To the Graduate Council:

I am submitting herewith a thesis written by Bumbae Kim entitled "Studies of a Numerical Method in Fluid Mechanics." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mechanical Engineering.

Trevor H. Moulden, Major Professor

We have read this thesis and recommend its acceptance:

Accepted for the Council:

Vice Chancellor
Graduate Studies and Research

A

STUDIES OF A NUMERICAL METHOD
IN FLUID MECHANICS

A Thesis

Presented for the

Master of Science

Degree

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Bumbae Kim

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ABSTRACT

A numerical investigation was carried out using the steady state Navier-Stokes equations. These steady solutions were achieved by an iteration procedure, which normally requires less computing time than do time marching procedures.

Primitive variables were used, since they are more suitable for extension to three-dimensional flows, turbulence, etc., than are the vorticity-stream function methods.

A decay scheme was chosen, since it has wider adaptability than either the central-difference, or the upwind-difference techniques.

The method was used to study flow in a cavity and the results were compared with published results. The second application was to model and study flow inside a radiant furnace. This problem was the main interest in the study. The numerical resutlts obtained were analyzed for accuracy and consistency.

TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

C	Weighting function
D	Dilation, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$
F, f	Function
ft	Feet
G_i	Decay function in x-direction
G_j	Decay function in y-direction
G_t	Decay function in time
lbf	Pound-force
p	Static pressure
$\hat{p}$	Static pressure between two calculation steps
q	Velocity vector
q*	Velocity vector between two calculation steps
Re	Reynolds number
$R\Delta x$	Grid Reynolds number in x-direction
$R\Delta y$	Grid Reynolds number in y-direction
S1,S2,S3	Stream function values
sec	Second
s_p	Source term of the pressure equation
t	Time
u	Velocity component in x-direction
v	Velocity component in y-direction
$\vec{v}$	Velocity vector
$\vec{v}_{surface}$	Velocity vector at surface

x	Horizontal coordinate
y	Vertical coordinate
α	Relaxation parameter, or Acceleration parameter
δ	Central-difference operator
Δ	Finite increment
ε	Convergence parameter
μ	Absolute viscosity, or Averaging operator
ν	Kinematic viscosity
ρ	Density
ψ	Stream function
ω	Vorticity

Subscripts and Superscripts

i	Grid number in x-direction
j	Grid number in y-direction
k,n	Iteration number

Abbreviations

ADI	Alternate Direction Implicit method
SMAC	Simplified Marker-And-Cell method
SOR	Successive-Overrelaxation method

CHAPTER I

INTRODUCTION

General Review

Numerical methods in fluid dynamics probably started with Thom's[1] hand calculation of the velocity field around a circular cylinder. Since that time several numerical methods have been developed and applied to fluid dynamics and parallel the use of numerical methods in other areas of Mechanics.

The finite difference method has been favored because it is straight forward and relatively simple. However, it has some difficulties in complex flow fields such as high Reynolds number flow and flows with arbitrary physical boundaries. High Reynolds number flows will generally induce large gradients and require an increased number of grid points for adequate resolution. The small eddies found in turbulent flows cannot be resolved. As a result more computer storage capacity and speed is needed before such flows can be studied. Three-dimensional flows and flows with turbulence seem to be still far beyond the capabilities of even the latest generation of computers, despite the almost revolutionary progress that has been made in the last two decades. Complex geometry may lead to errors in the calculation of flow variables, such as vorticity and pressure, on arbitrary walls. The errors produced at the boundaries will cause deterioration of the entire flow and the results will be poor.

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[1]Numbers in brackets refer to numbered references in the bibliography.

The inherent limitations and shortcomings of the finite difference method for flows with geometric complexity may be overcome by the application of other methods. These methods include the finite element method, which allows much more complex geometry, and the spectral method, which is in general a more accurate approximation to partial differential equations than is the finite difference method. However, the finite element method requires more complex matrix operations and the use of a meaningful variational formulation. Spectral methods also have difficulties due to their relative complexity and the implementation of complex boundary conditions.

According to Fasel[2], recent progress has been made in the finite difference method for the areas in which it is deficient. Other methods have seen similar development.

