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I am submitting herewith a thesis written by Yoko Shioji entitled "The Spectral Properties of Boundary Value Problems with Eigenvalue Parameter in the Boundary Conditions." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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**THE SPECTRAL PROPERTIES OF
BOUNDARY VALUE PROBLEMS WITH
EIGENVALUE PARAMETER IN THE
BOUNDARY CONDITIONS**

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Yoko Shioji
December 1995

DEDICATION

to my parents

Nobuko and Ryoichi Shioji

(塩路 良一・暢子)

also, to the memory of

Venkata Ramamohana Rao Uppuluri

who has been an unforgettable friend of mine and

my father in the United States

ACKNOWLEDGEMENT

The author wishes to express her sincere appreciation to Dr. Don B. Hinton for his support in the preparation of this thesis. She would also like to thank Shona Workman for typing assistance and Kenichi Namiki, the author's husband, for his help and encouragement.

ABSTRACT

We study three types of boundary value problems with eigenvalue parameter in the boundary conditions, called (*), (**), and (***). We develop the self-adjoint operator and the Green's function associated with each of (*), (**), and (***). We also demonstrate the convergence of the eigenfunction expansion graphically and numerically for (*). Based on these results, we then solve a nonhomogeneous heat flow problem associated with (*).

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CHAPTER 1.

INTRODUCTION

We consider here the eigenvalue problem with the parameter depending boundary condition

$$(*) \begin{cases} \tau(u) = \frac{1}{r}[-(pu')' + qu] = \lambda u, & (1.1) \\ (\cos \theta)u(a) + (\sin \theta)(pu')(a) = 0, & 0 \leq \theta < \pi, & (1.2) \\ -[\beta_1 u(b) - \beta_2 (pu')(b)] = \lambda[\beta'_1 u(b) - \beta'_2 (pu')(b)], & (1.3) \end{cases}$$

where r , p , and q are real continuous functions on an interval $[a, b]$ with r and p positive. The numbers β_1 , β_2 , β'_1 , β'_2 are real. We also consider the related problems with different boundary conditions such as

$$(**) \begin{cases} (1.1), \\ -[\alpha_1 u(a) - \alpha_2 (pu')(a)] = \lambda[\alpha'_1 u(a) - \alpha'_2 (pu')(a)], \\ \alpha_1, \alpha_2, \alpha'_1, \alpha'_2 \in \mathbf{R}, \\ (1.3), \end{cases} \quad (1.4)$$

and

$$(***) \begin{cases} (1.1), \\ (1.2), \\ [\gamma_0 + \gamma_1 \lambda + \gamma_2 \lambda^2]u(b) = [\delta_0 + \delta_1 \lambda + \delta_2 \lambda^2](pu')(b), \\ \gamma_0, \gamma_1, \gamma_2, \delta_0, \delta_1, \delta_2 \in \mathbf{R}. \end{cases} \quad (1.5)$$

We can find many problems of the form $(*)$ in physics and engineering. One example is the heat conduction of a bar with a finite amount of fluid at the end. This will be discussed in Chapters 3 and 5. As another example, consider the longitudinal displacement $z(x, t)$ of a bar of length L with a mass M attached at $x = L$ as shown in Figure A-1.¹ We assume that the cross sectional area S of the

¹All figures may be found in the appendix.

bar is constant and that the particles in the bar move only in the axial direction (see [6, p364-373, 387-395] for a more detailed description). Following the analysis by R. V. Churchill and J. W. Brown[1, p250][2, p7], $z(x, t)$ satisfies the partial differential equation

$$d(x) \frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial x} \left(E(x) \frac{\partial z}{\partial x} \right), \quad t > 0, \quad (1.6)$$

where $d(x)$ is the density and $E(x) > 0$ is the modulus of elasticity. If the bar is fixed at $x = 0$, then we have

$$z(0, t) = 0. \quad (1.7)$$

And by [1] and [2], the mass M at $x = L$ gives another boundary condition

$$M \frac{\partial^2 z}{\partial t^2}(L, t) = -SE(L) \frac{\partial z}{\partial x}(L, t). \quad (1.8)$$

By separation of variables, (1.6)-(1.8) lead to the ODE boundary value problem

$$-\frac{d}{dx}(E(x)X'(x)) = \lambda d(x)X(x), \quad (1.9)$$

$$X(0) = 0, \quad (1.10)$$

$$SE(L)X'(L) = \lambda M X(L). \quad (1.11)$$

We note that the above problem has the form (*) with $r(x) \equiv d(x)$, $p(x) = E(x)$, $q(x) \equiv 0$, $\theta = 0$, $\beta_1 = 0$, $\beta_2 = S$, $\beta'_1 = M$, and $\beta'_2 = 0$. Many other physical applications of the problem (*) are given in C. T. Fulton's paper[3].

Since (*), (**), and (***) have a parameter λ in the boundary condition, the domain of the operator τ depends on λ and thus τ is not a linear operator. Therefore, these problems are not self-adjoint by definition. However, J. Walter[5] and Fulton[3] have shown that a symmetric operator A_1 can be associated with (*) under certain conditions of β 's and the Green's function can be constructed by using the operator A_1 . Since the symmetry and existence of Green's function yields self-adjointness of A_1 , we can obtain the necessary results about λ for eigenfunction

expansions. D. B. Hinton has proved theorems for uniform and absolute convergence of the eigenfunction expansion for (*) in [4]. We refer the reader to [3] and [5] for other eigenfunction expansion theorems.

In this paper, we aim to solve a nonhomogeneous heat flow problem associated with (*). First, we will find the self-adjoint operator for each of the problems (*), (**), and (***) in Chapter 2. Next in Chapter 3, after introducing the nonhomogeneous heat flow problem, we will focus on the homogeneous part of the problem and demonstrate the convergence of the eigenfunction expansion graphically and numerically. Then in Chapter 4, we construct the Green's function for each of (*), (**), and (***), based on the discussions of Chapter 2. At last in Chapter 5, we will solve the nonhomogeneous heat flow problem by using the results from Chapters 3 and 4.

CHAPTER 2.

THE SELF-ADJOINT OPERATOR

Our interest in this chapter is to define a self-adjoint operator for each of (*), (**), and (***) so that each problem can be transformed to an equivalent self-adjoint problem. The transformation yields many necessary results about the eigenvalues for solving each problem.

For each of (*), (**), and (***), we associate a Hilbert space, define an operator in the Hilbert space, and show that there is an eigenvalue problem with the operator which is equivalent to each of (*), (**), and (***). Then we find a condition so that the operator is self-adjoint.

2.1 OPERATOR FOR (*)

Define a complex Hilbert space H^* to be the set of two-component vectors (H_1, H_2) which satisfy

- (i) H_1 is a function defined on $[a, b]$ where $\int_a^b r(x)|H_1(x)|^2 dx < \infty$,
- (ii) H_2 is a complex number.

Strictly speaking, H_1 is an equivalence class of Lebesgue measurable functions which satisfies $\int_a^b r(x)|H_1(x)|^2 dx < \infty$. We refer to this condition as $H_1 \in \mathcal{L}^2(r; a, b)$. A standard calculation shows that an inner product in H^* is defined by

$$\langle (F_1, F_2), (G_1, G_2) \rangle = \int_a^b r(x)F_1(x)\overline{G_1(x)}dx + kF_2\overline{G_2} \quad (2.1)$$

where $(F_1, F_2), (G_1, G_2) \in H^*$ and $k > 0$ is a constant.

Let the domain of the operator A_1 , denoted as $D(A_1)$, be the set of all $F =$

$(F_1, F_2) \in H^*$ which satisfy

- (i) F_1 is a function in $\mathcal{L}^2(r; a, b)$ so that $\tau(F_1)$ is defined (i.e., F_1, pF_1' are absolutely continuous) and in $\mathcal{L}^2(r; a, b)$ (e.g., see [4]),
- (ii) F_1 satisfies the boundary condition (1.2), i.e.,

$$(\cos \theta)F_1(a) + (\sin \theta)(p(a)F_1'(a)) = 0, \quad (2.2)$$

- (iii) F_2 is a number such that

$$F_2 = \beta_1' F_1(b) - \beta_2' (pF_1')(b). \quad (2.3)$$

Define $A_1 : D(A_1) \rightarrow H^*$ by

$$A_1 F = (\tau(F_1), -\beta_1 F_1(b) + \beta_2 (pF_1')(b)). \quad (2.4)$$

Now we show the following.

Theorem 2.1. *Let $F = (F_1, F_2) \in H^*$. Then F_1 satisfies (*) if and only if $F \in D(A_1)$ and $A_1 F = \lambda F$, i.e., λ is an eigenvalue of A_1 .*

Proof. Suppose F_1 satisfies (*). Define F_2 to be as (2.3). Then $F = (F_1, F_2) \in D(A_1)$ by definition. Since

$$\tau(F_1) = \lambda F_1 \quad (2.5)$$

and

$$-\beta_1 F_1(b) + \beta_2 (pF_1')(b) = \lambda [\beta_1' F_1(b) - \beta_2' (pF_1')(b)] = \lambda F_2, \quad (2.6)$$

we have

$$\begin{aligned} A_1 F &= (\tau(F_1), -\beta_1 F_1(b) + \beta_2 (pF_1')(b)) \\ &= (\lambda F_1, \lambda F_2) = \lambda (F_1, F_2) = \lambda F. \end{aligned}$$

Conversely, suppose $F \in D(A_1)$ and $A_1 F = \lambda F$. The reverse calculation shows (2.5) and (2.6). Since $F \in D(A_1)$, (2.2) holds. Thus F_1 satisfies (*). $\square$

To achieve self-adjointness of A_1 , we need to show that A_1 is symmetric and there exists a Green's function for the nonhomogeneous equation $A_1 F = \lambda F + G$ when λ is not an eigenvalue. As seen in Walter's work [5], symmetry of A_1 requires a certain condition of a constant $k > 0$ in terms of real numbers β_1 , β_2 , β'_1 , and β'_2 .

Theorem 2.2. A_1 is symmetric, i.e.,

$$\langle A_1 F, G \rangle = \langle F, A_1 G \rangle \quad \text{for all } F, G \in D(A_1)$$

if

$$\rho := \frac{1}{k} = \beta'_1 \beta_2 - \beta_1 \beta'_2 > 0. \quad (2.7)$$

Proof. Let $F = (F_1, F_2)$ and $G = (G_1, G_2)$. Since $D(A_1) \subset H^*$, the inner product of $A_1 F$ and G is

$$\langle A_1 F, G \rangle = \int_a^b r(x) \tau(F_1)(x) \overline{G_1(x)} dx + \frac{1}{\rho} [-\beta_1 F_1(b) + \beta_2 (pF'_1)(b)] \overline{G_2}$$

by (2.1). Using integration by parts, we get

$$\begin{aligned} & \int_a^b r(x) \tau(F_1)(x) \overline{G_1(x)} dx \\ &= \int_a^b [-(pF'_1)'(x) + (qF_1)(x)] \overline{G_1(x)} dx \\ &= \int_a^b [(pF'_1 \overline{G'_1})(x) + (qF_1 \overline{G_1})(x)] dx - (pF'_1 \overline{G_1})(b) + (pF'_1 \overline{G_1})(a) \end{aligned}$$

Since $G \in D(A_1)$, $\overline{G_2} = \beta'_1 \overline{G_1(b)} - \beta'_2 (p \overline{G'_1})(b)$. Hence,

$$\begin{aligned} \langle A_1 F, G \rangle &= \int_a^b [(pF'_1 \overline{G'_1})(x) + (qF_1 \overline{G_1})(x)] dx \\ &\quad - \frac{\beta_1 \beta'_1}{\rho} (F_1 \overline{G_1})(b) - \frac{\beta_2 \beta'_2}{\rho} (pF'_1 p \overline{G'_1})(b) \\ &\quad + \frac{\beta_1 \beta'_2}{\rho} (pF_1 \overline{G'_1})(b) + \frac{\beta'_1 \beta_2}{\rho} (pF'_1 \overline{G_1})(b) \\ &\quad - (pF'_1 \overline{G_1})(b) + (pF'_1 \overline{G_1})(a). \end{aligned} \quad (2.8)$$

Similarly, we get

$$\begin{aligned}
\langle F, A_1 G \rangle &= \int_a^b [(pF_1' \overline{G_1'})(x) + (qF_1 \overline{G_1})(x)] dx \\
&\quad - \frac{\beta_1 \beta_1'}{\rho} (F_1 \overline{G_1})(b) - \frac{\beta_2 \beta_2'}{\rho} (pF_1' p \overline{G_1'})(b) \\
&\quad + \frac{\beta_1' \beta_2}{\rho} (pF_1 \overline{G_1'})(b) + \frac{\beta_1 \beta_2'}{\rho} (pF_1' \overline{G_1})(b) \\
&\quad - (pF_1 \overline{G_1'})(b) + (pF_1 \overline{G_1'})(a).
\end{aligned} \tag{2.9}$$

Note that first three terms of (2.8) and (2.9) are the same.

Since $F, G \in D(A_1)$, we have

$$(\cos \theta) F_1(a) + (\sin \theta) (pF_1')(a) = 0 \tag{2.10}$$

and

$$(\cos \theta) \overline{G_1}(a) + (\sin \theta) (p \overline{G_1'})(a) = 0. \tag{2.11}$$

For $0 < \theta < \pi$, multiply (2.10) and (2.11) by $\overline{G_1}(a)$ and $F_1(a)$, respectively, and subtract. We get

$$\begin{aligned}
(\sin \theta) \left[(pF_1' \overline{G_1})(a) - (pF_1 \overline{G_1'})(a) \right] &= 0 \\
\therefore (pF_1' \overline{G_1})(a) &= (pF_1 \overline{G_1'})(a).
\end{aligned}$$

For $\theta = 0$, a similar calculation also shows $(pF_1 \overline{G_1'})(a) = (pF_1' \overline{G_1})(a)$. Thus, the last term of (2.8) and (2.9) match. We now have three terms left unmatched. Let those terms in (2.8) be K , i.e.,

$$K := \frac{\beta_1 \beta_2'}{\rho} (pF_1 \overline{G_1'})(b) + \frac{\beta_1' \beta_2}{\rho} (pF_1' \overline{G_1})(b) - (pF_1' \overline{G_1})(b).$$

Since $\rho = \beta_1' \beta_2 - \beta_1 \beta_2'$ by hypothesis, we get

$$\begin{aligned}
K &= \frac{\beta_1' \beta_2 - \rho}{\rho} (pF_1 \overline{G_1'})(b) + \frac{\beta_1' \beta_2}{\rho} (pF_1' \overline{G_1})(b) - \frac{\beta_1' \beta_2 - \beta_1 \beta_2'}{\rho} (pF_1' \overline{G_1})(b) \\
&= \frac{\beta_1' \beta_2}{\rho} (pF_1 \overline{G_1'})(b) - (pF_1 \overline{G_1'})(b) + \frac{\beta_1 \beta_2'}{\rho} (pF_1' \overline{G_1})(b)
\end{aligned}$$

which equals the three terms left in (2.9) and it completes the proof. $\square$

Here, we note that $\rho = \beta'_1\beta_2 - \beta_1\beta'_2 \neq 0$ is enough for the symmetry of A_1 , although $\rho = \frac{1}{k} > 0$ comes from the definition of the inner product in H^* on (2.1).

Since A_1 is symmetric for $\rho = \beta'_1\beta_2 - \beta_1\beta'_2 > 0$, all eigenvalues λ of $A_1F = \lambda F$, i.e., eigenvalues for (*), are real numbers.

Furthermore, when λ is not an eigenvalue, the existence of the Green's function will be shown in Chapter 4. By a theorem of functional analysis[7, p108], this shows that the operator A_1 associated with (*) is self-adjoint.

2.2 OPERATOR FOR (**)

Define a complex Hilbert space H^{**} by the set of three-component vectors (H_1, H_2, H_3) where H_1, H_2 are defined as for H^* in Section 2.1 and H_3 is a complex number. By a standard calculation, an inner product in H^{**} is

$$\langle (F_1, F_2, F_3), (G_1, G_2, G_3) \rangle = \int_a^b r(x)F_1(x)\overline{G_1(x)}dx + (F_2 \ F_3)Q\begin{pmatrix} \overline{G_2} \\ \overline{G_3} \end{pmatrix} \quad (2.12)$$

where $(F_1, F_2, F_3), (G_1, G_2, G_3) \in H^{**}$ and $Q = \begin{pmatrix} h & l \\ l & m \end{pmatrix}$ is a positive definite, real matrix. We will show that for a certain choice of Q , the operator A_2 will be symmetric.

