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I am submitting herewith a dissertation written by Rong-chang Chiang entitled "Generalization of Jacobi's Decomposition Theorem to the Rotation and Translation of a Solid in a Fluid". I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

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**Generalization of Jacobi's
Decomposition Theorem to the Rotation
and Translation of a Solid in a Fluid**

A Dissertation
presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Rong-chang Chiang

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Abstract

Jacobi found that the rotation of a symmetrical heavy top about a fixed point is composed of the two torque-free rotations of two triaxial bodies about their centers of mass. His discovery rests on the fact that the orthogonal matrix which represents the rotation of a symmetrical heavy top is decomposed into a product of two orthogonal matrices, each of which represents the torque-free rotations of two triaxial bodies. This theorem is generalized to the Kirchhoff's case of the rotation and translation of a symmetrical solid in a fluid.

This theorem requires the explicit computation, by means of theta functions, of the nine direction cosines between the rotating body axes and the fixed space axes. The addition theorem of theta functions makes it possible to decompose the rotational matrix into a product of similar matrices. This basic idea of utilizing the addition theorem is simple but the carry-through of the computation is quite involved and the full proof turns out to be a lengthy process of computing rather long and complex expressions.

For the translational motion we give a new treatment. The position of the center of mass as a function of the time is found by a direct evaluation of the elliptic integral by means of a new theta interpretation of Legendre's reduction formula of the elliptic integral.

For the complete solution of the problem we have added further the study of the physical aspects of the motion. Based on a complete examination of the all possible manifolds of the steady helical cases it is possible to obtain a full qualitative description of the motion. Many numerical examples and graphs are given to illustrate the rotation and translation of the solid in a fluid.

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Chapter 1

Introduction

The rotation of a rigid body about a fixed point has three degrees of freedom, and three independent variables are enough to describe the rotation, for example, the three Eulerian angles. Once the Eulerian angles as functions of time are found, the further computation of the orthogonal matrix whose nine elements are the direction cosines of the rotating body axes with respect to the fixed space axes, may seem to be an unnecessary tedious work without yielding any further new information. However, Jacobi has demonstrated that the orthogonal matrix M , which describes the motion of a heavy symmetrical top can be decomposed into a product of two orthogonal matrices P and Q , each of which represents the motion of a torque-free triaxial top[1] [2] [3]. Jacobi's decomposition theorem can only be found in his posthumous paper, and monographs only mention it[2], probably due to the extensive use of elliptic functions and theta functions. This paper is

concerned with the generalization of Jacobi's theorem to the Kirchhoff's case of a symmetrical solid rotating and translating in a fluid[4] [5] [6].

Only three cases of the rotation of a rigid body about a fixed point are solvable: the motion of a torque-free triaxial top, the motion of a symmetrical top about a fixed point under a uniform gravitational field, and the motion of the Kowalevski's top. The first one is known as Euler's case, the second one is known as Lagrange's case. In the posthumous paper Jacobi applied elliptic functions of his own invention to the problem of the rotation of a symmetrical top about a fixed point in a uniform gravitational field. When Jacobi went beyond the solution of the three angles as elliptic functions of the time and computed all the nine direction cosines between the moving body axes and the fixed reference axes, he discovered the decomposition theorem. The Kowalevski's top[7] [8] is the most involved and requires hyperelliptic functions and theta functions of two variables.

First in chapter two we introduce Jacobi's decomposition theorem for a symmetric top in uniform gravity. The main text follows Jacobi, but we present an alternative, more natural way of introducing the parameters in appendix A. Parameters defined in these two different ways are related to each other, and the sum of the parameter defined in Jacobi's way and the parameter defined in the alternative way is one half imaginary period of the elliptic integral. Appendix A gives the detailed discussions.

In chapter three we derive the solution for the Kirchhoff's case of the translation and rotation of a symmetrical solid in a fluid. We have added the study

of the physical aspects of the motion. The translational movement of the center of mass of a symmetrical solid in a fluid is a spiral curve between two surfaces of two concentric cylinders of different radii. Its forward position increases with time but is modulated with periodic zeta functions. Its projection on the perpendicular plane shows various cyclic patterns of movements. At the same time, the body rotates in a progressive or retrogressive precession with its symmetrical axis nutating between two inclination angles. A special case arises when the two limits of nutation merge into one, and its trajectory becomes a helix. The forward speed as well as the precession and the spinning speed become constant. It is interesting to note that the helical orbit is the same as the motion of a charged particle in a helical spiral in a uniform magnetic field. With respect to the advancing direction, the helical orbit can be right or left handed. And the constant inclination angle, constant precession and spinning speeds in the rotation are similar to the steady precession of a symmetric top in a field of uniform gravity. Our special case is sort of the combination of these two motions described above. The solid is neither charged nor is there any magnetic field. But the interaction of the motion of the solid with the induced flow of the surrounding fluid provide the necessary forces of constraint to make the center of mass move in a helical spiral, while the solid also simultaneously performs a steady precession. Whether the helix is right or left handed depends on the longitudinal mass, the transversal mass and the inclination angle. We have made a complete examination of all the possible manifolds for the steady helical cases and present the results in Appendix B. We also studied

the general case in which the symmetry axis nutates between two distinct limits and the results are described in Appendix C.

Finally in chapter four we generalize Jacobi's decomposition theorem to Kirchhoff's case. We find that the matrix R for the rotation of Kirchhoff's case can be decomposed into a product of two matrices S and T , each of which represents the motion of a torque-free triaxial body

$$R = S\tilde{T},$$

where $\tilde{T}$ is the transpose of T . Let us consider the motion R of the symmetrical solid in the fluid and the motion of a free triaxial top S , both of which are described with respect to a frame of reference moving with a certain constant precession or spinning rate in space. Then the motion of the torque-free triaxial top observed from the reference frame is described by the orthogonal matrix S and the same motion when observed from the frame of the symmetrical solid in the fluid (i.e. the observer moving with the symmetrical solid) is described by the orthogonal matrix T . The relative motion T turns out to be also representing the motion of a torque-free triaxial top.

Chapter five is a conclusion.

Chapter 2

Jacobi's Decomposition Theorem for the Motion of a Heavy Symmetrical Top

We shall first derive the rotational matrix M for a symmetrical heavy top in section one and similar but simpler matrices P, Q for torque-free rotations of triaxial bodies in section two. Jacobi's decomposition theorem $M = P\tilde{Q}$ will then be treated in detail in section three. The exposition is designed to facilitate the later generalization of the theorem to the case of the rotation and the translation of a solid in a fluid.

2.1 The Motion of a Symmetrical Top About a Fixed Point in a Uniform Gravitational Field

We denote the principal moments of inertia of a rigid body by (I_1, I_2, I_3) . For the motion of a symmetrical top $(I_1 = I_2)$ about a fixed point under the action of the uniform gravitational force, we have[3]

$$J = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2}I_3\omega_3^2 + mgl \cos \theta, \quad (2.1)$$

where J is the total energy, θ , ϕ and ψ are Eulerian angles. ω_3 is the angular velocity component along the axis of symmetry. m is the total mass and l is the length between the center of gravity and the fixed point. Figure 2.1 shows Eulerian angles. The Lagrangian is

$$L = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi} \cos \theta)^2 - mgl \cos \theta.$$

It will be noticed that ϕ and ψ do not appear explicitly in the lagrangian; they are therefore cyclic coordinates, indicating that the corresponding generalized momenta are constant in time. Since the momentum conjugate to a rotation angle is the component of the total angular momentum along the axis of rotation, which for ϕ is the vertical axis, and for ψ , the symmetrical axis in the body. We get two constants of the motion

$$P_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi} \cos \theta) = I_3\omega_3 = I_1 c_1$$

and

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \sin^2 \theta + I_3 \cos^2 \theta) \dot{\phi} + I_3 \dot{\psi} \cos \theta = I_1 c_2,$$

where c_1 and c_2 are two constants. We can also get some useful expressions

$$\dot{\phi} = \frac{c_2 - c_1 \cos \theta}{\sin^2 \theta},$$

$$\dot{\psi} = \frac{I_1}{I_3} c_1 - \cos \theta \frac{c_2 - c_1 \cos \theta}{\sin^2 \theta}$$

and

$$\dot{\psi} = c_1 \left(\frac{I_1}{I_3} - 1 \right) - \frac{c_2 \cos \theta - c_1}{\sin^2 \theta}.$$

We rewrite equation 2.1 as

$$J' = J - \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + mgl \cos \theta. \quad (2.2)$$

Now we introduce

$$v = \left(\frac{2mgl}{I_1} \right)^{1/2},$$

where v has the dimension of an angular velocity. From now on, we will always divide any angular velocity by v to make it dimensionless. In terms of v , the three dimensionless physical constants of the motion of a heavy top will be

$$\alpha' = J'/mgl,$$

c_1 = the angular momentum component along the axis of symmetry divided by vI_1 ,

c_2 = the angular momentum component along the space Z-axis (antiparallel to the gravity) divided by vI_1 .

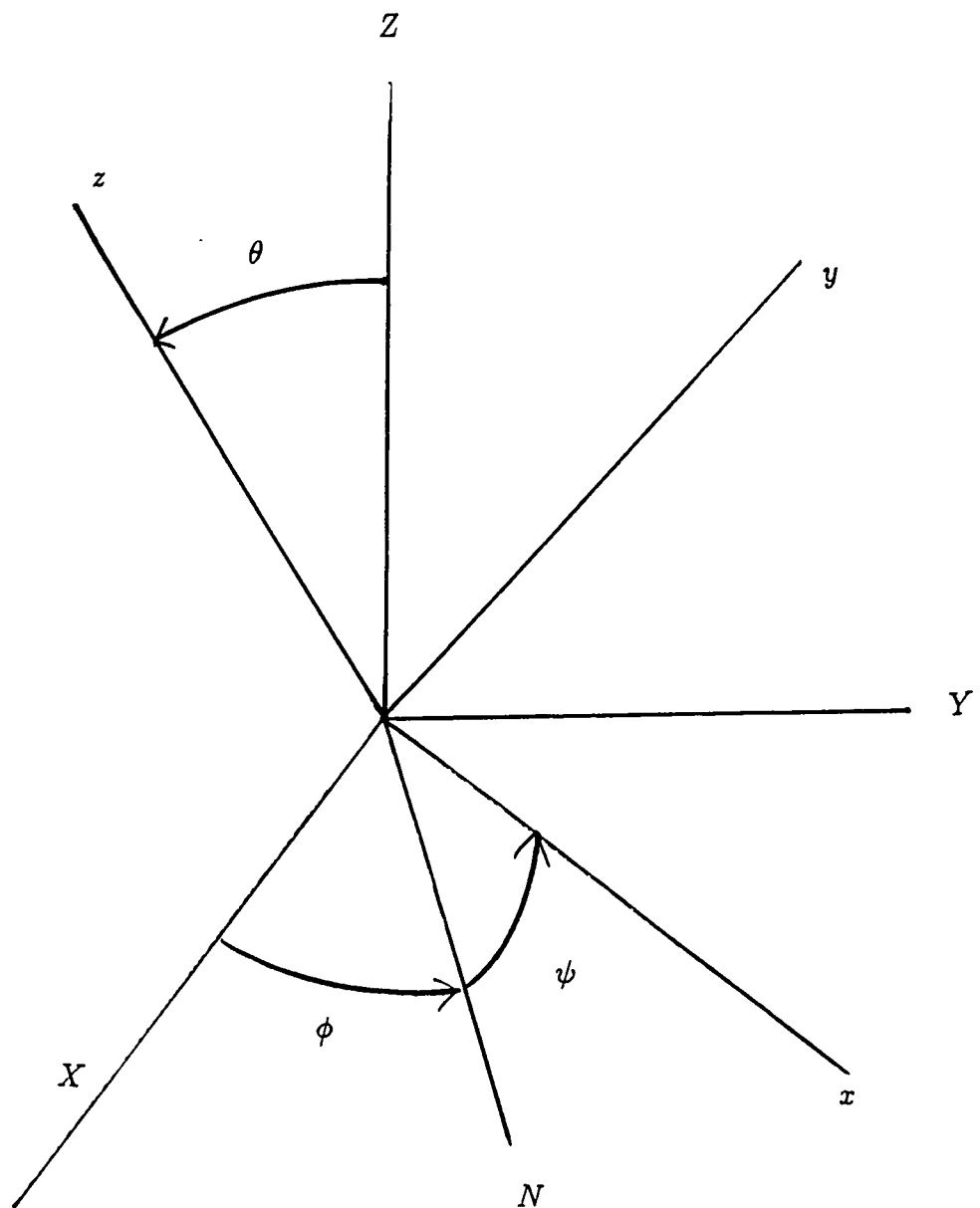


Figure 2.1: Eulerian angles. XYZ are space axes, xyz are body axes. N represents the nodal line.

The expression for the dimensionless angular velocity $\dot{\theta}$ of the nutation is given by

$$\dot{\theta}^2 \sin^2 \theta = (1 - \cos^2 \theta)(\alpha' - \cos \theta) - (c_2 - c_1 \cos \theta)^2.$$

If we set $x = \cos \theta$, then the above equation becomes a cubic equation in x

$$\begin{aligned} \dot{x}^2 &= (1 - x^2)(\alpha' - x) - (c_2 - c_1 x)^2 \\ &= (x - x_1)(x - x_2)(x - x_3) \\ &= f(x), \end{aligned} \tag{2.3}$$

where x_1, x_2, x_3 are the roots of the cubic equation $f(x) = 0$. It can be shown that the three roots are all real and thus supposed to satisfy the following relation

$$-1 < x_3 < x_2 < 1 < x_1.$$

The physical range of x is

$$x_3 < x < x_2.$$

Some useful relations can be gotten from equation 2.3. Let $x = 1$, then -1 ; we get two relations

$$-(c_2 - c_1)^2 = (1 - x_1)(1 - x_2)(1 - x_3),$$

and

$$-(c_2 + c_1)^2 = -(1 + x_1)(1 + x_2)(1 + x_3).$$

After taking the square root, they become

$$c_2 - c_1 = \pm[(x_1 - 1)(1 - x_2)(1 - x_3)]^{1/2}$$

and

$$c_2 + c_1 = [(1 + x_1)(1 + x_2)(1 + x_3)]^{1/2}.$$

To find the solution for x in equation 2.3, we set

$$x = x_3 + (x_2 - x_3) \sin^2 \xi.$$

Then it is seen that

$$x_2 - x = (x_2 - x_3) \cos^2 \xi,$$

$$x - x_3 = (x_2 - x_3) \sin^2 \xi,$$

$$x_1 - x = (x_1 - x_3) \left(1 - \frac{x_2 - x_3}{x_1 - x_2} \sin^2 \xi\right),$$

$$1 - x = (1 - x_3) \left(1 - \frac{x_2 - x_3}{1 - x_3} \sin^2 \xi\right),$$

$$1 + x = (1 + x_3) \left(1 + \frac{x_2 - x_3}{1 + x_3} \sin^2 \xi\right)$$

and

$$dx = 2(x_2 - x_3) \sin \xi \cos \xi d\xi.$$

From these equations, we get

$$\dot{x} = \sqrt{f(x)}$$

and

$$dt = \frac{dx}{\sqrt{f(x)}} = \frac{2d\xi}{\sqrt{x_1 - x_3} \Delta \xi},$$

where

$$\Delta^2 \xi = 1 - k^2 \sin^2 \xi$$

with

$$k^2 = \frac{x_2 - x_3}{x_1 - x_3}.$$

And then we set

$$du = \frac{d\xi}{\Delta\xi},$$

to get

$$u = \int \frac{d\xi}{\sqrt{1 - k^2 \sin^2 \xi}},$$

where the integral is an elliptic integral. Then the inverse of the above integral is

$\xi = am u$, and the differential of t becomes

$$dt = \frac{2}{\sqrt{x_1 - x_3}} du.$$

From these, we have the relation

$$u = \frac{1}{2} \sqrt{x_1 - x_3} t.$$

Here the solution for x is

$$x = x_3 + (x_2 - x_3) \sin^2 am u,$$

or in terms of elliptic functions[9]

$$x = x_3 + (x_2 - x_3) \operatorname{sn}^2 u.$$

To find other direction cosines, we need two other Eulerian angles which can be found by the following equations

$$\dot{\phi} = \frac{c_2 - c_1 x}{1 - x^2},$$

$$\dot{\psi} = c_1 \left(\frac{I_1}{I_3} - 1 \right) - \frac{c_2 x - c_1}{1 - x^2}.$$

By integration, we can find out ϕ and ψ . For convenience, let us define

$$\psi_1 = - \int \frac{c_2 x - c_1}{1 - x^2} dt.$$

Then we get

$$\phi - \psi_1 = \int \frac{c_2 - c_1}{1 - x} dt$$

and

$$\phi + \psi_1 = \int \frac{c_2 + c_1}{1 + x} dt.$$

From these, we have

$$\phi = \frac{1}{2} \left[\int \frac{c_2 - c_1}{1 - x} dt + \int \frac{c_2 + c_1}{1 + x} dt \right]$$

and

$$\psi_1 = -\frac{1}{2} \left[\int \frac{c_2 - c_1}{1 - x} dt - \int \frac{c_2 + c_1}{1 + x} dt \right].$$

To integrate the integral

$$\int \frac{c_2 - c_1}{1 - x} dt = (c_2 - c_1) \int \frac{2(x_1 - x_3)^{-1/2} du}{(1 - x_3)[1 - (x_2 - x_3)(1 - x_3)^{-1} \text{sn}^2 u]}, \quad (2.4)$$

Jacobi introduced two parameters a and b , which are expressed in terms of the three roots through the elliptic function sn

$$\text{sn}^2(ia) = -\frac{x_1 - 1}{1 - x_2}, \quad \text{sn}^2(ib) = -\frac{x_1 - x_3}{1 + x_3}. \quad (2.5)$$

Here we follow Jacobi in the introduction of the two parameters. But it seems to be more natural to introduce two parameters such that $x = 1$ and $x = -1$ which

correspond to the poles of the two integrals $(1 \pm x)^{-1}$. We give this alternative route in the appendix A and prove that Jacobi's ingenious introduction of the two parameters by equation 2.5 is essentially equivalent to these alternative parameters which correspond to the poles.

The periodicity of the elliptic function allows to choose a or b less than K' . Some other elliptic functions are expressed in terms of Jacobi's parameters in the following

$$cn^2(ia) = \frac{x_1 - x_2}{1 - x_2}, \quad cn^2(ib) = \frac{1 + x_1}{1 + x_3}$$

and

$$dn^2(ia) = \frac{(x_1 - x_2)(1 - x_3)}{(x_1 - x_3)(1 - x_2)}, \quad dn^2(ib) = \frac{1 + x_2}{1 + x_3}.$$

From above expressions, we have

$$\frac{x_2 - x_3}{1 - x_3} = \frac{k^2 cn^2(ia)}{dn^2(ia)} = \frac{k^2}{dn^2(a, k')},$$

$$\frac{x_2 - x_3}{1 + x_3} = -k^2 sn^2(ib) = k^2 tn^2(b, k'),$$

$$1 - x_3 = -\frac{2sn^2(ib)dn^2(ia)}{cn^2(ia) - sn^2(ib)dn^2(ia)}$$

and

$$1 + x_3 = \frac{2cn^2(ia)}{cn^2(ia) - sn^2(ib)dn^2(ia)}.$$

Then the integral 2.4 becomes[10]

$$\begin{aligned} \int \frac{c_2 - c_1}{1 - x} dt &= (c_2 - c_1) \int \frac{2(x_1 - x_3)^{-1/2} du}{(1 - x_3)[1 - (x_2 - x_3)(1 - x_3)^{-1} sn^2 u]} \\ &= 2 \int \frac{\sqrt{(x_1 - 1)(1 - x_2)}}{(x_1 - x_3)(1 - x_3)} \frac{dn^2(ia) du}{dn^2 u - k^2 sn^2(ia) cn^2 u} \end{aligned}$$

$$\begin{aligned}
&= 2 \int \frac{k'^2 tn(ia) dn(ia) du}{i[dn^2 u - k^2 sn^2(ia) cn^2 u]} \\
&= 2 \left[\frac{d \ln H_1(ia)}{da} u - \frac{1}{2i} \ln \frac{\Theta_1(u - ia)}{\Theta_1(u + ia)} \right],
\end{aligned}$$

where

$$\frac{k'^2 tn(ia)}{i dn(ia)} = \frac{\sqrt{(x_1 - 1)(1 - x_2)}}{\sqrt{(x_1 - x_3)(1 - x_3)}}$$

has been used. Similarly, the other integral is

$$\begin{aligned}
\int \frac{c_2 + c_1}{1 + x} dt &= (c_2 + c_1) \int \frac{2(x_1 - x_3)^{-1/2} du}{(1 + x_3)[1 + (x_2 - x_3)(1 + x_3)^{-1} sn^2 u]} \\
&= 2 \int \frac{\sqrt{(1 + x_1)(1 + x_2)}}{(x_1 - x_3)(1 + x_3)} \frac{du}{1 - k^2 sn^2(ib) sn^2 u} \\
&= 2 \int \frac{ictn(ib) dn(ib) du}{1 - k^2 sn^2(ib) sn^2 u} \\
&= 2 \left[\frac{d \ln H(ib)}{db} u + \frac{1}{2i} \ln \frac{\Theta(u + ib)}{\Theta(u - ib)} \right],
\end{aligned}$$

where the following relation has been used

$$ictn(ib) dn(ib) = i \frac{cn(ib)}{sn(ib)} dn(ib) = \sqrt{\frac{(1 + x_1)(1 + x_2)}{(x_1 - x_3)(1 + x_3)}}.$$

Now we find that

$$\phi = \left[\frac{d \ln H_1(ia)}{da} + \frac{d \ln H(ib)}{db} \right] u - \frac{1}{2i} \ln \frac{\Theta_1(u - ia)}{\Theta_1(u + ia)} + \frac{1}{2i} \ln \frac{\Theta(u + ib)}{\Theta(u - ib)}$$

and

$$\psi_1 = - \left[\frac{d \ln H_1(ia)}{da} - \frac{d \ln H(ib)}{db} \right] u + \frac{1}{2i} \ln \frac{\Theta_1(u - ia)}{\Theta_1(u + ia)} + \frac{1}{2i} \ln \frac{\Theta(u + ib)}{\Theta(u - ib)}.$$

If we only consider the periodic part of ϕ and ψ and denote them as ϕ' and ψ' ,

and choose the negative sign and put π as the initial angle, then we have

$$\phi' = \pi + \frac{1}{2i} \ln \frac{\Theta_1(u - ia)}{\Theta_1(u + ia)} - \frac{1}{2i} \ln \frac{\Theta(u + ib)}{\Theta(u - ib)}$$

and

$$\psi' = \pi - \frac{1}{2i} \ln \frac{\Theta_1(u - ia)}{\Theta_1(u + ia)} - \frac{1}{2i} \ln \frac{\Theta(u + ib)}{\Theta(u - ib)}.$$

After taking the exponential power on both sides of the above two equations, we have

$$e^{i\phi'} = -\sqrt{\frac{\Theta_1(u - ia)\Theta(u - ib)}{\Theta_1(u + ia)\Theta(u + ib)}} = -\sqrt{\frac{\Theta(u + K - ia)\Theta(u - ib)}{\Theta(u - K + ia)\Theta(u + ib)}} \quad (2.6)$$

and

$$e^{i\psi'} = -\sqrt{\frac{\Theta_1(u + ia)\Theta(u - ib)}{\Theta_1(u - ia)\Theta(u + ib)}} = -\sqrt{\frac{\Theta(u - K + ia)\Theta(u - ib)}{\Theta(u + K - ia)\Theta(u + ib)}} \quad (2.7)$$

In terms of these two parameters, we have

$$\begin{aligned} 1 - x &= (1 - x_3) \left(1 - \frac{k^2 cn^2(ia)}{dn^2(ia)} sn^2 u\right) \\ &= (1 - x_3) [1 - k^2 sn^2(K - ia) sn^2 u], \end{aligned}$$

where the following relations have been used

$$\begin{aligned} sn^2(K - ia) &= \frac{cn^2(ia)}{dn^2(ia)}, \\ \frac{x_2 - x_3}{1 - x_3} &= k^2 \frac{cn^2(ia)}{dn^2(ia)} = k^2 sn^2(K - ia). \end{aligned}$$

Similarly, we get

$$1 + x = (1 + x_3) [1 - k^2 sn^2(ib) sn^2 u].$$

Substituting the two expressions

$$1 - x_3 = -\frac{2sn^2(ib)}{sn^2(K - ia) - sn^2(ib)}$$

and

$$1 + x_3 = \frac{2sn^2(K - ia)}{sn^2(K - ia) - sn^2(ib)}.$$

into the expressions for $1 - x$ and $1 + x$, we get

$$1 - x = -\frac{2sn^2(ib)[1 - k^2sn^2(K - ia)sn^2u]}{sn^2(K - ia) - sn^2(ib)}$$

and

$$1 + x = \frac{2sn^2(K - ia)[1 - k^2sn^2(ib)sn^2u]}{sn^2(K - ia) - sn^2(ib)}.$$

Now we will transform the above two equations from elliptic functions into theta functions. Some useful formulas for transformations between elliptic functions and theta functions[11] are

$$\begin{aligned} 1 - k^2sn^2(K - ia)sn^2u &= \frac{\Theta^2(0)\Theta(u + K - ia)\Theta(u - K + ia)}{\Theta^2(K - ia)\Theta^2(u)}, \\ 1 - k^2sn^2(ib)sn^2u &= \frac{\Theta^2(0)\Theta(u + ib)\Theta(u - ib)}{\Theta^2(ib)\Theta^2(u)}, \\ k[sn^2(K - ia) - sn^2(ib)] &= \frac{\Theta^2(0)H(K - ia + ib)H(K - ia - ib)}{\Theta^2(K - ia)\Theta^2(ib)}, \\ ksn^2(ib) &= \frac{H^2(ib)}{\Theta^2(ib)}. \end{aligned}$$

Substituting these formulas into the expressions for $1 - x$ and $1 + x$ to transform them from elliptic functions into theta functions, we get

$$1 - x = -\frac{2H^2(ib)\Theta(u + K - ia)\Theta(u - K + ia)}{H(K - ia + ib)H(K - ia - ib)\Theta^2(u)} \quad (2.8)$$

and

$$1 + x = \frac{2H^2(K - ia)\Theta(u + ib)\Theta(u - ib)}{H(K - ia + ib)H(K - ia - ib)\Theta^2(u)}. \quad (2.9)$$

Adding up the above two equations, we get x which is $\cos \theta$.

$$x = \frac{H_1^2(ia)\Theta(u + ib)\Theta(u - ib) + H^2(ib)\Theta_1(u + ia)\Theta_1(u - ia)}{H_1(ia + ib)H_1(ia - ib)\Theta^2(u)}. \quad (2.10)$$

Multiplying equation 2.8 and 2.9, we get $1 - \cos^2 \theta$ which is $\sin^2 \theta$ and take the square root of it and choose the negative sign to make $\sin \theta$ always positive as the range of θ is from 0 to π

$$\sin \theta = -i \frac{2H(ib)H(K - ia)\sqrt{\Theta(u + K - ia)\Theta(u - K + ia)\Theta(u + ib)\Theta(u - ib)}}{H_1(ia + ib)H_1(ia - ib)\Theta^2(u)}. \quad (2.11)$$

The reason for taking the $-$ sign after taking square root is simply that $\sin \theta$ is positive in the range of θ from 0 to π . Therefore the right hand side has to be positive too. Consider those factors in the right hand side as follows

$$\Theta\left(\frac{2Ku}{\pi}\right) = 1 - 2q \cos 2u + 2q^4 \cos 4u - 2q^9 \cos 6u + \dots,$$

$$H\left(\frac{2Ku}{\pi}\right) = 2q^{1/4} \sin u - 2q^{9/4} \sin 3u + 2q^{25/4} \sin 5u - 2q^{49/4} \sin 7u + \dots,$$

where

$$q = \exp\left\{-\pi \frac{K'}{K}\right\}.$$

The variable u is a function of time t which is real, then u is real too

$$u = \frac{1}{2}\sqrt{x_1 - x_3}t.$$

Since $\Theta(u)$ is real, its square $\Theta^2(u)$ is positive. The product $\Theta(u + ib)\Theta(u - ib)$ is positive because $\Theta(u - ib)$ is the complex conjugate of $\Theta(u + ib)$. Similarly $\Theta(u + K - ia)\Theta(u - K + ia)$ is positive.

$$\Theta(ia) = 1 - q^{1-\lambda}(1 + q^{2\lambda}) + q^{4-2\lambda}(1 + q^{4\lambda}) - \dots,$$

where[10]

$$\lambda = \frac{a}{K'}.$$

Since $\Theta(ia)$ is real, $\Theta^2(ia)$ is positive. By the same reason $\Theta^2(ib)$ is positive also.

$$\begin{aligned} & H_1(ia + ib)H_1(ia - ib) \\ &= \frac{k}{k'}\Theta^{-2}(0)\Theta^2(ia)\Theta^2(ib)[cn^2(ib) - sn^2(ia)dn^2(ib)], \end{aligned}$$

where

$$\begin{aligned} & cn^2(ib) - sn^2(ia)dn^2(ib) \\ &= \frac{1+x_1}{1+x_3} + \frac{x_1-1}{1-x_2} \frac{1+x_2}{1+x_3} \\ &> 0, \end{aligned}$$

$$\Theta(0) = \sqrt{\frac{2k'K}{\pi}} > 0.$$

Hence

$$H_1(ia + ib)H_1(ia - ib) > 0.$$

$$\begin{aligned} H(K - ia) &= H_1(ia) = q^{\frac{1}{4} - \frac{1}{2}\lambda}[(1 + q^\lambda) + q^{2-\lambda}(1 + q^{3\lambda}) + \dots] \\ &> 0. \end{aligned}$$

$$-iH(ib) = q^{\frac{1}{4} - \frac{1}{2}\lambda}[(1 - q^\lambda) - q^{2-\lambda}(1 + q^{3\lambda}) + \dots] > 0.$$

Considering those factors in the expression of $\sin \theta$, it is necessary to choose the $-$ sign after taking square root.

