPTER 2

THE ENTROPY MAXIMIZATION MODEL WITH BENEFIT
AND ADMINISTRATION CONSTRAINTS2.1. Introduction

This chapter introduces a modified version of Wilson's doubly constrained entropy maximizing model, and proves the equivalence between this form of entropy and the three social welfare maximization models discussed in chapter 1. That is, the Modified Entropy Maximizing Model (MEMM) is equivalent to location-allocation models with accessibility or social welfare objectives. The MEMM is also used to construct a more logical, straightforward derivation of a spatial interaction model that considers economies of scale in attraction. That model is then related to theories of spatial interaction which consider both facility attractiveness and capacity. These models of spatial interaction, which have not been considered in the location-allocation context, can be related to theories of social welfare via the MEMM. Finally, there is a discussion of which objective function -modified entropy, users' benefit, consumer surplus, or log accessibility - to use in the case of location-allocation modeling on congested networks.

2.2. The Modified Entropy Maximizing Model

This section presents the mathematical formulation of the MEMM, the derivation of the MEMM, the objective function and constraints, and an economic interpretation of the MEMM.

2.2.1. Model Presentation

The formulation for the Modified Doubly Constrained Entropy Maximization Model (MDCMEMM) is as follows:

P1 :

$$\max_{\{s,w\}} - \sum_i \sum_j s_{ij} \ln s_{ij}, \quad (2.1)$$

s.t.:

$$\sum_j s_{ij} = O_i, \quad \text{for all } i, \quad (2.2)$$

$$\sum_i s_{ij} = D_j, \quad \text{for all } j, \quad (2.3)$$

$$\sum_i \sum_j c_{ij} s_{ij} = C, \quad (2.4)$$

$$\sum_i \sum_j s_{ij} \ln w_j = W, \quad (2.5)$$

$$g_k(w) \leq B_k, \quad \text{for any } k. \quad (2.6)$$

where:

s_{ij} : number of trips from i to j ,
 O_i : number of trips generated at i ,

D_j : number of trips arriving at j ,
 c_{ij} : transportation cost from i to j ,
 w_j : attractiveness at location j ,
 C_j : constant representing social cost,
 W : constant representing social benefit,
 $g_k(w)$: any possible administration constraint k ,
 such as a budget constraint.

The problem defined as:

P2 : (2.1) - (2.4),

is the classic doubly constrained entropy maximization model of Wilson [1970]. The problem,

P3 : (2.1) (2.2) (2.4),

is the standard production constrained entropy maximization model. It is reasonable, therefore, to call (P1) a Modified Doubly Constrained Entropy Maximization Model (MDCEMM), and:

P4 : (2.1), (2.2), (2.4), (2.5), (2.6),

a Modified Production Constrained Entropy Maximization Model (MPCEMM).

There are two major differences between problems P1 and P4 and Wilson's models, P2 and P3: the benefit and administration constraints (2.5) and (2.6); and, the fact that facility size is a decision variable. This later point transforms the entropy maximization problem to a location-allocation problem.

2.2.2. The Benefit and Administration Constraints

Wilson considers w_j to be a measure of the log scale of the facility size [1970, p. 48]. Wilson's term "benefit" should not be confused with users' benefit; attractiveness or quality of a facility, are perhaps better terms. Attractiveness is used to represent Wilson's benefit in this chapter.

Constraint (2.5) implies that the total log attractiveness gained by society is a constant. If (2.5) is divided by the total number of users, T , then:

$$\frac{\sum_i \sum_j s_{ij} \ln w_j}{T} = \frac{W}{T}, \quad (2.7)$$

which means that the average perception of log attractiveness by each user is constant within a region.

Constraints (2.4) and (2.5) are very similar. It has been argued that the total cost C in (2.4) is an index of transportation service [Wilson, 1970]. A higher cost C indicates an expensive and inconvenient transportation network, while a lower cost C implies a superior transportation system. Wilson [1967] showed there is a unique dual variable β associated with the cost constraint. The value of β is an index of travel

behavior. The higher the β value, the more unwilling people are to travel. The lower the β value, the more willing people are to travel.

Constraint (2.5) acts in a similar fashion. W is the measure of attractiveness. A high value of W implies that the total attractiveness generated by facilities is great. A small value of W denotes a low total attractiveness. Of course, there is also a dual variable, α , associated with this constraint. A high value of α ($\alpha > 1$) means there are increasing attractiveness returns, while a low α value indicates decreasing returns to attractiveness. If, for example, the attractiveness of a facility is a function of its size, then $\alpha > 1$ would indicate that doubling a facility's size would more than double its attractiveness.

Constraints (2.4) and (2.5) represent the spatial differentiation among regions. That is, the same size and quality facility, w_j , with the same distance or travel cost, c_{ij} , will generate different travel patterns, s_{ij} , depending on the total or average cost, C , and attractiveness, W , for any specific region.

In most previous research, such as Leonardi [1978], Harris [1978], and Wilson [1970], it is assumed that the quality of a facility, or the benefits derived by its users, can be measured in terms of its quantity or size.

The larger the facility, the more attractive it is. This is a reasonable assumption in most cases, but not always. A highly attractive facility does not always have enough space to accommodate all the people who want to use it. This creates a situation of spatial competition for use of the facility. The doubly constrained entropy model does not take this overcrowding condition into consideration. The addition of the benefit constraint, (2.5), to the doubly constrained entropy model enables it to compensate for this overcrowding situation.

The MDCEMM considers the attractiveness and capacity of a facility as separate entities (w_j and D_j). They are not necessarily equivalent nor proportional. It is possible to have a large facility which does not generate a high degree of attractiveness. It is, by the same token, also possible to have a small facility which is very attractive. Therefore, the benefit which is accrued may be a function of the quality of the service or commodity which is offered at the facility, rather than the size of the facility.

The administration constraint (2.6) is related to any policy decisions. The most common example is the budget constraint. Church's [1972] mandatory closeness constraint and Khumawala's [1973] maximum distance constraint discussed in Chapter 1 are other examples.

2.3. The MPCEMM and the Spatial Interaction Model

The name of constraint (2.5), the benefit constraint, was adopted from Wilson [1970], who tried to include attractiveness in his classic production entropy maximization problem. His argument is as follows:

[the spatial interaction model] can be derived using entropy maximizing principles and is indeed a version of the production-constrained model ... with appropriate changes in notation. [p. 65]

.....
We can build in such a hypothesis as follows: let v_j be the benefit in j arising from the use of a unit of [facility] compared with the use of that unit in any other region. Choose units so this can be offset directly against the transport cost c_{ij} .

Thus equation:

$$s_{ij} = A_i O_i \exp(-\beta c_{ij}), \quad (2.8)$$

can be rewritten as

$$s_{ij} = A_i O_i \exp(\mu v_j) \exp(-\beta c_{ij}), \quad (2.9)$$

$$\text{and } A_i = \left[\sum_j \exp(\mu v_j) \exp(-\beta c_{ij}) \right]^{-1}, \quad (2.10)$$

Now, how would we actually estimate v_j ? One possible argument is that such benefits as v_j are usually generated by scale economies in region j , and so a proxy variable would be the current scale of usage of [a facility] in j . The usual kind of assumption in such circumstances is that the

benefits are proportional to the log of the scale, so:

$$\mu v_j = \alpha \ln w_j, \quad (2.11)$$

where α is an appropriate parameter. Equation (2.9) can be transferred to [the spatial interaction model]. [Wilson, 1970, p. 48]

Equation (2.8) represents the spatial interaction model with fixed facility size. Wilson then introduces a benefit variable, v_j , into (2.8), which becomes (2.9).

Next, Wilson defines benefit (or attractiveness) to be proportional to the log of facility size (2.11). The log of facility size is substituted into (2.9) for the benefit variable v_j . The arbitrary introduction of this benefit variable, v_j , and the restrictive constraint, (2.11),

enable him to transform the fixed facility spatial interaction model into one which considers different facility sizes.

The benefit variable and the assumption that benefits and size are proportional can be eliminated from the derivation of the spatial interaction model with the introduction of constraints (2.5) and (2.6). This provides a more general case for the model because attractiveness and size are independent variables. There is also no need to introduce arbitrary variables such as v_j into the

spatial interaction model (2.9) with this revised formulation.

To make these points formally, we rewrite P3 as a Lagrangian:

$$\begin{aligned} \max L = & - \sum_i \sum_j s_{ij} \ln s_{ij} + \alpha (W - \sum_i \sum_j \ln w_j s_{ij}) \\ & + \beta (C - \sum_i \sum_j s_{ij} c_{ij}) + \sum_i A_i (\sum_j s_{ij} - O_i) \\ & + \sum_k \tau_k (B_k - g_k(w)). \end{aligned}$$

Using the maximum principle, the necessary condition with respect to w_j is:

$$D_{w_j} L = 0, \quad \text{for all } j.$$

Thus,

$$0 = \alpha \frac{\sum_j s_{ij}}{w_j} - \sum_k \tau_k D_{w_j} g_k(w). \quad (2.12)$$

Assume that there is only one administration constraint in (2.6), a budget constraint, which is:

$$\sum_j b_j w_j \leq B,$$

where:

b_j : unit cost of the quality of facility,

B : total budget.

Then (2.12) becomes:

$$\frac{\sum_j s_{ij}}{w_j} = \frac{\tau}{\alpha} b_j, \quad \text{for all } j.$$

If the cost to increase the facility quality is the same at each node, that is, b_j is constant for every j , then:

$$\frac{\sum_j s_{ij}}{w_j} = \frac{D_j}{w_j} = \text{constant}.$$

This is the assumption made by most previous researchers; attractiveness is related with facility size [Wilson, 1970; Leonardi, 1978; and Harris, 1978].

The necessary condition with respect to s_{ij} is that:

$$D_{s_{ij}} L = 0.$$

This means that:

$$0 = -\ln s_{ij} + \alpha \ln w_j - \beta c_{ij} + A_i.$$

Therefore,

$$s_{ij} = \exp(A_i) w_j^\alpha \exp(-\beta c_{ij}).$$

Summing over j on both sides yields:

$$s_{ij} = O_i \frac{w_j^\alpha \exp(-\beta c_{ij})}{\sum_j w_j^\alpha \exp(-\beta c_{ij})}.$$

This is the same result as Wilson's research [1970] discussed above without the notation substitution.

2.4. The Relationship among Social Welfare Maximization Models and the MPCEMM

In the previous chapter, it was shown that when the location-allocation problem is solved, users' benefit, consumer surplus, and accessibility are all optimized because each of these maximization models is equivalent. Coelho et al. [1976] showed that the users' benefit maximization model is equivalent to the consumer surplus maximization model. Leonardi also proved that the log accessibility model is equivalent to the consumer surplus maximization model [1978].

The relationship between entropy and accessibility was discussed by Erlander [1977]. Erlander showed how the objectives of accessibility and efficiency can be

formulated in a simple model. Instead of maximizing entropy, Erlander suggested a model to minimize the total social cost, subject to a minimal level entropy constraint:

$$\min C = \frac{1}{T} \sum_i \sum_j c_{ij} s_{ij},$$

s.t.:

$$\sum_j s_{ij} = O_i,$$

$$\sum_i s_{ij} = D_j,$$

$$-\sum_i \sum_j \frac{s_{ij}}{T} \ln \frac{s_{ij}}{T} \geq H_0,$$

where:

H_0 : the minimal level of entropy.

Erlander argued that average transportation cost is a function of entropy. Average transportation cost is used to measure efficiency. Therefore, efficiency is a function of entropy. Efficiency is directly related to accessibility. For instance, a very efficient network would create high accessibility for the users. Thus, accessibility is a function of entropy.

Erlander's research only discussed the relationship between accessibility and entropy in broad terms. However,

it can be proven directly that the MPCEMM is equivalent to users' benefit and log accessibility maximizing models.

To prove this equivalence, the following two lemmas are needed.

Lemma 1 : (Coelho and Wilson [1976, p. 163])

The following problems are equivalent.

P6:

$$\max_{\{x\}} f(x),$$

s.t.:

$$g_i(x) = b_i, \quad i = 1, \dots, n.$$

P7:

$$\max_{\{x, \beta\}} f(x) + \sum_{i=1}^p \beta_i (b_i - g_i(x)),$$

s.t.:

$$g_i(x) = b_i, \quad i = p+1, \dots, n.$$

Lemma 2 : (Coelho and Wilson [1978, p. 164])

If $f(x)$ and $g_k(x)$, $k = 1, \dots, l$ are convex functions,

then the following problems are equivalent and have a unique solution whenever the feasible solution set is not empty.

P8:

$$\max_{\{x,s\}} Z = f(x) + \sum_i \sum_j s_{ij} c_{ij},$$

s.t.:

$$s_{ij} = A_i B_j O_i D_j \exp(-\beta c_{ij}),$$

$$A_i = \left[\sum_j B_j D_j w_j \exp(-\beta c_{ij}) \right]^{-1},$$

$$B_j = \left[\sum_i A_i O_i w_i \exp(-\beta c_{ij}) \right]^{-1},$$

$$x \geq 0.$$

P9:

$$\max_{\{x,s\}} Z' = f(x) + \sum_i \sum_j s_{ij} c_{ij} - \frac{1}{\beta} \sum_i \sum_j s_{ij} \ln s_{ij},$$

s.t.:

$$g_k(x) \leq b_k,$$

$$\sum_j s_{ij} = O_i,$$

$$\sum_i s_{ij} = D_j,$$

$$x \geq 0.$$

Equivalence can now be proven.

Theorem 2.1: The Users' Benefit Maximizing Model (UBMM, P10) and MPCEMM (P11) are equivalent.