Let $D(A_2)$, the domain of the operator A_2 , be the set of all $F = (F_1, F_2, F_3) \in H^{**}$ which satisfy

- (i) F_1 is a function in $\mathcal{L}^2(r; a, b)$ so that $\tau(F_1)$ is defined and in $\mathcal{L}^2(r; a, b)$,
- (ii) F_2 is a number such that

$$F_2 = \alpha'_1 F_1(a) - \alpha'_2 (pF'_1)(a), \quad (2.13)$$

- (iii) F_3 is a number such that

$$F_3 = \beta'_1 F_1(b) - \beta'_2 (pF'_1)(b). \quad (2.14)$$

Define $A_2 : D(A_2) \rightarrow H^{**}$ by

$$A_2 F = (\tau(F_1), -\alpha_1 F_1(a) + \alpha_2(pF_1')(a), -\beta_1 F_1(b) + \beta_2(pF_1')(b)) \quad (2.15)$$

We now show:

Theorem 2.3. *Let $F = (F_1, F_2, F_3) \in H^{**}$. Then F_1 satisfies (**) if and only if $F \in D(A_2)$ and $A_2 F = \lambda F$.*

Proof. Suppose F_1 satisfies (**). Define F_2 and F_3 by (2.13) and (2.14) so that $(F_1, F_2, F_3) \in D(A_2)$. Since F_1 satisfies (**), we get

$$\begin{aligned} A_2 F &= (\tau(F_1), -\alpha_1 F_1(a) + \alpha_2(pF_1')(a), -\beta_1 F_1(b) + \beta_2(pF_1')(b)) \\ &= (\lambda F_1, \lambda F_2, \lambda F_3) = \lambda(F_1, F_2, F_3) = \lambda F. \end{aligned}$$

Conversely, suppose $A_2 F = \lambda F$ for $F = (F_1, F_2, F_3) \in D(A_2)$. Then for each component, we have

$$\begin{aligned} \tau(F_1) &= \lambda F_1, \\ -\alpha_1 F_1(a) + \alpha_2(pF_1')(a) &= \lambda F_2 = \lambda[\alpha'_1 F_1(a) - \alpha'_2(pF_1')(a)], \\ -\beta_1 F_1(b) + \beta_2(pF_1')(b) &= \lambda F_3 = \lambda[\beta'_1 F_1(b) - \beta'_2(pF_1')(b)]. \end{aligned}$$

Thus, F_1 satisfies (**). $\square$

We wish to find a condition on $Q = \begin{pmatrix} h & l \\ l & m \end{pmatrix}$ of (2.12) in terms of $\alpha_1, \alpha_2, \alpha'_1, \alpha'_2, \beta_1, \beta_2, \beta'_1$, and β'_2 so that the operator A_2 is symmetric, i.e.,

$$\langle A_2 F, G \rangle = \langle F, A_2 G \rangle \quad \text{for all } F, G \in D(A_1) \quad (2.16)$$

where the inner product is given by (2.12).

Let $F = (F_1, F_2, F_3), G = (G_1, G_2, G_3) \in D(A_2)$. By the definition of the inner product and the operator A_2 ,

$$\begin{aligned} \langle A_2 F, G \rangle &= \int_a^b [r(\tau(F_1)) \overline{G_1}](x) dx \\ &\quad + (-\alpha_1 F_1(a) + \alpha_2(pF_1')(a), -\beta_1 F_1(b) + \beta_2(pF_1')(b)) Q \begin{pmatrix} \overline{G_2} \\ \overline{G_3} \end{pmatrix} \end{aligned}$$

and

$$\langle F, A_2 G \rangle = \int_a^b [r F_1(\tau(\overline{G_1}))](x) dx + (F_2, F_3) Q \begin{pmatrix} -\alpha_1 \overline{G_1(a)} + \alpha_2 (p \overline{G'_1})(a) \\ -\beta_1 \overline{G_1(b)} + \beta_2 (p \overline{G'_1})(b) \end{pmatrix}$$

where $Q = \begin{pmatrix} h & l \\ l & m \end{pmatrix}$, $h, l, m \in \mathbb{R}$. Then by calculation, it follows that

$$\begin{aligned} \langle A_2 F, G \rangle &= \int_a^b [(p F'_1 \overline{G'_1})(x) + (q F_1 \overline{G_1})(x)] dx \\ &\quad + h(-\alpha_1 F_1(a) + \alpha_2 (p F'_1)(a)) (\alpha'_1 \overline{G_1(a)} - \alpha'_2 (p \overline{G'_1})(a)) \\ &\quad + l(-\alpha_1 F_1(a) + \alpha_2 (p F'_1)(a)) (\beta'_1 \overline{G_1(b)} - \beta'_2 (p \overline{G'_1})(b)) \\ &\quad + l(-\beta_1 F_1(b) + \beta_2 (p F'_1)(b)) (\alpha'_1 \overline{G_1(a)} - \alpha'_2 (p \overline{G'_1})(a)) \\ &\quad + m(-\beta_1 F_1(b) + \beta_2 (p F'_1)(b)) (\beta'_1 \overline{G_1(b)} - \beta'_2 (p \overline{G'_1})(b)) \\ &\quad - (p F'_1 \overline{G_1})(b) + (p F'_1 \overline{G_1})(a) \end{aligned}$$

and

$$\begin{aligned} \langle F, A_2 G \rangle &= \int_a^b [(p F'_1 \overline{G'_1})(x) + (q F_1 \overline{G_1})(x)] dx \\ &\quad + h(\alpha'_1 F_1(a) - \alpha'_2 (p F'_1)(a)) (-\alpha_1 \overline{G_1(a)} + \alpha_2 (p \overline{G'_1})(a)) \\ &\quad + l(\alpha'_1 F_1(a) - \alpha'_2 (p F'_1)(a)) (-\beta_1 \overline{G_1(b)} + \beta_2 (p \overline{G'_1})(b)) \\ &\quad + l(\beta'_1 F_1(b) - \beta'_2 (p F'_1)(b)) (-\alpha_1 \overline{G_1(a)} + \alpha_2 (p \overline{G'_1})(a)) \\ &\quad + m(\beta'_1 F_1(b) - \beta'_2 (p F'_1)(b)) (-\beta_1 \overline{G_1(b)} + \beta_2 (p \overline{G'_1})(b)) \\ &\quad - (p F_1 \overline{G'_1})(b) + (p F_1 \overline{G'_1})(a). \end{aligned}$$

To have $\langle A_2 F, G \rangle = \langle F, A_2 G \rangle$, the constants h, l, m must satisfy

$$h(\alpha'_1 \alpha_2 - \alpha_1 \alpha'_2) + 1 = 0, \quad (2.17)$$

$$m(\beta'_1 \beta_2 - \beta_1 \beta'_2) - 1 = 0, \quad (2.18)$$

$$l(\alpha'_1 \beta_1 - \alpha_1 \beta'_1) = 0, \quad (2.19)$$

$$l(\alpha'_1 \beta_2 - \alpha_1 \beta'_2) = 0, \quad (2.20)$$

$$l(\alpha'_2 \beta_1 - \alpha_2 \beta'_1) = 0, \quad (2.21)$$

$$l(\alpha'_2 \beta_2 - \alpha_2 \beta'_2) = 0. \quad (2.22)$$

By (2.17), we get

$$\rho_1 := \alpha'_1 \alpha_2 - \alpha_1 \alpha'_2 \neq 0, \quad \text{and} \quad h \neq 0.$$

Also, by (2.18), we get

$$\rho_2 := \beta'_1 \beta_2 - \beta_1 \beta'_2 \neq 0, \quad \text{and} \quad m \neq 0.$$

Moreover, we obtain $l = 0$.

Proof. Suppose $l \neq 0$. Then, by (2.19)–(2.22)

$$\alpha'_1 \beta_1 - \alpha_1 \beta'_1 = 0, \tag{a}$$

$$\alpha'_1 \beta_2 - \alpha_1 \beta'_2 = 0, \tag{b}$$

$$\alpha'_2 \beta_1 - \alpha_2 \beta'_1 = 0, \tag{c}$$

$$\alpha'_2 \beta_2 - \alpha_2 \beta'_2 = 0. \tag{d}$$

From $[(a) \times \beta'_2] - [(b) \times \beta'_1]$, we get $\alpha'_1 = 0$ since $\rho_2 \neq 0$. Then $\alpha'_2 \neq 0$ by $\rho_1 \neq 0$. But from $[(c) \times \beta'_2] - [(d) \times \beta'_1]$, we get $\alpha'_2 = 0$ since $\rho_2 \neq 0$ which is a contradiction. $\square$

Since Q is positive definite if and only if $h > 0$ and $hm - l^2 > 0$, it follows that

$$-\frac{1}{h} = \rho_1 = \alpha'_1 \alpha_2 - \alpha_1 \alpha'_2 < 0, \tag{2.23}$$

$$\frac{1}{m} = \rho_2 = \beta'_1 \beta_2 - \beta_1 \beta'_2 > 0, \tag{2.24}$$

$$l = 0. \tag{2.25}$$

(2.23)–(2.25) are the conditions for A_2 to be self-adjoint.

Since the Green's function for $A_2 F = \lambda F + G$ where λ is not an eigenvalue will be found in Chapter 4, we have just proved that A_2 is self-adjoint subject to (2.23)–(2.25).

2.3 OPERATOR FOR (***)

In the Hilbert space H^{**} defined in Section 2.2, we define an operator A_3 as follows. Let $D(A_3)$, the domain of the operator A_3 , be the set of all $F = (F_1, F_2, F_3) \in H^{**}$ which satisfy

- (i) F_1 is a function in $\mathcal{L}^2(r; a, b)$ so that $\tau(F_1)$ is defined and in $\mathcal{L}^2(r; a, b)$,
- (ii) F_1 satisfies the boundary condition (1.2), i.e.,

$$(\cos \theta)F_1(a) + (\sin \theta)(pF_1')(a) = 0, \quad \text{where } 0 \leq \theta < \pi, \quad (2.26)$$

- (iii) F_2 is a number such that

$$F_2 = -\gamma_2 F_1(b) + \delta_2(pF_1')(b), \quad (2.27)$$

- (iv) F_3 is a complex number.

Define $A_3 : D(A_3) \rightarrow H^{**}$ by

$$A_3 F := (\tau(F_1), F_3 - [-\gamma_1 F_1(b) + \delta_1(pF_1')(b)], \gamma_0 F_1(b) - \delta_0(pF_1')(b)). \quad (2.28)$$

Now we show the following statement.

Theorem 2.4. *Let $F = (F_1, F_2, F_3) \in H^{**}$. Then, F_1 satisfies (***) if and only if $F \in D(A_3)$ and $A_3 F = \lambda F$.*

Proof. Suppose F_1 satisfies (***), i.e.,

$$\tau(F_1) = \lambda F_1,$$

$$(\cos \theta)F_1(a) + (\sin \theta)(pF_1')(a) = 0,$$

$$(\gamma_0 + \gamma_1 \lambda + \gamma_2 \lambda^2)F_1(b) = (\delta_0 + \delta_1 \lambda + \delta_2 \lambda^2)(pF_1')(b).$$

Define F_2 by (2.27). Define F_3 by

$$F_3 = [-\gamma_1 F_1(b) + \delta_1(pF_1')(b)] + \lambda[-\gamma_2 F_1(b) + \delta_2(pF_1')(b)]. \quad (2.29)$$

Then $(F_1, F_2, F_3) \in D(A_3)$ by definition and we have

$$\text{1st component of } A_3 F = \lambda F_1,$$

$$\text{2nd component of } A_3 F = \lambda[-\gamma_2 F_1(b) + \delta_2(pF_1')(b)],$$

$$\text{3rd component of } A_3 F$$

$$= \lambda[-\gamma_1 F_1(b) + \delta_1(pF_1')(b)] + \lambda^2[-\gamma_2 F_1(b) + \delta_2(pF_1')(b)].$$

Therefore,

$$A_3 F = (\lambda F_1, \lambda F_2, \lambda F_3) = \lambda F.$$

Conversely, suppose $A_3 F = \lambda F$ for $F = (F_1, F_2, F_3) \in D(A_3)$, i.e.,

$$\tau(F_1) = \lambda F_1, \quad (2.30)$$

$$F_3 - [-\gamma_1 F_1(b) + \delta_1(pF_1')(b)] = \lambda F_2 = \lambda[-\gamma_2 F_1(b) + \delta_2(pF_1')(b)], \quad (2.31)$$

$$\gamma_0 F_1(b) - \delta_0(pF_1')(b) = \lambda F_3. \quad (2.32)$$

F_1 satisfies (1.1) by (2.30). F_1 also satisfies (1.2) by the definition of $D(A_3)$. Multiply (2.31) by λ and substitute (2.32) into (2.31) to get (1.5). Hence, F_1 satisfies (***). $\square$

Recall that the inner product of H^{**} is defined by

$$\langle (F_1, F_2, F_3), (G_1, G_2, G_3) \rangle = \int_a^b r(x) F_1(x) \overline{G_1(x)} dx + (F_2, F_3) Q \begin{pmatrix} \overline{G_2} \\ \overline{G_3} \end{pmatrix} \quad (2.12)$$

where $(F_1, F_2, F_3), (G_1, G_2, G_3) \in H^{**}$ and $Q = \begin{pmatrix} h & l \\ l & m \end{pmatrix}$ is a positive definite, real matrix. We will show that for a certain choice of Q the operator A_2 will be symmetric. This Q will be different from the Q in problem (**).

We now wish to find conditions to establish the symmetry of A_3 , i.e.,

$$\langle A_3 F, G \rangle = \langle F, A_3 G \rangle \quad (2.33)$$

for all $F, G \in D(A_3)$.

By definition, we have

$$\begin{aligned}\langle A_3 F, G \rangle &= \int_a^b [\tau(\tau(F_1))\overline{G_1}](x)dx \\ &\quad + (F_3 - [-\gamma_1 F_1(b) + \delta_1(pF'_1)(b)], \gamma_0 F_1(b) - \delta_0(pF'_1)(b))Q\left(\frac{\overline{G_2}}{\overline{G_3}}\right)\end{aligned}$$

and

$$\begin{aligned}\langle F, A_3 G \rangle &= \int_a^b [rF_1(\tau(\overline{G_1}))](x)dx \\ &\quad + (F_2, F_3)Q\left(\frac{\overline{G_3} - [-\gamma_1 \overline{G_1}(b) + \delta_1(p\overline{G'_1})(b)]}{\gamma_0 \overline{G_1}(b) - \delta_0(p\overline{G'_1})(b)}\right).\end{aligned}$$

By calculation, it follows that

$$\begin{aligned}\langle A_3 F, G \rangle &= \int_a^b (pF'_1 \overline{G'_1} + qF_1 \overline{G_1})(x)dx \\ &\quad - (pF'_1 \overline{G_1})(b) + (pF'_1 \overline{G_1})(a) \\ &\quad + (-h\delta_1\delta_2 - l\delta_0\delta_2)(pF'_1 p\overline{G'_1})(b) \\ &\quad + (h\delta_1\gamma_2 + l\delta_0\gamma_2)(pF'_1 \overline{G_1})(b) + (h\delta_2\gamma_1 + l\delta_2\gamma_0)(pF_1 \overline{G'_1})(b) \\ &\quad + (-l\delta_1 - m\delta_0)(pF'_1)(b)\overline{G_3} + h\delta_2(p\overline{G'_1})(b)F_3 \\ &\quad + (-h\gamma_1\gamma_2 - l\gamma_0\gamma_2)(F_1 \overline{G_1})(b) \\ &\quad + (l\gamma_1 + m\gamma_0)F_1(b)\overline{G_3} - h\gamma_2 \overline{G_1}(b)F_3 + lF_3 \overline{G_3}\end{aligned}$$

and

$$\begin{aligned}\langle F, A_3 G \rangle &= \int_a^b (pF'_1 \overline{G'_1} + qF_1 \overline{G_1})(x)dx \\ &\quad - (pF_1 \overline{G'_1})(b) + (pF_1 \overline{G'_1})(a) \\ &\quad + (-h\delta_1\delta_2 - l\delta_0\delta_2)(pF'_1 p\overline{G'_1})(b) \\ &\quad + (h\delta_2\gamma_1 + l\delta_2\gamma_0)(pF'_1 \overline{G_1})(b) + (h\delta_1\gamma_2 + l\delta_0\gamma_2)(pF_1 \overline{G'_1})(b) \\ &\quad + (h\delta_2)(pF'_1)(b)\overline{G_3} + (-l\delta_1 - m\delta_0)(p\overline{G'_1})(b)F_3 \\ &\quad + (-h\gamma_1\gamma_2 - l\gamma_0\gamma_2)(F_1 \overline{G_1})(b) \\ &\quad + (-h\gamma_2)F_1(b)\overline{G_3} + (l\gamma_1 + m\gamma_0)\overline{G_1}(b)F_3 + lF_3 \overline{G_3}.\end{aligned}$$

We know that $(pF_1'\overline{G_1})(a) = (pF_1'\overline{G_1'})(a)$ since we have

$$(pF_1'\overline{G_1})(a) = \left(-\frac{\cos \theta}{\sin \theta} F_1(a)\right) \overline{G_1(a)} = (p\overline{G_1'} F_1)(a) \quad \text{for } \theta \neq 0,$$

and

$$(pF_1'\overline{G_1})(a) = (pF_1')(a) \cdot 0 = 0 = (p\overline{G_1'})(a) \cdot 0 = (p\overline{G_1'} F_1)(a) \quad \text{for } \theta = 0$$

by (ii) of the definition of $D(A_3)$. Hence, in order to have $\langle A_3 F, G \rangle = \langle F, A_3 G \rangle$, h , l , and m must satisfy

$$\begin{cases} -1 + h\delta_1\gamma_2 + l\delta_0\gamma_2 = h\delta_2\gamma_1 + l\delta_2\gamma_0, \\ -l\delta_1 - m\delta_0 = h\delta_2, \\ l\gamma_1 + m\gamma_0 = -h\gamma_2, \end{cases}$$

or

$$P \begin{pmatrix} h \\ l \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{where } P := \begin{pmatrix} \delta_1\gamma_2 - \delta_2\gamma_1 & \delta_0\gamma_2 - \delta_2\gamma_0 & 0 \\ \delta_2 & \delta_1 & \delta_0 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix}. \quad (2.34)$$

We note that $\det P \neq 0$.

