An Optimization Framework for Integrating Variable Capacity and Pavement Thickness Requirements in Highway Cost Allocation

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ABSTRACT

The objective of Highway Cost Allocation (HCA) is to distribute or allocate in a fair and rational manner the cost of a transportation facility (either a highway or bridge) among all vehicle classes using it. The purpose of this dissertation is to study and enhance a model, known as the *least-core* model, to include both pavement thickness and traffic capacity requirements for all coalitions formed with a given group of vehicle classes. Considering vehicle classes as players and groups of vehicle classes as coalitions, it is possible to quantify the thickness and width of pavement needed to accommodate the vehicle classes in a coalition. Typically, thickness of the pavement depends upon the traffic load, which is measured in *18,000 lb. Equivalent Single Axle Load* (ESAL). Additionally, the width of the pavement is measured in terms of *traffic lanes*. The cost of the *grand coalition* (including all players) is allocated using a linear programming (LP) model with constraints defining set of allocations belonging to the *core* that represent three properties of completeness, marginality and rationality. The right hand side of the LP model is determined for a given coalition with known traffic load (ESAL) and traffic capacity requirements (lanes). The least-core model maximizes the lower bound on savings experienced by all vehicle classes as a result of joining the grand coalition. The resulting optimal solution can be a unique allocation or multiple allocations. If the optimal *basic feasible solution* is unique, it is known as the *nucleolus*. Otherwise, the nucleolus is the average of all optimal basic feasible solutions. Since enumeration of all optimal basic feasible solutions is considered inefficient, an existing algorithm -referred to in the literature as the sequential LP approach- is adapted to the HCA problem. The procedure converges to the nucleolus by solving a sequence of LP models using well-known complementary slackness conditions. Finally, necessary conditions are identified for a special case of the least-core model to occur, wherein a closed-form solution can be used to obtain the nucleolus.

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ABBREVIATIONS AND SYMBOLS

HCA	Highway Cost Allocation
DOT	Department of Transportation
FHWA	Federal Highway Administration
HCM	Highway Capacity Manual
GM	Generalized Method
LP	Linear Programming
FP	Flexible Pavement
LINDO	Linear, Interactive and Discrete Optimizer
AASHTO	American Association of State and Highway Transportation Officials
MEPDG	Mechanistic-Empirical Pavement Design Guide
ADT	Average Daily Traffic
VMT	Vehicle Miles Traveled
ESAL	Equivalent Single Axle Load
phf	Peak Hour Factor
PCE	Passenger Car Equivalent
f_{hv}	Heavy Vehicle Factor
DD	Directional Distribution
L_i	Number of Lanes needed by Vehicle Class *i*
VC	Vehicle Class
A-S	Aumann-Shapley
S	Coalition of groups of vehicle classes
N	Grand coalition of all vehicle classes
M	Power set of *N*
C(S)	Cost of designing a highway accommodating vehicle classes in *S*
C(N)	Cost of designing a highway accommodating vehicle classes in *N*
x_i	Cost allocated to vehicle class *i*
t_1	Decision variable used to maximize savings in least core LP model
CPFS	Corner Point Feasible Solution
t(x, S)	Saving achieved by Coalition *S* for participation in *N* for given allocation *x*
t^*_{max}	Maximum savings achieved when solving a least core LP model
L(x)	Savings achieved by all coalitions *S* for a given allocation *x* arranged in the increasing order.
X	Set of allocations satisfying completeness condition
$\mathcal{H}$	Any allocation *x* belonging to *X*
Y_S	Dual variable of coalitional rationality constraint of *S*
Y_N	Dual variable of coalitional rationality constraint of *N*

F_j	Set of coalitions whose savings cannot be improved
CFS	Closed-form Solution
X_N^*	An allocation *x* defining the nucleolus
λ_n	Scalar constant value
C_e	Cost per ESAL
C_l	Cost per Lane
E_N	Cumulative ESALs for grand coalition in set *N*
L_N	Number of lanes to accommodate vehicle classes in grand coalition *N*
L_S	Number of lanes to accommodate vehicle classes in grand coalition *S*
x_B	Matrix of basic variables
x_N	Matrix of non-basic variables
A	Matrix of coefficients
b	Right hand sides
Y_B	Dual basic variables
Y_N	Dual non-basic variables
Z	Primal objective function
W	Dual objective function
C_B	Coefficients of basic variables in objective function
C_N	Coefficients of non-basic variables in objective function
Y_B^*	Dual basic variables in optimal basis
Y_N^*	Dual non-basic variables in optimal basis
B	Matrix of basic variables
B^{-1}	Inverse of matrix of basic variables
N	Matrix of non-basic variables
N^{-1}	Inverse matrix of non-basic variables

CHAPTER I

INTRODUCTION

Background

The objective of Highway Cost Allocation (HCA) is to distribute in a fair and equitable manner the total cost of construction, rehabilitation, maintenance, and other related costs of up keeping highways and bridges facilities in a nation's transportation system among various categories of highway users. Public agencies such as the Federal Highway Administration (FHWA), state and local Departments of Transportation (DOT) look to HCA studies as a policy making instrument used for the purposes of recovering cost of road use. Development of a sustainable financing system wherein the users of the roadway pay their fair share of cost responsibility is what drives the need for periodic HCA studies conducted by state DOTs and FHWA. The end outcome of a comprehensive HCA study is to evaluate the relative closeness between the revenues generated from user fees and taxes paid by road users into a highway trust fund and the cost responsibility of each user towards the highway expenditures made by the DOTs. It is generally accepted among the HCA practitioners that the methodology vetted for use among the repository of available cost allocation methods for conducting an allocation must remain true to the fundamental principles of equity, fairness and rationality to ensure willful acceptance of the results by stakeholders.

Many traditional mathematical HCA procedures used for cost allocation have concentrated their efforts towards measuring highway system usage in terms of contribution of different vehicle classes towards the wear and tear of the pavement. However, several HCA studies have not deliberated in detail the resulting cost associated with capacity expansion by the way of constructing additional lanes to entertain higher levels of traffic volumes as a result of lighter vehicle classes such as passenger cars. The lack of amalgamation of these two competing factors into one HCA methodology contributes to ineffectual treatment of capacity related cost being occasioned on the highway facility.

Allocation of highway related cost based solely on damage caused to pavement will magnify the cost being attributed to heavier vehicle classes such as commercial heavy duty tractor trailer trucks. On the contrary, attribution of cost based solely on lane use, results in higher cost responsibility for lighter vehicle classes such as passenger cars due to their high traffic volumes. Hence, reliance on pavement damage as a solitary measure of highway usage might result in an allocation where heavier vehicles would end up paying more than their fair share of highway related capacity cost because the traffic volume patterns exhibited by these very same vehicle classes were considered to be an invariant or an equal factor in the allocation process.

Most methods used by FHWA and state DOTs for the purposes of cost allocation have assumed that all vehicle classes have analogous requirements for number of lanes in a highway facility. In light of this shortcoming, the spectrum of research conducted in this dissertation will be centered around developing a new methodology by means of enhancing existing procedures to account for this discrepancy. The underlining feature of this new method will be the way it divides the highway cost fairly among road users, wherein, the process involved in estimating cost responsibility is reticent to variable roadway capacity and pavement thickness design requirements of vehicle classes. This dissertation will showcase the efficacy of the new proposed methodology by specifically allocating cost of constructing new pavement among vehicle classes. However, with slight refinement, the proposed methodology could also be used for distributing cost among other expenditure classifications such as rehabilitation, maintenance, reconstruction and repair.

Therefore, this dissertation is a response to a growing need to develop an allocation transformation framework that is cognizant of complexities of cost expended on designing highway with appropriate pavement thickness in order to provide a high ride quality level for all highway user types. As well as, simultaneously being appreciative of the nuances of traffic flow theory, so as to promise an unbiased approach, that can realistically quantify the consumption of highway by also measuring the need for number of lanes so as to better provide satisfactory traffic flow services to certain types of vehicle classes. The procedure presented in this dissertation builds upon the previous work performed on the development of allocation methods based on the theory of cooperative games, particularly, the ones based on linear optimization. Game theory, which is a branch of mathematics, has been used extensively over the years to solve the HCA problem. The use of game theory for allocating joint costs has been of great interest to scholars due to its ability to model cost sharing situations where multiple entities are choosing to collaborate at their own accord or engaging in an instinctive cooperative behavior with one another. The intent for such a behavior stems from a desire of lowering cost responsibility by jointly sharing a transportation facility rather than paying for an exclusive use of the facility by individual road users.

In general, the most cost effective option for state DOTs and FHWA for meeting the mobility needs of all vehicle classes is to build highways that can accommodate all vehicles classes as opposed to building independent highways for few of them. Conversely, from the viewpoint of road users, it is also in their best interest to use a shared highway facility, as independent highway for each vehicle class would be a costly proposition. This behavior demonstrated by the highway agencies and road users in unison is referred to in economics as the rational choice or rationality. This theory applied to HCA assumes that when road users are left to their own free will, they would coalesce with all road users in congruence to use a joint facility to pay the lowest cost and achieve the greatest savings by not using alternative highways built for individual or groups of vehicle classes. Therefore, any type of HCA methodology, must adhere to an essential economic principle of rationality. The satisfaction of this property ensures that allocated cost do not exceed their respective opportunity or stand-alone costs in such a way that it would persuade road users to pay for exclusive highways built for fewer vehicle classes.

The vision for the proposed methodology in this dissertation strives to accord savings to vehicle classes for meeting their transportation needs in an efficient way. The procedure would incentivize vehicle classes to continue sharing by attributing them a lower cost in order to prevent the breaking up from the supposed grand collaborative behavior being exhibited by them. There are substantial precedents in the library of available HCA methods, namely, coalitional game theory based procedures that guarantee a socially acceptable allocation by optimizing the welfare of road users and enforcing the property of rationality. More importantly, this dissertation will recommend a fair cost division scheme, that provides an allocation on the basis of maximizing the savings for all the vehicle classes using a methodical programming technique. The favorable point of this scheme is that the proposed allocation would result in a fair savings distribution, that is considerably, unparalleled to any other allocation in the scholarship of HCA. This dissertation will strive to expand upon the body of work that have at length made the case for the implementation of game-theoretic cost allocation practices for HCA, serving as a potential replacement to traditional HCA methods. One such instance was an article written by Garcia-Diaz and Lee (2013) reemphasizing the importance of using game-theoretic based procedures in HCA studies.

Traditional HCA models have known to generate unstable allocations that yield different results when their fundamental assumptions are changed. This dissertation will spend some time debating the strengths and weaknesses of methods used extensively in recent HCA studies and highlight the importance of using the new proposed methodology. However, due to the existence of multitude of allocations from different procedures, along with subjective decision making involved on the basis of allocation in conventional models, only culminate in confounding the issue of fairness i.e. which procedure and the corresponding basis of allocation would remain non-partisan towards certain-vehicle types. Additionally, most HCA models tends to apportion cost based on contribution towards degradation of the pavement due to vehicle classes. Use of such a benchmark results in trucks being assigned a higher cost responsibility in spite of them having minimum demand for number of lanes on any given highway system. Moreover, standard models of cost allocation used by state DOTs and FHWA have extensively used allocation units or cost allocators obtained from pavement distress models, that rely on mechanistic-empirical procedures to estimate the distress share contribution of vehicle classes. As an alternative, this dissertation will recommend the development of a pavement cost function based on actual expenditures made by highway agencies on a given highway system. This function can be readily developed from a bid database consisting of historical documentations of projects conducted on specific section of the highway system. These stored databases are also referred to as pavement management systems maintained by several state DOTs and FHWA. This function will measure the true cost imposed on the highway, and not estimated theoretical cost obtained from empirical models.

This dissertation will also break new ground by introducing a new way of estimating costs imposed by vehicle classes for a given HCA problem, having the capability of assigning a cost to the demand for traffic lanes and pavement thickness requirements of vehicle classes. The new solution concept will be implemented within the right hand sides of a linear programming (LP) model whose constraints enforce fairness by satisfying certain appealing economic properties. In addition, the dissertation will also search for answers to several unresolved long-standing issues in conventional methods of highway cost allocation, one such being the issue of order dependency. Another important contribution of this dissertation will be to develop closed-form solutions, specific to LP models used in HCA. The functionality of these closed-form expressions will come from its ability to arrive at the optimal solution for a given LP model without using the simplex algorithm as developed by Dantzig (1963).

Motivation

Most of the prevalent game-theoretic procedures and conventional highway cost allocation methods have assumed all vehicle classes have similar requirements for number of lanes. This assumption remains inconsistent with the principles of traffic flow theory and has not been sufficiently addressed by HCA practitioners. In addition, supportive research to the main question will be performed to analyze several theoretical, computational and fairness issues still not satisfactorily addressed in the area of cooperative games based HCA. These are listed below:

1. The quantification of highway system usage among vehicle class can be better embarked upon by separation of highway cost into cost for building thicker pavements in order to support heavy road user groups, and the cost for constructing additional lanes to accommodate high traffic volumes of lighter road user groups. As of now, the highway expenditures are by and large attributed based on pavement thickness requirements. In this regard, this dissertation will advocate the use of a game-theoretic concept, that has a proven ability to provide an efficacious way of separating highway cost into pavement and lane costs. This concept, whose effectiveness has been proven in an earlier HCA method will be furthered by incorporating it into an LP model.
2. The cost estimated based on empirical models from a statistical perspective might have a tendency to overestimate pavement related cost for given ESALs. The development of a pavement cost equation strictly based on actual expenditures made by the transportation agencies serves as a better expression of relationship between cost and pavement damage.
3. The fairness criterion associated with the allocation of highway related costs among vehicle classes has the potential to be further strengthened by surveying and bringing in fairness inducing concepts from cooperative game theory to the area of HCA.
4. Exploration of possible computationally efficient ways of solving linear programming based HCA methodologies that are commonly associated with exponential number of constraints for a given number of vehicle classes. More importantly, this dissertation will also address a key limitation of existing LP formulations that rely on an ancillary criterion in order to break the ties in cases of multiple optimal solutions.

Since the acceptability of allocated costs is greatly influenced by the conflict of human interest, there is no HCA method that would be agreeable to all the stakeholders. However, one can alleviate the issue of human interest conflict by using proven mathematical cost allocation models that are bounded by principles of fairness and objectivity. It is of crucial importance that a cost allocation be not merely a solution to a technical problem, but a fair and equitable distribution of costs based on a precise and comprehensive representation of highway use (traffic loads, climatic zones, highway classification, highway location, thickness of pavements, traffic lanes, etc.) and consideration of relevant economic properties, such as completeness, marginality and rationality.

Relevant Concepts and Properties of Fair Allocation

The custodians of the US transportation infrastructure namely the state Departments of Transportation (DOT), the Federal Highway Administration (FHWA) and local transportation authority's such as Metropolitan Planning Organizations (MPO) make significant financial investments to build a world class transportation infrastructure to meet the daily commercial and private transportation needs of its constituents. A Highway Cost Allocation is an asset management modeling technique used to attribute the total expenditures made by highway agencies for Construction, Rehabilitation and Maintenance (CRM) of highways and bridges in a fair and equitable manner among the users of the road. The cost is apportioned among various categories of highway users known as vehicle classes. More importantly, the expenditures made for the upkeep of highways are used for the benefit of all vehicle classes. The term *coalition* is brought into focus in an HCA problem when vehicle classes are forming coalitions to meet their transportation needs by using an independent highway facility (i.e. single-vehicle class coalition) or a facility they can share with other vehicle classes (i.e. two-or-more vehicle class coalition). Since a highway facility is built to accommodate all vehicle classes, a fair allocation must be reflective of the inherent savings or excesses being gained by all vehicle classes for jointly sharing a facility as part of a *grand coalition,* where cost responsibilities are reduced by not partaking in coalitions. Three standard properties that enforce the notion of fairness are described below:

- *Rationality*: The cost allocated to a vehicle class when sharing a highway facility collectively amongst all vehicle classes should not exceed an amount it would have paid to use an exclusive highway facility (i.e. individual rationality), as well as, when using a partially exclusive highway facility with fewer than all vehicle classes (i.e. group rationality).

- *Completeness*: All classes of vehicles should bear the financial burden for highway expenditures.
- *Marginality*: Each vehicle class should pay for its marginal cost or minimum use fee in order to avoid issues of cross-subsidization of highway cost within vehicle classes.

The strength of an allocation procedure depends upon the benchmarks chosen to quantify usage based on which the allocation process is performed. The broad consensus among HCA practitioners is to use *cumulative 18,000lb Equivalent Single Axle Load* as a measure of pavement consumption for allocating highway related costs. This metric plays a pivotal role in American Association of State and Highway Transportation Officials (AASHTO) design equation for determining thickness of pavement (comprising of Base, Subbase, Grade and Subgrade) to support vehicle classes on which they traverse. The process of designing pavement of required thickness to sustain definite axle-wheel loadings is referred to as the *traffic loading requirements* of vehicle classes. However, an often-overlooked criterion is the cost needed to build lanes for highways to have sufficient capacity to entertain traffic volumes of these very vehicle classes. The process of designing a highway with specific number of lanes in order to maintain an acceptable Level of Service (LOS) is termed as the *traffic capacity requirements* of vehicle classes.

Truck vehicle classes are known to subject the roadways to heavy axle-loads resulting in substantial pavement deterioration and distress. Vehicle classes with low-axle weights such as passenger cars have negligible loading impact on the pavement. In point of fact, civil engineers build roads keeping mostly truck traffic in mind and focus on improving pavement tolerance by building thicker pavements to withstand heavy loads. Consequently, trucks end up paying for significant portion of pavement related cost. However, it can be argued that were the roadway built only to accommodate truck traffic, cost responsibilities for trucks would in effect decrease as they would in essence pay their fair share of pavement related cost but also capitalize on the savings achieved upon fewer lanes being built to maintain only truck traffic flow. From the perspective of traffic flow theory, truck classes constitute a fractional portion of overall highway traffic due to their characteristically low traffic volumes. For this reason, the dissertation aims to design a methodology that is attentive to the fact that additional lanes are generally constructed as a response to high traffic volumes of lighter vehicle classes upon their introduction into the traffic stream. The removal of passenger car traffic from a given traffic stream leaving only the heavier vehicle classes would result in a decrease in the number of lanes needed to provide an adequate level of service. This responsiveness is absent in all traditional methods of HCA. Therefore, as an acknowledgement of this fact, inequity in an allocation is compounded when trucks are financially held accountable for construction of lanes they don't end up using.

Problem Statement

The objective of this dissertation is to enhance an existing optimization based HCA method to account for variable capacity requirements of vehicle classes when allocating highway costs among vehicle classes. Additional goals of this dissertation are to address the computational complexity, multiple optimality and strengthening the notion of fairness. The work performed in this dissertation can be organized into the following six tasks.

1. *Propose an enhanced linear optimization approach responsive to variable roadway capacity and pavement thickness requirements in highway cost allocation*: This dissertation will propose an enhancement to an existing linear programming approach used in HCA known as the least core LP model to account for variable highway capacity requirements of vehicle classes. This will be facilitated by revising the right hand sides of the LP model using the concept of discrete Aumann-Shapley value. Implementation of this concept in a LP model will effectuate the delineation of highway cost into pavement and lane costs at the same time guarantee enforcement of properties of marginality, rationality and completeness.
2. *Apply the nucleolus solution concept to cooperative games based HCA*: A fair and an equitable cost allocation scheme known as the nucleolus will be modified for implementation in HCA. The cornerstone of this concept is that it maximizes the savings achieved by the least satisfied coalition consisting of individual or groups of vehicle classes in the increasing order of their magnitude. This allocation will be computed by using a prevalent algorithm that entails solving a sequence of LPs using complementary slackness conditions. It will also be demonstrated that the nucleolus is the average of all the optimal corner point feasible solutions to the least core LP model.
3. *Derive the normal equations for a pavement cost function with* $r = 0.5$: A pavement cost function in HCA is used to established a relationship between pavement damage and cost in dollars per lane-mile. A non-linear function with an exponential parameter of 0.5 has shown to be the best representation of relationship between pavement damage and cost occasioned in

a comprehensive HCA study conducted for the state of Texas. The normal equations used to obtain parameters *a* and *b* for this function will be derived for use in future HCA studies.

4. *Re-formulate Highway Cost Allocation Problem as a Dual:* This dissertation will investigate whether linear programming driven HCA procedures solved in the primal form are better off being formulated as a dual in an effort to reduce the well-known computational complexity of the least core LP model. The structure of a least LP model, entailing more number of constraints than variables lends itself to be a prime candidate for transformation to its dual. As the latter contains reduced number of constraints leading to faster processing times for solving a least core LP model. However, it will be shown that the sequential LP approach of computing the nucleolus precludes the ability of the dual to confirm uniqueness of nucleolus in the primal.
5. *Develop Closed-Form Solution to obtain optimal allocation without using simplex algorithm in a special case*: The dissertation will recommend the use of Closed-form Solution (CFS) that can be employed under specific conditions for obtaining the nucleolus. These necessary conditions will be supportive in characterization of a special case where certain variables are basic and non-basic in the optimal basis of the least core LP model with an assumption that simplex is being used to solve the LP. The closed-form solution will express the nucleolus as a collection of coalitions comprising of definite number of vehicle classes. These coalitions will be shown to play an active role in determining the nucleolus.
6. *Design a highway cost allocation decision support system for VC3-VC7***:** As part of this dissertation, an excel based spreadsheet software is developed that will have the capability of determining the cost allocation results for a good sample of available methodologies used for attributing cost among vehicle classes. The methods developed as part of this dissertation will also be programmed into the software. The software will be able to find cost allocations for HCA problems having three to seven vehicle classes.

Organization of the Dissertation

The dissertation is divided into six chapters along with four appendices that contain supplementary information regarding certain assumptions made during the dissertation. Chapter I lays out a theoretical and practical perspective on the issues that are being addressed in this dissertation. An extensive context is provided to the research question along with the modeling framework that is being used to address the HCA problem under examination. The importance of taking due consideration of roadway capacity and pavement thickness requirements in solving a highway cost allocation problem is discussed in detail. And, a brief synopsis is provided on the use of coalitional game theory to provide a justification for modeling collaborative behavior among vehicle classes. Chapter II contains the literature review comprising of a survey of traditional and non-traditional HCA methodologies along with a discussion on several inadequacies of traditional methods used for allocating highway costs. A special spotlight is placed on reviewing prior work performed on game theory based HCA models. All methods are evaluated based on the degree to which they enforce the notion of fairness defined in the context of satisfying the properties of marginality, rationality and completeness. This chapter also contains a brief review on available algorithms for computing the nucleolus and prior work performed on closed-form solutions.

Chapter III of this dissertation provides a more exhaustive and in-depth explanation of the procedure that was presented by Dubey and Garcia-Diaz (2016) involving use of the nucleolus and integration of variable roadway capacity and pavement thickness requirements in HCA. Chapter III describes the development of the new methodology based on an existing game-theoretic concept known as the least core LP model. The steps to implementing a sequential LP approach that starts with least core LP model and culminates with the computation of a unique allocation known as the nucleolus is discussed. A brief study is also conducted to assess the usefulness in conducting the sequential LP approach from the dual in obtaining the nucleolus. This chapter also motivates the implementation of a new characteristic function on the right hand sides of the LP where the issue of variable capacity will be addressed using discrete A-S values.

Chapter IV elaborates on the development of closed-form solutions for the nucleolus that can be used upon the confirmation that least core LP model behaves in a well-defined manner. The conditions that validate the evidence of such a behavior are obtained in this chapter. Chapter V is concerned with the comparative analysis of the proposed methodology against a sample of representative conventional models used for allocating highway related costs. The performance of the proposed methodology in terms of being receptive to the variable capacity requirements is deliberated in this chapter. Chapter VI outlines computerization of the proposed methodology. The computerization was performed on an excel based spreadsheet. Finally, the implications of the proposed enhancements to the field of HCA are summarized, conclusions are drawn and recommendations for future research work are presented in Chapter VII.

CHAPTER II

LITERATURE REVIEW ON HIGHWAY COST ALLOCATION

Incremental and Proportional Methods of HCA

The two most commonly used HCA procedures can be classified into traditional incremental method and the proportional method. Both these methods were used foremost in an HCA study conducted by Bureau of Public Roads (1961). Since then, several variations of these two methods have been used in subsequent HCA studies conducted by FHWA (1982,1997). The traditional incremental method is based on the principle of cost-occasioned that involves dividing the total thickness of a highway facility into a series of design increments based on an established sequence whose costs are attributed to vehicle classes that need them. The process of attributing the cost of an increment to specific vehicle class can be condensed into two steps as was done in Luskin et.al (2000). First, the initial cost of a highway facility than can support the lowest axle-weight vehicle class is calculated. The vehicle class with the lowest axle-weight is assigned the complete financial responsibility for this initial highway facility. Next, the cost of a highway facility that can accommodate two lowest axle-weight vehicle classes is calculated. The second lowest axle-weight vehicle class is assigned cost responsibility for the difference in the cost between the initial and the second highway facility, because without inclusion of the second vehicle class, this cost would not have existed. Subsequent costs of increments to support additional vehicle classes in the increasing order of their axle-weights are given to vehicle classes that need those thickness increments to support themselves on the highway. Proportional methods as the name implies allocates cost among vehicle classes based on a proportionality factor. These factors could be based on pavement damage such as Equivalent Single Axle Loads (ESAL), traffic travel levels such as Vehicle Miles Traveled (VMT) or size such as Passenger Car Equivalent (PCE) weighted-VMT.

Most of the recent HCA studies conducted by state and federal highway agencies have regarded the incremental method and proportional allocation schemes as the method of choice due its widespread adoption by practitioners. Balducci and Stowers (2008) have documented all past and contemporary methods used for allocating costs in federal and state HCA studies. A review of their works suggests that with exception of few state HCA studies, most state and federal HCA studies have used incremental, proportional or a combination of these two methods to allocate highway costs. Existence of multiple variations of incremental method is in large part due to the difficulty associated with answering the question that at what juncture does a heavy vehicle class become accountable for higher increments. There is also an economies of scale issue in HCA that one needs to recognize when allocating highway related costs. Fwa and Sinha (1985) demonstrated the non-linear relationship between traffic load and pavement thickness wherein a slight increase in thickness of pavement leads to a dramatic rise in load bearing capacity of the pavement. This economies of scale is an important issue that must be accounted for, lack of which, lighter vehicle classes end up subsidizing the cost of pavement for heavier vehicle classes. Fwa and Sinha proposed the use of thickness incremental method that allocated cost of base pavement proportionally based on VMT. The remaining pavement thickness were divided into a series of equal pavement increments. The cost of each incremental thickness was allocated proportionally based on relative ESALs of vehicle classes. Hong, Prozzi and Prozzi (2007) advocated for a new proportional way of allocating highway cost by assigning cost responsibility based on contribution to pavement distresses by vehicle classes. The distress shares for each vehicle class were estimated using the new mechanistic-empirical pavement design guide (MEPDG). From the notion of fairness, which is the focal point of this dissertation, the allocations produced by traditional incremental method, proportional method and nearly all their variations, lack consistency. Some of these shortcomings still persist in modern practice and are mentioned below:

1. The allocation obtained using incremental method changes depending upon the established order in which vehicle classes are introduced in the process of designing pavement cost increments for a given highway facility.
2. The incremental and proportional methods satisfy completeness but do not have the capability of enforcing rationality. Marginality is enforced in weak way due to their consideration of only one order of vehicle classes and not all possible groupings of vehicle classes with different marginal contributions. The proportional method has no mathematical means of ensuring vehicle classes pay their minimum marginal costs.
3. Finally, among above-mentioned limitations, a major drawback pertaining to this dissertation is the fact that both incremental and proportional methods implicitly assume all vehicle classes have similar requirements for number of lanes, despite the relatively large known differences in traffic volumes between heavy and lighter vehicle classes.

Methods Based on Cooperative Games

The first classical work that showed players exhibit rational behavior in strategic situations in order to maximize their utility stemmed from the concepts established in a book entitled *theory of games and economic behavior* written by Neumann and Morgenstern (1944). Since then, cooperative games have been used to allocate cost with the assumption of rational behavior in pricing of public utilities in the telecommunications networks, determination of airline landing fees and the allocation of joint costs in multipurpose reservoir developments. An extensive description of these applications are documented by Young (1985). This dissertation will prioritize reviewing previous research work conducted on cost allocation of highway expenditures based on cooperative games. A special emphasis will be placed towards the extent to which the nucleolus solution concept as proposed by Schmeidler (1969) was implemented.

Villarreal-Cavazos (1985) was the first in framing the HCA problem as an *n-person cooperative game*, where vehicle classes were considered as players and groups of vehicle classes as coalitions. The author developed a linear programming formulation known as the Generalized Method (GM) that satisfied the properties of marginality, rationality and completeness as a constraints of an LP model. The collection of constraints defined a feasible region from which an allocation belonging to *core* could be selected. This concept of core was first formulated by Gillies (1959) where every allocation in this bounded region has a counter-objection to an objection. The generalized method was founded on the concept of least core in game theory whose existence was shown by Shapley and Shubik (1966). The concept was later termed as the "least core" by Maschler, Peleg and Shapley (1979) wherein they analyzed the geometric properties of well-known solution concepts of kernel, core and the nucleolus. The cost function used to estimate the worth or cost of forming coalitions known as the *characteristic function* was a pavement cost equation that considered all vehicle classes to have fixed requirements for number of lanes. In addition, the GM procedure in principle used the least core allocation as a fair solution, which when unique is the nucleolus. But if not, the nucleolus must be obtained by implementation of a tie breaking rule that continues increasing the savings of the least satisfied coalition in a non-decreasing order leading to an eventual convergence to the nucleolus. However, when the GM method resulted in multiple optimal solutions, these solutions belonged to a set known as the reduced core or least core. At this stage, Villarreal-Cavazos recommended using second phase constraints to break the ties and arrive at an optimal solution based on the statistical cost effects of vehicle classes. Although the GM did not intend to compute the nucleolus, the dissertation notes that the nucleolus could have as well been obtained by extending GM to solve a sequence of LPs in order to increase the savings of all least satisfied coalitional groups of vehicle classes until a unique solution is obtained by the last LP model in the sequence. Garcia-Diaz and Villarreal (1985) also proposed the Modified Incremental Method (MIA) that improved the traditional incremental method by eradicating the need to introduce the vehicle classes in any specific order.

Castano-Pardo and Garcia-Diaz (1995) used the Shapley (1953) value solution concept to allocate highway expenditures among vehicle classes with players distinctly defined as the number of passes of 18-kip ESALs. The author suggested that by treating each pass as a player, the procedure would provide for a realistic assumption that road users are better represented as a single pass of ESAL than consolidated into vehicle classes. Their work also addressed the issue of order dependence of incremental method by determining the allocations in all possible orderings of vehicle classes and then acquiring the expected marginal cost contribution of each vehicle class. Another important significance of Castano-Pardo's work was the use of an innovative characteristic function of a game that optimized the pavement cost in the form of a non-linear programming model that was first proposed by Small and Winston (1988). This procedure satisfied marginality and completeness but did not guarantee adherence to rationality. Makrigeorgis (1991) revisited the Generalized Method and developed an LP based Optimal-durability HCA method.

Lee (2002) as emphasized in this dissertation attached great importance to the issue of not addressing the question of variable capacity requirements of vehicle classes in a HCA methodology. Lee proposed a non-optimization cost separation approach known as the discrete Aumann-Shapley value method that allocated cost among vehicle class by separating the total highway cost into pavement and capacity cost. The solution concept was originally developed by Aumann and Shapley (1974) for non-atomic games consisting of infinite number of players, where each player had an insignificant impact on the outcome of the game. The discrete A-S procedure also intrinsically used the Shapley value (1953) to allocate the number of lanes to vehicle classes, a rather robust and fair way of apportioning lane-use among vehicle classes. Although, this procedure resolved an important issue of recognizing variable capacity requirements of vehicle classes, it did not guarantee an allocation that would satisfy rationality. An added major contribution of this work was the development of a compact-form deduced from Redekop's (2000) formula and Moulin (1995) to obtain discrete A-S values. The compact form assisted in the easing of the computational issue of acquiring the discrete A-S values when considering large number of players in a given HCA problem. Next, as part of this dissertation, a literature search was conducted for the purpose of finding the most suitable algorithm for computing the nucleolus.

Existing Algorithms for Computing the Nucleolus and Closed-form Solutions

Cetiner (2013) did a detailed chronological review of algorithms available to compute the nucleolus. The author's review documented efforts of several prominent scholars making gradual but significant progress towards reducing the computational complexity of finding the nucleolus. Based on Cetiner's review, it was determined that the concept of engaging in a sequence of LPs to arrive at the nucleolus originated from Kopelowitz (1967). Notably, an algorithm exists such as the one presented by Puerto and Perea (2013) that allowed computation of the nucleolus by solving a single LP, however, the LP proposed by them has significantly large dimensions, rendering the procedure impractical for solving a typical moderate sized HCA problem. Behringer (1981) proposed an approach based on simplex algorithm founded by Dantzig (1963), which computed the nucleolus by solving a sequence of LPs using information from the dual. Fromen (1997) perfected Behringer's algorithm by further reducing the number of LPs needed to obtain the nucleolus by developing appropriate stopping conditions. Fromen's condition for termination were based on the observation that the nucleolus is obtained when a set of constraints in one of the linear programs of the sequence defined a unique solution. Based on an evaluation of available methods for computing the nucleolus, this dissertation concluded that in terms of algorithmic efficiency, obtaining nucleolus by progressively solving a sequence of LPs as proposed by Fromen is best suited for adaptation to the HCA problem being studied in this dissertation. Therefore, allocation of highway costs among vehicle classes will be performed using the concept of nucleolus as a basis for fair division. This dissertation will utilize Fromen's algorithm to compute the nucleolus and his procedure will be referred to as the sequential LP approach.

This dissertation would like to acknowledge researchers who have developed closed-form solutions for the nucleolus in various cost sharing problems. Littlechild (1974) demonstrated that the special nature of defining characteristic function for a *n*-player game allows for the nucleolus to be computed using efficient approximated algorithms where coalitional costs are based on the largest player in a coalition. Reinhardt (2004) developed closed-form solutions for the nucleolus in a class of cooperative games associated with allocation of congestion costs. The characteristic function suggested by Reinhardt assigned cost of a two or more player coalition as equal to the summation of the square root of cost of all single player exclusive coalitions that belong to the two or more coalition. Reinhardt and Littlechild capitalized on the symmetric nature of obtaining the costs of coalitions in their respective cost allocation problems in order to express the nucleolus in terms of closed-form solutions. Both have detailed extensive proof for their closed-form solutions.

Leng and Parlar (2010) developed analytical expressions for obtaining the nucleolus in a three-player cooperative game. The authors explicitly derived closed-form solutions for the nucleolus in both, empty and non-empty core situations. They outlined all possible scenarios in which the nucleolus for a three player cooperative game could be defined by specific coalitions irrespective of the type of characteristic function being used to get the cost of coalitions. Table II.1 depicts the properties that are satisfied by all the traditional and non-traditional HCA methods. The symbol of cross "✘" means that property is not guaranteed and the symbol of tick mark "✓" means that property is guaranteed. The abbreviation of C stands for Completeness, M for Marginality and R for Rationality, TL for Traffic Loading, VC for Variable Capacity.

Table II.1: Properties satisfied by HCA Methods

HCA Methods	C	M	R	TL	VC
Proportional Method (State and Federal)	✓	✘	✘	✓	✘
Incremental Method (State and Federal)	✓	✓	✘	✓	✘
Shapley value (Castano-Pardo and Garcia-Diaz)	✓	✓	✘	✓	✘
Modified Incremental Method (Villarreal-Cavazos and Garcia-Diaz)	✓	✓	✓	✓	✘
Generalized Method (Villarreal-Cavazos and Garcia-Diaz)	✓	✓	✓	✓	✘
Optimal Durability-Based Method (Makrigeorgis and Garcia-Diaz)	✓	✓	✓	✓	✘
Discrete A-S Value Method (Lee and Garcia-Diaz)	✓	✓	✘	✓	✓

CHAPTER III

PROPOSED METHODOLOGY

Let N be the set of vehicle classes *{1,2,....,n}* and M be the power set of all non-empty subsets of *N*. The characteristic function $c : M \rightarrow \Re$ assigns a cost *C(S)* to each coalition S$\in M$. The cost *C(S)* in this dissertation includes two components:

1. Pavement Thickness: Cost of highway facility with adequate thickness to sustain the traffic-loading requirements of vehicle classes in *S*.
2. Roadway Capacity: Cost of traffic lanes needed to accommodate vehicle classes in *S*

A cost allocation vector $x = (x_1, \ldots\ldots, x_n)$ assigns to each vehicle class *i* in grand coalition *N* an amount x_i such that:

a) The cost of designing a highway facility that can accommodate all vehicle classes should be paid for by members of *N* in entirety. This is known as the completeness condition or the efficiency criterion.

$$\sum_{i \in N} x_i = C(N) \qquad \text{(III-1)}$$

b) The cost allocated to each vehicle class in coalition *S* should not exceed the total cost paid by members of that coalition. This is known as the property of rationality. This can be represented as:

$$\sum_{i \in S} x_i \leq C(S) \qquad \text{(III-2)}$$

c) The sum of the cost allocated to each vehicle class in coalition *S* should at least cover its marginal cost. This is referred to as the marginality criterion:

$$\sum_{i \in S} x_i \geq C(N) - C(N - S) \qquad \text{(III-3)}$$

It has been proven by Young (1985) that properties of rationality (III-2) and marginality (III-3) are equivalent when completeness is held. Hence, marginality property does not need to be stated explicitly as a constraint in an LP model.

The Core

The set of allocations satisfying constraints (III-1) - (III-3) is known as the *Core*. Now, the savings $t(x, S)$ of coalition *S* at *x* is defined as the difference between the total cost *C(S)* paid by members of coalition *S* and actual amount being paid by partaking in the grand coalition *N*. In many cooperative game theory applications, this is also referred to as an excess gained in a cost sharing problem by coalitions for choosing to collaborate in a grand coalition *N*.

$$t(x, S) = C(S) - \sum_{i \in S} x_i \qquad \text{(III-4)}$$

Viewing relationship (III-4) as a measure of satisfaction. The preferred outcome in a HCA problem would be for vehicle classes to be awarded the lowest cost resulting in largest savings. Meaning, higher the savings of *S*, the more content coalition *S* would be with the cost allocation. The goal of the proposed HCA methodology is to come up with an equitable cost division among vehicle classes of grand coalition *C(N)* founded upon an established principle of fairness. This dissertation will advocate the use of the nucleolus solution concept as a fair distribution scheme for *C(N)*, where the savings vector of an allocation *x* is stated as $t(x, S) = (t(x, S_1), \dots \dots t(x, S_m))$, with $m = 2^n - 2$ coalitions. The allocation of interest known as the nucleolus sequentially maximizes the savings of successive sets of least satisfied coalitions in the increasing order of their magnitude. The procedure for obtaining the nucleolus for HCA involves solving sequence of LPs beginning first with the least core LP model. The next section describes the least core LP model.

Least Core Linear Programming Model

The objective of the linear programming model formulated in this section is to allocate costs among the vehicle classes (vehicle classes) using a common facility (highway) by maximizing savings achieved by all coalitions for participating in the grand coalition subject to completeness, marginality and rationality constraints as stated in (III-1) - (III-3). The mathematical formulation is shown below:

$$\max \ t \qquad \text{(III-5)}$$

Subject to

$$\sum_{i \in N} x_i = \mathrm{C(N)} \qquad \text{(III-6)}$$

$$\sum_{i \in S} x_i \leq \mathrm{C(S)} \ \ for\ all\ S \in N \qquad \text{(III-7)}$$

$$\mathrm{C(S)} - \sum_{i \in S} x_i \geq t \tag{III-8}$$

$$x_i \text{ , } t \geq 0 \text{ for all } i \in N \tag{III-9}$$

The objective function maximizes the lower bound t on cost savings for each coalition of vehicle classes. Constraint (III-6) represents the completeness condition. Constraint (III-7) is the formulation of the rationality condition that ensures that each vehicle class is allocated a lower cost by joining the grand coalition. The savings t enjoyed by coalition S when joining the grand coalition is given by the left-hand term of constraint (III-7). All decision variables are non-negative in this LP model as indicated in constraint (III-9). The least core LP model always has $2^n + n$ variables (decision, artificial and slack variables) and $2^n - 1$ constraints.

Sequential Approach to Computing the Nucleolus

Occasionally, the least-core model has infinitely many optimal solutions. In this case, the problem is not completely solved until further analysis identifies a unique solution, known as the *nucleolus*. A conflict of interest resulting from the existence of multiple solutions is demonstrated by the following assertion. Let $\hat{x}$ and $\hat{y}$ be two optimal allocations produced by the linear programming model. Furthermore, S and T are subsets of the grand coalition N. Coalition S has an "objection" to $\hat{x}$ if the savings $t(\hat{y}, S)$ and $t(\hat{x}, S)$ satisfy the following:

$$t(\hat{y}, S) > t(\hat{x}, S)$$

Additionally, T has a "counter objection" to $\hat{y}$ if:

$$t(\hat{y}, T) < t(\hat{x}, T)$$

$$t(\hat{y}, T) < t(\hat{x}, S)$$

The nucleolus serves as an allocation to which no coalition has sufficient reason to complain. This is the foundation on which this dissertation affirms that a reasonable allocation to the highway cost allocation problem should be the nucleolus. In order to find the nucleolus, to start with, the most dissatisfied coalition is identified, which is the one having least savings. Once identified, savings achieved by that coalition is increased and at the same time ensuring that the savings are being uniformly distributed among all the coalitions. The successive increase or maximization of savings for all coalitions in a HCA problem is accomplished by means of a mechanism that entails solving a sequence of linear programs using duality. This sequential LP approach was first developed by Behringer (1981) and then improved thereafter by Fromen (1997).

A strong argument in favor of implementing such an approach to the HCA problem comes from the seminal work conducted by Young, Okada and Hashimoto (1982) in allocation of joint costs for using a multipurpose reservoir facility. The authors of that study suggested computation of the nucleolus in cases where the least core LP model resulted in multiple optimal solutions. Although such a step was recommended and not required, review of relevant literature puts forward the view that progression towards computation of the nucleolus is mandated from the least core LP model if the prime objective is to search for the nucleolus.

An illustration of this sequential procedure is shown below for a three vehicle class HCA problem with characteristic function being defined as $C(\emptyset) = 0,\ C(1) = 58, C(2) = 44, C(3) = 31, C(12) = 65, C(13) = 71, C(23) = 57$ and $C(123) = 78$.

1st LP Model

The least core model for a three vehicle class HCA problem along with corresponding dual variables is formulated as follows:

$$\max t_1$$

Subject to

$$
\begin{aligned}
x_1 + t_1 &\le 58 && (Y_1) \\
x_2 + t_1 &\le 44 && (Y_2) \\
x_3 + t_1 &\le 31 && (Y_3) \\
x_1 + x_2 + t_1 &\le 65 && (Y_{12}) \\
x_1 + x_3 + t_1 &\le 71 && (Y_{13}) \\
x_2 + x_3 + t_1 &\le 57 && (Y_{23}) \\
x_1 + x_2 + x_3 &= 78 && (Y_{123}) \\
x_1, x_2, x_3, t_1 &\ge 0
\end{aligned}
$$

The first LP model was determined to have multiple optimal solutions because the objective function cost coefficient or the reduced cost for one non-basic variable in the optimal tableau was observed to be zero. The non-zero dual variables of the coalitional rationality constraints of {3} and {1,2} are $Y_3 = 0.5$ and $Y_{12} = 0.5$ respectively. Therefore, coalitions {3} and {1,2} are put into set $F_1 = \{\{3\},\{1,2\}\}$. One of the optimal basic feasible solution is determined be $x_1 = 40, x_2 = 16$ and $x_3 = 22$ with $t_1 = 9$. Pivoting on the non-basic variable with a reduced cost of zero, the other optimal basic feasible solution was found to be $x_1 = 30, x_2 = 26$ and $x_3 = 22$ with $t_1 = 9$. After recognizing the existence of multiple solutions, the procedure mandates that the optimal value t_1 from the 1st LP be made equal to the variable t_2 in the 2nd LP on account of binding coalitional rationality constraints in set F_1. This is carried out by subtracting the optimal t_1 value of 9 from *C(3)* and *C(12)* in the 2nd LP as shown below:

$$x_3 + 9 = 31$$
$$x_1 + x_2 + 9 = 65$$

Conceptually, the procedure is fixing the savings of coalition {3} and {1,2} as they cannot be increased any further. These coalitions also have the lowest savings and therefore are considered to be the first set of least satisfied coalitions. The 2nd LP attempts to further modify the first allocation so that it can continue increasing the savings of the next set of least satisfied coalitions.

2nd LP Model

$$\max t_2$$

Subject to

$$\begin{aligned} x_1 + t_2 &\leq 58 \\ x_2 + t_2 &\leq 44 \\ \boldsymbol{x_3} &= \mathbf{22} \\ \boldsymbol{x_1 + x_2} &= \mathbf{56} \\ x_1 + x_3 + t_2 &\leq 71 \\ x_2 + x_3 + t_2 &\leq 57 \\ \boldsymbol{x_1 + x_2 + x_3} &= \mathbf{78} \\ x_1, x_2, x_3, t_1 &\geq 0 \end{aligned}$$

The optimal solution given by the 2nd LP model is $x_1 = 35, x_2 = 21$ and $x_3 = 22$ with $t_2 = 14$. The solution is found to be unique as the cost coefficients or the reduced cost of non-basic variables in the optimal tableau was found to be greater than zero. Uniqueness of the 2nd LP implies that no further modification of this allocation would increase the savings of the remaining coalitions.

Table III.1: Computation of the nucleolus for three vehicle class HCA problem

Sequence of LPs		1^{st} LP *(j=0,n=1)*		2^{nd} LP *(j=1,n=2)*
DUAL		$Y_3 = 0.5$ $Y_{12} = 0.5$		$Y_{13} = 0.5$ $Y_{23} = 0.5$ (Termination Point)
Objection Function		$t_1 = 9$	$t_1 = 9$	$t_2 = 14$
PRIMAL	x_1	40	30	**35**
	x_2	16	26	**21**
	x_3	22	22	**22**
	C(N)	78	78	78

Hence, the sequential LP approach is terminated and there is no need for solving the 3rd LP. The results of the sequential LP approach are reported in Table III.1. A formal description along with a flowchart in Figure III.1 of the procedure previously illustrated is now given below:

(A) The formulation of the first LP model to be solved is the least core model shown in (III-5) - (III-8). The objective function value for this problem will be represented by t_1.

