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INVENTORY VALUATIONS—AN ANALYTIC TECHNIQUE*

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Recursive inventory decision processes, when formulated as dynamic programs, lead to functional equations. The unknown functions in these equations have arguments which range over the set of possible inventory conditions in the process, and values which represent the minimum possible expected present value of the costs of operating the process through the indefinite future. We call these functions "inventory valuations".

The realism of such formulations and the difficulties of solving for the implicitly defined functions lead to an alternative proposal: that approximate inventory valuations be induced more directly from the context of the recursive inventory process. Methods of developing such valuations are illustrated in sample analyses.

Introduction

The recognition of recursive structure in inventory problems leads to the implicit functional equation approach which is characteristic of dynamic programming [1]. But, as powerful a descriptive tool as these recursive decision structures are, they are extremely difficult to handle analytically. This difficulty is, in part, inherent in the very reality of the formulations. Simple analyses are not possible in highly nonlinear functional equations.

In view of these conflicting properties an alternative principle of analysis seems useful which is conceptually close to the functional equation approach of dynamic programming, but which incorporates many of its essential features in a more amenable framework of analysis. The idea is to ask slightly different questions of the realities of inventory problems. Instead of asking what identities certain unknown functions must satisfy, we ask directly what the functions must be. This leads to the construction of the unknown functions rather than to their implicit definition.

The alternate question we pose is not mathematically equivalent to the original. But it does not appear less realistic. There simply is no logical basis to favor one or the other on the basis of realism. That is a matter of the investigator's intuition and taste in the final analysis. It is believed, however, that the questions are practically equivalent in many inventory problems, and the alternate question often brings the matter within analytical reach not otherwise available.

Section 1 formulates a class of recursive inventory decision problems which seems well suited for describing a wide variety of inventory models. It also provides the framework for stating the dynamic programming point of view. Section 2 develops a characteristic feature of dynamic programming, namely "state valuations," or in the case at hand, "inventory valuations." In dynamic

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programming, inventory valuations are unknown functions (of inventory levels); our proposal is to construct similar inventory valuations directly from the knowledge of the real inventory situation. Section 3 then illustrates this approach in two specific problems.

1. Recursive Inventory Decision Problems

We formulate a recursive inventory decision problem as a mathematical system, M of two random variables R, S , each functions of two other variables s, d ,

$$M = [R(s, d), S(s, d)]$$

where we interpret

- s as a "state" of an operation at the beginning of an administrative time period—usually a vector, reflecting a more or less complex configuration of inventory in various locations, or conditions of availability
- d as a "decision" in the operation made during the time period—usually a vector representing a structure of more elementary actions of ordering, shipping, or disposing of inventory
- $R(s, d)$ as the "return" to the operation during the time period when decision d is employed in state s —the short term profit-cost (or cash flow) situation
- $S(s, d)$ as the "successor state" following s at the beginning of the succeeding time period if decision d is employed—a new vector (possibly the same) reflecting the new configuration of inventory.

As random variables, $R(s, d)$ and $S(s, d)$ represent returns and successor states subject to statistical uncertainties as is so often postulated in inventory problems. In order to express serially correlated relations, a state s may be a "history" rather than a "snapshot" of the configurations of the operation.

In ordinary parlance, M is the "model" of the situation under study; it contains all the kinematic necessary relations, such as material balances, disposition and evaluation of shortage conditions, costing of activities in the operation, etc.