Proof. We rewrite (2.34) as

$$\begin{cases} (\delta_1\gamma_2 - \delta_2\gamma_1)h + (\delta_0\gamma_2 - \delta_2\gamma_0)l = 1, & (a) \\ \delta_2h + \delta_1l + \delta_0m = 0, & (b) \\ \gamma_2h + \gamma_1l + \gamma_0m = 0. & (c) \end{cases}$$

Suppose $\det P = 0$. Then

$$\det P = (\delta_0\gamma_2 - \delta_2\gamma_0)^2 - (\delta_1\gamma_2 - \delta_2\gamma_1)(\delta_0\gamma_1 - \delta_1\gamma_0) = 0$$

$$\therefore (\delta_0\gamma_2 - \delta_2\gamma_0)^2 = (\delta_1\gamma_2 - \delta_2\gamma_1)(\delta_0\gamma_1 - \delta_1\gamma_0). \quad (d)$$

Case 1: $(\delta_0\gamma_2 - \delta_2\gamma_0) \neq 0$

We have

$$l = \frac{1 - (\delta_1\gamma_2 - \delta_2\gamma_1)h}{(\delta_0\gamma_2 - \delta_2\gamma_0)} \quad (e)$$

from (a). Since $\delta_1\gamma_0 - \delta_0\gamma_1 \neq 0$ from (d), we get from (b) and (c), that

$$\begin{pmatrix} l \\ m \end{pmatrix} = \frac{1}{\delta_1\gamma_0 - \delta_0\gamma_1} \begin{pmatrix} \gamma_0 & -\delta_0 \\ -\gamma_1 & \delta_1 \end{pmatrix} \begin{pmatrix} -\delta_2 h \\ -\gamma_2 h \end{pmatrix}.$$

Thus,

$$\begin{aligned} l &= \frac{\delta_0\gamma_2 - \delta_2\gamma_0}{\delta_1\gamma_0 - \delta_0\gamma_1} h \\ &= \frac{(\delta_0\gamma_2 - \delta_2\gamma_0)(\delta_1\gamma_2 - \delta_2\gamma_1)}{(\delta_1\gamma_0 - \delta_0\gamma_1)(\delta_1\gamma_2 - \delta_2\gamma_1)} h. \end{aligned}$$

From (d),

$$\begin{aligned} l &= \frac{(\delta_0\gamma_2 - \delta_2\gamma_0)(\delta_1\gamma_2 - \delta_2\gamma_1)}{-(\delta_0\gamma_2 - \delta_2\gamma_0)^2} h \\ &= \frac{-(\delta_1\gamma_2 - \delta_2\gamma_1)}{(\delta_0\gamma_2 - \delta_2\gamma_0)} h, \end{aligned}$$

which contradicts (e).

Case 2: $(\delta_0\gamma_2 - \delta_2\gamma_0) = 0$

From (a), we have $(\delta_1\gamma_2 - \delta_2\gamma_1)h = 1$. Thus, $\delta_1\gamma_2 - \delta_2\gamma_1 \neq 0$ and $h \neq 0$. Then, $\delta_0\gamma_1 - \delta_1\gamma_0 = 0$ from (d). From (b) and (c), we get

$$\begin{aligned} \begin{pmatrix} h \\ l \end{pmatrix} &= \frac{1}{\delta_2\gamma_1 - \delta_1\gamma_2} \begin{pmatrix} \gamma_1 & -\delta_1 \\ -\gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} -\delta_0 m \\ -\gamma_0 m \end{pmatrix} \\ \therefore h &= \frac{-\delta_0\gamma_1 + \delta_1\gamma_0}{\delta_2\gamma_1 - \delta_1\gamma_2} m = 0, \end{aligned}$$

which is a contradiction.

Hence, $\det P \neq 0$. $\square$

Thus, we have by Cramer's rule,

$$h = \frac{\delta_1\gamma_0 - \delta_0\gamma_1}{\det P}, \quad (2.35)$$

$$l = \frac{\delta_0\gamma_2 - \delta_2\gamma_0}{\det P}, \quad (2.36)$$

$$m = \frac{\delta_2 \gamma_1 - \delta_1 \gamma_2}{\det P}. \quad (2.37)$$

Since $Q = \begin{pmatrix} h & l \\ l & m \end{pmatrix}$ is real and positive definite,

$$h = \frac{\delta_1 \gamma_0 - \delta_0 \gamma_1}{\det P} > 0$$

and

$$hm - l^2 = \frac{1}{(\det P)^2} [(\delta_2 \gamma_1 - \delta_1 \gamma_2)(\delta_1 \gamma_0 - \delta_0 \gamma_1) - (\delta_0 \gamma_2 - \delta_2 \gamma_0)^2] = \frac{-1}{\det P} > 0.$$

Therefore, the conditions for A_3 to be symmetric are

$$\det P = (\delta_0 \gamma_2 - \delta_2 \gamma_0)^2 - (\delta_2 \gamma_1 - \delta_1 \gamma_2)(\delta_1 \gamma_0 - \delta_0 \gamma_1) < 0 \quad (2.38)$$

and

$$\delta_1 \gamma_0 - \delta_0 \gamma_1 < 0. \quad (2.39)$$

Symmetry of the operator A_3 and existence of the Green's function, which will be proved in Chapter 4, imply that A_3 is a self-adjoint operator for (***) .

CHAPTER 3.

EIGENFUNCTION EXPANSION

In this chapter, we first derive a nonhomogeneous heat equation where a rod has a finite amount of fluid attached at the right end. Then we show there are eigenvalue problems of the form (*), EVP1 and EVP2, associated with the homogeneous heat equation, and demonstrate eigenfunction expansions for several initial conditions for each problem.

3.1 HEAT FLOW PROBLEM

We consider a rod of length L with constant cross-sectional area S as illustrated in Figure A-2. We assume that the lateral surface area of the rod is insulated so that thermal energy can pass only in the x -direction. We also assume that the specific heat and mass density of the rod at x are $c(x)$ and $d(x)$, respectively, and that the rod has a heat source which generates heat energy $\Omega(x, t)$ per unit volume per unit time. Then by conservation of heat energy, the temperature $u(x, t)$ of the rod at x at time t satisfies a partial differential equation

$$c(x)d(x)\frac{\partial u}{\partial t}(x, t) = -\frac{\partial \phi}{\partial x}(x, t) + \Omega(x, t) \quad (3.1)$$

where $\phi(x, t)$ is the heat flux to the right. By Fourier's law of heat conduction, we know

$$\phi(x, t) = -K(x)\frac{\partial u}{\partial x}(x, t) \quad (3.2)$$

where $K(x)$ is the thermal conductivity of the rod at x . Substituting (3.2) into

(3.1) , we obtain the heat equation

$$c(x)d(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(K(x)\frac{\partial u}{\partial x} \right) + \Omega(x, t). \quad (3.3)$$

If the temperature of the left end ($x = 0$) is held at 0, then we have a boundary condition

$$u(0, t) = 0, \quad (3.4)$$

while if the left end of the rod is insulated, then we have

$$\frac{\partial u}{\partial x}(0, t) = 0. \quad (3.5)$$

For $x = L$, the right end of the rod, we suppose that the rod is in contact with a finite amount of fluid as shown in Figure A- 2. We assume the fluid is moving so that it has the same temperature $\sigma(t)$ at time t anywhere in the fluid. It is also assumed that the fluid container is insulated. By Newton's law of cooling, we have

$$\phi(L, t) = R[u(L, t) - \sigma(t)] \quad (3.6)$$

where the constant R is called the heat transfer coefficient. Partial differentiation with respect to t of both sides yields

$$\frac{\partial \phi}{\partial t}(L, t) = R \left[\frac{\partial u}{\partial t}(L, t) - \sigma'(t) \right]. \quad (3.7)$$

Since energy conservation holds, the heat flux $\phi(L, t)$ also satisfies

$$S\phi(L, t) = \frac{d}{dt}[d_F c_F v_F \sigma(t)] \quad (3.8)$$

where d_F , c_F , and v_F are mass density, specific heat, and volume of the fluid, respectively. Substituting $\sigma'(t)$ of (3.8) into (3.7), we get

$$\frac{\partial \phi}{\partial t}(L, t) = R \left[\frac{\partial u}{\partial t}(L, t) - \frac{S}{d_F c_F v_F} \phi(L, t) \right]. \quad (3.9)$$

By Fourier's law of heat conduction, i.e. (3.2), we have a boundary condition

$$-K(L)\frac{\partial^2 u}{\partial t \partial x}(L, t) = R \left[\frac{\partial u}{\partial t}(L, t) - \frac{S}{d_{FCFv_F}} \left(-K(L)\frac{\partial u}{\partial x}(L, t) \right) \right] \quad (3.10)$$

or

$$-K(L)\frac{\partial u}{\partial x}(L, t) = -\beta'_1 \frac{\partial u}{\partial t}(L, t) + \beta'_2 K(L)\frac{\partial^2 u}{\partial t \partial x}(L, t) \quad (3.11)$$

where

$$\beta'_1 = -\frac{d_{FCFv_F}}{S} < 0 \quad \text{and} \quad \beta'_2 = \frac{d_{FCFv_F}}{RS} > 0. \quad (3.12)$$

Further, we assume an initial condition at $t = 0$ is given by

$$u(x, 0) = g_1(x). \quad (3.13)$$

As a result, we obtain a nonhomogeneous initial-boundary value heat flow problem:

$$(P) \left\{ \begin{array}{l} c(x)d(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(K(x)\frac{\partial u}{\partial x} \right) + \Omega(x, t), \quad 0 < x < L, \quad t > 0, \\ u(x, 0) = g_1(x), \quad 0 \leq x \leq L, \\ u(0, t) = 0 \quad \text{or} \quad \frac{\partial u}{\partial x}(0, t) = 0, \quad t > 0, \\ -K(L)\frac{\partial u}{\partial x}(L, t) = -\beta'_1 \frac{\partial u}{\partial t}(L, t) + \beta'_2 K(L)\frac{\partial^2 u}{\partial t \partial x}(L, t), \quad t > 0, \\ \text{where } \beta'_1 = -\frac{d_{FCFv_F}}{S} < 0 \quad \text{and} \quad \beta'_2 = \frac{d_{FCFv_F}}{RS} > 0. \end{array} \right.$$

3.2 EIGENFUNCTION EXPANSION FOR A HOMOGENEOUS HEAT FLOW PROBLEM

In this section, we consider problem (P) with heat source $\Omega(x, t) \equiv 0$, namely,

$$(P_H) \left\{ \begin{array}{l} c(x)d(x)\frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left(K(x)\frac{\partial v}{\partial x} \right), \quad 0 < x < L, \quad t > 0, \quad (3.14) \\ v(x, 0) = g_1(x), \quad 0 \leq x \leq L, \quad (3.15) \\ v(0, t) = 0 \quad \text{or} \quad \frac{\partial v}{\partial x}(0, t) = 0, \quad t > 0, \quad (3.16) \\ -K(L)\frac{\partial v}{\partial x}(L, t) = -\beta'_1 \frac{\partial v}{\partial t}(L, t) + \beta'_2 K(L)\frac{\partial^2 v}{\partial t \partial x}(L, t), \quad t > 0, \quad (3.17) \end{array} \right.$$

where $v(x, t)$ represents the temperature of the rod.

Assuming problem (P_H) has solutions of the form

$$v(x, t) = X(x)T(t), \quad (3.18)$$

we apply the method of separation of variables. Substitution of (3.18) into (3.14) gives us

$$c(x)d(x)X(x)T'(t) = (K(x)X'(x))'T(t). \quad (3.19)$$

We divide both sides of (3.19) by $c(x)d(x)X(x)T(t) \neq 0$ and introduce a constant parameter λ as

$$\frac{1}{c(x)d(x)} (K(x)X'(x))' \frac{1}{X}(x) = \frac{T'(t)}{T(t)} := -\lambda.$$

The above equation yields two ordinary differential equations:

$$T'(t) = -\lambda T(t)$$

and

$$-\frac{1}{c(x)d(x)} (K(x)X'(x))' = \lambda X(x). \quad (3.20)$$

The product solutions, $v(x, t) = X(x)T(t)$, must also satisfy the two boundary conditions of problem (P_H) . From the boundary condition (3.16), we get

$$X(0) = 0 \quad \text{or} \quad X'(0) = 0. \quad (3.21)$$

For $x = L$, we have

$$-(KX')(L)T(t) = -\beta'_1 X(L)T'(t) + \beta'_2 (KX')(L)T'(t)$$

by (3.17). Since $\frac{T'(t)}{T(t)} = -\lambda$, we have

$$-(KX')(L) = \lambda[\beta'_1 X(L) - \beta'_2 (KX')(L)] \quad (3.22)$$

where $\beta'_1 < 0$ and $\beta'_2 > 0$.

As a result, we obtain the eigenvalue problem (3.20)-(3.22). Here we note that (3.20)-(3.22) has the form of the problem (*) with $a = 0$, $b = L$, $r(x) = c(x)d(x)$, $p(x) = K(x)$, $q(x) \equiv 0$, $\theta = 0$ or $\frac{\pi}{2}$, $\beta_1 = 0$, and $\beta_2 = -1$.

Since $\beta_1 = 0$, $\beta_2 = -1$, $\beta'_1 < 0$, and $\beta'_2 > 0$, we have $\rho = \beta'_1\beta_2 - \beta_1\beta'_2 > 0$ in the problem (3.20)-(3.22). Recall that a symmetric operator A_1 is associated with (*) when $\rho > 0$. Thus, we know all eigenvalues of (3.20)-(3.22) are real numbers.

We will find the eigenvalues and eigenfunctions for the two simpler cases of (3.20)-(3.22), namely, when $K(x) = c(x)d(x) = 1$, and demonstrate the convergence of the eigenfunction expansion for each case so that the problem (P_H) can be solved.

Eigenvalue Problem One.

The eigenvalue problem (EVP) we consider here is

$$\begin{cases} -X''(x) = \lambda X(x), & (3.23) \\ X(0) = 0, & (3.24) \\ -X'(L) = \lambda[\beta'_1 X(L) - \beta'_2 X'(L)], & (3.25) \end{cases}$$

where $\beta'_1 < 0$ and $\beta'_2 > 0$. We call this problem EVP1. The boundary condition $X(0) = 0$ is associated with $v(0, t) = 0$ of (3.16), i.e., the case where the left end temperature is held at zero. EVP1 has the form (*) with $p(x) = r(x) \equiv 1$, $q(x) \equiv 0$, $a = 0$, $b = L$, $\beta_1 = 0$, $\beta_2 = -1$, and $\theta = 0$.

We first show that all the eigenvalues of EVP1 are positive. In Hinton's paper [4, p.34-35] the following formula was proved.

Lemma 3.1. *Let A_1 be an operator defined as in Section 2.1. Let $F = (F_1, F_2)$,*

$G = (G_1, G_2) \in D(A_1)$. The function J is defined on $D(A_1) \times D(A_1)$ by

$$\begin{aligned} \beta'_2 = 0: \quad J(F, G) &= \int_a^b [pF'_1 \overline{G'_1} + qF_1 \overline{G_1}](x) dx - \cot \theta (F_1 \overline{G_1})(a) \\ &\quad - \frac{1}{\rho} \beta_1 \beta'_1 (F_1 \overline{G_1})(b), \\ \beta'_2 \neq 0: \quad J(F, G) &= \int_a^b [pF'_1 \overline{G'_1} + qF_1 \overline{G_1}](x) dx - \cot \theta (F_1 \overline{G_1})(a) \quad (3.26) \\ &\quad - \frac{\beta_1 \beta'_1}{\rho} (F_1 \overline{G_1})(b) + \frac{\beta_1 \beta'_2}{\rho} (F_1 p \overline{G'_1})(b) \\ &\quad + \frac{\beta_1 \beta'_2}{\rho} (p F'_1 \overline{G_1})(b) - \frac{\beta_2 \beta'_2}{\rho} (p F'_1 p \overline{G'_1})(b) \end{aligned}$$

where $\rho = \beta'_1 \beta_2 - \beta_1 \beta'_2$. For $\theta = 0$, the term $\cot \theta (F_1 \overline{G_1})(a)$ is deleted. Then,

$$J(F, G) = \langle F, A_1 G \rangle \quad (3.27)$$

where the inner product $\langle \cdot, \cdot \rangle$ is defined as (2.1) with $k = \frac{1}{\rho}$.

Now we prove the following theorem.