(B) The *n-th* LP model ($n > 1$) in the sequence is formulated below:

$$\max \ t_n \tag{III-10}$$

Subject to

$$\sum_{i \in N} x_i = \mathrm{C(N)} \tag{III-11}$$

$$\sum_{i \in S} x_i + t_n \leq \mathrm{C(S)} \ \ for\ all\ S \in N \tag{III-12}$$

$$\sum_{i \in S} x_i + t_j = \mathrm{C(S)} \ \ for\ all\ S \in F_j \quad j \in \{1, \ldots n - 1\} \tag{III-13}$$

$$x_i,\ t_n,\ t_j \geq 0 \text{ for all } i \in N$$

(C) If the model in (B) has multiple optimal solutions, then all coalitions whose dual values are non-zero are placed into set $\mathbf{F}_j$. Using the *complementary slackness conditions* of linear programming, the *primal* inequality constraints for the corresponding non-zero *dual* value are converted into equalities. Furthermore, the excess of the coalitions in $\mathbf{F}_j$ which is also the optimal excess of the *j-th* LP program, must be set equal to t_n in the *n-th* LP model in the series for all $n > j$, as shown in constraint (III-13). The process of solving a sequence of *n* LP models is continued until a unique cost allocation is obtained.

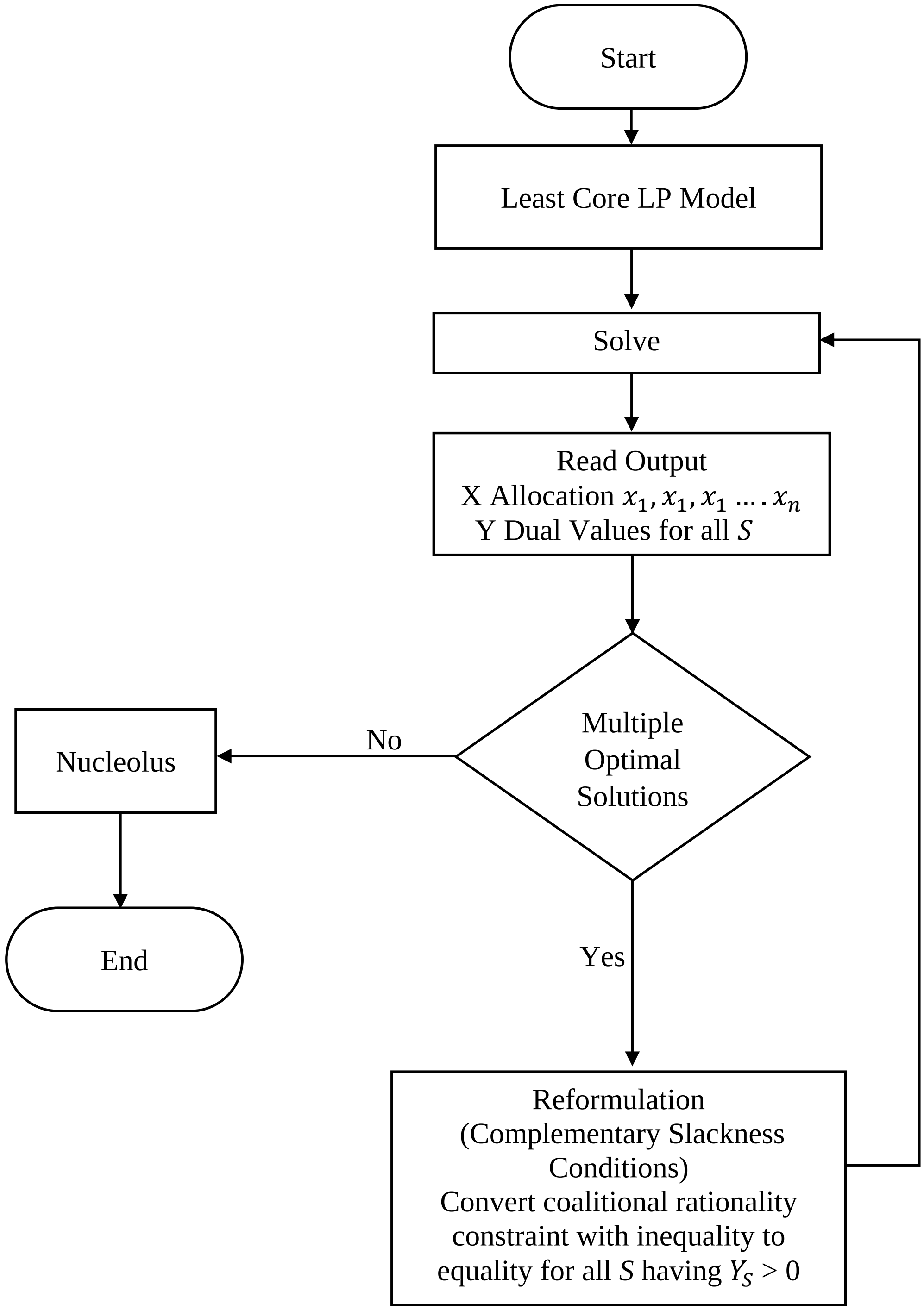

Figure III.1: Flowchart for Sequential LP Approach

Sequential LP Approach from the Dual

The purpose of this section is to investigate whether the sequential LP approach could be applied from dual as opposed to its customary implementation from the primal. The motivation for this arises from the observation that the least core LP model has more number of constraints than decision variables. The computational time of simplex algorithm depends more on the number of constraints than decision variables as the former controls the number of corner points that must be visited before an optimal solution is reached. Since the dual of the least core LP model has *n* constraints and $2^n - 1$ variables, recovery of the optimal solution from the dual would be advantageous in minimizing the processing time involved in solving exponential number of rationality constraints associated with the least core LP model.

As an illustration, consider an example HCA problem with characteristic function defined as $C(\emptyset) = 0$, $C(1) = 58$, $C(2) = 44$, $C(3) = 31$, $C(12) = 65$, $C(13) = 71$, $C(23) = 57$ and $C(123) = 78$,. In the previous section, the nucleolus for this example was obtained using a sequential LP approach applied from the primal. The following section will demonstrate the same procedure applied from the dual. The first dual LP model is formulated below:

1st Dual LP Model

$$\min\ 58Y_1 + 44Y_2 + 31Y_3 + 65Y_{12} + 71Y_{13} + 57Y_{23} + 78Y_{123}$$

Subject to

$$\begin{aligned} Y_1 + Y_{12} + Y_{13} + Y_{123} &\geq 0 \\ Y_2 + Y_{12} + Y_{23} + Y_{123} &\geq 0 \\ Y_3 + Y_{13} + Y_{23} + Y_{123} &\geq 0 \\ Y_1 + Y_2 + Y_3 + Y_{12} + Y_{13} + Y_{23} &\geq 1 \\ Y_1, Y_2, Y_3, Y_{12}, Y_{13}, Y_{23} &\geq 0 \\ Y_{123}\ &unrestricted \end{aligned}$$

The optimal solution is found to be $Y_3 = 0.5$, $Y_{12} = 0.5$, $Y_{123} = -0.5$ and $Y_{23} = 0$. The solution is noted to be a unique and degenerate. According to work performed by Sierksma (2001) regarding primal-dual relationships, this fact implies that the primal formulation has multiple optimal solutions. Degeneracy is of no consequence in this model and the positive dual variables are the only variables of interest. As expected, the objective function value was also found to be 9 equaling the primal objective function of $t_1 = 9$ due to strong duality. And so, dual variables Y_3 and Y_{12} are placed in special set $L_1 = \{Y_3, Y_{12}\}$. For the 2nd dual LP model, variables in L_1 will now remain unrestricted throughout the sequence of remaining dual LPs that would be solved. In addition, the variables of Y_3 and Y_{12} are eliminated from the final constraint. The objective function value of 9 is now subtracted from the cost co-efficient of Y_3 and Y_{12}, those being 31 and 65 respectively in the dual objective function. The second dual LP model will be written as follows:

2nd Dual LP Model

$$\min \ 58Y_1 + 44Y_2 + 22Y_3 + 56Y_{12} + 71Y_{13} + 57Y_{23} + 78Y_{123}$$

Subject to

$$Y_1 + Y_{12} + Y_{13} + Y_{123} \geq 0$$
$$Y_2 + Y_{12} + Y_{23} + Y_{123} \geq 0$$
$$Y_3 + Y_{13} + Y_{23} + Y_{123} \geq 0$$
$$Y_1 + Y_2 + Y_{13} + Y_{23} \geq 1$$
$$Y_1, Y_2, Y_{13}, Y_{23} \geq 0$$
$$Y_3, Y_{12}, Y_{123} \ unrestricted$$

The optimal solution is found to be $Y_3 = -0.5, Y_{13} = 0.5, Y_{123} = -0.5$ and $Y_{23} = 0.5$. The 2nd dual LP model was found to have multiple and non-degenerate solutions. The objective function value was found to be 14. And so, dual variables Y_{13} and Y_{23} are placed into set $L_2 = \{Y_{13}, Y_{23}\}$. For the 3rd dual LP model, these variables will now remain unrestricted throughout the sequence of the dual LPs along with existing variables in L_1. In addition, the variables of Y_{13} and Y_{23} are eliminated from the final constraint. The objective function value of 14 is now subtracted from the cost co-efficient of Y_{13} and Y_{23}, those being 71 and 57 respectively. However, due to the presence of multiple solutions that are non-degenerate, Sierksma shows that such a situation means that the primal must be unique. Therefore, one can conclude that the dual of the dual is in fact the nucleolus. And so, the next step is to recover the primal solution from the dual. With the knowledge of the co-efficient of basic variable C_B in the objective function and the inverse of matrix of basic variables B^{-1} for the 2nd dual LP model, the dual of the dual Y_B can be obtained using the following standardized relationship:

$$Y_B = C_B B^{-1}$$

$$= (22 \quad 71 \quad 78 \quad 57) * \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = (35 \quad 21 \quad 22 \quad 14)$$

It can be checked that the optimal allocation is $x_1 = 35, x_2 = 21$ and $x_3 = 22$ with $t_2 = 14$. However, as part of this dissertation, many HCA problems solved using the dual sequential LP approach revealed an absence of a well-defined point of termination. More specifically, two important insights were made as a direct consequence of relationships existing between the primal and dual optimal solutions. These are listed below:

1. When the dual model has multiple solutions that are non-degenerate, then, dual of the dual which is the primal, can be concluded to be unique. Hence, such a solution can then be declared to be the nucleolus.
2. When the dual model has multiple solutions that are degenerate, then, dual of the dual cannot be concluded to be unique because uniqueness of the primal cannot be established based on a LP model that has multiple and degenerate solutions.

The dissertation discovered that the second observation impedes the process of confirming whether the dual of the dual is in fact the nucleolus. The uniqueness of the nucleolus is an important component of affirming whether the optimal solution of the dual can in fact be used to get the nucleolus. One could in theory implement the sequential LP approach from the dual until the dual of the dual ceases to change.

However, such an approach leads to an extended computational time (more number of LPs) as compared to the sequential approach commenced from the primal that does have a clear criterion for termination. Therefore, this dissertation recommends that application of the sequential LP approach from the primal as the most prudent way of calculating the nucleolus.

Lexicographical Comparisons

The iterative maximization of the lower bound t_n performed by the least core LP model on all coalitions is expressed by arranging the savings achieved by each coalition *t(x,S)* in a non-decreasing order. Following this, an element-to-element comparison of savings of each coalition S in the nucleolus against any other allocation belonging to the core is conducted in order to demonstrate a unique lexicographic property of the nucleolus.

Let's say that if $X = \{\, x : \sum_{i=1}^{n} x_i = C(N) \,\}$ be a set of allocations satisfying completeness. $L(x)$ is an ordering convention known as the lexicographic order used to express the savings *t(x,S)* achieved by all coalitions $S \subseteq N$ assigned an allocation x that are arranged in the increasing order of their magnitude. It is said that for an allocation $x \in X$ designated as the nucleolus and for any other allocation $\boldsymbol{\mathcal{H}} \in X$, it is always the case that the savings vector defined by the nucleolus is lexicographically greater than any allocation that satisfies completeness. The definition of any allocation being lexicographically greater is represented by symbol "$\succcurlyeq_{lex}$" whose meaning is described below using an explanation motivated from a chapter written by Ferguson (2014). A vector $k = (k_1, \ldots\ldots, k_m)$ is lexographically greater than a vector $z = (z_1, \ldots\ldots, z_m)$, when:

- $k_1 > z_1$, or if $k_1 = z_1$ and $k_2 > z_2$,or
- when $k_1 = z_1$, $k_2 = z_2$ and $k_3 > z_3$

Specifically, it can be said that if $k_1 = z_1$, ..., k_{m-1}= $z_{m-1}$and $k_m > z_m$, the first element in which k and z are different, that element of k is greater than the corresponding element in z. Therefore, this greatness is denoted as $k \geq_{lex} z$ or $k \succcurlyeq z$ if either $k >_{lex} z$ or $k = z$. The nucleolus is a solitary allocation for a given HCA problem whose lexicographic order $L(x)$ is unique and always greater than any allocation that defines $L(\boldsymbol{\mathcal{H}})$. A more compact form of stating this is represented in notational form as follows:

$$L(x) \succcurlyeq_{lex} L(\boldsymbol{\mathcal{H}})$$

Notably, the sequential LP approach adapted for HCA in this dissertation that begins with the least core and converges to the nucleolus, by design, also maximizes the savings of coalitions in the lexicographic order.

As an illustration of the concept of lexicographical ordering, reconsider the following least core LP model with characteristic function defining cost of coalitions as $C(\emptyset) = 0$, $C(1) = 58$, $C(2) = 44$, $C(3) = 31$, $C(12) = 65$, $C(13) = 71$, $C(23) = 57$ and $C(123) = 78$. The nucleolus for this HCA was found to be $X_N^* = (35{,}21{,}22)$. Another allocation belonging to the core for this illustration is $x = (40{,}16{,}22)$. The lexicographical comparison between the two allocations are conducted as follows:

Let $L(x)$ be defined as the vector $k = (k_1, \ldots\ldots, k_m)$ of savings $t(X_n^*,S)$ arranged in the increasing order where X_n^* is the nucleolus. Next, defining $L(\boldsymbol{\mathcal{H}})$ as the vector $z = (z_1, \ldots\ldots, z_m)$ of savings $t(\boldsymbol{\mathcal{H}}, S)$ arranged in the increasing order where $\boldsymbol{\mathcal{H}} = (40{,}16{,}22)$. Next, an element-to-element comparison between the savings achieved by coalitions assigned the allocation of the nucleolus and the allocation of $\boldsymbol{\mathcal{H}}$ is conducted as shown in Table III.2. Comparing k_1 to z_1, k_2 to z_2 and k_3 to z_3, the following observations can be made:

$$k_1 = z_1 = 9$$

$$k_2 = z_2 = 9$$

$$k_3 = 14 > z_3 = 9$$

$$L(X_N^*) \succcurlyeq_{lex} L(\boldsymbol{\mathcal{H}})$$

Table III.2: Lexicographical order for savings

Sequential LP Approach (Nucleolus)		k	$L(X_N^*)$			$L(\boldsymbol{\mathcal{H}})$		z
			$X_N^* \in \sum_{i=1}^{n} x_i = 78\}$			x $\in \sum_{i=1}^{n} x_i = 78\}$		
			(35, 21, 22)			(40,16,22)		
			$t(X_n^*, S)$ = C(S)$-\sum_{i\in S} x_i$			$t(x, S)$ = C(S)$-\sum_{i\in S} x_i$		
n=1	$t_1 = 9$	k_1	{3}	9	=	9	{3}	z_1
		k_2	{1,2}	9	=	9	{1,2}	z_2
n=2	$t_2 = 14$	k_3	{1,3}	14	>	9	{1,3}	z_3
		k_4	{2,3}	14		18	{1}	z_4
n=3	$t_3 = 23$	k_5	{1}	23		19	{2,3}	z_5
n=4	$t_4 = 23$	k_6	{2}	23		28	{2}	z_6

Since $k_1 = z_1, k_2 = z_2$ and $k_3 > z_3$, vector $k = (k_1, \ldots\ldots, k_m)$ is acknowledged to be lexicographically greater than a vector $z = (z_1, \ldots\ldots, z_m)$. Therefore, $k \geq_{lex} z$ or $k \succcurlyeq z$. As a consequence, the nucleolus is affirmed to be an allocation that maximizes $L(x)$ in the lexicographic order. As an attestation to theoretical validity of the sequential LP approach, observation of Table III.2 discloses that the least core LP model also in essence maximizes savings with optimal value of $t_1 = 9$ for the 1st set of least satisfied coalitions {3} and {1,2} , then the 2nd set of least satisfied coalitions {1,3} and {2,3} with $t_2 = 14$, then the 3rd set of least satisfied coalitions {1} with optimal $t_3 = 23$ and finally the 4th set of least satisfied coalitions {2} with optimal value $t_4 = 23$. This entire process of solving a sequence of LPs is known as the lexicographical maximization.

Nucleolus: Average of all optimal basic solutions of least core LP model

If $Q = \{\, x : \sum_{i=1}^{n} x_i = C(N)\, ; \sum_{i \in S} x_i + t \leq \mathrm{C(S)}\ \forall S \subseteq N\}$ be the least core comprising of a set of allocations maximizing the lower bound on savings *t*. The average of all allocations in Q is the nucleolus and a single point solution.

$$X_N^* = \sum_{k=1}^{d} \lambda_k \, x_k^* \qquad (\mathrm{III} - 14)$$

where, $\lambda_k = \frac{1}{d}$

d = Number of corner point feasible solutions in Q

When the 1st LP model (least core) in a sequence of LPs needed to compute the nucleolus, results in multiple optimal solutions, then it was observed that the average of all the optimal corner point feasible solutions (CPFS) of the 1st LP revealed the nucleolus. This dissertation traces back this property to the axiomatic definition of the nucleolus. According to Young (1985), Schmeidler (1969) had proven uniqueness of the nucleolus by showing that if two allocations *x* and *y*, both of which are maximizing savings *t(x,S)*, then the average of the two savings vector for allocations *x* and *y* will be lexicographically greater than each one of them. Further to his point, in a two-person game, the nucleolus is the mid-point of the reduced core as shown in Young (1985). Also, for a three-player game, nucleolus is the mid-point of the least core as also discussed by Carter and Walker (1996). Therefore, a logical argument would be that taking the average of all lexicographic maximums obtained from the least core LP model will be a unique allocation (i.e. nucleolus). This allocation must also be lexicographically greater than each allocation in the least core and by extension any allocation in the core itself. This dissertation expresses this as following:

Consider a set E (core), consisting of allocations satisfying completeness and rationality constraints (III-1) and (III-2) :

$$E = \{ x : \sum_{i=1}^{n} x_i = C(N) ; \sum_{i \in S} x_i \leq \mathrm{C(S)} \quad \forall S \subseteq N \}$$

Consider another set Q (least core), consisting of allocations satisfying completeness and rationality constraints with variable t_1 maximizing the savings *t(x,S)* for all coalitions. Notably, if the set Q contains only one solution, then it is the nucleolus X_N^*. In cases of multiple optimal solutions, this set is constructed by enumerating all the optimal corner point feasible solutions of the least core LP model (III-5)-(III-9) and placing them into the set Q.

The number of optimal basic feasible solutions in Q represents the lexicographic maximums of the reduced core. Each distinct allocation $x_1^*, x_2^*, x_3^* \ldots\ldots x_d^*$, where d is the number of optimal basic feasible solutions in Q will also have a distinct savings vector defined by $t(x_1^*,S), t(x_2^*,S), t(x_3^*,S) \ldots. t(x_k^*,S)$. The set of allocations in least core set Q is represented as a vector of allocations x^* as following:

$$Q = \{ x^* : \sum_{i=1}^{n} x_i = C(N) ; \sum_{i \in S} x_i + t \leq \mathrm{C(S)} \quad \forall S \subseteq N \} \qquad Q \subset E$$

Next, as per Schmeidler, the average of all allocations in Q must be lexicographically greater than each allocation in Q. Since the average is a solution that is unique, therefore, taking the average of all optimal basic solutions in Q must reveal a unique point X_N^* as shown below.

$$X_N^* = \frac{x_1^* + x_2^* + x_3^* \ldots\ldots x_k^*}{d}$$

a) It is also quite evident that the least core allocations x^* defining $t(x^*,S)$ are lexicographically greater than the core allocations defining *t(x,S)* as shown below*:*

$$x^* \succcurlyeq_{lex} x$$

b) Further, the average of all the least core allocations defining $t(X_n^*,S)$ will be lexicographically greater than each allocation in the least core defining $t(x^*,S)$.

$$X_N^* \succcurlyeq_{lex} x^*$$

c) Consequently by (a) and (b), the average (i.e the nucleolus) is a unique point defining *t*(X_n^*,*S*) will be lexicographically greater than any allocation in the core defining *t(x,S)* .

$$X_N^* \succcurlyeq_{lex} x$$

Letting $\lambda_k = \frac{1}{d}$, where *d* is the number of corner point feasible solutions in Q. Then, the nucleolus can be obtained using the following relationship:

$$X_N^* = \sum_{k=1}^{d} \lambda_k \, x_k^*$$

Unfortunately, enumerating all the optimal basic feasible solutions in a linear program is considered to be time intensive and impractical for higher order HCA problems. Therefore, this dissertation recommends using the sequential LP approach as discussed in previous section to arrive at the nucleolus for any given highway cost allocation problem. As an illustration, it can be checked that upon enumerating all the optimal basic feasible solutions of the 1st LP for the example considered in the previous section, taking the average of optimal basic feasible solution I (40,16,22) and optimal basic feasible solution II (30,26,22) as shown in Table III.2 reveals the nucleolus. Using relationship (III-14) when *d = 2*, the nucleolus is obtained in the following way:

$$X_N^* = \left(\frac{40+30}{2}, \frac{16+26}{2}, \frac{22+22}{2} \right)$$

$$X_N^* = (35, 21, 22)$$

The next section describes the development of a new characteristic function for the least core LP model where the integration of variable capacity and pavement thickness requirements will be accomplished.

Characteristic Function: Discrete Aumann-Shapley value

The purpose of a characteristic function in the least core LP model is to assign costs to the formation of coalitions among vehicle classes. Pertaining to a HCA problem, this function is used to estimate the cost of designing a highway *C(S)* and *C(N)* that can accommodate vehicle classes in *S* and *N*. These costs of coalitions are represented in the right hand sides of the least core LP model as shown in (III-10) - (III-13). The new characteristic function proposed in this dissertation will first use a pavement cost function that will estimate the cost imposed on the highway facility by passage of specified amount of ESALs for a given number of lanes on a highway facility. This function will then be used to determine the marginal cost contribution made by every unit of ESAL and Lane consequent to consideration as distinct quantities being consumed by vehicle classes.

Traffic Loading: Pavement Cost Function

The cost function considered in this article has been developed for allocating new pavement construction cost. Based on fundamental principles of pavement design as described in the AASHTO (1993) pavement design guide, thickness of pavement is decided based upon the traffic loading measured in ESALs to be applied on the highway over its design life. The pavement cost function provides a *cost per lane-mile* for a specified level of *traffic loads* with a specified level of *traffic capacity*. Traffic loads are measured in terms of *18,000 lb Equivalent Single-Axle Load* (*18*-kip *ESAL*s) applications and traffic capacity in terms of *lanes* with a specified width. The proposed cost function is formulated in (III-15).

$$C(E) = l\ (a + bE^r) \tag{III-15}$$

Where $E \in \mathbf{N}$ is the number of *ESAL*s and $l \in \mathbf{N}$ is the number of lanes; $C(E)$ is the cost in dollars per lane-mile; and a, b, and r are positive parameters. The Texas HCA study conducted by Luskin et.al (2002) have extensively used function (III-15) to allocate new pavement costs. The study reported a significant goodness-of-fit between the costs for building new pavements at varying levels of applied ESALs. It was found that pavement cost function (III-15) with $r = 0.5$ provides a good fit to data on costs and ESALs.

Typically, a pavement cost function is developed using regression analysis based on the data collected from multiple existing projects. The cumulative traffic load ESALs applied over the design life of pavement causes a measure of ride quality referred to as *Present Serviceability Index* (PSI) to decrease. The function (III-15) is a cost versus ESALs characteristic function consistent with the AASHTO design guide since it recognizes the economies of scale for the increasing cost per lane-mile as ESALs increases. It is noted that the curve shown in Figure III-1 covers a range of ESALs from zero to E, where E is the cumulative ESALs applied over design period of the pavement.

Similarly, it covers the range of costs from a value A, corresponding to zero ESALs, to the total cost C for building a pavement that can accommodate traffic load imposed by E. The value of A can be viewed as an estimate of non-load cost caused by environmental factors.

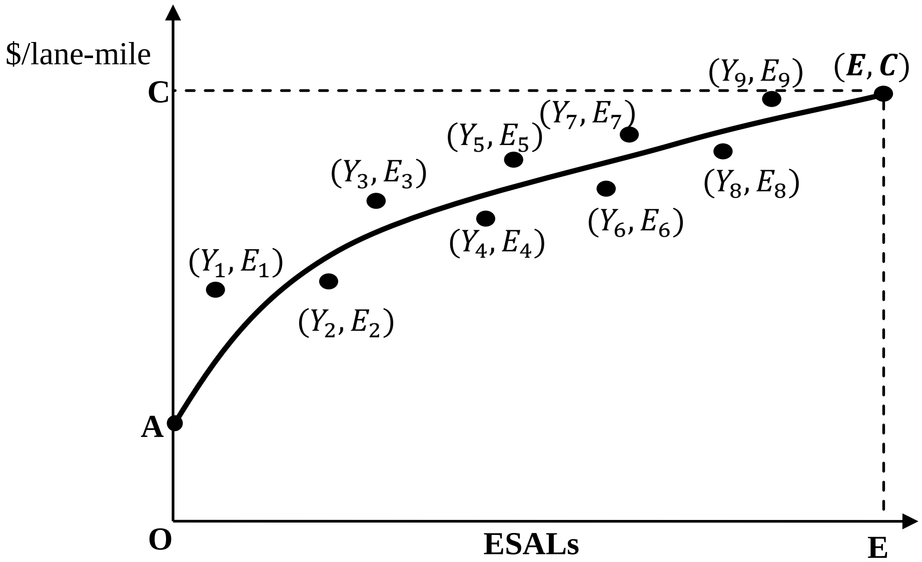

Figure III.1: Pavement Cost Function-Costs versus ESALs

Normal Equations for a, b with r = 0.5

This section summarizes the process of developing a cost function based on a sample set of pavement cost data representing various levels of applied ESALs. Specifically, normal equations are derived to estimate the parameters *a* and *b* for the cost function formulated in (III-15).The coordinates of the *i*th data point are the new pavement cost Y_i measured in cost per lane-mile for the corresponding level of ESALs E_i consisting of lanes *l*. where *n* is the sample size for this data set. Figure III.1 illustrates the regression analysis for a data set containing *n=10* pavement data points. The regression model is formulated as $Y_i = \frac{y_i}{l} = \left(a + b\sqrt{E_i}\right) + \epsilon_i$. The sum of squares to be minimized is shown below:

$$SS_\epsilon = \sum_{i=1}^{n}\left(Y_i - (a + b\sqrt{E_i}\right)^2$$

The corresponding normal equations for the sum of squares are the following

$$\frac{\partial \sum_i \epsilon_i^2}{\partial a} = 0 \text{ and } \frac{\partial \sum_i \epsilon_i^2}{\partial b} = 0$$

For the above regression model, the normal equations are:

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n}\sqrt{E_i}$$

$$\sum_{i=1}^{n} y_i\sqrt{E_i} = a\sum_{i=1}^{n}\sqrt{E_i} + b\sum_{i=1}^{n}\sqrt{E_i}^2$$

Solving for *a* and *b*, both of which are expressed as,

$$b = \frac{\sum_{i=1}^{n} y_i\sqrt{E_i} - \left(\frac{\sum_{i=1}^{n} y_i}{n}\right)\sqrt{E_i}}{\sum_{i=1}^{n} E_i - \frac{\left(\sum_{i=1}^{n}\sqrt{E_i}\right)^2}{n}}$$

Further simplification of the above relationship gives the following,

$$b = \frac{(n)\left(\sum_{i=1}^{n} y_i\sqrt{E_i}\right) - \left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n}\sqrt{E_i}\right)}{(n)\left(\sum_{i=1}^{n} E_i\right) - \left(\sum_{i=1}^{n}\sqrt{E_i}\right)^2} \quad \text{(III-16)}$$

$$a = \frac{\sum_{i=1}^{n} y_i}{n} - \left(\frac{\sum_{i=1}^{n} y_i\sqrt{E_i} - \left(\frac{\sum_{i=1}^{n} y_i}{n}\right)}{\sum_{i=1}^{n}\left(\sqrt{E_i}\right)^2 - \frac{\sum_{i=1}^{n}\left(\sqrt{E_i}\right)^2}{n}}\right)\frac{\sum_{i=1}^{n}\sqrt{E_i}}{n}$$

Further simplification of the above relationship gives the following,

$$a = \frac{(n)\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n}\sqrt{E_i}\right)\left(\sum_{i=1}^{n} y_i\sqrt{E_i}\right)}{(n)\left(\sum_{i=1}^{n} E_i\right) - \left(\sum_{i=1}^{n}\sqrt{E_i}\right)^2} \quad \text{(III-17)}$$

An Illustration for Pavement Cost Function

Consider the following hypothetical data set shown in Table III.3 with number of ESALs as the independent variable and the corresponding cost in dollars per lane-mile as the dependent variable. The parameters of *a* and *b* for the pavement cost function with *r=0.5* can determined as follows:

Table III.3: Sample data set for normal equations

(Y_i, E_i)	*i =1*	*i=2*	*i=3*	*i=4*	*i=5*	*i=6*	*i=7*	*n=7*
Y_i	12.00	13.18	14.25	17.00	17.81	18.58	21.37	$\sum_{i=1}^{n} y_i = 114.19$
E_i	4.00	5.00	6.00	9.00	10.00	11.00	15.00	$\sum_{i=1}^{n} E_i = 60.00$
$\sqrt{E_i}$	2.00	2.24	2.45	3.00	3.16	3.32	3.87	$\sum_{i=1}^{n}\sqrt{E_i} = 20.04$
$y_i\sqrt{E_i}$	24.00	29.47	34.90	51.00	56.33	61.63	82.75	$\sum_{i=1}^{n} y_i\sqrt{E_i} = 340.08$

The values for *a* and *b* for the data set in Table III.3 is obtained using (III-16) and (III-17) to get the following:

$$b = \frac{(340.08)(7) - (114.19)(20.04)}{(7)(60) - (20.04)^2} = 5.0058 \approx 5$$

$$a = \frac{(114.19)(60) - (20.04)(340.08)}{(7)(60) - (20.04)^2} = 1.9880 \approx 2$$

The pavement cost equation for the data set in this example is shown below:

$$C(E) = l\,(2 + 5 * E^{0.5})$$

Due to the extensive use of the pavement cost equation in this dissertation and several HCA studies, one can use the normal equations presented in this section to directly calculate the values for parameters *a* and *b* without always relying on statistical software to do so. The normal Equation (III-16) and (III-17) can be used to obtain the values of *a* and *b* whenever (III-15) is used as a pavement cost function for when $r = 0.5$. Continuing with the development of the characteristic function, the discrete Aumann-Shapley value concept is now used to break down the total cost *C(N)* into following components as shown in equation (III-18):

$$C(N) = E_N C_e + C_l\, L_N \tag{III-18}$$

where,

C_e: Average Cost per ESAL

C_l: Average Cost per Lane

E_N: Cumulative ESALs for the highway built for grand coalition *N*

L_N: Number of lanes for the highway built for grand coalition *N*

The total ESALs (E_N) and Lanes (L_N) in a highway facility for the grand coalition scenario are available to us as data. The Aumann-Shapley values are obtained by calculating the sum of marginal cost contributions made by each unit of ESAL and Lane in all possible sequences they can be arranged. Each ESAL (E) and Lane (L) is treated as a distinct unit. The consideration of all possible sequences can be understood to be a way of recognizing the interaction of several ESALs across multiple lanes of a highway facility.

Table III.4: Calculation of aumann-shapley values for 2 ESALs and 2 Lanes

No.	Sequences				Marginal Cost				C(N)
1	E	E	L	L	0.0000	0.0000	3.8284	3.8284	7.6569
2	E	L	E	L	0.0000	3.0000	0.8284	3.8284	7.6569
3	E	L	L	E	0.0000	3.0000	3.0000	1.6569	7.6569
4	L	E	E	L	1.0000	2.0000	0.8284	3.8284	7.6569
5	L	E	L	E	1.0000	2.0000	3.0000	1.6569	7.6569
6	L	L	E	E	1.0000	1.0000	4.0000	1.6569	7.6569

As an illustration, consider a hypothetical situation with 2 Lanes and 2 ESALs with the following pavement cost function:

$$C(E) = l\,(1 + 2\sqrt{E})$$

Table III.4 illustrates the calculation for A-S values of cost per ESAL (C_e) and cost per Lane (C_l) for a small-sized HCA problem with given ESALs and lanes.

Note that the following assumption are being made:

- $a = 1$
- $b = 2$
- $r\ = 0.5$
- $E_N = 2$
- $L_N = 2$

The A-S value for cost per ESAL (C_e) and cost per lane (C_l) is computed by determining the marginal cost contribution of each unit of E and L in all $\binom{4}{2}$ or $4!/2!\,2! = 6$ sequences, where an E stands for one unit of ESAL and L stands for one unit of Lane. Calculation of marginal cost for all sequences are shown in Table III.4. As an illustration, the marginal cost calculations using pavement cost function for sequence 3 (Colored Grey) are shown below:

$C_1(1{,}0) = 0 * \left(1 + 2\sqrt{1}\right) = 0.0000$ MC(E) = $C_1(1{,}0) - C_0(0{,}0)$= 0.000-0.000 = 0.0000

$C_2(1{,}1) = 1 * \left(1 + 2\sqrt{1}\right) = 3.0000$ MC(L) = $C_2(1{,}1) - C_1(1{,}0)$= 3.000-0.000 = 3.0000

$C_3(1{,}2) = 2 * \left(1 + 2\sqrt{1}\right) = 6.0000$ MC(L) = $C_3(1{,}2) - C_2(1{,}1)$= 6.000-3.000 = 3.0000

$C_4(2{,}2) = 2 * (1 + 2\sqrt{2}) = 7.6569$ MC(E) = $C_4(2{,}2) - C_3(1{,}2)$=7.6569-6.000 = 1.6569

The process outlined for sequence 3 is carried for the remaining five sequences to get the marginal cost contribution made by each type of unit in every one of the respective sequences. As the final step, A-S value of cost per ESAL and cost per Lane is the summation of marginal cost contributions of all E's averaged over the number of ESAL units in all sequences and the summation of marginal cost contributions of all L's averaged over the number of Lane units. The relationships for C_e and C_l are shown below:

$$C_e = \left\{\frac{Sum\ of\ Marginal\ Costs\ of\ ESAL\ players}{Number\ of\ ESAL\ players\ in\ all\ sequences}\right\}$$

$$C_l = \left\{\frac{Sum\ of\ Marginal\ Costs\ of\ Lane\ players}{Number\ of\ Lane\ players\ in\ all\ sequences}\right\}$$

Using the above formula, the values for C_e and C_l are obtained as follows:

$$C_l = \left\{\frac{\begin{pmatrix}3.8284 + 3.8284 + 3.0000 + 3.8284 + 3.0000 + 3.0000 + \\ 1.0000 + 3.8284 + 1.0000 + 3.0000 + 1.000 + 1.0000\end{pmatrix}}{2 + 2 + 2 + 2 + 2 + 2}\right\} = 2.609$$

$$C_e = \left\{\frac{\begin{pmatrix}0.000 + 0.000 + 0.0000 + 0.8284 + 0.0000 + 1.6569 + \\ 2.0000 + 0.8284 + 2.0000 + 1.6569 + 4.0000 + 1.6569\end{pmatrix}}{2 + 2 + 2 + 2 + 2 + 2}\right\} = 1.218$$

It can be computationally tedious to calculate the marginal cost contributions made by each unit using this approach when dealing with large number of units in a HCA problem. Especially since ESALs are in the magnitude of millions. Number of lane units tend to remain manageable in a HCA problem. In order to overcome this issue, a formative work conducted by Lee (2002) is used. Lee developed a compact form for calculating the discrete Aumann-Shapley value for situations where an HCA problem has large number of units for which cost contribution must be calculated.

Traffic Capacity: Revised Compact-Form for Discrete A-S Values for ESAL and Lane

A compact form developed by Lee (2002) is re-written for the purposes of easier implementation into the right hand sides of the proposed least core LP model being presented in this dissertation. For the determination of cost per ESAL and cost per Lane, compact form recommended by Lee is shown in (III-19):

$$x_2(q, C) = \frac{q_2}{q_1 + 1} \sum_{i=0}^{q_1} C(i, 1) \qquad \text{(III} - 19)$$

The dissertation proposes that in the above relationship, q_2 and q_1 can be expressed as the total number of ESALs in the grand coalition *N*, indicated by E_N, and total number of lanes in the grand coalition *N*, symbolized as L_N. The cost per lane-mile of highway represented by *C(N)* and *C(S)* for designing a highway facility for vehicle classes in set *S* and *N* can then be delineated into expenditures made to support traffic loading requirements of vehicle classes in *S* and the cost of constructing the number of lanes to meet demand for roadway capacity. Defining few additional notations as following:

E_S: The summation of ESAL requirements of vehicle classes in coalition *S*

L_S: Number of lanes needed by vehicle classes in coalition *S* based on instructions outlined in Highway Capacity Manual (2010)

A simplified representation based on the compact form (III-19) for cost per lane is shown in (III-20):

$$C_l = \left(\frac{\left(\frac{L_N}{\sum_{i \in N} E_i + 1}\right) * \left(\sum_{i=0}^{E_N} \left(a + b\sqrt{E_i}\right)\right)}{L_N} \right) \qquad \text{(III-20)}$$

As an illustration of the effectiveness of this re-written version of the compact-form, reconsider the hypothetical situation with 2 Lanes and 2 ESALs with similar pavement cost function in addition to the following assumptions.

- $a = 1$
- $b = 2$
- $r = 0.5$
- $E_N = 2$
- $L_N = 2$

The cost per lane can be calculated using (III-20) as follows:

$$C_l = \left(\frac{\left(\frac{2}{2+1}\right) * \left(\left(1+2\sqrt{0}\right) + \left(1+2\sqrt{1}\right) + \left(1+2\sqrt{2}\right)\right)}{2}\right)$$

$$C_l = \left(\frac{\left(\frac{2}{2+1}\right) * (7.8284)}{2}\right) = 2.6095$$

A simplified representation based on the compact form (III-19) for cost per ESAL is shown in (III-21):

$$C_e = \left(\frac{\left(L_N\left(a+b\sqrt{\sum_{i\in N} E_i}\right) - \left(\frac{L_N}{\sum_{i\in N} E_i + 1}\right)\left(\sum_{i=0}^{E_N}\left(a+b\sqrt{E_i}\right)\right)\right)}{\sum_{i\in N} E_i}\right) \tag{III-21}$$

- $a = 1$
- $b = 2$
- $r = 0.5$
- $E_N = 2$
- $L_N = 2$

$$C_e = \left(\frac{\left(2\left(1+2\sqrt{2}\right) - \left(\frac{2}{3+1}\right)\left(\left(1+2\sqrt{0}\right) + \left(1+2\sqrt{1}\right) + \left(1+2\sqrt{2}\right)\right)\right)}{2}\right)$$

$$= \left(\frac{7.6568 - 0.6667 * 7.8284}{2}\right) = 1.218$$

It can checked that the values of cost per ESAL (C_e) and cost per Lane (C_l) are equal to ones obtained by manual computation as was conducted in Table III.4. The next section describes the relationships used for estimating the requirements for number of lanes, L_S and L_N, and ESALs, E_S and E_N, in order to accommodate vehicle classes in coalitions *S* and *N*.

Equivalent Single Axle Loads for Each Vehicle Class

The ESALs are calculated by first obtaining the vehicle class traffic volume distributions that the highway in question will experience over a given time period. Then, depending upon the number of axles and the axle load-weight configuration of the vehicle classes, these different axle weight load configurations are converted into passes of equivalent number of 18,000lb single axle loads predicted to be applied over the pavement. More importantly, the interest is to calculate the cumulative ESALs for each individual vehicle class including the lighter vehicle classes such as passenger cars, which is generally avoided in pavement design due to their minimal impact on pavement deterioration. However, assessing number of ESALs for low-axle weight vehicle classes is an important consideration for the methodology being proposed in this dissertation. In order to forecast traffic loads, this dissertation will be guided by the procedures instituted in the handbook for project traffic forecasting as developed by Florida Department of Transportation (2014).

This dissertation recommends calculation of cumulative ESALs for each individual vehicle class by utilizing the ESAL design equation ($ESAL_D$) provided in the handbook. The equation would use Annual Average Daily Traffic (AADT) to measure traffic volume for each vehicle class, and a corresponding ESAL factor to convert traffic volumes of every vehicle class into applications of cumulative ESALs. The design equation as documented in the handbook, where *i* is the year of calculation and *n* is the expected design life of highway facility is shown below:

$$ESAL_D = \sum_{i=1}^{i=n} (AADT_i) * (L_{Fi}) * T_{24} * D_F * E_F * 365$$

where,

$AADT$	=	Annual Average Daily Traffic in the year *i*
T_{24}	=	% of Truck Traffic in a 24 hour time period
E_F	=	Load Equivalency Factor or ESAL factor
L_{Fi}	=	% of the total one-way 18-kip ESALs for the design lane (assumed to be 1.0)
D_F	=	% of total two-way peak hour traffic occurring in peak direction.

Number of Lanes for Each Vehicle Class

The number of lanes for each vehicle class *i* is ascertained using a relationship developed based on the Highway Capacity Manual (2010). This relationship was also used to estimate lane requirements for vehicle classes in the Texas HCA study conducted by Luskin et.al (2002).

$$L_i = \frac{K*ADT*DD}{phf*f_{hv}*MSF} \tag{III-22}$$

where,

K = Factor converting ADT to hourly traffic low

ADT = Average Daily Traffic

DD = Directional Distribution

f_{hv} = Heavy vehicle Factor

phf = Peak Hour Factor

The values for *K, ADT, DD* and f_{hv} and *phf* are available as data. Once the number of lanes for each vehicle class *i* using the highway capacity manual has been obtained, one can then proceed to use a rule to determine the number of lanes needed by vehicle classes in coalition *S*. This dissertation proposes that the number of lanes in coalition *S* should be equal to the largest requirement for number of lanes by any vehicle class member of coalition *S*.

The rule is defined as follows in (III-23) and (III-24):

$$L_S = \max(L_i) \qquad \forall\ i \in S \tag{III-23}$$

$$L_N = \max(L_i) \qquad \forall\ i \in N \tag{III-24}$$

Note: - It is viable to calculate the number of lanes for each coalition in *S* individually using the highway capacity manual. However, for large number of vehicle classes say *n*, the need for lane estimates increases in the order of magnitude of 2^n minus one. Therefore, it is a sound traffic approximation to get the lane requirements for each coalition in *S* directly using the relationship (III-23) and (III-24).

The New Right Hand Sides of Least Core LP Model

Once the pavement cost function (III-12) is utilized to calculate the average cost per ESAL (C_e) and average cost per Lane (C_l) using relationship (III-20) and (III-21), the cost of coalitions for *S* and *N* are obtained by multiplying C_e and C_l by summation of the ESALs of vehicle classes in coalition *S* and the number of lanes for coalition *S* denoted by L_S, obtained using relationship (III-22)-(III-24) to get *C(S)*. Similarly, C_e is multiplied by the summation of the ESALs of vehicle classes in coalition *N* and C_l is multiplied by the number of lanes in grand coalition *N* denoted by L_N, obtained using relationship (III-22)-(III-24) to get *C(N).* In the least core LP model, both *C(S)* and *C(N)* represent the right hand sides of the corresponding coalitional rationality constraints for *S* and *N* respectively.

The right hand sides of constraints (III-11) - (III-13) with modifications using discrete Aumann-Shapley value is shown in (III-25) and (III-26).

$$C(S) = C_e(\sum_{i\in S} E_i) + C_l L_S \tag{III-25}$$

$$C(N) = C_e(\sum_{i\in N} E_i) + C_l L_N \tag{III-26}$$

The Least Core LP Model with Discrete A-S Value

The least core LP formulation (III-5) - (III-8) is now updated with the new Aumann-Shapley based characteristic function to get the following:

$$\max \quad t_n \tag{III-27}$$

Subject to

$$\sum_{i\in S} x_i + t_n \leq C_e(\sum_{i\in S} E_i) + C_l L_S \qquad for\ all\ S \in N \tag{III-28}$$

$$\sum_{i\in S} x_i + t_j = C_e(\sum_{i\in S} E_i) + C_l L_S \quad for\ all\ S \in F_j\ \ j \in \{1, \dots n - 1\}\} \tag{III-29}$$

$$\sum_{i\in N} x_i = C_e(\sum_{i\in N} E_i) + C_l L_N \tag{III-30}$$

x_i, t_n, t_j for all $i \in N$

The above model will be hereafter referred to as the proposed methodology throughout the remaining chapters of this dissertation. Furthermore, the formulation (III-27) - (III-30) implies that the model in effect is engaged in solving a sequence of LPs in order to obtain a unique allocation known as the nucleolus.

An Illustration

For example, when N = {1,2,3} and S = {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, the least core LP model with an Aumann-Shapley value based characteristic function is formulated as follows

Step 1: Input Data

The expected number of applied ESALs for vehicle classes $E_{\{1\}}$, $E_{\{2\}}$ and $E_{\{3\}}$ are available as data and $L_{\{1\}}$, $L_{\{2\}}$ and $L_{\{3\}}$ are number of lanes obtained using relationship (III-22) for coalitions {1}, {2}, {3}. Relationships (III-23) and (III-24) are used to obtain lane requirements for coalitions {1.2}, {1,3}, {2,3} and {1,2,3}.