To illustrate these ideas, consider a specific inventory model as follows. At each time period t , we assume the state s is determined by inventory on hand and in transit to an inventory point—up to three periods—

$$s = (i_0, i_1, i_2, i_3),$$

where i_k is inventory to arrive at $t + k$. For now we allow i_0 to be negative, to reflect back order conditions, but will require i_1, i_2, i_3 to be non-negative. During a period a certain demand, say x , occurs. This demand is filled out of i_0 , if possible or if not, the remaining unfilled demand is added on to the present backlog of orders, if any. During the period, two decisions are to be made, ordering by a "fast" route, with 1 period of transit time, and ordering by a

“normal” route with 3 periods of transit time,

$$d = (q_1, q_3).$$

Finally, we shall assume four linear costs in the operation

c_1	cost per unit ordered over fast route
c_3	cost per unit ordered over normal route
h	cost per unit inventory held over
k	cost per unit backlog held over

We can summarize the situation explicitly with regard to how new states and returns arise out of old states and decisions, as

$$S(s, d) = (i_0 + i_1 - x, i_2 + q_1, i_3, q_3)$$

$$R(s, d) = -c_1q_1 - c_3q_3 - h \max(0, i_0 - x) - k \max(0, x - i_0)$$

where x is a given real random variable (possibly dependent on time). Thus

$$M = [S(s, d), R(s, d)]$$

completely describes our interest in the situation.

In order to hypothecate dynamical behavior in the situation under study, we adjoin a “decision policy” p to the model, relating states to decisions, in functional form as

$$d = p(s, t),$$

where we allow the decision to depend on the time of the choice as well as the state. When p is employed in the model, dynamic behavior is determined in terms of a stochastic process P , defined more precisely as follows. Beginning with an initial state, s_0 , we define a stochastic process

$$P = (s_0, d_0, r_0, s_1, d_1, r_1, \dots)$$

by the relations, for $t = 0, 1, 2, \dots$

$$(1) \quad d_t = p(s_t, t),$$

$$(2) \quad r_t = R(s_t, d_t),$$

$$(3) \quad s_{t+1} = S(s_t, d_t).$$

Given s_0 , it is clear that (1), (2), and (3) determine a random process P uniquely, since every partial sequence

$$P_k = (s_0, d_0, r_0, \dots, s_k, d_k, r_k)$$

has a probability distribution which can be constructed recursively from (1), (2), and (3).

2. Inventory Valuations

Ultimately, our interest in P is centered on the subsequence

$$r_0, r_1, \dots$$

of returns, generally on a discounted sum,

$$T = \sum_{j=0}^{\infty} \alpha^j r_j.$$

where $0 < \alpha < 1$. The random variable T describes, in the present value sense, the performance of the operation. We will consider T as an explicit function of s_0 and p , writing

$$T = T(s_0, p).$$

We define the "state valuations",

$$(4) \quad F(s) = \max_p E[T(s, p)].$$

The key relationship of dynamic programming is contained in the following theorem, given in more general form in [2].

Theorem. $F(s)$ satisfies (4) if and only if it satisfies

$$(5) \quad F(s) = \max_d E[R(s, d) + \alpha F(S(s, d))].$$

Proof. Notice, since

$$\begin{aligned} T(s, p) &= r_0 + \alpha r_1 + \alpha^2 r_2 + \dots \\ &= r_0 + \alpha(r_1 + \alpha r_2 + \dots) \end{aligned}$$

where $s = s_0$ and $r_0, r_1, \dots$, is a subsequence of P , that

$$T(s, p) = R(s, d) + \alpha T(S(s, d), p'),$$

where

$$d = p(s, o), p'(s, t) = p(s, t + 1).$$

Now, let $F(s)$ satisfy (4). Then

$$\begin{aligned} F(s) &= \max_p E[T(s, p)] \\ &= \max_p E[R(s, d) + \alpha T(S(s, d), p')] \\ &= \max_d E[R(s, d) + \alpha \max_{p'} E[T(S(s, d), p')]] \end{aligned}$$

whence

$$F(s) = \max_d E[R(s, d) + \alpha F(S(s, d))]$$

which is (5).