Here we like to make a comment which is useful in chapter three. Although Jacobi introduced two parameters in this way to make the integral integrable, however this is not the only way to do it. For example, we can introduce these

two parameters in the following way which makes the value of $\cos \theta$ equals 1 or -1 when the variable equals to $K - ia$ or ib . From the expression of x , for $x = 1$ and $u = K - ia$, it becomes

$$1 = x_3 + (x_2 - x_3)sn^2(K - ia);$$

similarly, for $x = -1$ and $u = ib$,

$$-1 = x_3 + (x_2 - x_3)sn^2(ib).$$

From the above two equations we get another way to define the two parameters which are eventually integrable for the integral and are equivalent to Jacobi's method

$$sn^2(K - ia) = \frac{1 - x_3}{x_2 - x_3}, \quad sn^2(ib) = \frac{-1 - x_3}{x_2 - x_3}.$$

The orthogonal matrix M for the motion of a heavy top is expressed in terms of the two parameters a and b as well as the variable u . The matrix element M_{33} is x . The other matrix elements can be found from the expressions of Eulerian angles. The Eulerian angles and the direction cosines have the following relations[12]

$$M_{31} = \sin \theta \sin \psi',$$

$$M_{32} = \sin \theta \cos \psi',$$

$$M_{13} = \sin \theta \sin \phi',$$

$$M_{23} = -\sin \theta \cos \phi',$$

$$M_{11} = \cos \psi' \cos \phi' - \cos \theta \sin \phi' \sin \psi',$$

$$M_{12} = -\sin \psi' \cos \phi' - \cos \theta \sin \phi' \cos \psi',$$

$$M_{21} = \cos \psi' \sin \phi' + \cos \theta \cos \phi' \sin \psi',$$

$$M_{22} = -\sin \psi' \sin \phi' + \cos \theta \cos \phi' \cos \psi',$$

$$M_{33} = \cos \theta = z.$$

By substituting the expressions 2.11 and 2.7 into the expression for M_{31}

$$\begin{aligned} M_{31} &= \sin \theta \sin \psi' = \sin \theta \frac{e^{i\psi'} - e^{-i\psi'}}{2i} \\ &= i \frac{2H(ib)H(K-ia)\sqrt{\Theta(u+K-ia)\Theta(u-K+ia)\Theta(u+ib)\Theta(u-ib)}}{H_1(ia+ib)H_1(ia-ib)\Theta^2(u)} \\ &= \frac{1}{2i} \left[\sqrt{\frac{\Theta(u-K+ia)\Theta(u-ib)}{\Theta(u+K-ia)\Theta(u+ib)}} - \sqrt{\frac{\Theta(u+K-ia)\Theta(u+ib)}{\Theta(u-K+ia)\Theta(u-ib)}} \right], \end{aligned}$$

we get

$$M_{31} = -H(ib)H_1(ia) \frac{\Theta_1(u-ia)\Theta(u+ib) - \Theta_1(u+ia)\Theta(u-ib)}{H_1(ia+ib)H_1(ia-ib)\Theta^2(u)}. \quad (2.12)$$

The other matrix elements M_{ij} of M can be found in a similar way,

$$\begin{aligned} M_{32} &= \sin \theta \cos \psi' = \sin \theta \frac{e^{i\psi'} + e^{-i\psi'}}{2} \\ &= i \frac{2H(ib)H(K-ia)\sqrt{\Theta(u+K-ia)\Theta(u-K+ia)\Theta(u+ib)\Theta(u-ib)}}{H_1(ia+ib)H_1(ia-ib)\Theta^2(u)} \\ &= \frac{1}{2} \left[\sqrt{\frac{\Theta(u-K+ia)\Theta(u-ib)}{\Theta(u+K-ia)\Theta(u+ib)}} + \sqrt{\frac{\Theta(u+K-ia)\Theta(u+ib)}{\Theta(u-K+ia)\Theta(u-ib)}} \right], \\ M_{32} &= \frac{i}{D_1} H(ib)H_1(ia) [\Theta_1(u-ia)\Theta(u+ib) + \Theta_1(u+ia)\Theta(u-ib)], \quad (2.13) \end{aligned}$$

$$\begin{aligned} M_{13} &= \sin \theta \sin \phi' = \sin \theta \frac{e^{i\phi'} - e^{-i\phi'}}{2i} \\ &= i \frac{2H(ib)H(K-ia)\sqrt{\Theta(u+K-ia)\Theta(u-K+ia)\Theta(u+ib)\Theta(u-ib)}}{H_1(ia+ib)H_1(ia-ib)\Theta^2(u)} \\ &= \frac{1}{2i} \left[\sqrt{\frac{\Theta(u+K-ia)\Theta(u-ib)}{\Theta(u-K+ia)\Theta(u+ib)}} - \sqrt{\frac{\Theta(u-K+ia)\Theta(u+ib)}{\Theta(u+K-ia)\Theta(u-ib)}} \right], \end{aligned}$$

$$M_{13} = \frac{-1}{D_1} H(ib) H_1(ia) [\Theta_1(u+ia)\Theta(u+ib) - \Theta_1(u-ia)\Theta(u-ib)], \quad (2.14)$$

$$\begin{aligned} M_{23} &= -\sin\theta \cos\phi' = -\sin\theta \frac{e^{i\phi'} + e^{-i\phi'}}{2} \\ &= -i \frac{2H(ib)H(K-ia)\sqrt{\Theta(u+K-ia)\Theta(u-K+ia)\Theta(u+ib)\Theta(u-ib)}}{H_1(ia+ib)H_1(ia-ib)\Theta^2(u)} \\ &\quad \frac{1}{2} \left[\sqrt{\frac{\Theta(u+K-ia)\Theta(u-ib)}{\Theta(u-K+ia)\Theta(u+ib)}} + \sqrt{\frac{\Theta(u-K+ia)\Theta(u+ib)}{\Theta(u+K-ia)\Theta(u-ib)}} \right], \end{aligned}$$

$$M_{23} = \frac{-i}{D_1} H(ib) H_1(ia) [\Theta_1(u+ia)\Theta(u+ib) + \Theta_1(u-ia)\Theta(u-ib)], \quad (2.15)$$

$$\begin{aligned} M_{11} &= \frac{1}{4}(1+\cos\theta)[e^{i(\psi'+\phi')} + e^{-i(\psi'+\phi')}] + \frac{1}{4}(1-\cos\theta)[e^{i(\psi'-\phi')} + e^{-i(\psi'-\phi')}] \\ &= \frac{1}{4} \frac{2H^2(K-ia)\Theta(u+ib)\Theta(u-ib)}{H(K-ia+ib)H(K-ia-ib)\Theta^2(u)} \left[\frac{\Theta(u-ib)}{\Theta(u+ib)} + \frac{\Theta(u+ib)}{\Theta(u-ib)} \right] \\ &\quad - \frac{1}{4} \frac{2H^2(ib)\Theta(u+K-ia)\Theta(u-K+ia)}{H(K-ia+ib)H(K-ia-ib)\Theta^2(u)} \left[\frac{\Theta(u-K+ia)}{\Theta(u+K-ia)} + \frac{\Theta(u+K-ia)}{\Theta(u-K+ia)} \right], \\ M_{11} &= \frac{1}{2D_1} \{ H_1^2(ia)[\Theta^2(u+ib) + \Theta^2(u-ib)] - H^2(ib)[\Theta_1^2(u-ia) + \Theta_1^2(u+ia)] \}, \end{aligned} \quad (2.16)$$

$$\begin{aligned} M_{12} &= i\frac{1}{4}(1+\cos\theta)[e^{i(\psi'+\phi')} - e^{-i(\psi'+\phi')}] + i\frac{1}{4}(1-\cos\theta)[e^{i(\psi'-\phi')} - e^{-i(\psi'-\phi')}] \\ &= i\frac{1}{4} \frac{2H^2(K-ia)\Theta(u+ib)\Theta(u-ib)}{H(K-ia+ib)H(K-ia-ib)\Theta^2(u)} \left[\frac{\Theta(u-ib)}{\Theta(u+ib)} - \frac{\Theta(u+ib)}{\Theta(u-ib)} \right] \\ &\quad - i\frac{1}{4} \frac{2H^2(ib)\Theta(u+K-ia)\Theta(u-K+ia)}{H(K-ia+ib)H(K-ia-ib)\Theta^2(u)} \left[\frac{\Theta(u-K+ia)}{\Theta(u+K-ia)} - \frac{\Theta(u+K-ia)}{\Theta(u-K+ia)} \right], \\ M_{12} &= \frac{-i}{2D_1} \{ H_1^2(ia)[\Theta^2(u+ib) - \Theta^2(u-ib)] - H^2(ib)[\Theta_1^2(u-ia) - \Theta_1^2(u+ia)] \}, \end{aligned} \quad (2.17)$$

$$\begin{aligned} M_{21} &= -i\frac{1}{4}(1+\cos\theta)[e^{i(\psi'+\phi')} - e^{-i(\psi'+\phi')}] + i\frac{1}{4}(1-\cos\theta)[e^{i(\psi'-\phi')} - e^{-i(\psi'-\phi')}] \\ &= -i\frac{1}{4} \frac{2H^2(K-ia)\Theta(u+ib)\Theta(u-ib)}{H(K-ia+ib)H(K-ia-ib)\Theta^2(u)} \left[\frac{\Theta(u-ib)}{\Theta(u+ib)} - \frac{\Theta(u+ib)}{\Theta(u-ib)} \right] \\ &\quad - i\frac{1}{4} \frac{2H^2(ib)\Theta(u+K-ia)\Theta(u-K+ia)}{H(K-ia+ib)H(K-ia-ib)\Theta^2(u)} \left[\frac{\Theta(u-K+ia)}{\Theta(u+K-ia)} - \frac{\Theta(u+K-ia)}{\Theta(u-K+ia)} \right], \end{aligned}$$

$$M_{21} = \frac{i}{2D_1} \{ H_1^2(ia) [\Theta^2(u+ib) - \Theta^2(u-ib)] + H^2(ib) [\Theta_1^2(u-ia) - \Theta_1^2(u+ia)] \}, \quad (2.18)$$

$$\begin{aligned} M_{22} &= \frac{1}{4} (1 + \cos \theta) [e^{i(\psi'+\phi')} + e^{-i(\psi'+\phi')}] - \frac{1}{4} (1 - \cos \theta) [e^{i(\psi'-\phi')} + e^{-i(\psi'-\phi')}] \\ &= \frac{1}{4} \frac{2H^2(K-ia)\Theta(u+ib)\Theta(u-ib)}{H(K-ia+ib)H(K-ia-ib)\Theta^2(u)} \left[\frac{\Theta(u-ib)}{\Theta(u+ib)} + \frac{\Theta(u+ib)}{\Theta(u-ib)} \right] \\ &\quad + \frac{1}{4} \frac{2H^2(ib)\Theta(u+K-ia)\Theta(u-K+ia)}{H(K-ia+ib)H(K-ia-ib)\Theta^2(u)} \left[\frac{\Theta(u-K+ia)}{\Theta(u+K-ia)} + \frac{\Theta(u+K-ia)}{\Theta(u-K+ia)} \right], \\ M_{22} &= \frac{1}{2D_1} \{ H_1^2(ia) [\Theta^2(u+ib) + \Theta^2(u-ib)] + H^2(ib) [\Theta_1^2(u-ia) + \Theta_1^2(u+ia)] \}, \end{aligned} \quad (2.19)$$

$$M_{33} = \frac{1}{D_1} [H_1^2(ia)\Theta(u+ib)\Theta(u-ib) + H^2(ib)\Theta_1(u+ia)\Theta_1(u-ia)], \quad (2.20)$$

where

$$D_1 = H_1(ia+ib)H_1(ia-ib)\Theta^2(u).$$

This completes the derivation of direction cosines for the motion of the heavy symmetrical top. We note that the rotation of a heavy symmetrical top is characterized by the two parameters which are defined by the three roots of the cubic equation. Before this motion is decomposed into two motions of free tops, we need to introduce the motion of a free top.

2.2 The Rotation of a Torque-Free Triaxial Top

We need the orthogonal matrix P for the rotation of a torque-free triaxial top,

and start from the two conservation equations

$$I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 = l^2$$

and

$$I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 = h,$$

where l is the magnitude of the total angular momentum and h is twice the total energy of the torque-free triaxial top. We set

$$\frac{l}{I_1} = s_1, \quad \frac{l}{I_2} = s_2, \quad \frac{h}{l} = s_3, \quad \frac{l}{I_3} = s_4.$$

From the above two equations, we get

$$\omega_1^2 = s_1^2 \frac{s_4 - s_3}{s_4 - s_1} \left[1 - \frac{s_4 - s_2}{s_4 - s_3} \frac{\omega_2^2}{s_2^2} \right]$$

and

$$\omega_3^2 = s_4^2 \frac{s_3 - s_1}{s_4 - s_1} \left[1 - \frac{s_2 - s_1}{s_3 - s_1} \frac{\omega_2^2}{s_2^2} \right].$$

Also from Euler's equation

$$I_2 \dot{\omega}_2 = (I_3 - I_1) \omega_3 \omega_1,$$

we get

$$\dot{\omega}_2 = \frac{s_2}{s_1 s_4} (s_1 - s_4) \omega_3 \omega_1. \quad (2.21)$$

Since elliptic functions have the relation

$$\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u,$$

we assume

$$\omega_2 = s_2 \sqrt{\frac{s_4 - s_3}{s_4 - s_2}} \sin \xi,$$

$$\omega_1 = -s_1 \sqrt{\frac{s_4 - s_3}{s_4 - s_1}} \cos \xi,$$

and

$$\omega_3 = s_4 \sqrt{\frac{s_3 - s_1}{s_4 - s_1}} \Delta \xi,$$

where

$$\Delta \xi = \sqrt{1 - k^2 \sin^2 \xi}$$

and

$$k^2 = \frac{(s_2 - s_1)(s_4 - s_3)}{(s_3 - s_1)(s_4 - s_2)}.$$

Then we substitute these expressions into Euler's equation to get

$$\frac{d\xi}{\Delta \xi} = \sqrt{(s_4 - s_2)(s_3 - s_1)} dt.$$

Hence,

$$\frac{d\xi}{\Delta \xi} = du = \sqrt{(s_4 - s_2)(s_3 - s_1)} dt.$$

Here we have

$$u = nt, \quad n = \sqrt{(s_4 - s_2)(s_3 - s_1)}.$$

Then we get the solutions, in terms of elliptic functions,

$$\omega_2 = s_2 \sqrt{\frac{s_4 - s_3}{s_4 - s_2}} snu,$$

$$\omega_1 = -s_1 \sqrt{\frac{s_4 - s_3}{s_4 - s_1}} cnu$$

and

$$\omega_3 = \pm s_4 \sqrt{\frac{s_3 - s_1}{s_4 - s_1}} dnu,$$

where the + sign corresponds to increasing sequence $s_1 < s_2 < s_3 < s_4$, and the - sign correspond to decreasing sequence $s_1 > s_2 > s_3 > s_4$. Here we will assume the increasing sequence to avoid ambiguity.

The projection of the fixed angular momentum l in the body axes yields

$$l \sin \theta \sin \psi = I_1 \omega_1,$$

$$l \sin \theta \cos \psi = I_2 \omega_2,$$

$$l \cos \theta = I_3 \omega_3.$$

Hence we have

$$\cos \theta = \frac{I_3}{l} \omega_3 = \sqrt{\frac{s_3 - s_1}{s_4 - s_1}} dnu.$$

Jacobi introduced a parameter p by

$$sn(ip) = i \left[\frac{s_3 - s_1}{s_4 - s_3} \right]. \quad (2.22)$$

Other related expressions are

$$cn^2(ip) = \frac{s_4 - s_1}{s_4 - s_3}, \quad dn^2(ip) = \frac{s_4 - s_1}{s_4 - s_2}.$$

and

$$k^2 sn^2(ip) = -\frac{s_2 - s_1}{s_4 - s_2}$$

It can be shown that

$$I_1 \omega_1^2 + I_2 \omega_2^2 = l \sin \theta \dot{\phi} \sin \theta.$$

From it, we have

$$\begin{aligned}
 \dot{\phi} &= \frac{I_1 \omega_1^2 + I_2 \omega_2^2}{l \sin^2 \theta} = l \frac{I_1 \omega_1^2 + I_2 \omega_2^2}{l^2 \sin^2 \theta} \\
 &= \frac{\omega_1^2/s_1 + \omega_2^2/s_2}{\omega_1^2/s_1^2 + \omega_2^2/s_2^2} \\
 &= s_1 + \frac{(s_2 - s_1)(s_4 - s_1)sn^2 u}{(s_4 - s_2)[1 - k^2 sn^2(ip)sn^2 u]}.
 \end{aligned}$$

Integration with respect to t , it becomes

$$\phi = s_1 t + \frac{(s_2 - s_1)(s_4 - s_1)}{s_4 - s_2} \int \frac{sn^2 u dt}{1 - k^2 sn^2(ip)sn^2 u}.$$

Since the variables t and u are related by

$$t = \frac{u}{\sqrt{(s_4 - s_2)(s_3 - s_1)}},$$

we replace t by u

$$\begin{aligned}
 \phi &= s_1 u \frac{1}{\sqrt{(s_4 - s_2)(s_3 - s_1)}} + \frac{(s_2 - s_1)(s_4 - s_1)}{(s_4 - s_2)\sqrt{(s_4 - s_2)(s_3 - s_1)}} \int \frac{sn^2 u du}{1 - k^2 sn^2(ip)sn^2 u} \\
 &= s_1 \frac{1}{\sqrt{(s_4 - s_2)(s_3 - s_1)}} \left[u + \frac{(s_2 - s_1)(s_4 - s_1)}{(s_4 - s_2)s_1} \int \frac{sn^2 u du}{1 - k^2 sn^2(ip)sn^2 u} \right].
 \end{aligned}$$

The above expression can be expressed in terms of Jacobi's parameter as

$$\frac{cn(ip)dn(ip)}{sn(ip)} = -i \frac{s_4 - s_1}{\sqrt{s_4 - s_2}\sqrt{s_3 - s_1}}$$

and

$$-ik^2 sn(ip)cn(ip)dn(ip) = \pm \frac{s_2 - s_1}{s_4 - s_2} \frac{s_4 - s_1}{\sqrt{(s_3 - s_1)(s_4 - s_2)}},$$

where $+$ sign for the increasing sequence $s_1 < s_2 < s_3 < s_4$ and $-$ sign for the decreasing sequence $s_1 > s_2 > s_3 > s_4$. Since the increasing sequence has been

assumed, we choose the + sign, then ϕ is expressed as

$$\phi = i \frac{s_1}{s_4 - s_1} \frac{cn(ip)dn(ip)}{sn(ip)} - ik^2 sn(ip)cn(ip)dn(ip) \int \frac{sn^2 u du}{1 - k^2 sn^2(ip)sn^2 u}.$$

Here we use the integration formula[10]

$$\int_0^u \frac{k^2 sn(a)cn(a)dn(a)sn^2 u du}{1 - k^2 sn^2(a)sn^2 u} = \frac{d \ln \Theta(a)}{da} u + \frac{1}{2} \ln \frac{\Theta(u - a)}{\Theta(u + a)},$$

and obtain

$$\phi = i \frac{s_1}{s_4 - s_1} \frac{cn(ip)dn(ip)}{sn(ip)} u - i \left[\frac{d \ln \Theta(ip)}{dip} u + \frac{1}{2} \ln \frac{\Theta(u - ip)}{\Theta(u + ip)} \right].$$

The derivative of $\ln sn(ip)$ is

$$\frac{d \ln sn(ip)}{dp} = i \frac{cn(ip)dn(ip)}{sn(ip)}.$$

Since the elliptic function $sn u$ can be expressed in terms of theta functions

$$sn u = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)},$$

then it follows that

$$\frac{d \ln sn(ip)}{dp} = \frac{d \ln H(ip)}{dp} - \frac{d \ln \Theta(ip)}{dp}.$$

After this relation has been substituted into the expression for ϕ , it follows that

$$\phi = [s_1 \frac{d \ln H(ip)}{dp} - s_4 \frac{d \ln \Theta(ip)}{dp}] (s_4 - s_1)^{-1} u + \frac{1}{2i} \ln \frac{\Theta(u - ip)}{\Theta(u + ip)}.$$

For the decomposition theorem, only the periodic part of ϕ is considered and

denoted as ϕ' ,

$$e^{i\phi'} = \sqrt{\frac{\Theta(u \mp ip)}{\Theta(u \pm ip)}},$$

where the upper or lower signs correspond to the increasing or decreasing sequence.

From the expressions for the angular velocities, three direction cosines can be found

$$P_{31} = \sin \theta \sin \psi = \frac{\omega_1}{s_1} = -\sqrt{\frac{s_4 - s_3}{s_4 - s_1}} cnu = -\frac{cnu}{cn(ip)} = -\frac{\Theta(ip)H_1(u)}{H_1(ip)\Theta(u)},$$

where the constants in the motion have been expressed in terms of the parameter p ;

$$P_{32} = \sin \theta \cos \psi = \frac{\omega_2}{s_2} = \sqrt{\frac{s_4 - s_3}{s_4 - s_2}} snu = \frac{dn(ip)snu}{cn(ip)} = \frac{\Theta_1(ip)H(u)}{H_1(ip)\Theta(u)}$$

and

$$P_{33} = \cos \theta = \frac{\omega_3}{s_4} = \sqrt{\frac{s_3 - s_1}{s_4 - s_1}} dnu = \frac{sn(ip)dnu}{icn(ip)} = -i \frac{H(ip)\Theta_1(u)}{H_1(ip)\Theta(u)}.$$

The other six direction cosines can be found in the following way. First we have for $\sin^2 \theta$

$$\begin{aligned} \sin^2 \theta &= 1 - \cos^2 \theta = 1 + \frac{sn^2(ip)dn^2u}{cn^2(ip)} \\ &= \frac{cn^2(ip) + sn^2(ip)dn^2u}{cn^2(ip)} = \frac{1 - k^2 sn^2(ip)sn^2u}{cn^2(ip)}. \end{aligned}$$

With the help of the formula[13]

$$1 - k^2 sn^2(ip)sn^2u = \frac{\Theta^2(0)\Theta(u+ip)\Theta(u-ip)}{\Theta^2(ip)\Theta^2(u)},$$

it follows that

$$\sin \theta = \frac{\Theta(0)\sqrt{\Theta(u+ip)\Theta(u-ip)}}{\Theta(ip)\Theta(u)cn(ip)} = \frac{H_1(0)\sqrt{\Theta(u+ip)\Theta(u-ip)}}{H_1(ip)\Theta(u)}.$$

Substituting these expressions into

$$P_{13} = \sin \theta \sin \phi' = \sin \theta \frac{e^{i\phi'} - e^{-i\phi'}}{2i},$$

we obtain

$$\begin{aligned} P_{13} &= \frac{H_1(0)\sqrt{\Theta(u+ip)\Theta(u-ip)}}{H_1(ip)\Theta(u)} \frac{1}{2i} \left[\sqrt{\frac{\Theta(u-ip)}{\Theta(u+ip)}} - \sqrt{\frac{\Theta(u+ip)}{\Theta(u-ip)}} \right] \\ &= \frac{i}{2D_2} H_1(0) [\Theta(u+ip) - \Theta(u-ip)], \end{aligned}$$

where

$$D_2 = H_1(ip)\Theta(u).$$

Similarly, for

$$P_{23} = -\sin \theta \cos \phi' = -\sin \theta \frac{e^{i\phi'} + e^{-i\phi'}}{2},$$

we get

$$P_{23} = \frac{-1}{2D_2} H_1(0) [\Theta(u+ip) + \Theta(u-ip)].$$

To find P_{11} , two addition theorems[13] are useful

$$\begin{aligned} \operatorname{sn}(u \pm ip) &= \frac{\operatorname{cn}(ip)\operatorname{dn}(ip)\operatorname{sn}u \pm \operatorname{sn}(ip)\operatorname{cn}u\operatorname{dn}u}{1 - k^2\operatorname{sn}^2(ip)\operatorname{sn}^2u}, \\ \operatorname{cn}(u \pm ip) &= \frac{\operatorname{cn}(ip)\operatorname{cn}u \mp \operatorname{sn}(ip)\operatorname{dn}(ip)\operatorname{sn}u\operatorname{dn}u}{1 - k^2\operatorname{sn}^2(ip)\operatorname{sn}^2u}. \end{aligned}$$

After expressing the parameter ip by Eulerian angles θ and ψ , it follows that

$$\begin{aligned} \operatorname{sn}(u \pm ip) &= \frac{\cos \psi \mp i \cos \theta \sin \psi}{\sin \theta}, \\ \operatorname{cn}(u \pm ip) &= \frac{-\sin \psi \mp i \cos \theta \cos \psi}{\sin \theta}. \end{aligned}$$

Substituting these into

$$2P_{11} = (\cos \psi + i \cos \theta \sin \psi)e^{i\phi'} + (\cos \psi - i \cos \theta \sin \psi)e^{-i\phi'},$$

$$2iP_{21} = (\cos \psi + i \cos \theta \sin \psi)e^{i\phi'} - (\cos \psi - i \cos \theta \sin \psi)e^{-i\phi'},$$

$$2P_{12} = (-\sin \psi + i \cos \theta \cos \psi)e^{i\phi'} + (-\sin \psi - i \cos \theta \cos \psi)e^{-i\phi'},$$

$$2iP_{22} = (-\sin \psi + i \cos \theta \cos \psi)e^{i\phi'} - (-\sin \psi - i \cos \theta \cos \psi)e^{-i\phi'},$$

we obtain

$$2P_{11} = \sin \theta e^{i\phi'} \operatorname{sn}(u - ip) + \sin \theta e^{-i\phi'} \operatorname{sn}(u + ip),$$

$$2iP_{21} = \sin \theta e^{i\phi'} \operatorname{sn}(u - ip) - \sin \theta e^{-i\phi'} \operatorname{sn}(u + ip),$$

$$2P_{12} = \sin \theta e^{i\phi'} \operatorname{cn}(u - ip) + \sin \theta e^{-i\phi'} \operatorname{cn}(u + ip),$$

$$2iP_{22} = \sin \theta e^{i\phi'} \operatorname{cn}(u - ip) - \sin \theta e^{-i\phi'} \operatorname{cn}(u + ip).$$

After substituting the expressions for $\sin \theta$, $e^{i\phi'}$ and the following two equations

$$\operatorname{sn}(u + ip) = \frac{\Theta_1(0)H(u + ip)}{H_1(0)\Theta(u + ip)}$$

and

$$\operatorname{cn}(u + ip) = \frac{\Theta(0)H_1(u + ip)}{H_1(0)\Theta(u + ip)}$$

into the expressions for P_{11} and the other three direction cosines, it follows that

$$\begin{aligned} P_{11} &= \frac{1}{2D_2} \Theta_1(0)[H(u + ip) + H(u - ip)], \\ P_{12} &= \frac{1}{2D_2} \Theta(0)[H_1(u + ip) + H_1(u - ip)], \\ P_{21} &= \frac{i}{2D_2} \Theta_1(0)[H(u + ip) - H(u - ip)], \\ P_{22} &= \frac{i}{2D_2} \Theta(0)[H_1(u + ip) - H_1(u - ip)]. \end{aligned} \quad (2.23)$$

Along with the direction cosine P_{33}

$$P_{33} = -i \frac{H(ip)\Theta_1(u)}{D_2},$$

all the nine direction cosines have been found. This completes the derivation of direction cosines for the rotation of the free top. We notice that the rotation of a free top is characterized by one parameter which is defined by the two constants in the motion of the system and the three moments of inertia of the free top. The other matrix Q in the decomposition theorem is basically the same expression as P except that the parameter is q , which may or may not be different from p .

When M is decomposed into P and Q , the parameters p and q of the matrices P and Q become

$$p = a + b,$$

$$q = a - b.$$

Thus the matrix elements P_{ij} of the matrix P for the motion of a free top in Jacobi's decomposition theorem are

$$\begin{aligned} P_{11} &= \frac{1}{2D_p} \Theta_1(0) [H(u + ia + ib) + H(u - ia - ib)], \\ P_{12} &= \frac{1}{2D_p} \Theta(0) [H_1(u + ia + ib) + H_1(u - ia - ib)], \\ P_{13} &= \frac{i}{2D_p} H_1(0) [\Theta(u + ia + ib) - \Theta(u - ia - ib)], \\ P_{21} &= \frac{i}{2D_p} \Theta_1(0) [H(u + ia + ib) - H(u - ia - ib)], \\ P_{22} &= \frac{i}{2D_p} \Theta(0) [H_1(u + ia + ib) - H_1(u - ia - ib)], \\ P_{23} &= \frac{-1}{2D_p} H_1(0) [\Theta(u + ia + ib) + \Theta(u - ia - ib)], \\ P_{31} &= -\frac{\Theta(ia + ib)H_1(u)}{D_p}, \end{aligned} \tag{2.24}$$

$$P_{32} = \frac{\Theta_1(ia + ib)H(u)}{D_p},$$

$$P_{33} = -i \frac{H(ia + ib)\Theta_1(u)}{D_p},$$

where

$$D_p = H_1(ia + ib)\Theta(u).$$

The matrix elements of another matrix Q for the motion of a free top in Jacobi's decomposition theorem are

$$\begin{aligned} Q_{11} &= \frac{1}{2D_q} \Theta_1(0)[H(u + ia - ib) + H(u - ia + ib)], \\ Q_{12} &= \frac{1}{2D_q} \Theta(0)[H_1(u + ia - ib) + H_1(u - ia + ib)], \\ Q_{13} &= \frac{i}{2D_q} H_1(0)[\Theta(u + ia - ib) - \Theta(u - ia + ib)], \\ Q_{21} &= \frac{i}{2D_q} \Theta_1(0)[H(u + ia - ib) - H(u - ia + ib)], \\ Q_{22} &= \frac{i}{2D_q} \Theta(0)[H_1(u + ia - ib) - H_1(u - ia + ib)], \\ Q_{23} &= \frac{-1}{2D_q} H_1(0)[\Theta(u + ia - ib) + \Theta(u - ia + ib)], \\ Q_{31} &= -\frac{\Theta(ia - ib)H_1(u)}{D_q}, \\ Q_{32} &= \frac{\Theta_1(ia - ib)H(u)}{D_q}, \\ Q_{33} &= -i \frac{H(ia - ib)\Theta_1(u)}{D_q}, \end{aligned} \tag{2.25}$$

where

$$D_q = H_1(ia - ib)\Theta(u).$$

2.3 The Derivation of the Decomposition Theorem

To show his decomposition theorem, Jacobi employed one of the many identity relations between theta functions all of which are his own invention

$$\begin{aligned}
 & 2\Theta(u)\Theta(u')H(u'')H(u''') \\
 = & \Theta(v)\Theta(v')H(v'')H(v''') - H(v)H(v')\Theta(v'')\Theta(v''') \\
 & + \Theta_1(v)\Theta_1(v')H_1(v'')H_1(v''') - H_1(v)H_1(v')\Theta_1(v'')\Theta_1(v'''),
 \end{aligned}$$

where

$$\begin{aligned}
 v &= \frac{1}{2}(u + u' + u'' + u'''), \\
 v' &= \frac{1}{2}(u + u' - u'' + u'''), \\
 v'' &= \frac{1}{2}(u - u' + u'' - u'''), \\
 v''' &= \frac{1}{2}(u - u' - u'' + u'''),
 \end{aligned}$$

and u, u', u'', u''' are four arbitrary quantities. Suppose in the above relation

$$u' = u, \quad u'' = u''' = a,$$

such that

$$v = u + a, \quad v' = u - a, \quad v'' = 0, \quad v''' = 0,$$

then it follows that

$$2\Theta^2(u)H^2(a) = H_1^2(0)\Theta_1(u+a)\Theta_1(u-a) - \Theta^2(0)H(u+a)H(u-a) - \Theta_1^2(0)H_1(u+a)H_1(u-a).$$

If firstly $u + K - ia$ replaces u , and ib replaces a , and secondly $u + ib$ replaces u , and $K - ia$ replaces a ; then two formulas follow

$$\begin{aligned}
& 2\Theta_1^2(u - ia)H^2(ib) \\
= & H_1^2(0)\Theta(u - ia + ib)\Theta(u - ia - ib) - \Theta^2(0)H_1(u - ia + ib)H_1(u - ia - ib) \\
& - \Theta_1^2(0)H(u - ia + ib)H(u - ia - ib),
\end{aligned}$$

$$\begin{aligned}
& 2\Theta^2(u + ib)H_1^2(ia) \\
= & H_1^2(0)\Theta(u - ia + ib)\Theta(u + ia + ib) + \Theta^2(0)H_1(u - ia + ib)H_1(u + ia + ib) \\
& + \Theta_1^2(0)H(u - ia + ib)H(u + ia + ib).
\end{aligned}$$

If a is replaced by $-a$, or b is replaced by $-b$, then two more formulas are obtained. These four formulas can be substituted into the expressions for the direction cosines $M_{11}, M_{12}, M_{21}, M_{22}$

$$\begin{aligned}
M_{11} = & \frac{1}{4D_1} \{ -H_1^2(0)[\Theta(u + ia - ib) - \Theta(u - ia + ib)][\Theta(u + ia + ib) \\
& - \Theta(u - ia - ib)] \\
& + \Theta^2(0)[H_1(u + ia - ib) + H_1(u - ia + ib)][H_1(u + ia + ib) \\
& + H_1(u - ia - ib)] \\
& + \Theta_1^2(0)[H(u + ia - ib) + H(u - ia + ib)][H(u + ia + ib) \\
& + H(u - ia - ib)] \},
\end{aligned}$$

$$M_{12} = \frac{-i}{4D_1} \{ H_1^2(0)[\Theta(u + ia - ib) + \Theta(u - ia + ib)][\Theta(u + ia + ib)$$

$$\begin{aligned}
& -\Theta(u - ia - ib)] \\
& -\Theta^2(0)[H_1(u + ia - ib) - H_1(u - ia + ib)][H_1(u + ia + ib) \\
& + H_1(u - ia - ib)] \\
& -\Theta_1^2(0)[H(u + ia - ib) - H(u - ia + ib)][H(u + ia + ib) \\
& + H(u - ia - ib)]\}, \\
M_{21} = & \frac{i}{4D_1}\{-H_1^2(0)[\Theta(u + ia - ib) - \Theta(u - ia + ib)][\Theta(u + ia + ib) \\
& + \Theta(u - ia - ib)] \\
& + \Theta^2(0)[H_1(u + ia - ib) + H_1(u - ia + ib)][H_1(u + ia + ib) \\
& - H_1(u - ia - ib)] \\
& + \Theta_1^2(0)[H(u + ia - ib) + H(u - ia + ib)][H(u + ia + ib) \\
& - H(u - ia - ib)]\}, \\
M_{22} = & \frac{1}{4D_1}\{H_1^2(0)[\Theta(u + ia - ib) + \Theta(u - ia + ib)][\Theta(u + ia + ib) \\
& + \Theta(u - ia - ib)] \\
& - \Theta^2(0)[H_1(u + ia - ib) - H_1(u - ia + ib)][H_1(u + ia + ib) \\
& - H_1(u - ia - ib)] \\
& - \Theta_1^2(0)[H(u + ia - ib) - H(u - ia + ib)][H(u + ia + ib) \\
& - H(u - ia - ib)]\}.
\end{aligned}$$