Step 2: Calculation of ESALs for Coalitions S and N

$E_{\{1\}} = 5,\ E_{\{2\}} = 10,\ E_{\{3\}} = 15,\ E_{\{1,2\}} = E_1 + E_2 = 15,\ \ E_{\{1,3\}} = E_1 + E_3 = 20,\ E_{\{2,3\}} = E_2 + E_3 = 25,\ E_{\{1,2,3\}} = E_1 + E_2 + E_3 = 30$

Step 3: Calculation of Number of Lanes for Coalitions S and N

$L_{\{1\}} = 3, L_{\{2\}} = \ 2,\ L_{\{3\}} = 1, L_{\{1,2\}} = max(L_1, L_2) = 3,\ L_{\{1,3\}} = max(L_1, L_2) = 3,\ L_{\{2,3\}} = max(L_2, L_3) = 2,\ L_{\{1,2,3\}} = max(L_1, L_2, L_3) = 3$

Step 4: Calculation of Aumann-Shapley Values:(a=3, b=5)

Specific Pavement Cost Function: $\left(a + b\sqrt{E_i}\right) = (2 + 3 * \sqrt{E_i})$

$$\text{Cost per ESAL, } C_e \ = \left(\frac{\left(L_N(a+b\sqrt{\sum_{i\in N} E_i}\right) - \left(\frac{L_N}{\sum_{i\in N} E_i + 1}\right)\left(\sum_{i=0}^{E_N}\left(a+b\sqrt{E_i}\right)\right)}{\sum_{i\in N} E_i}\right)$$

$$= \frac{\left(3(2+3\sqrt{\sum_{i\in N} E_i}\right) - \left(\frac{3}{30+1}\right)\left(\sum_{i=0}^{30}\left(2+3\sqrt{E_i}\right)\right)}{30}$$

$$= \$0.55849/ESAL$$

$$\text{Cost per Lane, } \ C_l \ = \left(\frac{\left(\frac{L_N}{\sum_{i\in N} E_i + 1}\right) * \left(\sum_{i=0}^{E_N}\ \left(a+b\sqrt{E_i}\right)\right)}{L_N}\right)$$

$$= \left(\frac{\left(\frac{3}{31}\right) * \left(\sum_{i=0}^{30}\ \left(2+3\sqrt{E_i}\right)\right)}{3}\right)$$

$$= \$12.84677/Lane$$

Step 5: Characteristic Function: Integration of Variable Capacity and Pavement Thickness

The costs of coalitions for this HCA problem will be obtained as follows:

$C(1) = E_1 C_e + L_1 C_l = \ \ 5 * 0.55849 + 3 * 12.84677 = 41.3328 \approx 41$

$C(2) = E_2 C_e + L_2 C_l = 10 * 0.55849 + 2 * 12.84677 = 31.2785 \approx 31$

$C(3) = E_3 C_e + L_3 C_l = 15 * 0.55849 + 1 * 12.84677 = 21.2241 \approx 21$

$C(12) = (E_1 + E_2) C_e + L_{12} C_l = 15 * 0.55849 + 3 * 12.84677 = \ 46.9177 \approx 47$

$$C(13) = (E_1 + E_3)C_e + L_{13}C_l = 20 * 0.55849 + 3 * 12.84677 = \ 49.7107 \approx 50$$

$$C(23) = (E_2 + E_3)C_e + L_{23}C_l = 25 * 0.55849 + 2 * 12.84677 = \ 39.6558 \approx 40$$

$$C(123) = (E_1 + E_2 + E_3)C_e + L_{123}C_l = 30 * 0.55849C_e + 3 * 12.84677 = 55.295 \approx 55$$

Step 6: Least Core LP Model with Discrete A-S Values:

The LP model will have *n+1* or four decision variables and $2^n - 1$ or 7 constraints.

$$\max t_1$$

Subject to

$$
\begin{aligned}
x_1 + t_1 &\le \boldsymbol{E_1 C_e + L_1 C_l} && (Y_1)\\
x_2 + t_1 &\le \boldsymbol{E_2 C_e + L_2 C_l} && (Y_2)\\
x_3 + t_1 &\le \boldsymbol{E_3 C_e + L_3 C_l} && (Y_3)\\
x_1 + x_2 + t_1 &\le \boldsymbol{(E_1 + E_2)C_e + L_{12} C_l} && (Y_{12})\\
x_1 + x_3 + t_1 &\le \boldsymbol{(E_1 + E_3)C_e + L_{13} C_l} && (Y_{13})\\
x_2 + x_3 + t_1 &\le \boldsymbol{(E_2 + E_3)C_e + L_{23} C_l} && (Y_{23})\\
x_1 + x_2 + x_3 &= \boldsymbol{(E_1 + E_2 + E_3)C_e + L_{123} C_l} && (Y_{123})\\
x_1, x_2, x_3, t_1 &\ge 0
\end{aligned}
$$

The final least core LP model will be formulated as follows:

$$\max t_1$$

Subject to

$$
\begin{aligned}
x_1 + t_1 &\le \mathbf{41} && (Y_1)\\
x_2 + t_1 &\le \mathbf{31} && (Y_2)\\
x_3 + t_1 &\le \mathbf{21} && (Y_3)\\
x_1 + x_2 + t_1 &\le \mathbf{47} && (Y_{12})\\
x_1 + x_3 + t_1 &\le \mathbf{50} && (Y_{13})\\
x_2 + x_3 + t_1 &\le \mathbf{40} && (Y_{23})\\
x_1 + x_2 + x_3 &= \mathbf{55} && (Y_{123})\\
x_1, x_2, x_3, t_1 &\ge 0
\end{aligned}
$$

The above least core LP formulation was solved using simplex algorithm. Upon solving 1st linear program, it was found that the dual variables of Y_3 and Y_{12} were equal to 0.5. Therefore, the coalition $\{3\}$ and $\{1,2\}$ were placed in F_1, this is represented as $F_1 = \{\{3\},\{1,2\}\}$ Further, the optimal objective function value t_1 was found to be equal to 6.5. Presence of multiple optimal solutions was detected as the reduced cost coefficient for one non-basic was observed to have a zero value. In view of this fact, sequential LP approach was initiated. For the 2nd LP program, the optimal value of t_1 is made equal to t_2 for rationality constraints of $\{3\}$ and $\{1,2\}$. This is accomplished by subtracting *C(3)* and *C(12)* from t_1 in the 2nd linear program. The 2nd LP is formulated as follows:

$$\max t_2$$

Subject to

$$x_1 + t_2 \le \mathbf{41}$$
$$x_2 + t_2 \le \mathbf{31}$$
$$x_3 = \mathbf{14.5}$$
$$x_1 + x_2 = \mathbf{40.5}$$
$$x_1 + x_3 + t_2 \le \mathbf{50}$$
$$x_2 + x_3 + t_2 \le \mathbf{40}$$
$$x_1 + x_2 + x_3 = \mathbf{55}$$
$$x_1, x_2, x_3, t_2 \ge 0$$

The second LP program with objective function t_2 was found have a unique optimal solution that led to $Y_{13} = Y_{23} = 0.5$ and set $F_2 = (\{1,3\},\{2,3\})$ with an optimal objective function value of $t_2 = 10.25$. The nucleolus for this problem was determined to be (25.25,15.25,14.50).

Analysis of Monotonic Behavior of Costs of Coalitions

The least core LP model as the name suggests is incumbent upon the presence of core or feasible region comprising of a set of allocations satisfying constraints (III-1) – (III-3). Without the existence of such a core, the LP model will be infeasible. This section offers an explanation as to the reason for the infeasibility of the LP model as pertaining to a HCA problem. Figure III.3 and Figure III.4 depicts the degree to which the economics of scale are being maintained by the two characteristic functions used in the proposed methodology.

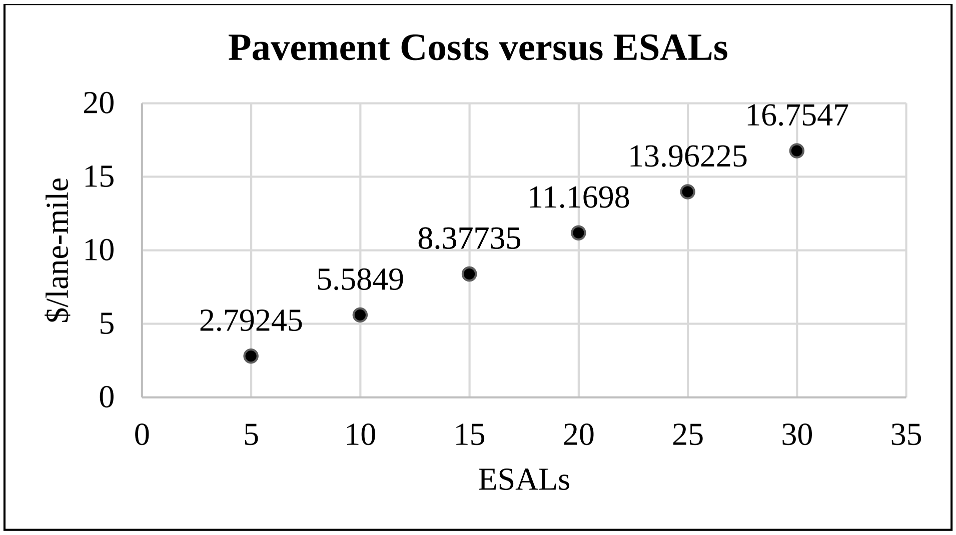

Figure III.3: Characteristic Function for Traffic Loading

Figure III.3 suggests a strong monotonic behavior for the illustration presented in this section. This is because the A-S value assigns a fixed unit price of \$0.55849 per ESAL to all coalitions. This means the vehicle class participating in these coalitions are paying the same price for using an ESAL irrespective of the member with whom the cooperation is taking place. Further, the rule that ESALs for coalition containing two or more vehicle classes is simply the addition of their individual ESAL requirements contributes to the monotonic behavior of the cost function.

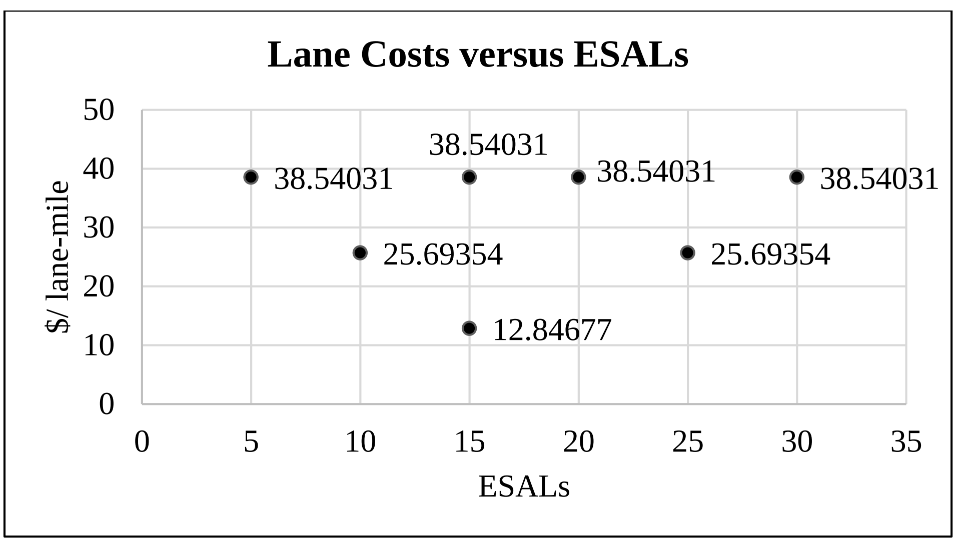

Figure III.4: Characteristic Function for Traffic Capacity

Figure III.4 shows a weaker monotonic behavior being maintained by the characteristic function associated with the traffic lanes. Each coalition pays a unit price of $12.84677 per lane-mile. As coalitions, {1}, {2} and {3} increase their demand for ESALs from 5,10 and 15, their corresponding lane costs decreases to $38 per lane-mile, $25 per lane-mile and $12 per lane-mile.

Infeasibility of the Least Core LP Model

The presence of feasible region for a linear programming model or the non-emptiness of the core is guaranteed in the LP model under a well-established convexity condition as shown in Young (1985) and discussed extensively by Shapley (1971). Any function that is demand monotonic in nature also satisfies convexity. A monotonic characteristic function is a function that does not decrease the cost assigned to coalitions, as the vehicle classes in these coalitions place a higher demand on the highway facility. The cost of coalitions *C(S)* and *C(N)* in the least core LP model must satisfy the condition of being submodular as shown in (III-31):

$$C(S \cup T) + C(S \cap T) \leq C(S) + C(T) \qquad S, T \subset N \tag{III-31}$$

As an illustration, the feasible region will exist for three-vehicle class HCA problem with $C(1), C(2), C(3), C(12), C(13), C(23)$ and $C(123)$ when all conditions (III-32)-(III-38) are satisfied.

$$C(12) \leq C(1) + C(2) \tag{III-32}$$

$$C(13) \leq C(1) + C(3) \tag{III-33}$$

$$C(23) \leq C(2) + C(3) \tag{III-34}$$

$$C(123) + C(1) \leq C(12) + C(13) \tag{III-35}$$

$$C(123) + C(2) \leq C(12) + C(23) \tag{III-36}$$

$$C(123) + C(3) \leq C(13) + C(23) \tag{III-37}$$

$$C(123) \leq C(1) + C(2) + C(3) \tag{III-38}$$

For example, consider a characteristic function defining cost of coalitions using a non-monotonic cost function for *C(N)* and *C(S)*, where $C(\emptyset) = 0, C(1) = 5,\ C(2) = 5,\ C(3) = 5,$ $C(12) = 12,\ C(13) = 14, C(23) = 18$,and $C(123) = 20$. Analyzing the two-vehicle class coalitional rationality constraints of {1,2}, {1,3} and {2,3} as shown below:

$$x_1 + x_2 \leq 12,\ x_1 + x_3 \leq 14, x_2 + x_3 \leq 18,$$

Addition of these constraints, divulges that $2(x_1 + x_2 + x_3) \leq 44$, which violates the completeness constraint of $x_1 + x_2 + x_3 = 20$. Furthermore, it is observed that conditions (III-35) - (III-37) are satisfied and conditions (III-32), (III-33), (III-34) and (III-37) are violated. This demonstrates that the use of a characteristic function that does not satisfy convexity condition as described by relationship (III-31) would result in least core LP model to be infeasible.

Convexity of Discrete Aumann-Shapley Based Characteristic Function

The convexity condition codifies a mathematical relationship, that defines whether savings accrued by vehicle classes for participating in the grand coalition N, does in fact increase with larger coalition size. The intent is to recognize the presence of motivation for vehicle classes jointly deriving utility out of using a common highway facility along with other vehicle classes. In most HCA applications core is existent. This is attributed to adding successive incremental pavement thickness to accommodate heavier vehicle classes by adding thickness to an already built pavement for lighter vehicle classes with low-axle weights and thereby accruing savings for not building the pavement from ground up. The least core LP model described in this dissertation acquires its right-hand sides from the discrete Aumann-Shapley value that does not guarantee presence of feasible region as it is unable to meet the criterion set out in (III-31). Meaning, their might not be an incentive for vehicle classes to collaborate when one considers the restriction on number of lanes used by certain-vehicle types. An example of such a situation would be when it makes more economic sense for trucks to use their own independent one-lane highway as opposed to using a three-lane highway where they pay substantial portion of lane cost they don't end up using and also pay for thicker pavement designs for all the lanes. This is represented as a relationship below:

$$C(Passenger\ Car\ Facility + Truck\ Facility{:}\ 3\ Lanes) \\ \geq C(Passenger\ Car\ Facility{:}\ 3\ Lanes) + C(Truck\ Facility{:}\ 1\ Lane)$$

Therefore, in an event where the problem is infeasible due to lack of core, it is plausible to use other non-optimization based game-theoretic solution concepts to get a desired cost division, such as using Shapley value. However, allocation generated by any other solution concept for a given costs of coalitions with empty core will not be able to satisfy rationality as the core does not exist. Therefore, if one chooses to seek an allocation that satisfies rationality, a core must be generated. Mathematically, this can be undertaken by relaxing the RHS of the least core model by increasing the cost of coalitions by a minimum amount until the core exists. Young et.al (1982) and Lemaire (1984) recommended solving a corresponding minimization problem to the maximization problem for games with empty-cores (infeasible). This dissertation uses a similar approach rather applied differently. The minimum amount by which the cost of coalition of all or few coalitions must be increased can be determined in three steps:

- Step 1: Set the variable t_1 as unrestricted for the least core model that is infeasible and resolve the problem.
- Step 2: Take the resulting absolute value of the optimal objective value attained and then add that number to all the RHS of coalitional constraints of *C(S)* except the grand coalition constraint *C(N)*. The outcome of this step will be a new least core LP model.
- Step 3: Reinstate the non-negativity constraint for variable t_1 in the new LP model.

The new least core LP model will now have a core from which the nucleolus can be obtained. It is important to note that if exactly this minimum amount is added to all costs of coalitions, the optimal objective function value for the modified least core LP model would be zero. Therefore, in practice, an amount greater than this minimum amount must be added to all coalitions to get a realistic allocation. The recourse option presented here allows the highway agencies to make an informed decision regarding incorporation of requisite additional cost of providing highway services. The justification behind this overture is to persuade the stakeholders to use a highway resource efficiently.

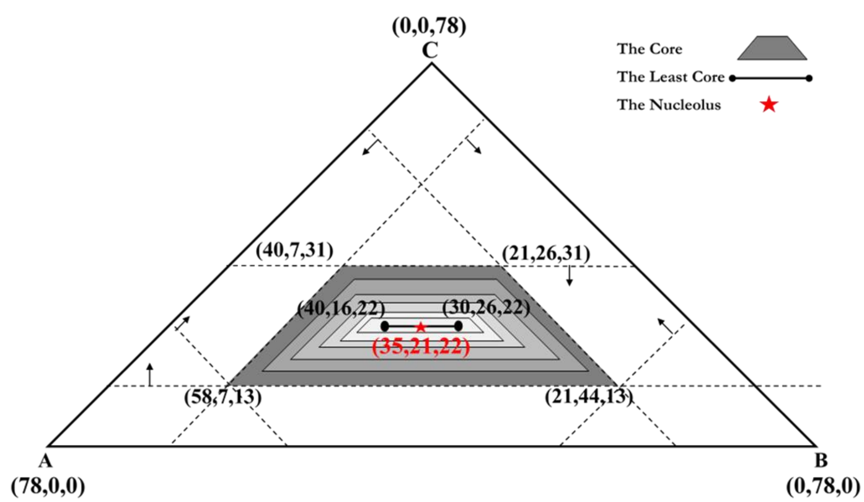

Figure III.5: Representation of Core, Least Core & Nucleolus using barycentric coordinates

CHAPTER IV

CLOSED-FORM SOLUTION FOR A SPECIAL CASE

The special case in this dissertation is referred to a scenario whereupon solving the least core LP model in a HCA problem, the nucleolus is defined by coalitions containing *n* and *n-1* vehicle classes. For such a case, nucleolus can be obtained using a closed-form solution. This chapter will establish a set of conditions that will evaluate all the cost of coalitions *C(S)* and *C(N)* for a given least core LP model, to foresee whether the output of the least core LP model would be consistent with the requirements for exhibiting the special case. Satisfaction of all conditions would indicate that closed-form solution can be used to obtain the nucleolus. The results presented in this chapter follow in line with work performed by Littlechild (1974) and Reinhardt (2004), but with an emphasis towards obtaining the nucleolus in closed-form from the least core LP model used to attribute highway costs among vehicle classes in the HCA problem.

Conditions for Special Case

1. Solve the coalitional rationality constraints of the grand coalition *C(N)* containing *n* vehicle classes and *C(S)* containing *n-1* vehicle classes as an independent system of linear equations outside the purview of least core LP model. This will result in solving a system of *n+1* equations with *n+1* variables having a unique solution.
2. Working under the supposition that simplex algorithm is being used to solve the LP model, the optimal basis is surveyed and an assertion is made that certain variables of the LP model need to be basic and non-basic for the model to show evidence of special case. The least core LP model has a total of $2^n + n$ variables that includes decision variables, slack variables and artificial variables. Exactly *n+1* out of $2^n + n$ must be non-basic in the optimal tableau. These non-basic variables are slack variables associated with coalitional rationality constraints of *S* with *n-1* vehicle classes including an artificial variable associated with grand coalition constraint *N*. The reasoning is that these constraints are being solved as a system of equations with implied assumption that their slacks are zero and as a result binding. The remaining $2^n - 1$ variables of the least core LP model must then be basic. The collection of these explicit basic and non-basic variables will serve to define the unique solution or the special case of the least core LP model.

3. The conditions for using closed-form solution will rely on the basic slack variables of the least core LP model as a yardstick to gauge which constraints must have slacks for the unique solution to a subdivision of *n+1* constraints to also be the unique optimal solution for the entire least core LP model containing $2^n - 1$ constraints.
4. The uniqueness of optimal solution will be verified by demonstrating the following:

   - Coefficients of exactly $n - m$ (where $n = 2^n + n$ and $m = 2^n - 1$) non-basic variables in the optimal objective function of the simplex tableau are strictly non-zero.
   - The dual variables for the corresponding coalitional rationality constraints will be strictly non-zero implying these constraints defining the allocation of the nucleolus are binding.

Once the uniqueness of the solution is confirmed using primal-dual properties, a unique solution to the 1st LP in the sequential LP approach is by definition the nucleolus.

Conceptual Approach to Closed-form Solution

A unique solution to a subgroup of coalitional rationality constraints containing *n* and *n-1* vehicle classes is found to have a unique solution that can be expressed in a closed-form. The closed-form solutions are not the nucleolus when the cost of coalitions *C(S)* and *C(N)* represented in the right hand sides of the LP model contravene the conditions set out for the proposed closed-form solution to be the unique optimal solution of the least core LP model. Lack thereof, a sequential LP approach must be implemented to obtain the nucleolus as described in Chapter III.

Let $X = \{\, x : \sum_{i=1}^{n} x_i = C(N) \,\}$ be the set of allocations satisfying completeness. The allocation vector $x^* \in X$ is the nucleolus and single point solution obtained using CFS. The CFS must have the following two properties in order for it to be the nucleolus:

1. For every $\boldsymbol{\mathcal{H}} \in X$, the lexicographical order of savings for allocation x^* is $L(\boldsymbol{\mathcal{H}}) \leq_{lex} L(x^*)$ with coalitions *C(S)* with $|S| = n - 1$ and *C(N)* with $|N| = n$. The least core LP model by definition ensures that the unique solution will satisfy the property of lexicographical ordering. Proving uniqueness will point toward the fact that the unique allocation will be lexicographically greater than any allocation.
2. These coalitions containing *n-1* vehicle classes have the lowest savings in the lexicographic order but maximum savings when solving the 1st LP (least core) in sequence of LPs needed to obtain the nucleolus.

The initial assertion is checked by solving a generalized representation of a three-vehicle class HCA problem. First a solution is obtained to a subgroup of coalition rationality constraints of {1,2}, {1,3}, {2,3} and {1,2,3} in the least core LP model followed by the development of conditions for this solution to be a unique optimal solution to the least core LP model.

Verification of Closed-form Solution using Three-Vehicle Class HCA Problem

The least core LP formulation is deconstructed for a three-vehicle class HCA problem with a generic characteristic function represented by $C(1)$, $C(2)$, $C(3)$, $C(12)$, $C(13)$, $C(23)$ and $C(123)$ as shown below:

Objective Function: $\max t_1$

Constraints:

$$x_1 + t_1 \leq C(1) \tag{IV-1}$$

$$x_2 + t_1 \leq C(2) \tag{IV-2}$$

$$x_3 + t_1 \leq C(3) \tag{IV-3}$$

$$x_1 + x_2 + t_1 \leq C(12) \tag{IV-4}$$

$$x_1 + x_3 + t_1 \leq C(13) \tag{IV-5}$$

$$x_2 + x_3 + t_1 \leq C(23) \tag{IV-6}$$

$$x_1 + x_2 + x_3 = C(123) \tag{IV-7}$$

The optimal solution for the above formulation will be obtained using the simplex algorithm. Next, for the sake of clarity, the name of the variable "t_1" is switched to "t^*_{max}". Then all the constraints are written in terms of the variable t^*_{max} instead of t_1 that emblematizes the maximum savings achieved upon solving the 1st LP where variables x^*_1, x^*_2 and x^*_3 stand for the optimal cost allocation. After this, the last four constraints of the LP formulation containing *n* and *n-1* vehicle classes are solved as a system of linear equations as shown in (IV-4) - (IV-7). This system of equations has four variables and four constraints having a unique solution that would define a unique allocation $x^* = \{ x^*_1, x^*_2, x^*_3, t^*_{max}\}$. By making this claim, it must also be assumed that the optimal slack variables S^*_4, S^*_5 and S^*_6 associated with coalitional rationality constraints for {1,2},{1,3} and {2,3} are zero as they have been converted from inequalities to equalities as per our assumption. The solution to system of equations (IV-4) - (IV-7) is found to be the following:

$$x^*_1 = 1/3\ C(12) + 1/3\ C(13) + 1/3\ C(123) - 2/3\ C(23)$$

$$x_2^* = 1/3\ C(12) + 1/3\ C(23) + 1/3\ C(123) - 2/3\ C(13)$$
$$x_3^* = 1/3\ C(13) + 1/3\ C(23) + 1/3\ C(123) - 2/3\ C(12)$$
$$t_{max}^* = 1/3\ C(12) + 1/3\ C(13) + 1/3\ C(23) - 2/3\ C(123)$$

Next, least core LP model (IV-1) - (IV-7) is converted into its Standard Equality Form (SEF). As stated earlier, since the slack variables S_4^*, S_5^* and S_6^* are assumed to be zero, they must be the non-basic variables in the optimal basis of the simplex algorithm. The constraint formulation of the LP problem along with slack variables is shown below:

$$t_{max}^* + S_1^* = C(1) - x_1^*$$
$$t_{max}^* + S_2^* = C(2) - x_2^*$$
$$t_{max}^* + S_3^* = C(3) - x_3^*$$
$$t_{max}^* + S_4^* = \boldsymbol{C(12)} - (x_1^* + x_2^*)$$
$$t_{max}^* + S_5^* = \boldsymbol{C(13)} - (x_1^* + x_3^*)$$
$$t_{max}^* + S_6^* = \boldsymbol{C(23)} - (x_2^* + x_3^*)$$
$$x_1^* + x_2^* + x_3^* = \boldsymbol{C(123)}$$

As mentioned earlier, the least core LP formulation maximizes the lower bound on the savings t_{max}^* as much as possible for all coalitions. Hence, in order for coalitions {1,2}, {1,3} and {2,3} to have the maximum savings in the 1st LP, the claim x^* is the nucleolus will only hold under the following conditions:

- $S_4^* = 0\ , S_5^* = 0\ and\ S_6^* = 0$
- $S_1^* > 0\ , S_2^* > 0\ and\ S_3^* > 0$

The variable t_{max}^* is now taken to the right hand side as an opening step towards developing a slack variable-based conditions for using closed-form solutions.

$$S_1^* = C(1) - x_1^* - t_{max}^* > 0 \quad \text{(IV-8)}$$
$$S_2^* = C(2) - x_2^* - t_{max}^* > 0 \quad \text{(IV-9)}$$
$$S_3^* = C(3) - x_3^* - t_{max}^* > 0 \quad \text{(IV-10)}$$
$$S_4^* = \boldsymbol{C(12)} - (x_1^* + x_2^*) - t_{max}^* = 0 \quad \text{(IV-11)}$$
$$S_5^* = \boldsymbol{C(13)} - (x_1^* + x_3^*) - t_{max}^* = 0 \quad \text{(IV-12)}$$
$$S_6^* = \boldsymbol{C(23)} - (x_2^* + x_3^*) - t_{max}^* = 0 \quad \text{(IV-13)}$$

$$\boldsymbol{x_1 + x_2 + x_3 = C(123)}$$

The solutions for x_1^*, x_2^*, x_3^*, and t_{max}^* obtained as part of solving a system of linear equations are substituted into the (IV-8)-(IV-13) equations to get relationships (IV-14)-(IV-19). These relationships assist in discerning the behavior of slack variables when the special case is being exhibited by least core LP model. Consequently, following relationships are obtained after substitution:

$$S_1^* = \left(C(1) - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(23)\right) - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) - \tfrac{2}{3}C(123)\right)\right) > 0 \qquad \text{(IV-14)}$$

$$S_2^* = \left(C(2) - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(23) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(13)\right) - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) - \tfrac{2}{3}C(123)\right)\right) > 0 \qquad \text{(IV-15)}$$

$$S_3^* = \left(C(3) - \left(\tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(12)\right) - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) - \tfrac{2}{3}C(123)\right)\right) > 0 \qquad \text{(IV-16)}$$

$$S_4^* = \left(\begin{array}{c} C(12) - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(23) + \tfrac{1}{3}C(12) + \tfrac{1}{3}C(23) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(13)\right) \\ - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) - \tfrac{2}{3}C(123)\right) \end{array}\right) = 0 \qquad \text{(IV-17)}$$

$$S_5^* = \left(\begin{array}{c} C(13) - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(23) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(12)\right) \\ - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) - \tfrac{2}{3}C(123)\right) \end{array}\right) = 0 \qquad \text{(IV-18)}$$

$$S_6^* = \left(\begin{array}{c} C(23) - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(23) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(13) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(12)\right) \\ - \left(\tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) - \tfrac{2}{3}C(123)\right) \end{array}\right) = 0 \qquad \text{(IV-19)}$$

At this juncture, it can said that the relationships for $S_1^*, S_2^*, S_3^*, S_4^*, S_5^*$ and S_6^* are in effect serving as the conditions for the unique solution to be a unique optimal solution for the least core LP formulation in a three-vehicle class HCA problem. Under this postulation, the relationship of slack variables S_1^*, S_2^* and S_3^* must be greater than zero and the relationship for slack variables S_4^*, S_5^* and S_6^* must be equal to zero. The cardinal objective of the conditions that are being developed in this section is to foretell the use of closed-form solution. Thus, the closed solution must also be able to invalidate the use of closed-form solutions even when special situations of simplex such as degeneracy, multiple optimality and infeasibility occur in the least core LP model.

Therefore, the dissertation observed that being only dependent on the behavior of slack variables as the conditions for using closed-form solution gives way to scenarios where the slack variable-based conditions would remain greater than zero for S_1^*, S_2^* and S_3^* and equal to zero for S_4^*, S_5^* and S_6^* for when the least core LP model exhibits special situations of simplex.

As a curative measure, further restrictions within the relationships of slack variable-based conditions must be placed in order to provide sufficient coverage against validating closed-form solution when special situations of simplex are known to have occurred. Each slack variable-based condition can be deemed consisting of components whose summation of allocations x_i^* and t_{max}^* should be less than *C(S)* for the slack variable-based condition to be positive. For instance, upon closer examination of one of the first slack variable-based condition say S_1^*. The sum of component $(x_1^* + t_{max}^*)$ must be less than *C(1)* for S_1^* to be positive.

One can also assume that the value for x_1^* and t_{max}^* must be positive by nature of the least core LP model where allocations and savings cannot be negative. However, the slack variable-based condition S_1^* could still be positive even if one of the following transpires:

- $x_1^* < 0$ and $t_{max}^* > 0$
- $x_1^* > 0$ and $t_{max}^* < 0$
- $x_1^* < 0$ and $t_{max}^* < 0$

Evaluation of above listed possibilities suggests that slack condition S_1^* can be positive even if x_1^* and t_{max}^* are negative. This argument can be extended to other slack variable-based conditions where S_2^* and S_3^* could be positive along with S_4^*, S_5^* and S_6^* could be equal to zero even if the decision variables x_2^* and x_3^* are negative. All such component wide conditions for three-vehicle class HCA problem are listed below:

- $C(1)\ must\ be\ greater\ than\ summation\ of\ (\boldsymbol{x_1^*} + \boldsymbol{t_{max}^*}) > \mathbf{0}\ for\ \ S_1^* > 0$
- $C(2)\ must\ be\ greater\ than\ summation\ of\ (\boldsymbol{x_2^*} + \boldsymbol{t_{max}^*}) > \mathbf{0}\ for\ \ S_2^* > 0$
- $C(3)\ must\ be\ greater\ than\ summation\ of\ (\boldsymbol{x_3^*} + \boldsymbol{t_{max}^*}) > \mathbf{0}\ for\ \ S_3^* > 0$
- $C(12)\ must\ be\ equal\ to\ the\ summation\ of\ \left((\boldsymbol{x_1^*} + \boldsymbol{x_2^*}) + \boldsymbol{t_{max}^*}\right) = 0\ for\ S_4^* = 0$
- $C(13)\ must\ be\ equal\ to\ the\ summation\ of\ \left((\boldsymbol{x_1^*} + \boldsymbol{x_3^*}) + \boldsymbol{t_{max}^*}\right) = 0\ for\ S_5^* = 0$
- $C(23)\ must\ be\ equal\ to\ the\ summation\ of\ \left((\boldsymbol{x_2^*} + \boldsymbol{x_3^*}) + \boldsymbol{t_{max}^*}\right) = 0\ \ for\ S_6^* = 0$

Therefore, auxiliary conditions must be added so that the validation of a closed-form solution does not result in negative allocations. A circumstance an LP formulation is able to avoid due to the non-negativity constraints.

Since the values of x_i^* and t_{max}^* are obtained as solutions to the system of equations containing *n* and *n-1* vehicle classes, a corrective measure would be to make the closed-form solutions as part of the main collection of conditions. These conditions would serve to verify the non-negativity of the decision variables before the usage of closed-form solutions can be validated for calculation of the nucleolus. Therefore, in addition to the 6 slack conditions as seen in (IV-14) - (IV-19) , four more conditions are added that signify that the all the cost allocation variables defining the closed-form must be positive as shown below represented by equation S_7, S_8, S_9, and S_{10}:

$$S_7 \Rightarrow t_{max}^* \;= \tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) - \tfrac{2}{3}C(123) > 0$$
$$S_8 \Rightarrow \;\; x_1^* \;\; = \tfrac{1}{3}C(12) + \tfrac{1}{3}C(13) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(23) > 0$$
$$S_9 \Rightarrow \;\; x_2^* \;\; = \tfrac{1}{3}C(12) + \tfrac{1}{3}C(23) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(13) > 0$$
$$S_{10} \Rightarrow \;\; x_3^* \;\; = \tfrac{1}{3}C(13) + \tfrac{1}{3}C(23) + \tfrac{1}{3}C(123) - \tfrac{2}{3}C(12) > 0$$

The use of closed-form solutions as condition themselves will provide a barrier against recommending use of closed-form expressions when the least core LP model defining a given cost of coalitions *C(S)* and *C(N)* is communicative of a special situation of simplex. Therefore, in total, one would need to evaluate 10 conditions for a three-vehicle class HCA problem in order to make the determination whether the closed-form solution can be used to get the nucleolus.

The next step is to show that the unique solution of system of equations is also the nucleolus of the least core LP model. In order to do this, this dissertation aims to show that for the given assumption of $S_1^*, S_2^*,\; S_3^*, x_1^*, x_2^*, x_3^*$ and t_{max}^* being non-zero and positive, their corresponding reduced cost coefficient of non-basic variables S_4^*, S_5^* and S_6^* must be strictly positive. The non-zero magnitude of optimal objective function coefficients of these non-basic variable will imply uniqueness because any further increase of these variables will lead to a decrease in the objective function value. For a least core LP model, this would be in contradiction to the goal of maximizing savings for all coalitions. This dissertation will make use of primal-dual properties of simplex and also use a standard relationship that establishes a connection between coefficients of non-basic variables in the optimal objective function of primal and the associated value of the optimal dual basic variables. This known relationship is shown below:

$$Y_B^* = (NC_B B^{-1} - C_N)x_n$$

$$Y_N^* = 0$$

The above relationship is a standard relationship that can be obtained from a generalized representation of an LP model. Further information about this relationship can be obtained from Hillier and Lieberman (2005). For any given least core LP problem demonstrative of the special case, if the cost coefficients corresponding to the non-basic variables are strictly positive in a final optimal simplex tableau, then the basic solution of the final tableau is unique. As per the primal-dual properties, the value of cost coefficients of non-basic variables in a final optimal simplex tableau are equal to the optimal values of basic variables of the dual. However, there is a slight exception to this rule because the completeness constraint requires an additional artificial variable "R" along with a large non-positive number "M" in the objective function.

The corresponding dual variable of this equality constraint will be a non-zero negative value in the dual. The cost coefficients for this artificial variable in the optimal tableau will appear as a big number M minus the dual variable. However, one needs to only show the cost coefficients as positive to prove uniqueness. The least core LP model has $2^n + n$ variables including the slack and artificial variables and $2^n - 1$ constraints. The forthcoming section will show that cost coefficients of exactly $n + 1$ non-basic variables in the primal are non-zero and the fact that these cost coefficients are also equal to the value of dual basic variables. The uniqueness of the closed-form solutions will be performed in two stages. Stage 1 will involve showing that the value of the objection function coefficient of non-basic variables S_4^*, S_5^* and S_6^* will be equal to 1/3, 1/3 and 1/3 respectively.

Also, the objection function coefficient of artificial non-basic variable will be M minus 2/3. These variables represent slack in coalitional rationality constraints of *C(12), C(13), C(23)* and *C(123)* whose associated dual variables are $\mathrm{Y}_{12}, \mathrm{Y}_{13}, \mathrm{Y}_{23}$ and Y_{123}. In Stage 2, optimal dual basic variables of Y_{12}, Y_{13}, Y_{23} and Y_{123} will be shown to be 1/3, 1/3 ,1/3 and negative 2/3 under the assumption S_1^*, S_2^*, S_3^*, x_1^*, x_2^*, x_3^* and t_{max}^* are basic *(m=7)* and S_4^*, S_5^*, S_6^* and t_{max}^* are non-basic *(n-m=11-7=4)*. Next, continuing with the example of a three-vehicle class HCA problem, verification of the magnitude of the non-zero dual basic variables is conducted by first obtaining the coefficients of non-basic variables of the primal. Table IV.1 standardizes the nomenclature of basic and non-basic variables of the primal and dual for a three-vehicle class least core LP model being considered in this section.

Re-stating the previously mentioned assumption that $S_1^* > 0, S_2^* > 0, S_3^* > 0, S_4^* = 0$, $S_5^* = 0$ and $S_6^* = 0$ along with the assumption that the cost allocation variables need to be $x_1^* > 0, x_2^* > 0, x_3^* > 0$ and $t_{max}^* > 0$. As a side, this assumption also implies non-degeneracy. Further, the least core LP model will also have an artificial variable R^* which must not be in the basis as that would imply infeasibility. However, it must be accounted for in the analysis. The basic variables in the optimal tableau must be $\boldsymbol{x_B} = \{ t_{max}^*, {x_2}^*, {S_3}^*, {x_1}^*, {x_3}^*, {S_1}^*, {S_2}^*\}$ and non-basic variables in optimal tableau must be $\boldsymbol{x_N} = \{{S_4}^*, {S_5}^*, {S_6}^*, R^*\}$. The matrix of basic variables *B* comprises of coefficients of basic variables $t_{max}^*, {x_2}^*, {S_3}^*, {x_1}^*, {x_3}^*, {S_1}^*$ & ${S_2}^*$ in least core LP model and is shown after Table IV.1.

Table IV.1: Naming convention for primal-dual basic and non-basic variables

Primal-Dual Basic and Non-Basic Variable Naming Convention			
j-Variables	Primal	*i-Constraints*	Dual
1	x_1	1	Y_1
2	x_2	2	Y_2
3	x_3	3	Y_3
4	t_1	4	Y_{12}
5	S_1	5	Y_{13}
6	S_2	6	Y_{23}
7	S_3	7	Y_{123}
8	S_4		
9	S_5		
10	S_6		
11	R		

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Inverse of matrix B is shown below:

$$B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 0 & 1/3 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & -2/3 & -2/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 0 & -2/3 & 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 & -2/3 & -2/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & -2/3 & 1/3 & -2/3 & 1/3 \end{pmatrix}$$

The matrix of non-basic variables N comprises of coefficients of non-basic variables S_4, S_5, S_6 and R in the least core LP model is shown below:

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix obtained by multiplication of B^{-1} and N is shown below:

$$B^{-1}N = \begin{pmatrix} 1/3 & 1/3 & 1/3 & -2/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ -2/3 & 1/3 & 1/3 & 1/3 \\ -2/3 & -2/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & -2/3 & 1/3 \end{pmatrix}$$

Now, since t_1 is the only basic decision variable in the objective function, the cost-coefficient of basic variable t_1 in optimal objective function will have a value of 1 and all other basic variables will have a value of 0. The row vector of basic variables $\boldsymbol{C_B}$ in the objective function is as follows:

$$C_B = (1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$$

Recalling the following primal-dual relationship,

$$Y_B = (NC_BB^{-1} - C_N)x_n$$

$$Y_B = (C_BB^{-1}N)\,x_n - (C_N)\,x_n$$

More importantly, an artificial non-basic variable R with a large negative coefficient "–M" is also present in the least core LP formulation. The large value of "–M" is to ensure that the artificial variable never appears in the basis. The row vector of non-basic variables $\boldsymbol{C_N}$ in the objective function is shown below:

$$C_N = (0 \quad 0 \quad 0 \quad -M)$$

Matrices of C_B and $(B^{-1}N)$ are multiplied to to get $\boldsymbol{Y_B}$ as shown below:

$$C_B \cdot \left(B^{-1}N\right) = (1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0) \cdot \begin{pmatrix} 1/3 & 1/3 & 1/3 & -2/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ -2/3 & 1/3 & 1/3 & 1/3 \\ -2/3 & -2/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & -2/3 & 1/3 \end{pmatrix} = (1/3 \quad 1/3 \quad 1/3 \quad -2/3)$$

The optimal value for the basic variables in the dual formulation $\boldsymbol{Y_B}$ is shown below after $\boldsymbol{C_B}B^{-1}N$ is subtracted from $\boldsymbol{C_N}$ to get the following as shown below:

$$\left(C_B B^{-1} N\right) x_N - C_N x_N = \begin{pmatrix} 1/3 & 1/3 & 1/3 & -2/3 \end{pmatrix} x_N - \begin{pmatrix} 0 & 0 & 0 & -M \end{pmatrix} x_N$$

$$Y_B = \begin{pmatrix} 1/3 & 1/3 & 1/3 & M - 2/3 \end{pmatrix} x_N$$

- Coefficient of non-basic variable S_4^* in optimal objective function is $z_8 - c_8 = \frac{1}{3}$
- Coefficient of non-basic variable S_5^* in optimal objective function is $z_9 - c_9 = \frac{1}{3}$
- Coefficient of non-basic variable S_6^* in optimal objective function is $z_{10} - c_{10} = \frac{1}{3}$
- Coefficient of non-basic variable R^* in optimal objective function is $z_{11} - c_{11} = M - \frac{2}{3}$

The non-basic variables S_4^*, S_5^*, S_6^* and R^* are each contained in the 4th, 5th, 6th and 7th rationality constraints of the least core LP formulation. The corresponding dual variables of the 4th, 5th,6th and 7th constraints are $\mathrm{Y}_{12}, \mathrm{Y}_{13}, \mathrm{Y}_{23}$ and Y_{123} respectively. Next, it is demonstrated that these dual variables will have an associated non-zero values of 1/3, 1/3, 1/3 and -2/3 respectively that would point to the fact that coalitional rationality constraints of {1,2}, {1,3}, {2,3} and {1,2,3} are binding. This is accomplished by solving a generalized representation of the dual of three-vehicle class least core LP model.

Restating, when N = {1,2,3}, the primal of least core LP model along with all its slack variables with their corresponding assumptions are shown below:

$$max\ Z = t_1 + 0S_1 + 0S_2 + 0S_3 + 0S_4 + 0S_5 + 0S_6$$

$\{S_1 > 0\}$	$x_1 + t_1 + S_1 = C(1)$	(Y_1)
$\{S_2 > 0\}$	$x_2 + t_1 + S_2 = C(2)$	(Y_2)
$\{S_3 > 0\}$	$x_3 + t_1 + S_3 = C(3)$	(Y_3)
$\{S_4 = 0\}$	$x_1 + x_2 + t_1 + S_4 = C(12)$	(Y_{12})
$\{S_5 = 0\}$	$x_1 + x_3 + t_1 + S_5 = C(13)$	(Y_{13})
$\{S_6 = 0\}$	$x_2 + x_3 + t_1 + S_6 = C(23)$	(Y_{23})
	$x_1 + x_2 + x_3 = C(123)$	(Y_{123})

Dual Least Core LP Formulation for Three-vehicle class HCA problem

$min \quad C(1)\,Y_1 + C(2)\,Y_2 + C(3)\,Y_3 + C(12)\,Y_{12} + C(13)\,Y_{13} + C(23)\,Y_{23} + C(123)\,Y_{123}$

Subject to

$$Y_1+Y_{12}+Y_{13}+Y_{123} \geq 0 \qquad (x_1 > 0)$$
$$Y_2+Y_{12}+Y_{23}+Y_{123} \geq 0 \qquad (x_2 > 0)$$
$$Y_3+Y_{13}+Y_{23}+Y_{123} \geq 0 \qquad (x_3 > 0)$$
$$Y_1+Y_2+Y_3+Y_{12}+Y_{13}+Y_{23} = 1 \qquad (t_1 > 0)$$

Y_{123} is unrestricted.