Objectives

Two-dimensional laminar, incompressible flows have been studied extensively by many investigators using the Navier-Stokes equations. Such studies were formulated in terms of vorticity and stream function as the dependent variables. Here, vorticity is defined as

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \qquad \text{or} \qquad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

for their specific flows. The stream function is defined for any two-dimensional flow field as

$$u = \frac{\partial \psi}{\partial y} \ \text{ and } \ v = -\frac{\partial \psi}{\partial x}\ , \ \text{or } u = -\frac{\partial \psi}{\partial y} \ \text{ and } \ v = \frac{\partial \psi}{\partial x}$$

in Cartesian coordinates for the flows stated above.

Roache[3] has given an excellent survey of the techniques for the solution of Navier-Stokes equations using the vorticity-stream function formulation. This formulation, which removes the troublesome pressure terms (which can be recovered later if needed), is very useful for two-dimensional flows in simply-connected regions. However, it has critical drawbacks in being unsuitable for extension to three-dimensional, compressible or turbulent flow calculations. The primitive variable approach may be a viable alternative for this formulation. Ghia and his associates[4] have made progress with the primitive variable approach and have obtained accurate and efficient results, especially for simple geometry such as the cavity problem. The unsteady equations were used. Here the cavity problem is defined as the rectangular cavity flow, generated by a moving wall on one side[5]. Figure 1 shows the geometry of the cavity problem.

For the present study, a method was developed for using primitive variables, i.e. (u,v,p), directly to obtain solutions of the steady Navier-Stokes equation. The steady flow method is based upon the assumption that a steady solution exists for the problem. The reason for choosing the steady flow equations in the (u,v,p) formulation is its advantage (over the unsteady approach) in computing time, plus its merits over the vorticity-stream function formulation. It has been observed by the author that the use of the primitive variables method for the steady flow equation has not been favored by previous investigations. However, Schulz[6] has used it successfully for problems involving engine flows.

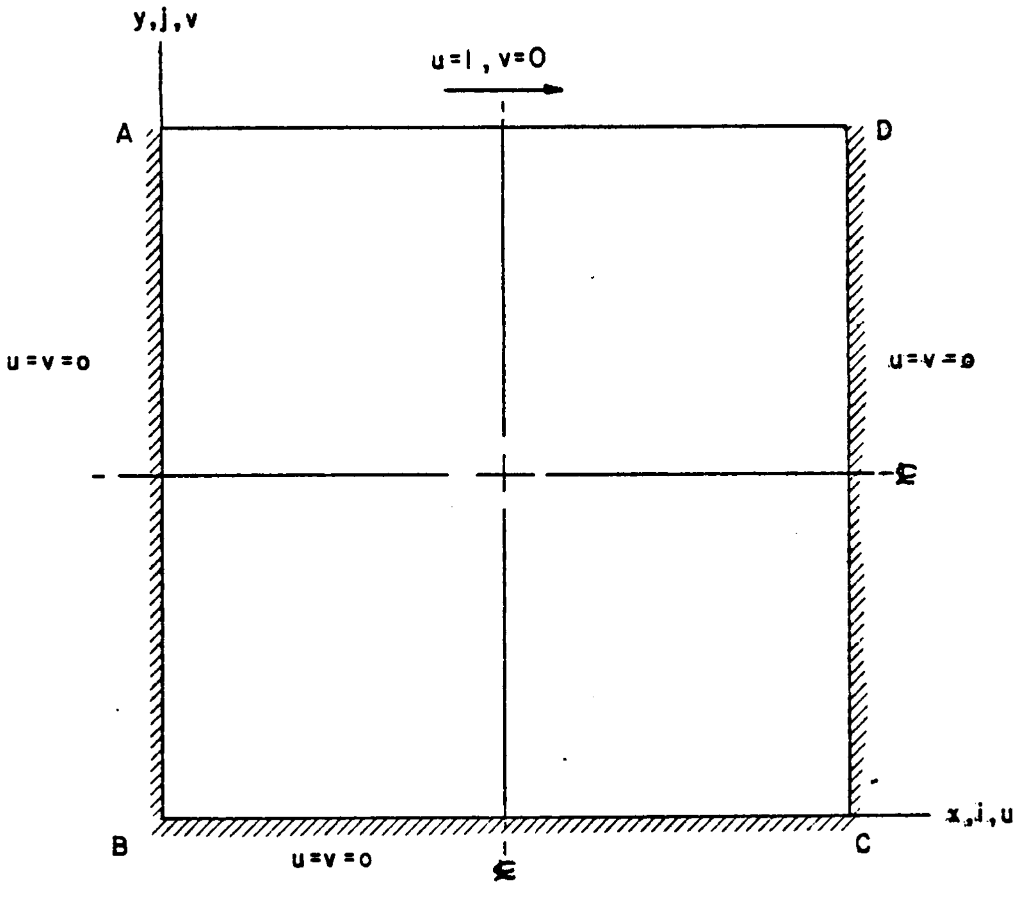

Figure 1. Geometry of the cavity problem.