At this moment, it can be found that

$$M_{11} = P_{11}Q_{11} + P_{12}Q_{12} + P_{13}Q_{13},$$

$$M_{12} = P_{11}Q_{21} + P_{12}Q_{22} + P_{13}Q_{23},$$

$$M_{21} = P_{21}Q_{11} + P_{22}Q_{12} + P_{23}Q_{13},$$

$$M_{22} = P_{21}Q_{21} + P_{22}Q_{22} + P_{23}Q_{23}.$$

Here it is found that the above four direction cosines $M_{11}, M_{12}, M_{21}, M_{22}$ have been decomposed into the product of the first or second row of the matrix P and the first or second row of the matrix Q . Since one row of a matrix is one column of its transpose matrix, it may be said that the above four direction cosines are the product of the first or second row of P and the first or second column of $\tilde{Q}$. Similar decompositions happen for the other direction cosines. To decompose the direction cosines $M_{31}, M_{32}, M_{13}, M_{23}$, some formulas are necessary. In the identity relation of theta functions, let $u + K - ia$ and $u + ib$ replace u and u' , and let $K - ia$ and ib replace u'' and u''' ; then the corresponding quantities v, v', v'', v''' are:

$$v = u + K - ia + ib,$$

$$v' = u,$$

$$v'' = K - ia - ib,$$

$$v''' = 0.$$

Substituting these quantities into the identity relation of theta functions, the following formula is obtained

$$2\Theta_1(u - ia)\Theta(u + ib)H_1(ia)H(ib)$$

$$\begin{aligned}
&= H_1(0)H(ia+ib)\Theta_1(u)\Theta(u-ia+ib) - \Theta(0)\Theta_1(ia+ib)H(u)H_1(u-ia+ib) \\
&\quad + \Theta_1(0)\Theta(ia+ib)H_1(u)H(u-ia+ib).
\end{aligned}$$

If a is replaced by $-a$, or b is replaced by $-b$, or both, then three more formulas are obtained. All these four formulas make the direction cosines $M_{31}, M_{32}, M_{13}, M_{23}$ expressible as

$$\begin{aligned}
M_{31} &= \frac{-1}{2D_1} \{ H_1(0)\Theta_1(u)H(ia+ib)[\Theta(u-ia+ib) - \Theta(u+ia-ib)] \\
&\quad - \Theta(0)H(u)\Theta_1(ia+ib)[H_1(u-ia+ib) + H_1(u+ia-ib)] \\
&\quad + \Theta_1(0)H_1(u)\Theta(ia+ib)[H(u-ia+ib) + H(u+ia-ib)] \},
\end{aligned}$$

$$\begin{aligned}
M_{32} &= \frac{i}{2D_1} \{ H_1(0)\Theta_1(u)H(ia+ib)[\Theta(u-ia+ib) + \Theta(u+ia-ib)] \\
&\quad - \Theta(0)H(u)\Theta_1(ia+ib)[H_1(u-ia+ib) - H_1(u+ia-ib)] \\
&\quad + \Theta_1(0)H_1(u)\Theta(ia+ib)[H(u-ia+ib) - H(u+ia-ib)] \},
\end{aligned}$$

$$\begin{aligned}
M_{13} &= \frac{1}{2D_1} \{ H_1(0)\Theta_1(u)H(ia-ib)[\Theta(u+ia+ib) - \Theta(u-ia-ib)] \\
&\quad + \Theta(0)H(u)\Theta_1(ia-ib)[H_1(u+ia+ib) + H_1(u-ia-ib)] \\
&\quad - \Theta_1(0)H_1(u)\Theta(ia-ib)[H(u+ia+ib) + H(u-ia-ib)] \},
\end{aligned}$$

$$\begin{aligned}
M_{23} &= \frac{i}{2D_1} \{ H_1(0)\Theta_1(u)H(ia-ib)[\Theta(u+ia+ib) + \Theta(u-ia-ib)] \\
&\quad + \Theta(0)H(u)\Theta_1(ia-ib)[H_1(u+ia+ib) - H_1(u-ia-ib)] \\
&\quad - \Theta_1(0)H_1(u)\Theta(ia-ib)[H(u+ia+ib) - H(u-ia-ib)] \},
\end{aligned}$$

they are

$$M_{31} = P_{31}Q_{11} + P_{32}Q_{12} + P_{33}Q_{13}$$

$$M_{32} = P_{31}Q_{21} + P_{32}Q_{22} + P_{33}Q_{23}$$

$$M_{13} = P_{11}Q_{31} + P_{12}Q_{32} + P_{13}Q_{33}$$

$$M_{23} = P_{21}Q_{31} + P_{22}Q_{32} + P_{23}Q_{33}.$$

Now in the identity relation for the theta functions let quantities u, u', u'', u''' be

$$u + K - ia, u - K + ia, ib, -ib,$$

and v, v', v'', v''' become quantities

$$u, u, K - ia + ib, K - ia - ib,$$

it follows that

$$\begin{aligned} & -2H^2(ib)\Theta_1(u + ia)\Theta_1(u - ia) \\ = & H_1(ia + ib)H_1(ia - ib)\Theta^2(u) - \Theta_1(ia + ib)\Theta_1(ia - ib)H^2(u) \\ & + H(ia + ib)H(ia - ib)\Theta_1^2(u) - \Theta(ia + ib)\Theta(ia - ib)H_1^2(u). \end{aligned}$$

In the above formula, let $K - ib$ replace ia , and $K - ia$ replace ib ,

$$\begin{aligned} & -2H_1^2(ia)\Theta(u + ib)\Theta(u - ib) \\ = & -H_1(ia + ib)H_1(ia - ib)\Theta^2(u) - \Theta_1(ia + ib)\Theta_1(ia - ib)H^2(u) \\ & + H(ia + ib)H(ia - ib)\Theta_1^2(u) - \Theta(ia + ib)\Theta(ia - ib)H_1^2(u). \end{aligned}$$

Add up the above two equations to get

$$\begin{aligned} M_{33} = & \frac{1}{D_1} [\Theta_1(ia + ib)\Theta_1(ia - ib)H^2(u) - H(ia + ib)H(ia - ib)\Theta_1^2(u) \\ & + \Theta(ia + ib)\Theta(ia - ib)H_1^2(u)] \end{aligned}$$

which is

$$M_{33} = P_{31}Q_{31} + P_{32}Q_{32} + P_{33}Q_{33}.$$

This completes the proof of Jacobi's decomposition theorem

$$M = P\tilde{Q},$$

where $\tilde{Q}$ is the transpose of Q . The physical meaning of the above equation is that the motion of a heavy symmetrical top can be decomposed into two motions of two torque-free triaxial tops. Let us consider the motion of a heavy symmetrical top M and the motion of a free triaxial top P , both of which are described with respect to a frame of reference moving with a certain constant precession or spinning rate in space. Then the motion of the torque-free triaxial top observed from the reference frame is described by the orthogonal matrix P and the same motion when observed from the frame of the heavy symmetrical top (i.e. the observer moving with the heavy symmetrical top) is described by the orthogonal matrix Q . The relative motion Q turns out to be also representing the motion of a torque-free triaxial top. Not only the dynamical parameters but also the kinematic parameter (e.g. moments of inertia) of the two triaxial tops underlying the motions P and Q are different.

Due to the different choices of conventions for ϕ and ψ , some of the matrix elements have opposite signs to that of Jacobi's. For example, in the orthogonal matrix for the free top, the first and second rows are all in the opposite sign with that in Jacobi's. This is due to the following two reasons. The first reason

is that the angles in Jacobi's direction cosines are not Eulerian angles such that their definitions are different from the definitions for Eulerian angles. Jacobi's expressions for direction cosines in terms of his angles are

$$P_{31}^J = -\sin \theta \sin \phi_J,$$

$$P_{32}^J = -\sin \theta \cos \phi_J,$$

$$P_{13}^J = \sin \theta \sin \psi'_J,$$

$$P_{23}^J = \sin \theta \cos \psi'_J,$$

$$P_{11}^J = \cos \phi_J \cos \psi'_J + \cos \theta \sin \phi_J \sin \psi'_J,$$

$$P_{12}^J = -\sin \phi_J \cos \psi'_J + \cos \theta \cos \phi_J \sin \psi'_J,$$

$$P_{21}^J = -\cos \phi_J \sin \psi'_J + \cos \theta \sin \phi_J \cos \psi'_J,$$

$$P_{22}^J = \sin \phi_J \sin \psi'_J + \cos \theta \cos \phi_J \cos \psi'_J,$$

$$P_{33}^J = \cos \theta = z,$$

where the superscript or subscript J denotes the direction cosines or the angles are used as defined by Jacobi. Comparing expressions for the direction cosines between Jacobi's and the way expressed in terms of Eulerian angles, the transformation to transform into each other is

$$\phi_J = \pi + \psi, \quad \psi'_J = \pi - \phi'.$$

Under this transformation they are equivalent to each other. In fact, there are some sign differences between ours and Jacobi's expressions for the direction cosines.

This is due to the second reason. The second reason is

$$\psi'_J = -\phi',$$

where ψ'_J represents the angle ψ' in Jacobi's expressions. This opposition in signs originates from the opposition of signs in the expressions for ϕ between ours and Jacobi's. However the sign difference can be made consistent by the following transformations

$$\phi_J = \pi + \psi, \quad \psi'_J = -\phi',$$

where the subscript J means the notation used in Jacobi's direction cosines. The first transformation in the above is completely due to the difference in the expressions between Jacobi's way and the way Eulerian angles are expressed. After the above transformations the sign difference are found in the way we have mentioned above. This is the explanation for the sign difference between our expressions and Jacobi's for the direction cosines of the rotation of a torque-free triaxial top.

For the orthogonal matrix of the heavy symmetrical top, $M_{31}, M_{32}, M_{13}, M_{23}$ all have opposite sign from that of Jacobi's. This is basically due to the different ways in choosing angles

$$\phi'_J = \psi', \quad \psi'_J = -\phi'.$$

If the above two equations are put into Jacobi's expressions, they will become identical to the expressions derived here.

Chapter 3

The Translation and Rotation of a Solid in a Fluid

In hydrodynamics, for a solid rotating and translating in a fluid, Kirchhoff found that the case of the symmetrical top can be solved by elliptic functions. Kopcke [14] continued the same problem and expressed the solutions in terms of theta functions. In this chapter we derive the solution for the Kirchhoff's case. First we solve the rotation of the solid in section one. The translation of the center of mass is treated in section two. The physical aspects with examples and graphs are given in appendices B and C, where B is for a special case, and C is for the general case. Appendix D gives the discussion about the effective masses and moments of inertia of the induced flow surrounding a solid.

3.1 The Rotation of a Symmetrical Solid in a Fluid

Notations of the coordinates are for a point in the body system xyz , in the space system XYZ , for the center of mass of the symmetrical top X_c, Y_c, Z_c . The coordinates transformation between the space system and the body system is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X_c \\ Y_c \\ Z_c \end{pmatrix} + \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The first term on the right describes the translational motion and the second term represents the rotational part. The nine direction cosines R_{ij} 's form an orthogonal matrix which will be denoted as R . The symmetrical axis of the top is chosen as the z -axis of the body axes. The equations of motion for the translation and rotation of a solid in a fluid [5] [6] [15] are

$$\frac{d}{dt} \frac{\partial T}{\partial u_1} = \omega_3 \frac{\partial T}{\partial u_2} - \omega_2 \frac{\partial T}{\partial u_3}, \quad (3.1)$$

$$\frac{d}{dt} \frac{\partial T}{\partial u_2} = \omega_1 \frac{\partial T}{\partial u_3} - \omega_3 \frac{\partial T}{\partial u_1}, \quad (3.2)$$

$$\frac{d}{dt} \frac{\partial T}{\partial u_3} = \omega_2 \frac{\partial T}{\partial u_1} - \omega_1 \frac{\partial T}{\partial u_2}, \quad (3.3)$$

$$\frac{d}{dt} \frac{\partial T}{\partial \omega_1} = u_3 \frac{\partial T}{\partial u_2} - u_2 \frac{\partial T}{\partial u_3} + \omega_3 \frac{\partial T}{\partial \omega_2} - \omega_2 \frac{\partial T}{\partial \omega_3}, \quad (3.4)$$

$$\frac{d}{dt} \frac{\partial T}{\partial \omega_2} = u_1 \frac{\partial T}{\partial u_3} - u_3 \frac{\partial T}{\partial u_1} + \omega_1 \frac{\partial T}{\partial \omega_3} - \omega_3 \frac{\partial T}{\partial \omega_1}, \quad (3.5)$$

$$\frac{d}{dt} \frac{\partial T}{\partial \omega_3} = u_2 \frac{\partial T}{\partial u_1} - u_1 \frac{\partial T}{\partial u_2} + \omega_2 \frac{\partial T}{\partial \omega_1} - \omega_1 \frac{\partial T}{\partial \omega_2}, \quad (3.6)$$

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 + \frac{1}{2} m_3 u_3^2; \quad (3.7)$$

where T is the total energy including both the body and the fluid, I_j and m_j are the moment of inertia and the mass respectively, u_j and ω_j are the velocity and angular velocity respectively. The effective masses and moments of inertia of the induced flow surrounding a solid are discussed in appendix D. In a symmetric top, I_1 equals I_2 and m_1 equals m_2 . From the above equations we get the following four first integrals:

$$\omega_3 = \text{constant}, \quad (3.8)$$

$$m_1^2 u_1^2 + m_2^2 u_2^2 + m_3^2 u_3^2 = \text{constant} = P^2, \quad (3.9)$$

$$I_1 \omega_1 m_1 u_1 + I_2 \omega_2 m_2 u_2 + I_3 \omega_3 m_3 u_3 = \text{constant} = n, \quad (3.10)$$

$$2T = \text{constant} = h. \quad (3.11)$$

We would choose the direction of the total linear momentum (equation 3.9) as the Z -direction of the space system. The projections of this momentum on the body system axes are

$$m_1 u_1 = P R_{31}, \quad (3.12)$$

$$m_2 u_2 = P R_{32}, \quad (3.13)$$

$$m_3 u_3 = P R_{33}. \quad (3.14)$$

Equation 3.10 means n is the scalar product of the angular momentum and the linear momentum. The projection component of the angular momentum along the

Z - axis l_P is n/P . Denoting the projection of the angular momentum along the z - axis as l_3 , along the antinodal line as l_{AN} , along the nodal line as l_N , then

$$l_3 \cos \theta + l_{AN} \sin \theta = l_P = \vec{l} \cdot \vec{P}/P = \frac{n}{P}, \quad (3.15)$$

where

$$l_3 = I_3 \omega_3, \quad l_N = I_1 \dot{\theta}, \quad l_{AN} = I_1 \dot{\phi} \sin \theta. \quad (3.16)$$

The symmetry of the top allows equation 3.7 be written as

$$h = P^2 \left(\frac{\cos^2 \theta}{m_3} + \frac{\sin^2 \theta}{m_1} \right) + \frac{l_3^2}{I_3} + \frac{l_{AN}^2 + l_N^2}{I_1}. \quad (3.17)$$

From equation 3.12 to equation 3.17, we obtain

$$(I_1 \sin \theta \dot{\theta})^2 = (1 - \cos^2 \theta)(\alpha - \beta \cos^2 \theta) - (l_P - l_3 \cos \theta)^2, \quad (3.18)$$

where

$$\alpha = I_1 \left(h - \frac{l_3^2}{I_3} - \frac{P^2}{m_1} \right), \quad \beta = I_1 P^2 \left(\frac{1}{m_3} - \frac{1}{m_1} \right). \quad (3.19)$$

In terms of Eulerian angles, the direction cosine R_{33} is

$$R_{33} = \cos \theta. \quad (3.20)$$

For convenience, we denote

$$z = \cos \theta, \quad c_4 = \alpha/\beta, \quad u \equiv \frac{\sqrt{\beta}}{I_1} t \quad (3.21)$$

Then equation 3.18 becomes

$$\begin{aligned} \left(\frac{dz}{du} \right)^2 &= (1 - z^2)(c_4 - z^2) - \frac{l_3^2}{\beta} \left(\frac{l_P}{l_3} - z \right)^2 \\ &= (z - a_1)(z - a_2)(z - a_3)(z - a_4) = g(z), \end{aligned} \quad (3.22)$$

or

$$\left(\frac{dz}{du}\right)^2 = z^4 - (c_4 + 1 + \frac{l_3^2}{\beta})z^2 + 2\frac{l_P l_3}{\beta}z + c_4 - \frac{l_P^2}{\beta} \quad (3.23)$$

Let us discuss the roots of equation 3.22. Case 1 : $m_1 > m_3$. Here u is real, which makes $(dz/du)^2$ or $g(z)$ positive somewhere between -1 and 1 when z has physical meaning. However from equation 3.22, we see $g(z)$ is negative at $z = -1$ and 1 . Hence $g(z)$ has two zeros at the domain between -1 and 1 . When z becomes infinite, $g(z)$ becomes positively infinite, which implies that there is one root less than -1 and also one root greater than 1 . All these four roots are real, we assume their order as

$$a_4 < -1 < a_3 < a_2 < 1 < a_1, \quad (3.24)$$

where a_1, a_2, a_3, a_4 denote the four roots. Case 2 : $m_1 < m_3$. Here u is imaginary which makes $(dz/du)^2$ or $g(z)$ negative somewhere between -1 and 1 when z has physical meaning. However from equation 3.22, we see $g(z)$ is positive at $z = -1$ and 1 . Hence $g(z)$ has at least two zeros at the domain between -1 and 1 . When z becomes infinite $g(z)$ becomes positively infinite which implies the other two roots may both be real or both be complex conjugates. Since the cubic term is missing from the quartic equation, this implies that the sum of the four roots is zero, if all of the four roots are real, they must be greater than -1 and less than 1 .

$$-1 < a_4 < a_3 < a_2 < a_1 < 1.$$

When two of the real roots coincide, the rotation becomes a steady precession and the center of mass moves on a helix. This case provides an analogy to the

steady precession of a symmetric top in a uniform gravity and we can gain a lot of insights into the motion by analyzing this special case. We have made an exhaustive analysis of this case and present it as appendix B.

In the general case, where all the roots are distinct, the symmetry axis nutates between two extreme inclination angles, and the nodal line precesses either progressively or retrogressively. The center of mass spirals up with various cyclic patterns between two concentric cylindrical surfaces. Discussion with examples and figures are given in appendix C.

A change of variables

$$z = \frac{a_3 - a_4(a_2 - a_3)(a_2 - a_4)^{-1}y^2}{1 - (a_2 - a_3)(a_2 - a_4)^{-1}y^2}$$

will reduce the integral

$$du = \frac{dz}{\sqrt{g(z)}}, \quad (3.25)$$

into the normal form

$$du = \frac{2dy}{\sqrt{(a_2 - a_4)(a_1 - a_3)}\sqrt{(1 - y^2)(1 - k^2y^2)}},$$

where

$$k^2 = \frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)} \quad (3.26)$$

is the modulus of the elliptic integral

$$dw = \frac{dy}{\sqrt{(1 - y^2)(1 - k^2y^2)}}$$

with

$$w = \frac{1}{2}\sqrt{(a_2 - a_4)(a_1 - a_3)}u.$$

t and w have a relation

$$t = \frac{I_1}{\epsilon\sqrt{\beta}}w, \quad (3.27)$$

where

$$\epsilon = \frac{1}{2}\sqrt{(a_2 - a_4)(a_1 - a_3)}.$$

Now we have

$$y = sn\ w$$

and

$$z = \frac{a_3 - a_4(a_2 - a_3)(a_2 - a_4)^{-1}sn^2w}{1 - (a_2 - a_3)(a_2 - a_4)^{-1}sn^2w}. \quad (3.28)$$

Since the range of snw is between -1 and 1 , the above equation shows the range of z is between a_3 and a_2 . Solving for sn^2w we get

$$sn^2w = \frac{(a_2 - a_4)(z - a_3)}{(a_2 - a_3)(z - a_4)}.$$

Next we define three parameters A, B and C as follows:

$$sn^2C = (a_2 - a_4)/(a_2 - a_3), \quad (3.29)$$

$$sn^2A/sn^2C = (1 - a_3)/(1 - a_4), \quad (3.30)$$

$$sn^2B/sn^2C = (1 + a_3)/(1 + a_4). \quad (3.31)$$

Because the value of sn^2C is always greater than 1 , we can choose

$$C = iK' + c, \quad (3.32)$$

where c is real, and K and K' are the complete elliptic integrals for the moduli k and k' with the relation

$$k^2 + k'^2 = 1.$$

Then

$$sn^2 C = \frac{a_1 - a_3}{a_1 - a_2} \frac{a_2 - a_3}{a_3 - a_4}. \quad (3.33)$$

Here we will perform some algebras to express z in terms of the three parameters.

Equations 3.31 and 3.30 implies

$$1 - a_3 = (1 - a_4) \frac{sn^2 A}{sn^2 C}$$

and

$$1 + a_3 = (1 + a_4) \frac{sn^2 B}{sn^2 C}.$$

Add up these two equations, we have

$$2 = \frac{sn^2 A}{sn^2 C} + \frac{sn^2 B}{sn^2 C} - a_4 \frac{sn^2 A}{sn^2 C} + a_4 \frac{sn^2 B}{sn^2 C}.$$

From this equation a_4 is solved as

$$a_4 = \frac{2sn^2 C - sn^2 A - sn^2 B}{sn^2 B - sn^2 A}.$$

It follows that

$$1 + a_4 = -2 \frac{sn^2 C - sn^2 A}{sn^2 A - sn^2 B}$$

and

$$1 - a_4 = 2 \frac{sn^2 C - sn^2 B}{sn^2 A - sn^2 B}.$$

Equation 3.28 of z can be rewritten as

$$z = \frac{a_3 sn^2 C - a_4 sn^2 w}{sn^2 C - sn^2 w}.$$

Now,

$$1 + z = (1 + a_4) \frac{sn^2 B - sn^2 w}{sn^2 C - sn^2 w},$$

$$1 - z = (1 - a_4) \frac{sn^2 A - sn^2 w}{sn^2 C - sn^2 w}.$$

So in terms of the parameters, we obtain the following two expressions

$$1 + z = -2 \frac{sn^2 C - sn^2 A}{sn^2 A - sn^2 B} \frac{sn^2 B - sn^2 w}{sn^2 C - sn^2 w},$$

$$1 - z = 2 \frac{sn^2 C - sn^2 B}{sn^2 A - sn^2 B} \frac{sn^2 A - sn^2 w}{sn^2 C - sn^2 w}.$$

Substituting the relation

$$k(sn^2 u - sn^2 v) = \frac{\Theta^2(0)H(u+v)H(u-v)}{\Theta^2(u)\Theta^2(v)}$$

into the above, we get

$$1 + z = 2 \frac{H(A-C)H(A+C)H(w-B)H(w+B)}{H(A-B)H(A+B)H(w-C)H(w+C)}, \quad (3.34)$$

$$1 - z = -2 \frac{H(B-C)H(B+C)H(w-A)H(w+A)}{H(A-B)H(A+B)H(w-C)H(w+C)}, \quad (3.35)$$

and hence

$$z = \frac{1}{N} [H(A-C)H(A+C)H(w-B)H(w+B) + H(B-C)H(B+C)H(w-A)H(w+A)], \quad (3.36)$$

where

$$N \equiv H(A-B)H(A+B)H(w-C)H(w+C).$$

To find the other eight direction cosines, we will use the three parameters A, B, C and the variable w to express Eulerian angles ϕ and ψ . Since the z -component of the angular velocity can be expressed as

$$\omega_3 = \dot{\psi} + \dot{\phi} \cos \theta, \quad (3.37)$$

and it can be shown that

$$I_1\omega_1m_1u_1 + I_1\omega_2m_1u_2 = I_1\dot{\phi}\sin\theta P\sin\theta.$$

Hence, from the above two equations and equation 3.10 we get

$$\dot{\phi}I_1\sin^2\theta + l_3\cos\theta = l_P.$$

It then follows

$$\dot{\phi} = \frac{l_P - l_3\cos\theta}{I_1(1 - \cos^2\theta)} \quad (3.38)$$

and

$$\dot{\psi} = \omega_3 - \frac{l_3}{I_1} - \frac{l_P\cos\theta - l_3}{I_1(1 - \cos^2\theta)}. \quad (3.39)$$

To get ϕ and ψ , we have to integrate with respect to t . But $\cos\theta$ is a function of w , then we have to change the variable from t to w . After the integration with respect to t , ψ becomes

$$\psi = \omega_3 t - \frac{l_3}{I_1}t - \int \frac{l_P\cos\theta - l_3}{I_1(1 - \cos^2\theta)}dt.$$

For convenience, we set

$$\psi_1 = - \int \frac{l_P\cos\theta - l_3}{I_1(1 - \cos^2\theta)}dt.$$

Then we get

$$\phi - \psi_1 = \frac{l_P - l_3}{I_1} \int \frac{dt}{1 - z},$$

$$\phi + \psi_1 = \frac{l_P + l_3}{I_1} \int \frac{dt}{1 + z}.$$

Let us start by integrating the integrals in the last two equations. To integrate, let us consider the following relation

$$\begin{aligned}\int \frac{dt}{1+z} &= \int \frac{sn^2 C - sn^2 w}{(1+a_4)(sn^2 B - sn^2 w)} dt \\ &= \frac{1}{(1+a_4)\epsilon_1} \int \left(1 + \frac{sn^2 C - sn^2 B}{sn^2 B - sn^2 w}\right) dw,\end{aligned}$$

where

$$\epsilon_1 = \frac{1}{2} \sqrt{(a_2 - a_4)(a_1 - a_3)} \sqrt{\frac{P^2}{I_1} \left(\frac{1}{m_3} - \frac{1}{m_1}\right)}.$$

The elliptic functions of the parameters are expressed in the following

$$\begin{aligned}sn^2 A &= \frac{1 - a_3}{1 - a_4} \frac{a_2 - a_4}{a_2 - a_3}, \quad sn^2 B = \frac{1 + a_3}{1 + a_4} \frac{a_2 - a_4}{a_2 - a_3}, \\ cn^2 A &= \frac{1 - a_2}{1 - a_4} \frac{a_4 - a_3}{a_2 - a_3}, \quad cn^2 B = \frac{1 + a_2}{1 + a_4} \frac{a_4 - a_3}{a_2 - a_3}, \\ dn^2 A &= \frac{1 - a_1}{1 - a_4} \frac{a_4 - a_3}{a_1 - a_3}, \quad dn^2 B = \frac{1 + a_1}{1 + a_4} \frac{a_4 - a_3}{a_1 - a_3}, \\ k^2 &= \frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)}, \\ k^2 sn^2 B &= \frac{(a_1 - a_4)(1 + a_3)}{(a_1 - a_3)(1 + a_4)}.\end{aligned}$$

From these expressions, we obtain the following relations between parameters and roots

$$\begin{aligned}snBcnBdnB &= \left\{ \frac{(1+a_3)(1+a_2)(1+a_1)(a_2-a_4)(a_4-a_3)^2}{(1+a_4)^3(a_2-a_3)^2(a_1-a_3)} \right\}^{1/2}, \\ sn^2 C - sn^2 B &= \frac{a_2 - a_4}{a_2 - a_3} \frac{a_4 - a_3}{1 + a_4}, \\ snAcnAdnA &= \left\{ \frac{(1-a_3)(1-a_2)(1-a_1)(a_2-a_4)(a_4-a_3)^2}{(1-a_4)^3(a_2-a_3)^2(a_1-a_3)} \right\}^{1/2},\end{aligned}$$

and

$$sn^2 C - sn^2 A = \frac{a_2 - a_4}{a_2 - a_3} \frac{a_3 - a_4}{1 - a_4}.$$

Using the following relation[10]

$$\int \frac{sn B cn B dn B}{sn^2 B - sn^2 w} dw = -\frac{d \ln \Theta(B)}{dB} w + \frac{1}{2} \ln \frac{H(B+w)}{H(B-w)},$$

we have

$$\begin{aligned} \int \frac{dt}{1+z} &= \frac{t}{1+a_4} + 2 \left\{ \frac{P^2}{I_1} \left(\frac{1}{m_3} - \frac{1}{m_1} \right) (1 \right. \\ &\quad \left. + a_1)(1+a_2)(1+a_3)(1+a_4) \right\}^{-1/2} \left\{ -\frac{d \ln \Theta(B)}{dB} w + \frac{1}{2} \ln \frac{H(B+w)}{H(B-w)} \right\}. \end{aligned}$$

Similarly,

$$\begin{aligned} \int \frac{dt}{1-z} &= \int \frac{sn^2 C - sn^2 w}{(1-a_4)(sn^2 A - sn^2 w)} dt \\ &= \frac{1}{(1-a_4)\epsilon_1} \int \left(1 + \frac{sn^2 C - sn^2 A}{sn^2 A - sn^2 w} \right) dw, \\ \int \frac{dt}{1-z} &= \frac{t}{1-a_4} - 2 \left\{ \frac{P^2}{I_1} \left(\frac{1}{m_3} - \frac{1}{m_1} \right) (1 \right. \\ &\quad \left. - a_1)(1-a_2)(1-a_3)(1-a_4) \right\}^{-1/2} \left\{ -\frac{d \ln \Theta(A)}{dA} w + \frac{1}{2} \ln \frac{H(A+w)}{H(A-w)} \right\}. \end{aligned}$$

If we set $z = 1$ and $z = -1$ separately in equation 3.22 we find

$$(1-a_1)(1-a_2)(1-a_3)(1-a_4) = -\frac{l_3^2}{\beta} \left(\frac{l_P}{l_3} - 1 \right)^2$$

and

$$(1+a_1)(1+a_2)(1+a_3)(1+a_4) = -\frac{l_3^2}{\beta} \left(\frac{l_P}{l_3} + 1 \right)^2.$$

After substituting these equations into the equations for ϕ and ψ , we get

$$\begin{aligned} \phi - \psi_1 &= \frac{t}{I_1} \frac{l_P - l_3}{1-a_4} + \frac{1}{i} \left[2 \frac{d \ln \Theta(A)}{dA} w - \ln \frac{H(A+w)}{H(A-w)} \right] \\ &\quad + \text{constant} \end{aligned}$$

and

$$\begin{aligned}\phi + \psi_1 &= \frac{t}{I_1} \frac{l_P + l_3}{1 + a_4} + \frac{1}{i} \left[-2 \frac{d \ln \Theta(B)}{dB} w + \ln \frac{H(B + w)}{H(B - w)} \right] \\ &\quad + \text{constant}\end{aligned}$$

After addition and subtraction of each other, we get

$$\begin{aligned}\phi &= \frac{t}{2I_1} \left[\frac{l_P + l_3}{1 + a_4} + \frac{l_P - l_3}{1 - a_4} \right] + \frac{1}{i} \left[\frac{d \ln \Theta(A)}{dA} - \frac{d \ln \Theta(B)}{dB} \right] w \\ &\quad - \frac{1}{2i} \ln \frac{H(A + w)}{H(A - w)} + \frac{1}{2i} \ln \frac{H(B + w)}{H(B - w)} + \text{initial value}\end{aligned}\quad (3.40)$$

and

$$\begin{aligned}\psi_1 &= \frac{t}{2I_1} \left[\frac{l_P + l_3}{1 + a_4} - \frac{l_P - l_3}{1 - a_4} \right] - \frac{1}{i} \left[\frac{d \ln \Theta(A)}{dA} + \frac{d \ln \Theta(B)}{dB} \right] w \\ &\quad + \frac{1}{2i} \ln \frac{H(A + w)}{H(A - w)} + \frac{1}{2i} \ln \frac{H(B + w)}{H(B - w)} + \text{initial value}.\end{aligned}$$

From the above two expressions, we notice that the precession angle ϕ or the spinning angle ψ consists of three parts: the initial value, the secular part and the periodic part. The initial value is due to the integration constant and corresponds to the value at the initial time. The secular part is the part which linearly increases with the time and will be denoted as ϕ_s or ψ_s .

$$\begin{aligned}\phi_s &= \frac{t}{2I_1} \left[\frac{l_P + l_3}{1 + a_4} + \frac{l_P - l_3}{1 - a_4} \right] + \frac{1}{i} \left[\frac{d \ln \Theta(A)}{dA} - \frac{d \ln \Theta(B)}{dB} \right] w \\ \psi_s &= \omega_3 t - \frac{l_3}{I_1} t + \frac{t}{2I_1} \left[\frac{l_P + l_3}{1 + a_4} - \frac{l_P - l_3}{1 - a_4} \right] - \frac{1}{i} \left[\frac{d \ln \Theta(A)}{dA} + \frac{d \ln \Theta(B)}{dB} \right] w.\end{aligned}$$

Now we introduce a coordinate system $(X', Y', Z' = Z)$ which rotates around the Z axis with a constant angular velocity ω_ϕ , and another coordinate system