Theorem 3.1. *The eigenvalues (λ 's) of EVP1 are all positive.*

Proof. Let $X_{n1}(x)$ be the n th eigenfunction of EVP1 associated with the n th eigenvalue λ_n . Let $X_n = (X_{n1}, X_{n2})$ where $X_{n2} = \beta'_1 X_{n1}(L) - \beta'_2 X'_{n1}(L)$. Then $X_n \in D(A_1)$ and $A_1 X_n = \lambda_n X_n$ by Theorem 2.1 since X_{n1} satisfies EVP1. Since we have $\rho = \beta'_1 \beta_2 - \beta_1 \beta'_2 > 0$, A_1 is symmetric, and thus all λ_n 's are real numbers. Hence,

$$\begin{aligned} \langle X_n, A_1 X_n \rangle &= \langle X_n, \lambda_n X_n \rangle \\ &= \int_0^L \lambda_n |X_{n1}(x)|^2 dx + \frac{1}{\rho} \lambda_n |X_{n2}|^2 \\ &= \lambda_n \left[\int_0^L |X_{n1}(x)|^2 dx + \frac{1}{\rho} |X_{n2}|^2 \right] \\ &= \lambda_n \langle X_n, X_n \rangle. \end{aligned}$$

Now, since $\beta_1 = 0$, $\beta_2 = -1$, $p(x) \equiv 1$, $q(x) \equiv 0$, and $\theta = 0$,

$$J(X_n, X_n) = \int_0^L |X'_{n1}(x)|^2 dx - \frac{1}{\rho} [-\beta'_2 |X_{n1}(L)|^2]$$

where $\rho > 0$, $\beta'_2 > 0$. We know $X'_{n1}(x) \not\equiv 0$ since $X_{n1}(x)$ is an eigenfunction and $X_{n1}(0) = 0$. Thus, $J(X_n, X_n) > 0$. By Lemma 3.1, we obtain

$$0 < J(X_n, X_n) = \langle X_n, A_1 X_n \rangle = \lambda_n \langle X_n, X_n \rangle.$$

Since $X_{n1}(x)$ is an eigenfunction, $X_{n1}(x) \not\equiv 0$. Thus, $\langle X_n, X_n \rangle = \int_0^L |X_{n1}(x)|^2 dx + \frac{1}{\rho} |X_{n2}|^2 > 0$ since $\rho > 0$. Hence, $\lambda_n > 0$. $\square$

Using the fact $\lambda_n > 0$, we now find eigenfunctions and eigenvalues for EVP1. Without loss of generality, we take the eigenfunctions to be real.

Since all λ 's are positive, the ODE (3.23) and the boundary condition (3.24) give the general solution

$$X(x) = C \sin \sqrt{\lambda} x \quad (3.28)$$

where C is a constant. Substituting (3.28) into the other boundary condition (3.25), we obtain

$$-C\sqrt{\lambda} \cos \sqrt{\lambda} L = \lambda \left[\beta'_1 C \sin \sqrt{\lambda} L - \beta'_2 C \sqrt{\lambda} \cos \sqrt{\lambda} L \right].$$

By calculation, it follows that

$$\frac{\beta'_2}{\beta'_1} \sqrt{\lambda} - \frac{1}{\beta'_1 \sqrt{\lambda}} = \tan \sqrt{\lambda} L \quad (3.29)$$

which is a transcendental equation. The graphs of $y = \frac{\beta'_2}{\beta'_1} \sqrt{\lambda} - \frac{1}{\beta'_1 \sqrt{\lambda}}$ and $y = \tan \sqrt{\lambda} L$ for $-\beta'_1 = \beta'_2 = 1$ and $L = 1$ are shown in Figure A-3. The λ -coordinate of each intersection of the two graphs indicates the value of the eigenvalue for EVP1. We call the eigenvalues, namely, the solutions of (3.29), $\lambda_1, \lambda_2, \lambda_3, \dots$ from the smallest.

For each eigenvalue λ_n , where $n = 1, 2, 3, \dots$, the associated eigenfunction is

$$X_{n1}(x) = C_n \sin \sqrt{\lambda_n} x \quad (3.30)$$

where C_n denotes a constant for each n . For the eigenfunction expansion, it is necessary for each X_n to be normalized in terms of the inner product (2.1) with $k = \frac{1}{\rho}$, i.e.,

$$\langle (X_{n1}, X_{n2}), (X_{n1}, X_{n2}) \rangle = \int_0^L |X_{n1}|^2 dx + \frac{1}{\rho} |X_{n2}|^2 = 1,$$

where $X_{n2} = \beta'_1 X_{n1}(L) - \beta'_2 X'_{n1}(L) = -\frac{1}{\lambda_n} X'_{n1}(L)$. By substitution of (3.30) into the above, each C_n must satisfy

$$C_n^2 = \frac{1}{\int_0^L \sin^2 \sqrt{\lambda_n} x dx + \frac{1}{\rho} \left| \frac{1}{\sqrt{\lambda_n}} \cos \sqrt{\lambda_n} L \right|^2} \quad (3.31)$$

and we take $C_n > 0$ without loss of generality. Let $-\beta'_1 = \beta'_2 = 1$, and $L = 1$. By calculation, it follows that

$$C_n = \sqrt{\frac{2}{1 + \left(1 + \frac{1}{\lambda_n}\right) \cos^2 \sqrt{\lambda_n}}}. \quad (3.32)$$

According to (3.29), (3.30), and (3.32), the first five eigenvalues, normalized coefficients, and associated eigenfunctions of EVP1 for $-\beta'_1 = \beta'_2 = 1$ and $L = 1$ are:

$$\begin{array}{lll} \lambda_1 = 0.4573, & C_1 = 0.8250, & X_{11} = 0.8250 \sin 0.6762x, \\ \lambda_2 = 4.482, & C_2 = 1.226, & X_{21} = 1.226 \sin 2.117x, \\ \lambda_3 = 24.22, & C_3 = 1.384, & X_{31} = 1.384 \sin 4.921x, \\ \lambda_4 = 63.69, & C_4 = 1.403, & X_{41} = 1.403 \sin 7.981x, \\ \lambda_5 = 122.91, & C_5 = 1.408, & X_{51} = 1.408 \sin 11.09x. \end{array} \quad (3.33)$$

In order to satisfy the initial condition of problem (P_H) , we need the convergence of the eigenfunction expansion.

There are general theorems about eigenfunction expansions as follows. For the next three theorems suppose $\{\lambda_n\}$ and $X_{n1}(x)$ are the eigenvalues and eigenfunctions of EVP1 or EVP2.

Theorem 3.2. Let $g = (g_1, g_2) \in H^*$. Then

$$g = (g_1, g_2) = \sum_{n=1}^{\infty} \langle g, X_n \rangle X_n \quad (3.34)$$

on $[a, b]$ in the norm of H^* . Namely,

$$\begin{aligned} g_1(x) &= \sum_{n=1}^{\infty} \langle g, X_n \rangle X_{n1}(x) \\ g_2 &= \sum_{n=1}^{\infty} \langle g, X_n \rangle X_{n2} \end{aligned} \quad (3.35)$$

on $[a, b]$.

Under an additional condition, Hinton proved the following in [4, p.39-41].

Theorem 3.3. If $g = (g_1, g_2) \in H^*$ and $\sum_{n=1}^{\infty} \lambda_n |\langle g, X_n \rangle|^2 < \infty$, then

$$g_1(x) = \sum_{n=1}^{\infty} \langle g, X_n \rangle X_{n1}(x),$$

absolutely and uniformly on $[a, b]$.

The class of functions for which $\sum_{n=1}^{\infty} \lambda_n |\langle g, X_n \rangle|^2 < \infty$ is described in [4]. In particular, the functions in the domain of A_1 satisfy this condition.

Hinton has also proved the following theorem in [4, p.41-42].

Theorem 3.4. If $\beta'_2 \neq 0$, $p(x) \equiv r(x) \equiv 1$, g_1 is absolutely continuous with $g'_1(x) \in \mathcal{L}^2(r; a, b)$, and $g_1(a) = 0$ if $\theta = 0$ in (1.2), then

$$g_1(x) = \sum_{n=1}^{\infty} \left(\int_a^b (g_1 X_{n1})(x) dx \right) X_{n1}(x) \quad (3.36)$$

absolutely and uniformly on $[a, b]$.

If $r(x) \equiv 1$ and $X_{n1}(x)$ is real, then by definition of the norm of H^* we have

$$\begin{aligned} \sum_{n=1}^{\infty} \langle g, X_n \rangle X_{n1}(x) &= \sum_{n=1}^{\infty} \left(\int_a^b (g_1 X_{n1})(x) dx \right) X_{n1}(x) \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{1}{\rho} g_2 X_{n2} \right) X_{n1}(x). \end{aligned} \quad (3.37)$$

Thus, Theorem 3.3 and Theorem 3.4 imply $\sum_{n=1}^{\infty} \left(\frac{1}{\rho} g_2 X_{n2} \right) X_{n1}(x) = 0$. If the sum is finite, however, $\sum_{n=1}^i \left(\int_a^b (g_1 X_{n1})(x) dx \right) X_{n1}(x)$ in Theorem 3.4 may give slower convergence than $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ in Theorem 3.2 since the terms $\left(\frac{1}{\rho} g_2 X_{n2} \right) X_{n1}(x)$ are missing. Now we see this comparison graphically.

We consider the cases of $g_1(x) = x, x^2, 1$ where $X_{n1}(x)$ is an eigenfunction of EVP1, namely, (3.30). Let $-\beta'_1 = \beta'_2 = 1$ and $L = 1$. Since $X_n = (X_{n1}, X_{n2}) \in D(A_1)$ by Theorem 2.1,

$$X_{n2} = -X_{n1}(1) - X'_{n1}(1) = -\frac{1}{\lambda_n} X'_{n1}(1). \quad (3.38)$$

For $g = (g_1, g_2)$ in $D(A_1)$, we have

$$g_2 = g_1(1) - g'_1(1) \quad (3.39)$$

so that g satisfies the hypothesis of Theorem 3.3.

For $g_1(x) = x$, by (3.29), (3.30), (3.38), and (3.39), it follows that

$$\left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x) = \frac{C_n^2}{\sqrt{\lambda_n}} \left(-2 + \frac{1}{\lambda_n} \right) (\cos \sqrt{\lambda_n}) (\sin \sqrt{\lambda_n} x)$$

and

$$\langle g, X_n \rangle X_{n1}(x) = \frac{C_n^2}{\sqrt{\lambda_n}} \left(\frac{1}{\lambda_n} \right) (\cos \sqrt{\lambda_n}) (\sin \sqrt{\lambda_n} x).$$

We substitute the values of the λ_n 's and C_n 's in (3.33) into the above and show the graphs of $\sum_{n=1}^i \left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x)$ and $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$ in Figure A-4 and Figure A-5, respectively. We see the convergence to $g_1(x) = x$ is faster in $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$.

For $g_1(x) = x^2$, we have

$$\left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x) = C_n^2 \left(\int_0^1 x^2 \sin \sqrt{\lambda_n} x dx \right) \sin \sqrt{\lambda_n} x$$

and

$$\langle g, X_n \rangle X_{n1}(x) = C_n^2 \left[\int_0^1 x^2 \sin \sqrt{\lambda_n} x dx + 3 \frac{1}{\sqrt{\lambda_n}} \cos \sqrt{\lambda_n} \right] \sin \sqrt{\lambda_n} x.$$

By calculating the above formula numerically, we obtain the graphs of $\sum_{n=1}^i \left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x)$ and $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, \dots, 5$ as shown in Figure A-6 and Figure A-7, respectively. Again, we see more rapid convergence to $g_1(x) = x^2$ in $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ than in $\sum_{n=1}^i \left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x)$.

For $g_1(x) = 1$, it follows that

$$\left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x) = \frac{C_n^2}{\sqrt{\lambda_n}} (1 - \cos \sqrt{\lambda_n}) (\sin \sqrt{\lambda_n} x)$$

and

$$\langle g, X_n \rangle X_{n1}(x) = \frac{C_n^2}{\sqrt{\lambda_n}} (\sin \sqrt{\lambda_n} x).$$

The graphs of $\sum_{n=1}^i \left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x)$ and $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$ are shown in Figure A-8 and Figure A-9, respectively. It is hard to recognize which convergence is more rapid between Figure A-8 and Figure A-9. This is because $g_1(x) = 1$ does not satisfy the hypothesis of Theorem 3.4.

Eigenvalue Problem Two.

Consider the eigenvalue problem with another boundary condition at $x = 0$, namely,

$$\begin{cases} -X''(x) = \lambda X(x), & (3.40) \end{cases}$$

$$\begin{cases} X'(0) = 0, & (3.41) \end{cases}$$

$$\begin{cases} -X'(L) = \lambda [\beta'_1 X(L) - \beta'_2 X'(L)], & (3.42) \end{cases}$$

where $\beta'_1 < 0$ and $\beta'_2 > 0$. We call this problem EVP2. The boundary condition $X'(0) = 0$ was derived from $v'(0, t) = 0$ of problem (P_H) , i.e., the case where the left end is insulated. We note that EVP2 is the problem (*) with $p(x) = r(x) \equiv 1$, $q(x) \equiv 0$, $a = 0$, $b = L$, $\beta_1 = 0$, $\beta_2 = -1$, and $\theta = \frac{\pi}{2}$.

We first show the following theorem.

Theorem 3.5. *All eigenvalues of EVP2 are non-negative.*

Proof. Let $X_{n1}(x)$ be the n th eigenfunction of EVP2 associated with the n th eigenvalue λ_n . Let $X_n = (X_{n1}, X_{n2}) \in H^*$ where $X_{n2} = \beta'_1 X_{n1}(L) - \beta'_2 X'_{n1}(L)$. Then by Theorem 2.1, $X_n \in D(A_1)$ and $A_1 X_n = \lambda_n X_n$. Here, $\rho = \beta'_1 \beta_2 - \beta_1 \beta'_2 = -\beta'_1 > 0$ and thus the λ_n 's are real numbers. We have

$$\begin{aligned} \langle X_n, A_1 X_n \rangle &= \langle X_n, \lambda_n X_n \rangle \\ &= \int_0^L \lambda_n |X_{n1}(x)|^2 dx + \frac{1}{\rho} \lambda_n |X_{n2}|^2 \\ &= \lambda_n \left[\int_0^L |X_{n1}(x)|^2 dx + \frac{1}{\rho} |X_{n2}|^2 \right] \\ &= \lambda_n \langle X_n, X_n \rangle. \end{aligned}$$

Now, consider $J(X_n, X_n)$ where the function J is defined as in Lemma 3.1. Here we have

$$J(X_n, X_n) = \int_0^L |X'_{n1}(x)|^2 dx - \frac{1}{\rho} [-\beta'_2 |X'_{n1}(L)|^2] \geq 0$$

since $\beta'_2 > 0$. (Note $J(X_n, X_n)$ can be zero (e.g., $X_n = \text{constant}$.) From Lemma 3.1, we obtain

$$0 \leq J(X_n, X_n) = \lambda_n \langle X_n, X_n \rangle.$$

We know $\langle X_n, X_n \rangle = \int_0^L |X_{n1}(x)|^2 dx + \frac{1}{\rho} |X_{n2}|^2 > 0$ since $X_{n1}(x) \not\equiv 0$. Hence, $\lambda_n \geq 0$. $\square$

Next, we compute the eigenvalues and eigenfunctions for EVP2. Without loss of generality, we take all the eigenfunctions to be real.

Since the eigenvalues are non-negative, by (3.40) it follows that

$$X(x) = Ax + B, \quad \text{if } \lambda = 0,$$

and

$$X(x) = C \cos \sqrt{\lambda}x + D \sin \sqrt{\lambda}x, \quad \text{if } \lambda > 0,$$

where A , B , C , and D are constants. Since $X'(0) = 0$, $A = D = 0$ above, i.e.,

$$X(x) = B, \quad \text{if } \lambda = 0,$$

and

$$X(x) = C \cos \sqrt{\lambda}x, \quad \text{if } \lambda > 0.$$

For $\lambda = 0$, $X(x) = B$ satisfies the other boundary condition (3.42). For $\lambda > 0$, we need

$$-(-C\sqrt{\lambda} \sin \sqrt{\lambda}L) = \lambda\beta'_1(C \cos \sqrt{\lambda}L) - \lambda\beta'_2(-C\sqrt{\lambda} \sin \sqrt{\lambda}L)$$

to satisfy (3.42). By calculation, we obtain a transcendental equation

$$\tan \sqrt{\lambda}L = \frac{\beta'_1 \sqrt{\lambda}}{1 - \beta'_2 \lambda} \quad (3.43)$$

for $\lambda > 0$. Figure A-10 shows the graphs of $y = \frac{\sqrt{\lambda}\beta'_1}{1 - \lambda\beta'_2}$ and $y = \tan \sqrt{\lambda}L$ for $-\beta'_1 = \beta'_2 = 1$ and $L = 1$. The λ -coordinate of each intersection indicates the value of a positive eigenvalue for EVP2. Let $\lambda_1 = 0$ and $\lambda_2, \lambda_3, \dots$ be the positive eigenvalues from the smallest.

For $n = 1$, the eigenfunction $X_{11}(x)$ associated with $\lambda_1 = 0$ is

$$X_{11} = C_1 := B$$

where the constant $C_1 > 0$ is normalized by

$$1 = \langle (X_{11}, X_{12}), (X_{11}, X_{12}) \rangle = \int_0^L |X_{11}|^2 dx + \frac{1}{\rho} |X_{12}|^2$$

where $X_{12} = \beta'_1 X_{11}(L) - \beta'_2 X'_{11}(L)$. For $-\beta'_1 = \beta'_2 = 1$ and $L = 1$, since $\rho = \beta'_1 \beta_2 - \beta_1 \beta'_2 = -\beta'_1 = 1$, it follows that

$$X_{11} = C_1 = \frac{1}{\sqrt{2}} = 0.7071. \quad (3.44)$$

For $n \geq 2$, the eigenfunction X_{n1} associated with the n th eigenvalue λ_n is

$$X_{n1} = C_n \cos \sqrt{\lambda_n} x, \quad n \geq 2 \quad (3.45)$$

where the constant $C_n > 0$ is normalized by

$$1 = \langle (X_{n1}, X_{n2}), (X_{n1}, X_{n2}) \rangle = \int_0^L |X_{n1}(x)|^2 dx + \frac{1}{\rho} |X_{n2}|^2$$

where $X_{n2} = \beta'_1 X_{n1}(L) - \beta'_2 X'_{n1}(L) = -\frac{1}{\lambda_n} X'(L)$. Then,

$$C_n^2 = \frac{1}{\int_0^L |\cos \sqrt{\lambda_n} x|^2 dx + \frac{1}{\rho} \left| \frac{1}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} L \right|^2}.$$

For $-\beta'_1 = \beta'_2 = 1$ and $L = 1$, by calculation, it follows that

$$C_n = \sqrt{\frac{2}{1 + \frac{1}{2} (\sin 2\sqrt{\lambda_n}) \frac{\lambda_n + 1}{\sqrt{\lambda_n}(\lambda_n - 1)}}}, \quad \text{for } n \geq 2.$$

As a result, we obtain the following.

$$\begin{aligned} \lambda_1 &= 0, & C_1 &= 0.7071, & X_{11} &= 0.7071, \\ \lambda_2 &= 1.459, & C_2 &= 0.8993, & X_{21} &= 0.8993 \cos 1.208x, \\ \lambda_3 &= 11.89, & C_3 &= 1.349, & X_{31} &= 1.349 \cos 3.448x, \\ \lambda_4 &= 41.49, & C_4 &= 1.397, & X_{41} &= 1.397 \cos 6.441x, \\ \lambda_5 &= 90.83, & C_5 &= 1.406, & X_{51} &= 1.406 \cos 9.541x. \end{aligned} \quad (3.46)$$

Now, we demonstrate the eigenfunction expansion for the initial condition $g_1(x) = x$ and x^2 . This time we only demonstrate $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $-\beta'_1 = \beta'_2 = 1$ ($\rho = 1$) and $L = 1$.