P10: (UBMM, [Coelho and Wilson, 1977, p. 414])

$$\max_{\{s,w\}} Z = \sum_i \sum_j s_{ij} \left(\frac{\alpha}{\beta} \ln w_j - c_{ij} \right),$$

s.t.:

$$\sum_j s_{ij} = A_i \quad \sum_j w_j^\alpha \exp(-\beta c_{ij}) = 1,$$

$$A_i = \left[\sum_j w_j^\alpha \exp(-\beta c_{ij}) \right]^{-1},$$

$$g_k(w) \leq B_k.$$

P11 : (Modified Production Constrained Entropy Maximizing Model).

$$\max_{\{s,w\}} Z' = - \sum_i \sum_j s_{ij} \ln s_{ij},$$

s.t.:

$$\sum_i \sum_j s_{ij} = O_i,$$

$$\sum_i \sum_j s_{ij} \ln w_j = W,$$

$$\sum_i \sum_j s_{ij} c_{ij} = C,$$

$$g_k(w) \leq B_k.$$

Proof :

From Lemma 1, P11 is equivalent to:

$$\begin{aligned} \max Z' = & - \sum_i \sum_j s_{ij} \ln s_{ij} + \alpha \left(\sum_i \sum_j \ln w_j s_{ij} - W \right) \\ & + \beta \left(C - \sum_i \sum_j s_{ij} c_{ij} \right), \end{aligned} \quad (2.13)$$

s.t.:

$$\sum_j s_{ij} = 0, \quad i$$

$$g_k(w) \leq B_k.$$

Rearrange the terms of (2.13) to yield:

$$\begin{aligned} \max Z' = & - \sum_i \sum_j s_{ij} \ln s_{ij} \\ & + \sum_i \sum_j s_{ij} (\alpha \ln w_j - \beta c_{ij}) - \alpha W + \beta C. \end{aligned} \quad (2.14)$$

From Lemma 2, P10 is equivalent to:

$$\max Z = \sum_{ij} s_{ij} \left(\frac{\alpha}{\beta} \ln w_j - c_{ij} \right) - \frac{1}{\beta} \sum_{ij} s_{ij} \ln s_{ij}, \quad (2.15)$$

s.t.:

$$\sum_j s_{ij} = 0.$$

Since β , α , W , and C are all constants, equation (2.14) and (2.15) are different only by a constant, this completes the proof that MPCEMM is equivalent to UBMM.

Corollary 2.2 The Log Accessibility Maximization Model (LAMM, P12) and the MPCEMM, (P11), with one administration constraint, a budget constraint, are equivalent.

P12: (Log Accessibility Maximization Model)

$$\max \sum_i O_i \ln \left(\sum_j w_j^\alpha \exp(-\beta c_{ij}) \right),$$

s.t.:

$$\sum_j \Gamma_j w_j^\tau \leq B, \quad (2.16)$$

where:

Γ_j : the construction cost at location j .

Previous researchers have proven that the Users' Benefit Maximizing Model is equivalent to the Consumer Surplus Maximizing Model and the Log Accessibility Maximizing Model [Coelho et al., 1976; Leonardi, 1978]. If there is only one administration constraint in the MPCEMM, and, it is defined as a budget constraint, (2.16), then MPCEMM is equivalent to Log Accessibility Maximization Model.

2.5. Facilities with Finite Capacity

The difference between MDCEMM, P1, and MPCEMM, P4, is that P4 assumes there is no competition among users for the services at facility sites. In other words, the facility is large enough to absorb all users. Therefore, $D_j = \infty$ for all j and constraint (2.3) was dropped from P1. If constraint (2.3), the destination constraint, is introduced into the modified model, facility j is no longer large enough to hold as many users as possible. The result will

be competition among users for services offered at location j . This means that:

$$\sum_i s_{ij} = D_j, \quad \text{where } D \text{ is a constant and } D \ll T.$$

Fotheringham [1985] suggested the following spatial competition model for this situation:

$$s_{ij} = \frac{O_i A_j^\tau w_j^\alpha \exp(-\beta c_{ij})}{\sum_j w_j^\alpha A_j^\tau \exp(-\beta c_{ij})}, \quad (2.17)$$

where:

A_j : competition variable at location j ,

τ : non-linear parameter of A_j .

In fact, this model can be derived directly from the MDCEMM. The necessary condition with respect to w_j is the same as (2.12). The necessary condition with respect to s_{ij} is:

$$0 = -\ln s_{ij} + \alpha \ln w_j - \beta c_{ij} + A_i + B_j,$$

therefore,

$$s_{ij} = \exp(A_i) \exp(B_j) w_j^\alpha \exp(-\beta c_{ij}). \quad (2.18)$$

Summing over j in (2.17) and simplifying, yields,

$$\exp(A_i) = \frac{O_i}{\sum_j w_j^\alpha \exp(-\beta c_{ij}) \exp(B_j)}. \quad (2.19)$$

Let:

$$A'_j = \exp\left(\frac{B_j}{\tau}\right),$$

then:

$$\exp(B_j) = A'_j^\tau \quad (2.20)$$

Substituting (2.19) and (2.20) into (2.18) yields:

$$s_{ij} = O_i \frac{w_j^\alpha \exp(-\beta c_{ij}) A'_j^\tau}{\sum_k w_k^\alpha \exp(-\beta c_{ik}) A'_k^\tau},$$

which is the same as the model suggested by Fotheringham (2.17).

2.6. The Solution of the MDCEMM

It is also possible to verify that the theoretical solution of the MDCEMM is the traditional spatial

interaction model with the addition of the attractiveness or benefit term.

The optimal solution of the MDCEMM is:

$$\alpha \frac{\sum_j s_{ij}}{w_j} - \sum_k \tau_k D_k g(w_k) = 0, \quad (2.21)$$

and

$$s_{ij} = A_i B_j w_j^\alpha \exp(-\beta c_{ij}), \quad (2.22)$$

$$A_i = [\sum_j B_j w_j^\alpha \exp(-\beta c_{ij})]^{-1},$$

$$B_j = [\sum_i A_i w_i^\alpha \exp(-\beta c_{ij})]^{-1}.$$

where A_i and B_j are balance factors to ensure:

$$\sum_j s_{ij} = 0,$$

$$\sum_i s_{ij} = D_j.$$

2.7. Concluding Remarks

This chapter considers the economic meaning of the term benefit. In the past, benefit has been solely a function of quantity or facility size. The MEMM can incorporate the quality of facilities to determine benefit or attractiveness. The structure of the Modified Doubly

Constrained Model, the connections between this modified model and other existing models, the equivalence of the entropy maximizing model and log accessibility model, and the theoretical solution of the modified model have also been discussed.

A significant result in this chapter is that the classic entropy maximization model can be transformed into a location-allocation model by the introduction of the benefit and administration constraints. The results, therefore, show that MEMM, UBMM, CSMM, and LAMM are all equivalent. These four models can be used to decide the sizes and locations of facilities. Since there is no difference between these four models, for ease in algorithm construction the next three chapters will utilize the Log Accessibility Maximization Model as an example and apply it to the location-allocation problem on capacitated networks.

CHAPTER 3

THE MAXIMUM ACCESSIBILITY MODEL ON BIPARTITE NETWORKS

Spatial interaction between two places, i and j , is determined by the attractiveness at location j and the resistance generated by travel cost between i and j . Most location-allocation models assume that the travel cost between two places is a constant. However, the travel cost between two places not only depends on distance, but also on the number of travelers on each link and the capacities of those links. The assumption of no congestion in a network may be reasonable in some instances, but in an urban setting, congestion is a crucial factor which impedes interaction. It should, therefore, be explicitly considered.

Congestion has become a very important factor in the transportation planning process [LeBlanc, 1975; Eash, 1979; Manheim, 1979; Boyce, 1981], but it is still not a significant factor in location-allocation modeling. However, Leonardi [1981] suggested a means by which link congestion could be incorporated into the location-allocation planning process.

Congestion found on the road links used to reach the destination also determines the choice [of facilities]. However, a model embedding both optimal location and allocation and optimal traffic

assignment ... will be the subject for future research. Such models could be developed, building upon the work of Evans [1976] and Boyce and LeBlanc [1979]. [Leonardi, 1981, p. 1014].

The next two chapters implement Leonardi's suggestion by developing models which integrate congestion into location-allocation modeling.

The method used to solve the location-allocation problem depends on the objectives of the problem studied. P-center, p-median, and social welfare maximizing models all have different solutions because they have distinct goals. The maximal social welfare location-allocation problem on congested networks is studied for two reasons. First, the social welfare maximization model selects the various facility sizes as well as their locations, while p-center and p-median problems are limited to facilities of the same fixed size. Secondly, the p-median and p-center problems usually assume that people go to the nearest facility, while the social welfare model distributes the users to various facilities based on a criterion of travel cost and facility attractiveness. Clearly, any of the measures of social welfare discussed in the previous chapters could be used. The choice of log accessibility merely reflects the author's algorithmic interests.

In this and the next chapter, a bipartite network is assumed. A non-bipartite network is considered in Chapter 5. A bipartite network has the property that only one link connects any pair of nodes. The capacitated bipartite network model adds a set of new constraints to the uncapacitated network model. These constraints insure that travel cost due to congestion is incorporated in the facility location decision making process. The necessary first order conditions for an optimal solution are derived directly, and a steepest ascent algorithm to find the optimal location of the facilities is developed.

Before studying the effects of link congestion, we review the major theoretical results developed by Leonardi [1978] for an uncapacitated network. Numerical analysis reveals the behavior of the model for various changes in the parameters. Theoretical and numerical results for a capacitated bipartite network are then presented.

3.1. The Log Accessibility Model on an Uncapacitated Bipartite Network

As a basis for comparison, consider Leonardi's [1978] model which maximizes the total log accessibility, subject to a budget constraint:

$$\max_{\{w\}} \Phi = \sum_i O_i \ln \left(\sum_j w_j^\alpha \exp(-\beta c_{ij}) \right), \quad (3.1)$$

s.t.:

$$\sum_j w_j^\tau \Gamma_j \leq B, \quad (3.2)$$

where:

O_i : population at i ,

w_j : facility size,

c_{ij} : travel cost,

Γ_j : construction cost at location j ,

τ : a construction cost economies of scale parameter,

α : a parameter which measures the attractiveness economies of scale,

β : spatial discount factor,

B : total budget.

The variable α measures the relationship between facility size and attraction. For example, if the capacity of a facility is doubled, an α value of 1 will mean that it's attractiveness is doubled. If $\alpha > 1$, the increase in attractiveness will more than double. If $\alpha < 1$, the increase in attractiveness will be less than double.

The β variable measures peoples' propensity to travel. In retail studies, β can be considered a surrogate for range of a good. A small value of β would indicate a large range of a good, while a large β would indicate a small range of a good.

The variable τ measures the construction cost for various facility sizes. The larger the τ , the more expensive it is to increase the size of the facility. A low τ indicates that expansion costs will be less expensive.

Although spatial interaction behavior is not explicitly incorporated in the log accessibility model, the optimal solution of this model will automatically result in spatial interaction allocations [Leonardi, 1978]. To illustrate this point consider the partial derivatives of (3.1). Multiplying both sides by w_j yields:

$$w_j \frac{D \Phi}{w_j} = \alpha \sum_i O_i \frac{w_j^\alpha \exp(-\beta c_{ij})}{\sum_j w_j^\alpha \exp(-\beta c_{ij})}. \quad (3.3)$$

By rearranging terms, we have:

$$\frac{D \Phi}{w_j} = \alpha \frac{\sum_i s_{ij}}{w_j}. \quad (3.4)$$

The right hand side of (3.3) is the attractiveness economies of scale parameter multiplied by the spatial interaction model. This states that the net change in accessibility for a marginal increase in facility size at j equals the amount of interaction multiplied by the economies of scale parameter.

Constraint (3.2) states that the total construction cost cannot exceed the budget. This also indicates that feasible solutions can be obtained without spending the entire budget. However, Leonardi proved that the optimal allocation and size of facilities is only obtained by completely spending the budget. Constraint (3.2), therefore, can be changed to:

$$\sum_j^{\alpha} w_j \Gamma_j = B.$$

Leonardi's model can be transformed to a simplified form by variable substitution.

Let:

$$x_j = \frac{w_j \Gamma_j}{B},$$

$$f_{ij} = \frac{\exp(-\beta c_{ij})}{\Theta \Gamma_j},$$

$$p_i = \frac{O_i}{T},$$

$$\Theta = \frac{\alpha}{\tau},$$

where:

T : total population,

x_j : proportion of budget spend at j ,

p_i : proportion of population living at i .

Therefore, the final form is:

$$\max \sum_i p_i \ln \left(\sum_j x_j f_{ij} \right), \quad (3.5)$$

s.t.:

$$\sum_j x_j \leq 1, \quad \text{for all } j, \quad (3.6)$$

$$x_j \geq 0, \quad \text{for all } j. \quad (3.7)$$

The maximal principle is used to solve the problem (3.5)-(3.7). Let μ and v_j be the Lagrangian multipliers of

(3.6) and (3.7) then:

$$L(x_j, \mu, v_j) = \sum_i p_i \ln \left(\sum_j x_j^\Theta f_{ij} \right) + \mu (1 - \sum_j x_j) + \sum_j x_j v_j. \quad (3.8)$$

The Kuhn-Tucker conditions for (3.8) are:

$$x_j = \Omega_j x_j, \quad (3.9)$$

where:

$$\Omega_j = \sum_i p_i \frac{x_j^{\Theta-1} f_{ij}}{\sum_k x_k^\Theta f_{ik}}.$$

The solution technique, a steepest ascent algorithm, finds x_j which satisfies the balance equation (3.9). The steepest ascent method cannot always find the optimal solution. It is dependent on the value of Θ . When $\Theta \leq 1$ the problem is concave, so a steepest ascent algorithm should reach the unique maximum. It should be noted, however, that when $\Theta = 1$, the objective function is very flat and a steepest ascent method may take a long time to converge. In such cases, one may wish to use a dual formulation to obtain a solution (O'Kelly and Forster, 1983). When $\Theta > 1$ the problem is not concave, and one must be thorough in searching the parameter space because there may be a number of local optima.

Consider the simplest case of deciding between two locations. The value of the objective function can be graphed against the proportion of budget spent at location 1 [Ralston and Liu, 1984]. Figure 1 illustrates the concavity when $\theta < 1$. Figure 2, while still concave, indicates the flatness of the objective function when $\theta = 1$. Figure 3 illustrates the undulating nature of the objective function when $\theta > 1$.

3.2. Parameter Sensitivity of LAMM on an Uncapacitated Bipartite Network

To study congestion's effects on the log accessibility model, a bipartite network is used. Figure 4 gives the general form of a bipartite network. Note that each node has a direct and independent connection to all other nodes. Table 1 presents the distance between the nodes and the population at each node. Node 3 has the largest population while node 5 has the smallest. In Figure 5, the nodes' relative positions based on the distances in Table 1 are shown. To maintain the bipartite network property, it must be assumed that the flow between each pair of nodes is not affected by any other flows. For example, even if capacities are finite, the flow from node 1 to node 7 is not affected by the flow from node 10 to node 7. (This assumption is relaxed in the following Chapter.)