On the other hand, let $F(s)$ satisfy (5). Then

$$\begin{aligned}
 F(s) &= \max_d E[R(s, d) + \alpha F(s, d)] \\
 &= \max_d E[R(s, d) + \alpha \max_{d_1} E[R(S(s, d), d_1) + \alpha F(S(s, d), d_1)]] \\
 &= \max_{d_0, d_1, d_2, \dots} E[R(s, d_0) + \alpha R(S(s, d_0), d_1) \\
 &\quad + \alpha^2 R(S(S(s, d_0), d_1), d_2) + \dots]
 \end{aligned}$$

whence

$$F(s) = \max_p E[T(s, p)]$$

which is (4).

The central point of difference between (4) and (5) is that $F(s)$ is described explicitly in (4) by means of a maximum operator on all future returns, over a policy space, while $F(s)$ is described implicitly in (5) by means of a maximum operator on events of the next period only, over a decision space. Thus in (5) a much more elementary maximum operation is used to implicitly characterize $F(s)$.

Equation (5) admits an appealing common sense interpretation by using the concept of a "valuation" on states, $F(s)$. The prescription in (5) is to

maximize (immediate returns + discounted state valuation),

a process which most experienced decision makers do quite intuitively.

The major question in dynamic programming is finding the unknown function $F(s)$ from equation (5), given the model M and discount factor α . However, the major operational question, and one usually answered by the previous one as a byproduct, is rather what policy $d = p(s)$ achieves the equality of (5); that is, for what p does

$$(6) \quad F(s) = E[R(s, p(s)) + \alpha F(S(s, p(s)))]$$

given the model M , discount factor α , and solution $F(s)$ to (5)?

In practical terms, solving equation (5) usually presents major analytical difficulties; in most realizations (5) represents a highly nonlinear functional equation with both maximum and expectation operators. If the problem has an especially simple structure, an approximating series construction may be possible on high speed computers, but even so simple and direct an approach as this soon breaks down under the sheer weight of computation and memory requirements.

In view of these difficulties an alternative proposal suggests itself, namely, that the determination of the state valuation function $F(s)$ be not regarded as a logical question arising from (5), but as an empirical question arising from the realities of the situation under study. This alternative reflects the fact that often, while we are at a complete loss to determine the precise form of $F(s)$

from (5), an examination of the realities of the situation under study suggests the approximate form which $F(s)$ should have almost immediately.

3. Two Illustrative Analyses

In support of the proposal above, we carry out two illustrative analyses. Situations similar to that described above will be convenient, though we will begin with a simpler case, and extend the analysis to a more complicated one. Consider, then, a model

$$M = [R(s, d), S(s, d)]$$

where $s = (i_0, i_1)$, $d = (q_2)$ and

$$(6) \quad R(s, d) = -c_2 q_2 - h \max(0, i_0 - x) - k \max(0, x - i_0).$$

$$(7) \quad S(s, d) = (i_0 + i_1 - x, q_2)$$

where x is a given real random variable (possibly dependent on time). On inspection (6) and (7) can be seen to represent conditions in an inventory model with one source of supply and a transit time of two periods, where costs of ordering, carrying and backordering inventory are taken to be linear.

We propose to build up a state and decision valuation, $V(s, d)$, as a sum of three terms, representing the value of state s and decision d in terms of

$V_a(s, d)$: inventory acquisition considerations

$V_h(s, d)$: inventory holding considerations

$V_b(s, d)$: backlog considerations

so

$$V(s, d) = V_a(s, d) + V_h(s, d) + V_b(s, d).$$

Before proceeding to our task of inducing these functions, we restate our point of view to minimize possibilities of confusion. We are interested in determining the combined value of state, s , and decision, d , regardless of how the state and decision came into being, in terms of their contributions to the long run returns in the operation. We are not interested in "costing" or "appraising" the state, or decision, at the moment.