The velocity boundary conditon is simple and always explicit, while the pressure boundary condition is determined by application of the basic equations with the viscous velocity boundary condition.

The pressure is a variable of smaller magnitude than the vorticity, which becomes rather large near a surface for high Reynolds number flows. Thus, a smaller error is expected from the primitive variable method in comparison with the vorticity method where the vorticity derivative terms near a surface are large. It is noted that the pressure gradient near a surface is related to vorticity gradient in some sense, as dictated by Navier-Stokes equations.

The cavity problem was selected as a test for the algorithm and the results can be compared with published results[7]. It is noted that the cavity problem requires the full Navier-Stokes equations, and not boundary layer equations.

The flow field of interest in the present study is that of a radiant furnace. The radiant furnace is a component of the DOE (Department of Energy) sponsored coal-fired Magnetohydrodynamic pilot-plant at The University of Tennessee Space Institute, Tullahoma, Tennessee. It is important to obtain information of the flow properties for improved design, development, analysis and commercial scale-up.

CHAPTER II

NAVIER-STOKES EQUATIONS AND DISCRETE SYSTEMS

Governing Equations

The nonlinear partial differential equations containing the postulates of a continuum fluid are the Navier-Stokes equations, together with the continuity equation. Since the discussion is restricted to fluids with constant physical properties, there is no reason for introducing an energy equation.

In the convective, i.e. non-conservative form,[1] the two-dimensional equations are:

$$\text{Continuity} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\text{x-momentum} \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{2}$$

$$\text{y-momentum} \qquad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \tag{3}$$

where ∇^2 is a Laplacian operator, i.e.,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

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[1]Conservative form of the convective term is defined as $\frac{\partial(uu)}{\partial x} + \frac{\partial(vu)}{\partial y}$ and $\frac{\partial(uv)}{\partial x} + \frac{\partial(vv)}{\partial y}$, while the non-conservative form is $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$ and $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$. The conservative form is also referred to as the divergence form of the equations.

and u, v the velocity components along the coordinate axes x, y. p is the fluid static pressure, u the first coefficient of viscosity, and ρ the fluid density.

Here, three points are noted. First, the coupled convective term makes the system nonlinear and makes an analytic approach difficult. A few exact solutions can be formed and other solutions obtained through linearization or by neglecting nonlinear terms. Second, different treatments of the convective term may produce different results. Bozeman and Dalton[8] compared different treatments of the convective terms and showed substantial differences in the results. This study considered the central-difference in non-conservative form, central-difference in conservative form, upwind-difference in the non-conservative form, and upwind-difference in conservative form. Nallasamy and Prasad[9] also emphasized the importance of different schemes when applied to vortex flow in a cavity. In general, the conservative form of the equations gives a more accurate result than does the non-conservative form. However, the non-conservative form, combined with the decay scheme, was used for the present study, providing the best approach. The decay scheme is described in Appendix A. Third, the pressure does not appear as a dominant variable in any of the above formulations. Pressure is determined from a Poisson equation, which in turn, is obtained from the appropriate momentum equations to correctly model the elliptic nature of the flow problem. One of the main reasons the primitive variable approach has not been applied widely is the cumbersome pressure boundary condition and manipulation of the pressure equation.

The time-dependent term has been disregarded for the following reasons. Crocco[10] apparently pioneered the "time-dependent technique" in his noted paper of 1965. Since then, it has been adopted by most workers. The principle of the time-dependent procedure is that unsteady flows converge after a long time to a physically stable steady-flow solution. There are advantages in using the time-dependent technique where the steady flow is approached. The transient flow could be obtained if small enough time steps were considered. Also, there is a widespread philosophy that a model of the actual physical process should be attempted. After all, any physical event is fundamentally time-dependent. And in some sense, the steady flow method is analogous to the unsteady flow method[3].

However, the computation time or modeled process rate will be different from the physical time scale or process rate. As a result, the computational transient solution is different, in general, from the physical transient solution unless great care is taken to form the temporal process numerically. The time needed to obtain a steady state solution is then different from that of the real process. More importantly, long computer run times are needed to obtain computational steady solution with the time-dependent technique.

In the present study, the steady flow method was used to take advantage of the shorter computing time for a converged solution.