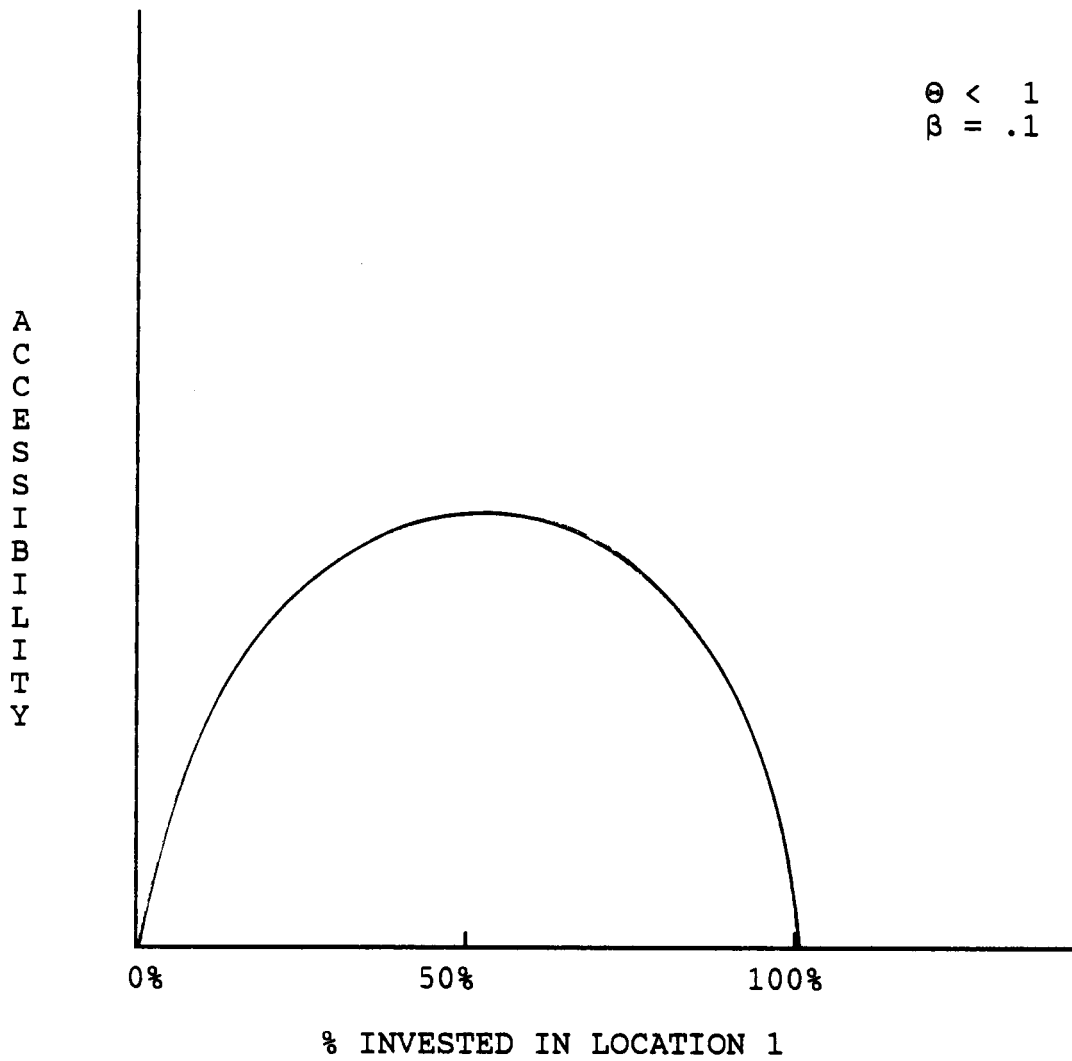


Figure 1 : Log Accessibility Maximization Model on an
Uncapacitated Network with Two Locations
when $\theta < 1$.

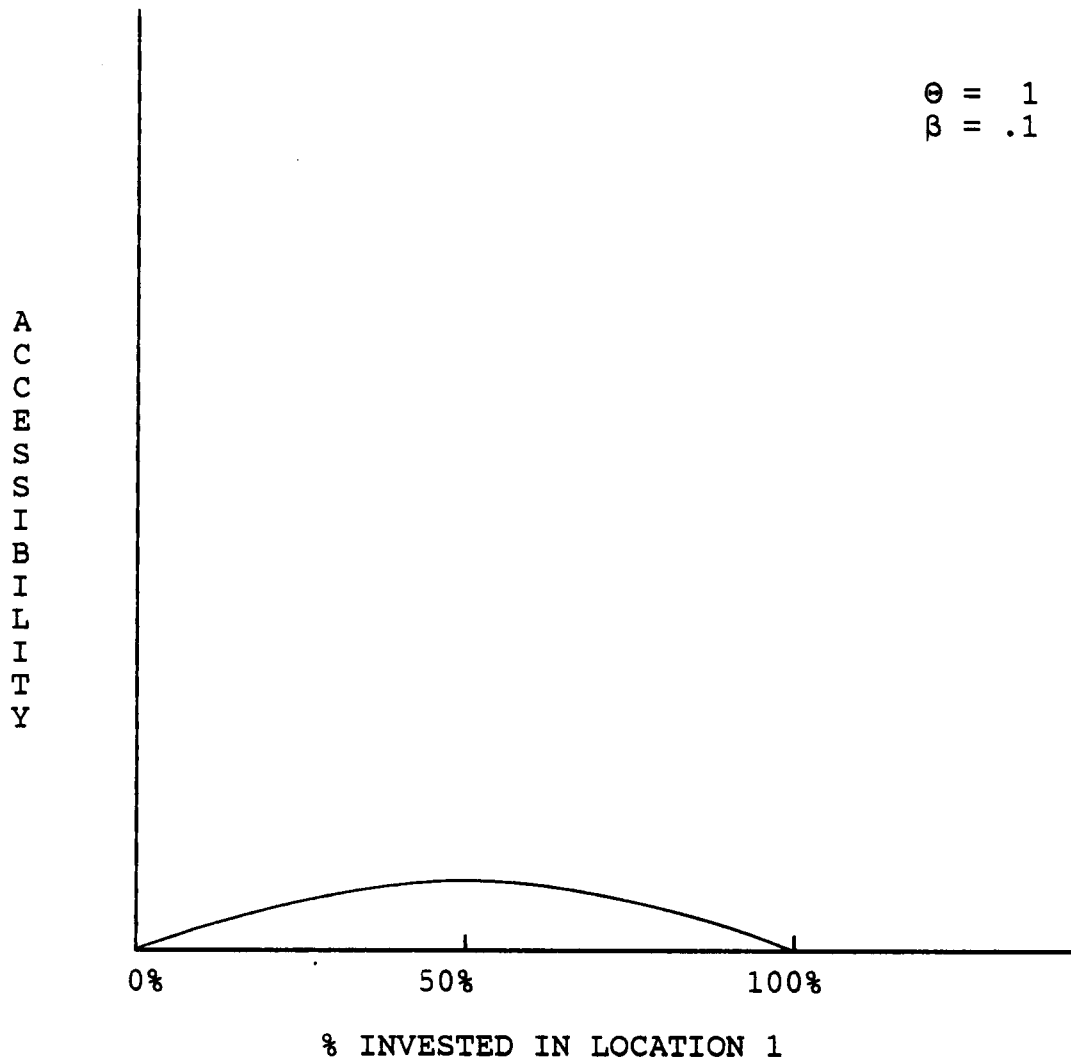


Figure 2 : Log Accessibility Maximization Model on an
Uncapacitated Network with Two Locations
when $\Theta = 1$.

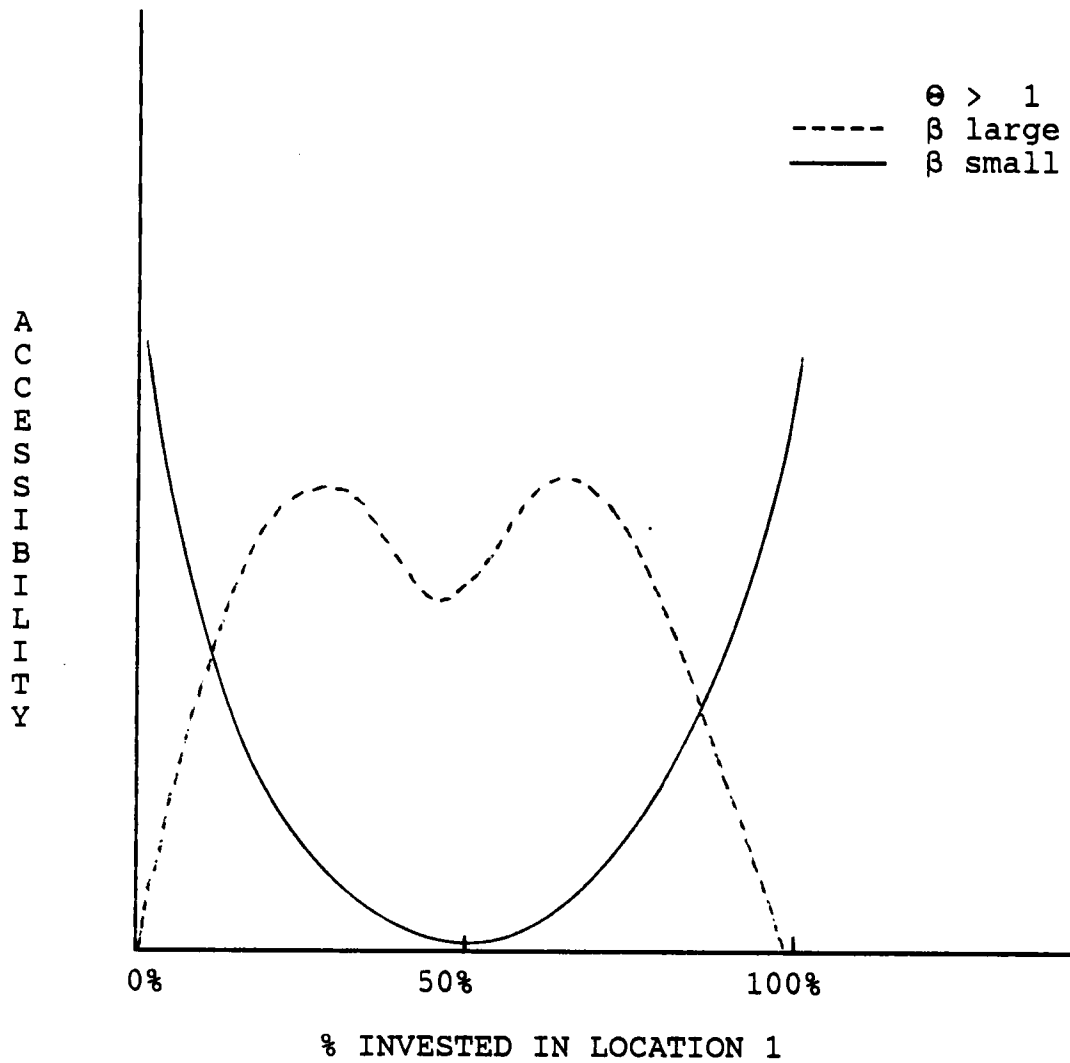


Figure 3 : Log Accessibility Maximization Model on an Uncapacitated Network with Two Locations when $\theta > 1$.

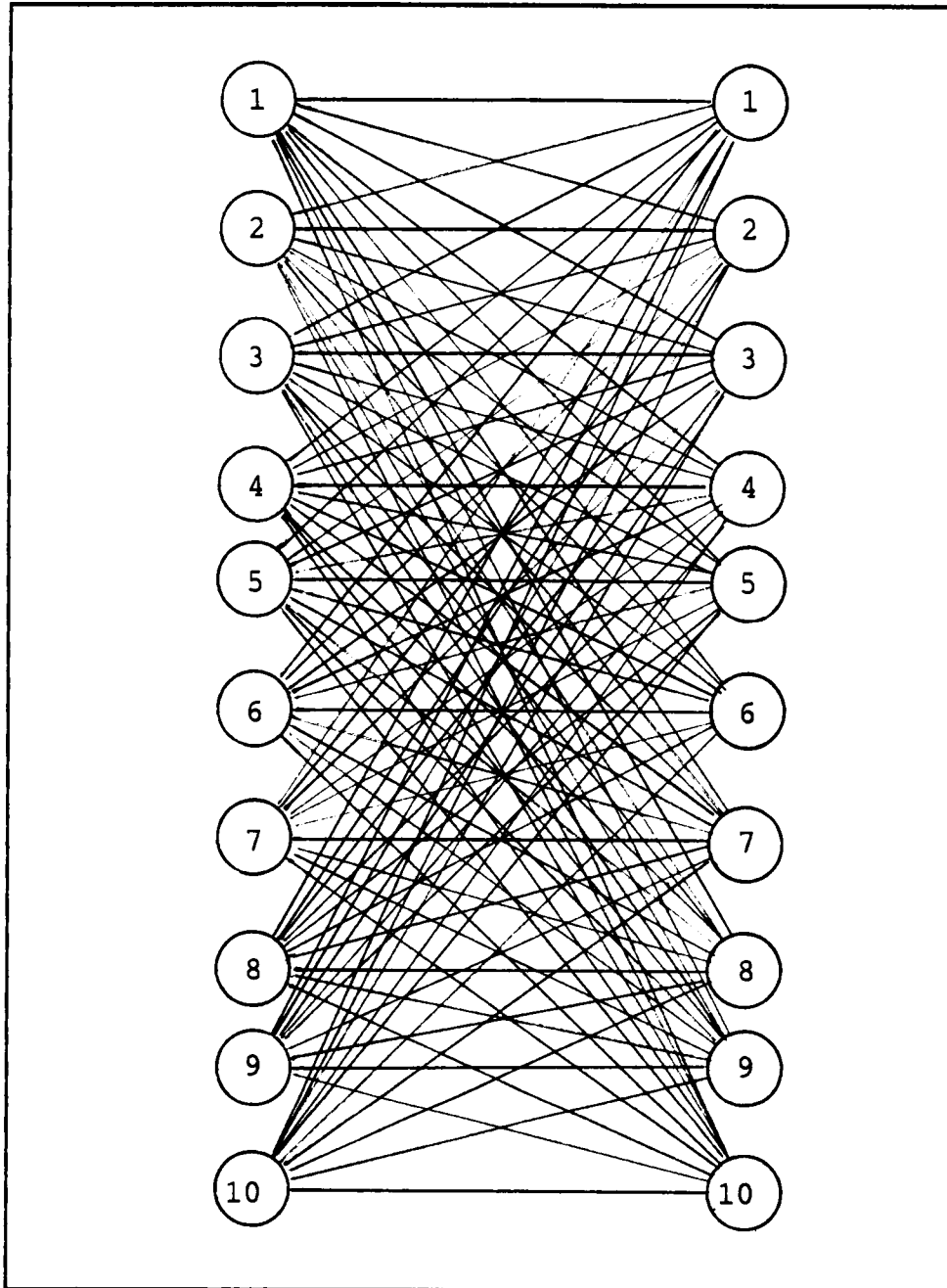


Figure 4 : A Sample Bipartite Network with
10 Nodes

Table 1

Distance Matrix and Populations
of a Bipartite Network

	Distance *										Population	
	1	2	3	4	5	6	7	8	9	10	0	p **
											i	i
1	10	246	396	611	496	648	385	418	376	208	913	.13
2	246	15	150	365	347	517	254	406	364	196	847	.12
3	396	150	10	215	197	428	193	368	490	242	2030	.28
4	611	365	215	10	153	283	264	439	511	441	569	.08
5	496	347	197	153	12	231	111	286	358	288	95	.01
6	648	517	428	283	231	10	263	355	427	440	225	.03
7	385	254	193	264	111	263	15	175	247	177	477	.07
8	418	406	368	439	286	355	175	10	72	210	851	.12
9	376	364	490	511	358	427	247	72	10	168	318	.04
0	208	196	242	441	288	440	177	210	168	10	906	.12

* Shortest distance is assumed.