For $g_1(x) = x$, by (3.37) it follows that

$$\begin{aligned} n = 1: & \quad \langle g, X_1 \rangle X_{11}(x) = \frac{5}{2} C_1^2 \\ n \geq 2: & \quad \langle g, X_n \rangle X_{n1}(x) = \frac{C_n^2}{\sqrt{\lambda_n}} \left[-\frac{1}{\lambda_n} \sin \sqrt{\lambda_n} - \frac{1}{\sqrt{\lambda_n}} \right] \cos \sqrt{\lambda_n} x. \end{aligned}$$

The graphs of $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, \dots, 5$ are shown in Figure A-11.

For $g_1(x) = x^2$, we get from (3.37) that

$$n = 1: \quad \langle g, X_1 \rangle X_{11}(x) = \frac{10}{3} C_1^2,$$

$$n \geq 2: \quad \langle g, X_n \rangle X_{n1}(x) = C_n^2 \left[\int_0^1 x^2 \cos \sqrt{\lambda_n} x dx - \frac{3}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} \right] \cos \sqrt{\lambda_n} x.$$

The graphs of $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, \dots, 5$ are shown in Figure A-12.

CHAPTER 4.

CONSTRUCTION OF GREEN'S FUNCTION

The purpose of this chapter is to construct the Green's Functions for (*), (**), and (***), and to solve the nonhomogeneous equations of the form $AF = \lambda F + G$ when λ is not an eigenvalue, where operator A for each problem is defined in Chapter 3.

4.1 GREEN'S FUNCTION FOR (*)

Consider the nonhomogeneous equation

$$A_1 F = \lambda F + G \tag{4.1}$$

where A_1 is the operator defined as in Section 2.1, $F = (F_1, F_2) \in D(A_1)$, $G = (G_1, G_2) \in H^*$, and λ is not an eigenvalue of $A_1 F = \lambda F$.

First, we show that the following statement is true.

Theorem 4.1. *Equation (4.1) holds if and only if F_1 , G_1 , and G_2 satisfy the nonhomogeneous problem associated with (*), i.e.,*

$$\begin{cases} \tau(F_1) = \lambda F_1 + G_1, & (4.2) \\ (\cos \theta)F_1(a) + (\sin \theta)(pF_1')(a) = 0, & (4.3) \\ -\beta_1 F_1(b) + \beta_2 (pF_1')(b) = \lambda[\beta_1' F_1(b) - \beta_2' (pF_1')(b)] + G_2. & (4.4) \end{cases}$$

Proof. Suppose that (4.1) holds. Since $F \in D(A_1)$, F_1 satisfies (4.3). Moreover, by the definition of A_1 , we get

$$\tau(F_1) = \lambda F_1 + G_1$$

and

$$-\beta_1 F_1(b) + \beta_2 (pF_1')(b) = \lambda F_2 + G_2 = \lambda [\beta_1' F_1(b) - \beta_2' (pF_1')(b)] + G_2.$$

Thus, F_1 , G_1 , and G_2 satisfy (4.2)–(4.4).

Conversely, suppose F_1 , G_1 , and G_2 satisfy (4.2)–(4.4). Define F_2 as $F_2 = \beta_1' F_1(b) - \beta_2' (pF_1')(b)$. Then the conclusion follows by the reverse procedure of the above. $\square$

Next, we seek a basis for the equation $\tau(u) = \lambda u$. Let $\phi_\lambda(x)$ be the solution of $\tau(u) = \lambda u$ with initial conditions

$$\phi_\lambda(a) = \sin \theta \quad \text{and} \quad (p\phi_\lambda')(a) = -\cos \theta. \quad (4.5)$$

Then, $\phi_\lambda(x)$ satisfies the boundary condition (1.2) of (*). Similarly, let $\chi_\lambda(x)$ be the solution of $\tau(u) = \lambda u$ with

$$\chi_\lambda(b) = \beta_2 + \beta_2' \lambda \quad \text{and} \quad (p\chi_\lambda')(b) = \beta_1 + \beta_1' \lambda. \quad (4.6)$$

Then $\chi_\lambda(x)$ satisfies condition (1.3) of (*). If λ is not an eigenvalue of (*), then $\phi_\lambda(x)$ does not satisfy (1.3) and $\chi_\lambda(x)$ does not satisfy (1.2) since $\phi_\lambda(x)$ and $\chi_\lambda(x)$ are not eigenfunctions of (*). Then, we have a Wronskian

$$W(a) = \begin{vmatrix} \phi_\lambda(a) & \chi_\lambda(a) \\ (p\phi_\lambda')(a) & (p\chi_\lambda')(a) \end{vmatrix} = \sin \theta (p\chi_\lambda')(a) + \cos \theta (\chi_\lambda'(a)) \neq 0 \quad (4.7)$$

since $\chi_\lambda(x)$ does not satisfy (1.2). Thus, $\phi_\lambda(x)$ and $\chi_\lambda(x)$ are linearly independent.

Define a fundamental matrix $\Phi_\lambda(x)$ by

$$\Phi_\lambda(x) = \begin{pmatrix} \phi_\lambda(x) & \chi_\lambda(x) \\ (p\phi_\lambda')(x) & (p\chi_\lambda')(x) \end{pmatrix}. \quad (4.8)$$

Let $\Delta(\lambda) := \det \Phi_\lambda(x)$. Then $\Delta(\lambda)$ does not depend on x since

$$\Delta(\lambda) = \phi_\lambda(x)(p\chi_\lambda')(x) - \chi_\lambda(x)(p\phi_\lambda')(x) \quad (4.9)$$

and

$$\begin{aligned}
\frac{d}{dx}\Delta(\lambda) &= \phi'_\lambda(x)(p\chi'_\lambda)(x) + \phi_\lambda(x)(p\chi'_\lambda)'(x) - \chi'_\lambda(x)(p\phi'_\lambda)(x) - \chi_\lambda(x)(p\phi'_\lambda)'(x) \\
&= \phi_\lambda(x)[(q\chi_\lambda)(x) - \lambda\chi_\lambda(x)] - \chi_\lambda(x)[(q\phi_\lambda)(x) - \lambda\phi_\lambda(x)] \\
&= 0
\end{aligned}$$

and thus $\Delta(\lambda)$ is a constant. We also note that $\Delta(\lambda) \neq 0$ by (4.7).

For the equation (4.2), we have the variation of constants formula

$$\begin{pmatrix} F_1(x) \\ pF'_1(x) \end{pmatrix} = \Phi_\lambda(x) \left[\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \int_a^x \Phi_\lambda^{-1}(s) \begin{pmatrix} 0 \\ -r(s)G_1(s) \end{pmatrix} ds \right]. \quad (4.10)$$

For $x = a$, we have

$$\begin{pmatrix} F_1(a) \\ pF'_1(a) \end{pmatrix} = \Phi_\lambda(a) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

By the boundary condition (4.3), we get

$$\begin{aligned}
0 &= (\cos \theta, \sin \theta) \begin{pmatrix} F_1(a) \\ (pF'_1)(a) \end{pmatrix} \\
&= (\cos \theta, \sin \theta) \begin{pmatrix} \phi_\lambda(a) & \chi_\lambda(a) \\ (p\phi'_\lambda)(a) & (p\chi'_\lambda)(a) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\
&= ((\cos \theta)\phi_\lambda(a) + (\sin \theta)(p\phi'_\lambda)(a), (\cos \theta)\chi_\lambda(a) + (\sin \theta)(p\chi'_\lambda)(a)) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.
\end{aligned}$$

Since $(\cos \theta)\phi_\lambda(a) + (\sin \theta)(p\phi'_\lambda)(a) = 0$ and $(\cos \theta)\chi_\lambda(a) + (\sin \theta)(p\chi'_\lambda)(a) \neq 0$, it follows that

$$C_2 = 0. \quad (4.11)$$

For $x = b$, we get

$$\begin{pmatrix} F_1(b) \\ pF'_1(b) \end{pmatrix} = \Phi_\lambda(b) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad (4.12)$$

where

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} := \begin{pmatrix} C_1 \\ 0 \end{pmatrix} + \int_a^b \frac{1}{\Delta(\lambda)} \begin{pmatrix} (p\chi'_\lambda)(s) & -\chi_\lambda(s) \\ -(p\phi'_\lambda)(s) & \phi_\lambda(s) \end{pmatrix} \begin{pmatrix} 0 \\ -r(s)G_1(s) \end{pmatrix} ds.$$

Since we have

$$G_2 = (-\beta_1 - \lambda\beta'_1, \beta_2 + \lambda\beta'_2) \begin{pmatrix} F_1(b) \\ (pF'_1)(b) \end{pmatrix}$$

from (4.4), we substitute (4.12) into the above to get

$$\begin{aligned}
G_2 &= (-\beta_1 - \lambda\beta'_1, \beta_2 + \lambda\beta'_2) \begin{pmatrix} \phi_\lambda(b) & \chi_\lambda(b) \\ (p\phi'_\lambda)(b) & (p\chi'_\lambda)(b) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \\
&= (-(p\chi'_\lambda)(b), \chi_\lambda(b)) \begin{pmatrix} \phi_\lambda(b) & \chi_\lambda(b) \\ (p\phi'_\lambda)(b) & (p\chi'_\lambda)(b) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \\
&= -\Delta(\lambda)V_1 + 0 \cdot V_2 \\
&= -\Delta(\lambda)V_1 \\
&= -\Delta(\lambda) \left[C_1 + \int_a^b \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds \right].
\end{aligned}$$

Hence,

$$C_1 = -\frac{G_2}{\Delta(\lambda)} - \int_a^b \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds. \quad (4.13)$$

Substituting C_1 and C_2 into the original formula (4.10), we obtain

$$\begin{aligned}
F_1(x) &= \left[-\frac{G_2}{\Delta(\lambda)} - \int_a^b \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds \right] \phi_\lambda(x) \\
&\quad + \left[\frac{1}{\Delta(\lambda)} \int_a^x \chi_\lambda(s)r(s)G_1(s)ds \right] \phi_\lambda(x) \\
&\quad - \left[\frac{1}{\Delta(\lambda)} \int_a^x \phi_\lambda(s)r(s)G_1(s)ds \right] \chi_\lambda(x),
\end{aligned}$$

or

$$\begin{aligned}
F_1(x) &= -\frac{G_2}{\Delta(\lambda)} \phi_\lambda(x) \\
&\quad + \int_a^x -\frac{1}{\Delta(\lambda)} \chi_\lambda(x) \phi_\lambda(s) r(s) G_1(s) ds \\
&\quad + \int_x^b -\frac{1}{\Delta(\lambda)} \phi_\lambda(x) \chi_\lambda(s) r(s) G_1(s) ds \\
&= -\frac{G_2}{\Delta(\lambda)} \phi_\lambda(x) + \int_a^x \mathcal{G}(x, s, \lambda) ds + \int_x^b \mathcal{G}(x, s, \lambda) ds. \quad (4.14)
\end{aligned}$$

Here we have constructed the Green's function $\mathcal{G}$ for (*) as follows:

$$\mathcal{G}(x, s, \lambda) = \begin{cases} -\frac{1}{\Delta(\lambda)} \chi_\lambda(x) \phi_\lambda(s), & a \leq s \leq x \leq b \\ -\frac{1}{\Delta(\lambda)} \phi_\lambda(x) \chi_\lambda(s), & a \leq x < s \leq b \end{cases} \quad (4.15)$$

where $\Delta(\lambda) = \phi_\lambda(x)(p\chi'_\lambda)(x) - \chi_\lambda(x)(p\phi'_\lambda)(x)$ and $\phi_\lambda(x)$ and $\chi_\lambda(x)$ are defined in (4.5) and (4.6).

4.2 GREEN'S FUNCTION FOR (**)

Consider the nonhomogeneous equation

$$A_2 F = \lambda F + G \quad (4.16)$$

where A_2 is the operator defined in Section 2.2, $F = (F_1, F_2, F_3) \in D(A_2)$, $G = (G_1, G_2, G_3) \in H^{**}$, and λ is not an eigenvalue of $A_2 F = \lambda F$.

First, we show the following.

Theorem 4.2. *Equation (4.16) holds if and only if F_1 , G_1 , G_2 , and G_3 satisfy the nonhomogeneous problem associated with (**), i.e.,*

$$\begin{cases} \tau(F_1) = \lambda F_1 + G_1, & (4.17) \end{cases}$$

$$\begin{cases} -\alpha_1 F_1(a) + \alpha_2 (pF'_1)(a) = \lambda[\alpha'_1 F_1(a) - \beta'_a (pF'_1)(a)] + G_2, & (4.18) \end{cases}$$

$$\begin{cases} -\beta_1 F_1(b) + \beta_2 (pF'_1)(b) = \lambda[\beta'_1 F_1(b) - \beta'_2 (pF'_1)(b)] + G_3. & (4.19) \end{cases}$$

Proof. Suppose F and G satisfy (4.16). Since $F \in D(A_2)$, we have

$$F_2 = \alpha'_1 F_1(a) - \alpha'_2 (pF'_1)(a)$$

and

$$F_3 = \beta'_1 F_1(b) - \beta'_2 (pF'_1)(b).$$

Thus, by the definition of A_2 , we get (4.17)–(4.19).

Conversely, suppose (4.17)–(4.19) hold. Define F_2 and F_3 as above. Then the reverse argument shows that (4.16) is true. $\square$

Let the solutions $\phi_\lambda(x)$ and $\chi_\lambda(x)$ of $\tau(u) = \lambda u$ satisfy

$$\phi_\lambda(a) = \alpha_2 + \alpha'_2 \lambda, \quad (p\phi'_\lambda)(a) = \alpha_1 + \alpha'_1 \lambda \quad (\text{then } \phi_\lambda \text{ satisfies (1.4)}) \quad (4.20)$$

and

$$\chi_\lambda(b) = \beta_2 + \beta'_2 \lambda, \quad (p\chi'_\lambda)(b) = \beta_1 + \beta'_1 \lambda \quad (\text{then } \chi_\lambda \text{ satisfies (1.3)}). \quad (4.21)$$

If λ is not an eigenvalue of (**), then by the same argument in Section 4.1, we obtain the following:

- (i) $\phi_\lambda(x)$ does not satisfy (1.3) and $\chi_\lambda(x)$ does not satisfy (1.4),
- (ii) $\phi_\lambda(x)$ and $\chi_\lambda(x)$ are linearly independent,
- (iii) If

$$\Phi_\lambda(x) := \begin{pmatrix} \phi_\lambda(x) & \chi_\lambda(x) \\ (p\phi'_\lambda)(x) & (p\chi'_\lambda)(x) \end{pmatrix},$$

then $\Delta(\lambda) := \det \Phi_\lambda(x)$ does not depend on x and $\Delta(\lambda) \neq 0$.