Poisson Equation for Pressure

The equations (1), (2) and (3) are not likely to provide the solution directly in the primitive variable approach, because the pressure is not a dominant variable in this formulation. Thus, another equation for pressure as a dominant variable is generally used in the primitive variable method.

In the formulation of the pressure equation from the unsteady motion equations, Ames[11] does not set the local dilation term to zero, but keeps it to enhance the numerical stability. However, in the present study, the local dilation term

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

is set to zero by continuity. The result:

$$\nabla^2 p = s_p \tag{4}$$

where $s_p = - 2p \, [(\frac{\partial u}{\partial y}) \, (\frac{\partial v}{\partial x}) - (\frac{\partial u}{\partial x}) \, (\frac{\partial v}{\partial y})]$

is obtained by taking the divergence of the appropriate momentum equations.

It is both important and difficult for the pressure terms to be treated as a solution to equation (4) per Neumann, i.e. derivative, boundary conditions.

It is also noted that the pressure is an important driving force in the physics of the flow as well as in calculation as vorticity is in vorticity-included formulation, and that pressure is smaller in magnitude and in gradient, by the author's observation of numerical result, than vorticity which has large gradients, especially near walls. As a result it is conjectured that the errors in the primitive variable approach are due to the velocities near the walls, if proper treatment of pressure is used, and the errors in the vorticity-stream function approach are caused mainly by vorticity boundary treatment.

Boundary Conditions

The boundary conditions in the primitive variable approach are discussed. The velocity boundary condition, the so-called no-slip condition is used for viscous flow at a non-porous wall where the mean free path of the fluid is small enough compared to some characteristic length (i.e., consistent with the continuum fluid assumption). The no-slip condition on the non-porous surface gives $\vec{v} = 0$ if the surface is stationary, or $\vec{v} = \vec{v}_{surface}$ if the surface is moving with the velocity, $\vec{v}_{surface}$. Here, by mean free path is meant the average length of each segment of the path between successive molecular collisions. For the pressure boundary condition, the normal pressure gradient is generally used, which is combined with the appropriate momentum equation. The boundary problem with a derivative boundary condition is called a Neumann problem.

Numerical Boundary Conditions

Five ways to treat the Neumann problem numerically has been found in the literature. Each has its own advantages and problems.

First, consider the surface normal pressure gradient which is calculated by a one-sided difference at the boundary. The Taylor expansion, used for approximating the derivatives at the walls, differs from that employed for the interior domain. In the present study the normal derivative $\frac{\partial p}{\partial y}$ at a typical wall, BC, shown in Figure 1, is

$$\left.\frac{\partial p}{\partial y}\right|_{i,1} = \mu \left.\frac{\partial^2 v}{\partial y^2}\right|_{i,1} = \frac{\mu}{\Delta y^2} \left[4\, v_{i,2} - \frac{1}{2} v_{i,3} - 3\Delta y \left.\frac{\partial v}{\partial y}\right|_{i,1}\right] \quad (5)$$

Here, μ is the absolute viscosity.

The term $\left.\frac{\partial v}{\partial y}\right|_{i,1}$ is then set to zero to further ensure satisfying the continuity equation. There is some difficulty in the treatment of the corner, since a singularity can appear. However, this singularity is local and does not propagate its effect into the interior domain. The continuity requirement can be ensured by the Taylor expansion at the solid boundary. This method is used for the present study like equation (5). However, it is possible that the inconsistent use of a difference technique between the interior domain and the boundaries, (i.e., central-difference for the domain and one-sided difference for boundaries) can cause the result to deteriorate (i.e., leading to different error magnitudes).

Second, consider Chorin[12], who used a "pressure iteration" method, i.e.,

$$p^{k+1}_{i,j} = p^{k}_{i,j} - \alpha \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]_{i,j}$$

where α is the acceleration parameter, and superscript k designates the pressure-iteration number. When the continuity requirement is satisfied it is possible to obtain a converged value. However, this method does not model the derivatives of the pressure throughout the domain. It is recommended by Ghia and his associates[4], that with a regular grid system, i.e., not a staggered grid system, the pressure iteration can only be used at boundaries if the Poisson equation is being solved in the interior.

Third, consider the gradients of pressure evaluated at the interior boundaries by calculating the complete momentum equations at these boundaries. The pressure at actual boundaries can be determined by using the pressure gradient and the complete momentum equation at the interior boundary. This method allows accurate central-difference through the domain and boundary. Ghia and his associates[4] used this method with great success.