** Proportion of total population at i

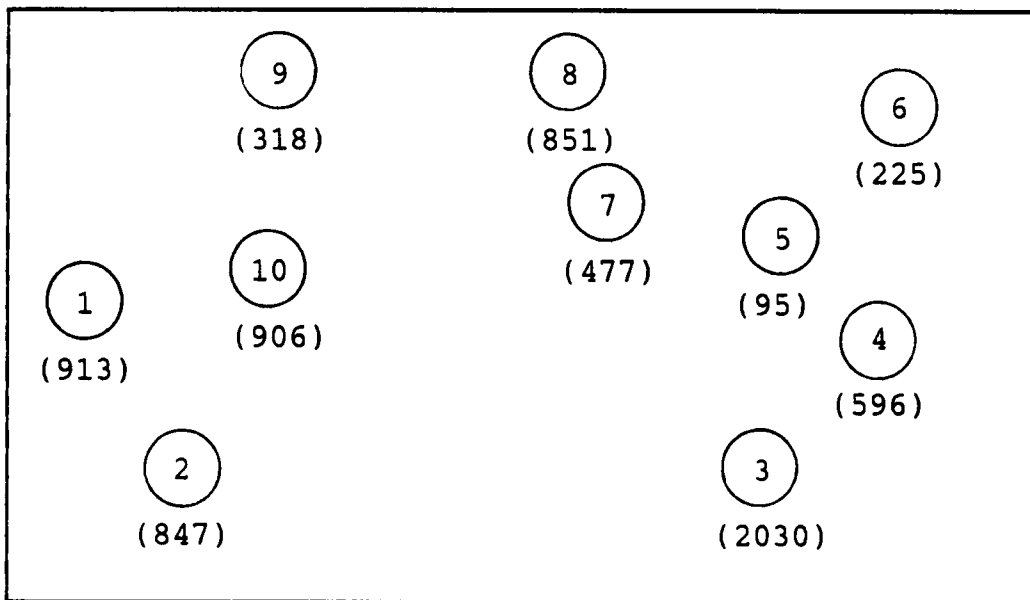


FIGURE 5 : A Sample Bipartite Network with
Population Distribution

(The numbers in parentheses are populations)

As the economies of scale and distance decay parameter, (θ and β), change, the location and size of the facilities on the network also vary. The value of θ is the ratio of α , the attractiveness parameter, and τ , the construction cost parameter. A high value of θ ($\theta > 1$) indicates higher attractiveness and lower construction cost for larger facilities. A low value of θ ($\theta < 1$) means low attractiveness and higher construction cost for larger facilities. Clearly, the larger the value of θ , the more likely a few large facilities will be built. The following three examples demonstrate the major points to be considered from Table 2.

Case 1: If the distance decay value is large and the economies of scale favor large facilities, the attractiveness of a large facility outweighs the users' aversion to traveling long distances. The result will be that one large facility is constructed at the most populous node (node 3).

Case 7: An intermediate case would result when both the distance decay value and economies of scale value are lower, but the latter is greater than 1. This means that people are more willing to travel and the attractiveness of a large facility is not as great as it was in the previous case. Therefore, two facilities will be built in the two populous and accessible nodes (node 3 and 10).

Table 2

The Optimal Solutions of the Log Accessibility
Maximization Model on an Uncapacitated
Network with Different Parameters

	β	θ	1	2	3	4	5	6	7	8	9	10
1	.01	2.0			1.00							
2	.01	1.1	.11		.56			.01		.16		.16
3	.01	1.0	.11	.02	.47	.03		.02		.16		.18
4	.01	0.9	.11	.08	.37	.05		.02	.05	.13	.02	.15
5	.01	0.5	.11	.13	.25	.07	.04	.03	.09	.10	.06	.13
C 6	.005	2.0			1.00							
A 7	.005	1.1			.53							.47
S 8	.005	1.0			.50				.04			.46
E 9	.005	0.9	.04	.11	.35	.01	.02		.14	.06	.02	.23
10	.005	0.5	.08	.13	.18	.07	.08	.03	.12	.09	.07	.14
11	.001	2.0			1.00							
12	.001	1.1			1.00							
14	.001	1.0		.09	.40				.20			
13	.001	0.9	.06	.14	.18	.05	.09	.02	.16	.08	.06	.16
15	.001	0.5	.09	.11	.12	.09	.10	.08	.11	.10	.09	.11

(The numbers measure the percent of the total budget spent
at each location.)

Case 5: If the propensity to travel is low ($\beta = 0.01$) and the advantage of a large facility is also low ($\theta = 0.5$), then a facility will be built at each node. The size of the facilities will vary according to the populations at each node.

Some important generalizations can be gained from an examination of the results presented in Table 2:

1. The place with the largest population will always have a facility.
2. The critical values are θ and β , which have the following general numerical result:

	β large	β small
$\theta > 1$	uncertain case	Build one large facility
$\theta \leq 1$	Build small facilities at each node	Build many small facilities

In general, if the economies of scale favor large facilities and people are willing to travel, a single large facility should be built. If, on the other hand, there are diseconomies to large facilities, and people are unwilling to travel, a small facility should be built at each location. The same result (i.e. build a facility at each

location) holds when small facilities are economical and people are willing to travel. The difference between cell (2,1) and (2,2) is the budget distribution among facilities. Cell (2,1) distributes the amount of budget to each facility according to the principle that most people will choose their own facility, while cell (2,2) distributes the budget by the rule that people may visit other facilities more often. No general solution can be determined for cell (1,1) because there is a trade off between economies of scale, θ , and distance decay, β . The solutions in this case will depend on the characteristics of the network.

3.3. Congestion Functions

If a network is capacitated, travel time on links will vary with the volume of traffic flow. This relationship is represented in the congestion function. The average travel time on a link is constant when the amount of flow is very low. The travel time increases slowly at first as the number of trips increases. However, as the flow begins to reach the design capacity of that link, the travel cost increases tremendously.

"Many different mathematical forms of the average travel time function are in common use throughout the world" [LeBlanc 1975, p. 311]. This research uses two

congestion functions, the U. S. Federal Highway Administration [FHWA] traffic assignment model and Davidson's congestion function [Boyce, 1981].

Let:

c_{ij}^0 : the travel time on the link ij when there is only one trip on the link.

The travel time can be presented as:

$$c_{ij} = c_{ij}^0 + \eta(s_{ij}), \quad (3.9)$$

where:

$\eta(s_{ij})$: the delay function [Branston, 1976, p. 229].

The FHWA congestion function, shown in Figure 6, defines the delay function as: [Boyce, et al., 1981, p. 223]

$$\eta_1(s_{ij}) = 0.15 c_{ij}^0 \left(\frac{s_{ij}}{cp_{ij}} \right)^4,$$

where:

s_{ij} : flows on link ij ,

cp_{ij} : capacity on link ij .

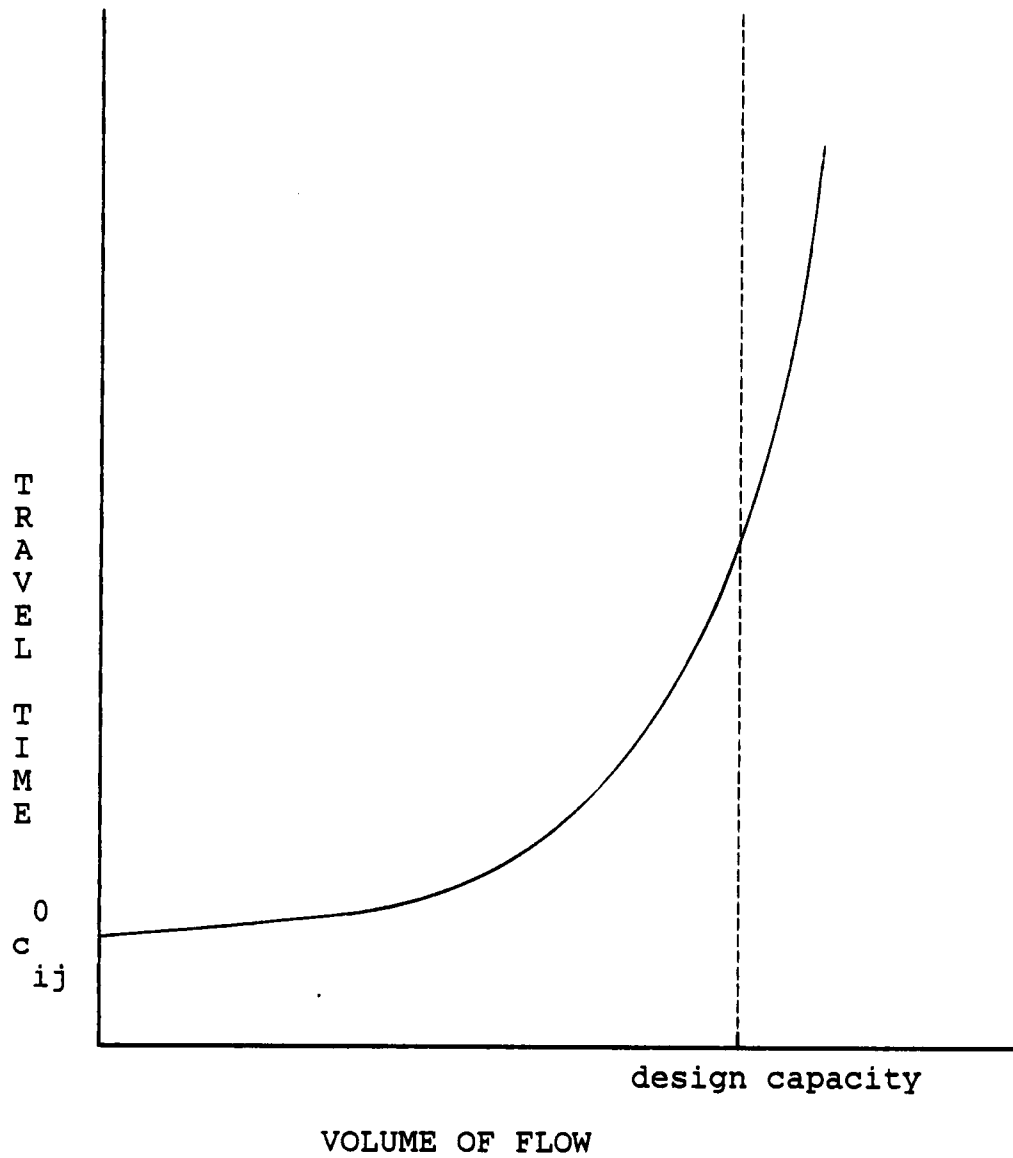


Figure 6 : The Federal Highway Administration
Congestion Function

Davidson's congestion function, shown in Figure 7, defines the delay function as [Boyce, et al., p. 224]:

$$\eta_{ij}^2(s_{ij}) = J c_{ij}^0 \left(\frac{s_{ij}}{c_{ij} - s_{ij}} \right),$$

where:

J : a factor that reflects the frequency of delay producing situations along a link.

In Davidson's function, travel time increases to infinity as link flow approaches capacity. Davidson argues that congestion functions should work in this manner to restrain flow below capacity.

The true link capacity function is only applicable at flows which do not exceed the capacity of a link. This is because flows predicted by an assignment procedure are only feasible if they are less than or equal to the steady-state capacity of a link. However, in most iterative assignment procedures it is possible to have predicted flows which, at least in the initial stages, exceed the steady-state capacities of some links. Under such circumstances these links are overloaded. [Branston, 1976, p. 227].

Therefore, for computational purposes, it is important to redefine Davidson's congestion function as:

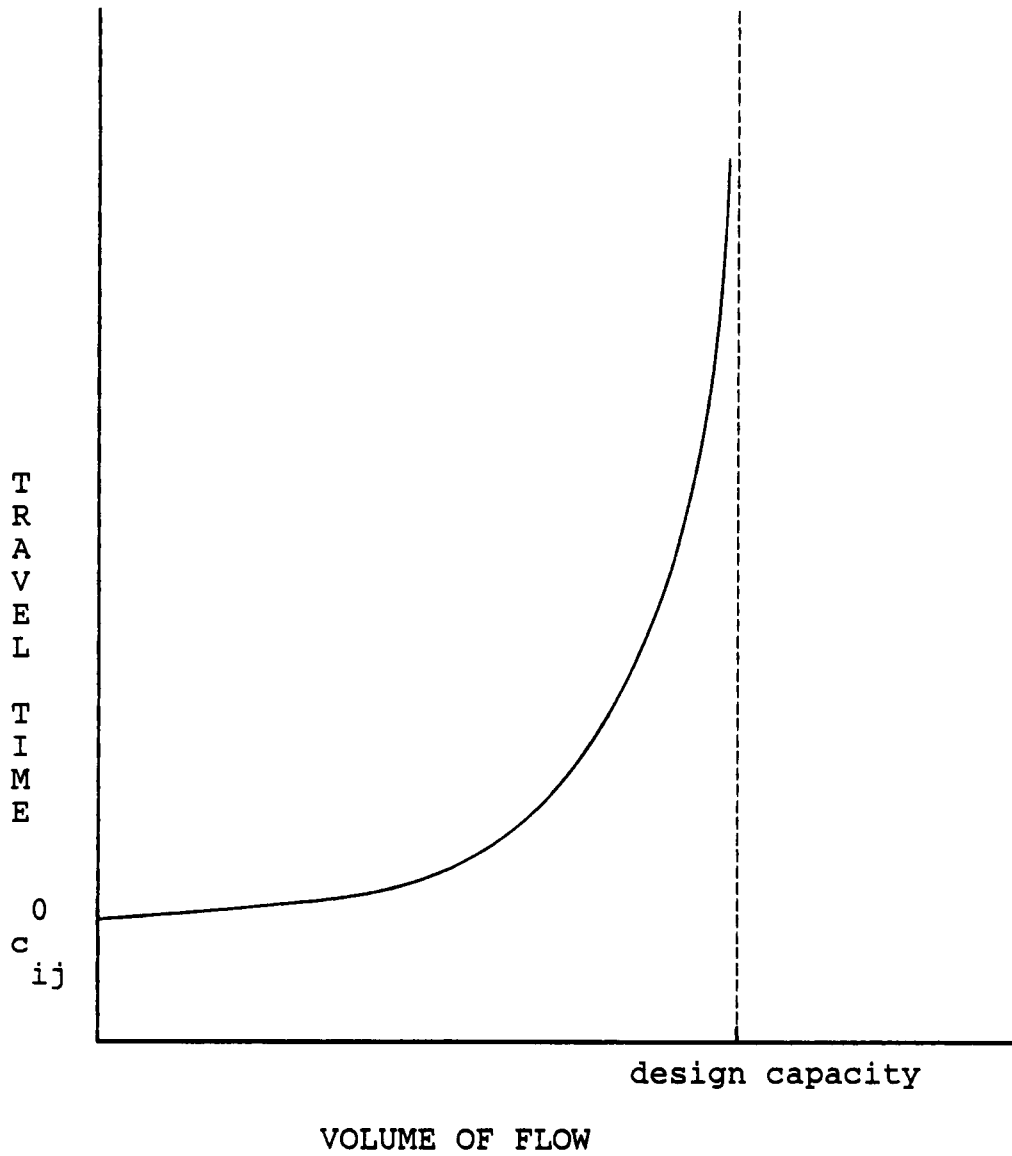


Figure 7: Davidson's Congestion Function

$$\eta_{2ij}(s_{ij}) = \begin{cases} J c_{ij} \left(\frac{s_{ij}}{c_{ij} - s_{ij}} \right) & \text{if } s_{ij} < c_{ij}, \\ M & \text{if } s_{ij} \geq c_{ij}. \end{cases}$$

where:

M is a large number.