$(x', y', z' = z)$ which rotates around the z axis with a constant angular velocity ω_ψ , where

$$\omega_\phi = \frac{\phi_s}{t}, \quad \omega_\psi = \frac{\psi_s}{t}.$$

To the $X'Y'Z'$ system, the secular term of the precession angle ϕ_s vanishes. Similarly, to the $x'y'z'$ system, the secular term of the spinning angle ψ_s is eliminated. The decomposition theorem which involves only periodic terms will be observed between $X'Y'Z'$ and $x'y'z'$ systems. We will denote the remaining parts as ϕ' or ψ' . It follows that

$$\phi' = -\frac{1}{2i} \ln \frac{H(A+w)}{H(A-w)} + \frac{1}{2i} \ln \frac{H(B+w)}{H(B-w)} + \phi_0 + \pi \quad (3.41)$$

and

$$\psi' = +\frac{1}{2i} \ln \frac{H(A+w)}{H(A-w)} + \frac{1}{2i} \ln \frac{H(B+w)}{H(B-w)} + \psi_0 + \pi, \quad (3.42)$$

where

$$\psi_0 = \frac{1}{2i} \ln \frac{H(A+C)H(B+C)}{H(A-C)H(B-C)} - \frac{\pi}{2K}(A+B) \quad (3.43)$$

and

$$\phi_0 = \frac{1}{2i} \ln \frac{H(A+C)H(B-C)}{H(A-C)H(B+C)} + \frac{\pi}{2K}(A-B). \quad (3.44)$$

We have chosen the initial value $\phi_0 + \pi$ for ϕ and $\psi_0 + \pi$ for ψ to simplify the later result. Then, we get

$$\phi' = \frac{1}{2i} \ln \frac{H(A+C)H(B-C)}{H(A-C)H(B+C)} - \frac{1}{2i} \ln \frac{H(A+w)H(B-w)}{H(A-w)H(B+w)} + \frac{\pi}{2K}(A-B) + \pi.$$

and

$$\psi' = \frac{1}{2i} \ln \frac{H(A+C)H(B+C)}{H(A-C)H(B-C)} + \frac{1}{2i} \ln \frac{H(A+w)H(B+w)}{H(A-w)H(B-w)} - \frac{\pi}{2K}(A+B) + \pi.$$

The nine matrix elements of R can be found by the relations between the Eulerian angles and the direction cosines.

$$R_{11} = \cos \psi' \cos \phi' - \cos \theta \sin \phi' \sin \psi', \quad (3.45)$$

$$R_{12} = -\sin \psi' \cos \phi' - \cos \theta \sin \phi' \cos \psi', \quad (3.46)$$

$$R_{21} = \cos \psi' \sin \phi' + \cos \theta \cos \phi' \sin \psi', \quad (3.47)$$

$$R_{22} = -\sin \psi' \sin \phi' + \cos \theta \cos \phi' \cos \psi', \quad (3.48)$$

$$R_{31} = \sin \theta \sin \psi', \quad (3.49)$$

$$R_{32} = \sin \theta \cos \psi', \quad (3.50)$$

$$R_{13} = \sin \theta \sin \phi', \quad (3.51)$$

$$R_{23} = -\sin \theta \cos \phi', \quad (3.52)$$

$$R_{33} = \cos \theta = z. \quad (3.53)$$

The first four can be rewritten as the following forms

$$R_{11} = \frac{1}{4}(1 + \cos \theta)[e^{i(\psi'+\phi')} + e^{-i(\psi'+\phi')}] + \frac{1}{4}(1 - \cos \theta)[e^{i(\psi'-\phi')} + e^{-i(\psi'-\phi')}], \quad (3.54)$$

$$R_{12} = i\frac{1}{4}(1 + \cos \theta)[e^{i(\psi'+\phi')} - e^{-i(\psi'+\phi')}] + i\frac{1}{4}(1 - \cos \theta)[e^{i(\psi'-\phi')} - e^{-i(\psi'-\phi')}], \quad (3.55)$$

$$R_{21} = -i\frac{1}{4}(1 + \cos \theta)[e^{i(\psi'+\phi')} - e^{-i(\psi'+\phi')}] + i\frac{1}{4}(1 - \cos \theta)[e^{i(\psi'-\phi')} - e^{-i(\psi'-\phi')}], \quad (3.56)$$

$$R_{22} = \frac{1}{4}(1 + \cos \theta)[e^{i(\psi'+\phi')} + e^{-i(\psi'+\phi')}] - \frac{1}{4}(1 - \cos \theta)[e^{i(\psi'-\phi')} + e^{-i(\psi'-\phi')}]. \quad (3.57)$$

In these expressions, $1 + \cos \theta$ and $1 - \cos \theta$ can be found from equations 3.34 and 3.35, $e^{i\phi'}$ and $e^{i\psi'}$ can be obtained by taking the exponential power of equations 3.41

and 3.42. The multiplication of equations 3.34 and 3.35 is $\sin^2 \theta$, after taking the square root of it we can obtain $\sin \theta$

$$\sin \theta = -\frac{2i}{N} [H(A+C)H(A-C)H(w-B)H(w+B)H(B-C)H(B+C)H(w+A)H(w-A)]^{1/2},$$

where we have chosen the minus sign after taking square root since $\sin \theta$ is always positive. Let us start from equations 3.41, 3.42 and leave ϕ_0, ψ_0 unspecified. Then we have

$$\psi' = -i \left[\frac{1}{2} \ln \frac{H(A+w)H(B+w)}{H(A-w)H(B-w)} + i\psi_0 \right] + \pi$$

and

$$\phi' = i \left[\frac{1}{2} \ln \frac{H(A+w)H(B-w)}{H(A-w)H(B+w)} - i\phi_0 \right] + \pi.$$

We take the exponential powers on both side of the above two equations, then it follows

$$e^{i\psi'} = -\sqrt{\frac{H(A+w)H(B+w)}{H(A-w)H(B-w)}} e^{i\psi_0}$$

and

$$e^{i\phi'} = -\sqrt{\frac{H(w-A)H(w+B)}{H(w+A)H(w-B)}} e^{i\phi_0}.$$

After we replace Eulerian angles by their expressions of parameters, we get direction cosines as

$$R_{31} = \frac{1}{N} \sqrt{H(A-C)H(A+C)H(B-C)H(B+C)} [e^{i\psi_0} H(w+A)H(w+B) - e^{-i\psi_0} H(w-A)H(w-B)],$$

$$R_{32} = \frac{i}{N} \sqrt{H(A-C)H(A+C)H(B-C)H(B+C)} [e^{i\psi_0} H(w+A)H(w+B) + e^{-i\psi_0} H(w-A)H(w-B)],$$

$$R_{13} = \frac{1}{N} \sqrt{H(A-C)H(A+C)H(B-C)H(B+C)} [e^{i\phi_0} H(w-A)H(w+B) - e^{-i\phi_0} H(w+A)H(w-B)],$$

$$R_{23} = \frac{-i}{N} \sqrt{H(A-C)H(A+C)H(B-C)H(B+C)} [e^{i\phi_0} H(w-A)H(w+B) + e^{-i\phi_0} H(w+A)H(w-B)],$$

$$R_{11} = \frac{H(A-C)H(A+C)}{2N} [e^{i(\phi_0+\psi_0)} H^2(w+B) + e^{-i(\phi_0+\psi_0)} H^2(w-B)] - \frac{H(B-C)H(B+C)}{2N} [e^{-i(\phi_0-\psi_0)} H^2(w+A) + e^{i(\phi_0-\psi_0)} H^2(w-A)],$$

$$R_{12} = \frac{iH(A-C)H(A+C)}{2N} [e^{i(\phi_0+\psi_0)} H^2(w+B) - e^{-i(\phi_0+\psi_0)} H^2(w-B)] - \frac{iH(B-C)H(B+C)}{2N} [e^{-i(\phi_0-\psi_0)} H^2(w+A) - e^{i(\phi_0-\psi_0)} H^2(w-A)],$$

$$R_{21} = \frac{-iH(A-C)H(A+C)}{2N} [e^{i(\phi_0+\psi_0)} H^2(w+B) - e^{-i(\phi_0+\psi_0)} H^2(w-B)] - \frac{iH(B-C)H(B+C)}{2N} [e^{-i(\phi_0-\psi_0)} H^2(w+A) - e^{i(\phi_0-\psi_0)} H^2(w-A)],$$

$$R_{22} = \frac{H(A-C)H(A+C)}{2N} [e^{i(\phi_0+\psi_0)} H^2(w+B) + e^{-i(\phi_0+\psi_0)} H^2(w-B)] + \frac{H(B-C)H(B+C)}{2N} [e^{-i(\phi_0-\psi_0)} H^2(w+A) + e^{i(\phi_0-\psi_0)} H^2(w-A)].$$

For abbreviation, we set

$$M_1 = \frac{H(A-C)H(B-C)}{H(A+C)H(B+C)}, \quad M_2 = \frac{H(A+C)H(B-C)}{H(A-C)H(B+C)}.$$

Then the above expressions can be rewritten as

$$\begin{aligned}
R_{11} = & \frac{1}{2N} \left[\frac{\sqrt{M_1}}{\sqrt{M_2}} e^{i(\phi_0 + \psi_0)} H^2(A + C) H^2(w + B) \right. \\
& + \frac{\sqrt{M_2}}{\sqrt{M_1}} e^{-i(\phi_0 + \psi_0)} H^2(A - C) H^2(w - B) \\
& - \sqrt{M_1} \sqrt{M_2} e^{i(\psi_0 - \phi_0)} H^2(B + C) H^2(w + A) \\
& \left. - \frac{1}{\sqrt{M_1} \sqrt{M_2}} e^{-i(\psi_0 - \phi_0)} H^2(B - C) H^2(w - A) \right],
\end{aligned}$$

$$\begin{aligned}
R_{12} = & \frac{i}{2N} \left[\frac{\sqrt{M_1}}{\sqrt{M_2}} e^{i(\phi_0 + \psi_0)} H^2(A + C) H^2(w + B) \right. \\
& - \frac{\sqrt{M_2}}{\sqrt{M_1}} e^{-i(\phi_0 + \psi_0)} H^2(A - C) H^2(w - B) \\
& - \sqrt{M_1} \sqrt{M_2} e^{i(\psi_0 - \phi_0)} H^2(B + C) H^2(w + A) \\
& \left. + \frac{1}{\sqrt{M_1} \sqrt{M_2}} e^{-i(\psi_0 - \phi_0)} H^2(B - C) H^2(w - A) \right],
\end{aligned}$$

$$\begin{aligned}
R_{21} = & \frac{-i}{2N} \left[\frac{\sqrt{M_1}}{\sqrt{M_2}} e^{i(\phi_0 + \psi_0)} H^2(A + C) H^2(w + B) \right. \\
& - \frac{\sqrt{M_2}}{\sqrt{M_1}} e^{-i(\phi_0 + \psi_0)} H^2(A - C) H^2(w - B) \\
& + \sqrt{M_1} \sqrt{M_2} e^{i(\psi_0 - \phi_0)} H^2(B + C) H^2(w + A) \\
& \left. - \frac{1}{\sqrt{M_1} \sqrt{M_2}} e^{-i(\psi_0 - \phi_0)} H^2(B - C) H^2(w - A) \right],
\end{aligned}$$

$$\begin{aligned}
R_{22} = & \frac{1}{2N} \left[\frac{\sqrt{M_1}}{\sqrt{M_2}} e^{i(\phi_0 + \psi_0)} H^2(A + C) H^2(w + B) \right. \\
& + \frac{\sqrt{M_2}}{\sqrt{M_1}} e^{-i(\phi_0 + \psi_0)} H^2(A - C) H^2(w - B) \\
& + \sqrt{M_1} \sqrt{M_2} e^{i(\psi_0 - \phi_0)} H^2(B + C) H^2(w + A) \\
& \left. + \frac{1}{\sqrt{M_1} \sqrt{M_2}} e^{-i(\psi_0 - \phi_0)} H^2(B - C) H^2(w - A) \right],
\end{aligned}$$

$$\begin{aligned}
R_{31} &= \frac{1}{N} [\sqrt{M_1} e^{i\psi_0} H(A+C)H(B+C)H(w+A)H(w+B) \\
&\quad - \frac{1}{\sqrt{M_1}} e^{-i\psi_0} H(A-C)H(B-C)H(w-A)H(w-B)], \\
R_{32} &= \frac{i}{N} [\sqrt{M_1} e^{i\psi_0} H(A+C)H(B+C)H(w+A)H(w+B) \\
&\quad + \frac{1}{\sqrt{M_1}} e^{-i\psi_0} H(A-C)H(B-C)H(w-A)H(w-B)], \\
R_{13} &= \frac{1}{N} [-\sqrt{M_2} e^{-i\phi_0} H(A-C)H(B+C)H(w+A)H(w-B) \\
&\quad + \frac{1}{\sqrt{M_2}} e^{i\phi_0} H(A+C)H(B-C)H(w-A)H(w+B)], \\
R_{23} &= \frac{-i}{N} [\sqrt{M_2} e^{-i\phi_0} H(A-C)H(B+C)H(w+A)H(w-B) \\
&\quad + \frac{1}{\sqrt{M_2}} e^{i\phi_0} H(A+C)H(B-C)H(w-A)H(w+B)], \\
R_{33} &= \frac{1}{N} [H(A+C)H(A \\
&\quad -C)H(w-B)H(w+B) + H(B-C)H(B+C)H(w+A)H(w-A)],
\end{aligned}$$

where

$$N = H(A+B)H(A-B)H(w+C)H(w-C).$$

Now if we choose ϕ_0 and ψ_0 as the value in equations (3.20) and (3.21) and substitute into the above expressions. After the substitution, we get

$$\begin{aligned}
R_{11} &= \frac{1}{2N} [E_2 E_1^{-1} H^2(A+C)H^2(w+B) + E_2^{-1} E_1 H^2(A-C)H^2(w-B) \\
&\quad - E_2^{-1} E_1^{-1} H^2(B+C)H^2(w+A) - E_2 E_1 H^2(B-C)H^2(w-A)], \\
R_{12} &= \frac{i}{2N} [E_2 E_1^{-1} H^2(A+C)H^2(w+B) - E_2^{-1} E_1 H^2(A-C)H^2(w-B) \\
&\quad - E_2^{-1} E_1^{-1} H^2(B+C)H^2(w+A) + E_2 E_1 H^2(B-C)H^2(w-A)],
\end{aligned}$$

$$R_{21} = \frac{-i}{2N} [E_2 E_1^{-1} H^2(A+C) H^2(w+B) - E_2^{-1} E_1 H^2(A-C) H^2(w-B) \\ + E_2^{-1} E_1^{-1} H^2(B+C) H^2(w+A) - E_2 E_1 H^2(B-C) H^2(w-A)],$$

$$R_{22} = \frac{1}{2N} [E_2 E_1^{-1} H^2(A+C) H^2(w+B) + E_2^{-1} E_1 H^2(A-C) H^2(w-B) \\ + E_2^{-1} E_1^{-1} H^2(B+C) H^2(w+A) + E_2 E_1 H^2(B-C) H^2(w-A)],$$

$$R_{31} = \frac{1}{N} [E_1^{-1} H(A+C) H(B+C) H(w+A) H(w+B) \\ - E_1 H(A-C) H(B-C) H(w-A) H(w-B)],$$

$$R_{32} = \frac{i}{N} [E_1^{-1} H(A+C) H(B+C) H(w+A) H(w+B) \\ + E_1 H(A-C) H(B-C) H(w-A) H(w-B)],$$

$$R_{13} = \frac{1}{N} [-E_2^{-1} H(A-C) H(B+C) H(w+A) H(w-B) \\ + E_2 H(A+C) H(B-C) H(w-A) H(w+B)],$$

$$R_{23} = \frac{-i}{N} [E_2^{-1} H(A-C) H(B+C) H(w+A) H(w-B) \\ + E_2 H(A+C) H(B-C) H(w-A) H(w+B)],$$

where

$$N = H(A+B) H(A-B) H(w+C) H(w-C),$$

$$E_1 = \exp\left\{\frac{\pi i}{2K}(A+B)\right\}, \quad E_2 = \exp\left\{\frac{\pi i}{2K}(A-B)\right\}.$$

The exponential parts E_1 and E_2 in the above direction cosines are due to the choice of the initial values in equation 3.43 and equation 3.44. We need these exponential parts later to cancel the exponential parts which will appear due to the transformation between theta functions.

3.2 The Translation of the Center of Mass

From equation 3.1 to 3.6, we can get[16] [17]

$$R_{11}\frac{\partial T}{\partial u_1} + R_{12}\frac{\partial T}{\partial u_2} + R_{13}\frac{\partial T}{\partial u_3} = p_1, \quad (3.58)$$

$$R_{21}\frac{\partial T}{\partial u_1} + R_{22}\frac{\partial T}{\partial u_2} + R_{23}\frac{\partial T}{\partial u_3} = p_2, \quad (3.59)$$

$$R_{31}\frac{\partial T}{\partial u_1} + R_{32}\frac{\partial T}{\partial u_2} + R_{33}\frac{\partial T}{\partial u_3} = p_3, \quad (3.60)$$

$$R_{11}\frac{\partial T}{\partial \omega_1} + R_{12}\frac{\partial T}{\partial \omega_2} + R_{13}\frac{\partial T}{\partial \omega_3} + Y_cp_3 - Z_cp_2 = l_1, \quad (3.61)$$

$$R_{21}\frac{\partial T}{\partial \omega_1} + R_{22}\frac{\partial T}{\partial \omega_2} + R_{23}\frac{\partial T}{\partial \omega_3} + Z_cp_1 - X_cp_3 = l_2, \quad (3.62)$$

$$R_{31}\frac{\partial T}{\partial \omega_1} + R_{32}\frac{\partial T}{\partial \omega_2} + R_{33}\frac{\partial T}{\partial \omega_3} + X_cp_2 - Y_cp_1 = l_3. \quad (3.63)$$

The first three equations are the components of the total linear momentum along the three space axes, the second three equations are the components of the total angular momentum projected in the three space axes. Since we had chosen the direction of the total linear momentum as the Z -axis, the X, Y components are zero.

$$p_1 = p_2 = 0, \quad p_3 = P. \quad (3.64)$$

Since the angular momentum depends on where the choice of the origin of the coordinates, the X, Y components of the total angular momentum can be made zero by properly choosing the origin of the space system axes. Hence

$$l_1 = 0, \quad l_2 = 0. \quad (3.65)$$

Equations 3.61, 3.62, 3.64 and 3.65 imply that

$$X_c = \frac{1}{P}(R_{21}I_1\omega_1 + R_{22}I_1\omega_2 + R_{23}I_3\omega_3), \quad (3.66)$$

$$Y_c = \frac{-1}{P}(R_{11}I_1\omega_1 + R_{12}I_1\omega_2 + R_{13}I_3\omega_3). \quad (3.67)$$

For a symmetrical body the component of the angular velocity along the symmetry axis, ω_3 , is a constant during the motion. For the other two components ω_1 and ω_2 , we have

$$\omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \quad (3.68)$$

and

$$\omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi. \quad (3.69)$$

If we set

$$\sigma \cos \Omega = \dot{\phi} \sin \theta, \quad \sigma \sin \Omega = \dot{\theta}, \quad (3.70)$$

then we can prove

$$\omega_1 = \sigma \sin(\psi + \Omega), \quad \omega_2 = \sigma \cos(\psi + \Omega), \quad (3.71)$$

where

$$\sigma^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \quad (3.72)$$

For eqs. 3.18 and 3.38, we get

$$\dot{\theta}^2 = \frac{\alpha - \beta z^2}{I_1^2} - \frac{(l_P - l_3 \cos \theta)^2}{I_1^2 \sin^2 \theta}, \quad (3.73)$$

$$\dot{\phi} = \frac{l_P - l_3 \cos \theta}{I_1(1 - \cos^2 \theta)}. \quad (3.74)$$

Then,

$$\sigma^2 = \frac{\alpha - \beta z^2}{I_1^2} \quad (3.75)$$

From equations 3.12 to 3.14, we get

$$u_1 = \frac{P}{m_1} \sin \theta \sin \psi = s \sin \psi, \quad (3.76)$$

$$u_2 = \frac{P}{m_1} \sin \theta \cos \psi = s \cos \psi, \quad (3.77)$$

where

$$s = \frac{P}{m_1} \sin \theta. \quad (3.78)$$

Since

$$\omega_1 u_1 + \omega_2 u_2 = s \sigma \cos \Omega, \quad (3.79)$$

together with equations 3.10 and 3.15, we get

$$s \sigma \cos \Omega = \frac{P}{m_1 I_1} (l_P - l_3 z). \quad (3.80)$$

From equations 3.75, 3.78 and 3.80, the expressions for ω_1 and ω_2 (3.71) follows.

Substitutions of $\omega_1, \omega_2, \omega_3$ into eqs. 3.66 - 3.67, then yield X_c, Y_c .

To find Z_c , let us consider the projection of the translational velocity on the Z -axis of the space system.

$$\begin{aligned} \frac{dZ_c}{dt} &= u_1 R_{31} + u_2 R_{32} + u_3 R_{33} = \frac{PR_{31}^2}{m_1} + \frac{PR_{32}^2}{m_1} + \frac{PR_{33}^2}{m_3} \\ &= \frac{P}{m_1} + P \left(\frac{1}{m_3} - \frac{1}{m_1} \right) z^2 \end{aligned} \quad (3.81)$$

Integrating, we get

$$Z_c = \frac{P}{m_1} t + \frac{\sqrt{\beta}}{P\epsilon} \int z^2 dw. \quad (3.82)$$

For the evaluation of the integral in the above equation, we rewrite equation 3.28 as

$$z = a_4 + (a_3 - a_4)\left(1 - \frac{a_2 - a_3}{a_2 - a_4}sn^2w\right)^{-1}. \quad (3.83)$$

Now, the integral becomes

$$\begin{aligned} \int z^2 dw &= a_4^2 w + 2a_4(a_3 - a_4) \int \frac{dw}{1 - (a_2 - a_3)(a_2 - a_4)^{-1}sn^2w} \\ &\quad + (a_3 - a_4)^2 \int \frac{dw}{[1 - (a_2 - a_3)(a_2 - a_4)^{-1}sn^2w]^2}. \end{aligned} \quad (3.84)$$

Legendre's reduction formula[18] is

$$\begin{aligned} \frac{2\delta}{r} \int \frac{dw}{[1 + rsn^2w]^2} &= \frac{dnwsnwcw}{1 + rsn^2w} - \frac{k^2}{r^2} \int (1 + rsn^2w)dw \\ &\quad + \left(1 + \frac{2 + 2k^2}{r} + \frac{3k^2}{r^2}\right) \int \frac{dw}{1 + rsn^2w}, \end{aligned} \quad (3.85)$$

where

$$\delta \equiv (1 + r)\left(1 + \frac{k^2}{r}\right),$$

and k is the modulus. If we set

$$r = -\frac{a_2 - a_3}{a_2 - a_4},$$

then

$$\begin{aligned} \delta &= -\frac{(a_3 - a_4)^2}{(a_1 - a_3)(a_2 - a_4)}, \quad \frac{\delta}{r} = \frac{(a_3 - a_4)^2}{(a_1 - a_3)(a_2 - a_3)}, \\ \frac{k^2}{r^2} &= \frac{a_1 - a_4}{a_1 - a_3} \frac{a_2 - a_4}{a_2 - a_3}. \end{aligned}$$

We also need

$$1 + \frac{2 + 2k^2}{r} + \frac{3k^2}{r^2} = \frac{(a_3 - a_4)(a_1 + a_2 + a_3 - 3a_4)}{(a_1 - a_3)(a_2 - a_3)}.$$

Hence

$$\begin{aligned}
& (a_3 - a_4)^2 \int \frac{dw}{[1 - (a_2 - a_3)(a_2 - a_4)^{-1}sn^2w]^2} = \\
& \frac{1}{2} \frac{dnwsnwcnw}{1 - (a_2 - a_3)(a_2 - a_4)^{-1}sn^2w} (a_1 - a_3)(a_2 - a_3) \\
& - \frac{1}{2} (a_1 - a_4)(a_2 - a_4) \int \left(1 - \frac{a_2 - a_3}{a_2 - a_4} sn^2w\right) dw \\
& + \frac{1}{2} (a_3 - a_4)(a_1 + a_2 + a_3 - 3a_4) \int \frac{dw}{1 - (a_2 - a_3)(a_2 - a_4)^{-1}sn^2w}.
\end{aligned}$$

Finally

$$\begin{aligned}
\int z^2 dw &= [a_4^2 - \frac{1}{2}(a_1 - a_4)(a_2 - a_4)]w \\
&+ \frac{1}{2}(a_1 - a_3)(a_2 - a_3) \frac{dnwsnwcnw}{1 - (a_2 - a_3)(a_2 - a_4)^{-1}sn^2w} \\
&+ \frac{1}{2}(a_1 - a_4)(a_2 - a_3) \int sn^2w dw.
\end{aligned}$$

The last integral in the above equation is [9]

$$\int sn^2w dw = \frac{1}{k^2} \left[w - \frac{E}{K} w - Z(w) \right], \quad (3.86)$$

where E is the complete elliptic integral of the second kind and $Z(w)$ is the zeta function. After combining the above several relations, we get

$$\begin{aligned}
\int z^2 dw &= a_4^2 w - \frac{1}{2}(a_2 - a_4)(a_3 - a_4)w + \frac{1}{2}(a_1 - a_3)(a_2 \\
&- a_3) \frac{snwcnw dnw}{1 - (a_2 - a_3)(a_2 - a_4)^{-1}sn^2w} - \frac{1}{2}(a_1 - a_3)(a_2 - a_4) \left[\frac{E}{K} w + Z(w) \right].
\end{aligned}$$

The result is

$$\begin{aligned}
Z_c &= \frac{P}{m_1} t + \frac{\sqrt{\beta}}{P\epsilon} \left\{ a_4^2 w - \frac{1}{2}(a_2 - a_4)(a_3 - a_4)w + \frac{1}{2}(a_1 - a_3)(a_2 \right. \\
&- a_3) \frac{snwcnw dnw}{1 - (a_2 - a_3)(a_2 - a_4)^{-1}sn^2w} - \frac{1}{2}(a_1 - a_3)(a_2 - a_4) \left[\frac{E}{K} w + Z(w) \right] \}. \quad (3.87)
\end{aligned}$$

We can also express Z_c in terms of the parameter c . From equations 3.33 and 3.26, we get

$$1 - \frac{a_2 - a_3}{a_2 - a_4} \operatorname{sn}^2 w = 1 - k^2 \operatorname{sn}^2 c \operatorname{sn}^2 w.$$

The formula in Jacobian theta functions[13]

$$1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v = \frac{\Theta^2(0)\Theta(u+v)\Theta(u-v)}{\Theta^2(u)\Theta(v)}$$

is useful, when we get

$$1 - k^2 \operatorname{sn}^2 c \operatorname{sn}^2 w = \frac{\Theta^2(0)\Theta(w+c)\Theta(w-c)}{\Theta^2(w)\Theta(c)}. \quad (3.88)$$

Since

$$\operatorname{sn} w \operatorname{cn} w \operatorname{dn} w = \frac{1}{2} \frac{d}{dw} \operatorname{sn}^2 w,$$

hence

$$\begin{aligned} \frac{\operatorname{sn} w \operatorname{cn} w \operatorname{dn} w}{1 - k^2 \operatorname{sn}^2 c \operatorname{sn}^2 w} &= -\frac{1}{2k^2 \operatorname{sn}^2 c} \frac{d}{dw} \ln(1 - k^2 \operatorname{sn}^2 c \operatorname{sn}^2 w). \\ &= -\frac{1}{2k^2 \operatorname{sn}^2 c} \frac{d}{dw} [\ln \Theta(w+c) + \ln \Theta(w-c) - 2 \ln \Theta(w)] \\ &= -\frac{1}{2k^2 \operatorname{sn}^2 c} [Z(w+c) + Z(w-c) - 2Z(w)]. \end{aligned}$$

The expression for Z_c is now

$$\begin{aligned} Z_c &= \frac{P}{m_1} t + \frac{\sqrt{\beta}}{P\epsilon} \left\{ a_4^2 w - \frac{1}{2} (a_2 - a_4)(a_3 - a_4) w \right. \\ &\quad \left. - \frac{1}{2} (a_1 - a_3)(a_2 - a_4) \frac{E}{K} w - \frac{1}{4} (a_1 - a_3)(a_2 - a_4) [Z(w+c) + Z(w-c)] \right\}. \end{aligned} \quad (3.89)$$

Chapter 4

Generalization of Jacobi's

Decomposition Theorem to the

Rotation and Translation of a

Solid in a Fluid

Jacobi has demonstrated that the orthogonal matrix M , which describes the motion of a heavy symmetrical top can be decomposed into a product of two orthogonal matrices P and Q , each of which represents the motion of a torque-free triaxial top.

In this chapter we generalize Jacobi's decomposition theorem to Kirchhoff's

case. We find that the matrix for the rotation of Kirchhoff's case can be decomposed into two matrices each of which represents the motion of a torque-free triaxial body. Our result is a generalization of Jacobi's decomposition theorem which applies to the rotation of a symmetrical heavy top about a fixed point. Our method is also a generalization of Jacobi's method where only two parameters are necessary.

Our basic formula[11] is

$$2H(w')H(x')H(y')H(z') = \Theta_1(w)\Theta_1(x)\Theta_1(y)\Theta_1(z) - H_1(w)H_1(x)H_1(y)H_1(z) \\ - \Theta(w)\Theta(x)\Theta(y)\Theta(z) + H(w)H(x)H(y)H(z),$$

where

$$w = \frac{1}{2}(w' + x' + y' + z'), \quad w' = \frac{1}{2}(w + x + y + z), \\ x = \frac{1}{2}(w' + x' - y' - z'), \quad x' = \frac{1}{2}(w + x - y - z), \\ y = \frac{1}{2}(w' - x' + y' - z'), \quad y' = \frac{1}{2}(w - x + y - z), \\ z = \frac{1}{2}(w' - x' - y' + z'), \quad z' = \frac{1}{2}(w - x - y + z),$$

and w', x', y', z' are four arbitrary quantities. This formula is one of the many addition theorems for theta functions. It is called the addition theorem because w', x', y', z' are obviously the additions of w, x, y, z .