Based on the assumption that the cost allocated to each vehicle class must be $x_1^* > 0, x_2^* > 0, x_3^* > 0$ and $t_{max}^* > 0$ (dual of the dual is primal) , all the constraints of dual must be binding. This information permits the conversion of an inequality from "≥" to equality "=". Next, the dual variables for coalitions containing *n-2* vehicle classes namely, Y_1 , Y_2 and , Y_3 are taken to the right hand sides. Since the sign of dual variable Y_{123} is unrestricted. The variable Y_{123} is written in terms of two non-negative variables of Y_{123}^+ and Y_{123}^- where $Y_{123} = Y_{123}^+ - Y_{123}^-$. The constraints of the dual formulation are shown below:

$$Y_{12}+Y_{13} + Y_{123}^+ - Y_{123}^- = -Y_1$$

$$Y_{12}+Y_{23} + Y_{123}^+ - Y_{123}^- = -Y_2$$

$$Y_{13}+Y_{23} + Y_{123}^+ - Y_{123}^- = -Y_3$$

$$Y_{12}+Y_{13}+Y_{23} = 1-Y_1-Y_2-Y_3$$

The above four equations are now added to get a relationship for Y_{12} in terms of all dual variables containing less than *n-1* vehicle classes, those being Y_1, Y_2 and Y_3 as follows:

$$3(Y_{12} + Y_{13} + Y_{23} + Y_{123}^+ - Y_{123}^-) = -Y_1 - Y_2 - Y_3 + 1 - Y_1 - Y_2 - Y_3$$

$$3(Y_{12} - Y_3) = -Y_1 - Y_2 - Y_3 + (1 - Y_1 - Y_2 - Y_3) \qquad since\ (Y_{13}+Y_{23} + Y_{123}^+ - Y_{123}^- = -Y_3)$$

$$3Y_{12} = -Y_1 - Y_2 + 2Y_3 + (1 - Y_1 - Y_2 - Y_3)$$

$$\boldsymbol{Y_{12} = \frac{1}{3}(-Y_1) + \frac{1}{3}(-Y_2) + \frac{2}{3}(Y_3) + \frac{1}{3}(1-Y_1-Y_2-Y_3)}$$

Similarly, a relationship is now obtained for Y_{13} in terms of all dual variables containing less than *n-1* vehicle classes. Those being Y_1, Y_2 and Y_3 as follows:

$$3(Y_{12} + Y_{13} + Y_{23} + Y_{123}^{+} - Y_{123}^{-}) = -Y_1 - Y_2 - Y_3 + 1 - Y_1 - Y_2 - Y_3$$

$$3(Y_{13} - Y_2) = -Y_1 - Y_2 - Y_3 + (1 - Y_1 - Y_2 - Y_3) \qquad since\ (Y_{12} + Y_{23} + Y_{123}^{+} - Y_{123}^{-} = -Y_2)$$

$$3Y_{13} = -Y_1 + 2Y_2 - Y_3 + (1 - Y_1 - Y_2 - Y_3)$$

$$\boldsymbol{Y_{13} = \frac{1}{3}(-Y_1) + \frac{2}{3}(Y_2) + \frac{1}{3}(-Y_3) + \frac{1}{3}(1 - Y_1 - Y_2 - Y_3)}$$

Likewise, a relationship is now obtained for Y_{23} in terms of all dual variables containing less than *n-1* vehicle classes, those being Y_1, Y_2 and Y_3 as follows:

$$3(Y_{12} + Y_{13} + Y_{23} + Y_{123}^{+} - Y_{123}^{-}) = -Y_1 - Y_2 - Y_3 + 1 - Y_1 - Y_2 - Y_3$$

$$3(Y_{23} - Y_1) = -Y_1 - Y_2 - Y_3 + (1 - Y_1 - Y_2 - Y_3) \qquad since\ (Y_{12} + Y_{13} + Y_{123}^{+} - Y_{123}^{-} = -Y_1)$$

$$3Y_{23} = 2Y_1 - Y_2 - Y_3 + (1 - Y_1 - Y_2 - Y_3)$$

$$\boldsymbol{Y_{23} = \frac{2}{3}(Y_1) + \frac{1}{3}(-Y_2) + \frac{1}{3}(-Y_3) + \frac{1}{3}(1 - Y_1 - Y_2 - Y_3)}$$

In the same way, a relationship is now obtained for Y_{123} in terms of all dual variables containing less than *n-1* vehicle classes, those being Y_1, Y_2 and Y_3 as follows:

$$3(Y_{12} + Y_{13} + Y_{23} + Y_{123}^{+} - Y_{123}^{-}) = -Y_1 - Y_2 - Y_3 + 1 - Y_1 - Y_2 - Y_3$$

$$\begin{aligned} 3(1 - Y_1 - Y_2 - Y_3 + Y_{123}^{+} - Y_{123}^{-} \\ &= -Y_1 - Y_2 - Y_3 \\ &+ (1 - Y_1 - Y_2 - Y_3) \qquad Since\ (Y_{12} + Y_{13} + Y_{23} = 1 - Y_1 - Y_2 - Y_3) \end{aligned}$$

$$3(Y_{123}^{+} - Y_{123}^{-}) = -Y_1 - Y_2 - Y_3 - 2(1 - Y_1 - Y_2 - Y_3)$$

$$Y_{123}^{+} - Y_{123}^{-} = \boldsymbol{\frac{1}{3}(-Y_1) + \frac{1}{3}(-Y_2) + \frac{1}{3}(-Y_3) - \frac{2}{3}(1 - Y_1 - Y_2 - Y_3)}$$

Since S_1^*, S_2^* and S_3^* are basic variables in the primal, by complementary slackness conditions, Y_1^*, Y_2^* and Y_3^* must be zero in the dual. Therefore, setting $Y_1^* = 0$, $Y_2^* = 0$ and $Y_3^* = 0$.

First, value of Y_{12} is calculated as follows,

$3Y_{12} = -Y_1 - Y_2 + 2Y_3 + 1 - Y_1 - Y_2 - Y_3$

$3Y_{12} = -2Y_1 - 2Y_2 + Y_3 + 1$

$$Y_{12} = \frac{-2Y_1 - 2Y_2 + Y_3 + 1}{3} = \frac{1}{3} \qquad (Y_1^* = 0, Y_2^* = 0 \; and \; Y_3^* = 0)$$

Second, value for $Y_{13}$is calculated as follows,

$3Y_{13} = -Y_1 + 2Y_2 - Y_3 + 1 - Y_1 - Y_2 - Y_3$

$3Y_{13} = -2Y_1 + Y_2 - Y_3 + 1$

$$Y_{13} = \frac{-2Y_1 + Y_2 - 2Y_3 + 1}{3} = \frac{1}{3} \qquad (Y_1^* = 0, Y_2^* = 0 \; and \; Y_3^* = 0)$$

Third, value for Y_{23} is calculated as follows,

$3Y_{23} = 2Y_1 - Y_2 - Y_3 + 1 - Y_1 - Y_2 - Y_3$

$3Y_{23} = Y_1 - 2Y_2 - 2Y_3 + 1$

$$Y_{23} = \frac{Y_1 - 2Y_2 - 2Y_3 + 1}{3} = \frac{1}{3} \qquad (Y_1^* = 0, Y_2^* = 0 \; and \; Y_3^* = 0)$$

To finish, value for Y_{123} is calculated as follows,

$3(Y_{123}^{+} - Y_{123}^{-}) = -Y_1 - Y_2 - Y_3 - 2 + 2Y_1 + 2\,Y_2 + 2Y_3$

$$\boldsymbol{Y_{123}} = Y_{123}^{+} - Y_{123}^{-} = \frac{Y_1 + Y_2 + Y_3 - 2}{3} = -\frac{2}{3} \qquad (Y_1^* = 0, Y_2^* = 0 \; and \; Y_3^* = 0)$$

$$Y_{123}^{-} = \frac{2}{3} \qquad \left(Assuming \; Y_{123}^{+} = 0, Y_{123}^{-} = \frac{2}{3} \right)$$

$$\boldsymbol{Y_{12} = Y_{13} = Y_{23} = \frac{1}{3}\,, Y_{123}^{-} = \frac{2}{3}}$$

The values of dual basic variables are equal to the cost coefficients of primal non-basic variables in optimal objective function row as shown below:

- $(\boldsymbol{z_8 - c_8})\, S_4{}^* = \boldsymbol{Y_{12}} = \boldsymbol{\frac{1}{3}}$
- $(\boldsymbol{z_9 - c_9})\, S_5{}^* = \boldsymbol{Y_{13}} = \boldsymbol{\frac{1}{3}}$

- $(\boldsymbol{z_{10}} - \boldsymbol{c_{10}})\, S_6{}^* = \boldsymbol{Y_{23}} = \boldsymbol{\frac{1}{3}}$
- $(\boldsymbol{z_{11}} - \boldsymbol{c_{11}})\, R^* = \boldsymbol{Y_{123}} = -\boldsymbol{\frac{2}{3}}$

The values of the dual basic variables Y_{12}, Y_{13}, Y_{23} and Y_{123} correspond to the coefficients of the corresponding slack variables $S_4{}^*$, $S_5{}^*$, $S_6{}^*$ and R^* in the optimal objective function of the primal. Based on the optimality conditions, it is concluded that the closed-form solution defining an allocation $x^* = (\, x_1^*, x_2^*, x_3^*, t_{\max}^*)$ is optimal and unique i.e. the nucleolus because the coefficients of *n-m (n=11 and m=7) or* four non-basic variables in the optimal objective function are non-zero and the corresponding value of exactly *n-m* or four basic variables of the dual are also non-zero with equal magnitude. Therefore, for a three-vehicle class HCA problem, $x^* = (\, x_1^*, x_2^*, x_3^*, t_{\max}^*)$ is the nucleolus where,

$$x_1^* = \frac{1}{3}C(12) + \frac{1}{3}C(13) + \frac{1}{3}C(123) - \frac{2}{3}C(23) \qquad \text{(IV-20)}$$

$$x_2^* = \frac{1}{3}C(12) + \frac{1}{3}C(23) + \frac{1}{3}C(123) - \frac{2}{3}C(13) \qquad \text{(IV-21)}$$

$$x_3^* = \frac{1}{3}C(13) + \frac{1}{3}C(23) + \frac{1}{3}C(123) - \frac{2}{3}C(12) \qquad \text{(IV-22)}$$

$$t_{max}^* = \frac{1}{3}C(12) + \frac{1}{3}C(13) + \frac{1}{3}C(23) - \frac{2}{3}C(123) \qquad \text{(IV-23)}$$

More importantly, from the perspective of highway cost allocation, the best possible optimal allocation that has maximized the savings of all coalitions exclusively for the special case has been determined. The dual variables $\boldsymbol{Y_S}$ for the corresponding coalitional rationality constraints of *C(S)* containing *n-1* vehicle classes and *n* vehicle classes will always be non-zero with magnitude of $1/n$ and $(n-1)/n$ respectively. In Appendix B, it has been shown that this would always be the case for any *n* vehicle-class dual least core LP formulation with the assumption that slack variables for all coalitional rationality constraints containing less than *n-1* vehicle classes will be zero as the corresponding slacks in the primal are non-zero.

Interpretation of Uniqueness of Closed-form Solutions

Due to insufficient slack, objective function value or savings can only be improved (maximized) by *1/3rd* if one relaxes the RHS or increases by one unit the cost of coalitions of *C(12)* ,*C(13)* ,*C(23)* and *C(123)*. For the least core LP model in a HCA problem, cost of coalitions cannot be changed once the solution is reached. Conversely, these positive dual variables serve as the optimal objective function coefficients or the reduced cost coefficient of non-basic slack variables S_4, S_5 and S_6 in the optimal basis of the primal formulation. Therefore, any attempt to bring these slack variables into the basis will lead to a decrease in the objective function value, which would be going against our goal of maximizing savings. However, doing so would be impossible as there is no slack left to increase and therefore the conditions for termination of the simplex algorithm have been reached.

Hence, the uniqueness of the solution translates to the nucleolus. This is because there exists no other allocation in a given least core LP model with a specified characteristic function that can further maximize the savings of the remaining coalitions in the lexicographic order.

For the purposes of using the closed-form, the coalitions of interest are the one's comprising of n and $n\text{-}1$ vehicle classes. The number of coalitional rationality constraints containing $n\text{-}1$ vehicle classes are always ${}_nC_{n-1} = n$. The left hand sides of these constraints will have a summation of all *n-1* decision variables x_i for all *i* in *S* plus one variable t_1 that comes out to *n* variables being in each coalitional rationality constraint *C(S)*. The grand coalition constraint will always have *n* decision variables x_i for all *i* in *N* excluding the savings variable of t_1.

$$\sum_{i \in S} x_i + t_1 = C(S) \quad \forall S \subset N \;\; |S| = n - 1 \text{ with } n \text{ equations \& } n \text{ variables}$$

$$\sum_{i \in S} x_i = C(N) \qquad |S| = n \text{ with } 1 \text{ equation \& } n \text{ variables}$$

- 1st Group: Collection of *n* rationality constraints for coalitions with *n-1* vehicle classes having a summation of *n* variables on the left hand sides.
- 2nd Group: A group of one constraint for the grand coalition having *n* variables.

The 1st and 2nd group of constraints will always exist in any *n*-vehicle class HCA problem that would define a system of $n+1$ linear equations with $n+1$ variables having a single solution that can be expressed in closed-form. This closed-form will define an allocation comprising of coalitions containing *n* vehicle classes and *n-1* vehicle classes. The question that remains is whether the closed-form would be in accordance with the nucleolus or not prior to solving the least core LP model. For which, one must develop requisite conditions. Although the analysis conducted in this section has been carried out for a three-vehicle class HCA problem. The analysis could just as well be extended to any higher order *n*-vehicle class HCA problem without any loss of generality. Next, the conditions for checking the concurrence of closed-form solutions with nucleolus are generalized for any *n*-vehicle class HCA problem.

Conditions for Using Closed-form Solutions

The compact conditions shown in (IV-24) - (IV-27) are all necessary conditions. The conditions state that slack variables in the optimal basis for coalitional constraints containing *n-1* vehicle classes must be equal to zero and all other slack variables for coalitional constraints containing less than *n-2* vehicle classes must be greater than zero for the proposed closed-form solution to reveal the nucleolus in the 1st LP or least core LP model. The conditions are labeled as S_u due to their derivation from slack variable-based relationships. These are listed below where $u = (1,2,3, \ldots., 2^n - 2)$.

$$S_{u=1} \Rightarrow \{C(S) - (\textstyle\sum_{i \in S} x^*{}_i - t_1^*)\} \geq 0 \qquad |S| = 1 \qquad \text{(IV−24)}$$

$$\vdots$$

$$S_{u=2^n-(n+2)} \Rightarrow \{C(S) - (\textstyle\sum_{i \in S} x^*{}_i - t_1^*)\} \geq 0 \qquad |S| = n - 2 \qquad \text{(IV−25)}$$

$$S_{u=2^n-(n+1)} \Rightarrow \{C(S) - (\textstyle\sum_{i \in S} x^*{}_i - t_1^*)\} = 0 \qquad |S| = n - 1 \qquad \text{(IV−26)}$$

$$\vdots$$

$$S_{u=2^n-2} \Rightarrow \{C(S) - (\textstyle\sum_{i \in S} x^*{}_i - t_1^*)\} = 0 \qquad |S| = n - 1 \qquad \text{(IV−27)}$$

As discussed earlier in the analysis of three-vehicle class HCA problem, relying only on slack variables to be positive was shown to be an inadequate line of approach as the conditions could be satisfied even when the decision variables were negative. Therefore, extra conditions must be added in order to ensure non-authentication of closed-form solution when costs of coalitions *C(S)* and *C(N)* leads to a situation defined by infeasibility or degeneracy in the least core LP model.

The dissertation has chosen to include the expression of cost allocation of closed-form solution themselves defined in terms of coalition *C(S)* and *C(N)* containing *n-1* and *n* vehicle classes into the main collection of conditions. These condition must also be evaluated for non-negativity. The additional set of conditions will be introduced that state in addition to satisfaction of all the necessary conditions (IV-24) – (IV-27), the variable $\boldsymbol{t_1^*}$ and $\boldsymbol{x^*{}_i}$ as described below must also be positive.

$$S_{u=2^n-1} \Rightarrow \sum_{\substack{i\in S\\ S\subseteq N}} \frac{1}{n} C\,(S) - \left(\left(\frac{n-1}{n}\right) C\,(N)\right) > 0 \qquad |S| = n-1 \qquad |N| = n$$

$$S_{u=2^n} \Rightarrow \sum_{\substack{i\in S\\ S\subseteq N}} \frac{1}{n} C\,(S) - \left(\left(\frac{n-1}{n}\right) C\,(S)\right)_{\substack{i\notin S\\ S\subseteq N}} > 0 \qquad |S| = n-1 \qquad |N| = n$$

$$\vdots$$

$$S_{u=2^n+n-1} \Rightarrow \sum_{\substack{i\in S\\ S\subseteq N}} \frac{1}{n} C\,(S) - \left(\left(\frac{n-1}{n}\right) C\,(S)\right)_{\substack{i\notin S\\ S\subseteq N}} > 0 \qquad |S| = n-1 \qquad |N| = n$$

For any *n*-vehicle class HCA problem using the least core LP model, one must verify that each and every one of the $2^n + n - 1$ conditions are satisfied before a judgement can be made whether the closed-form solution would equate to allocation of the nucleolus.

Closed-form Solutions for N-Vehicle Class HCA Problem

The allocation vector x^* is the nucleolus generalized for any *n*-vehicle class HCA problem where the cost allocated to each vehicle class $x^*{}_i$ and the optimal savings t_1^* can be obtained using relationships (IV-28) and (IV-29) with conditions for using closed-form solutions, serving as a mandatory prerequisite:

$$\boldsymbol{t_1^*} = \sum_{\substack{i\in S\\ S\subseteq N}} \frac{\mathbf{1}}{\boldsymbol{n}} \boldsymbol{C}\,(\boldsymbol{S}) - \left(\left(\frac{\boldsymbol{n-1}}{\boldsymbol{n}}\right) \boldsymbol{C}\,(\boldsymbol{N})\right) \qquad |S| = n-1 \qquad |N| = n \quad \text{(IV-28)}$$

$$\boldsymbol{x^*}_{\boldsymbol{i}} = \sum_{\substack{i\in S\\ S\subseteq N}} \frac{\mathbf{1}}{\boldsymbol{n}} \boldsymbol{C}\,(\boldsymbol{S}) - \left(\left(\frac{\boldsymbol{n-1}}{\boldsymbol{n}}\right) \boldsymbol{C}\,(\boldsymbol{S})\right)_{\substack{i\notin S\\ S\subseteq N}} \qquad |S| = n-1 \qquad |N| = n \quad \text{(IV-29)}$$

Note: The above closed-form solutions are the nucleolus if and only if all the conditions for using closed-form solutions are satisfied.

Conditions for Non-Usage of Closed-form Solutions

Let $X = \{\, x : \sum_{i=1}^{n} x_i = C(N)\}$ be the set of allocation satisfying completeness. The dissertation proposes that a vector $x^{**} \in X$ is the nucleolus and single point solution if for every $\boldsymbol{\mathcal{H}} \in X$ the lexicographical order of allocation x^{**} is $L(\boldsymbol{\mathcal{H}}) \leq_{lex} L(x^{**})$ and whose optimal savings t_{max}^{**} is lower than the optimal savings t_{max}^{*} obtained from closed-form solutions. Their might be an exception to this rule in situations when the least core LP model is infeasible or degenerate. In such instances allocations produced are mostly invalid.

For instance in a three-vehicle class HCA problem, if $x_1^{**},\ x_2^{**}\ and\ x_3^{**}$ are any allocations other than $x_1^{*},\ x_2^{*}$ and x_3^{*} then any one of the following must be true in cases where CFSs cannot be used:

$$x_1^{**} < x_1^{*} \quad \text{(IV-30)}$$

$$x_2^{**} < x_2^{*} \quad \text{(IV-31)}$$

$$x_3^{**} < x_3^{*} \quad \text{(IV-32)}$$

The condition (IV-33) states that when the closed-form solution cannot be used, the greatest lower bound on savings t_{max}^{*} obtained using closed-form solution is greater than the greatest lower bound on savings t_{max}^{**} generated by solving the first LP. This condition is always true when the closed-form solutions are not viable for implementation in order to get the nucleolus.

$$t_{max}^{**} < t_{max}^{*} \quad \text{(IV-33)}$$

Let *z* be a sequence defined as *z = 1, 2, 3…n-1.* $n < z$

$$t(x, S_m) = C(S) - \sum_{i \in S} x_i \quad |S| = n - 1 \quad \text{and} \quad m = 1,2,\dots,2^n - 2 \quad \text{(IV-34)}$$

In order for our analytical formulas to be valid, coalitions containing *n-1* and *n* members must have the lowest savings as shown in (IV-34). It is probable, that the proposed closed-form solutions might not work in circumstances when cost of coalitions *C(S)* containing between *n-2* and *n-(z-1)* vehicle classes become lower than the cost of coalition *C(S)* containing *n-1* and *n* vehicle classes.

$$t(x, S_m) = C(S) - \sum_{i \in S} x_i \quad 1 < |S| < n - (z - 1) \quad \text{and} \quad 1 < m < 2^n - n - 2$$

Henceforth, it can be stated categorically that for the closed-form solutions to be valid, the cost of coalitions *C(S)* for all coalitions must be greater than the RHS for the equation shown in (IV-35a) and (IV-35b), along with closed-form solutions of $x^{*}{}_i$ and t^{*}being positive.

$$C(S) > \left(\sum_{\substack{i \in S \\ S \subseteq N}} \frac{1}{n} C\,(S) - \left(\left(\frac{n-1}{n} \right) C\,(S) \right)_{\substack{i \notin S \\ S \subseteq N}} \right) + \left(\sum_{\substack{i \in S \\ S \subseteq N}} \frac{1}{n} C\,(S) - \left(\left(\frac{n-1}{n} \right) C\,(N) \right) \right) \quad \text{(IV-35a)}$$

For all, $1 < |S| < n - (z - 1) \qquad |N| = n \quad \forall S \in N$

$$C(S) = \left(\sum_{\substack{i\in S \\ S\subseteq N}} \frac{1}{n} C\,(S) - \left(\left(\frac{n-1}{n}\right) C\,(S)\right)_{\substack{i\notin S \\ S\subseteq N}}\right) + \left(\sum_{\substack{i\in S \\ S\subseteq N}} \frac{1}{n} C\,(S) - \left(\left(\frac{n-1}{n}\right) C\,(N)\right)\right) \quad \text{(IV-35b)}$$

For all, $|S| = n - 1 \qquad |N| = n \quad \forall S \in N$

To better understand the implementation of the closed-form solution, a simple example is illustrated. A more detailed use of the closed-form solution on a realistic data set can be found in Appendix A where Closed-form Solution were used in a four-vehicle class HCA problem.

Example 1: Usage of Closed-form Solution

As an illustration, consider an HCA problem with characteristic function defining cost of coalitions as $C(\emptyset) = 0$, $C(1) = 58$, $C(2) = 44$, $C(3) = 60$, $C(12) = 65$, $C(13) = 71$, $C(23) = 57$ *and* $C(123) = 78$. The least core LP model for this problem is shown below:

$$\max\ t_1$$

Subject to

$$\begin{aligned}
x_1 + t_1 &\le 58 && (Y_1)\\
x_2 + t_1 &\le 44 && (Y_2)\\
x_3 + t_1 &\le 60 && (Y_3)\\
x_1 + x_2 + t_1 &\le 65 && (Y_{12})\\
x_1 + x_3 + t_1 &\le 71 && (Y_{13})\\
x_2 + x_3 + t_1 &\le 57 && (Y_{23})\\
x_1 + x_2 + x_3 &= 78 && (Y_{123})\\
x_1, x_2, x_3, t_1 &\ge 0
\end{aligned}$$

First, the right hand sides or the cost of coalitions for all two and three member vehicle class coalitions of $C(1)$, $C(2)$, $C(3)$, $C(12)$, $C(13)$, $C(23)$ and $C(123)$ are substituted into $2^n + n - 1$ or 10 conditions as shown in (IV-14) - (IV-19) along with S_7, S_8, S_9 and S_{10}. The satisfaction of these conditions are reported in Table IV.2.

As seen in Table IV.2, all conditions for using closed-form solutions are satisfied wherein $S_1 > 0, S_2 > 0, S_3 > 0$, $S_4 = 0, S_5 = 0, S_6 = 0, S_7 > 0, S_8 > 0, S_9 > 0$ and $S_{10} > 0$. For this reason, the nucleolus can be obtained using the closed-form solution (IV-20) - (IV-23).

Table IV.2: Conditions for using Closed-form Solutions in Example 1

	C(S)	Conditions ($2^n + n - 1$)		Value	Status
C(1)	58	S_1	$C(1) - x_1^* - t_1^* > 0$	12.3333	Satisfied
C(2)	44	S_2	$C(2) - x_2^* - t_1^* > 0$	12.3333	Satisfied
C(3)	60	S_3	$C(3) - x_3^* - t_1^* > 0$	12.3333	Satisfied
C(12)	65	S_4	$C(12) - (x_1^* + x_2^*) - t_1^* = 0$	0	Satisfied
C(13)	71	S_5	$C(12) - (x_1^* + x_3^*) - t_1^* = 0$	0	Satisfied
C(23)	57	S_6	$C(13) - (x_2^* + x_3^*) - t_1^* = 0$	0	Satisfied
C(123)	78	S_7	$t_1^* > 0$	12.3333	Satisfied
		S_8	$x_1^* > 0$	33.3333	Satisfied
		S_9	$x_2^* > 0$	19.3333	Satisfied
		S_{10}	$x_3^* > 0$	25.3333	Satisfied

$$x_1 = \frac{1}{3}(65 + 71 + 78) - \frac{2}{3}(57) = 33.333$$

$$x_2 = \frac{1}{3}(65 + 57 + 78) - \frac{2}{3}(71) = 19.333$$

$$x_3 = \frac{1}{3}(71 + 57 + 78) - \frac{2}{3}(65) = 25.333$$

$$t_1 = \frac{1}{3}(65 + 71 + 57) - \frac{2}{3}(78) = 12.333$$

It can be verified that LINDO optimization software yields the same solution.

Example 2: Non-Usage of Closed-form Solution

Consider another HCA problem with characteristic function defining cost of coalitions $C(\emptyset) = 0$, $C(1) = 58$, $C(2) = 44$, $C(3) = 31$, $C(12) = 65$, $C(13) = 71$, $C(23) = 57$ and $C(123) = 78$. The least core LP model for this problem is shown below:

$$\max t_1$$

Subject to

$$x_1 + t_1 \leq 58 \qquad (Y_1)$$

$$x_2 + t_1 \leq 44 \qquad (Y_2)$$

$$x_3 + t_1 \leq 31 \qquad (Y_3)$$
$$x_1 + x_2 + t_1 \leq 65 \qquad (Y_{12})$$
$$x_1 + x_3 + t_1 \leq 71 \qquad (Y_{13})$$
$$x_2 + x_3 + t_1 \leq 57 \qquad (Y_{23})$$
$$x_1 + x_2 + x_3 = 78 \qquad (Y_{123})$$
$$x_1, x_2, x_3, t_1 \geq 0$$

First, the right hand sides or the cost of coalitions $C(1)$, $C(2)$, $C(3)$, $C(12)$, $C(13)$, $C(23)$ and $C(123)$ are substituted into the conditions (IV-14) - (IV-19) along with S_7, S_8, S_9 and S_{10}. The satisfaction of these conditions are reported in Table IV.3. As seen in Table IV.3, all conditions for using closed-form solutions are not satisfied wherein $S_1 > 0, S_2 > 0, \boldsymbol{S_3 < 0}, S_4 = 0, S_5 = 0, S_6 = 0, S_7 > 0, S_8 > 0, S_9 > 0$ and $S_{10} > 0$.

Therefore, the nucleolus cannot be obtained using the closed-form solution. The nucleolus for this HCA problem was previously found in chapter III whose results are reported again in Table IV.4.

Table IV.3: Conditions for using Closed-form Solutions in Example 2

	C(S)	Conditions ($2^n + n - 1$)		Value	Status
C(1)	58	S_1	$C(1) - x_1^* - t_1^* > 0$	12.3333	Satisfied
C(2)	44	S_2	$C(2) - x_2^* - t_1^* > 0$	12.3333	Satisfied
C(3)	31	S_3	$C(3) - x_3^* - t_1^* > 0$	-6.6667	Violated
C(12)	65	S_4	$C(12) - (x_1^* + x_2^*) - t_1^* = 0$	0	Satisfied
C(13)	71	S_5	$C(12) - (x_1^* + x_3^*) - t_1^* = 0$	0	Satisfied
C(23)	57	S_6	$C(13) - (x_2^* + x_3^*) - t_1^* = 0$	0	Satisfied
C(123)	78	S_7	$t_1^* > 0$	12.3333	Satisfied
		S_8	$x_1^* > 0$	33.3333	Satisfied
		S_9	$x_2^* > 0$	19.3333	Satisfied
		S_{10}	$x_3^* > 0$	25.3333	Satisfied

Table IV.4: Sequential LP approach for Example 2

Closed-Form Solution (Non-Nucleolus)			1^{st} LP		2^{nd} LP
			I	II	Nucleolus
$\boldsymbol{t^*_{max}}$	**12.333**	$\boldsymbol{t^{**}_{max}}$	**9**	**9**	14
x^*_1	33.333	x^{**}_1	40	30	35
x^*_2	19.333	x^{**}_2	16	26	21
x^*_3	25.333	x^{**}_3	22	22	22
C(N)	78	C(N)	78	78	78

Discussion on Closed-form Solutions

In Example 1, it can be seen that the lowest savings were received by coalitions containing *n-1* vehicle classes or two vehicle classes. These were *C(12), C(13)* and *C(23)* with a savings value of 12.333. Moreover, since the least core model maximizes the lower bound on savings for all coalitions. The optimal savings t^*_1 ,as expected, had a value of 12.333. In Example 2, it can be seen that the lowest savings were received by coalitions containing at least less than *n-z* vehicle classes with coalition *C(3)* containing one vehicle class and coalition *C(12)* containing two vehicle classes with each having a savings value of 9 as shown in Table IV.5.

Upon closer examination of Example 2, the reason for the invalidity of the closed-form solution is evident in Table IV.14. If $t^*_{max} = 12.3333$ is a savings value obtained using the closed-form solution and $t^{**}_{max} = 9$ is the lowest savings achieved by solving the 1st LP, CFSs could not be applied due to relationship (IV-25). The relationship states that the allocation generated using the closed-form solution are not representative of the lowest savings a least core LP model can accord to coalitions in a HCA problem.

Table IV.5: Closed-form solution and sequential LP approach

EXAMPLE 1		EXAMPLE 2	
x= (33.33,19.33,25.333) $t(x,S) = C(S) - \sum_{i \in S} x_i$		*x= (35,21,22)* $t(x,S) = C(S) - \sum_{i \in S} x_i$	
C(12)	**12.3334**	**C(3)**	**9**
C(13)	**12.3334**	**C(12)**	**9**
C(23)	**12.3334**	C(13)	14
C(1)	24.6667	C(23)	14
C(2)	24.6667	C(1)	23
C(3)	34.6667	C(2)	23

$$t^{**}_{max} < t^{*}_{max} \quad \text{(IV-25)}$$

Furthermore, the conditions that validate the of closed-form solutions, are also able to provide an insight into the minimum values the cost of coalitions *C(S)* needs to maintain in order for the closed-form solution to be the nucleolus. For instance, increasing the cost of coalition of *C(3)* by exactly 6.6667≈ 7, the least core LP model in Example 2 would have given the same solution as the closed-form solution. Table IV.6 illustrates the relationship (IV-35) displaying all the minimum values that right hand sides of the least core LP model must have in order for the closed-form solution to be valid.

In addition, it is important to ensure that closed-form solution of variable x^*_i and t^*_1 must also be positive. The next section capitalizes on the benefits offered by the conditions of using closed-from solution to recommend the nucleolus for a modified least core LP model that brings the initial least core LP model into compliance with the conditions of using CFS.

The Closed-Form Solution Based Approach

This approach offers a tangential interpretation of the closed-form solutions. The application of this method is envisioned in situations where allocations given by the closed-form solution is deemed to be fair, rational and equitable than the one obtained from the sequential LP approach. This procedure considers cost of coalitions *C(S)* and *C(N)* as an input obtained from a characteristic function.

Table IV.6: Minimum values for costs of coalitions

CLOSED-FORM SOLUTION: Cost of Coalitions					
Example 1: Usable			Example 2: Not Usable		
	C(S)	Conditions		C(S)	Condition
C(1)	58	45.66667	C(1)	58	45.66667
C(2)	44	31.66667	C(2)	44	31.66667
C(3)	60	37.66667	C(3)	31	37.66667
C(12)	65	65	C(12)	65	65
C(13)	71	71	C(13)	71	71
C(23)	57	57	C(23)	57	57

The procedure then increases the cost of certain or all coalitions depending upon the necessary conditions that have been violated. The end result of this procedure is the nucleolus for a modified form of the original HCA problem. Consider the following generalized representation of the least core LP model with costs of coalitions *C(S)* and *C(N)* estimated from any given characteristic function.

$$\max \ t_n$$

Subject to

$$\sum_{i \in N} x_i = \mathrm{C(N)}$$

$$\sum_{i \in S} x_i + t_n \leq \mathrm{C(S)} \ \ for\ all\ S \in N$$

$$\sum_{i \in S} x_i + t_j = \mathrm{C(S)} \ \ for\ all\ S \in F_j \quad j \in \{1, \dots n - 1\}$$

$$x_i\ ,\ t_n\ ,\ t_j \geq 0 \text{ for all } i \in N$$

This dissertation has identified five independent outcomes that would occur when the costs of coalitions *C(S)* and *C(N)* are substituted into the conditions for using closed-form solution. Let S_u be a counter for number of conditions, where $u = (1, 2, 3, \dots\dots, 2^n + n - 1)$. Satisfaction of all $2^n + n - 1$ conditions validates the use of closed-form solution along with x_1^* and t_1^* being positive, before initiating the use of closed-form solutions. For each of these outcomes, recourses are recommended that must be made to either few, none or all of the costs of coalitions *C(S)* in order for the least core LP model to generate a unique solution (i.e. the nucleolus) that concurs with the closed-form solution. These outcomes are listed below:

Outcome 1

For each, $S_{u=1\ to\ u=2^n-(n+1)} < 0$ and $t_1 < 0$, then $C^*(S) = C(S) + abs(S_u) + abs(t)$, for all $S \subset N$ and the violated condition S_u provided $S_{u=2^n-1\ to\ u=2^n+n-1} > 0$.

Outcome 2

For each, $S_{u=1\ to\ u=2^n-(n+1)} < 0$ and $t_1 > 0$, then $C^*(S) = C(S) + abs(S_u)$, for all $S \subset N$ and the violated condition S_u provided $S_{u=2^n-1\ to\ u=2^n+n-1} > 0$.

Outcome 3

For each, $S_{u=1\ to\ u=2^n-(n+1)} > 0$ and $t_1 > 0$, then $C^*(S) = C(S)$, for all $S \subset N$ provided $S_{u=2^n-1\ to\ u=2^n+n-1} > 0$.

Outcome 4

For each, $S_{u=1\ to\ u=2^n-(n+1)} > 0$ and $t_1 < 0$, then $C^*(S) = C(S) + abs(t)$, for all $S \subset N$ provided $S_{u=2^n-1\ to\ u=2^n+n-1} > 0$.

Outcome 5-No Recourse

For each, $S_{u=1\ to\ u=2^n-(n+1)} > 0$ or $S_{u=1\ to\ u=2^n-(n+1)} < 0$ and $t_1 > 0\ or\ t_1 < 0$ and even if one of the conditions of $S_{u=2^n-1\ to\ u=2^n+n-1} < 0$, then the least core model cannot be modified for the closed-form to concur with the nucleolus. This is concluded on the grounds that even if one were to modify the problem, the allocation to the modified problem would most likely be a degenerate solution with an allocation of zero being assigned to a vehicle class, a solution considered to be invalid in an HCA problem. Note: By definition the conditions of $S_{u=2^n-(n+1)\ to\ u=2^n-2} = 0$ are always satisfied. This dissertation would like to point out that the discussion regarding the use of closed-form solutions is exclusively limited to instances when the pavement cost function is used as the characteristic function for the least core LP model. Further, as a rule to avoid precision errors associated with solvers, it is a better practice to round up all numbers to the nearest integer when formulating the modified least core HCA problem. The least core LP model modified based on the observance of the above four outcomes is shown below:

$$\max\ t_n$$

Subject to

$$\sum_{i \in N} x_i = \mathrm{C(N)}$$

$$\sum_{i \in S} x_i + t_n \leq C^*(S)\ \ for\ all\ S \in N$$

$$\sum_{i \in S} x_i + t_j = C^*(S)\ \ for\ all\ S \in F_j \quad j\ \in\ \{1, \dots n\ -\ 1\}$$

$$x_i,\ t_n,\ t_j \geq 0 \text{ for all } i \in N$$

The genesis behind the development of the closed-form solutions was to obtain the nucleolus without the use of simplex algorithm needed to solve a linear program. However, this dissertation was able to circumvent the use of simplex only when the special case of least core LP model came about upon solving an HCA problem. With a slight modification to the conditions for using closed-form solutions, conceptually, the approach presented here, changes the right hand sides *C(S)* of the least core LP model to $C^*(S)$ in order to force the special case to be exhibited by the newer least core LP model. Thereby, rendering the solving of the new updated least core LP model as redundant and in the process validating the use of closed-form solution. However, it was observed that when the cost of coalitions for *C(S)* and *C(N)* containing *n-1* and *n* vehicle classes increases considerably, then the closed-form solution approach cannot be used. The closed solution approach has certain important characteristics that bear mentioning:

1. The nucleolus will always accord coalitions containing *n-1* vehicle classes with the lowest savings.
2. The conditions of using closed-form solutions that govern its usage are independent of the type of characteristic function defining costs of coalitions *C(S)* and *C(N)*.

As an illustration for the closed-form approach, consider an HCA problem with characteristic function defining cost of coalitions as $C(\emptyset) = 0$, $C(1) = 5$, $C(2) = 5$, $C(3) = 5$, $C(12) = 12$, $C(13) = 14$, $C(23) = 18$, and $C(123) = 20$. First, the right hand sides or the cost of coalitions $C(1)$, $C(2)$, $C(3)$, $C(12)$, $C(13)$, $C(23)$ and $C(123)$ are substituted into the conditions (IV-14) - (IV-19). The satisfaction of these conditions are reported in Table IV.17.

Table IV.7: Conditions for using closed-form solutions to modify costs of coalitions

	C(S)	Conditions $(2^n + n - 1)$		Value	Status
C(1)	5	S_1	$C(1) - x_1^* - t_1^* > 0$	0.3333	Satisfied
C(2)	5	S_2	$C(2) - x_2^* - t_1^* > 0$	-3.66667	Violated
C(3)	5	S_3	$C(3) - x_3^* - t_1^* > 0$	-5.66667	Violated
C(12)	12	S_4	$C(12) - (x_1^* + x_2^*) - t_1^* = 0$	0	Satisfied
C(13)	14	S_5	$C(12) - (x_1^* + x_3^*) - t_1^* = 0$	0	Satisfied
C(23)	18	S_6	$C(13) - (x_2^* + x_3^*) - t_1^* = 0$	0	Satisfied
C(123)	20	S_7	$t_1^* > 0$	1.3333	Satisfied
		S_8	$x_1^* > 0$	3.3333	Satisfied
		S_9	$x_2^* > 0$	7.3333	Satisfied
		S_{10}	$x_3^* > 0$	9.3333	Satisfied

It can be checked that the HCA problem has encountered Outcome 2. The approach recommends that cost of coalition *C(2)* be increased by *3.6666* and *C(3)* by *5.66667.* Next, all numbers are rounded up to the nearest integer. The new costs of coalition become, $C^*(1) = 5$, $\boldsymbol{C^*(2) = C(2) + 3.6666 = 8.666 \approx 9}$,$\boldsymbol{C^*(3) \ = \ C(3) + 5.66667 = 10.666 \approx 11}$, $C^*(12) = 12$, $C^*(13) = 14$, $C^*(23) = 18$,and $C(123) = 20$. Now, the closed-form solutions are used to obtain the following allocation:

$$x_1^* = \frac{1}{3}(12) + \frac{1}{3}(14) + \frac{1}{3}(20) - \frac{2}{3}(18) = 3.3333$$

$$x_2^* = \frac{1}{3}(12) + \frac{1}{3}(18) + \frac{1}{3}(20) - \frac{2}{3}(14) = 7.3333$$

$$x_3^* = \frac{1}{3}(14) + \frac{1}{3}(18) + \frac{1}{3}(20) - \frac{2}{3}(12) = 9.3333$$

$$t_1^* = \frac{1}{3}(12) + \frac{1}{3}(14) + \frac{1}{3}(18) - \frac{2}{3}(20) = 1.3333$$

It can be verified that LINDO optimization software yields the same solution.

Limitations of the Closed-form Solutions for HCA

The closed-form solution along with conditions of its usage relied on proving uniqueness of one optimal basic feasible solution to the 1^{st} least core LP model to confirm whether the closed-form solution was indeed the nucleolus. This optimal solution was characterized by coalitions containing *n* and *n-1* vehicle classes. However, this optimal solution represented only one possibility that merited further investigation as it was seen that it had a high propensity of occurrence when the right hand sides where obtained from the pavement cost function. The limitation of this postulation is that based on a given set of costs of coalitions *C(S)* without any particular restriction on the type of characteristic function being used, there could be other unique solutions to the least core LP model defined by a different set of coalitions containing different number of vehicle classes. This dissertation did not explore those scenarios because those unique solutions were obtained outside the purview of using pavement cost function as the characteristic function. The difference between the closed-form solution presented in this dissertation and other scholarly work are listed below:

1. Closed-form solutions are developed specific to cooperative games based HCA that satisfy convexity conditions.
2. Closed-form solutions can only be used when each and every one of the $2^n + n - 1$ necessary conditions are satisfied.

The dissertation acknowledges that the practical implementation of the closed-form solutions are quite constrained. Typical use of closed-form solutions in any n-vehicle class HCA problem with $2^n - 1$ constraints would be predicated on satisfying all $2^n + n - 1$ conditions. If a practitioner were to implement the closed-form solution in a 12 vehicle class HCA problem, one would need to check 4,107 conditions before making a determination whether the use of closed-form solution would yield the nucleolus. From a pessimistic viewpoint, it could be reasoned that using the sequential LP approach to get the nucleolus would be faster than checking each one of the 4,107 conditions, wherein, violation of even one condition would make closed-form solutions invalid for computation of the nucleolus. Also, the very act of verifying 4,107 conditions adds to the computational complexity of obtaining the nucleolus. Furthermore, the satisfaction of all 4,107 by itself could be argued to be a rare occurrence. From an optimistic viewpoint, if all conditions were to be satisfied, the practitioner would save upon significant computational time involved in solving a corresponding 4,095 constraints associated with a 12 vehicle class least core LP model.

CHAPTER V

COMPARATIVE ANALYSIS

This section compares the proposed methodology to two traditional methods known as the incremental method, proportional method, and three game-theoretic based methods known as the Shapley value method, generalized method and discrete aumann-shapley value method. The methods presented in this section are by no means an exhaustive list of all available methods used by state DOTs and FHWA for the purposes of highway cost allocation. However, this dissertation has chosen to conduct a comparative analysis on a sample of methods symbolic of distinct differences in the way the notion of fairness is being enforced among them. A special emphasis will be placed on the criterion of fairness being used to allocate costs among different vehicle classes. First, a brief description of procedures being considered are provided below:

Traditional Methods

1. *Traditional Incremental Method*: Vehicle classes are sequentially included and costs are calculated as groups are enlarged. At each step, the additional cost is charged to the last included. Allocations depend on the order of vehicle classes. A generalization of this method is the Shapley value.
2. *Proportional Method*: Total costs are proportionally distributed among vehicle classes according to a specified measure. The cost allocator could be vehicle miles of travel (VMT), equivalent single axle loads (ESAL) or passenger car equivalent (PCE).

Non-Traditional Methods

3. *Shapley value*: Assuming vehicle classes as players partaking in the grand coalition (comprising of all vehicle classes). Shapley value is the sum of the average marginal cost contribution made by each vehicle class in all possible sequences they can be introduced into the grand coalition. For any given HCA problem, there are $n!$ ways or $n!$ sequences of introducing vehicle classes to form the grand coalition.
4. *Generalized Method*: The procedure is an application of linear programming to coalitional game theory, where properties of marginality, rationality and completeness are satisfied as constraints. This method ensures that every vehicle class will be attributed a lower cost in the grand coalition, in relation to participation in other smaller coalitions (groups of vehicle

classes). The costs of coalitions *C(S)* and *C(N)* are estimated using the pavement cost function.

5. *Discrete Aumann-Shapley value*: The total cost of the grand coalition is divided into cost per ESAL (pavement thickness) and cost per traffic-lane (highway capacity). The number of lanes needed by each vehicle class is obtained by conducting a Shapley value allocation of lanes among vehicle classes in the grand coalition. Knowing number of ESALs required by vehicle classes as data and lane requirements from Shapley value allocation, the final cost attributed to a vehicle class is the summation of traffic loading cost, estimated by multiplication of cost per ESAL with number of ESALs needed by vehicle classes and the traffic capacity cost, estimated by multiplication of cost per lane with the number of lanes needed by respective vehicle classes.

All methods will be evaluated in the context of previously described fairness properties of highway cost allocation. These properties are completeness; which states that highway costs are entirely paid by participating vehicle classes, rationality; which states that by joining the grand coalition, vehicle classes do not pay more than they would if they chose to be part of any smaller coalition of vehicle classes and marginality; stating that vehicle classes are charged at least enough to cover their marginal costs. All five HCA procedures were used to allocate the same total cost of the highway facility. The data set used in the comparative analysis is not comprehensive and is used for illustrative purpose only. In order to keep the emphasis on the merits and demerits of the proposed approach in comparison to existing methods of highway cost allocation, this chapter has consolidated all the pertinent input data used for conducting the comparison into Table V.1. Appendix A contains back calculations and additional information regarding the following:

1. The calculation of aumann-shapley values of C_e and C_l using rewritten versions of compact form has been provided.
2. The calculation of number of lanes using a relationship from HCM that uses traffic volume data of respective vehicle classes measured in Average Daily Traffic (ADT).
3. Calculation of cumulative ESALs expected to be applied by each vehicle class over a two-year design period along with assumptions on certain traffic characteristics.
4. A detailed description of integrating the data set into the least core LP model and the five LPs solved in order to compute the nucleolus using the sequential LP approach.