The use of the upper bound M makes Davidson's function well defined for all values of s_{ij} , even those which exceed the design capacity.

3.4. A Log Accessibility Maximizing Model on a Capacitated Bipartite Network

A bipartite network means the nodes on the network can be separated into two subsets. One subset denotes origin nodes, such as population centers. The other subset represents destination nodes, such as facilities. The path between each pair of nodes within these two subsets is unique and no two paths can share the same link. Equivalently, we can assume each link has infinite capacity.

The location-allocation problem on a capacitated network is a systems equilibrium problem. Systems equilibrium, in this case, entails finding the balance between route congestion, travel cost, and flow patterns.

Since travel cost and travel pattern are interrelated, there is a recursive relationship between these two factors. They depend on the capacity of the network to move a certain volume of traffic through it. Since the travel pattern is a function of facility sizes and locations, there is a recursive relationship between facility size, locations, and travel cost. Two major questions need to be addressed: does a systems equilibrium exist; and if it does, is it unique?

3.4.1. The Maximum Log Accessibility Model with Flow Capacity Constraints

The objective function in this model maximizes log accessibility (Φ) which is defined as:

$$\Phi = \sum_i O_i \ln \left(\sum_j w_j^\alpha \exp(-\beta c_{ij}) \right), \quad (3.11)$$

where:

Φ : log accessibility,

O_i : population at location i ,

w_j : facility size at location j ,

c_{ij} : travel cost from original i to facility j ,

α : economies of scale,

β : distance discount factor.

This objective function has the same form as equation (3.2). The difference between this case and the uncapacitated problem is that the travel cost, c_{ij} , is not a constant, as in the uncapacitated network model, but a function of the number of trips and link capacity. The objective function can be maximized over all w_j and s_{ij} or equivalently - over all w_j and c_{ij} [Ralston and Liu, 1984]. The FHWA function will be used as the constraint for illustrative purposes, but other congestion functions, such as Davidson's, could be easily substituted in this model.

The FHWA congestion function is:

$$c_{ij} = c_{ij}^0 \left(1 + 0.15 \left(\frac{s_{ij}^4}{c_{ij}^4} \right) \right), \quad (3.12)$$

where :

s_{ij} : determined by the facility size (w_j)
and travel cost (c_{ij}).

It is assumed that a planning authority's funds are limited, for without the budget constraint the solution is

to build a large facility at every location. The form of the budget constraints is the same as above:

$$\sum_j w_j^\tau \Gamma_j \leq B \quad (3.13)$$

where:

τ : economies of scale in construction,

Γ_j : the construction cost at proposed location j .

Two additional sets of constraints (3.14) and (3.15) are introduced into this model:

$$s_{ij} = p_i \frac{w_j^\alpha \exp(-\beta c_{ij})}{\sum_k w_k^\alpha \exp(-\beta c_{ik})}, \quad (3.14)$$

$$c_{ij}^0 = c_{ij} (1 + 0.15 \left(\frac{s_{ij}}{c_{p_{ij}}} \right)^4). \quad (3.15)$$

These constraints (3.14) and (3.15) ensure the balance between flow pattern s_{ij} and travel cost c_{ij} . Constraint (3.14) insures that the trips between original i and destination j , s_{ij} , are functions of facility size w_j and the travel cost c_{ij} to these facilities. Constraint (3.15)

shows that the travel cost, c_{ij} , is also a function of the number of trips to these facilities, s_{ij} , and finite link capacity cp_{ij} . Values for s_{ij} and c_{ij} associated with facility size w_j must, therefore, be found which will satisfy both equations simultaneously and will lead to a systems equilibrium state.

A final constraint ensures that no negative facility size will be chosen:

$$w_j \geq 0. \quad (3.16)$$

In sum, this model will choose facility size, travel cost, and flow pattern to maximize log accessibility (3.11), subject to the budget constraint (3.13), flow equilibrium pattern (3.14 and 3.15), and non-negative facility size (3.16). The budget constraint (3.13) is initially an inequality constraint. This means that it is not necessary to spend all the budget. It can be proven that for optimality, all the budget should be spent, therefore the budget constraint can be changed to an equality.

Theorem 3.1 In order to obtain optimality, all the budget must be spent; i.e., the budget constraint is an equality:

$$\sum_j w_j^\beta \tau_j = B.$$

Proof: The proof is by contradiction.

Assume that there is an optimal solution, w^* , such that:

$$\sum_j w_j^* \tau_j < B.$$

Since w^* is optimal, there can be no feasible ϕ' where $\phi' > \phi^*$.

Let E equal the extra budget:

$$\sum_j w_j^* \tau_j + E = B,$$

and let:

$$w_j' = w_j^* (1 + \delta), \quad \text{for all } j, \quad (3.17)$$

where:

$$\delta = \frac{E}{\sum_j w_j^* \tau_j},$$

substitute (3.17) into (3.14)

$$s'_{ij} = O_i \frac{\sum_j \alpha w_j^* (1+\delta) f_{ij} \exp(-\beta b_{ij} r'_{ij})}{\sum_k \alpha w_k^* (1+\delta) f_{ik} \exp(-\beta b_{ik} r'_{ik})}, \quad (3.18)$$

$$\Rightarrow s'_{ij} = s^*_{ij}, \quad \text{for all } i, j,$$

$$\Rightarrow c'_{ij} = c^*_{ij}, \quad \text{for all } i, j.$$

$$\begin{aligned} \Phi' &= \sum_i O_i \ln \left(\sum_j \alpha w_j^* (1+\delta) \exp(-\beta c^*_{ij}) \right), \\ &= \sum_i O_i \ln \left(\sum_j \alpha w_j^* \exp(-\beta c^*_{ij}) \right) + \sum_i O_i \ln(1+\delta), \\ &= \Phi^* + \sum_i O_i \ln(1+\delta). \end{aligned}$$

Since O_i and $\ln(1+\delta) > 0$,

$$\Rightarrow \Phi' > \Phi^*.$$

This contradiction finishes the proof.

3.4.2. Model Transformation

Equations (3.15) and (3.16) show that flow patterns and travel cost are interrelated. A higher travel cost leads to lower travel flows. A lower travel cost encourages higher flows to a destination. The travel costs are also determined by the flows on the links. More flows

from origin i to destination j cause congestion and increase the travel cost.

There are two ways to find the stable equilibrium point. Substituting (3.16) into (3.15) and finding the solution, gives the balance flow patterns which satisfies the interrelationship between facility size attractiveness and travel cost resistance. Another way is to substitute (3.15) into (3.16) and find the solutions of travel costs which also will satisfy the balance condition. Both methods will have the same solution. The first method will be used here to find the solution.

The capacitated network model can be transformed into a simpler form.

Let:

$$r_{ij} = \frac{s_{ij}}{o_i},$$

where:

r_{ij} : the proportion of the trip from i to j respect to population at i ,

$$x_j = \frac{\beta \sum_j w_j \tau_j}{B},$$

where:

x_j : the proportion of budget spend at facility j ,

$$p_i = \frac{O_i}{T},$$

where:

p_i : the proportion of the total population at i ,

Define:

$$a_{ij} = \tau \exp(-\beta c_{ij}^0),$$

$$b_{ij} = \beta \cdot 0.15 \cdot c_{ij}^0 \left(\frac{O_i}{c_{ij}^0} \right)^4,$$

where:

a_{ij}, b_{ij} : constants.

Finally, let:

$$\theta = \frac{\alpha}{\beta}. \quad (3.19)$$

The cost constraint (3.14) becomes:

$$c_{ij} = c_{ij}^0 + b_{ij} \frac{r_{ij}^4}{c_{ij}^0}. \quad (3.20)$$

Substituting (3.19) - (3.20) into (3.14) yields:

$$r_{ij} = \frac{\theta \sum_j x_j a_{ij} \exp(-\beta c_{ij})}{\sum_k \theta x_k a_{ik} \exp(-\beta c_{ik})}, \quad (3.21)$$

The capacitated log accessibility model is then simplified to the following form:

$$\max \sum_i p_i \ln \left(\sum_j \theta x_j a_{ij} \exp(-b_{ij} r_{ij}^4) \right), \quad (3.22)$$

s.t.:

$$\sum_j x_j = 1, \quad (3.23)$$

$$r_{ij} = \frac{\theta \sum_j x_j a_{ij} \exp(-b_{ij} r_{ij}^4)}{\sum_k \theta x_k a_{ik} \exp(-b_{ik} r_{ik}^4)}, \quad \text{for all } ij, \quad (3.24)$$

$$x_j \geq 0, \quad \text{for all } j. \quad (3.25)$$

3.4.3. Existence and Uniqueness of a System Equilibrium

Two interesting questions arise from the transformation. Does a solution for equation (3.24) exist, and, if so, is it unique? If there is no solution for any arbitrary value x_j in equation (3.24), there is no stable

and balanced flow pattern. If there is more than one solution, it is difficult to determine which equilibrium flow pattern to choose. This section will prove that there exists a unique solution for the equation (3.24). To prove this, we need the following definitions and theorems.

Definition: A function G from $D = R^n$ into R^n is said to have a fixed point at $p \in D$ if $G(p) = p$.

The following theorem is a special case of the well-known Contraction Mapping Theorem [Burden, et al., 1981, p. 442].

The Contraction Mapping Theorem:

Part I - Existence

Let $D = \{(x_1, x_2, \dots, x_n) \mid a_i \leq x_i \leq b_i \text{ for each}$

$i = 1, 2, \dots, n\}$ for some collection of constants

$a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_n$. Suppose G is a continuous

function with continuous first partial derivatives from $D =$

R^n into R^n with the property that $G(x) \in D$ whenever $x \in D$.

Then G has a fixed point in D .

Part II - Uniqueness

If, in addition, a constant $K < 1$ exists with

$$\left| \begin{matrix} D \\ x_j \end{matrix} g_i(x) \right| \leq \frac{K}{n}, \quad \text{whenever } x \in D.$$

for each $j = 1, 2, \dots, n$, and each of the coefficient functions g_i , then the fixed points is unique.

Using the above theorem the theorem 3.2 can be proven.

Theorem 3.2 There is at least one solution set for equation (3.24)

Proof:

Let:

$$g_{ij} = r_{ij} = \frac{\theta x_j a_{ij} \exp(-b r_{ij}^4)}{\sum_k \theta x_k a_{ik} \exp(-b r_{ik}^4)}.$$

Since g_{ij} is continuously differentiable, its first derivative is also continuous.

Choose $a_i = 0$ and $b_i = 1$ for all i ,

then $D = \{ r_{ij} \mid 0 \leq r_{ij} \leq 1 \}$.

Since $\sum_j r_{ij} = 1$, for all i and $r_{ij} \geq 0$,

$\Rightarrow g_{ij}(x) \in D$, for all i and all j ,

and $x \in D$.

From the Contraction Mapping Theorem part I, there is at least one fixed point r_{ij} for equation (3.14).

Theorem 3.3: The fixed point for (3.24) is unique if b_{ij} is small.

To prove the uniqueness, take the first derivative of (3.14):

$$D_{r_{ij}} g_{ik} = \begin{cases} -4 b_{ij} r_{ij}^4 (1 - r_{ij}), & \text{if } j = k, \\ -4 b_{ij} r_{ij}^4 r_{ik}, & \text{if } j \neq k. \end{cases}$$

Since $r_{ij} \leq 1$ for all i and j , from part II of the contraction Mapping Theorem the fixed point is unique if b_{ij} is small enough, i.e.;

$$\max \{ |-4b_{ij} r_{ij}^4 (1 - r_{ij})|, |-4b_{ij} r_{ij}^4 r_{ik}| \} \leq 4 b_{ij} r_{ij}.$$

If we choose $b_{ij} \leq \frac{1}{4n}$, then:

$$\left| \begin{matrix} D \\ x_j \end{matrix} \begin{matrix} g_i(x) \end{matrix} \right| \leq \frac{K}{n}, \quad \text{whenever } x \in D.$$

Recall that:

$$b_{ij} = \beta \cdot 0.15 \cdot c_{ij}^0 \left(\frac{O_i}{cp_{ij}} \right)^4.$$

This implies that if the ratio of population, O_i , to link capacity, cp_{ij} , is small enough, the solution would be unique.

Theorem 3.2 indicates that there is at least one solution set for equation (3.24), and the solution set will be unique if the network is not highly congested. This result can be applied to any congestion function, if the first derivative of the congestion function exists and is small enough. However, the congestion function chosen here [FHWA] has a special property of uniqueness, regardless of the amount of congestion on the network. The following theorem proves this property.

Theorem 3.4: Equation (3.24) has a unique solution
if $b_{ij} > 0$ for all i and j .