Apply this addition theorem to the two terms in the expression of z , we get

$$2H(A + C)H(A - C)H(w + B)H(w - B)$$

$$\begin{aligned}
&= -2H(A+C)H(-A+C)H(w+B)H(w-B) \\
&= -\Theta_1(w+C)\Theta_1(w-C)\Theta_1(A+B)\Theta_1(A-B) \\
&\quad +H_1(w+C)H_1(w-C)H_1(A+B)H_1(A-B) \\
&\quad +\Theta(w+C)\Theta(w-C)\Theta(A+B)\Theta(A-B) \\
&\quad +H(w+C)H(w-C)H(A+B)H(A-B)
\end{aligned}$$

and

$$\begin{aligned}
&2H(B+C)H(B-C)H(w+A)H(w-A) \\
&= -2H(B+C)H(-B+C)H(w+A)H(w-A) \\
&= -\Theta_1(w+C)\Theta_1(w-C)\Theta_1(A+B)\Theta_1(A-B) \\
&\quad +H_1(w+C)H_1(w-C)H_1(A+B)H_1(A-B) \\
&\quad +\Theta(w+C)\Theta(w-C)\Theta(A+B)\Theta(A-B) \\
&\quad -H(w+C)H(w-C)H(A+B)H(A-B).
\end{aligned}$$

When we add up the above two equations, the fourth term in each equation cancels each other, we get an expression for z

$$\begin{aligned}
z = & \frac{1}{N} [\Theta(w+C)\Theta(w-C)\Theta(A+B)\Theta(A-B) \\
& +H_1(w+C)H_1(w-C)H_1(A+B)H_1(A-B) \\
& -\Theta_1(w+C)\Theta_1(w-C)\Theta_1(A+B)\Theta_1(A-B)].
\end{aligned}$$

For the direction cosine R_{31} , we need following decompositions

$$2H(A+C)H(B+C)H(w+A)H(w+B)$$

$$\begin{aligned}
&= \Theta_1(w + C + A + B)\Theta_1(w - C)\Theta_1(A - B)\Theta_1(0) \\
&\quad - H_1(w + C + A + B)H_1(w - C)H_1(A - B)H_1(0) \\
&\quad - \Theta(w + C + A + B)\Theta(w - C)\Theta(A - B)\Theta(0) \\
&\quad 2H(A - C)H(B - C)H(w - A)H(w - B) \\
&= \Theta_1(w + C - A - B)\Theta_1(w - C)\Theta_1(A - B)\Theta_1(0) \\
&\quad - H_1(w + C - A - B)H_1(w - C)H_1(A - B)H_1(0) \\
&\quad - \Theta(w + C - A - B)\Theta(w - C)\Theta(A - B)\Theta(0)
\end{aligned}$$

where H is an odd function, $H(0)$ is 0, therefore only three terms in each of the above two equations. We substitute these decomposed terms into the expression for R_{31} . Then

$$\begin{aligned}
R_{31} &= \frac{1}{2N} \{ \Theta_1(0)\Theta_1(A - B)\Theta_1(w - C)[E_1^{-1}\Theta_1(w + C + A + B) \\
&\quad - E_1\Theta_1(w + C - A - B)] \\
&\quad - H_1(0)H_1(A - B)H_1(w - C)[E_1^{-1}H_1(w + C + A + B) \\
&\quad - E_1H_1(w + C - A - B)] \\
&\quad - \Theta(0)\Theta(A - B)\Theta(w - C)[E_1^{-1}\Theta(w + C + A + B) \\
&\quad - E_1\Theta(w + C - A - B)] \},
\end{aligned}$$

Similarly, the other direction cosines need the following decompositions

$$\begin{aligned}
&2H(A - C)H(B + C)H(w + A)H(w - B) \\
&= -2H(A - C)H(-B - C)H(w + A)H(w - B)
\end{aligned}$$

$$\begin{aligned}
&= -\Theta_1(w - C + A - B)\Theta_1(w + C)\Theta_1(A + B)\Theta_1(0) \\
&\quad + H_1(w - C + A - B)H_1(w + C)H_1(A + B)H_1(0) \\
&\quad + \Theta(w - C + A - B)\Theta(w + C)\Theta(A + B)\Theta(0) \\
&\quad - H(w - C + A - B)H(w + C)H(A + B)H(0),
\end{aligned}$$

$$\begin{aligned}
&\quad 2H(A + C)H(B - C)H(w - A)H(w + B) \\
&= -2H(-A - C)H(B - C)H(w - A)H(w + B) \\
&= -\Theta_1(w - C - A + B)\Theta_1(w + C)\Theta_1(A + B)\Theta_1(0) \\
&\quad + H_1(w - C - A + B)H_1(w + C)H_1(A + B)H_1(0) \\
&\quad + \Theta(w - C - A + B)\Theta(w + C)\Theta(A + B)\Theta(0) \\
&\quad + H(w - C - A + B)H(w + C)H(A + B)H(0),
\end{aligned}$$

$$\begin{aligned}
&\quad 2H^2(A + C)H^2(w + B) \\
&= \Theta_1(w + C + A + B)\Theta_1(w - C - A + B)\Theta_1^2(0) \\
&\quad - H_1(w + C + A + B)H_1(w - C - A + B)H_1^2(0) \\
&\quad - \Theta(w + C + A + B)\Theta(w - C - A + B)\Theta^2(0) \\
&\quad - H(w + C + A + B)H(w - C - A + B)H^2(0),
\end{aligned}$$

$$\begin{aligned}
&\quad 2H^2(A - C)H^2(w - B) \\
&= \Theta_1(w + C - A - B)\Theta_1(w - C + A - B)\Theta_1^2(0) \\
&\quad - H_1(w + C - A - B)H_1(w - C + A - B)H_1^2(0)
\end{aligned}$$

$$\begin{aligned}
& -\Theta(w+C-A-B)\Theta(w-C+A-B)\Theta^2(0) \\
& -H(w+C-A-B)H(w-C+A-B)H^2(0), \\
& 2H^2(B+C)H^2(w+A) \\
= & \Theta_1(w+C+A+B)\Theta_1(w-C+A-B)\Theta_1^2(0) \\
& -H_1(w+C+A+B)H_1(w-C+A-B)H_1^2(0) \\
& -\Theta(w+C+A+B)\Theta(w-C-A+B)\Theta^2(0) \\
& -H(w+C+A+B)H(w-C-A+B)H^2(0), \\
& 2H^2(B-C)H^2(w-A) \\
= & \Theta_1(w+C-A-B)\Theta_1(w-C-A+B)\Theta_1^2(0) \\
& -H_1(w+C-A-B)H_1(w-C-A+B)H_1^2(0) \\
& -\Theta(w+C-A-B)\Theta(w-C-A+B)\Theta^2(0) \\
& -H(w+C-A-B)H(w-C-A+B)H^2(0),
\end{aligned}$$

They are found as:

$$\begin{aligned}
R_{32} = & \frac{i}{2N} \{ \Theta_1(0)\Theta_1(A-B)\Theta_1(w-C)[E_1^{-1}\Theta_1(w+C+A+B) \\
& + E_1\Theta_1(w+C-A-B)] \\
& - H_1(0)H_1(A-B)H_1(w-C)[E_1^{-1}H_1(w+C+A+B) \\
& + E_1H_1(w+C-A-B)] \\
& - \Theta(0)\Theta(A-B)\Theta(w-C)[E_1^{-1}\Theta(w+C+A+B) \\
& + E_1\Theta(w+C-A-B)] \},
\end{aligned}$$

$$\begin{aligned}
R_{13} = & \frac{1}{2N} \{ \Theta_1(0) \Theta_1(A+B) \Theta_1(w+C) [E_2^{-1} \Theta_1(w-C+A-B) \\
& - E_2 \Theta_1(w-C-A+B)] \\
& - H_1(0) H_1(A+B) H_1(w+C) [E_2^{-1} H_1(w-C+A-B) \\
& - E_2 H_1(w-C-A+B)] \\
& - \Theta(0) \Theta(A+B) \Theta(w+C) [E_2^{-1} \Theta(w-C+A-B) \\
& - E_2 \Theta(w-C-A+B)] \},
\end{aligned}$$

$$\begin{aligned}
R_{23} = & \frac{i}{2N} \{ \Theta_1(0) \Theta_1(A+B) \Theta_1(w+C) [E_2^{-1} \Theta_1(w-C+A-B) \\
& + E_2 \Theta_1(w-C-A+B)] \\
& - H_1(0) H_1(A+B) H_1(w+C) [E_2^{-1} H_1(w-C+A-B) \\
& + E_2 H_1(w-C-A+B)] \\
& - \Theta(0) \Theta(A+B) \Theta(w+C) [E_2^{-1} \Theta(w-C+A-B) \\
& + E_2 \Theta(w-C-A+B)] \},
\end{aligned}$$

$$\begin{aligned}
R_{11} = & \frac{1}{4N} \{ -\Theta_1^2(0) [E_1^{-1} \Theta_1(w+C+A+B) - E_1 \Theta_1(w+C-A-B)] \\
& [E_2^{-1} \Theta_1(w-C+A-B) - E_2 \Theta_1(w-C-A+B)] \\
& + H_1^2(0) [E_1^{-1} H_1(w+C+A+B) - E_1 H_1(w+C-A-B)] \\
& [E_2^{-1} H_1(w-C+A-B) - E_2 H_1(w-C-A+B)] \\
& + \Theta^2(0) [E_1^{-1} \Theta(w+C+A+B) - E_1 \Theta(w+C-A-B)] \\
& [E_2^{-1} \Theta(w-C+A-B) - E_2 \Theta(w-C-A+B)],
\end{aligned}$$

$$R_{12} = \frac{i}{4N} \{ -\Theta_1^2(0) [E_1^{-1} \Theta_1(w+C+A+B) + E_1 \Theta_1(w+C-A-B)]$$

$$\begin{aligned}
& [E_2^{-1}\Theta_1(w - C + A - B) - E_2\Theta_1(w - C - A + B)] \\
& + H_1^2(0)[E_1^{-1}H_1(w + C + A + B) + E_1H_1(w + C - A - B)] \\
& [E_2^{-1}H_1(w - C + A - B) - E_2H_1(w - C - A + B)] \\
& + \Theta^2(0)[E_1^{-1}\Theta(w + C + A + B) + E_1\Theta(w + C - A - B)] \\
& [E_2^{-1}\Theta(w - C + A - B) - E_2\Theta(w - C - A + B)], \\
\\
R_{21} = & \frac{i}{4N}\{-\Theta_1^2(0)[E_1^{-1}\Theta_1(w + C + A + B) - E_1\Theta_1(w + C - A - B)] \\
& [E_2^{-1}\Theta_1(w - C + A - B) + E_2\Theta_1(w - C - A + B)] \\
& + H_1^2(0)[E_1^{-1}H_1(w + C + A + B) - E_1H_1(w + C - A - B)] \\
& [E_2^{-1}H_1(w - C + A - B) + E_2H_1(w - C - A + B)] \\
& + \Theta^2(0)[E_1^{-1}\Theta(w + C + A + B) - E_1\Theta(w + C - A - B)] \\
& [E_2^{-1}\Theta(w - C + A - B) + E_2\Theta(w - C - A + B)], \\
\\
R_{22} = & \frac{-1}{4N}\{-\Theta_1^2(0)[E_1^{-1}\Theta_1(w + C + A + B) + E_1\Theta_1(w + C - A - B)] \\
& [E_2^{-1}\Theta_1(w - C + A - B) + E_2\Theta_1(w - C - A + B)] \\
& + H_1^2(0)[E_1^{-1}H_1(w + C + A + B) + E_1H_1(w + C - A - B)] \\
& [E_2^{-1}H_1(w - C + A - B) + E_2H_1(w - C - A + B)] \\
& + \Theta^2(0)[E_1^{-1}\Theta(w + C + A + B) + E_1\Theta(w + C - A - B)] \\
& [E_2^{-1}\Theta(w - C + A - B) + E_2\Theta(w - C - A + B)].
\end{aligned}$$

In order to match the mathematical expressions for the direction cosines of a triaxial body, we need some further transformations. These transformations[13]

[19]are

$$H(u) = -H_1(u + K) = H_1(u - K),$$

$$\Theta(u) = \Theta_1(u + K) = \Theta_1(u - K),$$

$$H(u) = (-i) \exp\left\{\frac{\pi i}{4K}(2u + iK')\right\} \Theta(u + iK') = i \exp\left\{-\frac{\pi i}{4K}(2u - iK')\right\} \Theta(u - iK'),$$

$$\Theta_1(u) = \exp\left\{\frac{\pi i}{4K}(2u + iK')\right\} H_1(u + iK') = \exp\left\{-\frac{\pi i}{4K}(2u - iK')\right\} H_1(u - iK'),$$

where K and K' are complete elliptic integrals of the first kind. We notice that the exponential part appears in the above transformations. Applying these transformations to z we find

$$N = H(w + C)H(w - C)H(A + B)H(A - B) = -\exp\left\{\frac{\pi i}{4K}(4w + 2iK')\right\} H_1(A + B + K)H_1(A - B + K)\Theta(w + C + iK')\Theta(w - C + iK'),$$

$$-\Theta_1(w + C)\Theta_1(w - C)\Theta_1(A + B)\Theta_1(A - B) = -\exp\left\{\frac{\pi i}{4K}(4w + 2iK')\right\} H_1(w + C + iK')H_1(w - C + iK')\Theta(A + B + K)\Theta(A - B + K),$$

$$\Theta(w + C)\Theta(w - C)\Theta(A + B)\Theta(A - B) = -\exp\left\{\frac{\pi i}{4K}(4w + 2iK')\right\} H(w + C + iK')H(w - C + iK')\Theta_1(A + B + K)\Theta_1(A - B + K),$$

and

$$H_1(w + C)H_1(w - C)H_1(A + B)H_1(A - B) = \exp\left\{\frac{\pi i}{4K}(4w + 2iK')\right\} \Theta_1(w + C + iK')\Theta_1(w - C + iK')H(A + B + K)H(A - B + K).$$

After these transformations, z becomes

$$z = \frac{1}{N_R} [\Theta(A+B+K)\Theta(A-B+K)H_1(w+C+iK')H_1(w-C+iK') \\ + \Theta_1(A+B+K)\Theta_1(A-B+K)H(w+C+iK')H(w-C+iK') \\ - H(A+B+K)H(A-B+K)\Theta_1(w+C+iK')\Theta_1(w-C+iK')],$$

where

$$N_R = H_1(A+B+K)H_1(A-B+K)\Theta(w+C+iK')\Theta(w-C+iK').$$

We note that the exponential part in the denominator and in the numerator cancel each other.

Let us now consider the transformation about R_{31} . The following transformations are used:

$$\Theta_1(A-B) = \Theta(A-B+K),$$

$$\Theta_1(w-C) = H_1(w-C+iK') \exp\left\{\frac{\pi i}{4K}[2(w-C)+iK']\right\},$$

$$\Theta_1(w+C+A+B) = H(w+C+iK'+A+B+K) \exp\left\{\frac{\pi i}{4K}[2(w+C+A+B)+iK']\right\},$$

$$\Theta_1(w+C-A-B) = -H(w+C+iK'-A-B-K) \exp\left\{\frac{\pi i}{4K}[2(w+C-A-B)+iK']\right\},$$

$$H_1(A-B) = H(A-B+K),$$

$$H_1(w-C) = \Theta_1(w-C+iK') \exp\left\{\frac{\pi i}{4K}[2(w-C)+iK']\right\},$$

$$H_1(w+C+A+B) = \Theta(w+C+iK'+A+B+K) \exp\left\{\frac{\pi i}{4K}[2(w+C+A+B)+iK']\right\},$$

$$H_1(w+C-A-B) = \Theta(w+C+iK'-A-B-K) \exp\left\{\frac{\pi i}{4K}[2(w+C-A-B)+iK']\right\},$$

$$\Theta(A - B) = \Theta_1(A - B + K),$$

$$\Theta(w - C) = -iH(w - C + iK') \exp\left\{\frac{\pi i}{4K}[2(w - C) + iK']\right\},$$

$$\Theta(w + C + A + B) = iH_1(w + C + iK' + A + B + K) \exp\left\{\frac{\pi i}{4K}[2(w + C + A + B) + iK']\right\},$$

$$\Theta(w + C - A - B) = -iH_1(w + C + iK' - A - B - K) \exp\left\{\frac{\pi i}{4K}[2(w + C - A - B) + iK']\right\}.$$

After substituting these transformations into the expression for R_{31} , we get

$$R_{31} =$$

$$\begin{aligned} & \frac{1}{2N_R} \{ \Theta_1(0) \Theta(A - B + K) H_1(w - C + iK') [-H(w + C + iK' + A + B + K) \\ & - H(w + C + iK' - A - B - K)] \\ & + \Theta(0) \Theta_1(A - B + K) H(w - C + iK') [H_1(w + C + iK' + A + B + K) \\ & + H_1(w + C + iK' - A - B - K)] \\ & + H_1(0) H(A - B + K) \Theta_1(w - C + iK') [\Theta(w + C + iK' + A + B + K) \\ & - \Theta(w + C + iK' - A - B - K)] \}, \end{aligned}$$

Similarly, for R_{32} , it has same terms except some different signs,

$$R_{32} =$$

$$\begin{aligned} & \frac{i}{2N_R} \{ \Theta_1(0) \Theta(A - B + K) H_1(w - C + iK') [-H(w + C + iK' + A + B + K) \\ & + H(w + C + iK' - A - B - K)] \\ & + \Theta(0) \Theta_1(A - B + K) H(w - C + iK') [H_1(w + C + iK' + A + B + K) \\ & - H_1(w + C + iK' - A - B - K)] \\ & + H_1(0) H(A - B + K) \Theta_1(w - C + iK') [\Theta(w + C + iK' + A + B + K) \\ & - \Theta(w + C + iK' - A - B - K)] \} \end{aligned}$$

$$+\Theta(w+C+iK'-A-B-K)]\}.$$

Here we note that the exponential part has been cancelled either by the exponential part from the denominator or by E_1 and E_2 from our earlier choice of ϕ_0 and ψ_0 . The transformation for R_{13} and R_{23} can be done by the following transformations at the same time, since they have the same items except some different plus or minus signs only.

$$\Theta_1(A+B) = \Theta(A+B+K),$$

$$\Theta_1(w+C) = H_1(w+C+iK') \exp\left\{\frac{\pi i}{4K}[2(w+C)+iK']\right\},$$

$$\Theta_1(w-C+A-B) = H(w-C+iK'+A-B+K) \exp\left\{\frac{\pi i}{4K}[2(w-C+A-B)+iK']\right\},$$

$$\Theta_1(w-C-A+B) = -H(w-C+iK'-A+B-K) \exp\left\{\frac{\pi i}{4K}[2(w-C-A+B)+iK']\right\},$$

$$H_1(A+B) = H(A+B+K),$$

$$H_1(w+C) = \Theta_1(w+C+iK') \exp\left\{\frac{\pi i}{4K}[2(w+C)+iK']\right\},$$

$$H_1(w-C+A-B) = \Theta(w-C+iK'+A-B+K) \exp\left\{\frac{\pi i}{4K}[2(w-C+A-B)+iK']\right\},$$

$$H_1(w-C-A+B) = \Theta(w-C+iK'-A+B-K) \exp\left\{\frac{\pi i}{4K}[2(w-C-A+B)+iK']\right\},$$

$$\Theta(A+B) = \Theta_1(A+B+K),$$

$$\Theta(w+C) = -iH(w+C+iK') \exp\left\{\frac{\pi i}{4K}[2(w+C)+iK']\right\},$$

$$\Theta(w-C+A-B) = iH_1(w-C+iK'+A-B+K) \exp\left\{\frac{\pi i}{4K}[2(w-C+A-B)+iK']\right\},$$

$$\Theta(w-C-A+B) = -iH_1(w-C+iK'-A+B-K) \exp\left\{\frac{\pi i}{4K}[2(w-C-A+B)+iK']\right\}.$$

After these transformations, R_{13} and R_{23} become

$$R_{13} =$$

$$\begin{aligned} & \frac{-1}{2N_R} \{ \Theta_1(0) \Theta(A+B+K) H_1(w+C+iK') [H(w-C+iK'+A-B+K) \\ & + H(w-C+iK'-A+B-K)] \\ & - H_1(0) H(A+B+K) \Theta_1(w+C+iK') [\Theta(w-C+iK'+A-B+K) \\ & - \Theta(w-C+iK'-A+B-K)] \\ & - \Theta(0) \Theta_1(A+B+K) H(w+C+iK') [H_1(w-C+iK'+A-B+K) \\ & + H_1(w-C+iK'-A+B-K)] \}, \end{aligned}$$

$$R_{23} =$$

$$\begin{aligned} & \frac{-i}{2N_R} \{ \Theta_1(0) \Theta(A+B+K) H_1(w+C+iK') [H(w-C+iK'+A-B+K) \\ & - H(w-C+iK'-A+B-K)] \\ & - H_1(0) H(A+B+K) \Theta_1(w+C+iK') [\Theta(w-C+iK'+A-B+K) \\ & + \Theta(w-C+iK'-A+B-K)] \\ & - \Theta(0) \Theta_1(A+B+K) H(w+C+iK') [H_1(w-C+iK'+A-B+K) \\ & - H_1(w-C+iK'-A+B-K)] \}, \end{aligned}$$

The transformations for R_{11} , R_{12} , R_{21} and R_{22} , can be done by those transformations above. The results are

$$R_{11} =$$

$$\frac{-1}{4N_R} \{ -[\Theta_1^2(0) H(w+C+iK'+A+B+K) + H(w+C+iK'-A-B-K)]$$

$$\begin{aligned}
& [H(w - C + iK' + A - B + K) + H(w - C + iK' - A + B - K)] \\
& + H_1^2(0)[\Theta(w + C + iK' + A + B + K) - \Theta(w + C + iK' - A - B - K)] \\
& [\Theta(w - C + iK' + A - B + K) - \Theta(w - C + iK' - A + B - K)] \\
& - \Theta^2(0)[H_1(w + C + iK' + A + B + K) + H_1(w + C + iK' - A - B - K)] \\
& [H_1(w - C + iK' + A - B + K) + H_1(w - C + iK' - A + B - K)],
\end{aligned}$$

$$R_{12} =$$

$$\begin{aligned}
& \frac{-i}{4N_R} \{ -[\Theta_1^2(0)H(w + C + iK' + A + B + K) - H(w + C + iK' - A - B - K)] \\
& [H(w - C + iK' + A - B + K) + H(w - C + iK' - A + B - K)] \\
& + H_1^2(0)[\Theta(w + C + iK' + A + B + K) + \Theta(w + C + iK' - A - B - K)] \\
& [\Theta(w - C + iK' + A - B + K) - \Theta(w - C + iK' - A + B - K)] \\
& - \Theta^2(0)[H_1(w + C + iK' + A + B + K) - H_1(w + C + iK' - A - B - K)] \\
& [H_1(w - C + iK' + A - B + K) + H_1(w - C + iK' - A + B - K)],
\end{aligned}$$

$$R_{21} =$$

$$\begin{aligned}
& \frac{-i}{4N_R} \{ -[\Theta_1^2(0)H(w + C + iK' + A + B + K) + H(w + C + iK' - A - B - K)] \\
& [H(w - C + iK' + A - B + K) - H(w - C + iK' - A + B - K)] \\
& + H_1^2(0)[\Theta(w + C + iK' + A + B + K) - \Theta(w + C + iK' - A - B - K)] \\
& [\Theta(w - C + iK' + A - B + K) + \Theta(w - C + iK' - A + B - K)] \\
& - \Theta^2(0)[H_1(w + C + iK' + A + B + K) + H_1(w + C + iK' - A - B - K)] \\
& [H_1(w - C + iK' + A - B + K) - H_1(w - C + iK' - A + B - K)],
\end{aligned}$$

$$R_{22} =$$

$$\begin{aligned} & \frac{1}{4N_R} \{ -[\Theta_1^2(0)H(w + C + iK' + A + B + K) - H(w + C + iK' - A - B - K)] \\ & [H(w - C + iK' + A - B + K) - H(w - C + iK' - A + B - K)] \\ & + H_1^2(0)[\Theta(w + C + iK' + A + B + K) + \Theta(w + C + iK' - A - B - K)] \\ & [\Theta(w - C + iK' + A - B + K) + \Theta(w - C + iK' - A + B - K)] \\ & - \Theta^2(0)[H_1(w + C + iK' + A + B + K) - H_1(w + C + iK' - A - B - K)] \\ & [H_1(w - C + iK' + A - B + K) - H_1(w - C + iK' - A + B - K)] \}. \end{aligned}$$

In the above, we have finished the necessary transformations for the direction cosines of the motion of a symmetrical solid in a fluid. Now we show those decomposed terms correspond to the product of two matrices each of which is a motion of a torque-free triaxial body. We will construct and denote these two orthogonal matrices as S and T . The orthogonal matrix Q representing the motion of a triaxial body has its matrix elements as:

$$\begin{aligned} Q_{11} &= \frac{1}{2D_q} \Theta_1(0) [H(u + iq) + H(u - iq)], \\ Q_{12} &= \frac{1}{2D_q} \Theta(0) [H_1(u + iq) + H_1(u - iq)], \\ Q_{13} &= \frac{i}{2D_q} H_1(0) [\Theta(u + iq) - \Theta(u - iq)], \\ Q_{21} &= \frac{i}{2D_q} \Theta_1(0) [H(u + iq) - H(u - iq)], \\ Q_{22} &= \frac{i}{2D_q} \Theta(0) [H_1(u + iq) - H_1(u - iq)], \\ Q_{23} &= \frac{-1}{2D_q} H_1(0) [\Theta(u + iq) + \Theta(u - iq)], \end{aligned} \tag{4.1}$$

$$Q_{31} = -\frac{\Theta(iq)H_1(u)}{D_q},$$

$$Q_{32} = \frac{\Theta_1(iq)H(u)}{D_q},$$

$$Q_{33} = -i\frac{H(iq)\Theta_1(u)}{D_q},$$

where

$$D_q = H_1(iq)\Theta(u).$$

Its variable is u and its parameter is iq . If we replace iq by $A - B + K$ and the variable u by $w - C + iK'$, then we get the matrix S . If we replace iq by $A + B + K$ and the variable u by $w + C + iK'$, then we get another matrix T . Here w is the variable and A, B and C are parameters. After these substitutions, the matrix elements of S are as follows:

$$S_{11} = \frac{\Theta_1(0)}{2N_S} [H(w - C + iK' + A - B + K) + H(w - C + iK' - A + B - K)],$$

$$S_{12} = \frac{\Theta(0)}{2N_S} [H_1(w - C + iK' + A - B + K) + H_1(w - C + iK' - A + B - K)],$$

$$S_{13} = i\frac{H_1(0)}{2N_S} [\Theta(w - C + iK' + A - B + K) - \Theta(w - C + iK' - A + B - K)],$$

$$S_{21} = i\frac{\Theta_1(0)}{2N_S} [H(w - C + iK' + A - B + K) - H(w - C + iK' - A + B - K)],$$

$$S_{22} = i\frac{\Theta(0)}{2N_S} [H_1(w - C + iK' + A - B + K) - H_1(w - C + iK' - A + B - K)],$$

$$S_{23} = -\frac{H_1(0)}{2N_S} [\Theta(w - C + iK' + A - B + K) + \Theta(w - C + iK' - A + B - K)],$$

$$S_{31} = -\frac{1}{N_S} \Theta(A - B + K)H_1(w - C + iK'),$$

$$S_{32} = \frac{1}{N_S} \Theta_1(A - B + K)H(w - C + iK'),$$

$$S_{33} = -\frac{i}{N_S} H(A - B + K) \Theta_1(w - C + iK'),$$

where

$$N_S = H_1(A - B + K) \Theta(w - C + iK').$$

For T we find

$$T_{11} = \frac{\Theta_1(0)}{2N_T} [H(w + C + iK' + A + B + K) + H(w + C + iK' - A - B - K)],$$

$$T_{12} = \frac{\Theta(0)}{2N_T} [H_1(w + C + iK' + A + B + K) + H_1(w + C + iK' - A - B - K)],$$

$$T_{13} = i \frac{H_1(0)}{2N_T} [\Theta(w + C + iK' + A + B + K) - \Theta(w + C + iK' - A - B - K)],$$

$$T_{21} = i \frac{\Theta_1(0)}{2N_T} [H(w + C + iK' + A + B + K) - H(w + C + iK' - A - B - K)],$$

$$T_{22} = i \frac{\Theta(0)}{2N_T} [H_1(w + C + iK' + A + B + K) - H_1(w + C + iK' - A - B - K)],$$

$$T_{23} = -\frac{H_1(0)}{2N_T} [\Theta(w + C + iK' + A + B + K) + \Theta(w + C + iK' - A - B - K)],$$

$$T_{31} = -\frac{1}{N_T} \Theta(A + B + K) H_1(w + C + iK'),$$

$$T_{32} = \frac{1}{N_T} \Theta_1(A + B + K) H(w + C + iK'),$$

$$T_{33} = -\frac{i}{N_T} H(A + B + K) \Theta_1(w + C + iK'),$$

where

$$N_T = H_1(A + B + K) \Theta(w + C + iK').$$

Now comparing the matrix elements of R , S , and T , we find that

$$R_{ij} = \sum_{k=1}^3 S_{ik} T_{jk},$$

or

$$R = S\tilde{T}, \quad (4.2)$$

where $\tilde{T}$ means the transpose of T matrix. This means that the motions of an intermediate system with respect to the $x'y'z'$ and to $X'Y'Z'$ both belong to the torque-free rotations of triaxial bodies.

Chapter 5

Conclusion

Jacobi found that the orthogonal matrix, which describes the motion of a symmetrical top about a fixed point under the action of the uniform gravitational force, can be decomposed into a product of two orthogonal matrices, each of which represents the motion of a torque-free triaxial top.

We have generalized the above Jacobi's decomposition theorem to the Kirchhoff's case of the rotation and translation of a symmetrical solid in a fluid, based on the following considerations:

For the motion of a symmetrical top about a fixed point under the action of the uniform gravitational force, we have the total energy

$$J = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2}I_3\omega_3^2 + mgl \cos \theta. \quad (5.1)$$

The last term

$$mgl \cos \theta$$

is the interaction potential and is of the first degree of $\cos \theta$. The equation for $\cos \theta$ is

$$\dot{\theta}^2 \sin^2 \theta = (1 - \cos^2 \theta)(\alpha' - \beta' \cos \theta) - (c_2 - c_1 \cos \theta)^2, \quad (5.2)$$

where

$$\alpha' = \frac{2J'}{I_1}, \quad \beta' = \frac{2mgl}{I_1}.$$

It is a cubic equation of $\cos \theta$.

For the motion of a solid in a fluid, the total energy is

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 + \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 + \frac{1}{2}m_3u_3^2. \quad (5.3)$$

We rewrite equation 5.3 as

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 + \frac{1}{2}\frac{P^2}{m_1} + \frac{1}{2}\left(\frac{1}{m_3} - \frac{1}{m_1}\right)P^2 \cos^2 \theta. \quad (5.4)$$

The last term

$$\frac{1}{2}\left(\frac{1}{m_3} - \frac{1}{m_1}\right)P^2 \cos^2 \theta$$

is the effective potential and is of the second degree of $\cos \theta$. The equation for $\cos \theta$ is

$$(I_1 \sin \theta \dot{\theta})^2 = (1 - \cos^2 \theta)(\alpha - \beta \cos^2 \theta) - (l_P - l_3 \cos \theta)^2, \quad (5.5)$$

where

$$\alpha = I_1\left(h - \frac{l_3^2}{I_3} - \frac{P^2}{m_1}\right), \quad \beta = I_1P^2\left(\frac{1}{m_3} - \frac{1}{m_1}\right). \quad (5.6)$$

It is a quartic equation.

Hence we applied Jacobi's method to our problem and found that the matrix R for the rotation of Kirchhoff's case can be decomposed into a product of two matrices S and T , each of which represents the motion of a torque-free triaxial body

$$R = S\tilde{T},$$

where $\tilde{T}$ is the transpose of T .

For the translational motion we give a new treatment. The position of the center of mass as a function of the time is found by a direct evaluation of the elliptic integral by means of a new theta function interpretation of Legendre's reduction formula of the elliptic integral. The forward advancing movement (Z_c component) is expressed by zeta functions. The X_c , Y_c components are expressed by angular velocities and direction cosines. The direction cosines are expressed by three parameters and theta functions. All these are different from other previous authors' treatments.

The literature of a solid moving in a fluid is mostly concerned with its mathematical solvability and its physical aspects are rarely studied. we have filled up this lacuna by adding two appendices covering the physical aspects of the problem.

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Appendices

Appendix A

The complementary parameters

In this appendix, we will show that it is natural to choose the parameters such that $x = +1$ and $x = -1$, which correspond to the poles of the two integrands $(1 \pm x)^{-1}$ in elliptic integrals. This is related with equation 2.5 in chapter two. Jacobi's ingenious introduction of the two parameters in his decomposition theorem is essentially equivalent to these alternative parameters which correspond to the poles.

The cubic equation for the heavy top is

$$\dot{x}^2 = (x - x_1)(x - x_2)(x - x_3), \quad x_3 < x_2 < x_1,$$

and for its solution we have

$$x = x_3 + (x_2 - x_3)sn^2 u.$$

with the modulus

$$k^2 = \frac{x_2 - x_3}{x_1 - x_3}.$$

Let us define the parameter A as the value of u for $x = +1$, and B for $x = -1$

$$1 = x_3 + (x_2 - x_3)sn^2 A.$$

$$-1 = x_3 + (x_2 - x_3)sn^2 B.$$

From these two equations, we obtain

$$sn^2 A = \frac{1 - x_3}{x_2 - x_3}, \quad sn^2 B = \frac{-1 - x_3}{x_2 - x_3}.$$

On the other hand, the two parameters introduced by Jacobi are

$$sn^2(ia) = -\frac{x_1 - 1}{1 - x_2}, \quad sn^2(ib) = -\frac{x_1 - x_3}{1 + x_3}. \quad (\text{A.1})$$

Now we will show that parameters A, B, ia, ib are related as

$$ia + A = iK' + K, \quad ib + B = iK',$$

or

$$ia = iK' + K - A, \quad ib = iK' - B.$$

To evaluate $sn^2(ia)$, we have the formula

$$sn^2(iK' + K - A) = \frac{dn^2 A}{k^2 cn^2 A}.$$

Since

$$cn^2 A = 1 - sn^2 A = 1 - \frac{1 - x_3}{x_2 - x_3} = \frac{x_2 - 1}{x_2 - x_3}$$

and

$$dn^2 A = 1 - k^2 sn^2 A = 1 - \frac{x_2 - x_3}{x_1 - x_3} \frac{1 - x_3}{x_2 - x_3} = \frac{x_1 - 1}{x_1 - x_3},$$

We get the result

$$sn^2(iK' + K - A) = -\frac{x_1 - 1}{1 - x_2} = sn^2(ia),$$

which is the same as equation A.1. Similarly,

$$sn^2(iK' - B) = \frac{1}{k^2 sn^2 B} = -\frac{x_1 - x_3}{1 + x_3} = sn^2(ib),$$

which is also the same result as equation A.1.