For the equation (4.17), we have the variation of constants formula

$$\begin{pmatrix} F_1(x) \\ pF'_1(x) \end{pmatrix} = \Phi_\lambda(x) \left[\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \int_a^x \Phi_\lambda^{-1}(s) \begin{pmatrix} 0 \\ -r(s)G_1(s) \end{pmatrix} ds \right]. \quad (4.10)$$

For $x = a$, we have

$$\begin{pmatrix} F_1(a) \\ pF'_1(a) \end{pmatrix} = \Phi_\lambda(a) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

By the boundary condition (4.18), we get

$$\begin{aligned} G_2 &= (-\alpha_1 - \lambda\alpha'_1, \alpha_2 + \lambda\alpha'_2) \begin{pmatrix} F_1(a) \\ (pF'_1)(a) \end{pmatrix} \\ &= (-\alpha_1 - \lambda\alpha'_1, \alpha_2 + \lambda\alpha'_2) \Phi_\lambda(a) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ &= (-(p\phi'_\lambda)(a), \phi_\lambda(a)) \begin{pmatrix} \phi_\lambda(a) & \chi_\lambda(a) \\ (p\phi'_\lambda)(a) & (p\chi'_\lambda)(a) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ &= (0, \Delta(\lambda)) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ &= C_2 \Delta(\lambda). \end{aligned}$$

Thus,

$$C_2 = \frac{G_2}{\Delta(\lambda)}. \quad (4.22)$$

For $x = b$, we have

$$\begin{pmatrix} F_1(b) \\ pF_1'(b) \end{pmatrix} = \Phi_\lambda(b) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

where

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} := \begin{pmatrix} C_1 \\ \frac{G_2}{\Delta(\lambda)} \end{pmatrix} + \int_a^b \frac{1}{\Delta(\lambda)} \begin{pmatrix} (p\chi'_\lambda)(s) & -\chi_\lambda(s) \\ -(p\phi'_\lambda)(s) & \phi_\lambda(s) \end{pmatrix} \begin{pmatrix} 0 \\ -r(s)G_1(s) \end{pmatrix} ds.$$

From (4.19), we have

$$\begin{aligned} G_3 &= (-\beta_1 - \lambda\beta'_1, \beta_2 + \lambda\beta'_2) \begin{pmatrix} F_1(b) \\ (pF_1')(b) \end{pmatrix} \\ &= (-(p\chi'_\lambda)(b), \chi_\lambda(b)) \begin{pmatrix} \phi_\lambda(b) & \chi_\lambda(b) \\ (p\phi'_\lambda)(b) & (p\chi'_\lambda)(b) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \\ &= (-\Delta(\lambda), 0) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \\ &= -\Delta(\lambda)V_1 \\ &= -\Delta(\lambda) \left[C_1 + \int_a^b \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds \right]. \end{aligned}$$

Thus,

$$C_1 = -\frac{G_3}{\Delta(\lambda)} - \int_a^b \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds. \quad (4.23)$$

Substituting C_1 and C_2 into the original formula (4.10), we obtain

$$\begin{aligned} F_1(x) &= \left[-\frac{G_3}{\Delta(\lambda)} - \int_a^b \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds \right] \phi_\lambda(x) + \frac{G_2}{\Delta(\lambda)} \chi_\lambda(x) \\ &\quad + \left[\int_a^x \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds \right] \phi_\lambda(x) \\ &\quad - \left[\int_a^x \frac{1}{\Delta(\lambda)} \phi_\lambda(s)r(s)G_1(s)ds \right] \chi_\lambda(x), \end{aligned}$$

namely,

$$\begin{aligned} F_1(x) &= -\frac{G_3}{\Delta(\lambda)} \phi_\lambda(x) + \frac{G_2}{\Delta(\lambda)} \chi_\lambda(x) \\ &\quad + \int_a^x -\frac{1}{\Delta(\lambda)} \chi_\lambda(x) \phi_\lambda(s)r(s)G_1(s)ds \\ &\quad + \int_x^b -\frac{1}{\Delta(\lambda)} \phi_\lambda(x) \chi_\lambda(s)r(s)G_1(s)ds. \end{aligned} \quad (4.24)$$

Hence, the Green's function $\mathcal{G}$ for (**) is

$$\mathcal{G}(x, s, \lambda) = \begin{cases} -\frac{1}{\Delta(\lambda)} \chi_\lambda(x) \phi_\lambda(s), & a \leq s \leq x \leq b \\ -\frac{1}{\Delta(\lambda)} \phi_\lambda(x) \chi_\lambda(s), & a \leq x < s \leq b \end{cases} \quad (4.25)$$

where $\Delta(\lambda) = \phi_\lambda(x)(p\chi'_\lambda)(x) - \chi_\lambda(x)(p\phi'_\lambda)(x)$ and $\phi_\lambda(x)$, $\chi_\lambda(x)$ are defined by (4.20) and (4.21).

4.3 GREEN'S FUNCTION FOR (***)

Consider the nonhomogeneous equation

$$A_3 F = \lambda F + G \quad (4.26)$$

where A_3 is the operator defined in Section 2.3, $F = (F_1, F_2, F_3) \in D(A_3)$, $G = (G_1, G_2, G_3) \in H^{**}$, and λ is not an eigenvalue of $A_3 F = \lambda F$.

We first show the following.

Theorem 4.3. *Equation (4.26) holds if and only if F_1 , G_1 , G_2 , and G_3 satisfy the nonhomogeneous problem associated with (***), i.e.,*

$$\begin{cases} \tau(F_1) = \lambda F_1 + G_1, & (4.27) \\ (\cos \theta) F_1(a) + (\sin \theta)(pF'_1)(a) = 0, & (4.28) \\ (\gamma_0 + \gamma_1 \lambda + \gamma_2 \lambda^2) F_1(b) = (\delta_0 + \delta_1 \lambda + \delta_2 \lambda^2)(pF'_1)(b) + \lambda G_2 + G_3. & (4.29) \end{cases}$$

Proof. Suppose F and G satisfy (4.26). Since $F \in D(A_3)$, (4.27) and (4.28) follow, and

$$F_2 = -\gamma_2 F_1(b) + \delta_2 (pF'_1)(b).$$

The second component of (4.26) gives that

$$F_3 = [-\gamma_1 F_1(b) + \delta_1 (pF'_1)(b)] + \lambda [-\gamma_2 F_1(b) + \delta_2 (pF'_1)(b)] + G_2.$$

Then by the third component of (4.26), (4.29) follows.

Conversely, suppose (4.27)–(4.29) hold. Define F_2 and F_3 as above. Then the reverse argument shows that (4.26) is true. $\square$

Define the solution $\phi_\lambda(x)$ of $\tau(u) = \lambda u$ by

$$\phi_\lambda(a) = \sin \theta \quad \text{and} \quad (p\phi'_\lambda)(a) = -\cos \theta \quad (4.30)$$

so that $\phi_\lambda(x)$ satisfies (1.2). Also, define the solution $\chi_\lambda(x)$ of $\tau(u) = \lambda u$ by

$$\chi_\lambda(b) = \delta_0 + \delta_1\lambda + \delta_2\lambda^2 \quad \text{and} \quad (p\chi'_\lambda)(b) = \gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2 \quad (4.31)$$

so that $\chi_\lambda(x)$ satisfies (1.5).

If λ is not an eigenvalue of (**), then by an analogous argument as in Section 4.1, we obtain the following:

- (i) $\phi_\lambda(x)$ does not satisfy (1.5) and $\chi_\lambda(x)$ does not satisfy (1.2),
- (ii) $\phi_\lambda(x)$ and $\chi_\lambda(x)$ are linearly independent,
- (iii) If

$$\Phi_\lambda(x) := \begin{pmatrix} \phi_\lambda(x) & \chi_\lambda(x) \\ (p\phi'_\lambda)(x) & (p\chi'_\lambda)(x) \end{pmatrix},$$

then $\Delta(\lambda) := \det \Phi_\lambda(x)$ does not depend on x and $\Delta(\lambda) \neq 0$.

For equation (4.27), we have the variation of constants formula

$$\begin{pmatrix} F_1(x) \\ pF'_1(x) \end{pmatrix} = \Phi_\lambda(x) \left[\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \int_a^x \Phi_\lambda^{-1}(s) \begin{pmatrix} 0 \\ -r(s)G_1(s) \end{pmatrix} ds \right]. \quad (4.10)$$

For $x = a$, we have

$$\begin{pmatrix} F_1(a) \\ pF'_1(a) \end{pmatrix} = \Phi_\lambda(a) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

By the boundary condition (4.28), we get

$$\begin{aligned} 0 &= (\cos \theta, \sin \theta) \begin{pmatrix} F_1(a) \\ (pF'_1)(a) \end{pmatrix} \\ &= (\cos \theta, \sin \theta) \begin{pmatrix} \phi_\lambda(a) & \chi_\lambda(a) \\ (p\phi'_\lambda)(a) & (p\chi'_\lambda)(a) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ &= (0, \Delta(\lambda)) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ &= \Delta(\lambda)C_2. \end{aligned}$$

Since $\Delta(\lambda) \neq 0$,

$$C_2 = 0. \quad (4.32)$$

For $x = b$, we have

$$\begin{pmatrix} F_1(b) \\ pF_1'(b) \end{pmatrix} = \Phi_\lambda(b) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

where

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} := \begin{pmatrix} C_1 \\ 0 \end{pmatrix} + \int_a^b \frac{1}{\Delta(\lambda)} \begin{pmatrix} (p\chi'_\lambda)(s) & -\chi_\lambda(s) \\ -(p\phi'_\lambda)(s) & \phi_\lambda(s) \end{pmatrix} \begin{pmatrix} 0 \\ -r(s)G_1(s) \end{pmatrix} ds.$$

From (4.29), we have

$$\begin{aligned} \lambda G_2 + G_3 &= (\gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2, -(\delta_0 + \delta_1\lambda + \delta_2\lambda^2)) \begin{pmatrix} F_1(b) \\ (pF_1')(b) \end{pmatrix} \\ &= ((p\chi'_\lambda)(b), -\chi_\lambda(b)) \begin{pmatrix} \phi_\lambda(b) & \chi_\lambda(b) \\ (p\phi'_\lambda)(b) & (p\chi'_\lambda)(b) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \\ &= (\Delta(\lambda), 0) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \\ &= \Delta(\lambda)V_1 \\ &= \Delta(\lambda) \left[C_1 + \int_a^b \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds \right]. \end{aligned}$$

Thus,

$$C_1 = \frac{\lambda G_2 + G_3}{\Delta(\lambda)} - \int_a^b \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds. \quad (4.33)$$

Substituting C_1 and C_2 into the original formula (4.10), we obtain

$$\begin{aligned} F_1(x) &= \left[\frac{\lambda G_2 + G_3}{\Delta(\lambda)} - \int_a^b \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds \right] \phi_\lambda(x) \\ &\quad + \left[\int_a^x \frac{1}{\Delta(\lambda)} \chi_\lambda(s)r(s)G_1(s)ds \right] \phi_\lambda(x) \\ &\quad - \left[\int_a^x \frac{1}{\Delta(\lambda)} \phi_\lambda(s)r(s)G_1(s)ds \right] \chi_\lambda(x), \end{aligned}$$

namely,

$$\begin{aligned} F_1(x) &= \frac{\lambda G_2 + G_3}{\Delta(\lambda)} \phi_\lambda(x) \\ &\quad + \int_a^x -\frac{1}{\Delta(\lambda)} \chi_\lambda(x)\phi_\lambda(s)r(s)G_1(s)ds \\ &\quad + \int_x^b -\frac{1}{\Delta(\lambda)} \phi_\lambda(x)\chi_\lambda(s)r(s)G_1(s)ds. \end{aligned} \quad (4.34)$$

Hence, the Green's function $\mathcal{G}$ for (***) is

$$\mathcal{G}(x, s, \lambda) = \begin{cases} -\frac{1}{\Delta(\lambda)} \chi_\lambda(x) \phi_\lambda(s), & a \leq s \leq x \leq b \\ -\frac{1}{\Delta(\lambda)} \phi_\lambda(x) \chi_\lambda(s), & a \leq x < s \leq b \end{cases} \quad (4.35)$$

where $\Delta(\lambda) = \phi_\lambda(x) (p\chi'_\lambda)(x) - \chi_\lambda(x) (p\phi'_\lambda)(x)$ and $\phi_\lambda(x)$, $\chi_\lambda(x)$ are defined by (4.30) and (4.31).

CHAPTER 5.

THE NONHOMOGENEOUS HEAT FLOW PROBLEM

In this chapter, our goal is to solve a nonhomogeneous heat flow problem (P) with heat source $\Omega(x, t) = e^{-\mu t}\omega(x)$, which we call problem (P_μ).

We first show that the above problem has a solution in the form of the sum of two functions $v(x, t)$ and $\Gamma(x)$ where each function satisfies a particular boundary value problem. Next, we solve both of them based on the knowledge we obtained in Chapters 3 and 4, and finally find the solution to (P_μ).

5.1 TECHNIQUE TO SOLVE (P_μ)

Here we consider the problem (P) with $u(0, t) = 0$, $c(x)d(x) = K(x) = 1$, and heat source $\Omega(x, t) = e^{-\mu t}\omega(x)$, namely,

$$(P_\mu) \left\{ \begin{array}{l} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-\mu t}\omega(x), \\ \mu \neq \text{eigenvalue of EVP1}, \quad 0 < x < L, \quad t > 0, \end{array} \right. \quad (5.1)$$

$$u(0, t) = 0, \quad t > 0, \quad (5.2)$$

$$-\frac{\partial u}{\partial x}(L, t) = -\beta'_1 \frac{\partial u}{\partial t}(L, t) + \beta'_2 \frac{\partial^2 u}{\partial t \partial x}(L, t), \quad (5.3)$$

$$t > 0, \quad \beta'_1 < 0, \quad \beta'_2 > 0, \quad (5.4)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L.$$

Choose $U(x, t) = e^{-\mu t}\Gamma(x)$ such that

$$\left\{ \begin{array}{l} -\Gamma''(x) = \mu\Gamma(x) + \omega(x), \end{array} \right. \quad (5.5)$$

$$\left\{ \begin{array}{l} \Gamma(0) = 0, \end{array} \right. \quad (5.6)$$

$$\left\{ \begin{array}{l} -\Gamma'(L) = \mu[\beta'_1\Gamma(L) - \beta'_2\Gamma'(L)]. \end{array} \right. \quad (5.7)$$

Choose $v(x, t)$ such that

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, & 0 < x < L, \quad t > 0, \end{cases} \quad (5.8)$$

$$v(0, t) = 0, \quad t > 0, \quad (5.9)$$

$$\begin{cases} -\frac{\partial v}{\partial x}(L, t) = -\beta'_1 \frac{\partial v}{\partial t}(L, t) + \beta'_2 \frac{\partial^2 v}{\partial t \partial x}(L, t), \\ t > 0, \quad \beta'_1 < 0, \quad \beta'_2 > 0, \end{cases} \quad (5.10)$$

$$v(x, 0) = f(x) - \Gamma(x), \quad 0 \leq x \leq L. \quad (5.11)$$

Now, we show the following theorem.

Theorem 5.1. $u(x, t) = v(x, t) + e^{-\mu t} \Gamma(x)$ solves (P_μ) .

Proof. We have

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} \right) + [(-\mu)e^{-\mu t} \Gamma(x) - e^{-\mu t} \Gamma''(x)] \\ &= 0 + e^{-\mu t} [-\mu \Gamma(x) - \Gamma''(x)] \\ &= e^{-\mu t} \omega(x). \end{aligned}$$

For $x = 0$, we get

$$u(x, 0) = v(x, 0) + e^{-\mu t} \Gamma(0) = 0.$$

If $x = L$, then

$$\begin{aligned} & -\frac{\partial u}{\partial x}(L, t) + \beta'_1 \frac{\partial u}{\partial t}(L, t) - \beta'_2 \frac{\partial^2 u}{\partial t \partial x}(L, t) \\ &= \left[-\frac{\partial v}{\partial x}(L, t) + \beta'_1 \frac{\partial v}{\partial t}(L, t) - \beta'_2 \frac{\partial^2 v}{\partial t \partial x}(L, t) \right] \\ &\quad + [-e^{-\mu t} \Gamma'(L) + \beta'_1 (-\mu) e^{-\mu t} \Gamma(L) - \beta'_2 (-\mu) e^{-\mu t} \Gamma'(L)] \\ &= 0 + e^{-\mu t} [-\Gamma'(L) - \mu [\beta'_1 \Gamma(L) - \beta'_2 \Gamma'(L)]] \\ &= 0. \end{aligned}$$

Moreover, when $t = 0$, we have

$$u(x, 0) = v(x, 0) + \Gamma(x) = f(x) - \Gamma(x) + \Gamma(x) = f(x).$$

Hence, $u(x, t) = v(x, t) + e^{-\mu t} \Gamma(x)$ solves (P_μ) . $\square$

5.2 FINDING $\Gamma(x)$

Since the nonhomogeneous problem (5.5)–(5.7) for $\Gamma(x)$ is associated with problem (*), it can be solved by using the Green's function found in Section 4.1.