Table V.1: Input data for comparative analysis

DATA INPUT

	LEF	ADT
VC1	0.00834	15,158
VC2	0.00277	10,158
VC3	0.08299	5,500
VC4	0.03148	2,900

$Cost\ per\ ESAL\ (C_e) = \$0.18910892$

$Cost\ per\ Lane\ (C_l) = \$89{,}950.09047$

					CHARACTERISTIC FUNCTION		
					Pavement Cost Function (Fixed Lanes)		Discrete A-S Value (Variable Lanes)
Coalition	E_S	ESALs	L_S		$Y_{FP} = L_N\ (58{,}432 + 66.86 * E^{0.5})$		$(E_s * C_e + L_s * C_l)$
{1}	E_1	90,000	$L_{\{1\}}$	6	C(1)	$470,940	$556,720
{2}	E_2	20,000	$L_{\{2\}}$	4	C(2)	$407,325	$363,583
{3}	E_3	325,000	$L_{\{3\}}$	2	C(3)	$579,288	$241,361
{4}	E_4	65,000	$L_{\{4\}}$	1	C(4)	$452,868	$102,242
{1,2}	$E_1 + E_2$	110,000	$L_{\{1,2\}} = max(L_1, L_2)$	6	C(12)	$483,642	$560,503
{1,3}	$E_1 + E_3$	415,000	$L_{\{1,3\}} = max(L_1, L_3)$	6	C(13)	$609,021	$618,181
{1,4}	$E_1 + E_4$	155,000	$L_{\{1,4\}} = max(L_1, L_4)$	6	C(14)	$508,529	$569,012
{2,3}	$E_2 + E_3$	345,000	$L_{\{2,3\}} = max(L_2, L_3)$	4	C(23)	$586,220	$425,043
{2,4}	$E_2 + E_4$	85,000	$L_{\{2,4\}} = max(L_2, L_4)$	4	C(24)	$467,549	$375,875
{3,4}	$E_3 + E_4$	390,000	$L_{\{3,4\}} = max(L_3, L_4)$	2	C(34)	$601,116	$253,653
{1,2,3}	$E_1 + E_2 + E_3$	435,000	$L_{\{1,2,3\}} = max(L_1, L_2, L_3)$	6	C(123)	$615,175	$621,963
{1,2,4}	$E_1 + E_2 + E_4$	175,000	$L_{\{1,2,4\}} = max(L_1, L_2, L_4)$	6	C(124)	$518,409	$572,795
{1,3,4}	$E_1 + E_3 + E_4$	480,000	$L_{\{1,3,4\}} = max(L_1, L_3, L_4)$	6	C(134)	$628,524	$630,473
{2,3,4}	$E_2 + E_3 + E_4$	410,000	$L_{\{2,3,4\}} = max(L_2, L_3, L_4)$	4	C(234)	$607,460	$437,335
{1,2,3,4}	$E_1 + E_2 + E_3 + E_4$	500,000	$L_{\{1,2,3,4\}} = max(L_1, L_2, L_3, L_4)$	6	C(1234)	$634,255	$634,255

The Traditional Incremental Method

This method of allocation relies strictly upon the sequence in which vehicle classes are introduced for determination of incremental cost necessitated by them. Changing the order of introducing the vehicle classes, results in a completely different allocation, thereby, making the procedure order dependent. As a convention for the scenario being considered in this section, vehicle class with the largest ESAL factor is assumed to have the heaviest axle-weight and the vehicle class with the smallest ESAL factor has the lightest axle-weight. Therefore, the order of vehicle classes in the increasing order of axle-weight would be II, I, IV, III or 2-1-4-3. The incremental cost allocated to each vehicle class is shown below:

Vehicle Class II $= C(2) - 0 = \$407{,}325 - 0 = \$407{,}325$

Vehicle Class I $= C(12) - C(2) = \$483{,}642 - \$407{,}325 = \$76{,}317$

Vehicle Class IV $= C(124) - C(12) = \$518{,}409 - \$483{,}642 = \$34{,}767$

Vehicle Class III $= C(1234) - C(124) = \$634{,}255 - \$518{,}409 = \$115{,}846$

Furthermore, it can be noted that Vehicle Class II with a minimal pavement thickness requirement of 20,000 ESALs has been assigned the highest cost responsibility of $407,325. On the other hand, Vehicle Class III with the highest pavement thickness requirement of 325,000 ESALs has been assigned the 2nd highest cost responsibility of $115,846. This allocation brings to light an important unfairness associated with this procedure's ability to minimize the negative effects of economies of scale. Vehicle classes I, II and IV have already paid for 175,000 ESALs out of 325,000 ESALs needed by vehicle III, leaving vehicle class III to only pay for the final increment comprising of 150,000 ESALs. This is a known drawback of traditional incremental method where heavier vehicle classes gain the benefits of incremental thickness already paid for by the lighter vehicle classes. Perhaps, a better approach would be to introduce the vehicle classes from heavy-to-low axle weight or perhaps, writing out all possible sequences vehicle classes could be introduced and then taking the average of all the corresponding allocations generated in each of those sequences.

Rationality is satisfied by the allocation generated by the traditional incremental method for the example presented in this section. However, many coalitions were at the precipice of violation of rationality. The incentives for three coalitions to disengage from the grand coalition were found to be very high. These instances are cited below:

i. The cost of coalition *C(2)* for vehicle class II is $407,325, which is equal to the actual amount allocated to Vehicle Class II.
ii. The combined cost allocated to vehicle class I and vehicle class II is $483,642, which is equal to the cost vehicle class I and vehicle class II would have paid to be part of a smaller coalition {1,2} amounting to $483,642.
iii. The combined cost allocated to Vehicle Class I, Vehicle Class II and Vehicle Class IV is $518,409, which is equal to the cost Vehicle Class I, Vehicle Class II and Vehicle Class IV would have paid to be in a smaller coalition {1,2,4} amounting to $518,409.

The Proportional Method

The proportional method used in this section allocates total cost of the highway facility proportionally based on 18,000lb ESALs. The proportional method is by far the simplest and expedient way of allocating cost among vehicle classes. As a matter of fact, its simplicity has in many ways contributed to its widespread adoption in several highway cost allocation studies. The cost allocated to each vehicle class using ESAL as the proportionality factor is shown below:

$$\text{Vehicle Class I} = \$634{,}255 * \left(\frac{90{,}000}{500{,}000}\right) = \$114{,}166$$

$$\text{Vehicle Class II} = \$634{,}255 * \left(\frac{20{,}000}{500{,}000}\right) = \$25{,}370$$

$$\text{Vehicle Class III} = \$634{,}255 * \left(\frac{325{,}000}{500{,}000}\right) = \$412{,}266$$

$$\text{Vehicle Class IV} = \$634{,}255 * \left(\frac{65{,}000}{500{,}000}\right) = \$82{,}453$$

This method allows for any selection of a proportionality factor that can precisely quantify highway system usage by vehicle classes. For instance, if one were to use the Average Daily Traffic (ADT) as a measure of highway use, Vehicle Class I would be allocated a higher share of the cost responsibility of the total cost due to its high daily traffic counts. Therefore, any allocation produced by this method is heavily dependent on the proportionality factor chosen.

Property of marginality can be checked by calculating the marginal cost that each vehicle class must pay in accordance with cost associated with the formation of a coalition. Taking the difference between the cost of grand coalition and the cost of coalition not containing that specific vehicle class gives the marginal cost for providing highway service to an additional vehicle class. The values of cost of coalitions $C(1234), C(234), C(134), C(124)$ and $C(123)$ are obtained from the pavement cost function as shown in Table V.1.

$$\text{Vehicle Class I} = \mathrm{C}(1234) - \mathrm{C}(234) = \$634{,}255 - \$607{,}460 = \$26{,}795$$

$$\text{Vehicle Class II} = \mathrm{C}(1234) - \mathrm{C}(134) = \$634{,}255 - \$628{,}524 = \$5{,}731$$

$$\text{Vehicle Class III} = \mathrm{C}(1234) - \mathrm{C}(124) = \$634{,}255 - \$518{,}409 = \$115{,}846$$

$$\text{Vehicle Class IV} = \mathrm{C}(1234) - \mathrm{C}(123) = \$634{,}255 - \$615{,}175 = \$19{,}080$$

It can be seen that the cost allocation produced by the proportional method ensures that vehicle classes I, II, III and IV pay their minimal marginal cost. There are two important observations that mask the weakness of the proportional method.

i. The total cost of $645,255 is divided among vehicle classes with a percentage split of 18%, 4%, 65% and 13% among vehicle classes I, II, III and IV respectively. This split is obtained from the relative distribution of ESALs among vehicle classes for the example in this section.
ii. However, if the total cost of $645,255 were divided among vehicle classes with a percentage split of 40%, 30%, 10% and 20% among the four vehicle classes. Assuming, the split has been obtained from a sample proportionality factor, cost allocation would come out to be $253,702, $190,278, $63,426 and $126,851. It can be seen that Vehicle Class III would pay less than its marginal cost.

Therefore, the proportional method does not guarantee that all vehicle classes would pay their marginal cost of using the highway facility. The procedure also fails to consider rationality. The allocation produced by the proportional method in this example does in fact satisfy rationality but the procedure has no mechanism for enforcing this property. For instance, using a sample allocation of $253,702, $190,278, $63,426 and $126,851 attributed to four vehicle classes based on 40/30/10/20 split, will lead to violation of rationality for coalition $\{1,2,4\}$. The combined cost of Vehicle Class I, Vehicle Class II, Vehicle Class IV would come out to be $570,830, which is more than the cost these members would have paid to be part of a smaller coalition {1,2,4} amounting to $518,409. Albeit, the strength of the proportional method lies in its ease-of-use, the application of this method is not the best means of enforcing fairness.

The Shapley value Method

The Shapley value considers the incremental cost of additional thickness needed to support vehicle classes by calculating the same incremental cost in all possible ways the vehicle classes could be introduced to form the grand coalition. For the problem discussed in this section, Shapley value calculates the cost allocation for all 4! or 24 sequences and then determines the average marginal cost contribution made by each vehicle class in all sequences. Certainly, by doing so, the procedure diminishes the inherent unfairness associated with choosing the allocations produced from only one sequence. Estimating the costs of coalitions *C(S)* and *C(N)* from pavement cost function, marginal cost contribution made by each vehicle class in all sequences is shown in Table V.2 whose Shapley value allocation are found to be:

$\text{Vehicle Class I} \quad = \$146{,}845$

$\text{Vehicle Class II} \;\; = \$107{,}988$

$\text{Vehicle Class III} = \$245, 172$

$\text{Vehicle Class IV} = \$134{,}250$

Shapley value satisfies marginality as it can be seen that the allocations are clearly more than the minimum marginal cost of $26,795, $5,731, $115,846 and $19,080 for vehicle classes I, II, III and IV. This approach is fairer than the incremental method whose allocations relied upon a single sequence of low-to-heavy axle-weight based design increments. Interestingly, the allocation obtained for traditional method is in fact one of the permutations of the Shapley value namely 2-4-1-3, which is the 8th sequence in Table V.2. The satisfaction of the marginality property can be seen in Table V.2.

1. The allocation of Vehicle Class I as seen in 3rd column of all permutations does not fall below its minimum marginal cost of $26,795.

2. The allocation of Vehicle Class II as seen in 4th column in all permutations does not fall below the minimum marginal cost of $5,731.
3. The allocation of Vehicle Class III as seen in 5th column in all permutations does not fall below the minimum marginal cost of $115,846.
4. The allocation of Vehicle Class IV as seen in the 6th column in all permutation does not fall below the minimum marginal cost of $19,080.

Despite a compelling mathematical structure of this solution concept, the satisfaction of rationality is not always guaranteed. However, in most cases of HCA, Shapley value does satisfy rationality as long as the costs of coalitions adheres to the principle of demand monotonicity.

Table V.2: Shapley value for four-vehicle class HCA problem

	Sequences	I	II	III	IV
1	1234	$470,940	$12,702	$131,533	$19,080
2	1243	$470,940	$12,702	$115,846	$34,768
3	1324	$470,940	$6,154	$138,081	$19,080
4	1342	$470,940	$5,731	$138,081	$19,503
5	1423	$470,940	$9,880	$115,846	$37,589
6	1432	$470,940	$5,731	$119,995	$37,589
7	2134	$76,317	$407,325	$131,533	$19,080
8	2143	$76,317	$407,325	$115,846	$34,768
9	2314	$28,955	$407,325	$178,896	$19,080
10	2341	$26,795	$407,325	$178,896	$21,240
11	2413	$50,860	$407,325	$115,846	$60,225
12	2431	$26,795	$407,325	$139,911	$60,225
13	3124	$29,733	$6,154	$579,288	$19,080
14	3142	$29,733	$5,731	$579,288	$19,503
15	3214	$28,955	$6,932	$579,288	$19,080
16	3241	$26,795	$6,932	$579,288	$21,240
17	3412	$27,407	$5,731	$579,288	$21,828
18	3421	$26,795	$6,343	$579,288	$21,828
19	4123	$55,661	$9,880	$115,846	$452,868
20	4132	$55,661	$5,731	$119,995	$452,868
21	4213	$50,860	$14,681	$115,846	$452,868
22	4231	$26,795	$14,681	$139,911	$452,868
23	4312	$27,407	$5,731	$148,248	$452,868
24	4321	$26,795	$6,343	$148,248	$452,868
	Min (Marginal Cost)	$26,795	$5,731	$115,846	$19,080
	Shapley value	$146,845	$107,988	$245,172	$134,250

Meaning, if the cost of coalitions *C(S)* and *C(N)* satisfies the convexity conditions as outlined in Chapter III, then it has been demonstrated by Shapley (1971), that the allocation generated, will always be in the core. This is similar to stating that rationality will be satisfied. Since the example presented in this section adheres to the convexity conditions, it can be verified that the solution is in the core and so satisfies rationality. From the perspective of fairness, this procedure satisfies marginality, completeness but does not guarantee rationality.

The Generalized Method

In contrast to incremental method and Shapley value, this approach is independent of the order in which vehicle classes are introduced to form the grand coalition. The procedure also eliminates the computational complexity in determining marginal cost contributions of every vehicle class in all $n!$ sequences of a HCA problem. A convincing point in support of using generalized method comes from the strategic use of constraints in a linear programming model to guarantee an allocation that will satisfy the properties of rationality, marginality and completeness. The generalized method facilitates the allocation for the example presented in this section using an LP model with 4 decision variables and 14 coalitional rationality constraints.

Objective Function: $\max t$

Constraints:

$$
\begin{aligned}
x_1 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_1}}\right)\\
x_2 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_2}}\right)\\
x_3 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_3}}\right)\\
x_4 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_4}}\right)\\
x_1 + x_2 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_1} + \boldsymbol{E_2}}\right)\\
x_1 + x_3 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_1} + \boldsymbol{E_3}}\right)\\
x_1 + x_4 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_1} + \boldsymbol{E_4}}\right)\\
x_2 + x_3 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_2} + \boldsymbol{E_3}}\right)\\
x_2 + x_4 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_2} + \boldsymbol{E_4}}\right)\\
x_3 + x_4 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_3} + \boldsymbol{E_4}}\right)\\
x_1 + x_2 + x_3 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_1} + \boldsymbol{E_2} + \boldsymbol{E_3}}\right)\\
x_1 + x_2 + x_4 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_1} + \boldsymbol{E_2} + \boldsymbol{E_4}}\right)\\
x_1 + x_3 + x_4 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_1} + \boldsymbol{E_3} + \boldsymbol{E_4}}\right)\\
x_2 + x_3 + x_4 + t &\leq \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_2} + \boldsymbol{E_3} + \boldsymbol{E_4}}\right)\\
x_1 + x_2 + x_3 + x_4 &= \boldsymbol{L_N}\left(\boldsymbol{a} + \boldsymbol{b}\sqrt{\boldsymbol{E_1} + \boldsymbol{E_2} + \boldsymbol{E_3} + \boldsymbol{E_4}}\right)
\end{aligned}
$$

The right hand sides for the LP model shown above are directly obtained from the pavement cost function. The values of L_N , a, b, E_1, E_2, E_3, and E_4 are obtained from Table V.1. The optimal unique solution for the above LP model was found to be the following with $t =$ $116,700.5.

Vehicle Class I = $143,495.75

Vehicle Class II = $122,431.75

Vehicle Class III = $232,546.75

Vehicle Class IV = $135,780.75

It can be verified that the allocation of generalized method is well over the minimum marginal cost for all the vehicle classes using the highway facility. More importantly, this approach adds another dimension to the notion of fairness called savings. In addition to enforcing the properties of marginality, rationality and completeness, this method generates an allocation that recognizes the savings achieved for sharing the highway with all other vehicle classes in the grand coalition.

Table V.3: Generalized Method: Savings

Generalized Method: New Fairness Dimension			Savings
x_1 = $143,495.75		x_3 = $232,546.75	$\left(1 - \frac{\sum_{i \in S} x_i}{C(S)}\right) * 100$
x_2 = $122,431.75		x_4 = $135,780.75	
C(S)		C(S)− $\sum_{i \in S} x_i$	
C(1)	$470,940	$327,444	69.53%
C(2)	$407,325	$284,893	69.94%
C(3)	$579,288	$346,742	59.86%
C(4)	$452,868	$317,087	70.02%
C(12)	$483,642	$217,714	45.02%
C(13)	$609,021	$232,979	38.25%
C(14)	$508,529	$229,252	45.08%
C(23)	$586,220	$231,242	39.45%
C(24)	$467,549	$209,337	44.77%
C(34)	$601,116	$232,789	38.73%
C(123)	$615,175	$116,701	18.97%
C(124)	$518,409	$116,701	22.51%
C(134)	$628,524	$116,701	18.57%
C(234)	$607,460	$116,700	19.21%

Savings can be perceived as incentives, that can be given to vehicle classes for choosing the most economically feasible option. The objective function of the LP ensures that savings obtained for all coalitions are maximized. Table V.3 shows savings obtained by each coalition. As seen in Table V.3, coalition {4} has achieved a 70.02% savings for participating in the grand coalition. Also, coalition {1} and {2} have both achieved a proportional savings of 69.53% and 69.54%. Coalition {3} has achieved a savings of 59.86% for participating in the grand coalition.

Highway Capacity Requirements: The traffic capacity analysis performed for this example suggests that in order for the highway facility to provide a satisfactory level of service, Vehicle Class I would have a requirement of 6 lanes, Vehicle Class II would have a requirement of 4 lanes, Vehicle Class III would have a requirement of 2 lanes and Vehicle Class IV would have a requirement of 1 lane. These were calculated based on their corresponding data on traffic volumes. In spite of this fact, all previously discussed methods namely, the incremental method, proportional method, Shapley value and generalized method operated under the assumption that all six lanes of the highway facility are being used equally by all vehicle classes. This supposition contradicts the guidelines established in the highway capacity manual, wherein the number of lanes for a roadway facility is dependent upon the traffic volumes measured in average daily traffic. Equal lane assumption implies that all vehicle classes have similar ADT values. Generally, lighter vehicle classes such as passenger cars have high ADT values in comparison to heavier vehicle classes such as trucks. In the example presented in this section, vehicle classes I, II, III and IV all have varying ADT values as shown in Table V.1. However, this fact had no bearing on the allocation generated by traditional methods and non-traditional methods including Shapley value and generalized method. It is well known among traffic engineers, that heavy vehicle classes such as trucks constitute a small portion of the total daily traffic volume level for a specific time period. Therefore, the assumption of equal lane utilization is incompatible with the principles of roadway capacity design. Next, the discrete A-S value method and the proposed methodology, both of which address the issue of variable capacity requirements of vehicle classes in HCA are discussed.

The Discrete Aumann-Shapley Value Method

The discrete Aumann-Shapley value is a non-optimization HCA methodology that divides the total highway cost into pavement and capacity costs. This is the first known procedure that measures the highway system usage in terms both, traffic lanes and pavement thickness requirements. In the example presented in this section, the pavement cost was measured in cost per ESAL and was determined to be \$0.1891 per ESAL (C_e) and capacity costs was found to be \$89,950.0905 per lane (C_l). Foundationally, the dissimilarity between Shapley value and A-S value is that the A-S value considers vehicle classes consuming two distinct goods, namely lanes and ESALs, while Shapley value considers only one type of good being consumed by vehicle classes namely, ESALs, serving as a measure of pavement damage with an implicit fixed lane assumption. The discrete A-S values of cost per ESAL and cost per Lane is the average of the sum of all the marginal cost contributions made by each unit of ESAL and each unit of Lane in all possible ways the two units can be arranged in sequences. Furthermore, the cost per lane estimate is a new roadway capacity measure, adding to other available library of capacity related measures based on-Vehicle Miles Traveled (VMT), Average Daily Traffic and Passenger Car Equivalent weighted VMTs.

The final allocation for this procedure is obtained in two steps. First, traffic loading costs are obtained by multiplying the cost per ESAL with the respective number of ESALs for each vehicle class. Second, lane costs are assigned to vehicle classes by multiplication of cost per lane with the Shapley value of lanes for each vehicle class. The lane requirements for single vehicle coalitions were determined to be 6 lanes, 4 lanes, 2 lanes and 1 lane and the number of lanes to accommodate vehicle classes in all other coalitions have been determined using relationship (III-23) and (III-24). The Shapley value of lanes are shown in Table V.4.

The Shapley values for lanes are the following:

Vehicle Class I $= 3.58333$ Lanes

Vehicle Class II $= 1.58333$ Lanes

Vehicle Class III $= 0.58333$ Lanes

Vehicle Class IV $= 0.25000$ Lanes

The discrete Aumann-Shapley value allocates cost using the following relationship:

$$x_i = E_i C_e + C_l L_i$$

$$x_1 = 90{,}000\ ESAL * \frac{\$0.1891}{\text{ESAL}} + \frac{\$89{,}950.0905}{Lane} * 3.58333\ Lanes\ = \$339{,}341$$

$$x_2 = 20{,}000\ ESAL * \frac{\$0.1891}{\text{ESAL}} + \frac{\$89{,}950.0905}{Lane} * 1.58333\ Lanes = \$146{,}203$$

$$x_3 = 325{,}000\ ESAL * \frac{\$0.1891}{\text{ESAL}} + \frac{\$89{,}950.0905}{Lane} * 0.58333\ Lanes = \$113{,}931$$

$$x_4 = 65{,}000\ ESAL * \frac{\$0.1891}{\text{ESAL}} + \frac{\$89{,}950.0905}{Lane} * 0.250000\ Lanes = \$34{,}780$$

The above allocation is highly depictive of not only the requirements for ESALs by vehicle classes I, II, III and IV, but has also taken due consideration of the number of lanes needed by these individual vehicle classes. In light of this, it can be seen that Vehicle Class I bears substantial portion of the cost responsibility, as it has the highest lane requirements among all vehicle classes.

Table V.4: Shapley value for lanes in a four-vehicle class HCA problem

	Permutation	I	II	III	IV
1	1234	6	0	0	0
2	1243	6	0	0	0
3	1324	6	0	0	0
4	1342	6	0	0	0
5	1423	6	0	0	0
6	1432	6	0	0	0
7	2134	2	4	0	0
8	2143	2	4	0	0
9	2314	2	4	0	0
10	2341	2	4	0	0
11	2413	2	4	0	0
12	2431	2	4	0	0
13	3124	4	0	2	0
14	3142	4	0	2	0
15	3214	2	2	2	0
16	3241	2	2	2	0
17	3412	4	0	2	0
18	3421	2	2	2	0
19	4123	5	0	0	1
20	4132	5	0	0	1
21	4213	2	3	0	1
22	4231	2	3	0	1
23	4312	4	0	1	1
24	4321	2	2	1	1
Shapley value		3.58333	1.58333	0.58333	0.25000

More significantly, one notices a dramatic shift in cost distribution among vehicle classes upon comparing the results of this procedure with incremental method, proportional method, Shapley value and the generalized method. This procedure does not satisfy rationality. For the example presented in this section, two instances can be cited where the property of rationality has been violated:

i. First, the combined cost allocated to Vehicle Class I and Vehicle Class II comes out to be $485,544. This is more than the cost of coalition Vehicle Class I and Vehicle Class II would have paid to be in a smaller coalition {1,2} amounting to $483,642.
ii. Second, the combined cost allocated to Vehicle Class I, Vehicle Class II and Vehicle Class IV comes out to be $520,324. This is more than the cost of coalition Vehicle Class I, Vehicle Class II and Vehicle Class IV would have paid to be in the smaller coalition {1,2,4} amounting to $518,409.

In conclusion, the A-S value method guarantees satisfaction of marginality and completeness, but not rationality.

The Proposed Methodology: Least Core LP Model with A-S Values

The proposed methodology represents the state-of-the-art in the area of cooperative games based HCA that combines the capabilities offered by three existing solutions concepts into one cost allocation procedure. These concepts along with value they add are listed below:

1. *Least Core LP Model*: Guaranteed enforcement of marginality, rationality and completeness using constraints.
2. *Discrete Aumann-Shapley value*: Separation of the total highway cost into capacity costs and pavement thickness costs in the problem of highway cost allocation.
3. *The Nucleolus*: The implementation of a fair division concept that involves solving a sequence of linear programs in order to maximize the savings of all coalitions in a non-decreasing order as a means of attributing the largest savings possible to the least satisfied coalition.

The proposed methodology facilitates the allocation for the example in this section by using an LP model containing 4 decision variables and 15 coalitional rationality constraints where the right hand sides are obtained from discrete A-S values as discussed in Chapter III. The complete least core LP model formulation is shown below:

$$\max t_1$$

Subject to

$$x_1 + t_1 \leq \boldsymbol{E_1 C_e} + \boldsymbol{L_{\{1\}} C_l} \qquad (Y_1)$$
$$x_2 + t_1 \leq \boldsymbol{E_2 C_e} + \boldsymbol{L_{\{2\}} C_l} \qquad (Y_2)$$
$$x_3 + t_1 \leq \boldsymbol{E_3 C_e} + \boldsymbol{L_{\{3\}} C_l} \qquad (Y_3)$$
$$x_4 + t_1 \leq \boldsymbol{E_4 C_e} + \boldsymbol{L_{\{4\}} C_l} \qquad (Y_4)$$
$$x_1 + x_2 + t_1 \leq (\boldsymbol{E_1} + \boldsymbol{E_2})\boldsymbol{C_e} + \boldsymbol{L_{\{12\}} C_l} \qquad (Y_{12})$$
$$x_1 + x_3 + t_1 \leq (\boldsymbol{E_1} + \boldsymbol{E_3})\boldsymbol{C_e} + \boldsymbol{L_{\{13\}} C_l} \qquad (Y_{13})$$
$$x_1 + x_4 + t_1 \leq (\boldsymbol{E_1} + \boldsymbol{E_4})\boldsymbol{C_e} + \boldsymbol{L_{\{14\}} C_l} \qquad (Y_{14})$$
$$x_2 + x_3 + t_1 \leq (\boldsymbol{E_2} + \boldsymbol{E_3})\boldsymbol{C_e} + \boldsymbol{L_{\{23\}} C_l} \qquad (Y_{23})$$
$$x_2 + x_4 + t_1 \leq (\boldsymbol{E_2} + \boldsymbol{E_4})\boldsymbol{C_e} + \boldsymbol{L_{\{24\}} C_l} \qquad (Y_{24})$$
$$x_3 + x_4 + t_1 \leq (\boldsymbol{E_3} + \boldsymbol{E_4})\boldsymbol{C_e} + \boldsymbol{L_{\{34\}} C_l} \qquad (Y_{34})$$
$$x_1 + x_2 + x_3 + t_1 \leq (\boldsymbol{E_1} + \boldsymbol{E_2} + \boldsymbol{E_3})\boldsymbol{C_e} + \boldsymbol{L_{\{123\}} C_l} \qquad (Y_{123})$$
$$x_1 + x_2 + x_4 + t_1 \leq (\boldsymbol{E_1} + \boldsymbol{E_2} + \boldsymbol{E_4})\boldsymbol{C_e} + \boldsymbol{L_{\{124\}} C_l} \qquad (Y_{124})$$
$$x_1 + x_3 + x_4 + t_1 \leq (\boldsymbol{E_1} + \boldsymbol{E_3} + \boldsymbol{E_4})\boldsymbol{C_e} + \boldsymbol{L_{\{134\}} C_l} \qquad (Y_{134})$$
$$x_2 + x_3 + x_4 + t_1 \leq (\boldsymbol{E_2} + \boldsymbol{E_3} + \boldsymbol{E_4})\boldsymbol{C_e} + \boldsymbol{L_{\{234\}} C_l} \qquad (Y_{234})$$
$$x_1 + x_2 + x_3 + x_4 = (\boldsymbol{E_1} + \boldsymbol{E_2} + \boldsymbol{E_3} + \boldsymbol{E_4})\boldsymbol{C_e} + \boldsymbol{L_{\{1234\}} C_l} \qquad (Y_{1234})$$

The values of $a, b, E_1, E_2, E_3, E_4, C_e$ and C_l are obtained from Table V.1. The nucleolus for the above least core LP model was obtained after solving five LPs in a sequence as documented in Table V.6. The last LP in the sequence resulted in the following allocation:

Vehicle Class I $=$ \$320,601.5

Vehicle Class II $=$ \$127,463.5

Vehicle Class III $=$ \$128,923.0

Vehicle Class IV $=$ \$57,267.0

The marginal cost calculation will be different for the least core model because the value of *C(S)* and *C(N)* are obtained from a new characteristic function as shown in TableV.1.

Vehicle Class I $= \mathrm{C}(1234) - \mathrm{C}(234) = \$634{,}255 - \$437{,}335 = \$196{,}920$

Vehicle Class II $= \mathrm{C}(1234) - \mathrm{C}(134) = \$634{,}255 - \$630{,}473 = \$3{,}782$

$$\text{Vehicle Class III} = \text{C}(1234) - \text{C}(124) = \$634{,}255 - \$572{,}795 = \$61{,}460$$

$$\text{Vehicle Class IV} = \text{C}(1234) - \text{C}(123) = \$634{,}255 - \$621{,}963 = \$12{,}292$$

The allocation produced by proposed methodology are clearly more than their respective marginal costs. This procedure obtains similar A-S values of $0.1891per ESAL and capacity costs of $89,950.0905 per lane as the data set used were similar for all procedures that are being compared in this section. Next, the responsiveness of the new characteristic function to estimate cost of coalition *C(S)* and *C(N)* used in the proposed methodology is assessed against characteristic function used in generalized method where the point of divergence is in their respective abilities to variate costs of coalitions depending upon the number of lanes used by vehicle classes. Assuming that L_S and L_N are the lane requirements of coalition *S* and grand coalition *N*, the generalized method by design assumes all vehicle classes have similar requirements for number of lanes.

Table V.5: Responsiveness towards variable capacity requirements

Variable Roadway Capacity Requirements: Integration of Highway Capacity Costs Optimization Methods of Highway Cost Allocation					
		Generalized Method		The Proposed Methodology	
Coalitions		Fixed	RHS	Variable	RHS
	ESAL	Lane	Non-Responsive to Lanes		Responsive to Lanes
C(1)	90,000	6 Lanes	$470,940	6 Lanes	$556,720
C(2)	20,000	6 Lanes	$407,325	4 Lanes	$363,583
C(3)	325,000	6 Lanes	$579,288	2 Lanes	$241,361
C(4)	65,000	6 Lanes	$452,868	1 Lanes	$102,242
C(12)	110,000	6 Lanes	$483,642	6 Lanes	$560,503
C(13)	415,000	6 Lanes	$609,021	6 Lanes	$618,181
C(14)	155,000	6 Lanes	$508,529	6 Lanes	$569,012
C(23)	345,000	6 Lanes	$586,220	4 Lanes	$425,043
C(24)	85,000	6 Lanes	$467,549	4 Lanes	$375,875
C(34)	390,000	6 Lanes	$601,116	2 Lanes	$253,653
C(123)	435,000	6 Lanes	$615,175	6 Lanes	$621,963
C(124)	175,000	6 Lanes	$518,409	6 Lanes	$572,795
C(134)	480,000	6 Lanes	$628,524	6 Lanes	$630,473
C(234)	410,000	6 Lanes	$607,460	4 Lanes	$437,335
C(1234)	500,000	6 Lanes	$634,255	6 Lanes	$634,255

This is enforced by means of a constant L_N denoting the total number of lanes in the grand coalition. As a consequence of this, cost of coalitions are estimated with an assumption that each coalition has lane requirements equal to the grand coalition ($L_S = L_N$). The concern is that such an assumption results in an over-design where certain type of vehicle classes such as trucks don't end up using all the lanes in a given highway facility but nonetheless, are obligated to pay for highway capacity they do not utilize. At its foundation, generalized method uses the least core LP model and obtains the cost of coalition represented in RHS using the pavement cost function, where constant L_N forces all coalitions S to have the same lanes as N . On the other hand, the proposed methodology uses the same least core LP model as its foundation and obtains the costs of coalitions represented in RHS using a new discrete A-S value based characteristic function containing constant value of L_S for each coalition S. This approach empowers the least core LP model to increases and decreases the cost of the hypothetical highway facility depending upon the number of lanes needed by each coalition S. Table V.5 illustrates this in the following way:

1. The cost of coalition *C(4)* or the stand alone cost of Vehicle Class IV with 65,000 ESALs and an assumption of L_N =6 lanes results in a cost of $452,868 as obtained using pavement cost function. However, when using the A-S value to estimate the same cost of coalition, its value decreases from $452,868 to $102,242 due to the recognition that coalition {4} has a requirement for 1 lane only.
2. Similarly, the cost of coalition *C(3)* or the stand alone cost for vehicle class III with 325,000 ESALs and an assumption 6 lanes results in cost of $579,288. However, when using the A-S value to estimate the cost of coalition *C(3)*, the cost for forming this coalition decreases from $579,288 to $241,361 due to the recognition that coalition {3} has a requirement for only 2 lanes.

Similar reductions in cost of coalitions for $C(1), C(2), C(23), C(24), C(34)$ and $C(234)$ can be observed in Table V.5 for each of these coalitions when their need for lanes L_S is lower than the maximum number of lanes in the highway facility L_N. The grey colored rows in Table V.5 shows the corresponding cost of coalitions whose values have decreased when factoring the reduced requirement for number of lanes.

One of the fundamental enhancements made to the least core LP model was the use of nucleolus solution concept as paradigm for allocation of highway costs among vehicle classes. The division of cost generated by this concept is determined by successively maximizing the savings of all coalitions, conducted via solving a sequence of linear programs. The algorithmic process of obtaining the nucleolus involves using multiple LPs to engage in a systemic refinement of allocation that improves the savings of all the least satisfied coalitions when savings are arranged in increasing order of their magnitude. The novelty of this approach is that it uses the duality of linear programming to detect these least satisfied coalitions. Further, the concept eliminates any possibility of coalitions receiving an inferior or sub-optimal allocation. The procedure as a natural step uses the least core LP model as the starting point and terminates to a unique allocation leaving no prospect for multiple optimal solutions at the end of the process. The latter considered a recurrent issue in least core LP formulation. The process of determining the nucleolus for the example in this section has been tabulated in Table V.6 and summarized below:

1st Linear Program (n=1, j=0)

Upon solving the 1st LP, the optimal solution was found to be x = (\$421,796, \$48,757, \$106,435, \$57,267). Defining set F_1= $\{\{4\},\{1,2,3\}\}$ as values of dual variables Y_4 and Y_{123} are positive non-zero. Also, the 1st LP model was found to have contain multiple optimal solutions. Coalitions $\{4\}$ and $\{1,2,3\}$ have received an optimal savings of $t_1 =$ \$44,975. This means that savings of coalition $\{4\}$ cannot be increased without decreasing the savings of either Vehicle class I, II or III. Conversely, savings for coalition $\{1,2,3\}$ cannot be increased without decreasing savings of Vehicle class IV. Therefore, savings for both the coalitions in the 2nd LP are fixed by setting $t_1 = t_2$ and subtracting *C(4)* from t_1 and *C(123)* from t_1. The corresponding inequality of the coalitional rationality constraint associated with $\{4\}$ and $\{1,2,3\}$ are converted into equality. Next step is to proceed with the maximization of the second set of least satisfied coalitions with t_2 as the objective function for the 2nd LP model.

2nd Linear Program (n=2, j=1)

Upon solving the 2nd LP, the optimal solution was found to be x = (\$376,820.0, \$71,245.0, \$128,923, \$57,267). Defining set F_2= $\{\{1,2,4\}, \{3,4\}\}$ as values of dual variables Y_{124} and Y_{34} are greater than zero. Further, the 2nd LP model was found to have multiple optimal solutions.

Table V.6: Computation of the nucleolus with sequential LP approach

COMPUTATION OF NUCLEOLUS							
INPUT				C(S)	$t(X_n^*, S) = C(S) - \sum_{i \in S} x_i$	Sequences of LP	$x_i = (x_1, x_2, x_3, x_4)$
1	C(1)	\$556,720		C(123)	$t_1 = \$44,975$	1st LP: Multiple $F_1 = \{Y_{123}, Y_4\}$	(\$421,796.0, \$48,757.0, \$106,435, \$57,267)
2	C(2)	\$363,583		C(4)	$t_1 = \$44,975$		
3	C(3)	\$241,361		C(124)	$t_2 = \$67,463$	2nd LP: Multiple $F_2 = \{Y_{124}, Y_{34}\}$	(\$376, 820.0, \$71, 245.0, \$128,923, \$57,267)
4	C(4)	\$102,242		C(34)	$t_2 = \$67,463$		
5	C(12)	\$560,503		C(12)	$t_3 = \$112,438$	3rd LP: Multiple $F_3 = \{Y_{12}\}$	(\$331,845.0, \$116, 220.0, \$128,923, \$57,267)
6	C(13)	\$618,181		C(3)	$t_4 = \$112,438$	4th LP: Multiple $F_4 = \{Y_3\}$	(\$331, 845.0, \$116, 220.0, \$128,923 , \$57,267)
7	C(14)	\$569,012		C(134)	$t_5 = \$123,681.5$	5th LP: Unique $F_5 = \{Y_{134}, Y_{234}\}$ $(Termination\ Point)$	**(\$320, 601.5, \$127, 463.5, \$128,923, \$57,267)**
8	C(23)	\$425,043		C(234)	$t_5 = \$123,681.5$		
9	C(24)	\$375,875		C(13)	\$168,656.50		(\$320, 601.5, \$127, 463.5, \$128,923, \$57,267)
10	C(34)	\$253,653		C(23)	\$168,656.50		(\$320, 601.5, \$127, 463.5, \$128,923, \$57,267)
11	C(123)	\$621,963		C(14)	\$191,143.50		(\$320, 601.5, \$127, 463.5, \$128,923, \$57,267)
12	C(124)	\$572,795		C(24)	\$191,144.50		(\$320, 601.5, \$127, 463.5, \$128,923, \$57,267)
13	C(134)	\$630,473		C(1)	\$236,118.50		(\$320, 601.5, \$127, 463.5, \$128,923, \$57,267)
14	C(234)	\$437,335		C(2)	\$236,119.50		(\$320, 601.5, \$127, 463.5, \$128,923, \$57,267)

Coalitions {1,2,4} and {3,4} have received an optimal savings of $t_2 =$ \$67,463. This means that savings of coalition {1,2,4} cannot be increased without decreasing the savings of either vehicle class III or IV. Conversely, savings for coalition {3,4} cannot be increased without decreasing savings of vehicle classes I, II or IV. Therefore, savings for both the coalitions in the 3rd LP are fixed by setting $t_2 = t_3$ and subtracting *C(124)* from t_2 and *C(34)* from t_2. The corresponding inequality of the coalitional rationality constraint associated with {1,2,4} and {3,4} are converted into equality. Next step is to proceed with the maximization of the third set of least satisfied coalitions with t_3 as the objective function for the 3rd LP model.

3st Linear Program (n=3, j=2)

Upon solving the 3rd LP, the optimal solution was found to be *x* = (\$331,845.0, \$116,220.0, \$128,923, \$57,267). Defining set F_3= {{1,2}} as value of dual variable Y_{12} is greater than zero. In addition, the 3rd LP model was found to have multiple optimal solutions. Coalition {1,2} has received an optimal savings of $t_3 =$\$112,438. The savings of coalition {1,2} cannot be increased any further. Therefore, savings for coalition {1,2} is fixed in the 4th LP by setting $t_3 = t_4$ and subtracting *C(12)* from t_3 . The corresponding inequality of the coalitional rationality constraint associated with {1,2} is converted into equality. Next step is to proceed with the maximization of the fourth set of least satisfied coalitions with t_4 as the objective function for the 4th LP model.

4th Linear Program (n=4, j=3)

Upon solving the 4th LP, the optimal solution was found to be *x* = (\$331,845.0, \$116, 220.0, \$128,923, \$57,267). Defining set F_4= {{3}} due to value of dual variable Y_3 being greater than zero. Also, the 4th LP model was found to have multiple of optimal solutions. Coalition {3} has received an optimal savings of $t_4 =$\$112,438. The savings of coalition {3} cannot be increased any further. Therefore, saving for coalition {3} is fixed in the 5th LP by setting $t_4 = t_5$ and subtracting *C(3)* from t_3. The corresponding inequality of the coalitional constraint associated with {3} is converted into equality. Next step is to proceed with the maximization of the fifth set of least satisfied coalitions with t_5 as the objective function for the 5th LP model.

5th LP Linear Program (n=5, j=4)

Upon solving the 5th LP, the optimal solution is found to be *x* = (\$320, 601.5, \$127, 463.5, \$128,923, \$57,267). Defining set F_5= {{1,3,4}, {2,3,4}} due to values of dual variables Y_{134} and

Y_{234} being greater than zero. Coalitions {1,3,4} and {2,3,4} have received an optimal savings of $t_5 =$ \$123,681.5. This means that savings of coalition {1,3,4} cannot be increased without decreasing the savings of either Vehicle Class II, III or IV. Conversely, savings for coalition {2,3,4} cannot be increased without decreasing savings of vehicle classes I, III or IV. Therefore, savings for both coalition are fixed in the 6th LP by setting $t_5 = t_6$ and subtracting *C(134)* from t_5 and *C(234)* from t_5. The corresponding inequality of the coalitional rationality constraint associated with {1,3,4} and {2,3,4} are converted into equality. Next step is to proceed with the maximization of the sixth set of least satisfied coalitions with t_6 as the objective function for the 6th LP model. However, since the optimal solution of this LP is unique, the savings for all other remaining coalitions can no longer be improved even if one continues with the sequential approach. Therefore, the LP sequential approach is terminated and the solution x_N^* = (\$320, 601.5, \$127, 463.5, \$128,923, \$57,267) is confirmed to be the nucleolus. The strength of this allocation is further cemented in the fact that it is also lexicographically unique. In Appendix A, it has been shown that the average of all optimal basic feasible solutions of the 1st LP revealed the nucleolus. Case in point, the nucleolus was obtained in the 5th LP as described in this section.

The lexicographical comparison described here is outlined in Table V.7. The lexicographical order $L(x)$ is defined as the vector *k* = $(k_1, \dots\dots, k_m)$ of savings *t*(x_N^*,*S*) arranged in the increasing order. Next, the lexicographical order defining $L(\boldsymbol{\mathcal{H}})$ as the vector $z = (z_1, \dots\dots, z_m)$ of savings *t(x,S)* arranged in the increasing order where *x* = (\$331, 845.0, \$116, 220.0, \$128,923 , \$57,267); a solution to the 4th LP in the sequence. An element to element analysis of the two allocation is conducted as following:

$$k_1 = z_1 = \$44{,}975$$

$$k_2 = z_2 = \$44{,}975$$

$$k_3 = z_3 = \$67{,}463$$

$$k_4 = z_4 = \$67{,}463$$

$$k_5 = z_5 = \$112{,}438$$

$$k_6 = z_6 = \$112{,}438$$

$$k_7 = \$123{,}681.5 > z_7 = \$112{,}438$$

Table V.7: Lexicographical Comparisons

LEXICOGRAPHICAL ORDERING						
		$L(X_N^*)$		$L(\boldsymbol{\mathcal{H}})$		
		$X_n^* \in \sum_{i=1}^{n} x_i = C(N)\}$		$\mathrm{x} \in \sum_{i=1}^{n} x_i = C(N)\}$		
	($320, 601.5, $127, 463.5, $128,923, $57,267)			($331, 845.0, $116, 220.0, $128,923 , $57,267)		
k		$t(X_n^*, S) = C(S) - \sum_{i \in S} x_i$		$t(x, S) = C(S) - \sum_{i \in S} x_i$		z
k_1	C(123)	$44,975	=	$44,975	C(123)	z_1
k_2	C(4)	$44,975	=	$44,975	C(4)	z_2
k_3	C(124)	$67,463	=	$67,463	C(124)	z_3
k_4	C(34)	$67,463	=	$67,463	C(34)	z_4
k_5	C(12)	$112,438	=	$112,438	C(12)	z_5
k_6	C(3)	$112,438	=	$112,438	C(3)	z_6
k_7	**C(134)**	**$123,681.5**	≥	**$112,438**	**C(134)**	z_7
k_8	C(234)	$123,681.5		$134,925	C(234)	z_8
k_9	C(13)	$168,656.50		$157,413	C(13)	z_9
k_{10}	C(23)	$168,656.50		$179,900	C(23)	z_{10}
k_{11}	C(14)	$191,143.50		$179,900	C(14)	z_{11}
k_{12}	C(24)	$191,144.50		$202,388	C(24)	z_{12}
k_{13}	C(1)	$236,118.50		$224,875	C(1)	z_{13}
k_{14}	C(2)	$236,119.50		$247,363	C(2)	z_{14}

Since, $k_1 = z_1, k_2 = z_2, k_3 = z_3, k_4 = z_4, k_5 = z_5, k_6 = z_6$ and , $k_7 > z_7$. It is concluded that k = $(k_1, \ldots\ldots, k_m)$ is lexographically greater than a vector $z = (z_1, \ldots\ldots, z_m)$. Therefore , it is stated that $k \geq_{lex} z$ or $k \succcurlyeq z$ and so $L(X_N^*) \succcurlyeq_{lex} L(\boldsymbol{\mathcal{H}})$. The nucleolus is one and only allocation that maximizes $L(X_n^*)$ in the lexicographic order for any allocation that satisfies completeness. The savings achieved by all coalitions for the allocation defined by the nucleolus are reported in Table V.8. Coalitions {2} and coalitions {2,4} receive the largest percentage savings of 64.94% and 50.85% for participating in the grand coalition. More importantly, if one were to closely inspect coalitions {4} and {1,2,3}, both of these coalitions have received the same savings value of $44,975 but the proportional savings for one of them is 43.99% and for the other is 7.23%. This is because the savings measured by the nucleolus is the absolute difference between the cost of coalition *C(S)* and the allocated amount for participating in the grand coalition. This representation of the savings in the nucleolus does not capture the savings in relation to participation in smaller coalitions. For such an instance, the proportional nucleolus can be used that maximizes the relative savings of all coalitions with respect to not participating in smaller coalitions.