We will call the two parameters whose sum is iK' or $iK' + K$ as the complementary parameter to each other. For the generalization of Jacobi's decomposition theorem, we will use the parameters which correspond to $x = \pm 1$. A third parameter is also required as was shown in chapter three.

Appendix B

The Helical Motion

There are four roots for the equation 3.22 in chapter three. In this appendix, we will demonstrate the special case when the two real roots coincide. It is a steady precession and the center of mass moves on a helix. We will call it the helical motion.

We are quite familiar with the following two types of motion. First, a charged particle moves in a helical spiral in a uniform magnetic field[20]. With respect to the advancing direction, the helical orbit can be right- or left-handed. Second, a symmetrical top in a field of uniform gravity can rotate in a steady precession in which the inclination angle θ , the spin rate $\dot{\psi}$ and the precession rate $\dot{\phi}$ all remain constant[21]. The translation and rotation of the helical motion is the combination of the two motions described above. The solid is neither charged nor is there any magnetic field. But the interaction of the motion of the solid with the induced

flow of the surrounding liquid provides the necessary forces of constraint to make the center of mass move in a helical spiral, while the solid also simultaneously performs a steady precession.

Equations 3.8, 3.9, 3.10, 3.15 and 3.16 show that l_3 and l_P are constant. From equation 3.28, we see the range of z is between a_3 and a_2 because the range of sn^2w is between 0 and 1. When a_2 equals a_3 , the upper and lower limits of the nutation are equal such that no nutation happens, and $\dot{\theta}$ is zero, this is the steady motion which had been discussed by Kirchhoff. In this motion, the inclination angle θ is a constant. Follow that, equations 3.38 and 3.39 show that $\dot{\phi}$ and $\dot{\psi}$ are constant. All these mean that during the motion, with respect to the center of mass, the body spins around its symmetric axis with a constant spinning rate and precesses in space with a constant precession rate, while at the same time the inclination angle remains unchanged. There is no nutation.

The translational movement of the center of mass is described by X_c, Y_c and Z_c . Equation 3.81 shows that the velocity in Z component dZ_c/dt is constant because z is constant. The body is always moving forward. Z_c increases linearly with time.

$$Z_c = \frac{P}{m_1}(1 - z^2)t + \frac{P}{m_3}z^2t. \quad (\text{B.1})$$

The property of X_c and Y_c can be investigated as follows. Since $\dot{\theta}$ is zero, angular velocity components in the body system become

$$\omega_1 = \dot{\phi} \sin \theta \sin \psi, \quad (\text{B.2})$$

$$\omega_2 = \dot{\phi} \sin \theta \cos \psi, \quad (\text{B.3})$$

$$\omega_3 = \dot{\psi} + \dot{\phi} \cos \theta. \quad (\text{B.4})$$

Then in the expressions equations 3.66 and 3.67, the first two terms become

$$R_{11}\omega_1 + R_{12}\omega_2 = -\dot{\phi} \cos \theta \sin \theta \sin \phi \quad (\text{B.5})$$

and

$$R_{21}\omega_1 + R_{22}\omega_2 = \dot{\phi} \cos \theta \sin \theta \cos \phi. \quad (\text{B.6})$$

After substituting equations B.5 and B.6 into equations 3.66 and 3.67, we find that

$$X_c = \frac{1}{P}(I_1\dot{\phi} \cos \theta - I_3\omega_3) \sin \theta \cos \phi \quad (\text{B.7})$$

and

$$Y_c = \frac{1}{P}(I_1\dot{\phi} \cos \theta - I_3\omega_3) \sin \theta \sin \phi. \quad (\text{B.8})$$

If we set

$$R = \frac{1}{P}(I_1\dot{\phi} \cos \theta - I_3\omega_3) \sin \theta = \frac{1}{P} \frac{l_P \cos \theta - l_3}{\sin \theta}, \quad (\text{B.9})$$

then

$$X_c = R \cos \phi, \quad Y_c = R \sin \phi. \quad (\text{B.10})$$

Equation B.10 shows that the motion of the center of mass in the space XY plane is a circle with the radius R and the phase angle ϕ which is the same as the precession angle for the rotation of the body. So the center of mass rotates around the center of the circle at the same speed as the body precesses around its local Z axis. The time needed to move around a complete circle for the center of mass is equal to

the precession period for the body. Whether it is in counterclockwise direction or clockwise direction depends on $\dot{\phi}$ increases or decreases and does not depend on the positive or negative value of R . If R is negative, then X_c, Y_c become $-X_c, -Y_c$, which changes the starting point but not the clockwise direction. Therefore, the translational movement of the center of mass is in a helical orbit while the body rotates in a constant precession rate. In other words, if $\dot{\phi}$ is positive, the helical motion is right hand screw forward and the body precesses in positively azimuthal direction; if $\dot{\phi}$ is negative, the helical motion is left hand screw forward and the body precesses in negatively azimuthal direction.

Figure B.1 and B.2 shows how R and P vary with θ for $\dot{\phi} = \sqrt{2}$ and $\dot{\psi} = 1$ as the initial conditions. This is a right hand helix because $\dot{\phi}$ is positive. For the body, we have assumed that $I_1 = I_2 = 6, I_3 = 3, m_1 = m_2 = 1.5, m_3 = 3$. For $m_1 < m_3$, it happens only for θ from 0 to 45 degrees. R is zero at 0 and 45 degrees. P is finite at 0 degree and zero at 45 degrees.

Figure B.3 and B.4 shows how R and P vary with θ for $\dot{\phi} = \sqrt{2}$ and $\dot{\psi} = 1$ as the initial conditions. This is a right hand helix because $\dot{\phi}$ is positive. For the body, we have assumed that $I_1 = I_2 = 6, I_3 = 3, m_1 = m_2 = 3, m_3 = 1.5$. For $m_1 > m_3$, it happens only for θ from 45 to 90 degrees. R is zero at 45 and 90 degrees. P is zero at 45 degrees and infinite at 90 degrees. It is interesting that P has three different values at three different θ values (0 , 45, 90) where R 's are all zero.

The procedures of calculation are as follows. When $\theta, \dot{\phi}$ and $\dot{\psi}$ are given, l_P

and l_3 can be determined from equations 3.15 and 3.16. Now because $a_2 = a_3$, we have the following two equations

$$(1 - \cos^2 \theta)(\alpha - \beta \cos^2 \theta) - (l_P - l_3 \cos \theta)^2 = 0, \quad (\text{B.11})$$

$$-\beta \cos \theta(1 - \cos^2 \theta) - \cos \theta(\alpha - \beta \cos^2 \theta) + (l_P - l_3 \cos \theta)l_3 = 0. \quad (\text{B.12})$$

From equations B.11, B.12 and 3.15, we obtain

$$\beta = \frac{(l_3 - l_P \cos \theta)l_{AN} \sin \theta}{\cos \theta \sin^4 \theta}. \quad (\text{B.13})$$

Comparing with equation 3.19, we find that

$$P^2 = \frac{(l_3 - l_P \cos \theta)\dot{\phi}}{(m_3^{-1} - m_1^{-1}) \cos \theta \sin^2 \theta}. \quad (\text{B.14})$$

Equation 3.17 implies that

$$h = \frac{P^2}{m_1} + \frac{\beta}{I_1} \cos^2 \theta + \frac{l_3^2}{I_3} + \frac{(l_P - l_3 \cos \theta)^2}{I_1 \sin^2 \theta}. \quad (\text{B.15})$$

Also, we can get

$$l_3 - l_P \cos \theta = \sin^2 \theta [I_3 \dot{\psi} + (I_3 - I_1) \dot{\phi} \cos \theta] \quad (\text{B.16})$$

If $I_3 \dot{\psi} + (I_3 - I_1) \dot{\phi} \cos \theta > 0$, then

$$R = -\sin \theta \sqrt{I_3 \dot{\psi} + (I_3 - I_1) \dot{\phi} \cos \theta} \left[\frac{(m_3^{-1} - m_1^{-1}) \cos \theta}{\dot{\phi}} \right]^{1/2}.$$

This equation implies that for R to have physical meaning, it requires

$$\dot{\phi} > 0 \text{ when } m_1 > m_3;$$

or

$$\dot{\phi} < 0 \text{ when } m_1 < m_3;$$

When $\dot{\phi}$ is positive it means right hand helix, and negative for left hand helix. If

$I_3\dot{\psi} + (I_3 - I_1)\dot{\phi} \cos \theta < 0$, then

$$R = \sin \theta \sqrt{-[I_3\dot{\psi} + (I_3 - I_1)\dot{\phi} \cos \theta] \left[\frac{(m_3^{-1} - m_1^{-1}) \cos \theta}{-\dot{\phi}} \right]^{1/2}}.$$

This equation requires that

$$\dot{\phi} > 0 \text{ when } m_1 < m_3.$$

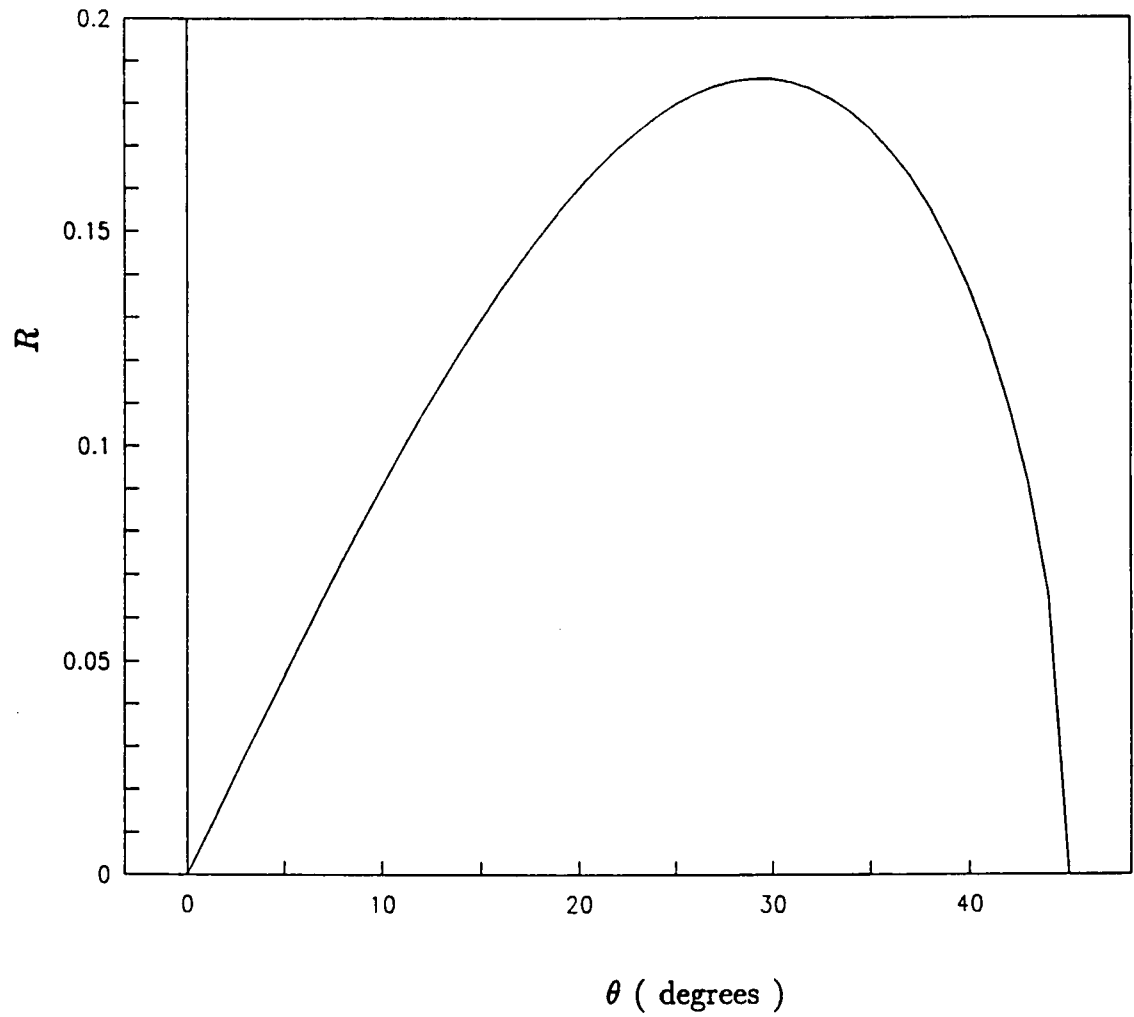


Figure B.1: The helical motion. The radius R varies with the inclination angle θ from 0 to 45 degrees for $m_1 < m_3$. At 0 and 45 degrees, R is zero.

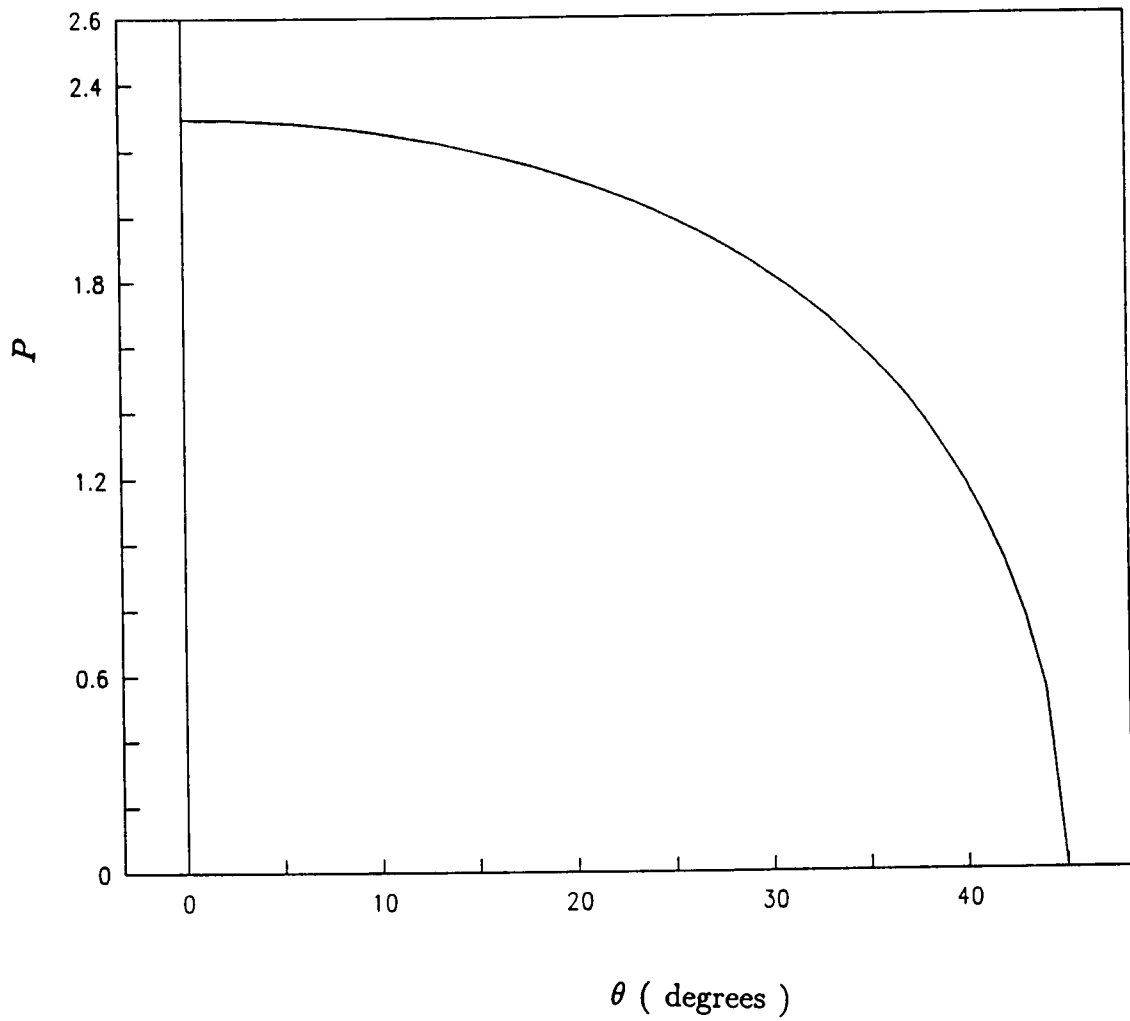


Figure B.2: The helical motion. The momentum P varies with the inclination angle θ from 0 to 45 degrees for $m_1 < m_3$. At 0 degree, P is finite. At 45 degrees, P is zero.

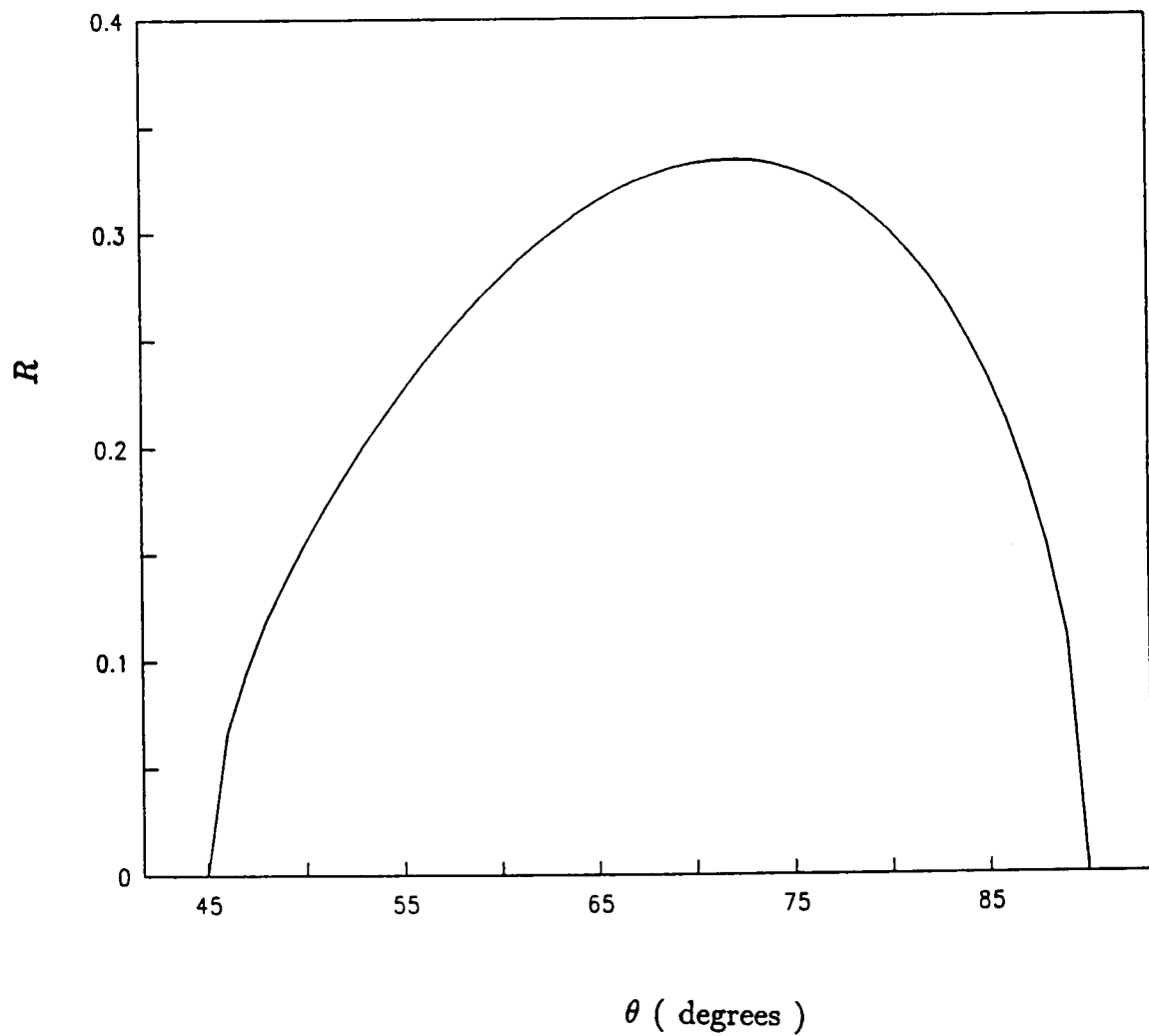


Figure B.3: The helical motion. The radius R varies with the inclination angle θ from 45 to 90 degrees for $m_1 > m_3$. At 45 and 90 degrees, R is zero.

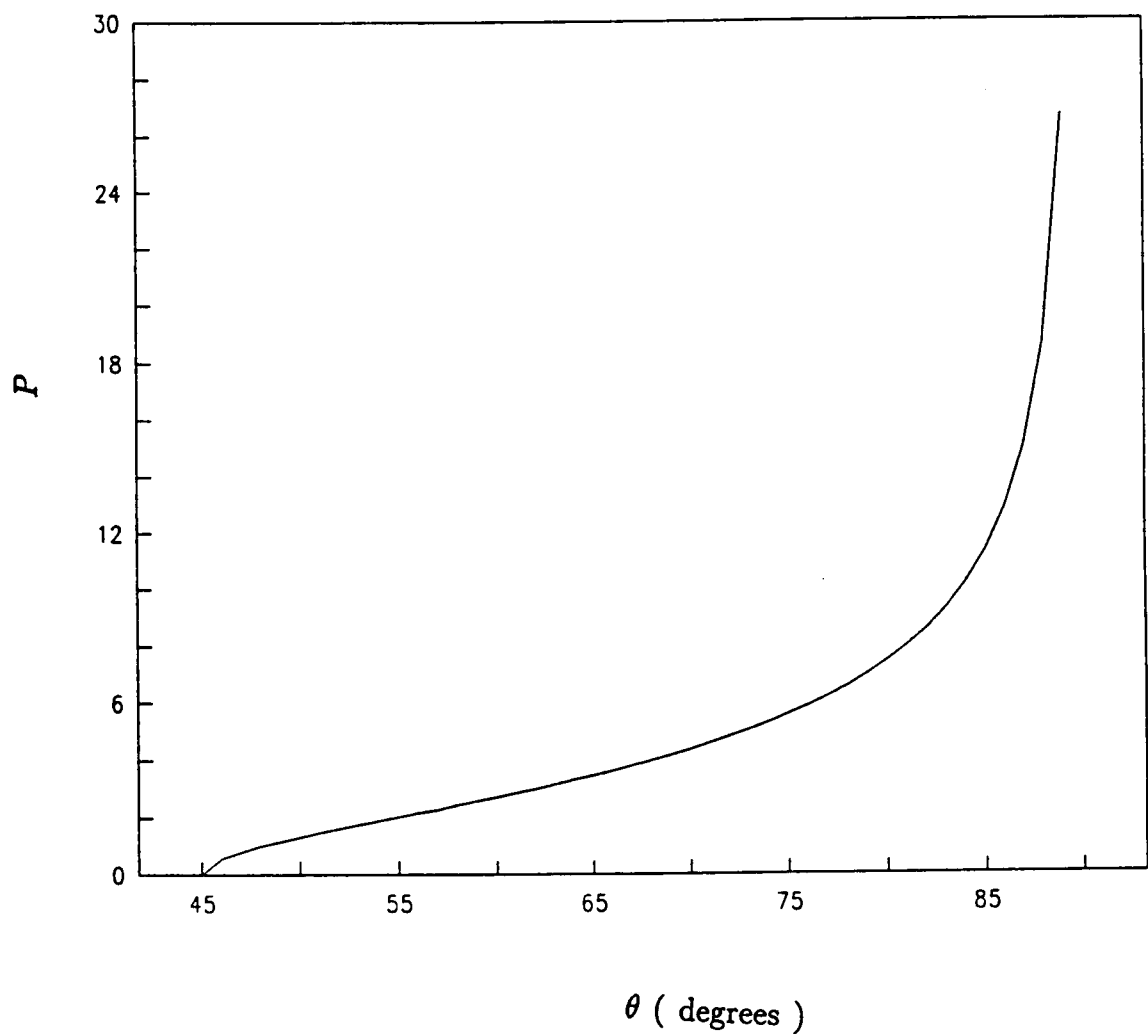


Figure B.4: The helical motion. The momentum P varies with the inclination angle θ from 45 to 90 degrees for $m_1 > m_3$. At 45 degrees, P is zero. At 90 degrees, P is infinite.

Appendix C

The General Case

In this appendix we discuss the physical aspects of the general case for the rotation and translation of a solid in a fluid, which we have derived its formulas in chapter three. The z -axis nutates between two extreme inclination angles. The line of nodes precesses progressively or retrogressively. The center of mass spirals up with various cyclic patterns (Figures C.1, C.4, C.7, C.8 and C.12 to C.15) between two concentric cylindrical surfaces. The period of the inclination angle is used as a unit of time, i.e. the unit of time in Figure C.3, C.6 and C.11 . For each period the movement completes a cycle of the pattern although it does not return to the starting place, but instead starts the next round of the same pattern. Some patterns of the movement of the center of mass are shown by curves of the forward translation versus time (Figure C.3, C.6 and C.11) and cyclic figures of the center of mass projected on the perpendicular plane (Figures C.1, C.4, C.7, C.8 and C.12

to C.15) . The rotation is shown by figures of the inclination angle versus the precession angle (Figures C.2, C.5, C.9 and C.10) . For the inclination angle less than 90 degrees, the larger radius of the two concentric cylinders corresponds to the lower limit of the inclination angle, and the smaller radius corresponds to the upper limit.

From equation 3.28, we see the range of z is between a_3 and a_2 because the range of $sn^2 w$ is between 0 and 1. The corresponding inclination angles will be denoted as θ_l and θ_u . The variation of θ is restricted between these two angles. While the axis of the symmetry of the top nutates between these two limits, its center of mass moves between surfaces of two concentric cylinders whose radii R_u and R_l are given by

$$R_i = \frac{l_P \cos \theta_i - l_3}{P \sin \theta_i}, \quad i = u \text{ or } l, \quad (\text{C.1})$$

where the subscript i can be u or l , indicating the upper or lower limit.

We have four degrees of manifolds and for an arbitrary chosen l_3 , l_P , θ_u and θ_l . The quantities P^2 and h are determined and can be computed as follows. At the upper limit, $\dot{\theta}$ is zero, so does l_N . Substituting equation 3.15 into equation 3.17, we obtain

$$h = \frac{P^2}{m_1} + \frac{\beta}{I_1} \cos^2 \theta_u + \frac{l_3^2}{I_3} + \frac{(l_P - l_3 \cos \theta_u)^2}{I_1 \sin^2 \theta_u}. \quad (\text{C.2})$$

Similarly, for θ_l

$$h = \frac{P^2}{m_1} + \frac{\beta}{I_1} \cos^2 \theta_l + \frac{l_3^2}{I_3} + \frac{(l_P - l_3 \cos \theta_l)^2}{I_1 \sin^2 \theta_l}. \quad (\text{C.3})$$

The subtraction between the above two equations results in

$$P^2 = -\left[\left(\frac{l_P - l_3 \cos \theta_u}{\sin \theta_u}\right)^2 - \left(\frac{l_P - l_3 \cos \theta_l}{\sin \theta_l}\right)^2\right] \left[I_1 \left(\frac{1}{m_3} - \frac{1}{m_1}\right) (\cos^2 \theta_u - \cos^2 \theta_l)\right]^{-1}, \quad (\text{C.4})$$

from that

$$h = P^2 \left(\frac{\cos^2 \theta_u}{m_3} + \frac{\sin^2 \theta_u}{m_1} \right) + \frac{l_3^2}{I_3} + \frac{(l_P - l_3 \cos \theta_u)^2}{I_1 \sin^2 \theta_u}. \quad (\text{C.5})$$

Equation 3.38 shows that $\dot{\phi}$ is always positive when $l_P/l_3 > a_2$. This is a progressive precession. The retrogressive precession will happen if $a_2 > l_P/l_3 > a_3$. Because $\dot{\phi}$ is negative when $\theta_u < \theta < \theta_P$, where $\theta_P = \cos^{-1}(l_P/l_3)$; and $\dot{\phi}$ is positive when $\theta_P < \theta < \theta_l$. During the period of θ , ϕ will increase and then decrease and then increase again. If $a_2 = l_P/l_3$, then $\dot{\phi}$ is zero at the upper limit. This is the cusp case and it can only happen at the upper limit. When the rotation is a cusp precession, the corresponding movement of the center of mass will be a cusped curve.

The expressions of X_c and Y_c can be further simplified after substituting the expressions for $R_{21}, R_{22}, R_{23}, R_{11}, R_{12}, R_{13}, \omega_1, \omega_2, \omega_3$.

$$X_c = \frac{1}{P} [(I_1 \sigma \cos \Omega \cos \theta - I_3 \sin \theta \omega_3) \cos \phi + I_1 \sigma \sin \Omega \sin \phi]$$

and

$$Y_c = \frac{1}{P} [(I_1 \sigma \cos \Omega \cos \theta - I_3 \sin \theta \omega_3) \sin \phi - I_1 \sigma \sin \Omega \cos \phi],$$

where σ and Ω are functions of θ . The above two equations show that for each period of θ , ϕ precesses a certain amount, X_c and Y_c will perform cyclic patterns.

To show the rotating and translation of the symmetrical solid in the fluid, we show many figures in three categories. The first category (Figures C.1, C.4, C.7, C.8 and C.12 to C.15) shows the projection of the translation of the center of mass on the plane perpendicular to its advancing trajectory, i.e. the XY plane in the space axes. X_c and Y_c are computed from equations 3.66 and 3.67. The second category (Figures C.2, C.5, C.9 and C.10) shows progressive precessions or retrogressive precessions. Figures are plotted to show the variation of θ versus ϕ , where θ is computed from equation 3.28 and ϕ from equation 3.40. The third category (Figure C.3, C.6 and C.11) shows the forward advancing of the translation of the center of mass, i.e. the Z - coordinate in the space axes system, versus time. Z_c is computed from equation 3.87. The computer software package Lotus 123 has been used to plot the figures.

Figure C.1 shows the movement of the center of mass on the XY plane. The initial conditions are: The upper and lower limits of inclination angle $\theta_u = 50, \theta_l = 80$ degrees. $l_p = 9.0, l_3 = 4.3$. The conditions for the body are: $I_1 = I_2 = 6, I_3 = 3, m_1 = m_2 = 3, m_3 = 1.5$. Equations 3.38 and C.1 show that $\dot{\phi} > 0$. $R_u = 0.8519448, R_l = 1.221415$. The movement of the center of mass starts from $X_c = -1.221415, Y_c = 0.001798$. The time used is one and a quarter periods of θ . As we can see the center of mass is moving between two circles with radii R_u and R_l , and is rotating in counterclockwise direction. At the same time the body precesses about its local Z axis while the inclination angle varies from the lower limit to the upper limit. The relation θ versus ϕ is shown in figure C.2 where θ

(the vertical) has completed one and a quarter periods. In the figure the angle is in units of the radian angle. In the Z component, the body moves always forward because equation 3.81 is always positive. However, equation 3.89 shows that there is a periodic modulation from the zeta function terms in addition to the linearly forward movement of the secular terms. Figure C.3 show how Z_c varies with time. The time used is also one and a quarter periods of θ . Whether R_u or R_l is larger can be determined by the following way. From equations C.1 and C.4, we can get

$$R_l^2 - R_u^2 = \frac{I_1(m_3^{-1} - m_1^{-1})(\cos^2 \theta_u - \cos^2 \theta_l)}{\sin^2 \theta_l \sin^2 \theta_u}. \quad (C.6)$$

The above equation shows that $R_l > R_u$ for $m_1 > m_3$ if θ_u or θ_l less than 90 degrees, and $R_l < R_u$ for $m_1 < m_3$.

In figure C.4 the initial conditions are: $\theta_u = 45, \theta_l = 70$ degrees. $l_P = 6.75, l_3 = 4.5$. The I_i and m_i of the body are the same as in figure C.1. The movement of the center of mass starts from $X_c = -0.8874848$, $Y_c = 0$. $\dot{\phi} > 0$. $R_u = 0.8519448$, $R_l = 1.221415$. The time used is also one and a quarter periods of θ . Figure C.5 shows how θ varies with ϕ at the same time. Figure C.6 shows the movement of Z component.

In figure C.7 the initial conditions are: $\theta_u = 20, \theta_l = 40$ degrees. $l_P = 0.9, l_3 = 1.0$. For the body, $I_1 = I_2 = 6, I_3 = 5$; and $m_1 = m_2 = 3, m_3 = 2$. $R_u = 1.418272$, $R_l = 1.519111$. The movement of the center of mass starts from $X_c = -1.519111$, $Y_c = 0.001175$. $\dot{\phi}$ sometimes is positive and sometimes is negative. The time used is one and a quarter periods of θ . Figure C.8 shows the same

movement for three periods of time. Figure C.9 shows how θ varies with ϕ for one and a quarter periods of time. Figure C.10 shows for three periods of time. Figure C.11 shows the movement in Z component for three periods of time.