Fourth, consider Spalding's SIMPLE method,[2] which is used to solve the pressure-linked Poisson equation. This SIMPLE procedure is summarized by Hefner[7] as follows:

1. Assume an initial value, p_o, for pressure.
2. Calculate u and v based upon the assumed p_o by any method.
3. Let the calculated velocity vector be q^*.
4. Assume that the actual velocity vector is $q = q^* + \nabla\hat{p}$

   The divergence of both sides of this equation, along with the continuity equation, gives

[2]More detail for SIMPLE procedure is provided by Caretto, L.S., A.D. Gosman, S.V. Patanker, and D.B. Spalding. Two Calculation Procedures for Steady Three-Dimensional Flows with Recirculation. Rep. No. EF/TN/A/47, Dep. Mech. Eng., Imperial College Sci. and Technol., June, 1972.

$$\nabla \cdot q = 0 = \nabla \cdot q^* + \nabla^2 \hat{p}$$

$$\text{i.e., } \nabla^2 \hat{p} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

The surface normal pressure gradients are assumed to be zero due to the identity $q = q^*$ on the boundaries.

5. The actual pressure, p, is then defined as $p = p_o + \hat{p}$ Once the pressure is obtained, then the velocity vector is corrected by $q = q^* + \nabla\hat{p}$.

Here, q, q*, $\hat{p}$ and p are dimensionless.
This method also gives quite good results, because it tries to satisfy the continuity equation at every iteration or time step. However, the assumption of zero normal pressure gradient at the surface is not valid.

Fifth, consider the fictitious rows and columns, outside the actual boundaries, which may be used for determination of derivatives at the wall[11]. However, some values at the fictitious points must be calculated.

So far, five methods for the Neumann problem for the pressure field have been discussed. The major difficulties arise in the implementation of the pressure boundary condition, and this is important for the success of the calculation. This is particularly true when the primitive variable approach is adopted.

For the cavity problem, a no-slip condition was used for the velocity boundary condition, and the first of the above five methods stated above was used for the pressure boundary condition.

In the furnace problem, the flow in the inlet is not made part of the calculation just for the convenience of calculation. In general, the corner gives a singularity problem and more complex matrices. The velocity no-slip boundary condition was applied on the walls. The global continuity constraint was used to obtain the outlet velocity boundary condition. The outlet velocity profile was assumed to be cubic (i.e., no u-component), such that the total mass coming from the inlet went out through the outlet, such that the velocities at each exit wall were zero, and that $\partial v/\partial x$ was zero at the recirculating side wall. It is noted that this assumed profile will affect the recirculating production. The first method was used for the pressure boundary condition as stated earlier.

Discretization Scheme

When solving the Navier-Stokes equation by finite-difference methods, it is desirable to have a scheme which is both stable and accurate for a wide Reynolds number range. The grid Reynolds number is defined as (local velocity) x (grid step size)/(kinematic viscosity). The central-difference formulation has second-order accuracy and is stable at a low grid Reynolds number; however, it begins to diverge at high Reynolds number values. Upwind-differencing is an alternate formulation and gives a stable solution at high grid Reynolds number. Unfortunately, it only has first order accuracy.

Here, upwind-differencing for the convection term is described as below, with a conservative form for the velocity-vorticity formulation.

$$\left.\frac{\partial(u\,\omega)}{\partial x}\right|_{i,j}^{n} = \begin{cases} (u\,\omega)_{i+1,j}^{n} - (u\,\omega)_{i,j}^{n} & \text{if} \quad u < 0 \\ (u\,\omega)_{i,j}^{n} - (u\,\omega)_{i-1,j}^{n} & \text{if} \quad u > 0 \end{cases}$$

$$\left.\frac{\partial(v\,\omega)}{\partial y}\right|_{i,j}^{n} = \begin{cases} (v\,\omega)_{i,j+1}^{n} - (v\,\omega)_{i,j}^{n} & \text{if} \quad v < 0 \\ (v\,\omega)_{i,j}^{n} - (v\,\omega)_{i,j-1}^{n} & \text{if} \quad v > 0 \end{cases}$$

It is noted that the one-sided difference gives first-order accuracy in the convection term.

Chien[13,14] suggested the use of a decay function to get a stable and accurate scheme which retains the advantages of both the central difference and the upwind scheme. The present study adopts this decay scheme and extends it to the primitive variable formulation. Details of decay theory extension are presented in Appendix A.