Case $V_a(s, d)$

A triple (i_0, i_1, q_2) represents inventory which will not have to be acquired in future operations. The alternative to having (i_0, i_1, q_2) in the operation is to eventually order that amount. Without discounting, this value is $c_2(i_0 + i_1 + q_2)$ but if $(i_0 + i_1 + q_2)$ is large, the order may be put off some time, so that the cost of ordering should be discounted. On the average, the delay in ordering can be put off $(i_0 + i_1 + q_2)/\mu$ periods, when μ is the mean of x . This has a present value of

$$c_2(i_0 + i_1 + q_2)\alpha^{(i_0 + i_1 + q_2)/\mu}$$

or, approximately,

$$(8) \quad V_a(s) = c_2(i_0 + i_1 + q_2) \left(1 - \frac{\beta(i_0 + i_1 + q_2)}{\mu} \right)$$

where $\beta = 1 - \alpha$ is small, so that terms of 2nd order and higher in the binomial expansion of $(1 - \beta)^z$ may be neglected where $z = (i_0 + i_1 + q_2)/\mu$.

Case $V_h(s, d)$

A triple (i_0, i_1, q_2) represents inventory which will be held over until required. This inventory will be held over $(i_0 + i_1 + q_2)/\mu$ periods on the average, and the average amount of this particular inventory in the operation will be $(i_0 + i_1 + q_2)/2$ for that time and an average of

$$\frac{(i_0 + i_1 + q_2)^2}{2\mu} \text{ unit periods}$$

of inventory will be held over. Thus, the effective amount of chargeable hold-over costs is taken to be (these are negative, being costs)

$$(9) \quad V_h(s, d) = -h \left[\frac{(i_0 + i_1 + q_2)^2}{2\mu} - i_1 - 2q_2 \right].$$

The negative terms inside the brackets arise since material in i_1 will not be subject to hold over costs for one period, and material of q_2 for two periods.

Case $V_b(s, d)$

In the case of backlog valuations, we invoke a "horizon principle." We do not consider possible backlogs past the longest in-transit-time in the operation, for later decisions may be used to diminish or eliminate such eventualities. Rather, we shall compute the expected number of backlogs over the horizon of the longest in-transit-time in the operation. In this case they become for the current, next, and second periods, respectively

$$\begin{aligned} & \int_0^\infty \max(0, x - i_0) f(x) dx, \\ & \int_0^\infty \max(0, y - i_0 - i_1) g(y) dy, \\ & \int_0^\infty \max(0, z - i_0 - i_1 - q_2) h(z) dz, \end{aligned}$$

where the frequency distribution of demands x, y, z for the next, next two, and next three, periods are $f(x), g(y), h(z)$, respectively. Discounting the resulting shortage costs linearly at rate β , as above, we have then (and restating the

integrals)

$$\begin{aligned}
 V_b(s, d) = & -k \int_{i_0}^{\infty} (x - i_0) f(x) dx \\
 (10) \quad & - k(1 - \beta) \int_{i_0+i_1}^{\infty} (y - i_0 - i_1) g(y) dy \\
 & - k(1 - 2\beta) \int_{i_0+i_1+q_2}^{\infty} (z - i_0 - i_1 - q_2) h(z) dz.
 \end{aligned}$$

Combining (8), (9) and (10), now, we obtain

$$\begin{aligned}
 V(s, d) = & c_2 i_0 + (c_2 + h) i_1 + (c_2 + 2h) q_2 \\
 & - \left(\frac{h + 2\beta}{2\mu} \right) (i_0 + i_1 + q_2)^2 - k \int_{i_0}^{\infty} (x - i_0) f(x) dx \\
 (11) \quad & - k(1 - \beta) \int_{i_0+i_1}^{\infty} (y - i_0 - i_1) g(y) dy \\
 & - k(1 - 2\beta) \int_{i_0+i_1+q_2}^{\infty} (z - i_0 - i_1 - q_2) h(z) dz.
 \end{aligned}$$