Recall that the nonhomogeneous problem

$$\begin{cases} \tau(F_1) = \lambda F_1 + G_1, & (4.2) \\ (\cos \theta) F_1(a) + (\sin \theta)(pF_1')(a) = 0, & (4.3) \\ -\beta_1 F_1(b) + \beta_2(pF_1')(b) = \lambda[\beta_1' F_1(b) - \beta_2'(pF_1')(b)] + G_2 & (4.4) \end{cases}$$

is solved by

$$\begin{aligned} F_1(x) = & -\frac{G_2}{\Delta(\lambda)} \phi_\lambda(x) \\ & + \int_a^x -\frac{1}{\Delta(\lambda)} \chi_\lambda(x) \phi_\lambda(s) r(s) G_1(s) ds \\ & + \int_x^b -\frac{1}{\Delta(\lambda)} \phi_\lambda(x) \chi_\lambda(s) r(s) G_1(s) ds. \end{aligned} \quad (4.14)$$

where $\phi_\lambda(x)$ and $\chi_\lambda(x)$ are defined by (4.5) and (4.6), and $\Delta(\lambda) = \phi_\lambda(x)(p\chi'_\lambda)(x) - \chi_\lambda(x)(p\phi'_\lambda)(x)$. Here we set $F_1(x) = \Gamma(x)$, $r(x) = p(x) \equiv 1$, $q(x) \equiv 0$, $\theta = 0$, $\beta_1 = 0$, $\beta_2 = -1$, $a = 0$, $b = L$, $\lambda = \mu$, $G_1(x) = \omega(x)$, and $G_2 = 0$. Then it follows that

$$\Gamma(x) = \int_0^x -\frac{1}{\Delta(\mu)} \chi_\mu(x) \phi_\mu(s) \omega(s) ds + \int_x^L -\frac{1}{\Delta(\mu)} \phi_\mu(x) \chi_\mu(s) \omega(s) ds, \quad (5.12)$$

where $\phi_\mu(x)$ and $\chi_\mu(x)$ are solutions of $-u'' = \mu u$ such that

$$\phi_\mu(0) = 0, \quad \phi'_\mu(0) = -1 \quad (5.13)$$

$$\chi_\mu(L) = -1 + \beta_2' \mu, \quad \text{and} \quad \chi'_\mu(L) = \beta_1' \mu; \quad (5.14)$$

and

$$\Delta(\mu) = \det \begin{pmatrix} \phi_\mu(x) & \chi_\mu(x) \\ \phi'_\mu(x) & \chi'_\mu(x) \end{pmatrix}. \quad (5.15)$$

Consider the case of $\mu = 0$. Then we have

$$\phi_\mu(x) = -x, \quad \chi_\mu(x) = -1, \quad \text{and} \quad \Delta(\mu) = -1$$

from (5.13)–(5.15). Thus, it follows that

$$\Gamma(x) = \int_0^x s\omega(s)ds + \int_x^L x\omega(s)ds. \quad (5.16)$$

In the case of $\mu = 1$, we have

$$\phi_\mu(x) = -\sin x, \quad \chi_\mu(x) = C_3 \cos x + C_4 \sin x$$

where

$$\begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} \cos L & -\sin L \\ \sin L & \cos L \end{pmatrix} \begin{pmatrix} -1 + \beta'_2 \\ \beta'_1 \end{pmatrix}$$

and

$$\Delta(\mu) = C_3 = (-1 + \beta'_2) \cos L - \beta'_1 \sin L.$$

Thus, by calculation, it follows that

$$\begin{aligned} \Gamma(x) = & \int_0^L \frac{(-1 + \beta'_2) \sin L + \beta'_1 \cos L}{(-1 + \beta'_2) \cos L - \beta'_1 \sin L} \sin x \sin s\omega(s)ds \\ & + \int_0^x \cos x \sin s\omega(s)ds + \int_x^L \cos s \sin x\omega(s)ds. \end{aligned} \quad (5.17)$$

5.3 FINDING $v(x, t)$

Since the problem (5.8)-(5.11) is (P_H) in Section 3.2 with $c(x)d(x) = K(x) = 1$ and $g_1(x) = f(x) - \Gamma(x)$, we know that the problem (5.8)-(5.10) is associated with EVP1 which has eigenfunctions

$$X_{n1}(x) = C_n \sin \sqrt{\lambda_n} x, \quad n = 1, 2, 3, \dots, \quad (5.18)$$

where the coefficient C_n is normalized by (3.32) and $\lambda_n > 0$ satisfies

$$\frac{\beta'_2}{\beta'_1} \sqrt{\lambda_n} - \frac{1}{\beta'_1 \sqrt{\lambda_n}} = \tan \sqrt{\lambda_n} L, \quad n = 1, 2, 3, \dots \quad (5.19)$$

We also know that the time-dependent function $T_n(t)$ satisfies

$$T'_n(t) = -\lambda_n T_n(t), \quad (5.20)$$

and thus the general solution of $T_n(t)$ is

$$T_n(t) = C e^{-\lambda_n t} \quad (5.21)$$

where C is a constant. Therefore, by the extended superposition principle, we find $v(x, t)$ in the form of (with $C = 1$)

$$v(x, t) = \sum_{n=1}^{\infty} B_n X_{n1}(x) T_n(t) = \sum_{n=1}^{\infty} B_n C_n (\sin \sqrt{\lambda_n} x) e^{-\lambda_n t} \quad (5.22)$$

where B_n is a constant for each n . Consider when $t = 0$. We want (5.22) to satisfy (5.11). Namely, we want

$$f(x) - \Gamma(x) = v(x, 0) = \sum_{n=1}^{\infty} B_n C_n \sin \sqrt{\lambda_n} x. \quad (5.23)$$

By Theorem 3.2, the above equation holds if

$$B_n = \langle g, X_n \rangle = \int_0^L (g_1 X_{n1})(x) dx - \frac{1}{\beta'_1} g_2 X_{n2}, \quad (5.24)$$

where

$$g = (g_1, g_2) = (f(x) - \Gamma(x), \quad \beta'_1 [f(L) - \Gamma(L)] - \beta'_2 [f'(L) - \Gamma'(L)]),$$

and

$$\begin{aligned} X_n &= (X_{n1}, X_{n2}) \\ &= \left(C_n \sin \sqrt{\lambda_n} x, \beta'_1 \left(C_n \sin \sqrt{\lambda_n} L \right) - \beta'_2 \left(C_n \sqrt{\lambda_n} \cos \sqrt{\lambda_n} L \right) \right). \end{aligned}$$

Hence, we conclude that $v(x, t)$ is as in (5.22) where λ_n satisfies (5.19) and

$$\begin{aligned} B_n &= \int_0^L [f(x) - \Gamma(x)] \left(C_n \sin \sqrt{\lambda_n} x \right) dx \\ &\quad - \beta'_1 (f(L) - \Gamma(L)) \left(C_n \sin \sqrt{\lambda_n} L \right) \\ &\quad + \beta'_2 (f(L) - \Gamma(L)) \left(C_n \sqrt{\lambda_n} \cos \sqrt{\lambda_n} L \right) \\ &\quad + \beta'_2 (f'(L) - \Gamma'(L)) \left(C_n \sin \sqrt{\lambda_n} L \right) \\ &\quad - \frac{\beta'^2_2}{\beta'_1} (f'(L) - \Gamma'(L)) \left(C_n \sqrt{\lambda_n} \cos \sqrt{\lambda_n} L \right). \end{aligned} \quad (5.25)$$

When $\mu = 0$, $\Gamma(x)$ is

$$\Gamma(x) = \int_0^x s\omega(s)ds + \int_x^L x\omega(s)ds. \quad (5.16)$$

5.4 THE SOLUTION OF (P_μ)

From the results we obtained in Section 5.2 and Section 5.3, we conclude that the solution $u(x, t)$ of the problem (P_μ) with $\mu = 0$ as constructed above is, by Theorem 5.1,

$$u(x, t) = \left[\sum_{n=1}^{\infty} B_n C_n (\sin \sqrt{\lambda_n} x) e^{-\lambda_n t} \right] + \left[\int_0^x s\omega(s)ds + \int_x^L x\omega(s)ds \right] \quad (5.26)$$

where B_n is as in (5.25).

In the case where $\omega(x) \equiv 1$ and $\mu = 0$, i.e., the heat source is 1 throughout the rod at any time, (5.26) becomes

$$u(x, t) = \left[\sum_{n=1}^{\infty} B_n C_n (\sin \sqrt{\lambda_n} x) e^{-\lambda_n t} \right] + \left[x(L - x) + \frac{x^2}{2} \right]. \quad (5.27)$$

As $t \rightarrow \infty$, we get

$$\lim_{t \rightarrow \infty} u(x, t) = x(L - x) + \frac{x^2}{2}, \quad (5.28)$$

which shows that the temperature of the rod at $x = L$ approaches $\frac{L^2}{2}$ as time goes on. This result corresponds to the fact that the temperature of the fluid $\sigma(t)$ reaches $\frac{L^2}{2}$ as $t \rightarrow \infty$, since from (3.6) and (3.2) we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \sigma(t) &= \lim_{t \rightarrow \infty} \left[u(L, t) + \frac{1}{R} \phi(L, t) \right] \\ &= \lim_{t \rightarrow \infty} \left[u(L, t) + \frac{1}{R} \frac{\partial u}{\partial x}(L, t) \right] \\ &= \lim_{t \rightarrow \infty} \left[\left(\sum_{n=1}^{\infty} B_n C_n (\sin \sqrt{\lambda_n} L) e^{-\lambda_n t} + \frac{L^2}{2} \right) \right. \\ &\quad \left. + \frac{1}{R} \left(\sum_{n=1}^{\infty} B_n C_n \sqrt{\lambda_n} (\cos \sqrt{\lambda_n} L) e^{-\lambda_n t} + 0 \right) \right] \\ &= \frac{L^2}{2}. \end{aligned} \quad (5.29)$$

LIST OF REFERENCES

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APPENDIX

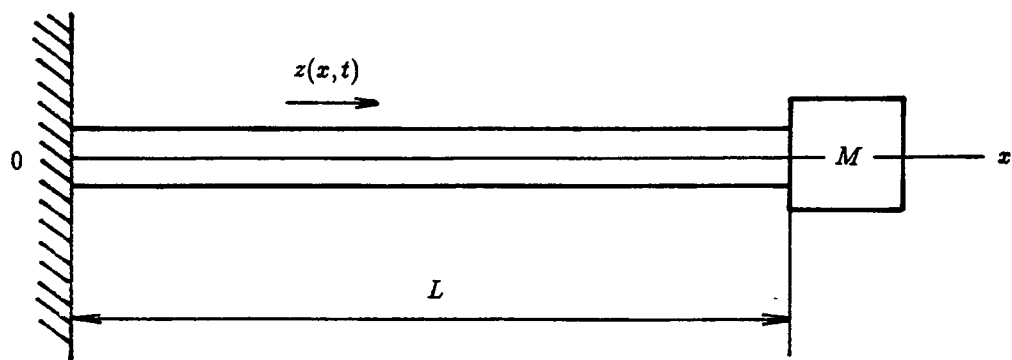


Figure A-1. Longitudinal vibration z of a bar with a mass attached at the end.

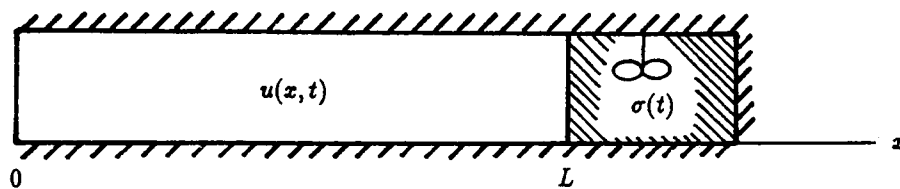


Figure A-2. Temperature u of a bar with a finite amount of fluid attached at the end.

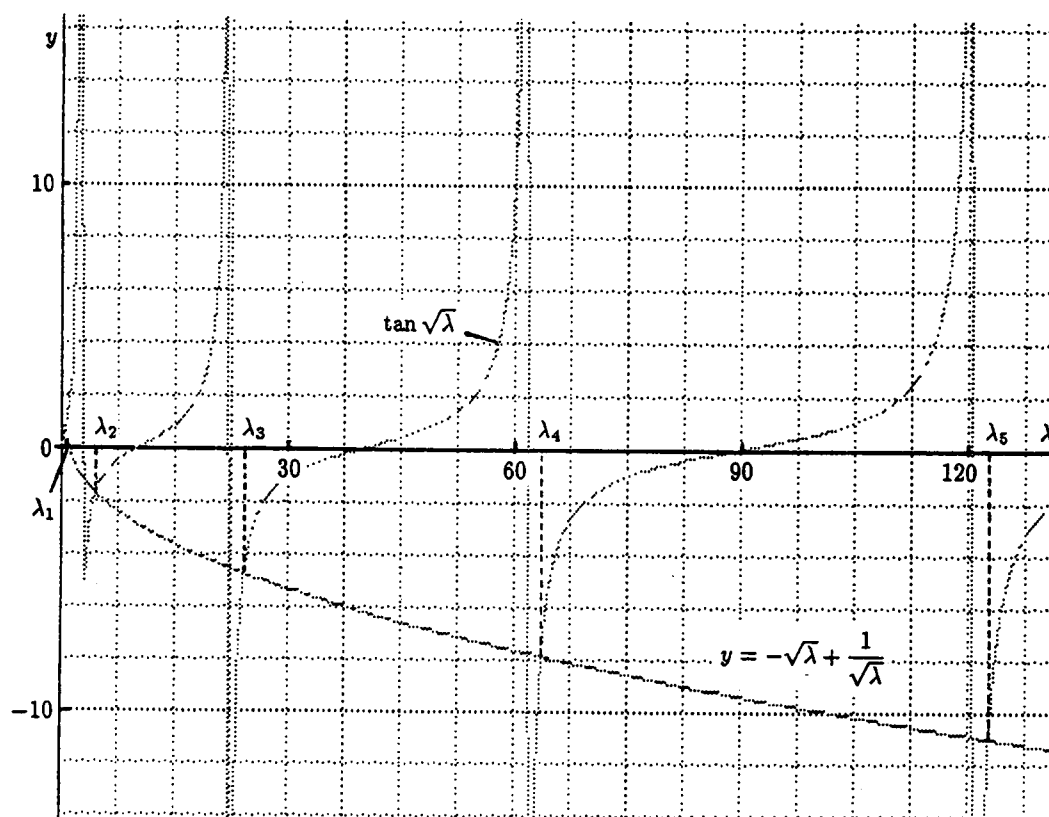


Figure A-3. The graph of $y = \frac{\beta'_2}{\beta'_1} \sqrt{\lambda} - \frac{1}{\beta'_1 \sqrt{\lambda}}$ and $y = \tan \sqrt{\lambda} L$
for $-\beta'_1 = \beta'_2 = 1$ and $L = 1$.

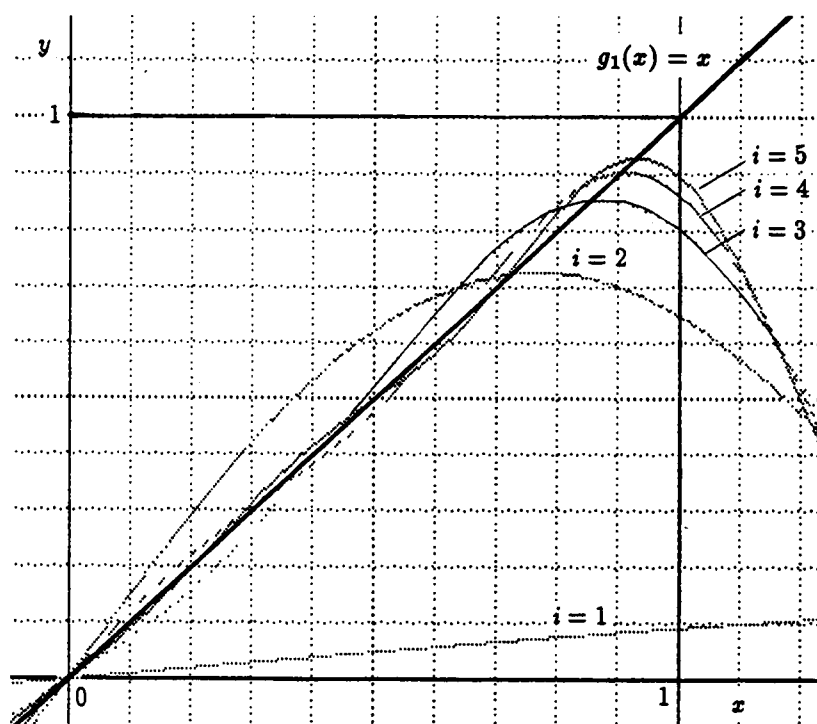
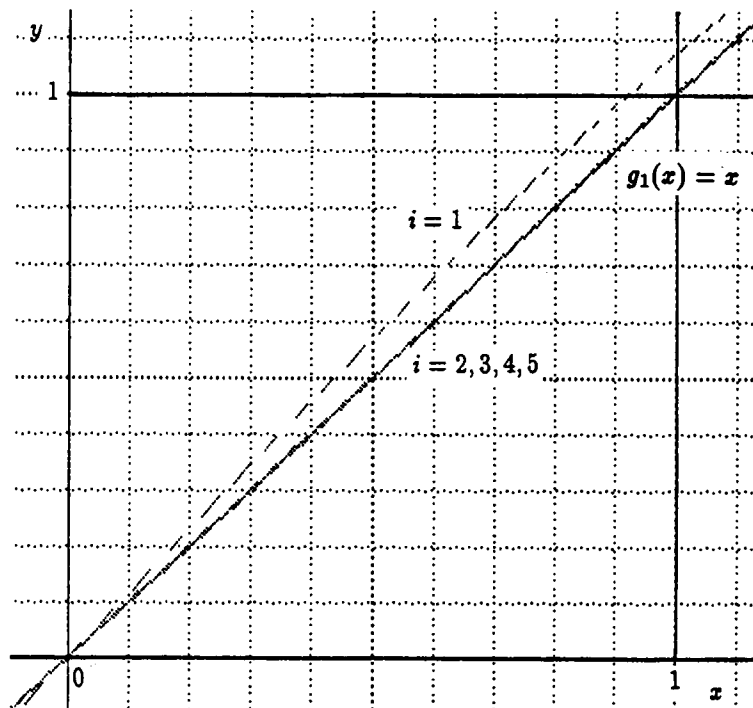


Figure A-4. The graphs of $\sum_{n=1}^i \left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$
 where $g_1(x) = x$ and $X_{n1}(x) = C_n \sin \sqrt{\lambda_n} x$.



	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
$g(x)$	0.00000	0.20000	0.40000	0.60000	0.80000	1.00000
$i = 1$	0.00000	0.23146	0.45868	0.67753	0.88401	1.07434
$i = 2$	0.00000	0.19765	0.39703	0.59893	0.80234	1.00402
$i = 3$	0.00000	0.20042	0.40011	0.59956	0.79996	1.00076
$i = 4$	0.00000	0.19993	0.40013	0.60005	0.79991	1.00028
$i = 5$	0.00000	0.20004	0.40001	0.60010	0.79998	1.00015

Figure A-5. The graphs of $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$ where $g_1(x) = x$ and $X_{n1}(x) = C_n \sin \sqrt{\lambda_n} x$.