Proof:

Note that for any two demand points i and k , r_{ij} and r_{kj} defined by (3.14) are independent. Thus proving uniqueness for any value of i will complete the proof. Since the existence of a unique solution is not dependent on the value of i , the first subscript can be ignored. After eliminating all the constant terms, we have:

$$r_j = \frac{a_j \exp(-b_j r_j^4)}{\sum_k a_k \exp(-b_k r_k^4)}, \quad \text{for all } j.$$

Assume that there are two solutions, r' and r'' . Then without loss of generality there is a integer variable p , $1 \leq p \leq n$, such that:

$$r'_p > r''_p.$$

Thus,

$$\frac{r'_p}{r''_p} > 1.$$

Since:

$$\sum_k r'_k = 1, \text{ and } \sum_k r''_k = 1,$$

there is at least an m such that:

$$\frac{r'_m}{r''_m} < 1. \quad (3.26)$$

Let:

$$\Phi = \sum_j a_j \exp(-b r_j^4).$$

Then,

$$\begin{aligned} \frac{r'_p}{r''_p} &= \frac{a_p \exp(-b r_p'^4) \Phi''}{a_p \exp(-b r_p''^4) \Phi'} \\ &= \exp b \left(-r_p'^4 + r_p''^4 \right) \frac{\Phi''}{\Phi'} > 1. \end{aligned}$$

Since $r'_p > r''_p$ and $b > 0$,

and $\exp b \left(-r_p'^4 + r_p''^4 \right) < 1$,

$$\Rightarrow \frac{\Phi''}{\Phi'} > 1,$$

$$\frac{\frac{r'_m}{r''_m}}{\frac{a_m \exp(-r'^4_m) \Phi''}{a_m \exp(-r''^4_m) \Phi'}} = \frac{r'_m}{r''_m} \cdot \frac{\exp(-r''^4_m) \Phi'}{\exp(-r'^4_m) \Phi''}.$$

Since $r'_m < r''_m$ and $b_m > 0$,

$$\Rightarrow \exp b_m (-r'^4_m + r''^4_m) > 1,$$

and $\frac{\Phi''}{\Phi'} > 1,$

$$\Rightarrow \frac{r'_m}{r''_m} > 1.$$

This contradicts (3.26) and the proof is complete.

3.5. Concluding Remarks

This chapter reviewed the theoretical results of the LAMM on a bipartite network suggested by Leonardi. A numerical solution for this model is also presented to explore the properties of the LAMM. The LAMM is extended to the case of capacitated network by introducing a balanced flow capacity constraint. The congestion function used in the flow capacity constraint is FHWA function. Other congestion functions can be easily substituted into this model, if the first derivative is well defined.

Chapter 4 will use these results to develop a solution technique for the model on a capacitated bipartite network. Numerical analysis will be used to explore the effects of link capacity on the location-allocation process. These results will be compared to those presented in this chapter.

CHAPTER 4

AN ALGORITHM FOR MAXIMIZING LOG ACCESSIBILITY
ON CAPACITATED BIPARTITE NETWORKS

This chapter extends the theoretical results presented in the previous chapter by developing a solution technique for the Log Accessibility Maximization Model. The uniqueness of the flow pattern constraints, (3.14) and (3.15), with respect to any set of facility sizes reduces the problem to a non-linear objective function with one linear constraint. By generating numerical solutions on uncapacitated and capacitated bipartite networks, the effect of congestion on the location-allocation decision process can be better understood.

4.1. The Maximum Principle

Before deriving the solution algorithm, the following transformations are made:

Let:

ϕ_{ij} : the interaction between nodes i and j . That is,

$$\phi_{ij} = x_j^{\theta} a_{ij} \exp(-b_{ij} r_{ij}^4).$$

ϕ_i : the accessibility at location i ,

$$\phi_i = \sum_k x_k a_{ik} \exp(-b_{ik} r_{ik}^4);$$

$\eta(r_{ij})$: the congestion function,

$$\eta(r_{ij}) = 0.15 c_{ij} \left(\frac{r_{ij}^0}{c_{ij}} \right)^4 r_{ij}^4 = b_{ij} r_{ij}^4;$$

and,

$$r_{ij} = \frac{\phi_{ij}}{\phi_i}.$$

Therefore the congested network log accessibility model (3.22)-(3.25) becomes:

$$\max \Phi = \sum_i p_i \phi_i, \quad (4.1)$$

s.t.:

$$\sum_k x_k = 1, \quad (4.2)$$

$$r_{ij} = \frac{\phi_{ij}}{\phi_i}, \quad \text{for all } i, j, \quad (4.3)$$

$$x_j \geq 0, \quad \text{for all } j. \quad (4.4)$$

Let μ , w_{ij} , and v_j be dual variables associated with (4.2),

(4.3), and (4.4) respectively. The Lagrangian function is:

$$L = \Phi + \mu (1 - \sum_k x_k) + \sum_{ik} \sum_{ik} w_{ik} \left(\frac{\phi_{ik}}{\phi_i} - r_{ik} \right) + \sum_k v_k x_k. \quad (4.5)$$

Within the capacitated network model, there are two decision variables, x and r . The optimal solution is the feasible solution satisfying the following two set of system equations:

$$D_{x_j} L = 0, \quad (4.6)$$

and

$$D_{r_{ij}} L = 0. \quad (4.7)$$

The first condition, (4.6), is to find the partial derivative of L for the variable x_j and set it to zero:

$$D_{x_j} L = D_{x_j} \Phi - \mu + v_j + \sum_{ik} \sum_{ik} w_{ik} D_{x_j} \left(\frac{\phi_{ik}}{\phi_i} \right) = 0. \quad (4.8)$$

Let:

$$\delta_{ik}^j = D_{x_j} \left(\frac{\phi_{ik}}{\phi_i} \right).$$

Equation (4.8) becomes:

$$D_x L = D_x \Phi + \mu + v + \sum_j \sum_{ik} w_{ik} \delta_{ik}^j. \quad (4.9)$$

Multiply both sides of equation (3.34) by x_j ,

$$x_j D_x L = D_x \Phi x_j + \mu x_j + x_j v + \sum_j \sum_{ik} w_{ik} \delta_{ik}^j x_j \quad (4.10)$$

Summing over j in equation (4.10),

$$\sum_j x_j D_x L = \sum_j D_x \Phi x_j + \mu \sum_j x_j + \sum_j \sum_{ik} w_{ik} \delta_{ik}^j x_j. \quad (4.11)$$

Since $\sum_j x_j = 1$, equation (4.11) becomes:

$$\mu = \sum_j D_x \Phi x_j + \sum_j \sum_{ijk} w_{ijk} \delta_{ik}^j x_j. \quad (4.12)$$

Substitute (4.12) into (4.10) and it becomes:

$$x_j = \frac{1}{\mu} \left(\sum_k D_k \Phi x_j + \sum_j \sum_{ik} w_{ik} \delta_{ik}^j x_j \right). \quad (4.13)$$

The first term in equation (4.13) is:

$$\begin{aligned} D_x \Phi &= D_x \left(\sum_i p_i \sum_k x_i a_{ik} \exp(\eta(r_{ik})) \right), \\ &= \sum_i p_i \frac{\sum_j \theta_j x_j a_{ij} \exp(\eta(r_{ij}))}{\phi_i}. \end{aligned}$$

Multiply x_j on both sides to obtain:

$$x_j D_x \phi = \Theta \sum_i p_i r_{ij}. \quad (4.14)$$

The second term in the right hand side of equation (4.13) is:

$$\delta_{ik}^j = D_x \left(\frac{\phi_{ik}}{\phi_i} \right),$$

$$= \left[\frac{\Theta x_j^{\Theta-1} a_{ij} \exp(n(r_{ij}))}{\phi_i} - \frac{\phi_{ij} \Theta x_j^{\Theta-1} a_{ij} \exp(n(r_{ij}))}{\phi_i^2}, \quad \text{if } k=j, \right.$$

$$\left. \frac{\phi_{ik}}{\phi_i} \frac{\Theta x_j^{\Theta-1} a_{ij} \exp(n(r_{ij}))}{\phi_i^2}, \quad \text{if } k \neq j. \right]$$

Multiply x_j on both sides,

$$x_j D_x \left(\frac{\phi_{ik}}{\phi_i} \right)$$

$$= \left[\begin{array}{ll} \Theta (r_{ij}^2 - r_{ij}), & \text{if } k = j, \\ \Theta r_{ij} r_{ik}, & \text{if } k \neq j. \end{array} \right] \quad (4.15)$$

Substitute (4.14) and (4.15) into (4.13) to obtain:

$$x_j = \frac{\theta}{\mu} \left(\sum_i p_{ij} r_{ij} + \sum_i \sum_{k \neq j} w_{ik} r_{ij} r_{ik} + \sum_i w_{ij} r_{ij} (1 - r_{ij}) \right), \text{ for all } j. \quad (4.16)$$

Summing over j gives:

$$\mu = \sum_{ij} p_{ij} r_{ij} + \theta \sum_i \sum_{k \neq j} w_{ik} r_{ij} r_{ik} + \sum_{ij} w_{ij} r_{ij} (1 - r_{ij}).$$

Note that,

$$\sum_{k \neq j} w_{ik} r_{ij} r_{ik} = w_{ij} r_{ij} (1 - r_{ij}).$$

Therefore,

$$\mu = \theta (1 + 2 \sum_{ij} w_{ij} r_{ij} (1 - r_{ij})). \quad (4.17)$$

Equations (4.16) and (4.17) are used to construct an algorithm to find x_j and r_{ij} .

The second condition (4.7) to be satisfied is to take the partial derivative of L respect to the variable r_{ij} and

set it to zero:

$$D_{r_{ij}} L = D_{r_{ij}} \Phi + \sum_{ik} w_{ik} D_{r_{ij}} \left(\left(\frac{\phi_{ik}}{\phi_i} \right) - r_{ij} \right) = 0. \quad (4.18)$$

The first term on the right hand side of equation (4.18) is:

$$D_{r_{ij}} \Phi = D_{r_{ij}} \left(\sum_i p_i \ln \left(\sum_k x_k^{\Theta} a_{ik} \exp(\eta(r_{ik})) \right) \right),$$

$$= \begin{cases} D_{r_{ij}} (\eta(r_{ij})) r_{ij}, & \text{if } j = k, \\ -4 b_{ij}^4 r_{ij}, & \text{if } j \neq k. \end{cases} \quad (4.19)$$

The second term of (4.18) is:

$$D_{r_{ij}} \left(\frac{\phi_{ik}}{\phi_i} - r_{ij} \right)$$

$$= \begin{cases} -4b_{ij}^3 r_{ij} \left(\frac{x_j^{\Theta} a_{ij} \exp(\eta(r_{ij}))}{\phi_i} - \frac{x_j^{\Theta} a_{ij} \exp(\eta(r_{ij}))}{\phi_i^2} \right) - 1, & \text{if } k = j, \\ -4b_{ij}^3 r_{ij} \left(x_j^{\Theta} a_{ij} \exp(\eta(r_{ij})) \frac{x_j^{\Theta} a_{ij} \exp(\eta(r_{ij}))}{\phi_i^2} \right), & \text{if } k \neq j. \end{cases}$$

$$= \begin{cases} -4 b_{ij}^4 r_{ij} (1 - r_{ij}), & \text{if } k = j, \\ -4 b_{ij}^4 r_{ij} r_{ik}, & \text{if } k \neq j. \end{cases} \quad (4.20)$$

Let:

$$K_{ij} = 4 b_{ij} r_{ij}^4.$$

Substituting (4.19) and (4.20) to (4.18) gives:

$$-K_{ij} + \sum_{k < j} w_{ik} K_{ik} r_{ik} + w_{ij} ((K_{ij} (1 - r_{ij})) - 1) = 0,$$

for all i . (4.21)

The optimal solution $\{x^*\}$ should satisfy the simultaneous equations (4.6) and (4.7). This problem can be solved by finding solutions for equations (4.16), (4.17), and (4.21).

4.2. The Solution Technique for a Capacitated Bipartite Network

An algorithm is used to find the optimal solution, $\{x^*, r^*\}$. The first step is to choose any feasible solution set x_j which satisfies the budget constraint (4.2) and non-negativity constraint (4.4). The second step is to find the flow pattern which is associated with this facility size. Since it has been proven that there is a unique solution for (4.3), there is an unique set of solutions for the flow pattern r_{ij} . Step 3 is to find the dual value of w_{ij} by solving the linear system equations (3.47). After substituting r_{ij} and w_{ij} into equation (3.46) in step 4, a

new set of x_j can be found. Repeat these procedures until this algorithm converges to a solution. Formally, the algorithm is as follows:

Step 1: Find any initial guess point X where:

$$\sum_k x_k = 1 \quad \text{and} \quad x_k > 0, \quad \text{for all } k.$$

Step 2: Calculate the flow patterns r_{ij} using the current values of x_j by:

$$r_{ij} = \frac{\theta_j a_{ij} \exp(-\beta b_{ij} r_{ij}^4)}{\sum_k \theta_k a_{ik} \exp(-\beta b_{ik} r_{ik}^4)}. \quad (4.22)$$

Step 3: Find the value w_{ij} by solving the following linear system of equations:

$$A^i W^i = B^i, \quad \text{for all } i, \quad (4.23)$$

where:

$$A^i = [a_{jk}^i],$$

$$a_{jk}^i = \begin{cases} -K_{ij} (1 - r_{ij}), & \text{if } k = j, \\ -K_{ij} r_{ik}, & \text{if } k \neq j. \end{cases}$$

$$W^i = \begin{bmatrix} w_{i1} \\ \vdots \\ w_{in} \end{bmatrix} \quad B^i = \begin{bmatrix} -K_{i1} \\ \vdots \\ -K_{in} \end{bmatrix}$$

Step 4: Calculate the new set of X by using the following equation:

$$x_j^{(m+1)} = \frac{\Theta}{\mu} \left(\sum_i p_{ij}^{(m)} r_{ij}^{(m)} + \sum_i \sum_{k \neq j} w_{ik}^{(m)} r_{ik}^{(m)} r_{ij}^{(m)} + \sum_i w_{ij}^{(m)} r_{ij}^{(m)} (1 - r_{ij}^{(m)}) \right). \quad (4.24)$$

Repeat step 2 to 4,

until $|x_j^{m+1} - x_j^m| \leq \epsilon$, for all j.

4.3. Feasibility and Convenience of the Solution Algorithm

It is now necessary to prove that the algorithm developed above is feasible and convenient. An algorithm is feasible if the solutions obtained in every iteration satisfy all constraints. It is convenient if the objective value is monotonic in the proper direction every time the solution is updated. The proof of feasibility and

convenience for the above algorithm is aided by the following transformations.