With this state and decision valuation we restate our dynamic operating problem as

$$\max_{q_2 \geq 0} [V(i_0, i_1, q_2) - c_2 q_2].$$

We take this maximum value, a function only of $s = (i_0, i_1)$, to be a good approximation to the inventory valuation $F(s)$ of (5); i.e.,

$$(12) \quad F(s) = \max_{q_2 \geq 0} [V(i_0, i_1, q_2) - c_2 q_2].$$

Necessary conditions on q_2 for maximizing this value are [3]

$$\frac{\partial V(i_0, i_1, q_2)}{\partial q_2} - c_2 \begin{cases} \leq 0 & \text{if } q_2 = 0 \\ = 0 & \text{if } q_2 > 0 \end{cases}$$

Differentiating, when $q_2 > 0$, these conditions reduce to

$$2h - \frac{(h + 2\beta)}{\mu} (i_0 + i_1 + q_2) + k(1 - 2\beta) \int_{i_0+i_1+q_2}^{\infty} h(z) dz = 0;$$

or

$$\begin{aligned}
 \int_{i_0+i_1+q_2}^{\infty} h(z) dz &= \frac{h}{\mu k(1 - 2\beta)} \left[i_0 + i_1 + q_2 - 2\mu + \frac{2\beta}{h} (i_0 + i_1 + q_2) \right], \\
 \int_{\bar{q}}^{\infty} h(z) dz &= A(\bar{q} - B), \quad \text{say, where } \bar{q} = q_2 + i_0 + i_1
 \end{aligned}$$

The left side is the probability of a backlog at the end of the period $t + 2$ while the right side is linear in q_2 or $\bar{q}$: thus graphical solutions are easily constructed

from the graph of $h(z)$, of the form shown in Figure 1. One picks $\bar{q}$ so the areas graphed are equal. It is of interest to note that the only relevant demand statistic in this model is that of three period totals—no others enter into the decision.

As further illustration of the concept and use of inventory valuations, consider a second model

$$M = [R(s, d), S(s, d)]$$

where $s = (i_0, i_1, i_2, i_3)$ $d = (q_2, q_4)$ and

$$(13) \quad R(s, d) = -c_2 q_2 - c_4 q_4 - h \max(0, i_0 - x) - k \max(0, x - i_0)$$

$$(14) \quad S(s, d) = (i_0 + i_1 - x, i_2 + q_2, i_3, q_4),$$

and x is a given real random variable (possibly dependent on time). On inspection (13) and (14) can be seen to represent conditions in an inventory model with two routes of supply—2 period and 4 period in transit times—where cost of ordering either way, carrying and backordering inventory are taken to be linear. As before, we build up a valuation on the following basis.

$$V_a(s, d) = (\alpha_2 c_2 + \alpha_4 c_4)(i_0 + i_1 + i_2 + q_2 + i_3 + q_4) \cdot \left[\left(1 - \frac{\beta}{\mu} (i_0 + i_1 + i_2 + q_2 + i_3 + q_4) \right) \right]$$

where α_2 and α_4 are assumed to be the (unknown optimal) fractions of material procured via the two routes ($\alpha_2 + \alpha_4 = 1$).

$$V_h(s, d) = -h \left[\frac{(i_0 + i_1 + i_2 + q_2 + i_3 + q_4)^2}{2\mu} - i_1 - 2i_2 - 2q_2 - 3i_3 - 4q_4 \right]$$

$$V_b(s, d) = -k \left[\int_{z_0}^{\infty} (x_0 - z_0) f_0(x_0) dx_0 + (1 - \beta) \int_{z_1}^{\infty} (x_1 - z_1) f_1(x_1) dx_1 \right. \\ \left. + (1 - 2\beta) \int_{z_2}^{\infty} (x_2 - z_2) f_2(x_2) dx_2 + (1 - 3\beta) \right. \\ \left. \cdot \int_{z_3}^{\infty} (x_3 - z_3) f_3(x_3) dx_3 + (1 - 4\beta) \int_{z_4}^{\infty} (x_4 - z_4) f_4(x_4) dy_4 \right]$$

where

$$z_0 = i_0$$

$$z_1 = i_0 + i_1$$

$$z_2 = i_0 + i_1 + i_2 + q_2$$

$$z_3 = i_0 + i_1 + i_2 + i_3 + q_2$$

$$z_4 = i_0 + i_1 + i_2 + i_3 + q_2 + q_4.$$

We consider, then, the dynamic operating problem

$$\max_{q_2 \geq 0, q_4 \geq 0} [V(i_0, i_1, i_2, i_3, q_2, q_4) - c_2 q_2 - c_4 q_4],$$