Table V.8: Nucleolus Solution Concept: The Standard of Fairness

The Least Core LP Model: Nucleolus Solution			$\left(1 - \frac{\sum_{i \in S} x_i}{C(S)}\right) * 100$
$x_1 = \$320{,}601.5$		$x_3 = \$128{,}923$	
$x_2 = \$127{,}463.5$		$x_4 = \$57{,}267$	
C(S)		*t(x, S)* = C(S)$-\sum_{i \in S} x_i$	*Percent Savings*
C(1)	$556,720	$236,119	42.41%
C(2)	$363,583	$236,119	64.94%
C(3)	$241,361	$112,438	46.58%
C(4)	$102,242	$44,975	43.99%
C(12)	$560,503	$112,438	20.06%
C(13)	$618,181	$168,656	27.28%
C(14)	$569,012	$191,144	33.59%
C(23)	$425,043	$168,656	39.68%
C(24)	$375,875	$191,144	50.85%
C(34)	$253,653	$67,463	26.60%
C(123)	$621,963	$44,975	7.23%
C(124)	$572,795	$67,463	11.78%
C(134)	$630,473	$123,681	19.62%
C(234)	$437,335	$123,681	28.28%

Concluding Remarks on Comparative Analysis

As stated in the introduction, the primary objective of this dissertation was to develop a new method that divides the highway cost among vehicle classes by means of facilitating an integration of variable roadway capacity and pavement thickness requirements. A concerted effort was made to put into practice a fair-division concept that would make the allocation generated by proposed methodology more acceptable to the stakeholders. This section also did a head-to-head comparison of all the traditional and non-traditional approaches used to divide highway cost among vehicle classes and examined their respective abilities in satisfying the properties of marginality, rationality and completeness. The cost allocations generated by all procedures are documented in Table V.10 along with their respective assumptions on the number of lanes. In order to check whether the proposed methodology met the objectives it set out accomplish. The following observations were made:

1. *Sensitivity of cost allocation to Lanes and ESALs*: The most important result emerging out of the comparative analysis was that the discrete A-S value method and the proposed methodology overwhelmingly allocated large portions of the total highway cost to Vehicle Class I. At the same time, the cost allocations to vehicle classes with large number of ESALs did not decrease to a level warranting a statement of negligible impact. This seems to be reasonable result in the sense that the incremental method, proportional method, Shapley value and generalized method all had an assumption of fixed lane requirements where the pavement damage metric of 18,000lb ESAL was the mainspring behind assigning cost responsibility to Vehicle Classes I, II, III and IV. As evidenced, when variable lane requirements were taken into consideration, both A-S method and proposed methodology responded by shifting cost responsibility to Vehicle Class I due to its relatively high roadway capacity requirement of six lanes.
2. *Variable Capacity and Rationality*: The proposed methodology used the discrete aumann-Shapley value as the characteristic function and was shown to be the only method that accounts for variable capacity and pavement thickness requirements of different types of vehicles along with guaranteed satisfaction of marginality, rationality and completeness. This twofold feature is absent in all prevalent methods of highway cost allocation.

Table V.9: Response of cost allocation methods towards number of lanes

			Fixed Number of Lanes				Variable Number of Lanes		
VC	ESAL	L_i	Incremental Method	Proportional Method	Shapley Value	Generalized Method	L_i	Discrete A-S Value Method	Least Core with A-S Value
I	**90,000**	**6**	**$76,317**	**$114,166**	**$146,845**	**$143,495.75**	**6**	**$339,341**	**$320,601.5**
II	20,000	**6**	$407,325	$25,370	$107,988	$122,431.75	**4**	$146,203	$127,463.5
III	**325,000**	**6**	**$115,846**	**$412,266**	**$245,172**	**$232,546.75**	**2**	**$113,931**	**$128,923**
IV	65,000	**6**	$34,767	$82,453	$134,250	$135,780.75	**1**	$34,780	$57,267
	C(N)		$634,255	$634,255	$634,255	$634,255.00		$634,255	$634,255

3. *Lessening of costs of coalitions due to reduced number of lanes*: The use of A-S values on the right hand sides of the proposed methodology was shown to be highly effective in integration of pavement thickness and highway capacity requirements of vehicle classes. It was demonstrated in Table V.5 that a decrease in number of lanes needed by any given coalition led to a decrease in the cost of coalition for members belonging to that specific coalition.
4. *Nucleolus as a basis of fair division*: The proposed methodology produced an allocation that was not only consistent with the defined notions of fairness, but the allocation was an output of a sequential programming technique, that ensured, savings achieved by each and every coalition is maximized. The theoretical prominence of this approach in curtailing objections of coalitions by according savings to the least satisfied coalition first, will greatly facilitate recognition of this method as being fair to the stakeholders, whose acceptance and opinions are an important factor in the adoption of new methods for cost allocation by highway agencies.
5. *Mitigation of long-standing issues*: Several of the following shortcomings were seen in traditional methods and few of the non-traditional methods, but rather non-existent in proposed methodology:

    a. Order dependency
    b. Multitude of allocation factors generating different cost allocations
    c. Violation of rationality
    d. Cases where certain methodologies did not guaranteed satisfaction of marginality
    e. Assumption of fixed lanes in traditional methods and few of the non-traditional methods

The results of the comparative analysis were found to be in concordance with previous research studies conducted on the ability of classical cooperative game theory models to address structural anomalies of traditional methods. The comparative analysis also showed that using theory of coalition formation enforces the notion of fairness to a greater degree than traditional methods. Continued use of traditional methods by HCA practitioners should be predicated by the recognition of their limitations. With the Nucleolus as a basis of fair cost-division along with the use of discrete A-S to address the issue of variable capacity requirements, the methodology presented in this dissertation should be given due consideration in allocation of highway cost among vehicle classes in future state and federal HCA studies.

CHAPTER VI

HIGHWAY COST ALLOCATION DECISION SUPPORT SYSTEM

Program features

The purpose of the HCA decision support system is to serve an academic exercise within this dissertation for comparing cost allocation results generated by different HCA methodologies. The Microsoft Excel-based software would require more upgrades before it could be used in an actual HCA study. The software in **FILE 1** consists of the following HCA procedures:

1. Modified Incremental Method for Three Vehicle Class Only
2. Incremental Method
3. Proportional Method
4. Shapley value: Characteristic Function Based on Pavement Cost Function
5. The Discrete Aumann-Shapley Value Method
6. Generalized Method: Right Hand Sides Based on Pavement Cost Function
7. The Least Core LP Model: Right Hand Sides Based on Discrete A-S Value
8. Closed-form Solution
9. Shapley value: Characteristic Function Based on Discrete Aumann-Shapley value

The spreadsheet based software contains five modules that can determine cost allocations for VC3, VC4, VC5, VC6 and VC7. However, these modules can be expanded to solve more than seven-vehicle class problem by experienced excel users with prior experience with visual basic macros. As designed, the software will allow the user to divide total cost of grand coalition in a HCA problem comprising of 3, 4, 5, 6 and 7 number of vehicle classes. The software enables the user to enter the relevant data such as ESALs and Lanes for each vehicle class. The number of lanes and ESAL requirement for coalitions are obtained using special rules as discussed in Chapter III. A pavement cost function with $r = 0.5$ is then used to estimate costs of coalitions and also develop discrete A-S value based characteristic function. Few prominent features included in the Highway Cost Allocation software are listed below:

1. Ability to choose between the use of pavement cost function as the characteristic function or the discrete Aumann-Shapley value based characteristic function as described in this dissertation. The cost of coalitions could also be estimated independently

2. The option of calculating the number of lanes using the shapley value is introduced into the software for use in conjunction with the discrete Aumann-Shapley value based characteristic function.
3. Computation of the Nucleolus using a solution procedure that involves solving a sequence of LPs in order to arrive at a unique solution. At this time, the user would need to engage in the sequential LP approach on the LINDO software itself and import the data back into the excel file. Closed-form Solutions and the corresponding associated conditions that must be satisfied for it to be the nucleolus.

In order to use HCA procedures that are formulated based on Linear Programming, LINDO optimization software must be available on the computer system that is running the Excel-driven HCA decision support system.

Preview of HCA Spreadsheet Based Software

This section will briefly describe features associated with each spreadsheet of the module. The **FILE 1** can be found as an attachment to this dissertation. Figure VI.I shows the list of modules available to the user for the purposes of determining cost allocation for appropriate number of vehicle classes. Each module comprises of seven spreadsheets. The activation button needs to be clicked once prior to using the module. Navigation among spreadsheets can be done by either using colored tabs at the bottom of the screen or command buttons provided on the sheet

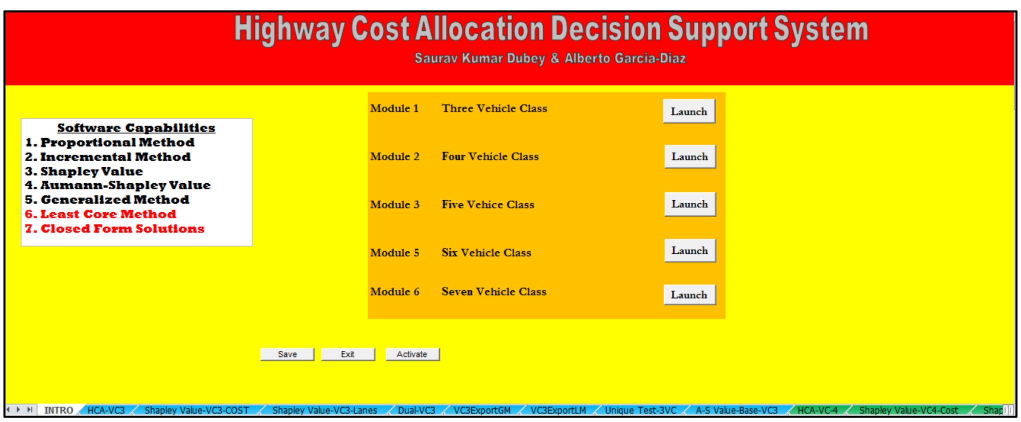

Figure VI.I: Main menu of the HCA software

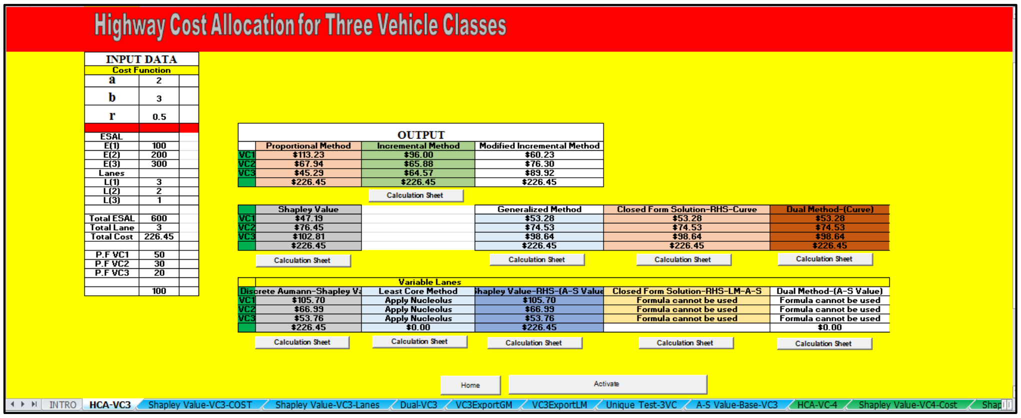

Figure VI.2: HCA Input /Output Page for three vehicle class

Figure VI.2 shows the spreadsheet that displays the input and the output on the same page. The calculations associated with each HCA method can be accessed by clicking on the command button titled "calculation sheet". The input data table would require the ESALs and Lanes for each vehicle class and values for parameters *a* and *b*. At this time, the software is only restricted to use of pavement cost function with $r = 0.5$.

Figure VI.3 shows the excel sheet in a module for three vehicle class being used for calculating the allocation acquired from the Shapley value, when cost of coalitions are obtained from pavement cost function and aumann-shapley value based characteristic function. The cell in this sheet are linked to the input/output sheet and hence are filled automatically based on initial data of the user. However, if necessary, the user can change the number in the cells as needed.

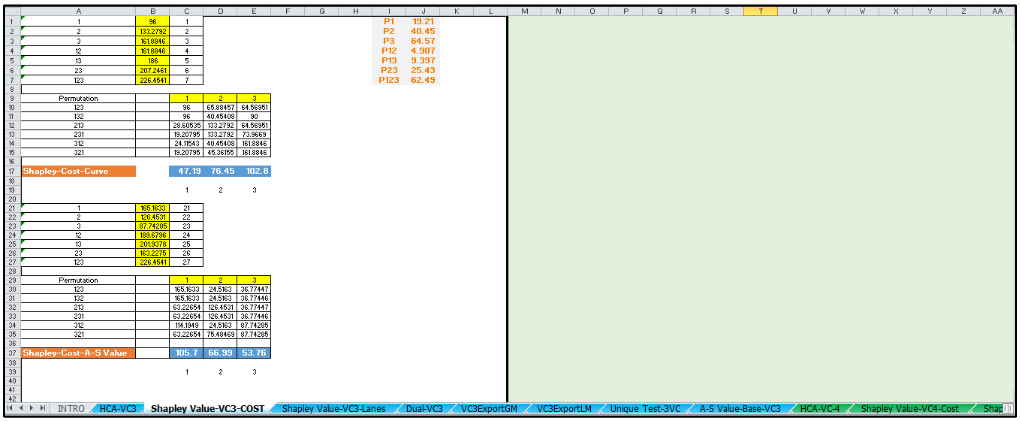

Figure VI.3: Shapley Value with Pavement Cost Function for three vehicle class

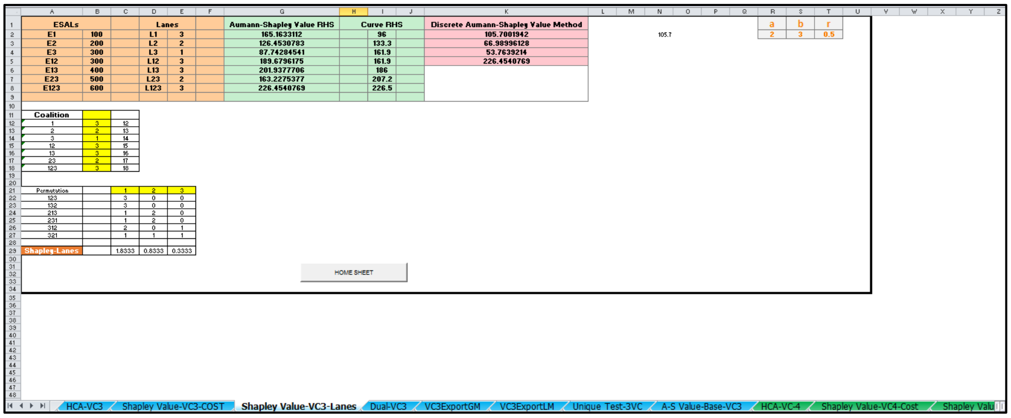

Figure VI.4: Shapley Value for Lanes for three vehicle class

Figure VI.4 shows the excel sheet that is used for calculating the Shapley value for lanes and also the results for the discrete aumann-shapley value method. The cells in this sheet are also linked to the input/output sheet and hence are filled automatically based on input data. However, the user is allowed to change data directly if required.

Figure VI.5 shows the excel sheet that is used for dividing highway expenditures among vehicle classes using least core LP model with RHS from aumann-shapley values. The excel file allows the user to export data to LINDO where sequential LP approach is carried out. Once sequential LP approach is completed, output of LINDO can be imported back into the excel file using a LINDO text file. The clear output button can be used to erase the previous data set.

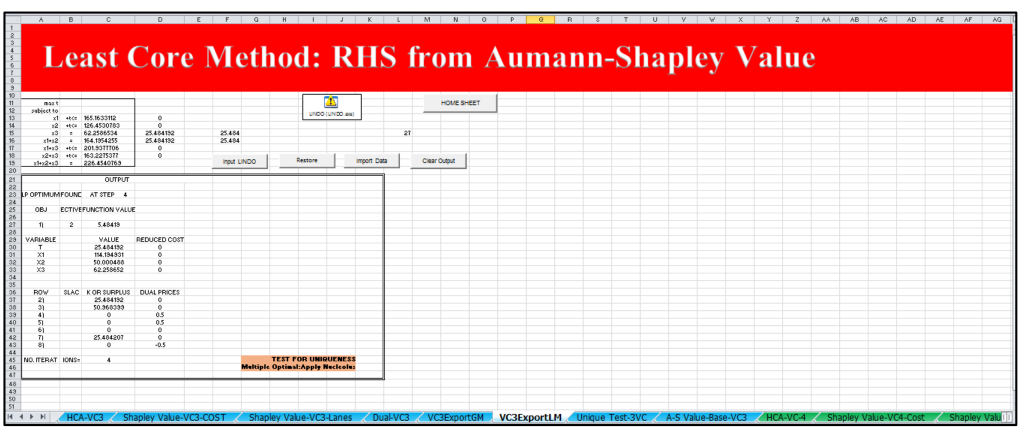

Figure VI.5: Least Core LP Model with Aumann-Shapley values

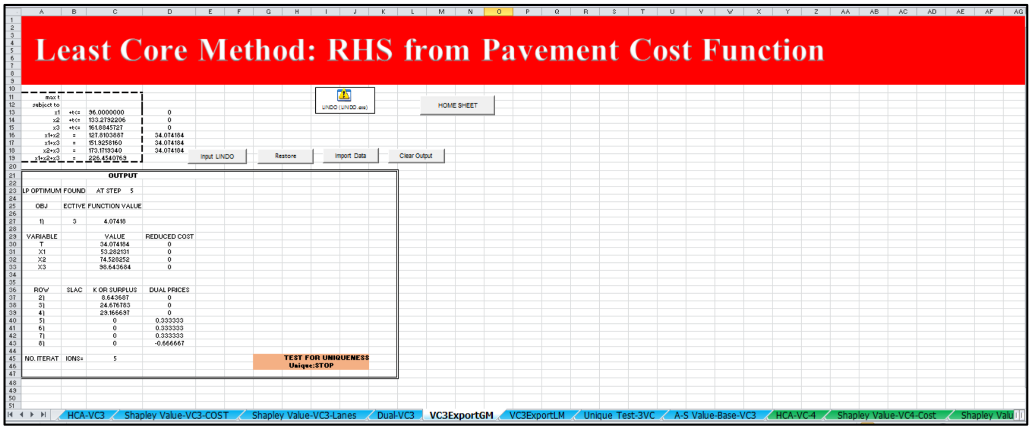

Figure VI.6: Least Core LP Model with Pavement Cost Function

Figure VI.6 shows the excel sheet that is used for dividing highway expenditures among vehicle classes using least core LP model with pavement cost function. The excel file allows the user to export data to LINDO where LP model is solved. After solving the LP model, output of LINDO can be imported back into the excel file by storing it in **FILE 2**. The process of solving a sequence of LPs and confirmation of uniqueness of the optimal solution must be carried out by examination of the optimal tableau of every LP that is being solved in the sequence.

Figure VI.7 shows the spreadsheet located in the module for four vehicle class HCA that checks whether each and every one of the 19 conditions for using the closed-form solution are satisfied. Post verification, the excel file will display a “Yes”, meaning all conditions are satisfied and closed -form solution can be used or “No”, meaning not all conditions are satisfied and closed-form solution cannot be used.

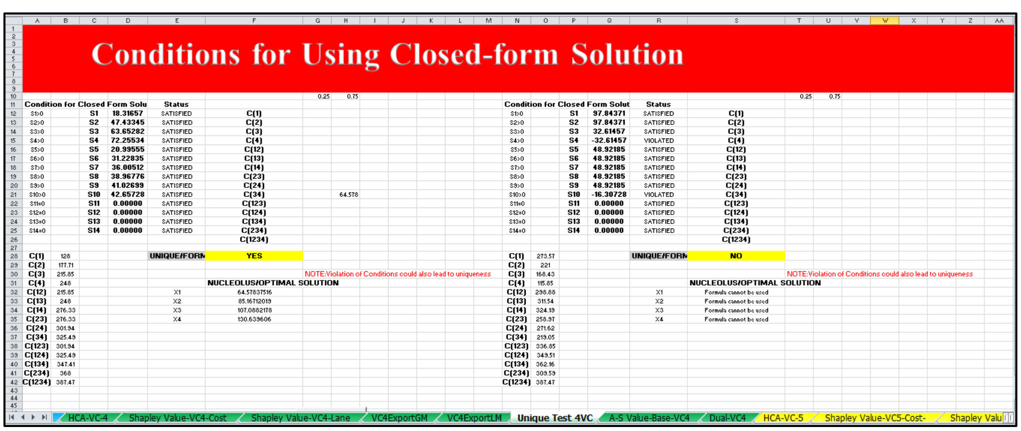

Figure VI.7: Conditions for using closed form solutions for four vehicle class

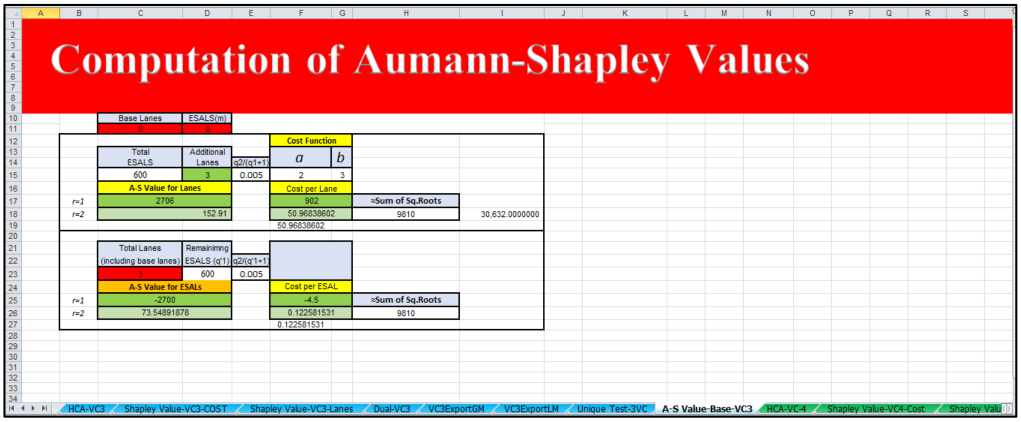

Figure VI.8: Computation of aumann-shapley values

Figure VI.8 shows the excel file used for calculating the Aumann-Shapley values of cost per ESAL and cost per Lane. This excel file requires the total number of ESALs and Lanes in the grand coalition which is automatically filled based on the data in the input/output page. Figure VI.9 shows a flowchart that illustrates the system flow diagram of the HCA software. More importantly, it shows that methods based on linear programming have an additional iterative process that involves checking whether the optimal solution to a LP model is unique or one of many optimal solutions. If multiple solutions exist, sequential LP approach must be initiated.

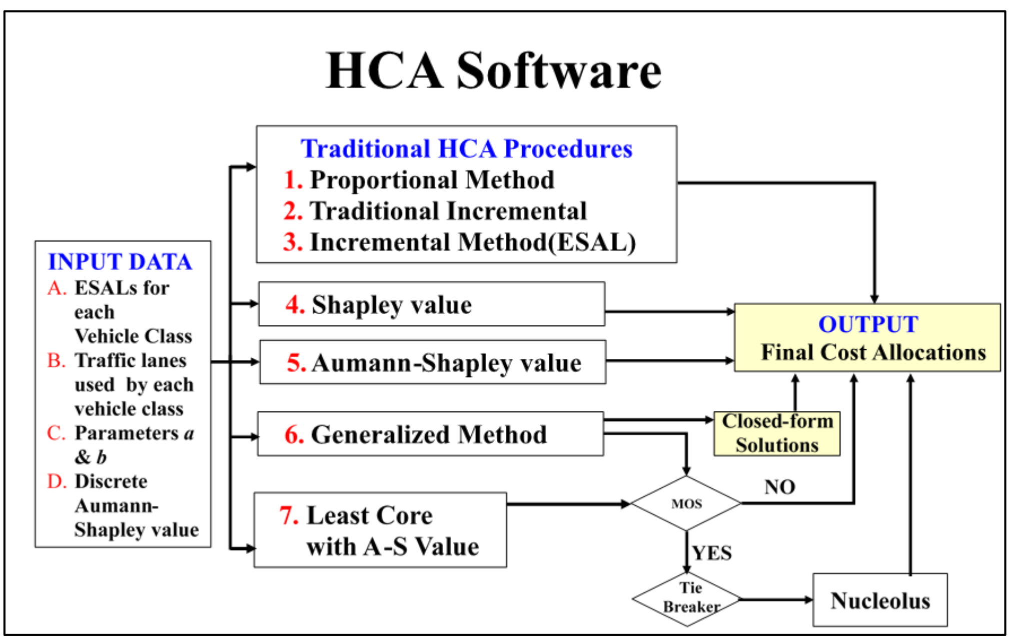

Figure VI.9: Flowchart for HCA Decision Support System

CHAPTER VII

SUMMARY, CONCLUSIONS AND RECOMMENDATION

Summary

Limited research has been performed in the field of HCA to pursue advancement of prevailing allocation procedures that would allocate highway expenditures among vehicle classes by taking into account two important highway design considerations namely, the number of lanes used, and pavement thickness requirements. The principle of cost-occasioned, which is usually synonymous with pavement damage, gives a partial picture of the cost being imposed on the highway facility. The results of this dissertation support the view that a high priority should be given to consideration of cost occasioned due to traffic capacity requirements among high volume road user groups. These groups impose sizable cost for building additional lanes on a transportation facility. Acknowledging this research gap, the central tenet of this dissertation has been to elaborate on the development of a new optimization based HCA methodology that is sensitive to traffic lane use and whose fairness is put into effect by invoking a classical solution concept from the principle of cooperative games known as the nucleolus. The methodology proposed in this dissertation enhanced an existing method in order to improve its capability to be reactive to the different demand for traffic lanes placed by vehicle classes on a given highway facility. The proposed methodology was devised based on the least core LP model with the goal of using a new characteristic function that would break down the total highway cost into two major components, each of which being a measure of highway system usage. These are listed below:

I. Pavement thickness requirements: Cost occasioned due to damage to the pavement.
II. Highway lane capacity: The number of lanes needed by vehicle classes.

The integration of the pavement and lane cost was made permissible using the concept of discrete Aumann-Shapley value. The A-S value concept was then used to develop a new characteristic function that assigned a fixed cost to each Lane and 18,000lb. equivalent single axle load unit in effect treating them as distinct quantities being used by vehicle classes. This function in turn was used to obtain the right hand sides of the least core LP model where the costs of forming coalitions among vehicle classes were allowed to vary depending upon the appropriate traffic capacity and traffic loading requirements of vehicle classes in respective coalitions. The study concluded that fairness in an HCA problem is best administered in the form of a LP model where properties of marginality, rationality and completeness represented as constraints promise an allocation within the envisioned notion of fairness. Such a mechanism of enforced fairness is not present in many traditional HCA methods. The dissertation developed the proposed methodology with a stated belief that road users categorized as vehicle classes, behave as rational drivers, who would like to pay the lowest cost for using the highway facility by forming a grand coalition with all vehicle classes. Modeling the HCA problem as such allows the practitioners to evaluate each allocation based on the savings that vehicle classes are given by foregoing the strategic alternatives of using exclusive highways. Measuring the strength of an allocation using this incentive based paradigm, expands the definition of fairness that traditional methods are unable to quantify.

The incentives in the proposed methodology were measured in terms of savings or excesses defined by the difference between the cost vehicle classes would have paid, had they hypothetically engaged in formation of smaller coalitions consisting of individual or groups of vehicle classes, and the amount they are being allocated to use the highway facility all together. The fairness of a single allocation (i.e nucleolus) was then expressed by means of a distribution of savings among all coalitions.

The operating principle of the new methodology was to ensure that such savings must be increased to the best extent possible for all coalitions in order to minimize any possible resistance from vehicle classes towards the objection of the proposed allocation. In this regard, a unique allocation that corresponded to a unique allotment of savings among the coalitions was obtained by solving multiple linear programs in a sequence that increased the savings of vehicle classes in the order of least satisfied coalition to the greatest satisfied coalition. In any given allocation, the least satisfied coalition possess the utmost inclination to object. Thereby, the nucleolus offers a solitary allocation, that is able to attribute the largest savings to the least satisfied coalition. Based on lexicographical comparisons, it has been shown that for a given HCA problem, nucleolus is the only allocation that is lexicographically greater than any other allocation whose highway costs are being divided. The dissertation concludes that drawing upon the concept of the nucleolus as a starting point for engaging in future comparison of allocations would transform the way fairness of allocations generated by several HCA methods are evaluated.

The dissertation has also shed some light on the relevance of choosing a methodology that must be based on a modeling framework that remain immune to significant changes in allocations when specific underlying assumptions are modified. Particularly, traditional methods such as incremental method and all its variations generate allocations that lead to substantial alteration of cost responsibility due to their dependence on a chosen sequence of introducing vehicle classes for designing cost increments. Similarly, discretionary based practice of selecting cost allocators in proportional methods enforces fairness to varying degrees of impartiality. In contrast, the proposed methodology rooted in cooperative games was demonstrated to be a stable approach that was neither dependent on sequences nor overly reliant on any specific cost allocator. The use of linear programming to obtain the nucleolus along with guaranteed satisfaction of properties of marginality, completeness and rationality provides a more scientific approach of allocating highway related cost among vehicle classes.

At a peripheral level, the dissertation also investigated plausible opportunities of improving the computational complexity of calculating the nucleolus in the least core LP. Generally, nucleolus is obtained by solving a series of linear programs. The optimal solution to these linear programs are obtained by using the simplex algorithm. Furthermore, when considering n number of vehicle classes, the number of constraints contained in the least core model increase in the order of magnitude $2^n - 1$. The multiple pathways identified in this dissertation for obtaining the nucleolus for least core LP model are classified into two categories namely, applied and dissertation methods, the latter being independent of simplex algorithm. These methods are listed below:

Applied Methods

1. The Least Core LP Model: Computation of the nucleolus using a sequential LP approach.
2. The Least Core LP Model: If solution to the least core model is unique, then it is the nucleolus, else, the nucleolus can be obtained by taking the average of all optimal basic feasible solutions.

Dissertation Methods

1. The Least Core LP Model: Computation of the nucleolus using closed-form solutions upon verification that the given costs of coalitions would exhibit a special case of least core defined by necessary conditions that must be met.
2. The Least Core LP Model: Computation of the nucleolus by engaging in the sequential LP approach from the dual. Once can only use this approach under the exclusive understanding that the dual of the dual can be verified to be the nucleolus.
3. The Least Core LP Model: Computation of the nucleolus using an approach based on the closed-form, that updates the cost of coalitions for certain coalitions, leading to a new least core LP model whose optimal solution (i.e. nucleolus) concurs with the closed-form solution. The implementation of this approach for attributing cost among vehicle classes must be contingent upon the appropriate justification for coalitions whose costs have been increased. However, it was found that in few cases, this approach cannot be used.

The dissertation acknowledges that the challenge in developing closed-form solutions for the nucleolus in the least core LP model comes from recognizing in advance, which coalitional rationality constraints will determine the nucleolus. The dissertation found that the way a characteristic function is recommended for an HCA problem provides an important insight into which coalitions are playing a critical role in the formation of the nucleolus. However, there are innumerable ways the characteristic function can be recommended for HCA, more often than not, without observable pattern. Therefore, in majority of the cases, computation of the nucleolus using the sequential LP approach is needed.

Conclusions

The research effort documented in this dissertation has brought together concepts from linear programming, cooperative game theory and pavement design in order to enhance the capability of an existing HCA methodology. Moreover, presented work resolved some of the computational problems and issues regarding multiple optimality associated with LP based cost allocation procedures. The important contributions of this dissertation to the area of highway cost allocation are listed below:

1. *An Enhanced Linear Programming Model Reactive to Variable Capacity Requirements*: This dissertation proposed an improvement to the existing HCA methodology known as the least core LP model. A new discrete aumann-shapley value based characteristic function was recommended to be placed on the right hand sides of the least core LP model that permitted the separation of highway cost into pavement and lanes costs. The use of least core LP model and discrete A-S value in aggregation facilitated the merging of variable roadway capacity and pavement thickness requirements of vehicle classes along with a guaranteed satisfaction of marginality, rationality and completeness.
2. *Computation of Nucleolus using a Sequential LP approach*: The dissertation addressed the issue of multiple optimal solutions in the least core LP model by recommending the application of the nucleolus solution concept. A sequential LP approach based on an established algorithm was used to compute the nucleolus for the HCA problem. The dissertation found the application of this concept beneficial, as it uses the complementary slackness conditions in order to progressively solve a sequence of LPs in order to converge to a unique solution termed the nucleolus. This unique allocation was shown be a standard bearer of fairness for dividing highway costs, as it produced an allocation that maximized the savings of the least satisfied coalition as much as possible in a non-decreasing order for all coalitions. The nucleolus was concluded to be a fair, rational and equitable allocation that would garner high level of confidence among the stakeholders of highway cost allocation than traditional methods and even to previous solution concepts used in cooperative games based HCA.
3. *Closed-form Solution for Special Case*: The dissertation concluded that when the right hand sides of the least core LP model are obtained using the pavement cost function, the LP model tends to behave in a predictive manner, wherein, it is tractable for the unique optimal solution to be expressed as a closed-form solution. This dissertation proposed a closed-form solution for situations when nucleolus of the allocation is uniquely defined by a pre-determined collection of constraints in the least core LP model. More specifically, in general, the dissertation found the nucleolus being defined by $n+1$ coalitions consisting of cost of coalitions $C(S)$ with $n-1$ vehicle classes and $C(N)$ consisting of n vehicle classes. Furthermore, necessary conditions for application of these closed-form solutions were also developed.

4. *Duality Based Highway Cost Allocation*: A nominal investigation was performed on whether the sequential LP approach could be applied from the dual of the least core LP model instead of the primal. The purpose of doing so was to ease the process of computing the nucleolus due to the advantage offered by the reduced number of constraints. The dissertation concluded that in situations when one can guarantee that the dual of the dual is in fact the nucleolus, only then the application of the sequential LP approach from the dual is reasonable. Otherwise, the sequential LP approach applied via the primal is the best way to obtain the nucleolus. As a foundation for a future study, an illustration was performed that demonstrated the possibility of obtaining the nucleolus by implementing the sequential LP approach from the dual. An observation was noted that if the dual model has multiple solutions that are non-degenerate, then the uniqueness of the primal can be confirmed. However, if the dual model has multiple solutions that are degenerate, then, uniqueness of the primal cannot be established with certainity.
5. *Development of normal equations for a pavement cost function with $r = 0.5$*: Based on regression analysis performed on several highway expenditure data sets in the Texas HCA study, a good model of relationship between number of ESAL applications and the cost occasioned in $/lane-mile was found to be a non-linear function with an exponential parameter of r being 0.5. In light of this, this dissertation derived normal equations from which the values for parameters a and b of pavement cost function can be determined without relying on a statistical software.
6. *Highway Cost Allocation Decision Support System*: As part of this dissertation, the procedures listed below were programmed into a Microsoft Excel spreadsheet-based software where calculation of certain parameters were done using macros developed from visual basic routines. The list of methods available on software is shown below:

   a) Modified Incremental Method for Three Vehicle Class Only
   b) Incremental Method
   c) Proportional Method
   d) Shapley value: Characteristic Function Based on Pavement Cost Function
   e) The Discrete Aumann-Shapley Value Method
   f) Generalized Method: Right Hand Sides Based on Pavement Cost Function
   g) The Least Core LP Model: Right Hand Sides Based on Discrete A-S Value

h) Closed-form Solution

i) Shapley value: Characteristic Function Based on Discrete Aumann-Shapley value

The VBA application has the capability to solve a three-vehicle class (VC3), four-vehicle class (VC4), five-vehicle class (VC5), six-vehicle class (VC6) and seven-vehicle class (VC7) HCA problem. In addition, the sequential LP approach of obtaining the nucleolus is accomplished by creating a LINDO – EXCEL interface, where the data can be readily imported and exported between the two software applications using an object text file.

Recommendations for Future Research

The section will outline three potential future research ideas that might be of interest to academic scholars, who would like to explore possible avenues of further broadening the scope of cooperative games based highway cost allocation. These ideas are briefly described below:

1. *Termination criterion for sequential LP approach from the Dual*: The dissertation demonstrated that it is possible to obtain the nucleolus by implementing the sequential LP approach from the dual. However, the difficulty of this approach remains in cementing an exact set of stopping conditions in order to gauge whether the allocation of the nucleolus (i.e. dual of the dual) has been obtained. Since the least core LP model has exponential number of constraints for a given HCA problem, taking advantage of the reduced size of the dual LP model offers a good substitute to the primal sequential LP approach. This line of approach could serve as a significant area of future research.

2. *Programming optimization procedures into HCA decision support system*: The optimization based procedures in the HCA decision support system namely, Least Core LP model with A-S values and Generalized Method, must rely on a commercial software to engage in the sequential LP approach. Although the attached software is able to import and export data to an optimization software, more work needs to be done to improve the user interface and automate the process of solving a sequence of linear programs and confirming the nucleolus.

3. *External Costs:* The existing Least Core LP Model quantifies the use of a highway facility by vehicle classes in terms of pavement damage and traffic capacity. However, there are certain external costs imposed by vehicle classes that are not accounted for in our proposed methodology. Some of these include:

1. Cost of congestion
2. Cost of safety or crash costs
3. Cost of highway noise
4. Cost of carbon emissions

Inclusion of these costs into the least core LP model would better reflect all the costs that are being imposed on the highway and the community at large. The non-trivial challenge of this research question will be to assess whether these cost could be better incorporated by either adding more constraints to the LP model or including these costs in the characteristic function itself. The addition of these costs must be done in accordance with well-defined notion of fairness enforced by the game-theoretic concepts.

4. *Highway Cost Allocation Cost-Revenue Optimization Model:* The objective of a highway cost allocation is to determine whether the vehicle classes are paying their fair share of cost responsibility. This payment from vehicle classes comes in the form of user fees and taxes paid by the highway users in the form of gas tax, sales tax, registration fees, weight-distance tax to name a few. This dissertation has concentrated its efforts on improving the direct pavement related cost allocation process. However, an HCA study comprises of another equally important component known as the revenue attribution, that involves, determination of the total revenue contribution made by each vehicle class in terms of road users fees and taxes. The end result of a comprehensive HCA study is an equity ratio between percent share cost imposed by each vehicle class, obtained as part of the cost allocation study, and the revenue share attributed to each vehicle class. If this ratio is greater that one, then a vehicle class is paying more than their fair share of cost responsibility and if it is less than one, then the vehicle class is paying less than their fair share. It would be an avenue worth exploring to combine both the revenue and cost allocation aspects into a single least core LP model. For example, the model would aid practitioners in answering policy related questions such as say in order to be within an acceptable specific equity range of 0.9-1.1.

   a. By how much should the revenue be increased?

   b. Or conversely, how much should vehicle classes at minimum be attributed in order for them to be within an acceptable equity range?

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APPENDIX

APPENDIX A

DATA DEVELOPMENT AND CALCULATIONS FOR COMPARATIVE ANALYSIS

The scope of this appendix will be to describe in detail the data and model development process for the comparative analysis conducted in Chapter V. A hypothetical data set was created from a recent highway cost allocation study in order to validate the proposed methodology. The proposed procedure is divided into four stages. These stages are described below:

Stage I: Data Classification

This process involves collecting the necessary data on which the analytical tool will be implemented. The first step in this process is to choose the type of pavement for which the cost needs to be allocated. The two most widely used pavement design types in the US are rigid and flexible pavement design. The example presented in this dissertation will consider the flexible pavement design type. The second step involves choosing the type of total cost that needs to be distributed among the vehicle classes. Some of the major cost classifications include Construction, Rehabilitation, Repair, Maintenance and Reconstruction of the pavement. The example in this section will allocate costs measured in dollars per mile of building a new flexible pavement among various vehicle classes. The third step involves obtaining the Equivalent Single-Axle Loads for the corresponding vehicle classes that are classified by axle weights. This information is readily obtained by Weigh-In-Motion (WIM) sensor data available with the state departments of transportation. A comprehensive process of estimating ESALs from WIM sites can be found in the Texas HCA study conducted by Luskin et all (2002). The fourth step is to determine the number of lanes used by each vehicle class. This information can be extrapolated from the Highway Capacity Manual released periodically by the Transportation Research Board using (III-22). As a HCA practitioner, it should be noted that all the aforementioned relevant information are available to practitioners as data when working with civil engineers in a HCA study. Since the objective of this dissertation is to demonstrate the efficacy of the proposed methodology, discussion regarding the data consolidation and development process will be kept brief and succinct.

Stage II: Development of a Least Core Primal and Dual Linear Programming Model

This stage will involve the implementation of a new characteristic function to the least core LP model that gives us the cost of construction/rehabilitation/maintenance for a subset of vehicle classes that are part of the grand coalition. As part of this dissertation, usage of a new aumann-shapley based characteristic is demonstrated by estimating costs of coalitions descriptive of both cost imposed due to axle-load impact on pavement and the cost for building additional lanes to meet the capacity requirements of the vehicle classes. A cost equation as shown in (III-15) is developed by performing regression analysis on a set of representative highway projects with cumulative ESALs as the independent variable and cost in dollars per mile as the dependent variable. This equation will then be materialized for determining the A-S values of cost per ESAL (C_e) and cost per Lane (C_e) infused with the given ESAL requirements of each vehicle class using (III-20) and (III-21) and lane requirements as mandated by the combination of vehicle classes using (III-22),(III-23) and (III-24) contained in the coalition. These coalitional costs will then be used to get the Right Hand Sides (RHS) of the least core LP formulation as defined in (III-27) – (III-30).

Stage III: Computation of the Nucleolus

This stage will involve obtaining the cost attributed to each vehicle class. The standard least core linear programming model will be formulated as described in Chapter III. At this phase, multiple LPs will be needed in order to arrive at the desired allocation known as the nucleolus. The focus will remain to describe in detail the following features developed and observed as part of this dissertation as described in chapter III and Chapter IV:

A. *The Nucleolus: Average of Optimal Corner Point Solutions*: After solving the first LP, if presence of multiple optimal allocations are detected, then enumerating all corner point feasible solutions (CPFS) and then taking the average of all of them will give the nucleolus as defined by relationship (III-14). This result will then be verified by demonstrating that the sequential LP approach will also converge to the same solution as the one obtained by taking the average of all CPFS.

B. *The Nucleolus: Sequential LP approach*: This dissertation will use the Linear, Interactive, and Discrete Optimizer software also known as classic LINDO to perform the algorithm for obtaining the nucleolus. This software offered by LINDO systems is best positioned for implementation of this algorithm due to its ability to simultaneously show the corresponding dual values when solving the least core LP formulation. Therefore, it will be assumed that for every primal least core LP problem, a corresponding dual problem is solved simultaneously. Termination Criteria: The sequential LP approach will be terminated when the set of constraints in the linear programming model define a unique solution.

C. *Lexicographical Ordering:* Another criterion will be used to define the strength of the nucleolus. It will be demonstrated that the nucleolus is lexicographically greater than any allocation that at minimum satisfies the completeness condition. The purpose of doing this is to use an axiomatic criterion for defining the nucleolus.

D. *Application of Closed-form Solution*: Given the cost of coalitions are calculated directly from the pavement cost function. The robustness of the analytical formulas will be demonstrated by calculating the nucleolus by first solving the LP using simplex algorithm and then comparing the obtained result with allocations derived from the closed-form solutions as described in (IV-23) and (IV-24). The intent is to demonstrate that closed-form solution must only be used when each and every one of the $2^n + n - 1$ conditions are satisfied.

E. *Closed-form Solution Non-Usage in Proposed Methodology*: Given the cost of coalition *C(S)* and *C(N)* are obtained from the discrete A-S value based characteristic function. It will be demonstrated that due to violation of even one of the conditions for using closed-form solutions, application of closed-form solution is invalidated. This section will also demonstrate when the closed-form solution should not be implemented. Here, the purpose is to verify relationships presented in (IV-25) and (IV-26).

Stage IV: A Comparative Analysis of Representative HCA Methods

The final stage will involve a comparison of the cost allocation results obtained from the proposed methodology and other broad sample of representative HCA methodologies. The emphasis will be on demonstrating the effectiveness of the conceptual approach used to develop methods of traditional incremental method, proportional method, Shapley value, discrete aumann-shapley value, least core LP model with pavement cost function defining the cost of coalitions *C(S)* and *C(N)* and finally the least core LP model with discrete aumann-shapley value based characteristic function defining the cost of coalitions *c(s)* and *c(n)*. Figure A.1 gives a schematic representation of the four stage implementation process.

STAGE I: DATA CLASSIFICATION

1. Pavement Cost Function based on Method of Least Squares
2. Assumption of Traffic Characteristics
3. Data on ESAL per Vehicle Class and Average Daily Traffic
4. Discrete A-S Value Based on Compact Form

STAGE II: DEVELOPMENT OF LEAST CORE LP MODEL

1. Least Core LP Formulation
2. Dual of Least Core LP Formulation

STAGE III: COMPUTATION OF THE NUCLEOLUS
(Illustrations A, B, C and D)

APPLIED METHODS
A. Average of all Optimal Solutions
B. Sequential LP Approach

DISSERTATION METHODS
C. Use of Closed-form Solution
D. Non-Use of Closed-form Solution

STAGE IV: COMPARATIVE ANALYSIS

1. Incremental Method
2. Proportional Method
3. Shapley Value
4. Generalized Method
5. Discrete Aumann-Shapley Method
6. Least Core LP Model

Figure A.1: Four Stage Implementation

Stage I: Data Classification

The aim of this section will be to describe the formation of the hypothetical data set being used in order to test the efficacy of the proposed procedure to allocate cost among vehicle classes. A pavement cost equation is used that was developed as part of the study conducted in Texas highway cost allocation study for allocating flexible new pavement construction cost. The authors of that study used a software known as Flexible Pavement Design System or FPS19 that determines the cost of building new pavement in dollars per square yard that can sustain repetitive applications of a specified number of 18,000lb Equivalent Single-Axle Loads. The software is a based on a sophisticated pavement performance system that estimates costs by taking due account of the types of construction material used among different layers of the pavement for a specified initial and terminal present serviceability index (PSI). The pavement cost equation shown below is a flexible pavement cost function that represents a mathematical relationship between pavement damage measured in ESALs and cost imposed in dollar per mile based on actual expenditure data.

$$Y_{FP} = L_N\,(58{,}432 + 66.86 * E^{0.5}) \tag{A-1}$$

The regression parameter for Equation (V-1) are the following and have been obtained using (III-16) and (III-17):

$a = 58{,}432$

$b = 66.86$

$r = 0.5$

The total number of lanes in the highway facility will be considered to be 6 lanes. Therefore, the value of L_N is equal to six. The non-load related cost can be obtained by substituting the value of E to be zero. Doing so implies the cost imposed on the highway facility when it is not subjected to any 18-kip single-axle load applications. Some examples of non-load related costs are the cost of right of way, grading and drainage, pavement marking and other such costs that do not vary in relation to the load applications of different types of vehicle classes.

$Y_{FP} = 6 * (58{,}432 + 66.86 * (0)^{0.5})$

$Y_{FP-Non\,Load} = 6 * 58{,}432$

$Y_{FP-Non\,Load} = \$350{,}592$ per mile

The non-load related cost for the highway facility comes out to be $350,592 per mile.