The next four figures simply show the movement patterns for the body conditions as that of figure C.7.

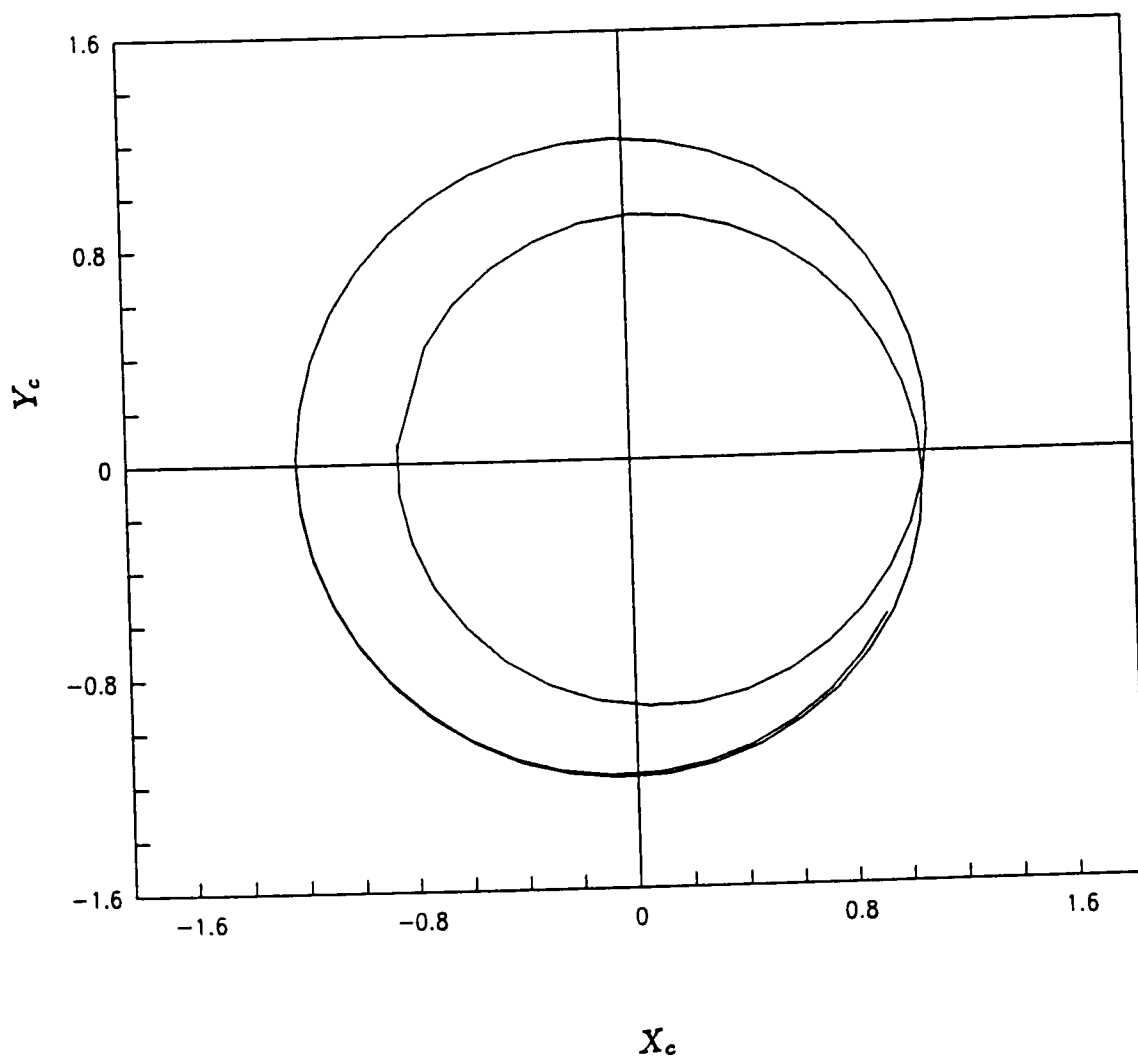


Figure C.1: The movement of the center of mass on the XY plane. $\theta_u = 50, \theta_l = 80$ degrees. $l_P = 9.0, l_3 = 4.3$. $\dot{\phi} > 0$. $R_u = 0.8519448, R_l = 1.221415$.

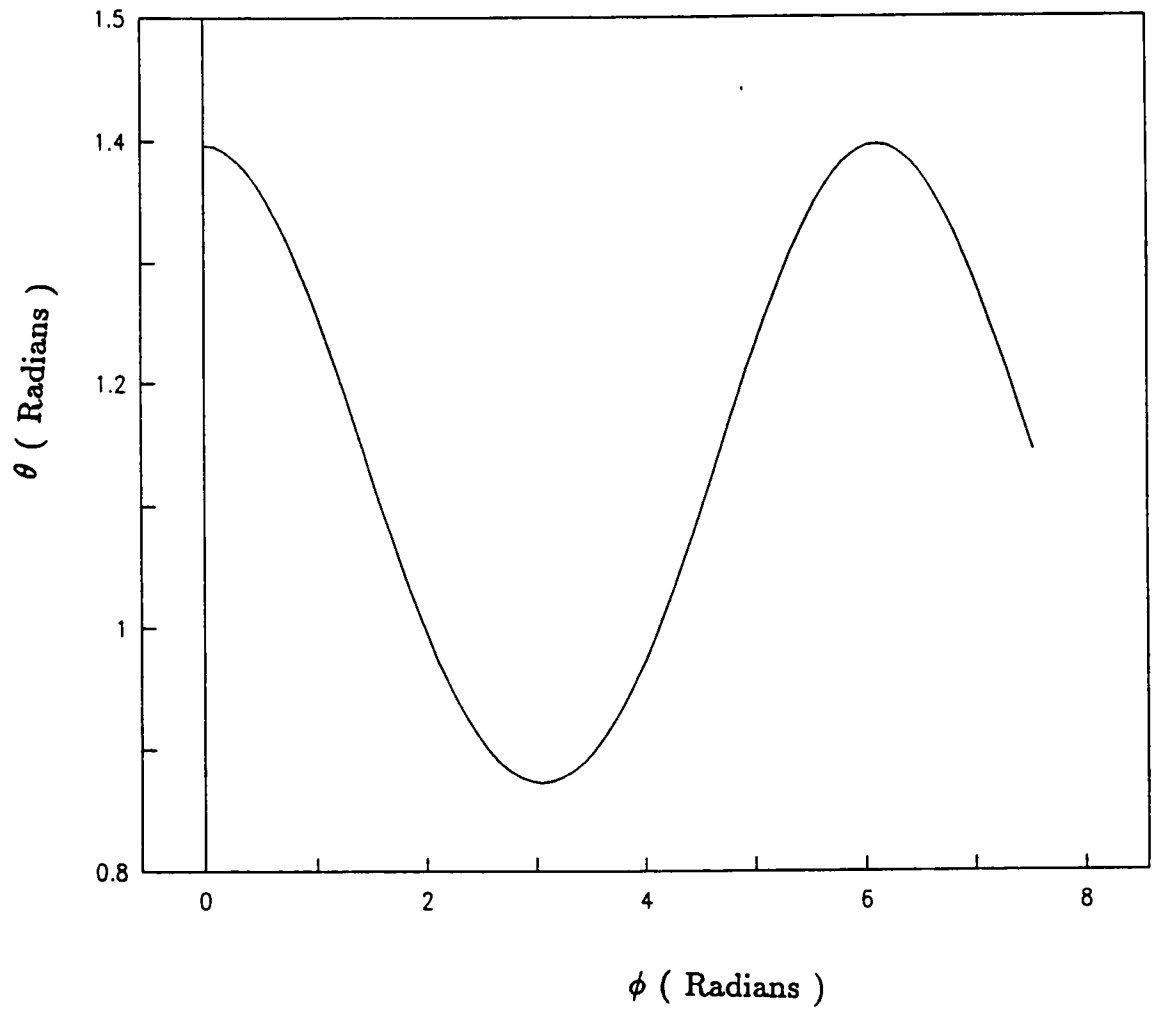


Figure C.2: θ (the vertical) vs. ϕ in the unit of radian angles. $\theta_u = 50, \theta_l = 80$ degrees. $l_P = 9.0, l_3 = 4.3$. $\dot{\phi} > 0$.

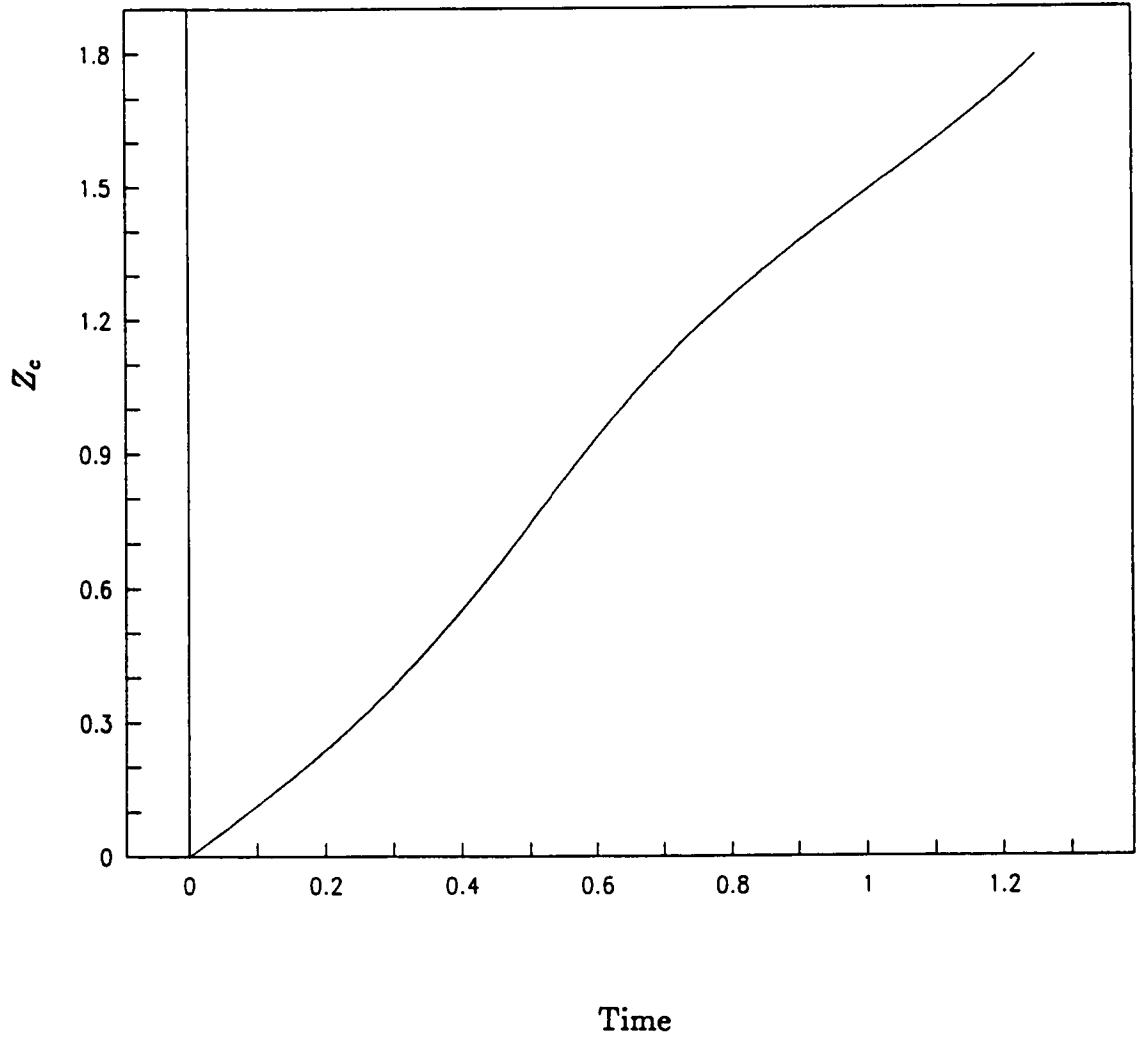


Figure C.3: The movement of the center of mass in the Z component.

$\theta_u = 50, \theta_l = 80$ degrees. $l_P = 9.0, l_3 = 4.3$. $\dot{\phi} > 0$.

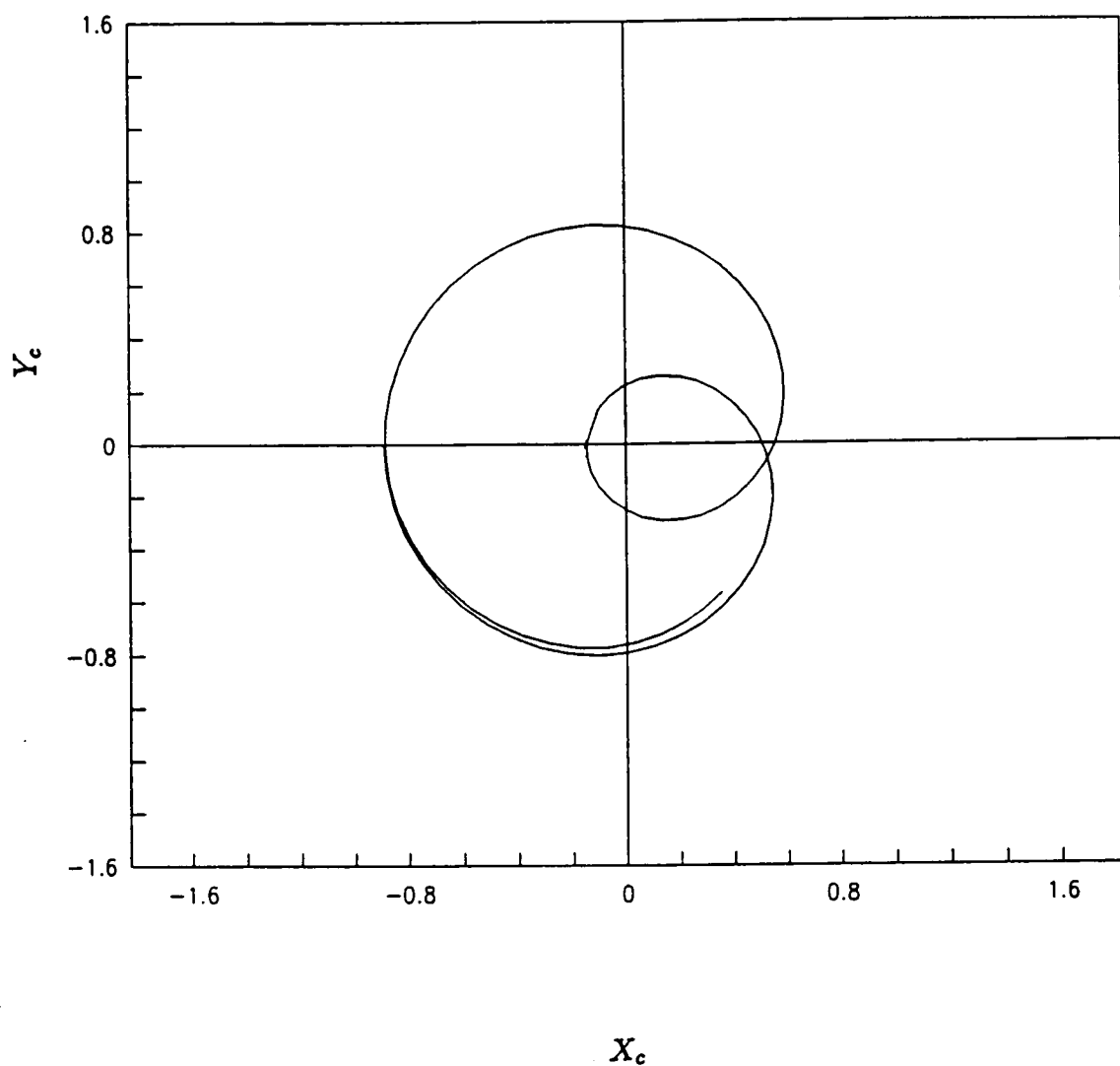


Figure C.4: The movement of the center of mass on the XY plane. $\theta_u = 45, \theta_l = 70$ degrees. $l_P = 6.75, l_3 = 4.5$. $\dot{\phi} > 0$. $R_u = 0.1469137, R_l = 0.8874846$.

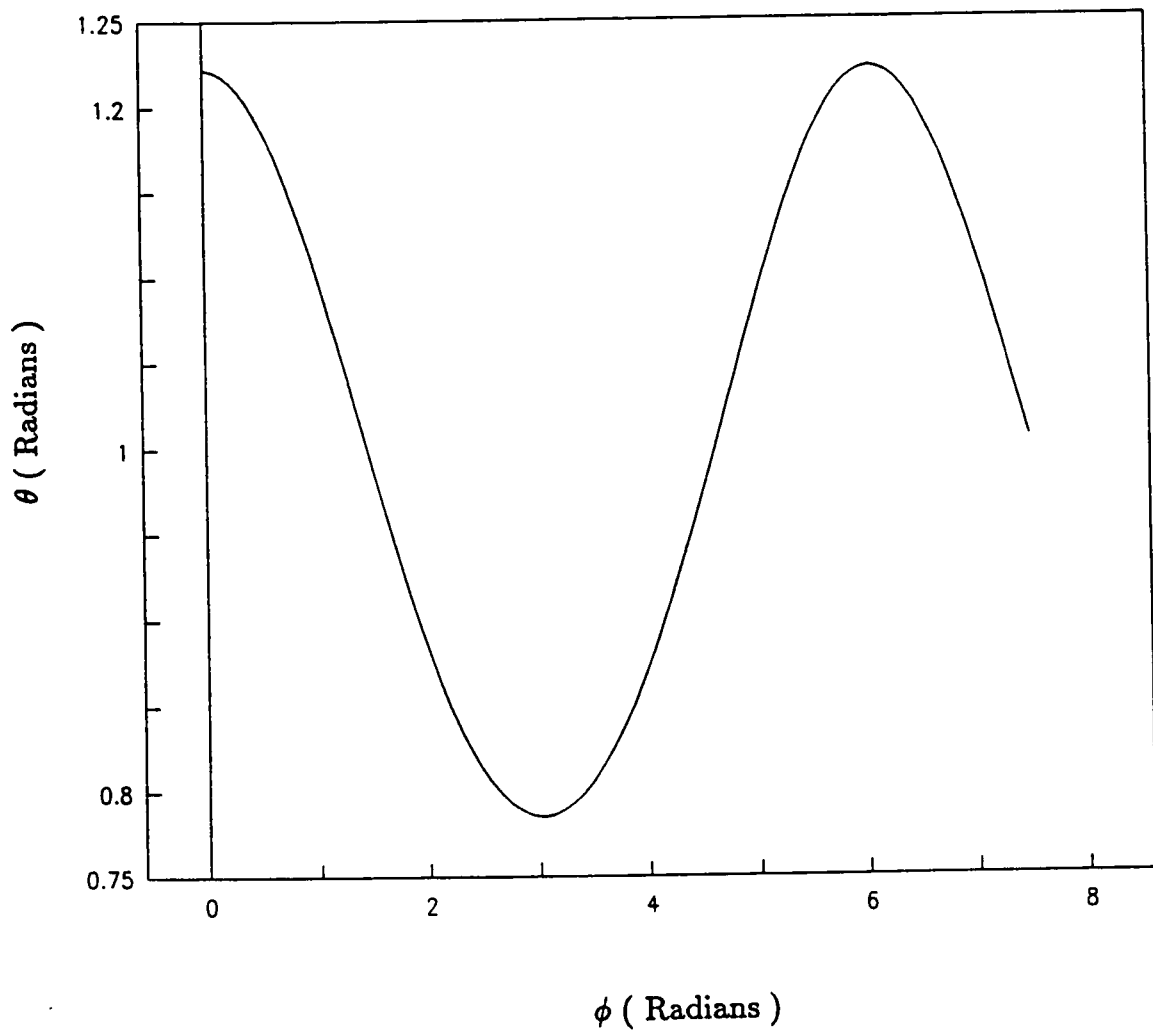


Figure C.5: θ (the vertical) vs. ϕ in the unit of radian angles. $\theta_u = 45, \theta_l = 70$ degrees. $l_P = 6.75, l_3 = 4.5$.

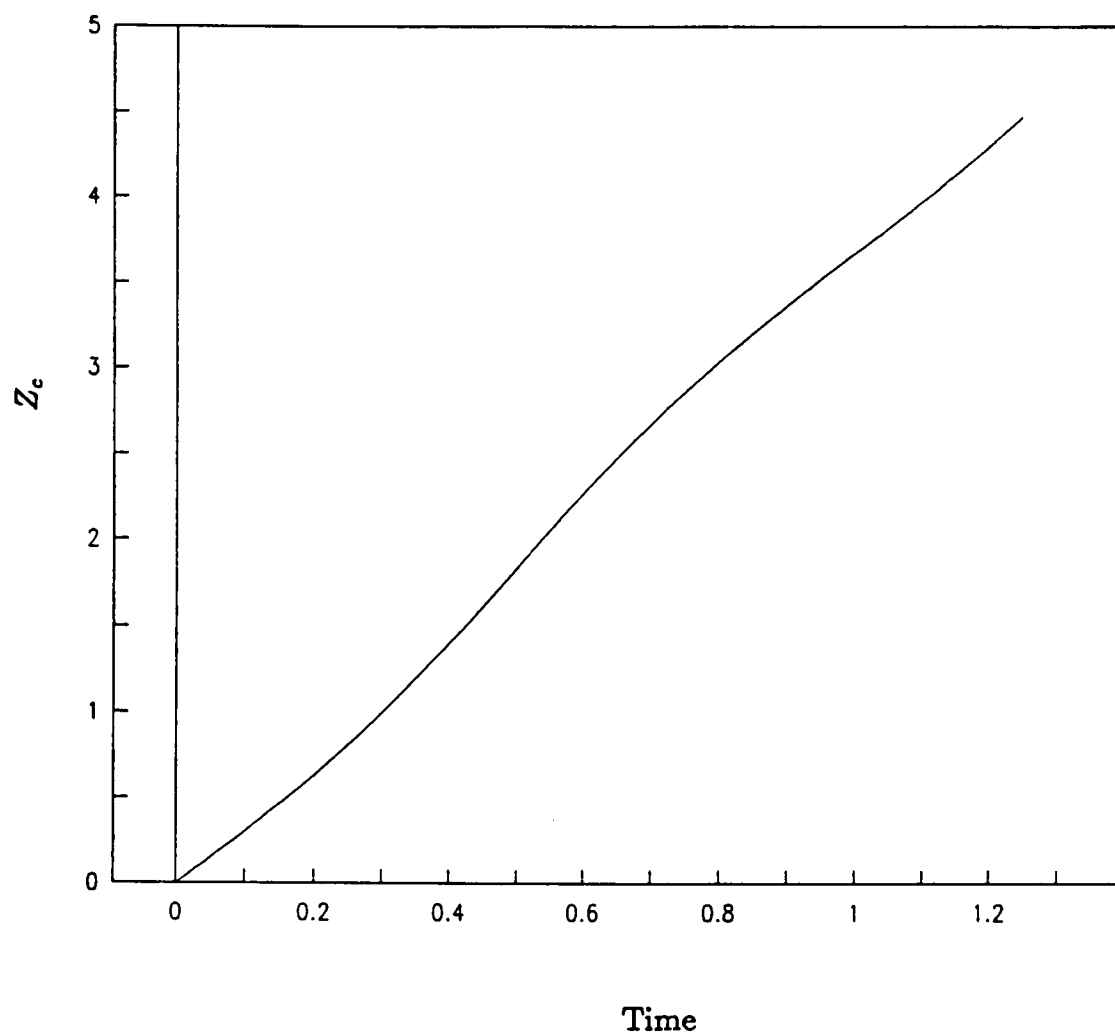


Figure C.6: The movement of the center of mass in the Z component.

$\theta_u = 45, \theta_l = 70$ degrees. $l_P = 6.75, l_3 = 4.5$.

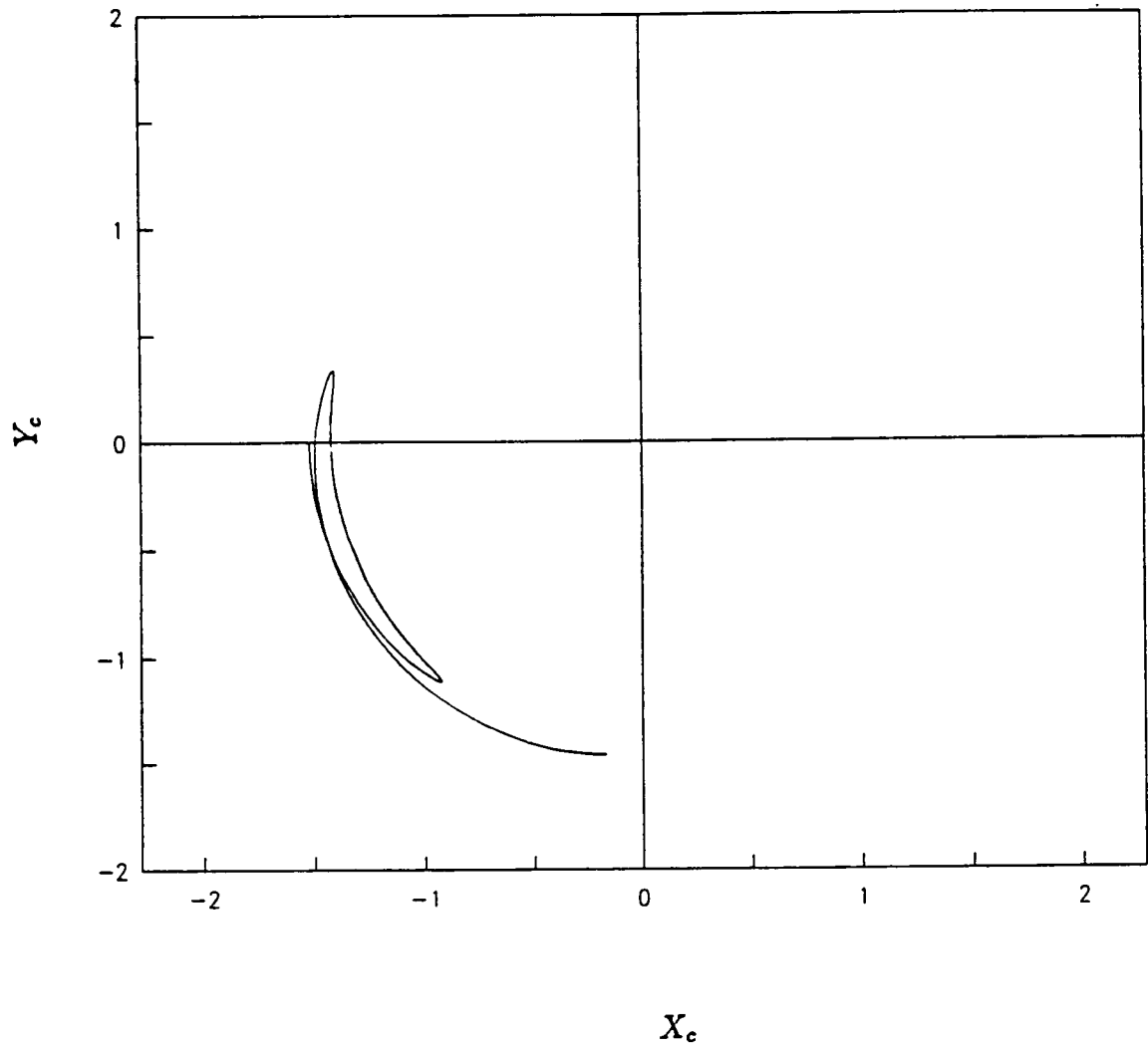


Figure C.7: The movement of the center of mass on the XY plane for one and a quarter periods of time. $\theta_u = 20, \theta_l = 40$ degrees. $l_P = 0.9, l_3 = 1.0$. $R_u = 1.418272, R_l = 1.519111$.

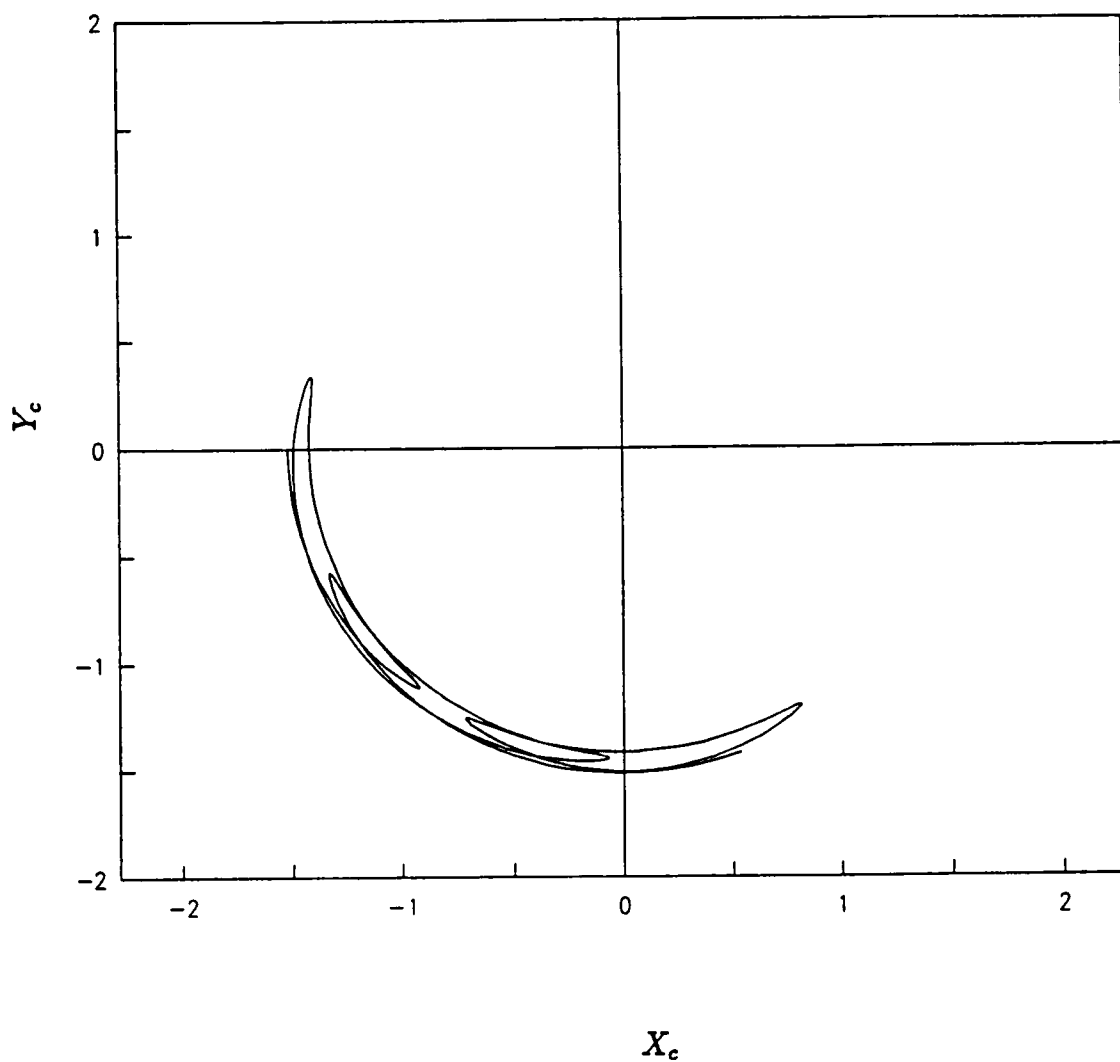


Figure C.8: The movement of the center of mass on the XY plane for three periods of time. $\theta_u = 20, \theta_l = 40$ degrees. $l_p = 0.9, l_3 = 1.0$. $R_u = 1.418272$, $R_l = 1.519111$.