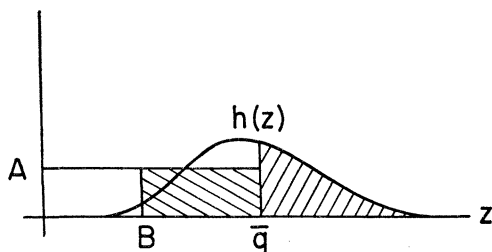


FIGURE 1

with necessary conditions

$$\frac{\partial V(s, d)}{\partial q_2} - c_2 \begin{cases} = 0 & \text{if } q_2 > 0 \\ \leq 0 & \text{if } q_2 = 0 \end{cases}$$

$$\frac{\partial V(s, d)}{\partial q_4} - c_4 \begin{cases} = 0 & \text{if } q_4 > 0 \\ \leq 0 & \text{if } q_4 = 0 \end{cases}$$

Now,

$$\begin{aligned} \frac{\partial V(s, d)}{\partial q_2} - c_2 &= -(1 - \alpha_2)c_2 + \alpha_4 c_4 + 2h \\ &\quad - \left[\frac{\beta(\alpha_2 c_2 + \alpha_4 c_4) + h}{\mu} \right] (i_0 + i_1 + i_2 + i_3 + q_2 + q_4) \\ &\quad + k \left[(1 - 2\beta) \int_{z_2}^{\infty} f_2(x_2) dx_2 + (1 - 3\beta) \int_{z_3}^{\infty} f_3(x_3) dx_3 \right. \\ &\quad \left. + (1 - 4\beta) \int_{z_4}^{\infty} f_4(x_4) dx_4 \right]. \\ \frac{\partial V(s, d)}{\partial q_4} - c_4 &= \alpha_2 c_2 - (1 - \alpha_4)c_4 + 4h - \left[\frac{\beta(\alpha_2 c_2 + \alpha_4 c_4) + h}{\mu} \right] \\ &\quad \cdot (i_0 + i_1 + i_2 + i_3 + q_2 + q_4) + k(1 - 4\beta) \int_{z_4}^{\infty} f_4(x_4) dx_4. \end{aligned}$$

When $q_2 > 0$, $q_4 > 0$ in a solution, these necessary conditions can be reduced to

$$(15) \quad \int_{z_4}^{\infty} f_4(x_4) dx_4 = A(q_2 + q_4 + \beta)$$

$$(16) \quad \int_{z_2}^{\infty} f_2(x_2) dx_2 + C \int_{z_3}^{\infty} f_3(x_3) dx_3 = D(q_2 + q_4 + E)$$

for some A, B, C, D, E . Then (15) can be solved for $q_2 + q_4$, graphically, in the manner above, so $q_2 + q_4 = q_T$, and (16) becomes

$$(17) \quad \int_{z_2}^{\infty} f_2(x_2) dx_2 + C \int_{z_3}^{\infty} f_3(x_3) dx_3 = F(q_2 + G)$$

for some new F , G depending on D , E and q_T . Now, by casting the previous graphical methods into forms using cumulative rather than frequency probability functions, (17) can also be solved graphically for q_2 .

It is interesting to note the sequential nature of this solution process: first, in effect, q_T is determined, as though only the slower route were available; and with q_T determined, the expedited shipment q_2 is then found. An examination of the structure of the problem, shows this sequential process is quite general, being valid in such situations regardless of the specific transit times, or even the number of ways of receiving shipments.

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