The HCA problem is now examined for attribution of cost among four types of vehicle classes as listed below:

- TYPE I
- TYPE II
- TYPE III
- TYPE IV

In this example, following assumptions regarding the traffic characteristics of the flexible pavement have been made whose cost is divided among vehicle classes.

Assumption of Traffic Characteristics

- The flexible pavement is in a rural county situated in a mountainous terrain.
- The required number of lanes were calculated based on highway facility providing a Level of Service (LOS) A as per guidelines established in Highway Capacity Manual.
- The Maximum Service flow (MSF) rate comes out to be 400 vehicles per hour-lane which gives us the maximum number of lanes allowable to enter a lane in an hour.
- The Peak Hour Factor (phf) for rural environment is taken to be 0.9 which gives the ratio of total hourly volume to the maximum 15-minute rate of flow within the hour.
- The value of K represent the percentage of ADT experienced at the peak hour and was chosen to be 0.175.
- Directional Distribution (DD) that describes percentage of the traffic in the peak directional flow was assumed to have a split of 75-25.
- Table A.1 shows the ESAL factor for Vehicle classes I, II, III and IV and the Average Daily Traffic Values used for each of them.

Table A.1: ESAL Factors and ADT of Vehicle Classes

VEHICLE CLASSES	ESAL FACTOR	Average Daily Traffic (ADT)
TYPE I	0.00834	15,158
TYPE II	0.00277	10,158
TYPE III	0.08299	5,500
TYPE IV	0.03148	2,900

Highway Capacity Calculations

The number of lanes required by Vehicle Class I:

$$L_{TYPE\ I} = \frac{K * ADT * DD}{phf * f_{hv} * MSF} = \frac{0.175 * 15{,}158 * 0.75}{0.9 * 1 * 400} = 5.52\ \approx 6\ Lanes$$

The number of lanes required by Vehicle Class II:

$$L_{TYPE\ II} = \frac{K * ADT * DD}{phf * f_{hv} * MSF} = \frac{0.175 * 10{,}158 * 0.75}{0.9 * 1 * 400} = 3.7\ \approx 4\ Lanes$$

The number of lanes required by Vehicle Class III:

$$L_{TYPE\ III} = \frac{K * ADT * DD}{phf * f_{hv} * MSF} = \frac{0.175 * 5{,}500 * 0.75}{0.9 * 1 * 400} = 2.02\ \approx 2\ Lanes$$

The number of lanes required by Vehicle Class III:

$$L_{TYPE\ IV} = \frac{K * ADT * DD}{phf * f_{hv} * MSF} = \frac{0.175 * 2{,}900 * 0.75}{0.9 * 1 * 400} = 1.06\ \approx 1\ Lane$$

Traffic Loading Calculations

The cumulative design ESALs expected to be applied over two years by Vehicle Class I:

$$ESAL_{\,TYPE\ I} = (356\ days/year * 15{,}158\ vehicle/day * 2\ year$$
$$* 0.008339127\ ESAL/\ TYPE\ I) = 90{,}000\ ESALs$$

The cumulative design ESALs expected to be applied over two years by Vehicle Class II:

$$ESAL_{\,TYPE\ II} = (356\ days/year * 10{,}158\ vehicle/day * 2\ years$$
$$* 0.002765297\ ESAL/\ TYPE\ II) = 20{,}000\ ESALs$$

The cumulative design ESALs expected to be applied over two years by Vehicle Class III:

$$ESAL_{\,TYPE\ III} = (356\ days/year * 5{,}500\ vehicle/day * 2\ years$$
$$* 0.08299285\ ESAL/\ TYPE\ III) = 325{,}000\ ESALs$$

The cumulative design ESALs expected to be applied over two years by Vehicle Class IV:

$$ESAL_{TYPE\ IV} = (356\ days/year * 2{,}900\ vehicle/day * 2\ years * 0.031480046\ ESAL/\ TYPE\ IV) = 65{,}000\ ESALs$$

Miscellaneous Notes-

- The value for f_{hv} was chosen to be one because the effect of heavy vehicle on the traffic stream was already considered when coming up with the value of the service flow rate of 400 vehicles per hour-lane.
- The analysis period considered for this example is two years. And so, cumulative ESAL represents the predicted number of 18-kip single-axle load application over period of two years.

Now, two important pieces of information needed to implement the proposed methodology have now been ascertained. These are listed below:

1. Number of Lanes needed by Vehicle Classes I, Vehicle Classes II, Vehicle Classes III, Vehicle Classes IV are $L_1, L_2, L_3$and , L_4.
2. Number of 18,000lb Equivalent Single-Axle Loads: The cumulative number of 18-kip single-axle load applications Vehicle Classes I, Vehicle Classes II, Vehicle Classes III, Vehicle Classes IV are $E_1, E_2, E_3$and , E_4.

Next, discrete Aumann-Shapley values of cost per ESAL (C_e) and cost per Lane (C_l) are determined by using the compact form (III-20) and (III-21)developed as part of this dissertation.

The results are reported in Table A.2 as shown below:

Table A.2: Cumulative ESAL and Number of Lanes based on HCM

VEHICLE CLASSES	TRAFFIC LOADING		TRAFFIC CAPACITY	
	ESAL		LANES	
TYPE I	E_1	90,000	L_1	6
TYPE II	E_2	20,000	L_2	4
TYPE III	E_3	325,000	L_3	2
TYPE IV	E_4	65,000	L_4	1

PAVEMENT COST FUNCTION

$$Y_{FP} = L_N\,(58{,}432 + 66.86 * E^{0.5})$$

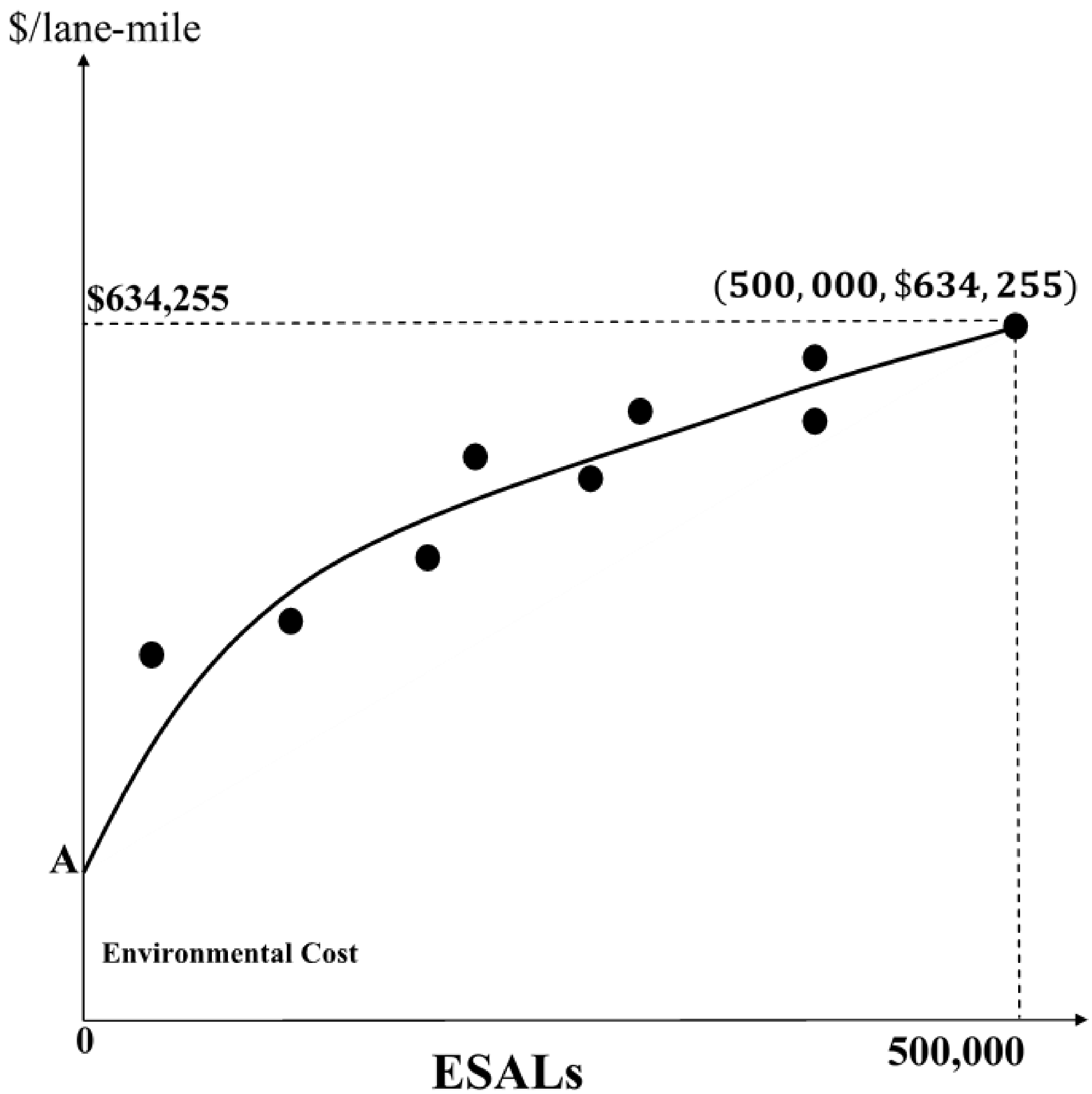

Figure A.2: Example Pavement Cost Function for Flexible Pavement

Determination of Discrete Aumann-Shapley value Using Compact Form

$$C_l = \left(\frac{\left(\frac{L_N}{\sum_{i\in N} E_i + 1}\right) * \left(\sum_{i=0}^{E_N}\left(a + b\sqrt{E_i}\right)\right)}{L_N}\right)$$

$$C_l = \left(\frac{\left(\frac{6}{500{,}000 + 1}\right) * \left(\sum_{i=0}^{500{,}000}\ \left(58{,}432 + 66.86\sqrt{E_i}\right)\right)}{6}\right)$$

$$C_l = \left(\frac{\left(\frac{6}{500{,}001}\right) * \left(\sum_{i=0}^{500{,}000}\left(58{,}432 + 66.86\sqrt{E_i}\right)\right)}{6}\right)$$

$$C_l = \left(\frac{0.000011999976000048 * \left(\sum_{i=0}^{500{,}000} 6\left(58{,}432 + 66.86\sqrt{E_i}\right)\right)}{6}\right)$$

$$C_l = \frac{0.000011999976000048 * 44975135186.6992000}{6}$$

$$C_l = \frac{539{,}700.54}{6}$$

$$\boldsymbol{C_l = \$89{,}950.09\ per\ Lane}$$

The formula for Cost Per ESAL is shown below:

$$C_e = \left(\frac{\left(L_N\left(a + b\sqrt{\sum_{i\in N} E_i}\right) - \left(\frac{L_N}{\sum_{i\in N} E_i + 1}\right)\left(\sum_{i=0}^{E_N}\left(a + b\sqrt{E_i}\right)\right)\right)}{\sum_{i\in N} E_i}\right)$$

$$C_e = \left(\frac{6\left(58{,}432 + 66.86\sqrt{\sum_{i\in N} E_i}\right) - \left(\frac{6}{500{,}000 + 1}\right)\left(\sum_{i=0}^{500{,}000}\left(58{,}432 + 66.86\sqrt{E_i}\right)\right)}{500{,}000}\right)$$

$$C_e = \left(\frac{634{,}255 - 539{,}700.54}{500{,}000}\right) = \frac{94{,}554.46}{500{,}000}$$

$$\boldsymbol{C_e = 0.18910892\ per\ ESAL}$$

Stage II: Development of a Least Core Primal & Dual Linear Programming Model

In this section, the working principle behind the sequential LP approach is briefly summarized. As a recap from Chapter III, the algorithm comes up with a payoff that maximizes the savings for each coalition $t\{x, \{S\}\}$ by solving sequence of LPs. Upon solving the 1st LP, If unique, stop, else, the corresponding coalitional rationality constraint of *S* whose associated dual variable Y_S is non-zero, is converted to an equality by means of subtraction of *C(S)* from the optimal t_1^* using the complementary slackness conditions. The coalitions *S* whose savings have been maximized in subsequent LPs with optimal t_n^* to the greatest degree possible are stored in set F_n. Several LPs will then be solved successively whose purpose will be to maximize the next set of least satisfied coalitions to the greatest amount possible. This process of solving LPs is completed when the last LP in the sequence has collection of constraints that defines a unique allocation. The primal least core LP Model for N = {1,2,3,4} with 15 constraints and 4 decision variables of x_1, x_2, x_3 and t_1.

Objective function $\quad$ $max\ t_1$

Subject to

$$x_1 + t_1 \leq C(1)$$
$$x_2 + t_1 \leq C(2)$$
$$x_3 + t_1 \leq C(3)$$
$$x_4 + t_1 \leq C(4)$$
$$x_1 + x_2 + t_1 \leq C(12)$$
$$x_1 + x_3 + t_1 \leq C(13)$$
$$x_1 + x_4 + t_1 \leq C(14)$$
$$x_2 + x_3 + t_1 \leq C(23)$$
$$x_2 + x_4 + t_1 \leq C(24)$$
$$x_3 + x_4 + t_1 \leq C(34)$$
$$x_1 + x_2 + x_3 + t_1 \leq C(123)$$
$$x_1 + x_2 + x_4 + t_1 \leq C(124)$$
$$x_1 + x_3 + x_4 + t_1 \leq C(134)$$
$$x_2 + x_3 + x_4 + t_1 \leq C(234)$$
$$x_1 + x_2 + x_3 + x_4 = C(1234)$$

The RHS for the above least core LP model will be obtained from two types of characteristic function:

1. Proposed Methodology: Estimation of cost of coalitions *C(S)* and *C(N)* from the discrete aumann-shapley based characteristic function.
2. Estimation of cost of coalitions *C(S)* and *C(N)* directly from the pavement cost function.

The Dual Least Core LP Model for N = {1, 2, 3, 4} with 4 Constraints and 15 decision variables

The dual of the least core LP formulation for a four vehicle class problem is shown in this section. The dual formulation will be used considerably in the remaining parts of this appendix for carrying out the Sequential LP Approach.

Objective Function

$$\text{Min } C(1)\,Y_1 + C(2)Y_2 + C(3)Y_3 + C(4)Y_4 + C(12)Y_{12} + C(13)Y_{13} + C(14)Y_{14} + C(23)Y_{23} + C(24)\,Y_{24} + C(34)\,Y_{34} + C(123)\,Y_{123} + C(124)\,Y_{124} + C(134)\,Y_{134} + C(234)\,Y_{234} + C(1234)\,Y_{1234}$$

Subject to

Constraints

$$Y_1+Y_{12}+Y_{13}+Y_{14}+Y_{123}+Y_{124}+Y_{134} + Y_{1234} \geq 0 \qquad (x_1)$$

$$Y_2+Y_{12}+Y_{23}+Y_{24}+Y_{123}+Y_{124}+Y_{234} + Y_{1234} \geq 0 \qquad (x_2)$$

$$Y_3+Y_{13}+Y_{23}+Y_{34}+Y_{123}+Y_{134}+Y_{234} + Y_{1234} \geq 0 \qquad (x_3)$$

$$Y_4+Y_{14}+Y_{24}+Y_{34}+Y_{124}+Y_{134}+Y_{234} + Y_{1234} \geq 0 \qquad (x_4)$$

$$Y_1 + Y_2 + Y_3 + Y_4 + Y_{12} + Y_{13} + Y_{14} + Y_{23} + Y_{24} + Y_{34} + Y_{123} + Y_{124} + Y_{134} + Y_{234} \geq 1 \qquad (t_1)$$

$$Y_{1234}\ unrestricted$$

Computation of the Nucleolus Using Sequential LP Approach

Least Core LP Model with Discrete Aumann-Shapley values: Illustration A&B

The RHS or the costs of coalition *C(S)* and *C(N)* are obtained from the rules described extensively in Chapter III. Reiterating the substance for such a function, the RHS using the discrete A-S considers both the pavement thickness requirements and demand for highway capacity occasioned by vehicle classes. The complete least core LP formulation for four vehicle class HCA problem along with new characteristic function is shown below:

Objective Function $\max t_1$

Subject to

Constraints

$$x_1 + t_1 \leq E_1 C_e + L_1 C_l$$
$$x_2 + t_1 \leq E_2 C_e + L_2 C_l$$
$$x_3 + t_1 \leq E_3 C_e + L_3 C_l$$
$$x_4 + t_1 \leq E_4 C_e + L_4 C_l$$
$$x_1 + x_2 + t_1 \leq (E_1 + E_2) C_e + L_{12} C_l$$
$$x_1 + x_3 + t_1 \leq (E_1 + E_3) C_e + L_{13} C_l$$
$$x_1 + x_4 + t_1 \leq (E_1 + E_4) C_e + L_{14} C_l$$
$$x_2 + x_3 + t_1 \leq (E_2 + E_3) C_e + L_{23} C_l$$
$$x_2 + x_4 + t_1 \leq (E_2 + E_4) C_e + L_{24} C_l$$
$$x_3 + x_4 + t_1 \leq (E_3 + E_4) C_e + L_{34} C_l$$
$$x_1 + x_2 + x_3 + t_1 \leq (E_1 + E_2 + E_3) C_e + L_{123} C_l$$
$$x_1 + x_2 + x_4 + t_1 \leq (E_1 + E_2 + E_4) C_e + L_{124} C_l$$
$$x_1 + x_3 + x_4 + t_1 \leq (E_1 + E_3 + E_4) C_e + L_{134} C_l$$
$$x_2 + x_3 + x_4 + t_1 \leq (E_2 + E_3 + E_4) C_e + L_{234} C_l$$
$$x_1 + x_2 + x_3 + x_4 = (E_1 + E_2 + E_3 + E_4) C_e + L_{1234} C_l$$

Next, the appropriate number of lanes L_S for coalition S and ESAL distribution data for each vehicle class is calculated using the highway capacity manual and reported in Table A.2. The number of lanes and ESAL distribution for each coalition for this example is obtained using the rules described in (III-22), (III-23) and (III-24). The final data needed to develop the least core LP formulation can be obtained from Table A.3.

Table A.3: Input Data for Proposed Methodology

Cost per ESAL (C_e) = \$0.18910892, Cost per Lane (C_l) = \$89,950.09047							
		ESALs	E(S)	L(S)		$(E_i * C_e + L_i * C_l)$	
1	{1}	E_1	90,000	L_1	6	C(1)	\$556,720
2	{2}	E_2	20,000	L_2	4	C(2)	\$363,583
3	{3}	E_3	325,000	L_3	2	C(3)	\$241,361
4	{4}	E_4	65,000	L_4	1	C(4)	\$102,242
5	{1,2}	$E_1 + E_2$	110,000	$L_{12} = max(L_1, L_2)$	6	C(12)	\$560,503
6	{1,3}	$E_1 + E_3$	415,000	$L_{13} = max(L_1, L_3)$	6	C(13)	\$618,181
7	{1,4}	$E_1 + E_4$	155,000	$L_{14} = max(L_1, L_4)$	6	C(14)	\$569,012
8	{2,3}	$E_2 + E_3$	345,000	$L_{23} = max(L_2, L_3)$	4	C(23)	\$425,043
9	{2,4}	$E_2 + E_4$	85,000	$L_{24} = max(L_2, L_4)$	4	C(24)	\$375,875
10	{3,4}	$E_3 + E_4$	390,000	$L_{34} = max(L_3, L_4)$	2	C(34)	\$253,653
11	{1,2,3}	$E_1 + E_2 + E_3$	435,000	$L_{123} = max(L_1, L_2, L_3)$	6	C(123)	\$621,963
12	{1,2,4}	$E_1 + E_2 + E_4$	175,000	$L_{124} = max(L_1, L_2, L_4)$	6	C(124)	\$572,795
13	{1,3,4}	$E_1 + E_3 + E_4$	480,000	$L_{134} = max(L_1, L_3, L_4)$	6	C(134)	\$630,473
14	{2,3,4}	$E_2 + E_3 + E_4$	410,000	$L_{234} = max(L_2, L_3, L_4)$	4	C(234)	\$437,335
15	{1,2,3,4}	$E_1 + E_2 + E_3 + E_4$	500,000	$L_{1234} = max(L_1, L_2, L_3, L_4)$	6	C(1234)	\$634,255

Implementation: Least Core 1st Linear Programming Model: Illustration A (j=0 and n=1)

After obtaining the RHS, the process of applying the proposed algorithm can now begin. Each constraint in the LP model is hereafter referred to as a coalitional rationality constraint for corresponding coalition *S*.

$$\max t_1$$

Subject to

$$\begin{aligned}
x_1 + t_1 &\leq 556{,}720 \\
x_2 + t_1 &\leq 363{,}583 \\
x_3 + t_1 &\leq 241{,}361 \\
x_4 + t_1 &\leq 102{,}242 \\
x_1 + x_2 + t_1 &\leq 560{,}503 \\
x_1 + x_3 + t_1 &\leq 618{,}181 \\
x_1 + x_4 + t_1 &\leq 569{,}012 \\
x_2 + x_3 + t_1 &\leq 425{,}043 \\
x_2 + x_4 + t_1 &\leq 375{,}875 \\
x_3 + x_4 + t_1 &\leq 253{,}653 \\
x_1 + x_2 + x_3 + t_1 &\leq 621{,}963 \\
x_1 + x_2 + x_4 + t_1 &\leq 572{,}795 \\
x_1 + x_3 + x_4 + t_1 &\leq 630{,}473 \\
x_2 + x_3 + x_4 + t_1 &\leq 437{,}335 \\
x_1 + x_2 + x_3 + x_4 &= 634{,}255
\end{aligned}$$

Upon solving the 1st LP model, the payoff allocation as reported in Table A.4 is found to be $x = (\$421{,}796, \$48{,}757, \$106{,}435, \$57{,}267)$. The optimal value for this model is $t_1^* = \$44{,}975$.

Table A.4: Optimal Solution for Primal 1st LP Model in Sequential LP Approach

t_1	\$44,975
x_1	\$421,796
x_2	\$48,757
x_3	\$106,435
x_4	\$57,267
Total	\$634,255

The Dual Least Core LP Model for 1st Linear Programming Model

This section shows the formulation of the dual for the least core LP formulation consisting of four vehicle classes. The dual formulation will be used considerably in this algorithm in order to identify coalitions whose savings cannot be maximized any further. The objective function and the corresponding constraints in the dual of the least core LP formulation is shown below:

$$\text{Min } 556{,}720\,\mathrm{Y}_1 + 363{,}583\,\mathrm{Y}_2 + 241{,}361\,\mathrm{Y}_3 + 102{,}242\,\mathrm{Y}_4 + 560{,}503\,\mathrm{Y}_{12} + 618{,}181\,\mathrm{Y}_{13} + 569{,}012\,\mathrm{Y}_{14} + 425{,}043\mathrm{Y}_{23} + 375{,}875\,\mathrm{Y}_{24} + 253{,}653\,\mathrm{Y}_{34} + 621{,}963\,\mathrm{Y}_{123} + 572{,}795\,\mathrm{Y}_{124} + 630{,}473\,\mathrm{Y}_{134} + 437{,}335\,\mathrm{Y}_{234} + 634{,}255\,\mathrm{Y}_{1234}$$

Subject to

$$Y_1+Y_{12}+Y_{13}+Y_{14}+Y_{123}+Y_{124}+Y_{134} + Y_{1234} \geq 0$$

$$Y_2+Y_{12}+Y_{23}+Y_{24}+Y_{123}+Y_{124}+Y_{234} + Y_{1234} \geq 0$$

$$Y_3+Y_{13}+Y_{23}+Y_{34}+Y_{123}+Y_{134}+Y_{234} + Y_{1234} \geq 0$$

$$Y_4+Y_{14}+Y_{24}+Y_{34}+Y_{124}+Y_{134}+Y_{234} + Y_{1234} \geq 0$$

$$Y_1 + Y_2 + Y_3 + Y_4 + Y_{12} + Y_{13} + Y_{14} + Y_{23} + Y_{24} + Y_{34} + Y_{123} + Y_{124} + Y_{134} + Y_{234} \geq 1$$

$$Y_{1234}\ unrestricted$$

Based on the optimal values of dual variables reported in Table A.5, it can be concluded that the savings $t(x,\{4\})$ and $t(x,\{1,2,3\})$ cannot be further maximized as the dual variables Y_4 and Y_{123} are non-zero. Coalition {4} and {1,2,3} is placed in the set F_1. In the 2nd LP, coalitional rationality constraints of {4} and {1,2,3}are converted into equalities by making $t_2 = t_1$ and subtracting *C(4)* and *C(123)* from t_1. Before proceeding with the 2nd LP, it will be demonstrated that the average of all optimal basic feasible solutions of the 1st LP is the nucleolus. The optimal solution for the dual model is shown below:

Table A.5: Optimal Solution for Dual 1st LP Model in Sequential LP Approach

DUAL								Conversion Constraints
Y_1	0.0	Y_{12}	0.0	Y_{24}	0.0	Y_{134}	0.0	$F_1 = \{\{4\},\{1,2,3\}\}$
Y_2	0.0	Y_{13}	0.0	Y_{34}	0.0	Y_{234}	0.0	
Y_3	0.0	Y_{14}	0.0	$\boldsymbol{Y_{123}}$	**0.5**	Y_{1234}	-0.5	
$\boldsymbol{Y_4}$	**0.5**	Y_{23}	0.0	Y_{124}	0.0			

Illustration A: The Nucleolus: Average of all Optimal Corner Point Feasible Solutions

At this point, the presence of multiple optimal solutions are detected. Hence, in order to verify the average property, all the optimal corner point feasible solutions are enumerated using the "Pivot" function in LINDO optimization software.

$$x_1^{**} = \frac{(\$421{,}796 + \$376{,}820 + \$241{,}895 + \$241{,}895)}{4} = \$320{,}601.5$$

$$x_2^{**} = \frac{(\$48{,}757 + \$48{,}757 + \$228{,}658 + \$183{,}682)}{4} = \$127{,}463.5$$

$$x_3^{**} = \frac{(\$106{,}435 + \$151{,}411 + \$106{,}435 + \$151{,}411)}{4} = \$128{,}923$$

$$x_4^{**} = \frac{(\$57{,}267 + \$57{,}267 + \$57{,}267 + \$57{,}267)}{4} = \$57{,}267$$

The nucleolus solution for this HCA problem is $\{ x_1^{**}, x_2^{**}, x_3^{**}, x_4^{**}\}$

$$X_N^* = (\$\mathbf{320, 601.5}, \$\mathbf{127, 463.5}, \$\mathbf{128, 923}, \$\mathbf{57, 267})$$

Now, the above allocation is verified by checking whether on solving a sequence of LPs as mandated by the sequential LP approach would also result in convergence to the allocation X_N^*. Resuming the implementation of the sequential LP approach, the 2nd LP model in the sequence is solved. Based on our assertion, the last LP in the sequential LP approach must be equal to N^*. All the optimal Corner Point Feasible Solutions (CPFS) are listed in Table A.6 as shown below:

Table A.6: List of Optimal Corner Point Feasible Solutions

Multiple Optimal Basic Feasible Solutions (Four Corner Point Feasible Solutions)					
VEHICLE CLASSES	I	II	III	IV	NUCLEOLUS (Average)
1	$421,796	$376,820	$241,895	$241,895	$320,601.5
2	$48,757	$48,757	$228,658	$183,682	$127,463.5
3	$106,435	$151,411	$106,435	$151,411	$128,923
4	$57,267	$57,267	$57,267	$57,267	$57,267
Total	$634,255	$634,255	$634,255	$634,255	$634,255

Illustration B: The Nucleolus: Sequential LP approach

The 2nd Primal Least Core Linear Programming Model: (j=1 and n=2)

The formulation of the 2nd linear programming model for the least core LP model is shown below:

$$\max t_2$$

Subject to

$$x_1 + t_2 \leq 556{,}720$$
$$x_2 + t_2 \leq 363{,}583$$
$$x_3 + t_2 \leq 241{,}361$$
$$\boldsymbol{x_4 = 57{,}267}$$
$$x_1 + x_2 + t_2 \leq 560{,}503$$
$$x_1 + x_3 + t_2 \leq 618{,}181$$
$$x_1 + x_4 + t_2 \leq 569{,}012$$
$$x_2 + x_3 + t_2 \leq 425{,}043$$
$$x_2 + x_4 + t_2 \leq 375{,}875$$
$$x_3 + x_4 + t_2 \leq 253{,}653$$
$$\boldsymbol{x_1 + x_2 + x_3 = 576{,}988}$$
$$x_1 + x_2 + x_4 + t_2 \leq 572{,}795$$
$$x_1 + x_3 + x_4 + t_2 \leq 630{,}473$$
$$x_2 + x_3 + x_4 + t_2 \leq 437{,}335$$
$$x_1 + x_2 + x_3 + x_4 = 634{,}255$$

The payoff allocation for the 2nd LP as reported in Table A.7 is $x = (\$376{,}820, \$71{,}245, \$128{,}923, \$57{,}267)$. The optimal $t_2^* = \$67{,}463$. Based on the tableau, the 2nd LP comprises of multiple optimal solutions.

Table A.7: Optimal Solution for Primal 2nd LP Model in Sequential LP Approach

t_2	\$67,463
x_1	\$376,820
x_2	\$71,245
x_3	\$128,923
x_4	\$57,267
Total	\$634,255

The Dual Least Core LP Model for 2nd Linear Programming Model

The dual of the 2nd least core LP formulation for a four vehicle class problem is shown below. This LP model is solved simultaneously to gain requisite information regarding constraints that are binding.

Objective Function

Min $556{,}720\ Y_1 + 363{,}583\ Y_2 + 241{,}361\ Y_3 + 57{,}267\ Y_4 + 560{,}503\ Y_{12} + 618{,}181\ Y_{13} + 569{,}012\ Y_{14} + 425{,}043 Y_{23} + 375{,}875\ Y_{24} + 253{,}653\ Y_{34} + 576{,}988\ Y_{123} + 572{,}795\ Y_{124} + 630{,}473\ Y_{134} + 437{,}335\ Y_{234} + 634{,}255\ Y_{1234}$

Subject to

Constraints

$$Y_1 + Y_{12} + Y_{13} + Y_{14} + Y_{123} + Y_{124} + Y_{134} + Y_{1234} \geq 0$$
$$Y_2 + Y_{12} + Y_{23} + Y_{24} + Y_{123} + Y_{124} + Y_{234} + Y_{1234} \geq 0$$
$$Y_3 + Y_{13} + Y_{23} + Y_{34} + Y_{123} + Y_{134} + Y_{234} + Y_{1234} \geq 0$$
$$Y_4 + Y_{14} + Y_{24} + Y_{34} + Y_{124} + Y_{134} + Y_{234} + Y_{1234} \geq 0$$
$$Y_1 + Y_2 + Y_3 + Y_{12} + Y_{13} + Y_{14} + Y_{23} + Y_{24} + Y_{34} + Y_{124} + Y_{134} + Y_{234} \geq 1$$

$$Y_4, Y_{123}, Y_{1234}\ unrestricted$$

The optimal solution for the dual model is shown in Table A.8. In addition, the savings $t(x, \{3,4\})$ and $t\{x, \{1,2,4\}$ cannot be further maximized based on the fact that dual variables Y_{34} and Y_{124} are greater than zero. As a consequence, for the 3rd LP model, coalition {3,4} and {1,2,4} are added to the set F_2 and their coalitional rationality constraints are converted into equalities by subtracting *C(34)* and *C(124)* from optimal t_2^*.

Table A.8: Optimal Solution for Dual 2nd LP Model in Sequential LP Approach

DUAL								Conversion Constraints
Y_1	0.0	Y_{12}	0.0	Y_{24}	0.0	Y_{134}	0.0	$F_2 = \{\{3,4\}, \{1,2,4\}\}$
Y_2	0.0	Y_{13}	0.0	Y_{34}	**0.5**	Y_{234}	0.0	
Y_3	0.0	Y_{14}	0.0	Y_{123}	-0.5	Y_{1234}	0.0	
Y_4	-1.0	Y_{23}	0.0	Y_{124}	**0.5**			

The 3rd Primal Least Core Linear Programming Model (j=2 and n=3)

Continuing, the 3rd linear programming model for the least core LP formulation is solved as the 2nd LP problem had multiple optimal solutions.

$$\max t_3$$

Subject to

$$x_1 + t_3 \leq 556{,}720$$
$$x_2 + t_3 \leq 363{,}583$$
$$x_3 + t_3 \leq 241{,}361$$
$$x_4 = 57{,}267$$
$$x_1+x_2 + t_3 \leq 560{,}503$$
$$x_1+x_3 + t_3 \leq 618{,}181$$
$$x_1+x_4 + t_3 \leq 569{,}012$$
$$x_2+x_3 + t_3 \leq 425{,}043$$
$$x_2+x_4 \leq 375{,}875$$
$$\boldsymbol{x_3+x_4 = 186{,}190}$$
$$x_1+x_2+x_3 = 576{,}988$$
$$\boldsymbol{x_1+x_2+x_4 = 505{,}332}$$
$$x_1+x_3+x_4 + t_3 \leq 630{,}473$$
$$x_2+x_3+x_4 + t_3 \leq 437{,}335$$
$$x_1+x_2+x_3+x_4 = 634{,}255$$

The payoff allocation for the 3rd LP as reported in Table A.9 is found to be $x = (\$331{,}845, \$116{,}220, \$128{,}923, \$57{,}267)$. The optimal $t_3^* = \$112{,}438$. Based on the Tableau, the 3rd LP also has multiple optimal solutions.

Table A.9: Optimal Solution for Primal 3rd LP Model in Sequential LP Approach

t_3	\$112,438
x_1	\$331,845
x_2	\$116,220
x_3	\$128,923
x_4	\$57,267
Total	\$634,255

The Dual Least Core LP Model for 3rd Linear Programming Model

The dual model for the 3rd least core LP formulation for a four vehicle class problem is shown below.

Objective Function

Min $556{,}720\, Y_1 + 363{,}583\, Y_2 + 241{,}361\, Y_3 + 57{,}267\, Y_4 + 560{,}503\, Y_{12} + 618{,}181\, Y_{13} + 569{,}012\, Y_{14} + 425{,}043 Y_{23} + 375{,}875\, Y_{24} + 186{,}190\, Y_{34} + 576{,}988\, Y_{123} + 505{,}332\, Y_{124} + 630{,}473\, Y_{134} + 437{,}335\, Y_{234} + 634{,}255\, Y_{1234}$

Subject to

Constraints

$$Y_1+Y_{12}+Y_{13}+Y_{14}+Y_{123}+Y_{124}+Y_{134}+Y_{1234} \geq 0$$
$$Y_2+Y_{12}+Y_{23}+Y_{24}+Y_{123}+Y_{124}+Y_{234}+Y_{1234} \geq 0$$
$$Y_3+Y_{13}+Y_{23}+Y_{34}+Y_{123}+Y_{134}+Y_{234}+Y_{1234} \geq 0$$
$$Y_4+Y_{14}+Y_{24}+Y_{34}+Y_{124}+Y_{134}+Y_{234}+Y_{1234} \geq 0$$
$$Y_1+Y_2+Y_3+Y_{12}+Y_{13}+Y_{14}+Y_{23}+Y_{24}+Y_{134}+Y_{234} \geq 1$$

$$Y_{34}, Y_{124}, Y_4, Y_{123}, Y_{1234}\ unrestricted$$

The optimal solution for the dual model is shown in Table A.10. In addition, the savings $t(x,\{1,2\})$ cannot be further maximized based on the fact that Y_{12} is greater than zero. The coalition {1,2} is added to the set F_3. As a consequence, for the 4th LP model, coalitional rationality constraint of {1,2} is converted into an equality by subtracting corresponding *C(12)* from the optimal t_3^*.

Table A.10: Optimal Solution for Dual 3rd LP Model in Sequential LP Approach

DUAL								Conversion Constraints
Y_1	0.0	$\boldsymbol{Y_{12}}$	1.0	Y_{24}	0.0	Y_{134}	0.0	$F_3 = \{\{1,2\}\}$
Y_2	0.0	Y_{13}	0.0	Y_{34}	0.0	Y_{234}	0.0	
Y_3	0.0	Y_{14}	0.0	Y_{123}	0.0	Y_{1234}	0.0	
Y_4	1.0	Y_{23}	0.0	Y_{124}	-1.0			

The 4th Primal Least Core Linear Programming Model (j=3 and n=4)

Continuing, the 4th linear programming model for least core LP formulation is solved as the 3nd LP problem had multiple optimal solutions.

$$\max t_4$$

Subject to

$$x_1 + t_4 \leq 556{,}720$$
$$x_2 + t_4 \leq 363{,}583$$
$$x_3 + t_4 \leq 241{,}361$$
$$x_4 = 57{,}267$$
$$\boldsymbol{x_1 + x_2 = 448{,}065}$$
$$x_1 + x_3 + t_4 \leq 618{,}181$$
$$x_1 + x_4 + t_4 \leq 569{,}012$$
$$x_2 + x_3 + t_4 \leq 425{,}043$$
$$x_2 + x_4 \leq 375{,}875$$
$$x_3 + x_4 = 186{,}190$$
$$x_1 + x_2 + x_3 = 576{,}988$$
$$x_1 + x_2 + x_4 = 505{,}332$$
$$x_1 + x_3 + x_4 + t_4 \leq 630{,}473$$
$$x_2 + x_3 + x_4 + t_4 \leq 437{,}335$$
$$x_1 + x_2 + x_3 + x_4 = 634{,}255$$

The payoff allocation for the 4th LP as reported in Table A.11 is found to be $\hat{x} = (\$331{,}845, \$116{,}220, \$128{,}923, \$57{,}267)$. The optimal $t_4^* = \$112{,}438$. Based on the tableau, the 4th LP has multiple optimal solutions.

Table A.11: Optimal Solution for Primal 4th LP Model in Sequential LP Approach

t_4	\$112,438
x_1	\$331,845
x_2	\$116,220
x_3	\$128,923
x_4	\$57,267
Total	\$634,255

The Dual Least Core LP Model for 4th Linear Programming Model

The dual model for the 4th least core LP formulation for a four vehicle class HCA problem is shown below.

Objective Function

Min $556{,}720\ Y_1 + 363{,}583\ Y_2 + 241{,}361\ Y_3 + 57{,}267\ Y_4 + 448{,}065\ Y_{12} + 618{,}181\ Y_{13} + 569{,}012\ Y_{14} + 425{,}043 Y_{23} + 375{,}875\ Y_{24} + 186{,}190\ Y_{34} + 576{,}988\ Y_{123} + 505{,}332\ Y_{124} + 630{,}473\ Y_{134} + 437{,}335\ Y_{234} + 634{,}255\ Y_{1234}$

Subject to

Constraints

$$Y_1 + Y_{12} + Y_{13} + Y_{14} + Y_{123} + Y_{124} + Y_{134} + Y_{1234} \geq 0$$

$$Y_2 + Y_{12} + Y_{23} + Y_{24} + Y_{123} + Y_{124} + Y_{234} + Y_{1234} \geq 0$$

$$Y_3 + Y_{13} + Y_{23} + Y_{34} + Y_{123} + Y_{134} + Y_{234} + Y_{1234} \geq 0$$

$$Y_4 + Y_{14} + Y_{24} + Y_{34} + Y_{124} + Y_{134} + Y_{234} + Y_{1234} \geq 0$$

$$Y_1 + Y_2 + Y_3 + Y_{13} + Y_{14} + Y_{23} + Y_{24} + Y_{134} + Y_{234} \geq 1$$

$$Y_{12}, Y_{34}, Y_{124}, Y_4, Y_{123}\ , Y_{1234}\ unrestricted$$

The optimal solution for the dual model is shown in Table A.12. In addition, the savings $t(x, \{3\})$ cannot be further maximized based on the fact that Y_3 is greater than zero. The coalition $\{3\}$ is added to set F_4. As a consequence, for the 5th LP model, the coalitional rationality constraint of $\{3\}$ is converted to an equality by subtracting the corresponding C(3) from optimal t_4^*.

Table A.12: Optimal Solution for Dual 4th LP Model in Sequential LP Approach

DUAL								Conversion Constraints
Y_1	0.0	Y_{12}	0.0	Y_{24}	0.0	Y_{134}	0.0	$F_4 = \{\{3\}\}$
Y_2	0.0	Y_{13}	0.0	Y_{34}	-1.0	Y_{234}	0.0	
$\boldsymbol{Y_3}$	**1.0**	Y_{14}	0.0	Y_{123}	0.0	Y_{1234}	0.0	
Y_4	1.0	Y_{23}	0.0	Y_{124}	0.0			

Least Core 5th Linear Programming Model (j=4 and n=5)

Continuing, the 5th linear programming model for the least core LP model is now solved as the 4th LP problem had multiple optimal solutions.

$$\max t_5$$

Subject to

$$x_1 + t_5 \leq 556{,}720$$
$$x_2 + t_5 \leq 363{,}583$$
$$\boldsymbol{x_3 = 128{,}923}$$
$$x_4 = 57{,}267$$
$$x_1 + x_2 = 448{,}065$$
$$x_1 + x_3 + t_5 \leq 618{,}181$$
$$x_1 + x_4 + t_5 \leq 569{,}012$$
$$x_2 + x_3 + t_5 \leq 425{,}043$$
$$x_2 + x_4 + t_5 \leq 375{,}875$$
$$x_3 + x_4 = 186{,}190$$
$$x_1 + x_2 + x_3 = 576{,}988$$
$$x_1 + x_2 + x_4 = 505{,}332$$
$$x_1 + x_3 + x_4 + t_5 \leq 630{,}473$$
$$x_2 + x_3 + x_4 + t_5 \leq 437{,}335$$
$$x_1 + x_2 + x_3 + x_4 = 634{,}255$$

The payoff allocation for the 5th LP as reported in Table A.13 is found to be $\hat{x} = (\$320{,}601.5, \$127{,}463.5, \$128{,}923, \$57{,}267)$ The optimal $t_5^* = \$123{,}681.5$. Based on the Tableau, the 5th LP has a unique optimal solution.

Table A.13: Optimal Solution for Primal 5th LP Model in Sequential LP Approach

t_5	\$123,681.5
x_1	\$320,601.5
x_2	\$127,463.5
x_3	\$128,923
x_4	\$57,267
Total	\$634,255

The Dual Least Core LP Model for 5th Linear Programming Model (j=4 and n=5)

The dual model for the 5th least core LP formulation for a four vehicle HCA class problem is shown below. 605-484-2652

Objective Function

Min $556{,}720\, Y_1 + 363{,}583\, Y_2 + 128{,}923\, Y_3 + 57{,}267\, Y_4 + 448{,}065\, Y_{12} + 618{,}181\, Y_{13} + 569{,}012\, Y_{14} + 425{,}043 Y_{23} + 375{,}875\, Y_{24} + 186{,}190\, Y_{34} + 576{,}988\, Y_{123} + 505{,}332\, Y_{124} + 630{,}473\, Y_{134} + 437{,}335\, Y_{234} + 634{,}255\, Y_{1234}$

Subject to

Constraints

$$Y_1 + Y_{12} + Y_{13} + Y_{14} + Y_{123} + Y_{124} + Y_{134} + Y_{1234} \geq 0$$
$$Y_2 + Y_{12} + Y_{23} + Y_{24} + Y_{123} + Y_{124} + Y_{234} + Y_{1234} \geq 0$$
$$Y_3 + Y_{13} + Y_{23} + Y_{34} + Y_{123} + Y_{134} + Y_{234} + Y_{1234} \geq 0$$
$$Y_4 + Y_{14} + Y_{24} + Y_{34} + Y_{124} + Y_{134} + Y_{234} + Y_{1234} \geq 0$$
$$Y_1 + Y_2 + Y_{13} + Y_{14} + Y_{23} + Y_{24} + Y_{134} + Y_{234} \geq 1$$

$$Y_3, Y_{12}, Y_{34}, Y_{124}, Y_4, Y_{123}, Y_{1234}\ unrestricted$$

The optimal solution for the dual model is shown in Table A.14. In addition, the savings $t(x,\{134\})$ and $t(x,\{234\})$ cannot be further maximized based on the fact that Y_{134} and Y_{134} are greater than zero. The coalitions {1,3,4} and {2,3,4} are added to the set F_5. As a consequence, for the 6th LP, coalitional rationality constraints of {1,3,4} and {2,3,4} are converted into equalities by subtracting the corresponding C(134) and C(234) the RHS from optimal t_5^*.

Table A.14: Optimal Solution for Dual 5th LP Model in Sequential LP Approach

DUAL								Conversion Constraints
Y_1	0.0	Y_{12}	-0.5	Y_{24}	0.0	$\boldsymbol{Y_{134}}$	**0.5**	$F_5 = \{\{134\},\{234\}\}$
Y_2	0.0	Y_{13}	0.0	Y_{34}	0.0	$\boldsymbol{Y_{234}}$	**0.5**	
Y_3	-1.0	Y_{14}	0.0	Y_{123}	0.0	Y_{1234}	0.0	
Y_4	-1.0	Y_{23}	0.0	Y_{124}	0.0			

However, solving of the 6th least core LP model is not necessary because the set of constraints in the 5th LP model defined a unique solution. Therefore, the sequential LP approach is terminated and it is concluded that the solution to the 5th LP (last LP in sequence) is the nucleolus.

The nucleolus solution for this HCA problem is $\{ x_1^{**}, x_2^{**}, x_3^{**}, x_4^{**}\}$

$$X_N^* = (\$\mathbf{320,601.5}, \$\mathbf{127,463.5}, \$\mathbf{128,923}, \$\mathbf{57,267})$$

Discussion on Termination Criterion:

It is important to only terminate the sequential LP approach when the last LP model is confirmed to be a unique optimal solution. The best way to ensure that this is in fact the case is by constantly evaluating the tableau of every LP model in the sequence that is being solved. Commercial software's such as LINDO have the capability to display the Tableaus for every LP model being solved.

Therefore, LINDO software allows the user to evaluate the optimal tableau for every nth least core LP model under consideration. The dissertation recommends scanning the tableau and looking for the magnitude of coefficients of all the non-basic variables in the objective function. If the costs coefficients or the reduced cost coefficient of these non-basic variables are strictly greater than zero for a maximization problem, then the optimality conditions have been satisfied and thus the simplex algorithm has arrived at a unique optimal solution. Since the solution is unique, the allocation is confirmed to be the nucleolus.

Discussion

Analyzing the structure of the four vehicle class Primal Least Core LP formulation as discussed in the example presented in this dissertatio, there are a total of 11 variables including the decision and slack variables.