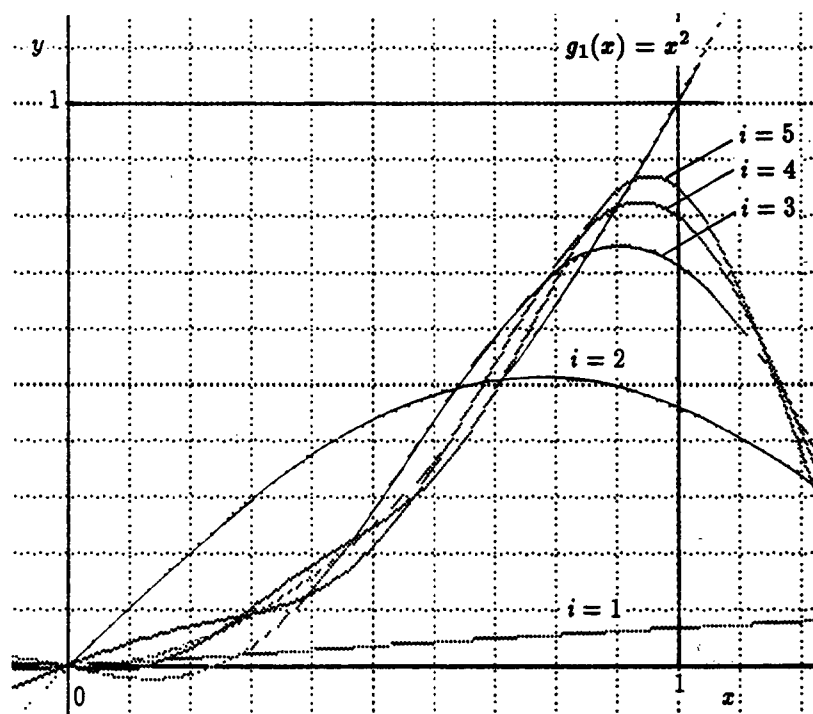
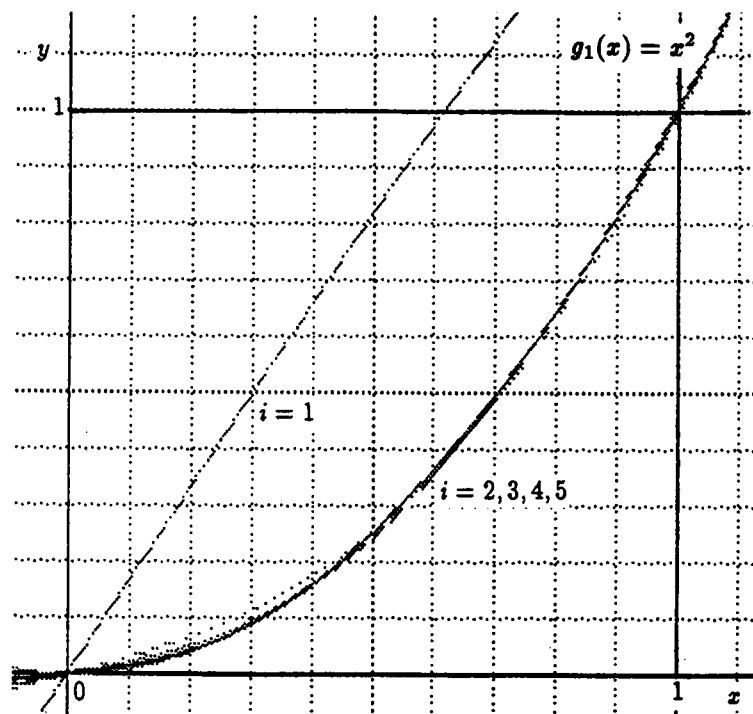


Figure A-6. The graphs of $\sum_{n=1}^i \left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$ where $g_1(x) = x^2$ and $X_{n1}(x) = C_n \sin \sqrt{\lambda_n} x$.



	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
$g(x)$	0.00000	0.04000	0.16000	0.36000	0.64000	1.00000
$i = 1$	0.00000	0.33228	0.65848	0.97267	1.26908	1.54232
$i = 2$	0.00000	0.06687	0.17454	0.35565	0.62797	0.99033
$i = 3$	0.00000	0.05122	0.15722	0.35212	0.64138	1.00871
$i = 4$	0.00000	0.04152	0.15773	0.36179	0.64036	0.99909
$i = 5$	0.00000	0.03961	0.16003	0.36093	0.63910	1.00147

Figure A-7. The graphs of $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$
where $g_1(x) = x^2$ and $X_{n1}(x) = C_n \sin \sqrt{\lambda_n} x$.

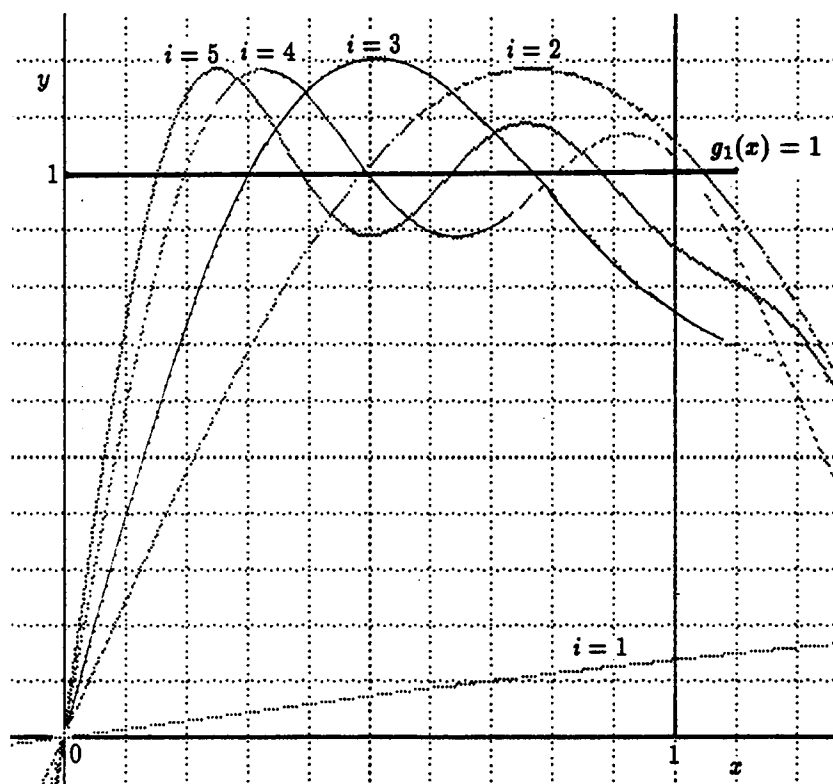


Figure A-8. The graphs of $\sum_{n=1}^i \left(\int_0^1 (g_1 X_{n1})(x) dx \right) X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$
 where $g_1(x) = 1$ and $X_{n1}(x) = C_n \sin \sqrt{\lambda_n} x$.

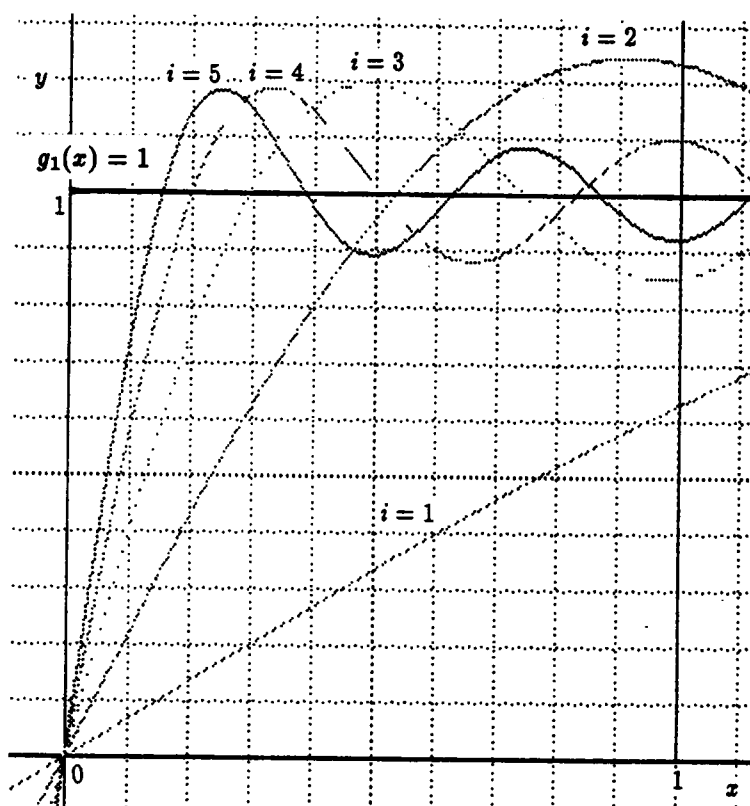


Figure A-9. The graphs of $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$
 where $g_1(x) = 1$ and $X_{n1}(x) = C_n \sin \sqrt{\lambda_n} x$.

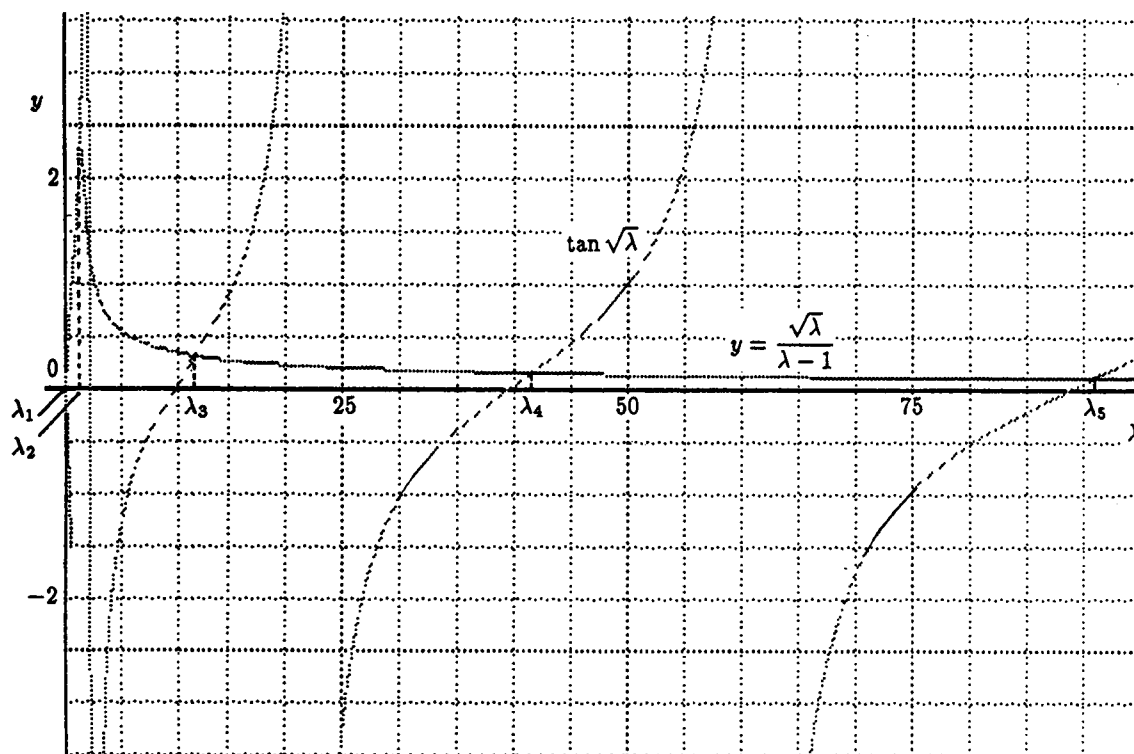


Figure A-10. The graph of $y = \frac{\beta'_1 \sqrt{\lambda}}{1 - \beta'_2 \lambda}$ and $y = \tan \sqrt{\lambda} L$ for $-\beta'_1 = \beta'_2 = 1$ and $L = 1$.

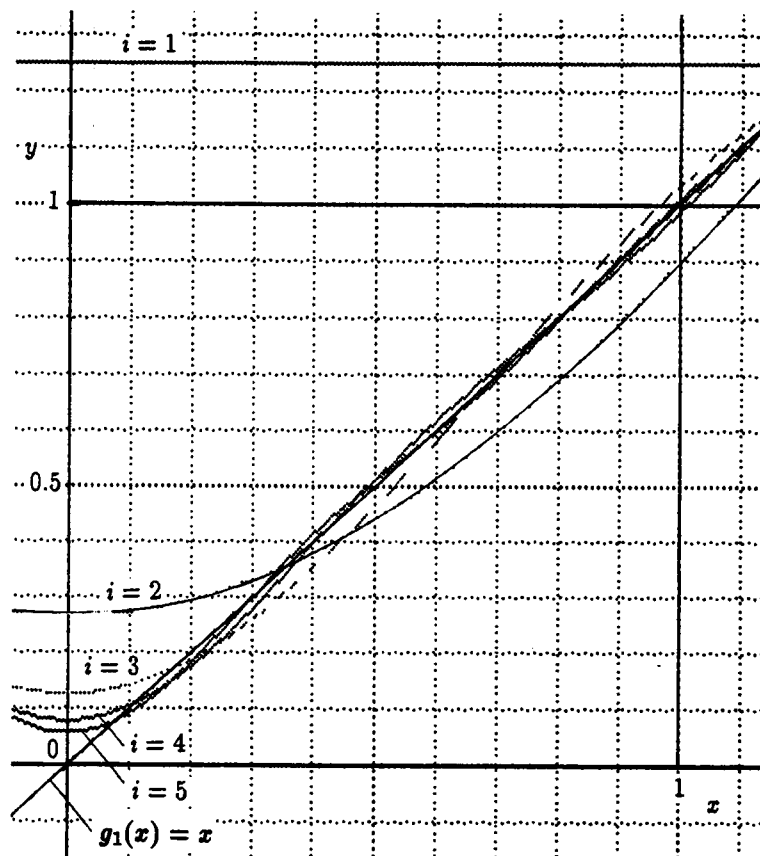
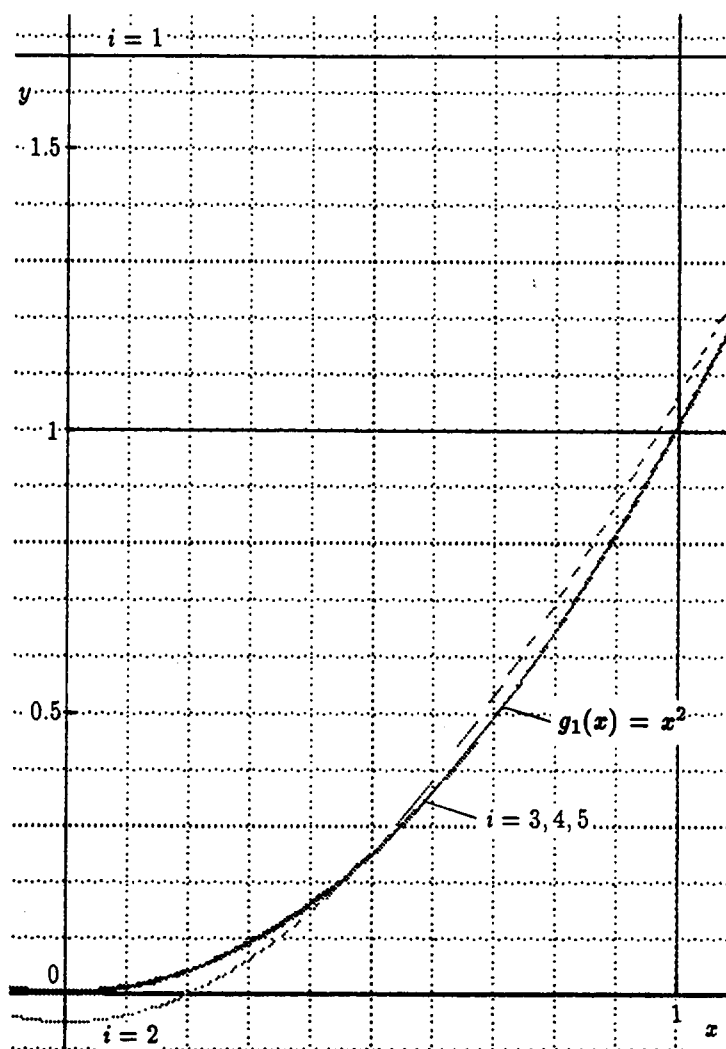


Figure A-11. The graphs of $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$
 where $g_1(x) = x$ and $X_{n1}(x) = C_n \cos \sqrt{\lambda_n} x$.



	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
$g(x)$	0.00000	0.04000	0.16000	0.36000	0.64000	1.00000
$i = 1$	1.66667	1.66667	1.66667	1.66667	1.66667	1.66667
$i = 2$	-0.04983	0.00000	0.14662	0.38150	0.69101	1.05718
$i = 3$	0.00371	0.04131	0.15681	0.35592	0.64135	1.00613
$i = 4$	-0.00088	0.04003	0.16069	0.35936	0.63939	1.00159
$i = 5$	-0.00184	0.04035	0.16145	0.35855	0.63917	1.00255

Figure A-12. The graphs of $\sum_{n=1}^i \langle g, X_n \rangle X_{n1}(x)$ for $i = 1, 2, 3, 4, 5$
where $g_1(x) = x^2$ and $X_{n1}(x) = C_n \cos \sqrt{\lambda_n} x$.

VITA

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