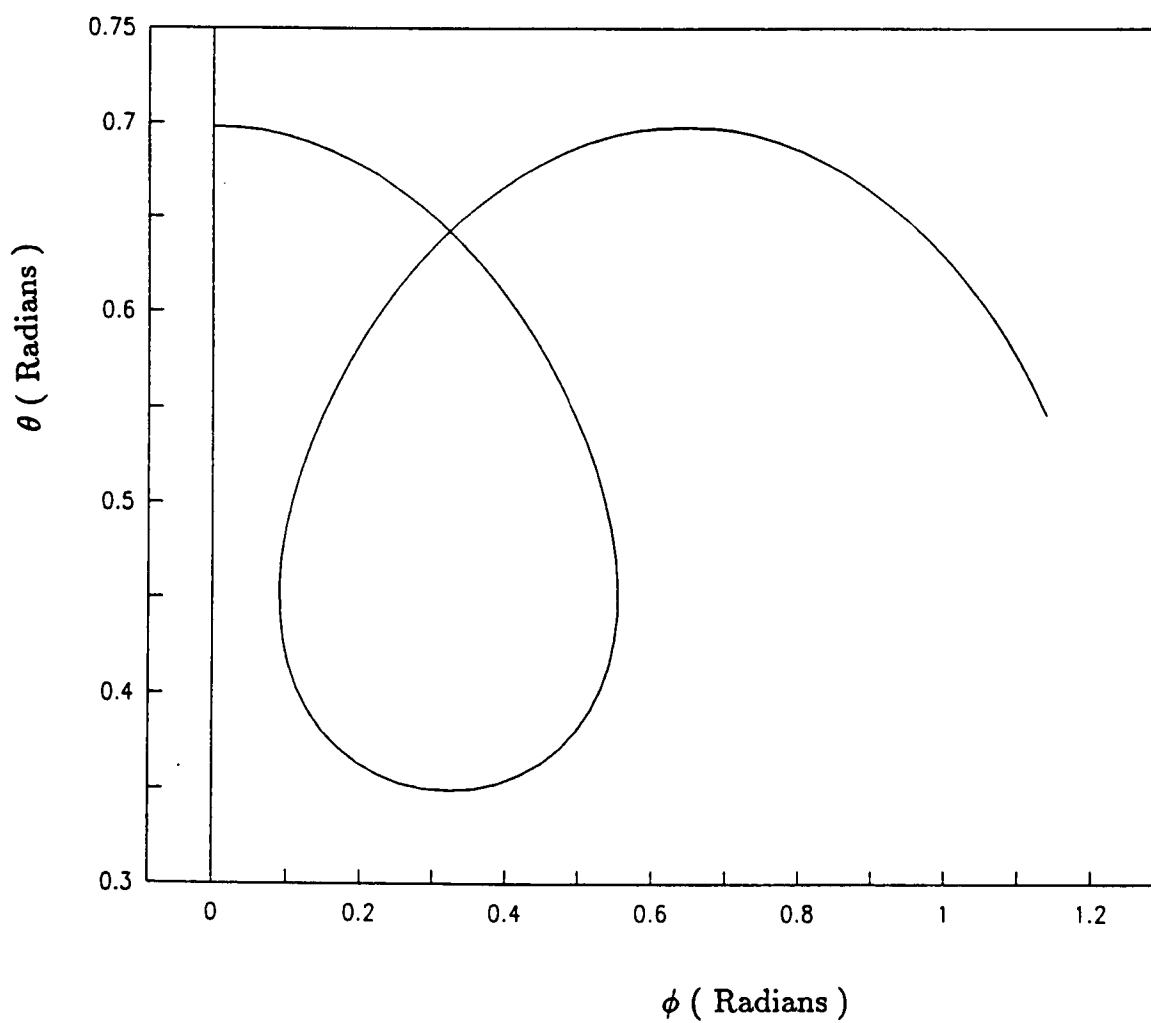


Figure C.9: θ (the vertical) vs. ϕ in the unit of radian angles for one and a quarter periods of time. $\theta_u = 20, \theta_l = 40$ degrees. $l_P = 0.9, l_3 = 1.0$.

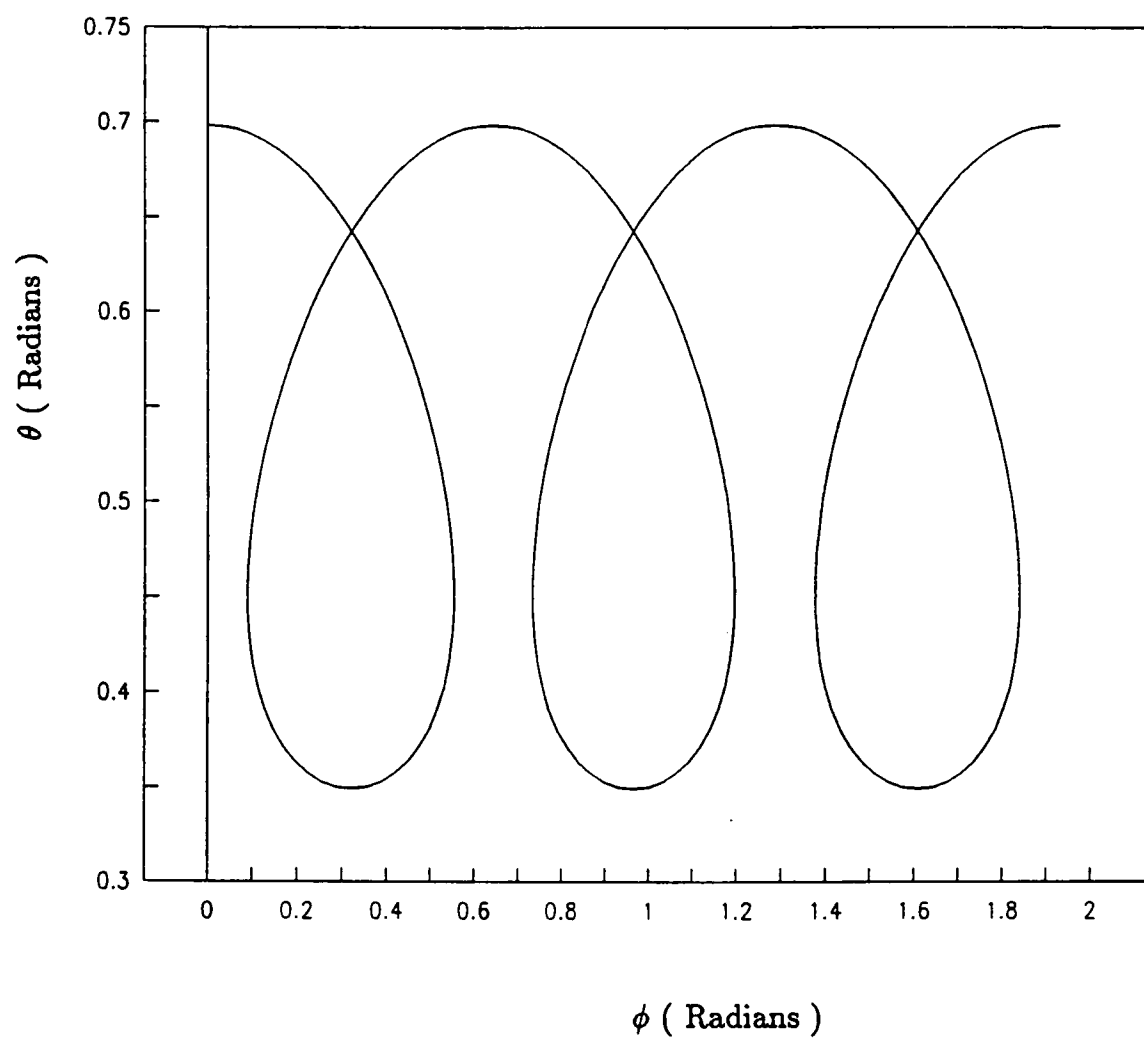


Figure C.10: θ (the vertical) vs. ϕ in the unit of radian angles for three periods of time. $\theta_u = 20, \theta_l = 40$ degrees. $l_P = 0.9, l_3 = 1.0$.

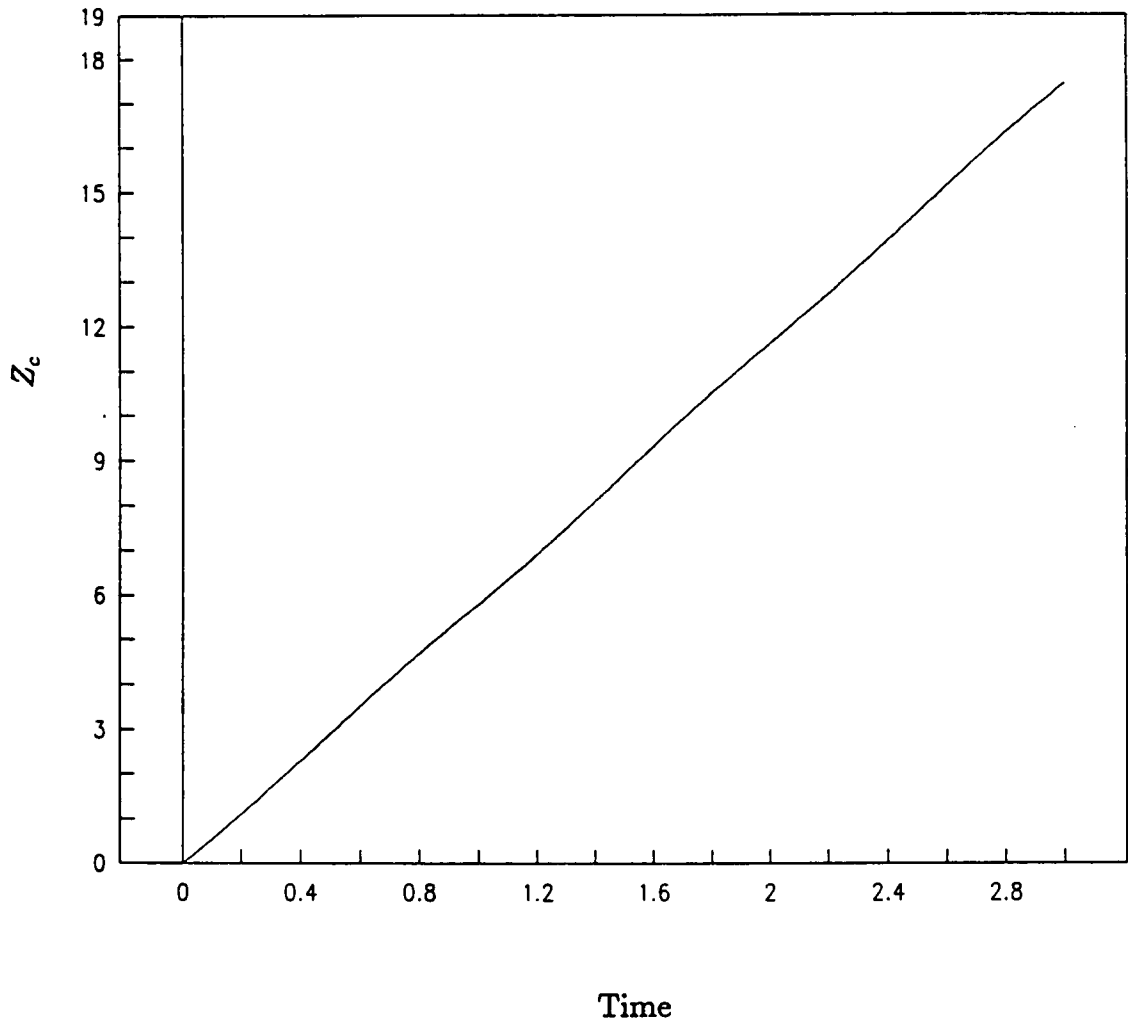


Figure C.11: The movement of the center of mass in the Z component.

$\theta_u = 20, \theta_l = 40$ degrees. $l_P = 0.9, l_3 = 1.0$.

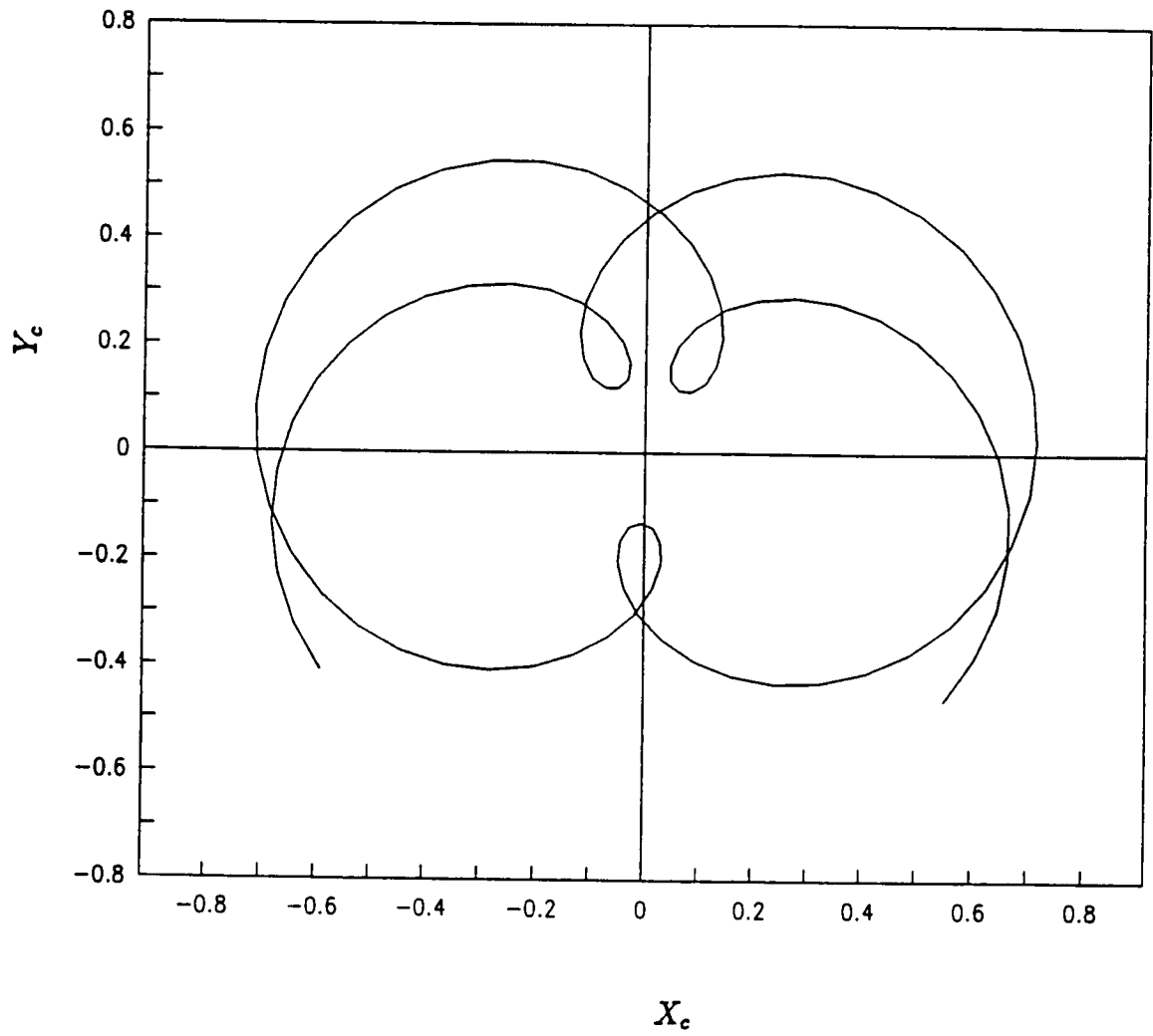


Figure C.12: The movement of the center of mass on the XY plane for three periods of time. $\theta_u = 30, \theta_l = 60$ degrees. $l_P = 1.1, l_3 = 1.0$. $R_u = 0.1311285$, $R_l = 0.7191625$.

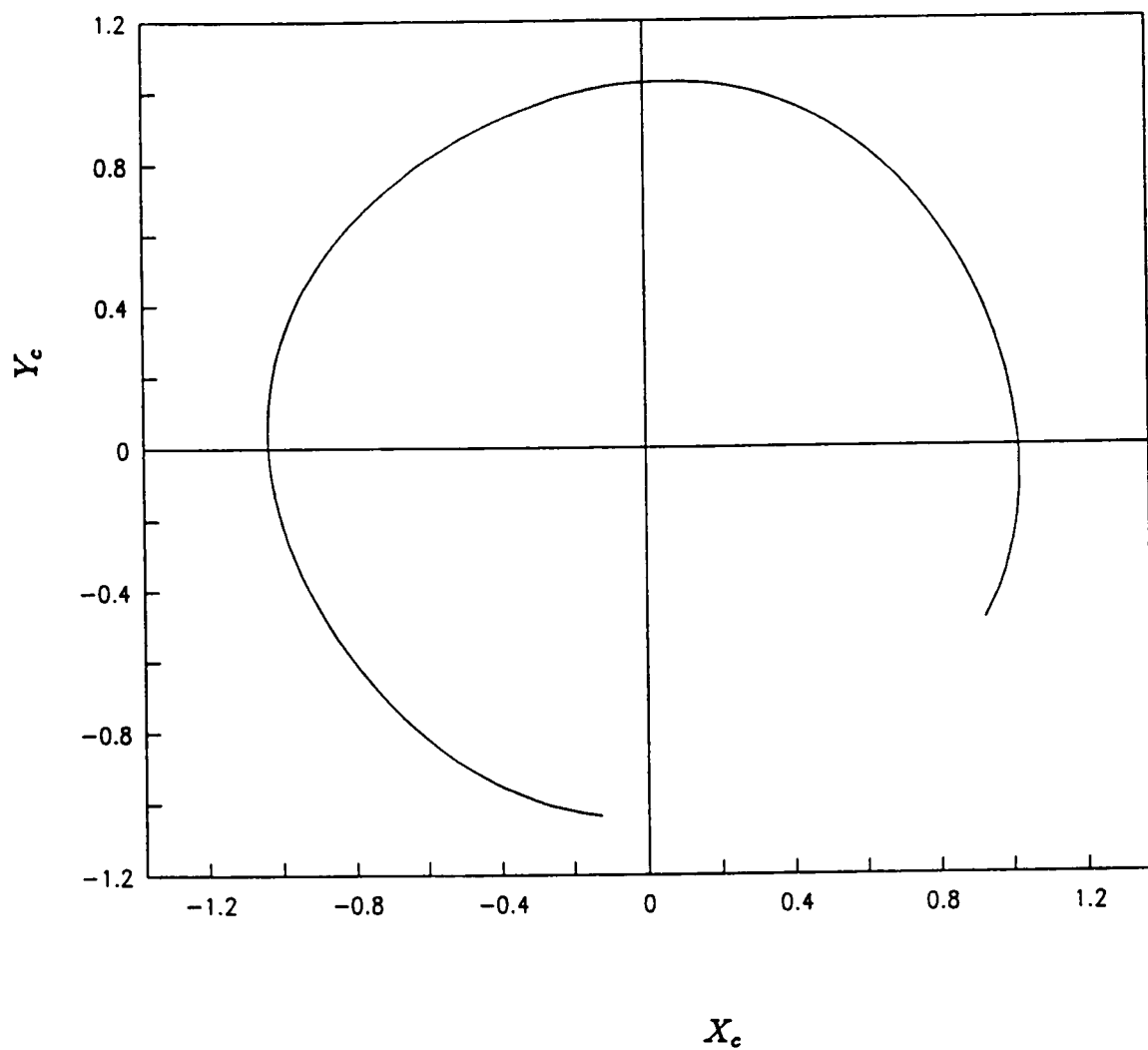


Figure C.13: The movement of the center of mass on the XY plane for three periods of time. $\theta_u = 30, \theta_l = 35$ degrees. $l_P = 0.9, l_3 = 1.0$. $R_u = 1.003987$, $R_l = 1.042583$.

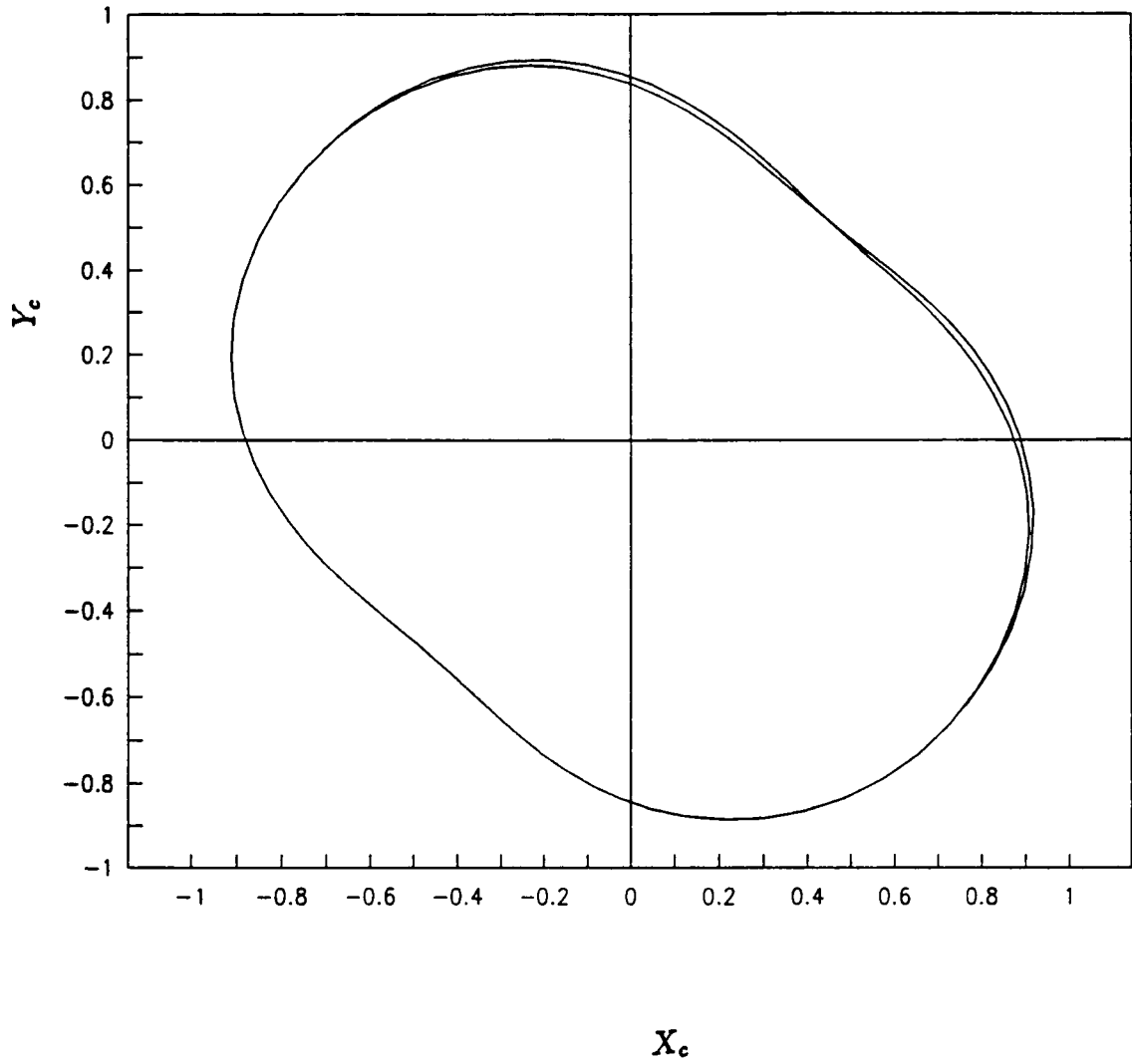


Figure C.14: The movement of the center of mass on the XY plane for three periods of time. $\theta_u = 30, \theta_l = 60$ degrees. $l_P = 0.9, l_3 = 1.0$. $R_u = 0.682806$, $R_l = 0.982967$.

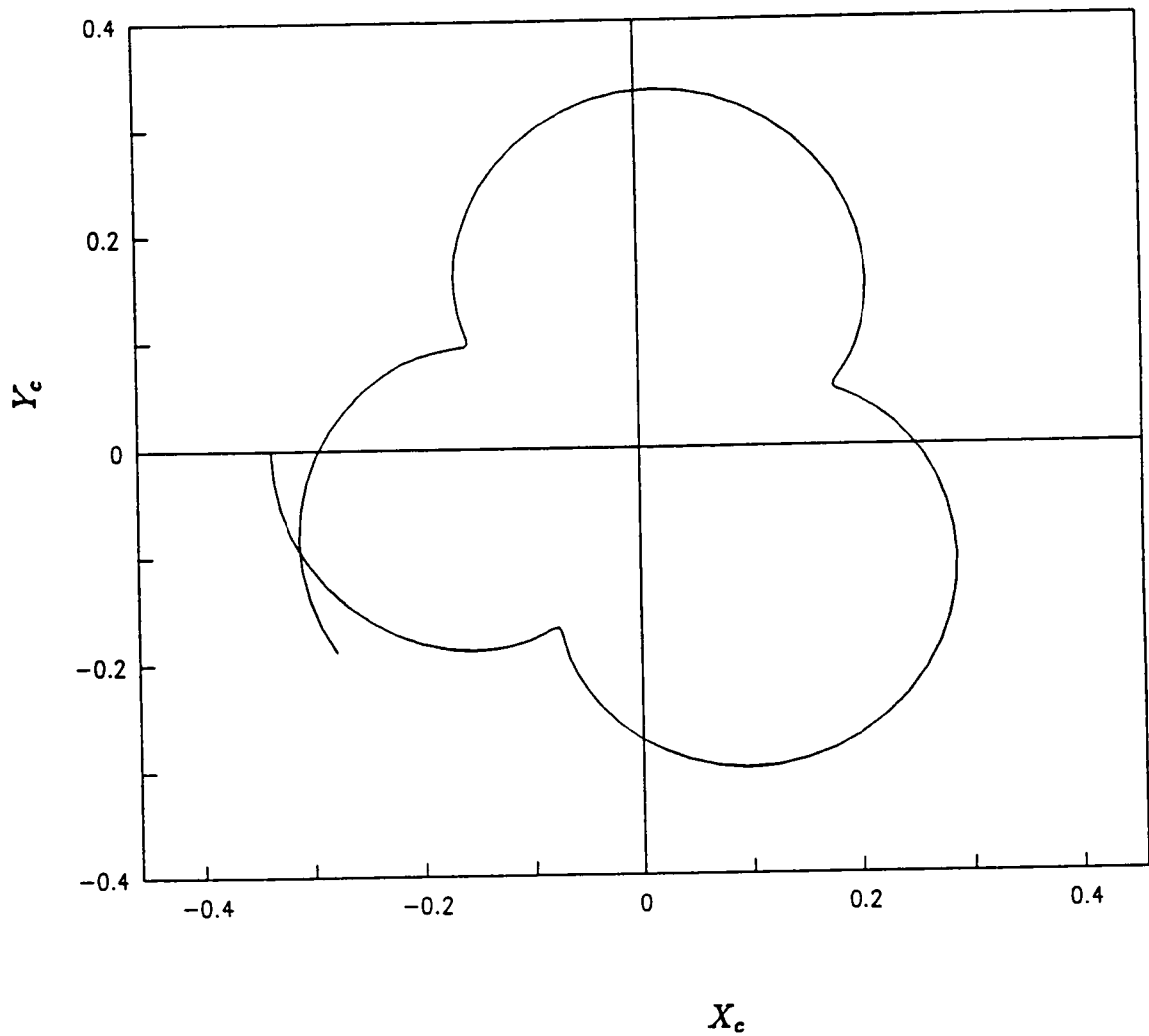


Figure C.15: The movement of the center of mass on the XY plane for three periods of time. $\theta_u = 30, \theta_l = 35$ degrees. $l_p = 1.1, l_3 = 1.0$. $R_u = 0.1847462$, $R_l = 0.3363351$.

Appendix D

The Effective Masses and Moments of Inertia of the Induced Flow Surrounding a Solid

In this appendix, we will discuss the effective masses and moments of inertia of the induced flow surrounding a solid as mentioned in equation 3.7 of chapter three.

When an ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \quad (\text{D.1})$$

moves in the direction of the x_i - axis with velocity u_i , the kinetic energy of the

induced flow is[22]

$$T_L = \frac{1}{2} M_L \frac{\alpha_i}{2 - \alpha_i} u_i^2 \quad (\text{D.2})$$

where M_L is the mass of liquid displaced by the ellipsoid and

$$\alpha_i \equiv a_1 a_2 a_3 \int_0^\infty \frac{d\lambda}{(a_i^2 + \lambda) k_\lambda} \quad (\text{D.3})$$

with $k_\lambda^2 = (a_1^2 + \lambda)(a_2^2 + \lambda)(a_3^2 + \lambda)$.

Thus the effective masses of the induced flow surrounding the solid along the three principal axes of the ellipsoid are given by

$$A_i = M_L \frac{\alpha_i}{2 - \alpha_i}. \quad (\text{D.4})$$

Similarly when the ellipsoid rotates with angular velocity ω_i around the x_i - axis, the kinetic energy of the induced flow is

$$T_L = \frac{1}{10} \frac{(a_j^2 - a_k^2)^2 (\alpha_k - \alpha_j)}{2(a_j^2 - a_k^2) - (a_j^2 + a_k^2)(\alpha_k - \alpha_j)} M_L \omega_i^2, \quad (\text{D.5})$$

where i, j, k is in the cyclic order of 1, 2, 3. Therefore the effective moments of inertia are

$$B_i = \frac{M_L}{5} \frac{(a_j^2 - a_k^2)^2 (\alpha_k - \alpha_j)}{2(a_j^2 - a_k^2) - (a_j^2 + a_k^2)(\alpha_k - \alpha_j)}. \quad (\text{D.6})$$

Here we must emphasize that the mass of the rotating ellipsoid which induces the flow can have any value from zero to infinity, i.e. it can be an empty shell or a massive ellipsoid. Its mass comes into play only when we consider the reaction of the induced flow on the motion of the solid. The computation of the two sets of formulas quoted above involves only the familiar harmonics satisfying Laplace's

equation $\nabla^2\phi = 0$, but the complete problem of solving the induction of the flow and its reaction on the motion of the solid requires hyperelliptic functions of two variables for the general case and elliptic functions for the simpler symmetric cases.

Now we add the general formula for computing effective masses and moments of inertia. To describe the conditions to be satisfied at the boundary of the solid, take a frame of reference R' fixed relatively to the solid, say an origin O' and three cartesian axes $O'x, O'y, O'z$. The motion of the solid is then specified by a velocity $\vec{u}$ of the origin and an angular velocity $\vec{\omega}$. Thus at the point $\vec{r}$ of the boundary the velocity is $\vec{u} + \vec{\omega} \times \vec{r}$, and if $\vec{n}$ is the unit outward normal vector at this point to the surface of the solid, the boundary condition is

$$-\frac{\partial\phi}{\partial n} = \vec{n} \cdot (\vec{u} + \vec{\omega} \times \vec{r}) = \vec{u} \cdot \vec{n} + \vec{\omega} \cdot (\vec{r} \times \vec{n}), \quad (\text{D.7})$$

where

$$\frac{\partial}{\partial n} \equiv \vec{\nabla} \cdot \vec{n}.$$

This condition may be satisfied by writing

$$\phi = \vec{u} \cdot \vec{\varphi} + \vec{\omega} \cdot \vec{\chi}, \quad (\text{D.8})$$

where $\vec{\varphi}, \vec{\chi}$ are vectors whose components are solutions of Laplace's equation and which satisfy the boundary conditions

$$-\frac{\partial\vec{\varphi}}{\partial n} = \vec{n}, \quad -\frac{\partial\vec{\chi}}{\partial n} = \vec{r} \times \vec{n}, \quad (\text{D.9})$$

so that $\vec{\varphi}, \vec{\chi}$ depend solely on the shape of the solid and not on its motion.

The kinetic energy of the liquid is given by

$$T_L = -\frac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n} dS = \frac{1}{2}\rho \int_S (\vec{u} \cdot \vec{\varphi} + \vec{\omega} \cdot \vec{\chi})(\vec{n} \cdot [\vec{u} + \vec{\omega} \times \vec{r}]) dS, \quad (D.10)$$

and shows that T_L is a homogeneous quadratic function of the vectors $\vec{u}, \vec{\omega}$. When written in full it will consist of 21 terms

$$T_L = \frac{1}{2} \sum_{i,k} A_{ik} u_i u_k + \frac{1}{2} \sum_{j,l} B_{jl} \omega_j \omega_l + \frac{1}{2} \sum_{i,l} C_{il} u_i \omega_l, \quad (D.11)$$

where the coefficients

$$A_{ik} = -\rho \int_S \varphi_i \frac{\partial \varphi_k}{\partial n} dS, \quad (D.12)$$

$$B_{jl} = -\rho \int_S \chi_j \frac{\partial \chi_l}{\partial n} dS, \quad (D.13)$$

$$C_{ik} = -\rho \int_S [\varphi_i \frac{\partial \chi_k}{\partial n} + \chi_k \frac{\partial \varphi_i}{\partial n}] dS, \quad (D.14)$$

and $u_i, \omega_i, \varphi_i, \chi_i$ are all components of vectors $\vec{u}, \vec{\omega}, \vec{\varphi}$ and $\vec{\chi}$.

Vita

Rong-chang Chiang was born in Taiwan on January 20, 1949. He received a Bachelor of Science degree in Physics in July 1972, and reentered the graduate school in September 1976 and received a Master of Science degree in Physics in July 1978 from National Taiwan Normal University.

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