$m = 15$ constraints

$n = 20$ decision variables

$n - m = 5\; non - basic\; variables$

The above property can be generalized for any *n-vehicle* class HCA problem and say the following. For any HCA problem. Let n be total number of variables consisting of decision variables, slack variables and artificial variables and m be the total number of constraints in the model.

$n = 2^n - 1 + n + 1$ $Decision\ variables$

$m = 2^n - 1$ $Constraints$

$n - m = 2^n - 1 + n + 1 - (2^n - 1) = 2^n - 1 + n + 1 - 2^n + 1$

$n - m = n + 1$ $non-basic\ variables$

Stopping Rule: If the cost coefficients of $n + 1$ non-basic variables in the optimal tableau of the least core LP formulation are strictly non-zero, then it can be concluded that an unique allocation has been determined for the least core LP formulation. However, if any one of the cost coefficients of non-basic variables are equal to zero for a maximization problem, then the algorithm must be continued to a point when one of the subsequent least core LPs defines a unique solution.

Further, the nucleolus obtained using the sequential approach is the same as the solution obtained by taking the average of all the optimal corner point feasible solutions as reported in Table A.15. Although, it was possible to obtain the nucleolus for a four vehicle class HCA problem by simply taking the average of all optimal basic feasible solutions of the 1st LP, doing so would be computationally inefficient for large scale highway cost allocation problems. In large-scale problems, it is much faster to implement the sequential LP approach.

Table A.15: The Nucleolus with Average of CPFS and Sequential LP Approach

Illustration A	Illustration B
Average CPFS	Sequential LP Approach
$320,601.5	$320,601.5
$127,463.5	$127,463.5
$128,923	$128,923
$57,267	$57,267
$634,255	$634,255

Illustration C: Lexicographical Comparison

The nucleolus is now characterized based on the definition of lexicographical ordering of savings achieved by all coalitions. In order to do this comparison, two allocations are needed that satisfy completeness. For this illustration, one allocation is chosen to be the nucleolus X_N^* and other allocation is selected to be the allocation generated by the 4th LP (III-3). Let X_N^* be the nucleolus obtained as an optimal solution to the 5th least core LP formulation as shown below:

$$X_N^* = (\$320{,}601.5, \$127{,}463.5, \$128{,}923, \$57{,}267)$$

Another obtained as an optimal solution to the 4th primal least core LP formulation is used as a basis for facilitating the lexicographical comparison. It is important to note that this allocation was one of the multiple optimal solutions of the 4th LP. The allocation acquired from the 4th LP is shown below-

$$x = (\$331{,}845, \$116{,}220, \$128{,}923, \$57{,}267)$$

The vector of savings $t_m(x, s)$ for allocation x is represented by the following relationship

$$t_m(x, s) = C(S) - \sum_{i \in S} x_i$$

The vector of savings $t_m(N^*, s)$ for allocation X_N^* is represented by the following relationship

$$t_m(X_N^*, s) = C(S) - \sum_{i \in S} x_i$$

The savings achieved by each coalition are arranged in the increasing order as shown in Table A.16. An element to element comparison of savings obtained by respective coalitions is performed. On performing the comparison, the first coalition that has a higher savings in one allocation is deemed lexicographically greater than another allocation. The concept behind lexicographical comparison is that by ordering the savings in the increasing order, the level of satisfaction for each coalition with their respective allocations is examined. For instance, consider allocation A and allocation B, if the magnitude of savings achieved by least satisfied coalition in A is comparatively larger than the magnitude of savings achieved by the least satisfied coalition B, then overall the allocation A is much more stable in the sense that the most dissatisfied coalition in A is more content then the least satisfied coalition in B. Further, if allocation A is the nucleolus, then the least satisfied coalition in allocation A is as content as he can ever become relative to any other plausible allocations that at minimum satisfies the completeness criterion.

Table A.16: Lexicographical Order of Savings of Coalitions

Lexicographical Order of Nucleolus					
m	$t_m(X_N^*, s)$		$t_m(x, s)$		m
	$X_N^* >_{lex} x$				
1	C(123)	\$44,975	\$44,975	C(123)	1
2	C(4)	\$44,975	\$44,975	C(4)	2
3	C(124)	\$67,463	\$67,463	C(124)	3
4	C(34)	\$67,463	\$67,463	C(34)	4
5	C(12)	\$112,438	\$112,438	C(12)	5
6	C(3)	\$112,438	\$112,438	C(3)	6
7	C(134)	\$123,681	\$112,438	C(134)	7
8	C(234)	\$123,681	\$134,925	C(234)	8
9	C(13)	\$168,656	\$157,413	C(13)	9
10	C(23)	\$168,656	\$179,900	C(23)	10
11	C(14)	\$191,144	\$179,900	C(14)	11
12	C(24)	\$191,144	\$202,388	C(24)	12
13	C(1)	\$236,119	\$224,875	C(1)	13
14	C(2)	\$236,119	\$247,363	C(2)	14

Table A.16 shows the lexicographical comparison of savings achieved by coalitions with allocation *x* and X_N^*. The savings vectors are now arranged in the increasing order of magnitude as shown in Table A.16 to make an element to element comparison for the savings obtained by vehicle classes in both the allocations.

$$t_1(X_N^*, S) = t_1(x, S) = \$44{,}975$$

$$t_2(X_N^*, S) = t_2(x, S) = \$44{,}975$$

$$t_3(X_N^*, S) = t_3(x, S) = \$67{,}463$$

$$t_4(X_N^*, S) = t_4(x, S) = \$67{,}463$$

$$t_5(X_N^*, S) = t_5(x, S) = \$112{,}438$$

$$t_6(X_N^*, S) = t_6(x, S) = \$112{,}438$$

$$t_7(X_N^*, S) = \$123{,}681 > t_7(x, S) = \$112{,}438$$

The least satisfied coalition {1,3,4} in X_N^* has a savings of \$123,681 and the least satisfied coalition {1,3,4} in *x* has a lower savings of \$112,438. Therefore, it is concluded that the nucleolus is lexicographically greater than the payoff allocation *x*.

$$X_N^* \succcurlyeq_{\boldsymbol{lex}} \boldsymbol{x}$$

It can also be verified that for any payoff allocation *x* belonging to set X ={ $\sum_{i \in N} x_i = C(N)$}. The nucleolus X_N^* is unique and is always lexicographical greater than any other allocation *x* in *X*.

Illustration D: Usage of Closed-form Solution

This section will demonstrate the efficacy of using the closed-form solutions proposed in this dissertation to get the nucleolus. This dissertation will first solve a formulation of Least Core LP Model in which the pavement cost function is being used to estimate the cost of coalitions C(S) and C(N). Following this, the nucleolus will be calculated by solving the LP model using the simplex algorithm.

The closed-form solution will then be authenticated by inspecting whether these solutions are similar to the ones obtained by solving an LP.

Objective Function $\max t_1$

Subject to

Constraints

$$x_1 + t_1 \leq L_N\left(a + b\sqrt{E_1}\right)$$

$$x_2 + t_1 \leq L_N\left(a + b\sqrt{E_2}\right)$$

$$x_3 + t_1 \leq L_N\left(a + b\sqrt{E_3}\right)$$

$$x_4 + t_1 \leq L_N\left(a + b\sqrt{E_4}\right)$$

$$x_1 + x_2 + t_1 \leq L_N\left(a + b\sqrt{E_1 + E_2}\right)$$

$$x_1 + x_3 + t_1 \leq L_N\left(a + b\sqrt{E_1 + E_3}\right)$$

$$x_1 + x_4 + t_1 \leq L_N\left(a + b\sqrt{E_1 + E_4}\right)$$

$$x_2 + x_3 + t_1 \leq L_N\left(a + b\sqrt{E_2 + E_3}\right)$$

$$x_2 + x_4 + t_1 \leq L_N\left(a + b\sqrt{E_2 + E_4}\right)$$

$$x_3 + x_4 + t_1 \leq L_N\left(a + b\sqrt{E_3 + E_4}\right)$$

$$x_1 + x_2 + x_3 + t_1 \leq L_N\left(a + b\sqrt{E_1 + E_2 + E_3}\right)$$

$$x_1 + x_2 + x_4 + t_1 \leq L_N\left(a + b\sqrt{E_1 + E_2 + E_4}\right)$$

$$x_1 + x_3 + x_4 + t_1 \leq L_N\left(a + b\sqrt{E_1 + E_3 + E_4}\right)$$

$$x_2 + x_3 + x_4 + t_1 \leq L_N\left(a + b\sqrt{E_2 + E_3 + E_4}\right)$$

$$x_1 + x_2 + x_3 + x_4 = L_N\left(a + b\sqrt{E_1 + E_2 + E_3 + E_4}\right)$$

The right hand sides of the above formulation are obtained based on the input Table A.17 as shown on the next page. Next, Table A.17 represents the necessary input data required for formulating the least core LP formulation when using the pavement cost function as the characteristic form of the game.

Table A.17: Input Data Least Core LP Model with Pavement Cost function

	S	ESALs	E(S)	L(S)	*C(S)*	$L_N(a + b\sqrt{E_i})$
1	{1}	E_1	90,000	6	C(1)	$470,940
2	{2}	E_2	20,000	6	C(2)	$407,325
3	{3}	E_3	325,000	6	C(3)	$579,288
4	{4}	E_4	65,000	6	C(4)	$452,868
5	{1,2}	$E_1 + E_2$	110,000	6	C(12)	$483,642
6	{1,3}	$E_1 + E_3$	415,000	6	C(13)	$609,021
7	{1,4}	$E_1 + E_4$	155,000	6	C(14)	$508,529
8	{2,3}	$E_2 + E_3$	345,000	6	C(23)	$586,220
9	{2,4}	$E_2 + E_4$	85,000	6	C(24)	$467,549
10	{3,4}	$E_3 + E_4$	390,000	6	C(34)	$601,116
11	{1,2,3}	$E_1 + E_2 + E_3$	435,000	6	C(123)	$615,175
12	{1,2,4}	$E_1 + E_2 + E_4$	175,000	6	C(124)	$518,409
13	{1,3,4}	$E_1 + E_3 + E_4$	480,000	6	C(134)	$628,524
14	{2,3,4}	$E_2 + E_3 + E_4$	410,000	6	C(234)	$607,460
15	{1,2,3,4}	$E_1 + E_2 + E_3 + E_4$	500,000	6	C(1234)	$634,255

Least Core LP Model with Pavement Cost Function

The RHS or cost of coalition *C(S)* and *C(N)* come from the rules described in Chapter IV & Table V.1.

Objective Function max t_1

Subject to

Constraints

$$x_1 + t_1 \leq \$470{,}940$$
$$x_2 + t_1 \leq \$407{,}325$$
$$x_3 + t_1 \leq \$579{,}288$$
$$x_4 + t_1 \leq \$452{,}868$$
$$x_1 + x_2 + t_1 \leq \$483{,}642$$
$$x_1 + x_3 + t_1 \leq \$609{,}021$$
$$x_1 + x_4 + t_1 \leq \$508{,}529$$
$$x_2 + x_3 + t_1 \leq \$586{,}220$$
$$x_2 + x_4 + t_1 \leq \$467{,}549$$
$$x_3 + x_4 + t_1 \leq \$601{,}116$$
$$x_1 + x_2 + x_3 + t_1 \leq \$615{,}175$$
$$x_1 + x_2 + x_4 + t_1 \leq \$518{,}409$$
$$x_1 + x_3 + x_4 + t_1 \leq \$628{,}524$$
$$x_2 + x_3 + x_4 + t_1 \leq \$607{,}460$$
$$x_1 + x_2 + x_3 + x_4 = \$634{,}255$$

Upon solving the 1st LP, the payoff allocation as reported in Table A.18 is found to be $\hat{x} = (\$143{,}495.75, \$122{,}431.75, \$232{,}546.75, \$135{,}780.75)$. The optimal $t_1^* = \$116{,}700.8$. Based on the Tableau, this LP has a unique optimal solution.

Table A.18: Optimal Solution for Primal 1st LP Model in GM

t_1	\$116,700.75
x_1	\$143,495.75
x_2	\$122,431.75
x_3	\$232,546.75
x_4	\$135,780.75
Total	\$634, 255

The Dual Least Core LP Model for 1st Linear Programming Model

The dual model for the least core LP formulation in a four vehicle class problem is shown below.

Objective Function

Min $\$470{,}940\ Y_1 + \$407{,}325\ Y_2 + \$579{,}288\ Y_3 + \$452{,}868\ Y_4 + \$483{,}642 Y_{12} +$
$\$609{,}021\ Y_{13} + \ \$508{,}529\ Y_{14} + \$586{,}220\ Y_{23} + \$467{,}549\ Y_{24} + \$601{,}116\ Y_{34} +$
$\$615{,}175\ Y_{123} + \$518{,}409\ Y_{124} + \$628{,}524\ Y_{134} + \ \$607{,}460\ Y_{234} + \$634{,}255\ Y_{1234}$
Subject to

Constraints

$$Y_1+Y_{12}+Y_{13}+Y_{14}+Y_{123}+Y_{124}+Y_{134} + Y_{1234} \geq 0$$
$$Y_2+Y_{12}+Y_{23}+Y_{24}+Y_{123}+Y_{124}+Y_{234} + Y_{1234} \geq 0$$
$$Y_3+Y_{13}+Y_{23}+Y_{34}+Y_{123}+Y_{134}+Y_{234} + Y_{1234} \geq 0$$
$$Y_4+Y_{14}+Y_{24}+Y_{34}+Y_{124}+Y_{134}+Y_{234} + Y_{1234} \geq 0$$
$$Y_1 + Y_2 + Y_3 + Y_4 + Y_{12} + Y_{13} + Y_{14} + Y_{23} + Y_{24} + Y_{34} + Y_{123} + Y_{124} + Y_{134} + Y_{234} \geq 1$$
$$Y_{1234}\ unrestricted$$

The optimal solution for the dual model is shown Table A.19. Based on the dual model, the savings $t(x,\{1,2,4\})$, $t(x,\{1,2,3\})$, $t(x,\{1,3,4\})$ and $t(x,\{2,3,4\})$ cannot be further maximized based on the fact that Y_{123}, Y_{124} Y_{134} and Y_{234} are strictly greater than zero. Hence, coalition {1,2,4}), {1,2,3}, {1,3,4} and {2,3,4} are added to the set F_1. As a consequence, for the 2nd LP model, the next step in the algorithm would be to convert the coalitional rationality constraint of {1,2,4}, {1,2,3}, {1,3,4} and {2,3,4} into equalities by subtracting its corresponding *C(124)*,*C(123)*,*C(134)* and *C(234)* from optimal t_1^*. However, this will not be required as the 1st LP model represents a system of constraints defining a unique solution.

Table A.19: Optimal Solution for Dual 1st LP Model in GM

DUAL								Conversion Constraints
Y_1	0.00	Y_{12}	0.00	Y_{24}	0.00	$\boldsymbol{Y_{134}}$	**0.25**	$F_1 = \{\{1,2,3\},\{1,2,4\},$ $\{1,2,4\},\{1,3,4\}\}$
Y_2	0.00	Y_{13}	0.00	Y_{34}	0.00	$\boldsymbol{Y_{234}}$	**0.25**	
Y_3	0.00	Y_{14}	0.00	$\boldsymbol{Y_{123}}$	**0.25**	$\boldsymbol{Y_{1234}}$	**- 0.75**	
Y_4	0.00	Y_{23}	0.00	$\boldsymbol{Y_{124}}$	**0.25**			

Based on the termination criterion defined in the earlier section, the above solution is concluded to be a unique optimal solution as cost coefficients of five non-basic variables in the optimal tableau were strictly non-zero. Therefore, the nucleolus X_N^* for this problem is shown below:

$$X_N^* = (\$\mathbf{143,495.75}, \$\mathbf{122,431.75}, \$\mathbf{232,546.75}, \$\mathbf{135,780.75})$$

Since a unique optimal solution was attained when the 1st LP was solved, the next step is to corroborate whether the closed-form solution could have been used to obtain the same optimal allocation without solving the linear program using a software such as LINDO. In other words, is it possible to avoid using the simplex algorithm to get the nucleolus for a least core LP model.

Conditions for Using Closed-form Solutions

The conditions state that the slack variables for coalitions containing 3 vehicle classes and 4 vehicle classes must be equal to zero and the slack variables for all other coalitions containing less than 3 vehicle classes be greater than zero. This is based on the assumption of the greatest lower bound given by the closed-form solutions in Chapter IV. In addition, the closed-form cost allocations relationships must also be positive. In total, $2^n + n - 1$ conditions or 19 conditions must be checked for a four vehicle class HCA problem.

The conditions along with their status of satisfaction is reported in Table A.20. The closed-form solutions give the following allocations to each vehicle class where *C(S)* and *C(N)* have been obtained from Table A.17. The conditions for using CFS will verify whether the solution would be an optimal solution for an LP.

The complete set of conditions have been illustrated in Table A.20. It can be checked that each and every one of the 19 conditions as shown in Table A.20 are satisfied and therefore the closed-forms solution can be used to get the optimal solution as shown in the next section.

Table A.20: Conditions for Using CFS for Least Core with Pavement Cost Function

C(S)	C(S)	Conditions $(2^n + n - 1)$			Status
C(1)	\$470,940	S_1	$C(1) - x_1^* - t_1^* > 0$	210,743.5	Satisfied
C(2)	\$407,325	S_2	$C(2) - x_2^* - t_1^* > 0$	168,192.5	Satisfied
C(3)	\$579,288	S_3	$C(3) - x_3^* - t_1^* > 0$	230,040.5	Satisfied
C(4)	\$452,868	S_4	$C(4) - x_4^* - t_1^* > 0$	200,386.5	Satisfied
C(12)	\$483,642	S_5	$C(12) - (x_1^*+x_2^*) - t_1^* > 0$	101,013.75	Satisfied
C(13)	\$609,021	S_6	$C(13) - (x_1^*+x_3^*) - t_1^* > 0$	116,277.75	Satisfied
C(14)	\$508,529	S_7	$C(14) - (x_1^*+x_4^*) - t_1^* > 0$	112,551.75	Satisfied
C(23)	\$586,220	S_8	$C(23) - (x_2^*+x_3^*) - t_1^* > 0$	114,540.75	Satisfied
C(24)	\$467,549	S_9	$C(24) - (x_2^*+x_4^*) - t_1^* > 0$	92,635.75	Satisfied
C(34)	\$601,116	S_{10}	$C(34) - (x_3^*+x_4^*) - t_1^* > 0$	116,087.75	Satisfied
C(123)	\$615,175	S_{11}	$C(123) - (x_1^*+x_2^*+x_3^*) - t_1^* = 0$	0	Satisfied
C(124)	\$518,409	S_{12}	$C(124) - (x_1^*+x_2^*+x_4^*) - t_1^* = 0$	0	Satisfied
C(134)	\$628,524	S_{13}	$C(134) - (x_1^*+x_3^*+x_4^*) - t_1^* = 0$	0	Satisfied
C(234)	\$607,460	S_{14}	$C(234) - (x_2^*+x_3^*+x_4^*) - t_1^* = 0$	0	Satisfied
		S_{15}	$x_1^* > 0$	\$143,495.75	Satisfied
		S_{16}	$x_2^* > 0$	\$122,431.75	Satisfied
		S_{17}	$x_3^* > 0$	\$232,546.75	Satisfied
		S_{18}	$x_4^* > 0$	\$135,780.75	Satisfied
		S_{19}	$t_1^* > 0$	\$116,700.75	Satisfied

Determination of the Nucleolus Using Closed-form Solutions

The closed-form solution for a four vehicle class HCA problem can be obtained as follows:

$$x_1 = \frac{1}{4}\{C(123) + C(124) + C(134) + C(1234)\} - \frac{3}{4}\{C(234)\}$$

$$= \frac{1}{4}\{\$615{,}175 + \$518{,}409 + \$628{,}524 + \$634{,}255\} - \frac{3}{4}\{\$607{,}460\}$$

$$= \$143{,}495.75$$

$$x_2 = \frac{1}{4}\{C(123) + C(124) + C(234) + C(1234)\} - \frac{3}{4}\{C(134)\}$$

$$= \frac{1}{4}\{\$615{,}175 + \$518{,}409 + \$607{,}460 +, \$634{,}255\} - \frac{3}{4}\{\$628{,}524\}$$

$$= \$122{,}431.75$$

$$x_3 = \frac{1}{4}\{C(123) + C(134) + C(234) + C(1234)\} - \frac{3}{4}\{C(124)\}$$

$$= \frac{1}{4}\{\$615{,}175 + \$628{,}524 + \$607{,}460 + \$634{,}255\} - \frac{3}{4}\{\$518{,}409\}$$

$$= \$232{,}546.75$$

$$x_4 = \frac{1}{4}\{C(124) + C(134) + C(234) + C(1234)\} - \frac{3}{4}\{C(123)\}$$

$$= \frac{1}{4}\{\$518{,}409 + \$628{,}524 + \$607{,}460 + \$634{,}255\} - \frac{3}{4}\{\$615{,}175\}$$

$$= \$135{,}780.75$$

$$t = \frac{1}{4}\{C(124) + C(134) + C(234) + C(123)\} - \frac{3}{4}\{C(1234)\}$$

$$= \frac{1}{4}\{\$615{,}175 + \$518{,}409 + \$628{,}524 + \$607{,}460\} - \frac{3}{4}\{\$634{,}255\}$$

$$= \$116{,}700.75$$

The nucleolus solution for this allocation is $(x_1^*, x_2^*, x_3^*, x_4^*)$ with optimal $t_1 = \$116{,}700.75$

$$\boldsymbol{X_N^*} = (\ \$\mathbf{143{,}495.75}, \$\mathbf{122{,}431.75}, \$\mathbf{232{,}546.75}, \$\mathbf{135{,}780.75})$$

The findings are reported in Table A.21 as shown on the next page.

Table A. 21: Verification of Closed-form Solutions

VEHICLE CLASSES	Generalized Method $X_N^* = (x_1^{**}, x_2^{**}, x_3^{**}, x_4^{**}\}$	Closed-form Solutions $X_N^* = (x_1^*, x_2^*, x_3^*, x_4^*\}$
TYPE I	\$143,495.75	\$143,495.75
TYPE II	\$122,431.75	\$122,431.75
TYPE III	\$232,546.75	\$232,546.75
TYPE IV	\$135,780.75	\$135,780.75
Total	\$634,255	\$634,255

Table A.21 confirms the findings that the nucleolus obtained by solving the least core LP model with pavement cost function as the characteristic function is equal to the allocation obtained using the closed-form solution as proposed in this dissertation.

Illustration E: Non-Usage of Closed-form Solution

The importance of understanding the conditions under which the closed-form solution is able and not able to provide the nucleolus that is desired. Therefore, it is of paramount importance that closed-form solutions are not used when they are not applicable for a given HCA problem. The example presented in this section where the cost of coalitions *C(S)* and *C(N)* or the RHS for the least core LP model are obtained from Table A.3, the nucleolus had to be obtained by solving five LP problems and not a single LP problem. Hence, the capability of the conditions for using closed-form solutions to recognize non-usage is stressed in this section. The input data or the RHS from Table A.3 is substituted into the 19 conditions as shown on the next page. In order to illustrate this non-usage of the proposed analytical formulas, all the 19 conditions must be satisfied in order for the closed-form solutions to be valid. Even if one condition is violated, then the closed-form solution cannot be used. The status regarding the satisfaction of all the conditions of using closed-form solutions is reported in Table A.22.

It can be checked that condition S_4 and S_{10} are not satisfied and since at least one of the 19 conditions are not satisfied, the proposed closed-form solution cannot be used to obtain the optimal allocation for the least core LP model that uses A-S value to define the costs of coalitions.

Table A.22: Conditions for Using CFS for Least Core with Discrete A-S Value

	C(S)		Conditions ($2^n + n - 1$)		Status
C(1)	\$556,720	S_1	$C(1) - x_1^* - t_1^* > 0$	179,899.5	Satisfied
C(2)	\$363,583	S_2	$C(2) - x_2^* - t_1^* > 0$	179,900.5	Satisfied
C(3)	\$241,361	S_3	$C(3) - x_3^* - t_1^* > 0$	0.5	Satisfied
C(4)	\$102,242	S_4	$C(4) - x_4^* - t_1^* > 0$	-89,950.5	Violated
C(12)	\$560,503	S_5	$C(12) - (x_1^*+x_2^*) - t_1^* > 0$	89,950.25	Satisfied
C(13)	\$618,181	S_6	$C(13) - (x_1^*+x_3^*) - t_1^* > 0$	89,950.25	Satisfied
C(14)	\$569,012	S_7	$C(14) - (x_1^*+x_4^*) - t_1^* > 0$	89,949.25	Satisfied
C(23)	\$425,043	S_8	$C(23) - (x_2^*+x_3^*) - t_1^* > 0$	89,950.25	Satisfied
C(24)	\$375,875	S_9	$C(24) - (x_2^*+x_4^*) - t_1^* > 0$	89,950.25	Satisfied
C(34)	\$253,653	S_{10}	$C(34) - (x_3^*+x_4^*) - t_1^* > 0$	-89,949.75	Violated
C(123)	\$621,963	S_{11}	$C(123) - (x_1^*+x_2^*+x_3^*) - t_1^* = 0$	0	Satisfied
C(124)	\$572,795	S_{12}	$C(124) - (x_1^*+x_2^*+x_4^*) - t_1^* = 0$	0	Satisfied
C(134)	\$630,473	S_{13}	$C(134) - (x_1^*+x_3^*+x_4^*) - t_1^* = 0$	0	Satisfied
C(234)	\$437,335	S_{14}	$C(234) - (x_2^*+x_3^*+x_4^*) - t_1^* = 0$	0	Satisfied
		S_{15}	$x_1^* > 0$	286,870.25	Satisfied
		S_{16}	$x_2^* > 0$	93,732.25	Satisfied
		S_{17}	$x_3^* > 0$	151,410.25	Satisfied
		S_{18}	$x_4^* > 0$	102,242.25	Satisfied
		S_{19}	$t_1^* > 0$	\$89,950.25	Satisfied

Expanding on the meaning of the closed-form solutions, solving of the proposed methodology using the simplex algorithm, would culminate in optimal savings in effect being lower than the savings generated by the closed-form solutions. This is illustrated as follows:

$$t^*_{max} = \frac{1}{4}\{C(124) + C(134) + C(234) + C(123)\} - \frac{3}{4}\{C(1234)\}$$

$$= \frac{1}{4}\{621{,}963 + 572{,}795 + 630{,}473 + 437{,}335\} - \frac{3}{4}\{634{,}255\} = \$89{,}950.25$$

$$t^{**}_{max} = \$44{,}975$$

$$\$44{,}975.00 < \$89{,}950.25$$

$$t^{**}_{max} < t^*_{max}$$

Discussion

The coalition that has the lowest savings t^{**}_{max} obtained using the sequential LP approach according to Table A.16 is *C(123)*, which shows that it has achieved a savings value of $44,975. Now, our conditions for using CFS are based on the principle that when the conditions for using closed-form solutions are satisfied, then this translates to the fact that coalition containing only *n-1* vehicle classes have received the lowest savings and so thereby play an important role in deciding the nucleolus. If solving the first LP leads to presence of multiple optimal solutions, then the coalitions receiving the lowest savings are not only the *n-1* vehicle classes, but also could be any one of the remaining coalitions containing less than *n-2* vehicle classes. This fact is validated in the above example because of the following two observations:

- *Least Core LP Model with Pavement cost function:* The coalitions that received the lowest savings when the least core LP model was solved with pavement cost function as the characteristic function were *C(124), C(134), C(234)* and *C(123)*. Each of them consisting of exactly three vehicle classes or *n-1* vehicle classes.
- *Least Core LP Model with Discrete A-S*: Whereas for the case in the proposed methodology having the least core LP model with the discrete Aumann-Shapley value based characteristic function, the coalition that received the lowest savings were in fact *C(123)* and *C(4)*, the latter being coalitions encompassing *n-3* vehicle classes, which is less than *n-2* vehicle classes. The results of this section are summarized in Table A.23 and Table A.24.

Table A.23: Usage and Non-Usage of Closed-form Solution

Usage and Non-Usage of Closed-form Solutions (CFS)		
	Usage	Non-Usage
Conditions for Using CFS	All $2^n + n - 1$ condition are satisfied	At least one of the $2^n + n - 1$ condition is violated
Lowest Savings	*S* containing exactly *n-1* vehicle classes and *N* containing *n* vehicle classes	*S* containing less than *n-1* vehicle classes and *N* containing *n* vehicle classes
Computation of Nucleolus	Use Closed-form Solutions	Sequential LP Approach

Table A.24: Conditions for Using Closed-form Solutions

Least Core LP Model for Four Vehicle Class Dissertation Example		
Characteristic Function	Pavement Cost Function	Discrete Aumann-Shapley value
Condition for Using Closed-form Solution	SATISFIED	NOT-SATISFIED

APPENDIX B

OPTIMAL VALUES OF DUAL BASIC VARIABLES IN SPECIAL CASE

The purpose of this appendix is to show that the values of dual basic variables when the special case is being exhibited will be $1/n$ for coalitional rationality constraints containing *n-1* vehicle classes and *n-1/n* for coalitions containing *n* vehicle classes. The dual of the least core LP formulation can be expressed in a generalized notation as following:

Objective function

Min $\sum_{S\in K} C(S) * Y_S$ (C-1)

Constraints

$\sum_{S\in M:\,\forall i\in S} Y_S \geq 0 \qquad \forall j \in N$ (C-2)

$\sum_{S\in M\backslash N} Y_S = 1$ (C-3)

Next, all the dual variables Y_S of coalitional rationality constraints of *S* with *n-2* vehicle classes are taken to the right hand sides as shown in (C-5).

$\sum_{S\in M:\,\forall i\in S} Y_S \geq 0 \qquad \forall j \in N$ (C-4)

$\sum_{S\in M\backslash N} Y_S = 1 - \sum_{S\in M:\,\forall i\in S} Y_S$ LHS→ $|S| = n - 1$, RHS→ $|S| = n - 2$ (C-5)

It is known that the dual of the dual is the primal basic variables defined by the allocations x_i and optimal savings t_1 for all $i \in N$. These primal basic variables must be $x_i > 0$ and $t_i > 0$. Based on this reason, the symbols of "≥" are converted into "=" as per the complementary slackness conditions. The dual variables for coalitions containing *n-2* vehicle classes in (C-4) and (C-5) are shifted to the RHS as shown in (C-6) and (C-7).

$\sum_{S\in M:\,\forall i\in S} Y_S = -Y_S \qquad \forall j \in N$ (C-6)

where,

- LHS→ $|S| = n - 1$
- RHS → for all coalitions with 1< $|S| < n - (z - 1)$ where *z=1,2,.....,n*

$\sum_{S\in M\backslash N} Y_S = 1 - \sum_{S\in M:\,\forall i\in S} Y_S$ (C-7) where,

- LHS→ $|S| = n - 1$
- RHS → for all coalitions with 1< $|S| < n - (z - 1)$ where *z=1,2,…..,n*

The equations represented by (C-6) and (C-7) are added to get (C-8).

$$n \sum_{S \in N} Y_S = \sum_{S \in M:\, \forall i \in S} Y_S + 1 - \sum_{S \in M:\, \forall i \in S} Y_S \tag{C-8}$$

Next, the dual variables Y_S for coalition S containing *n-1* vehicle classes and the dual variables Y_N for coalition *N* with *n* vehicle classes is written in terms of the summation of all other dual variables containing less than *n-2* vehicle classes as shown in (C-9) and (C-10).

$$Y_S = \sum_{S \in M \setminus N} \frac{1}{n} Y_S \qquad \text{LHS} \rightarrow |S| = n - 1\text{ , RHS} \rightarrow |S| = n - 2 \tag{C-9}$$

where,

- LHS→ $|S| = n - 1$
- RHS → for all coalitions with 1< $|S| < n - (z - 1)$ where *z=1,2,…..,n*

$$Y_N = \sum_{S \in M \setminus N} \frac{n-1}{n} Y_S \qquad \text{LHS} \rightarrow |S| = n - 1\text{ , RHS} \rightarrow |S| = n - 2 \tag{C-10}$$

where,

- LHS→ $|S| = n - 1$
- RHS → for all coalitions with 1< $|S| < n - (z - 1)$ where *z=1,2,…..,n*

As a final step, equation (C-8) can be further simplified using (C-9) and (C-10) to get a representation of dual variable Y_S with *n-1* vehicle classes in terms of all other dual variables containing less than *n-2* vehicle classes for a given *n-vehicle* class dual least core LP formulation.

Using the fact that the slack variables in the primal are all greater than zero for coalitional rationality constraint containing less than *n-2* vehicle class. It can be assumed that the dual variables Y_S for coalition *S* containing fewer than *n-2* vehicle classes must also be zero in the dual.

Therefore, setting $Y_S = 0$ for all *S* with 1< $|S| < n - (z - 1)$ where *z=1,2,…..,n* in (C-8)

Solving the above equations, the optimal values for the dual variables for coalitions containing *n-1* vehicle classes and *n* vehicle classes in the least core LP formulation using pavement cost function as the characteristic function is represented as the following:

$$Y_{\substack{S\in M:\,\forall i\in S\\ |S|=n-1}} = \frac{1}{n} \tag{C-11}$$

$$Y_{\substack{S\in M:\,\forall i\notin S\\ |S|=n}} = \frac{n-1}{n} \tag{C-12}$$

The above relationship essentially states that the dual variables for coalition *S* containing *n-1* and grand coalition *N* containing *n* vehicle classes will be of the magnitude of $\frac{1}{n}$ and $\frac{n-1}{n}$ respectively. Therefore, the solutions would be unique due to the fact that the value of basic variables of the dual Y_S for coalitional rationality constraints *S* for *n-1* vehicle classes and *n* vehicle classes would always be non-zero.

APPENDIX C

CONDITIONS FOR USING CLOSED-FORM SOLUTIONS FOR FOUR VEHICLE CLASSES

$$S_1 \Rightarrow C(1)-\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(1234)-\frac{3}{4}C(234)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_2 \Rightarrow C(2)-\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(134)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_3 \Rightarrow C(3)-\left(\frac{1}{4}C(123)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(124)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_4 \Rightarrow C(4)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(123)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_5 \Rightarrow C(12)-\left(\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(1234)-\frac{3}{4}C(234)\right)+\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(134)\right)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_6 \Rightarrow C(13)-\left(\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(1234)-\frac{3}{4}C(234)\right)+\left(\frac{1}{4}C(123)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(124)\right)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_7 \Rightarrow C(14)-\left(\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(1234)-\frac{3}{4}C(234)\right)+\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(123)\right)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_8 \Rightarrow C(23)-\left(\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(134)\right)+\left(\frac{1}{4}C(123)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(124)\right)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_9 \Rightarrow C(24)-\left(\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(134)\right)+\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(123)\right)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_{10} \Rightarrow C(34)-\left(\left(\frac{1}{4}C(123)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(124)\right)+\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(123)\right)\right)-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)>0$$

$$S_{11} \Rightarrow C(123)-\left(\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(1234)-\frac{3}{4}C(234)\right)+\left(\frac{1}{4}C(123)+\frac{1}{4}C(124)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(134)\right)+\left(\frac{1}{4}C(123)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(1234)-\frac{3}{4}C(124)\right)\right)$$
$$-\left(\frac{1}{4}C(124)+\frac{1}{4}C(134)+\frac{1}{4}C(234)+\frac{1}{4}C(123)-\frac{3}{4}C(1234)\right)=0$$

$$\mathbf{S_{12}} \Rightarrow \mathbf{C(124)} - \left(\left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(234)}\right) + \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(134)}\right)\right.$$

$$\left. + \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(123)}\right)\right) - \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(1234)}\right) = \mathbf{0}$$

$$\mathbf{S_{13}} \Rightarrow \mathbf{C(134)} - \left(\left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(234)}\right) + \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(124)}\right)\right.$$

$$\left. + \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(123)}\right)\right) - \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(1234)}\right) = \mathbf{0}$$

$$\mathbf{S_{14}} \Rightarrow \mathbf{C(234)} - \left(\left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(134)}\right) + \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(124)}\right)\right.$$

$$\left. + \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(123)}\right)\right) - \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(1234)}\right) = \mathbf{0}$$

$$\mathbf{S_{14}} \Rightarrow \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(234)}\right) > 0$$

$$\mathbf{S_{15}} \Rightarrow \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(134)}\right) > 0$$

$$\mathbf{S_{16}} \Rightarrow \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(124)}\right) > 0$$

$$\mathbf{S_{17}} \Rightarrow \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(1234)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(123)}\right) > 0$$

$$\mathbf{S_{18}} \Rightarrow \left(\frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(124)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(134)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(234)} + \frac{\mathbf{1}}{\mathbf{4}}\mathbf{C(123)} - \frac{\mathbf{3}}{\mathbf{4}}\mathbf{C(1234)}\right) > 0$$

APPENDIX D

HIGHWAY COST ALLOCATION FOR SEVEN VEHICLE CLASSES

(C_e= \$0.002426641 per ESAL & C_l=\$19,831.9083 per Lane)

Nr.	S	L_S	Annual ESALs	Cost (\$/lane-mile)	Nr.	S	L_S	Annual ESALs	Cost (\$/lane-mile)	Nr.	S	L_S	Annual ESALs	Cost (\$/lane-mile)
1	1	3	300,000	$60,223.72	43	1,6,7	3	1,750,000	$63,742.35	85	2,3,4,6	3	2,200,000	$64,834.34
2	2	3	400,000	$60,466.38	44	2,3,4	3	1,500,000	$63,135.69	86	2,3,4,7	3	2,250,000	$64,955.67
3	3	3	500,000	$60,709.05	45	2,3,5	2	1,650,000	$43,667.77	87	2,3,5,6	3	2,250,000	$64,955.67
4	4	2	600,000	$41,119.80	46	2,3,6	3	1,600,000	$63,378.35	88	2,3,5,7	3	2,300,000	$65,077.00
5	5	2	650,000	$41,241.13	47	2,3,7	3	1,650,000	$63,499.68	89	2,3,6,7	3	2,350,000	$65,198.33
6	6	2	700,000	$41,362.47	48	2,4,5	3	1,650,000	$63,499.68	90	2,4,5,6	3	2,350,000	$65,198.33
7	7	1	750,000	$21,651.89	49	2,4,6	3	1,700,000	$63,621.01	91	2,4,5,7	3	2,400,000	$65,319.66
8	1,2	3	700,000	$61,194.37	50	2,4,7	3	1,750,000	$63,742.35	92	2,4,6,7	3	2,450,000	$65,441.00
9	1,3	3	800,000	$61,437.04	51	2,5,6	3	1,750,000	$63,742.35	93	2,5,6,7	3	2,500,000	$65,562.33
10	1,4	3	900,000	$61,679.70	52	2,5,7	3	1,800,000	$63,863.68	94	3,4,5,6	3	2,450,000	$65,441.00
11	1,5	3	950,000	$61,801.03	53	2,6,7	3	1,850,000	$63,985.01	95	3,4,5,7	3	2,500,000	$65,562.33
12	1,6	3	1,000,000	$61,922.37	54	3,4,5	2	1,750,000	$43,910.44	96	3,4,6,7	3	1,800,000	$63,863.68
13	1,7	2	1,050,000	$42,211.79	55	3,4,6	3	1,800,000	$63,863.68	97	3,5,6,7	3	2,600,000	$65,804.99
14	2,3	3	900,000	$61,679.70	56	3,4,7	3	1,850,000	$63,985.01	98	4,5,6,7	3	2,700,000	$66,047.66
15	2,4	3	1,000,000	$61,922.37	57	3,5,6	3	1,850,000	$63,985.01	99	1,2,3,4,5	3	2,450,000	$65,441.00
16	2,5	3	1,050,000	$62,043.70	58	3,5,7	2	1,900,000	$44,274.43	100	1,2,3,4,6	3	2,500,000	$65,562.33
17	2,6	3	1,100,000	$62,165.03	59	3,6,7	3	1,950,000	$64,227.67	101	1,2,3,4,7	3	2,550,000	$65,683.66
18	2,7	2	1,150,000	$42,454.45	60	4,5,6	3	1,950,000	$64,227.67	102	1,2,3,5,6	3	2,550,000	$65,683.66
19	3,4	3	1,100,000	$62,165.03	61	4,5,7	2	2,000,000	$44,517.10	103	1,2,3,5,7	3	2,600,000	$65,804.99
20	3,5	3	1,150,000	$62,286.36	62	4,6,7	3	2,050,000	$64,470.34	104	1,2,3,6,7	3	2,650,000	$65,926.32
21	3,6	2	1,200,000	$42,575.79	63	5,6,7	3	2,100,000	$64,591.67	105	1,2,4,5,6	3	2,650,000	$65,926.32
22	3,7	2	1,250,000	$42,697.12	64	1,2,3,4	3	1,900,000	$64,106.34	106	1,2,4,5,7	3	2,700,000	$66,047.66
23	4,5	3	1,250,000	$62,529.03	65	1,2,3,5	3	1,850,000	$63,985.01	107	1,2,4,6,7	3	2,750,000	$66,168.99
24	4,6	3	1,300,000	$62,650.36	66	1,2,3,6	3	1,900,000	$64,106.34	108	1,2,5,6,7	3	2,800,000	$66,290.32
25	4,7	2	1,350,000	$42,939.78	67	1,2,3,7	3	1,950,000	$64,227.67	109	1,3,4,5,6	3	2,750,000	$66,168.99
26	5,6	3	1,350,000	$62,771.69	68	1,2,4,5	3	1,950,000	$64,227.67	110	1,3,4,5,7	3	2,800,000	$66,290.32
27	5,7	2	1,400,000	$43,061.11	69	1,2,4,6	3	2,000,000	$64,349.01	111	1,3,4,6,7	3	2,850,000	$66,411.65
28	6,7	2	1,450,000	$43,182.45	70	1,2,4,7	3	2,050,000	$64,470.34	112	1,3,5,6,7	3	2,900,000	$66,532.98
29	1,2,3	3	1,200,000	$62,407.69	71	1,2,5,6	3	2,050,000	$64,470.34	113	1,4,5,6,7	3	3,000,000	$66,775.65
30	1,2,4	3	1,300,000	$62,650.36	72	1,2,5,7	3	2,100,000	$64,591.67	114	2,3,4,5,6	2	2,850,000	$46,579.74
31	1,2,5	2	1,350,000	$42,939.78	73	1,2,6,7	3	2,150,000	$64,713.00	115	2,3,4,5,7	3	2,900,000	$66,532.98
32	1,2,6	3	1,400,000	$62,893.02	74	1,3,4,5	2	2,050,000	$44,638.43	116	2,3,4,6,7	2	2,950,000	$46,822.41
33	1,2,7	3	1,450,000	$63,014.35	75	1,3,4,6	3	2,100,000	$64,591.67	117	2,3,5,6,7	3	3,000,000	$66,775.65
34	1,3,4	3	1,400,000	$62,893.02	76	1,3,4,7	3	2,150,000	$64,713.00	118	2,4,5,6,7	3	3,100,000	$67,018.31
35	1,3,5	3	1,450,000	$63,014.35	77	1,3,5,6	3	2,150,000	$64,713.00	119	3,4,5,6,7	3	3,200,000	$67,260.98
36	1,3,6	3	1,500,000	$63,135.69	78	1,3,5,7	3	1,700,000	$63,621.01	120	1,2,3,4,5,6	3	3,150,000	$67,139.64
37	1,3,7	3	1,550,000	$63,257.02	79	1,3,6,7	3	2,250,000	$64,955.67	121	1,2,3,4,5,7	3	3,200,000	$67,260.98
38	1,4,5	3	1,550,000	$63,257.02	80	1,4,5,6	3	2,250,000	$64,955.67	122	1,2,3,4,6,7	3	3,250,000	$67,382.31
39	1,4,6	3	1,600,000	$63,378.35	81	1,4,5,7	3	2,300,000	$65,077.00	123	1,2,3,5,6,7	3	3,300,000	$67,503.64
40	1,4,7,	3	1,650,000	$63,499.68	82	1,4,6,7	3	2,350,000	$65,198.33	124	1,2,4,5,6,7	3	3,400,000	$67,746.30
41	1,5,6,	3	1,650,000	$63,499.68	83	1,5,6,7	2	2,400,000	$45,487.76	125	1,3,4,5,6,7	3	3,500,000	$67,988.97
42	1,5,7,	3	1,700,000	$63,621.01	84	2,3,4,5	3	2,150,000	$64,713.00	126	2,3,4,5,6,7	2	3,600,000	$48,399.72

Nr.	S	L_n	Annual ESALs	
Total Cost of Grand Coalition				
127	1,2,3,4,5,6,7	3	3,900,000	$68,959.62

RESULTS

Vehicle Classes	**Least Core LP Model-I**	**Least Core LP Model-II (Nucleolus)**
	Discrete A-S Approach	**Discrete A-S Approach**
	Cost Allocation (Multiple)	**Cost Allocation (Unique Solution)**
Savings	t=\$2,832.857	t=\$2,832.863
Vehicle Class-1	\$23,392.857	\$23,392.863
Vehicle Class-2	\$12,303.429	\$12,303.432
Vehicle Class-3	\$4,046.857	\$4,046.863
Vehicle Class-4	\$9,954.571	\$9,954.561
Vehicle Class-5	\$4,410.857	\$4,410.850
Vehicle Class-6	\$4,531.857	\$4,531.863
Vehicle Class-7	\$10,319.571	\$10,319.568
Total Cost	\$68,960.000	\$68,960.000

Environmental Costs

$$C(X) = l(13522.64631 + 4.792231426\sqrt{W_m}),\ W_m = 0,$$

$$\boldsymbol{C(X) = \$13,522.64631}\ \textbf{per lane mile}$$

(No. of lanes L_S for each coalition were determined independently)

Least Core Discrete A-S based LP Model

max t

Subject to

$$\textstyle\sum_{i\in N} x_i = C(N)$$

$$\textstyle\sum_{i\in S} x_i \le C(S)\ \ for\ all\ S \in N$$

$$c(S) = E_i C_e + E_i C_l$$

VITA

Mr. Saurav Kumar Dubey completed his Ph.D. in Industrial Engineering from the University of Tennessee at Knoxville in August 2017. He plans to continue his research work in academia or industry as opportunities present themselves in the area of Strategic Cost Management, Game Theory, Cost/Benefit Optimization and Transportation Economics. He earned a Master of Science in Mechanical Engineering from Boston University in Boston, Massachusetts. In addition, he holds a Bachelor of Science in Mechanical Engineering and a Bachelor of Science in Mathematics from Southern Methodist University in Dallas, Texas.