The model (4.1)-(4.4) has two decision variables, x_j , the proportion of the budget spent at j , and r_{ij} , the percentage of the population traveling from i to j . However, r_{ij} is dependent on x_j . By Implicit Function Theory [Trench, 1978, p. 439], the flow balance constraint (4.3) can be incorporated into the objective function (4.1) and the x_j become the only decision variables. That is, the problem can be written as:

$$\max_{\{x\}} \Phi(x, r(x)) = \sum_i p_i \ln \left(\sum_j x_j a_{ij}^{\Theta} \exp(-\beta r_{ij}(x)) \right)^4, \quad (4.25)$$

s.t.

$$\sum_j x_j = 1, \quad (4.26)$$

$$x_k \geq 0, \quad \text{for all } k. \quad (4.27)$$

From the Chain Rule we have:

$$\begin{aligned} D_{x_j} \Phi(x, r(x)) \\ = D_{x_j} \left(\sum_i p_i \ln \left(\sum_j x_j a_{ij}^{\Theta} \exp(-\beta r_{ij}(x)) \right)^4 \right), \end{aligned}$$

$$\begin{aligned}
& \frac{\sum_i p_i \left(\sum_j \theta x_j a_{ij} \exp(-\beta r_{ij}(x)) + \sum_j \theta x_j a_{ij} \exp(-\beta r_{ij}(x)) D_j x_{ij}^4 \right)}{\sum_j \theta x_j a_{ij} \exp(-\beta r_{ij}(x))}, \\
& \hspace{15em} (4.28).
\end{aligned}$$

It can be verified, after tedious algebra, that these two approaches, (4.1)-(4.4) and (4.25)-(4.27), are equivalent and have same solution algorithm [Ralston and Liu 1984]. It is easier to use the implicit function presentation, a non-linear objective function with a linear constraint, to prove feasibility and convenience.

Definition [Leonardi, 1978, p. 1298]: A point $X = (x_j)$ is feasible for problem (4.1)-(4.4), if it meets constraints (4.2), (4.3) and (4.4). The feasible region is defined to be the set of points:

$$G = \{y : \sum y = 1, y \geq 0, \text{ for all } j\}.$$

Thus, X is feasible if $X \in G$.

Definition [Leonardi, 1978, p. 1298]: A direction $d = (d_j)$ is feasible for problem (4.1)-(4.4) if, given $x \in G$ and a sufficiently small $\delta > 0$, the point:

$$y = x + \delta d,$$

is feasible, that is, $y \in G$.

Definition [Leonardi, 1978, p 1298]: A feasible direction d is convenient for problem (3.1)-(3.4). If given $x \in G$,

$$\nabla \Phi(x) = D_{x_j} \Phi,$$

and the inequality,

$$[\nabla \Phi(x)]^T d \geq 0,$$

holds. In other words, d is convenient if the directional derivative of the objective function along d is monotonically increasing.

The following arguments prove that the algorithm suggested in the previous section is feasible.

From step 1, the solution set satisfies the first constraint (4.2).

From step 2, the solution set satisfied second constraint (4.3).

In step 4,

let:

$$\Omega_j = \left(\sum_j D_{x_j} \Phi + \sum_{i,k} w_{ik} \delta_{ik} \right) x_j, \quad (3.18)$$

and substitute (3.18) into (3.9). Then:

$$x_j^{(m+1)} = \Omega_j x_j^{(m)},$$

where the superscript (m) indicates the m^{th} iteration.

Let d_j be the direction to search:

$$\begin{aligned} d_j &= x_j^{m+1} - x_j^m \\ &= (\Omega_j - 1) x_j^m \\ \sum_j d_j &= \sum_j \Omega_j x_j^m - \sum_j x_j^m = 0. \end{aligned}$$

Thus x_j^{m+1} is in G and feasibility is proven.

The following proves convenience.

Consider:

$$\begin{aligned} & [\Phi]^T d \\ &= \sum_j D_j \Phi_j d_j, \\ &= \sum_j \Omega_j d_j = \sum_j (\Omega_j - 1 + 1) (\Omega_j - 1) x_j^m, \\ &= \sum_j (\Omega_j - 1)^2 x_j^m + \sum_j (\Omega_j - 1) x_j^m \geq 0. \end{aligned}$$

The first term in the above equation is obviously non-negative, and the second term was proven equal to zero in the feasibility case. Thus, convenience is proven.

4.4. Numerical Solutions

The same network in section 3.2 is used for studying the effects of congestion. It consists of 10 nodes and 10 different populations. The largest population is in node 3; the smallest in node 5. Solutions were first derived for varying parameter values when there was no congestion in the system. A list of which nodes received investment is provided in Table 3.

The same parameters are used for the congested network, but with capacity restrictions for all flows into nodes 3 and 8. These nodes were chosen since they appear in all solutions where investment takes place in two or more locations.

Restricting capacity into these nodes has several effects, some intuitively obvious and some seemingly counterintuitive. In the first and third cases, investment in a capacitated area is decreased while investment in a nearby uncapacitated area is increased. In the third, fourth, and fifth cases, restricting access to nodes 3 and 8 causes new investment in other areas in the system. Both these trends seem intuitively pleasing. In the second, fourth and fifth cases, however, investment in an area of restricted access actually increases. The reason for this seems to be as follows: These are important places in

TABLE 3

The Comparison of the Log Accessibility Maximization
Model on Uncapacitated and Capacitated Networks
Restricting Capacity into Nodes 3 and 8

θ	β	uncapacitated network build in	capacitated network build in	decrease investment in	increase investment in
.4	.09	every place	every place	8	9
1.4	.01	1,3,8	1,3,8	1,8	3
1.4	.04	every place but 9	every place	8	9
1.6	.01	3,8	1,3,8	3	1,8
1.8	.01	3	1,3,8,10	3	1,8,10

terms of size and location. Investment is increased in order to insure their continued role in serving other nodes in the system, even in the face of higher transportation costs. Put another way, increased investment increases attractiveness while the restricted capacities have the opposite effects. It should also be noted that in these case, β is small (compared to the other cases) so that only a slight increase in investment in locations 3 and 8 can overcome the effects of congestion.

4.5. Concluding Remarks

This chapter, which derived a feasible and convenient four step solution technique for the LAMM on a capacitated network, presented numerical solutions on uncapacitated and capacitated networks. There are two general solutions. When congestion occurs at one or two locations and the propensity to travel is low, the budget should be distributed to other facilities to decrease the congestion and increase the accessibility. On the other hand, when congestion occurs at very important locations (in terms of population size and distance to other nodes) and the propensity to travel is high, more of the budget should be spent at these large, central locations. The expansion should increase the accessibility to an extent that some of the adverse consequences or disutility of traveling to the

facility will be overcome. This is clearly reminiscent of Coelho's [1976] discussion of users' benefits.

CHAPTER 5

THE LOG ACCESSIBILITY MODEL ON CAPACITATED NON-BIPARTITE
NETWORKS WITH TREE STRUCTURE

A non-bipartite network represents a realistic transportation system in which more than one path can connect any pair of nodes, and flows between different nodes can move on the same link. A non-bipartite network is clearly more complex than the bipartite network considered in the previous chapters. The results of the log accessibility model on a bipartite network presented in previous chapters are extended to non-bipartite networks with tree structure in this chapter. A numerical example is presented to explore the property of the log accessibility model on that type network.

5.1. Flow Pattern on Tree Networks

A tree network is defined as: "a connected graph ... that contains no cycles. Thus a tree has the property that there is a unique chain or path joining each pair of nodes" [Ford and Fulkerson, 1974]. Figure 8 presents a tree network with 6 nodes.

The bipartite and the tree network have the same property that the path between any pair of locations is unique. The difference between them is that the links in a

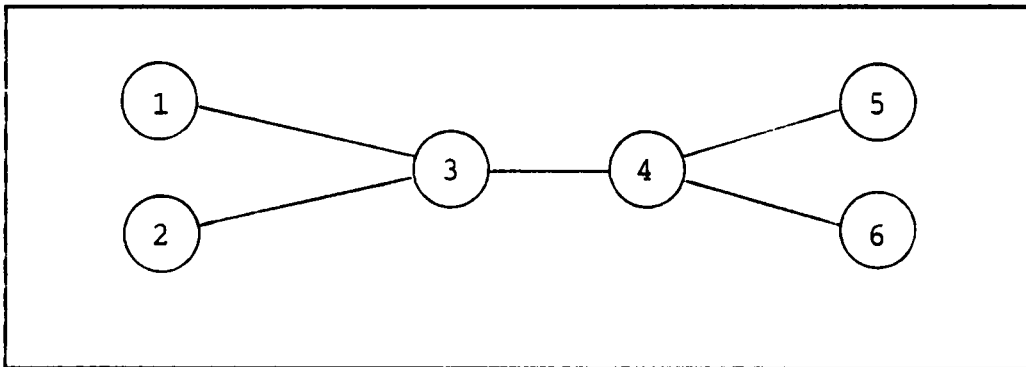


Figure 8 : A Tree-Structured Network

tree network can be shared by many origins and destinations, while the links on a bipartite network are independent. The flow between any two node pairs can be affected by the flow between any other two nodes. For example the flows from 1 to 6 and 3 to 4 in Figure 8 all share a common link.

Although the log accessibility model on a tree network has the same form as the model discussed in chapter 3, the travel cost, c_{ij} , is different, and it remains to be proven that an equilibrium flow pattern exists. The travel cost on the tree network is the summation of all travel costs on each link along the path P_{ij} from origin i to destination j . Furthermore, the number of trips on each link is not only dependent on one pair of nodes, but on many pairs of origins and destinations. That is:

$$c_{ij} = \sum_{lk \in P_{ij}} c_{lk}, \quad (5.1)$$

where:

P_{ij} : the path from location i to location j ,

c_{ij} : travel cost along path P_{ij} ,

c_{lk} : travel cost on link lk .

Let v_{lk} represent the volume of flow on the link lk , then:

$$c_{lk} = c_{lk}^0 \left(1 + 0.15 \left(\frac{v_{lk}^4}{c_{lk}^{p_{lk}}} \right) \right), \quad (5.2)$$

where:

$$v_{lk} = \sum_{pq \in A_{lk}} s_{pq},$$

s_{pq} : trips from location p to location q ,

$$A_{lk} = \{P_{mn} \mid \text{link } (l,k) \text{ is an element of } P_{mn}\}.$$

We now substitute (5.1)-(5.2) into (3.14) and (3.15).

Let:

$$a_{ij} = \exp \left(\sum_{lk \in P_{ij}} c_{lk}^0 \right),$$

$$b_{ij}^k = 0.15 \beta c_{ij}^0 \left(\frac{o_k}{c_{ij}^{p_{ij}}} \right),$$

and

$$\delta_{ij} = \sum_{lk \in P_{ij}} \left(\sum_{pq \in A_{lk}} b_{ij}^p r_{pq} \right)^4.$$

The balance constraint between flow pattern, s_{ij} , and

travel cost, c_{ij} , becomes:

$$r_{ij} = \frac{\theta x_j a_{ij} \exp(-\delta_{ij})}{\sum_k \theta x_k a_{ik} \exp(-\delta_{ik})}, \quad (5.3)$$

Equation (5.3) can be simplified to:

$$g_{ij} = r_{ij} = \frac{\phi_{ij}}{\Phi_i}, \quad (5.4)$$

where:

$$\phi_{ij} = \theta x_j a_{ij} \exp(-\delta_{ij}),$$

$$\Phi_i = \sum_j \phi_{ij}.$$

The same technique used in the previous chapters to find the balanced flow pattern between travel cost and the number of trips can also be extended to the case of tree networks. To prove this, observe that:

$$\sum_j r_{ij} = \sum_j \frac{\phi_{ij}}{\Phi_i} = 1,$$

and

$$0 \leq r_{ij} \leq 1, \quad \text{for all } i, j,$$

still hold for tree network. The existence of the solution can be proved by the Contraction Mapping Theory Part I, presented in Chapter 3.

The necessary condition for the uniqueness of equation (5.3) is that:

$$\left| \begin{matrix} D_{r_{ij}} & g_{lk} \end{matrix} \right| \leq \frac{K}{n} < \frac{1}{n},$$

and

$$D_{r_{ij}} g_{lk} = D_{r_{ij}} g_{lk} \frac{1}{\Phi_{lk}} - r_{lk} D_{r_{ij}} \Phi_{lk} \frac{1}{\Phi_{lk}}. \quad (5.5)$$

The first term of the right hand side of (5.5) is:

$$D_{r_{ij}} g_{lk} \frac{1}{\Phi_{lk}} = \begin{cases} 0 & : \text{if } P_{ij} \cap P_{lk} = \text{empty} \\ r_{ij} \sum_{pq \in A} b_{pq}^p \left(\frac{v_{pq}}{c_{pq}^p} \right)^3 & : \text{otherwise} \end{cases} \quad (5.6)$$

where:

v_{pq} = total trips on the link (pq)

$$b_{pq}^p = 0.15 \beta^4 c_{pq}^0 \left(\frac{p}{c_{pq}^p} \right)$$

$$A = \{ (pq) \mid (pq) \in P_{ij} \cap P_{lk} \}$$

It is easy to verify that the solution of equation (5.3) will be unique if the tree network is not highly congested as was the case on chapter 3. Further, the proof of uniqueness for the particular FHWA congestion function presented in theorem 3.4 can also be extended here because the property:

$$\sum_j r_{ij} = \sum_j \frac{\phi_{ij}}{\phi_i} = 1,$$

holds.

5.2. The Log Accessibility Model on a Tree Network

Having proven that a unique system equilibrium flow pattern does exist, the log accessibility model can now be solved. The solution technique developed in the last chapter can be extended to the tree-structured network by modifying step 3 of the solution techniques presented in section 4.2.

The log accessibility model on a tree-structured network is:

max Φ ,

s.t.:

$$\sum_j x_j = 1,$$

$$g_{ij} = r_{ij} = \frac{\phi_{ij}}{\phi_i}, \quad \text{for all } ij,$$

$$x_j \geq 0, \quad \text{for all } j.$$

The Lagrangian function is:

$$L = \Phi + \mu (1 - \sum_j x_j) + \sum_{ij} w_{ij} (g_{ij} - r_{ij}) + \sum_j v_j x_j.$$

It is easy to verify that the necessary condition:

$$D_{x_j} L = 0,$$

has the same form as was developed in Chapter 3. However, the necessary condition:

$$D_{r_{ij}} L = 0,$$

has a different form which requires that:

$$\sum_{lk} w_{lk} D_{r_{ij}} (g_{lk} - r_{lk}) = D_{r_{ij}} \Phi, \quad \text{for all } i,j.$$

This can be rewritten in matrix form as:

$$(A - I) W = B, \quad (5.7)$$

where:

$$A = \begin{bmatrix} D_{r_{11}}^g & D_{r_{11}}^g & \dots & D_{r_{11}}^g \\ D_{r_{11}}^g & D_{r_{11}}^g & \dots & D_{r_{11}}^g \\ \vdots & \vdots & \ddots & \vdots \\ D_{r_{mn}}^g & D_{r_{mn}}^g & \dots & D_{r_{mn}}^g \end{bmatrix}$$

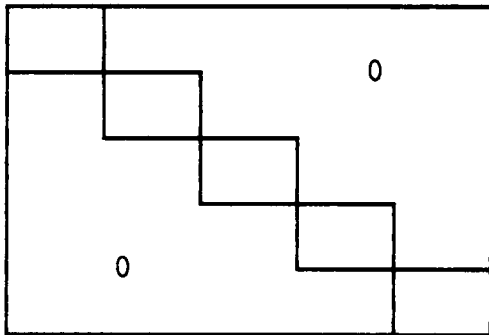
$$B = \begin{bmatrix} D_{r_{11}}^{\Phi} \\ D_{r_{11}}^{\Phi} \\ \vdots \\ D_{r_{mn}}^{\Phi} \\ D_{r_{mn}}^{\Phi} \end{bmatrix} \quad W = \begin{bmatrix} w_{11} \\ w_{12} \\ \vdots \\ w_{mn} \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & & 1 \\ & & & & & & 1 \\ & 0 & & & & & & 1 \end{bmatrix}$$

It is worth noting that in the bipartite network, all paths are independent and no links can be shared. This means:

$$D_{r_{ij}} g_{lk} = 0 \quad \text{if } i \neq l$$

Therefore, matrix A in the bipartite network has the following form:



The block form in matrix A indicates that the system of equations (5.7) can be decomposed into n separate problems. This is why the system equations to find the value of w_{ij} in a bipartite network can be decomposed as an

n system equations. In the tree network problem, however, the matrix A cannot be decomposed. Therefore, a large matrix must be maintained in the iterative process. The size of this matrix will be $(m \cdot n) \times (m \cdot n)$. This means that a 100×100 matrix is needed for a 10 node network problem.

The proof of feasibility and convenience of this algorithm is straightforward. In the bipartite network discussed in previous chapter, the LAMM with an implicit function form has only one set of decision variables, x_j .

The transportation cost, c_{ij} , and number of trips, s_{ij} , are unique with respect to any set of x_j . Since this property still holds on the tree-structured network, the proof of feasibility and convenience is the same here, mutatis mutandis.

5.3. Numerical Results

The tree network presented at Figure 8 is chosen for numerical analysis. Table 4 lists the information for each link. For the purpose of simplicity, the population at each node is set to 10 and the construction cost are equal in each node. All travel costs on external links are set to 10 and all internal links are set to 5. The capacities in each direction on every link are set to a large number, 100, except for each direction between nodes 3 and 4. These are set to 10, a low capacity. The numerical solution with different parameter values (θ and β) are presented at Table 5. For each set of parameter values, the problem is first solved with all capacities set to infinity. Then the problem is resolved with the capacities presented in Table 4.

In case 1, the value of θ is large, 2.0, indicating that the attractiveness economies of scale parameter is larger than the construction cost economies of scale parameter. This clearly favors building large facilities.

Table 4

Information for Each Link on a Tree Network

Nodes From To		Travel Cost	Capacity	Nodes From To		Travel Cost	Capacity
1	1	5	100	4	3	10	10
1	3	10	100	4	4	5	100
2	2	5	100	4	5	10	100
2	3	10	100	4	6	10	100
3	1	10	100	5	4	10	100
3	2	10	100	5	5	5	100
3	3	5	100	6	4	10	100
3	4	10	10	6	6	5	100

Table 5

The Optimal Solutions on Uncapacitated and Capacitated
Tree-Structured Networks

	β	θ	1	2	3	4	5	6	network type
C	1	.01 2.0				1.00			uncapacitated
						1.00			capacitated
A	2	.01 1.4				1.00			uncapacitated
					.50	.50			capacitated
S	3	.20 1.0			.50	.50			uncapacitated
			.01	.01	.48	.49	.01		capacitated
E	4	.10 0.9	.02	.02	.46	.46	.02	.02	uncapacitated
			.03	.04	.42	.46	.03	.02	capacitated
E	5	.01 0.9	.13	.13	.24	.24	.13	.13	uncapacitated
			.13	.16	.18	.26	.14	.13	capacitated
E	6	.10 0.5	.13	.13	.25	.25	.13	.13	uncapacitated
			.13	.13	.24	.25	.13	.12	capacitated
E	7	.01 0.5	.16	.16	.18	.18	.16	.16	uncapacitated
			.16	.17	.16	.19	.16	.16	capacitated

(The numbers are the percent of the total budget)

The optimal solution in uncapacitated network is to build a large facility either at location 3 or location 4, which are the centers of the network. In the capacitated case, β is small, 0.01, and the congestion raised on the capacitated links (3,4) and (4,3) does not change the solution set.

In case 2, the value of θ is decreased to 1.4. The attractiveness for a large facility is not as strong as in case 1, but it still plays a major role in the decision making process. The optimal solution in an uncapacitated network is the same as in case 1; build a large facility at location 3 or 4. However, the congestion on the links between 3 and 4 does change the optimal solution. The optimal solution on the capacitated network is to split the budget by half and build two equal sized facilities at nodes 3 and 4.

In case 3, θ equals 1 and β increases to 0.2. Congestion becomes an important factor. The solution of the uncapacitated network is to build two facilities of the same size at location 3 and 4. On the capacitated network, the solution is to decrease the facility sizes both at locations 3 and 4, and build small facilities at locations 1, 2, and 5.

In cases 4 and 6, θ is less than 1 indicating that the construction cost for a large facility is higher than the

attractiveness it can generate. The optimal solution on an uncapacitated network is to build two facilities at location 3 and 4 and a small facility on each other site. The solutions on a capacitated network is to decrease the size of location 3 and increase the size of location 2. The reason is that a facility at location 3 will attract trips from locations 4, 5, and 6. These trips will travel on the highly capacitated link between 3 and 4. The congestion raised on this link decreases accessibility. The facility at location 2 is far enough away from locations 4, 5, and 6 to avoid this problem.

Cases 5 and 7 are more interesting than the others. The facility size at location 4 is actually increased in the capacitated network case. This increases the congestion on the capacitated links. However, since the β value is relatively low, people are willing to travel to this facility. In case 7, the optimal solution on the capacitated network has a hierarchical structure. A large facility at location 4 as the center of the network, a second large facility at location 2 as the local center and a small facility at each other location.

In general, the solution on the uncapacitated network is symmetric; half of the budget is spent at node 1, 2, 3 (west part of the network) and half is spent at nodes 4, 5, and 6 (east part of the network). The solution on the

capacitated network is quite different. The budget is still split in half for the east and west parts, but the budget distribution at each site is no longer symmetric. In the east part of the network, location 4 receives most of the budget while locations 5 and 6 receive small facilities. In the west part, locations 2 and 3 receive roughly same amount of investment and act as two local centers.

5.4. Concluding Remarks

The tree-structured network presents a clear and simple case for the study of shared capacity. Most previous location-allocation studies assume that people travel along the shortest path despite other alternative routes [Francis and White 1974; Handler and Mirchandani; Tansel, et al. 1983a,b]. That is, the network is treated as an uncapacitated tree. The same solution technique for a bipartite network is extended here to the case of a tree structure network by a modifying the solution procedure of finding the dual variables w_{ij} . All the solution techniques

and theoretical results developed in Chapters 3 and 4 for the bipartite network are extended to the case of the tree structure network. The numerical solutions show that when the sample network is uncapacitated the budget is uniformly divided between its east and west halves. However, if the

network is capacitated, some solutions show a hierarchical structure. Different size facilities would be built at various locations so that accessibility is maximized on the congested network.

CHAPTER 6

CONCLUSIONS

6.1. Conclusions

The primary purposes of this research have been to consider the objective functions most commonly used in social welfare approaches to location-allocation modeling, and to study their application to capacitated networks. It has been proven that location-allocation models which maximize social welfare--Consumer Surplus Maximization Model, Users' Benefit Maximization Model, and Log Accessibility Maximization Model--generate the same solutions. This research has shown that the classic entropy maximization model is equivalent to those social welfare maximization models when the facility sizes are fixed. If facility sizes vary, the equivalence between the various social welfare functions and a modified entropy maximization location-allocation model was proven. The Modified Entropy Maximization Model converts the classic entropy model into a location-allocation model which determines facility sizes, locations, and network flow pattern. It also can be used to connect other models, which do not consider social welfare, such as Fotheringham's [1985], to the family of social welfare maximization models.

Since these four models, CSMM, UBMM, LAMM, and MEMM are equivalent, any one of them can be chosen to study location-allocation modeling on uncapacitated and capacitated networks. This research uses the Log Accessibility Maximization Model because it has the simplest form and it is the easiest to solve.

The unique balance point between the travel cost and travel pattern proven in this research indicates that the LAMM on a capacitated network can be converted, via implicit function theory, to a non-linear objective function with one linear constraint. This transformation simplifies the method of finding solutions. Indeed, a four step solution procedure was developed for this model.

The theoretical results and solution technique developed in this research can be used on bipartite and tree networks. Since many variables are involved in the LAMM model--population, facility size, facility location, construction cost, budget, economies of scale on attractiveness and construction cost, link capacity, transportation cost, flow pattern, and network structure--it is very difficult to obtain generalized results. In the numerical results presented, most variables are set equal in order to study the effect of congestion on the location-allocation decision process.

6.2. Further Studies

Three extensions of this research are: the LAMM on a general capacitated non-bipartite network with trip assignment model; elastic demand of LAMM on capacitated networks; and investments in facilities and links.

6.2.1. The Log Accessibility Model on a General Capacitated Non-Bipartite Network

The log accessibility model on a general non-bipartite network is more complex than on a tree or bipartite network. This section presents the model formulation and discusses the difficulty in solving this problem.

The log accessibility model on a general non-bipartite network can be presented as:

$$\max \quad \Phi, \quad (6.1)$$

s.t.:

$$\sum_j x_j = 1 \quad (6.2)$$

$$s_{ij} = \frac{\phi_{ij}}{\phi_i} \quad (6.3)$$

$$c_{ij} = \eta(v^*) \quad (6.4)$$

where:

x_j , s_{ij} , ϕ_{ij} , ϕ_i are defined as (5.4),

$\eta(\cdot)$ is the congestion function,

v^* : the optimal flow pattern which satisfied the network equilibrium model [LeBlanc et al., 1975].

Since there is only one path from any origin to any destination in the bipartite or the tree network, the trip assignment problem distributes all users to unique routes. In a general non-bipartite network there is more than one path between any pair of origins and destinations. This is discussed by LeBlanc et al. [1975] as a network equilibrium model.

Assuming that the number of trips from i to j , s_{ij} , is known, the network equilibrium model which assign flows to paths is:

$$\max_{\{x\}} \sum_{ij \in A} \int_0^{v_{ij}} \eta(x) dx, \quad (6.5)$$

s.t.:

$$\sum_{j \in A} v_{ji} - \sum_{i \in A} v_{ij} = \begin{cases} D_j & \text{if } j \text{ is source,} \\ 0 & \text{otherwise,} \\ -D_j & \text{if } j \text{ is sink,} \end{cases} \quad (6.6)$$

where:

v_{ij} : number of trips on link ij .

This model has a non-linear objective function and linear constraints. LeBlanc et al. [1975] present a very efficient algorithm to solve this problem for a large network.

In the general non-bipartite network, a direct derivation of the first order condition for optimality is very difficult, because the property of (6.4). The existence and uniqueness of the system equilibrium of flow pattern (6.3) and travel cost (6.4) is also difficult to prove. Even if these would be proven, it is not clear that a similar algorithm to that developed in Chapter 4 could be derived. Nor does it follow that such an algorithm would be feasible and convenient.

There is a good deal of literature on ways to induce flow pattern equilibrium on a capacitated network [Putman, 1983]. However, the proofs and methods chosen all rest on the assumption that the desired number of trips between any two nodes is known and fixed. In the location-allocation context, this assumption cannot be made. Theoretically speaking, any step of an iterative program to find a balanced flow pattern will induce a new set of location investments. In short, we are faced with an optimization

problem where one of the constraints is itself an optimization problem. A rigorous treatment of this topic must wait further research. The solution technique developed in Chapter 4 cannot be applied in this case without suitable modification.

6.2.2. Elastic Demand in Location-Allocation Models

The proof that all budgets should be spent results from the assumption that demand is fixed. That is, no matter how much of the budget is spent, the total number of users in the society is always the same. If elastic demand is introduced into the model, then the budget expenditure conditions change. In this situation, it is not necessary to spend the entire budget because a larger facility will increase attraction but also heighten congestion. Some people may, therefore, choose to stay home. The LAMM with elastic demand thus becomes:

$$\max \Phi = \sum_i e_i P_i \ln \left(\sum_j w_j^\alpha \exp(-\beta c_{ij}) \right),$$

s.t.:

$$\sum_j \Gamma_j w_j^\tau \leq B,$$

$$s_{ij} = \frac{e_i P_i w_j^\alpha \exp(-\beta c_{ij})}{\sum_j w_j^\alpha \exp(-\beta c_{ij})},$$

$$c_{ij} = \eta(s),$$

$$s, c, w \geq 0,$$

where:

P_i : the potential users at location i ,

e_i : the proportion of users generated from the potential users,

others are the same as before.

Since all the budget is not necessarily spent, the solution technique needs to be reformulated.

6.2.3. Investment at Facilities and Links

In this research, the budget is assumed to be spent exclusively on facilities. On the other hand, it may also be possible to spend the budget on the links leading to the facilities and increase the capacity of congested links. Since the LAMM on a capacitated network can determine the facility size and flow pattern, it is very easy to locate the congested links. These links can be improved to increase the accessibility to the facility. However, these two processes, investment in facilities and links, should

be considered simultaneously. The general formulation for this problem is as follows:

max Φ ,

s.t.:

$$b_1(w) + b_2(\delta) \leq B,$$

$$s = f(w, c),$$

$$c = g(s, \delta),$$

$$w, s, \delta, c \geq 0,$$

where:

w : facility size,

δ : improved link,

s : flow pattern on the network,

c : travel cost for each link,

$b_1(w)$: budget spend on the facilities,

$b_2(\delta)$: budget spend on the links,

B : total budget.

6.3. Final Comments

A spatial theory of social welfare is necessary if we are to better understand the effects of investment decisions on the economic landscape. In this research an attempt has been made to unite the work of others on social

welfare (most notably Wilson, Leonardi, and Coelho), and to extend the various social welfare approaches in location-allocation modeling to more realistic settings. The proof of the existence and uniqueness of solutions on bipartite and tree networks is another step in theory development. The derivation of a solution algorithm and the presentation of numerical results should lead to a better understanding of how congestion effects on travel behavior should affect investment decisions. However, as the problems presented in this chapter indicate, there are still many more interesting questions in the development of a spatial theory of social welfare to be addressed.

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