The governing equations (2), (3) and (4) to be solved can be stated in the following form:

$$A_1 \frac{\partial^2 F}{\partial x^2} + A_2 \frac{\partial^2 F}{\partial y^2} - B_1 \frac{\partial F}{\partial x} - B_2 \frac{\partial F}{\partial y} + D = 0$$

where

$$F = \begin{bmatrix} u \\ v \\ p \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} u/\nu \\ u/\nu \\ 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} v/\nu \\ v/\nu \\ 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} -\,1/\mu\ \partial p/\partial x \\ -\,1/\mu\ \partial p/\partial y \\ -\,s_p \end{bmatrix}$$

The finite-difference equations are solved by a point-by-point Liebman iteration (or Gauss-Seidel iteration) method,

$$F_{i,j}^{n+1} = C_{i+1,j}\, F_{i+1,j}^{n} + C_{i-1,j}\, F_{i-1,j}^{n+1} + C_{i,j+1}\, F_{i,j+1}^{n} + C_{i,j-1}\, F_{i,j-1}^{n+1} + D_{i,j}^{n}/CR_{i,j}$$

where n is the inner iteration number, and where $C_{i+1,j}$, $C_{i-1,j}$, $C_{i,j+1}$, and $C_{i,j-1}$ are weighting functions defined as below, where

$$C_{i+1,j} = \frac{\dfrac{A_1}{(\Delta x)^2 G_i} - \dfrac{B_1}{2\Delta x}}{CR_{i,j}}$$

$$C_{i-1,j} = \frac{\dfrac{A_1}{(\Delta x)^2 G_i} + \dfrac{B_1}{2\Delta x}}{CR_{i,j}}$$

$$C_{i,j+1} = \frac{\dfrac{A_2}{(\Delta y)^2 G_j} - \dfrac{B_2}{2\Delta y}}{CR_{i,j}}$$

$$C_{i,j-1} = \frac{\dfrac{A_2}{(\Delta y)^2 G_j} + \dfrac{B_2}{2\Delta y}}{CR_{i,j}}$$

$$CR_{i,j} = \frac{2}{(\Delta x)^2 G_i} + \frac{2}{(\Delta y)^2 G_j}$$

$$G_i = \begin{cases} 1.0 - 0.0625\, R^2\Delta x & \text{if } |R\Delta x| < 2 \\ \dfrac{2}{|R\Delta x|} - \dfrac{1}{(R\Delta x)^2} & \text{if } |R\Delta x| \geq 2 \end{cases}$$

$$G_j = \begin{cases} 1.0 - 0.0625\, R\Delta y^2 & \text{if } |R\Delta y| < 2 \\[2ex] \dfrac{2}{|R\Delta y|} - \dfrac{1}{(R\Delta y)^2} & \text{if } |R\Delta y| \geq 2 \end{cases}$$

The equation for the u-calculation can be written as:

$$u^{n+1}_{i,j} = \frac{\dfrac{1}{(\Delta x)^2 G_i} + \dfrac{R\Delta x}{2(\Delta x)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}} u^{n+1}_{i-1,j} + \frac{\dfrac{1}{(\Delta x)^2 G_i} - \dfrac{R\Delta x}{2(\Delta x)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}}$$

$$u^{n}_{i+1,j} + \frac{\dfrac{1}{(\Delta y)^2 G_j} - \dfrac{R\Delta y}{2(\Delta y)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}} u^{n}_{i,j+1} + \frac{\dfrac{1}{(\Delta y)^2 G_j} + \dfrac{R\Delta y}{2(\Delta y)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}}$$

$$u^{n+1}_{i,j-1} + \frac{1}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}} \left[- \frac{P_{i+1,j} - P_{i-1,j}}{2(\Delta x)\mu} \right] \tag{8}$$

where $R\Delta x = \dfrac{u \cdot \Delta x}{\nu}$, $R\Delta y = \dfrac{v \cdot \Delta y}{\nu}$, G_i and G_j are based upon $u^n_{i,j}$ and the most recent value of $v_{i,j}$ that for the v-calculation as

$$v^{n+1}_{i,j} = \frac{\dfrac{1}{(\Delta x)^2 G_i} + \dfrac{R\Delta x}{2(\Delta x)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}} v^{n+1}_{i-1,j} + \frac{\dfrac{1}{(\Delta x)^2 G_i} - \dfrac{R\Delta x}{2(\Delta x)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}}$$

$$v^{n}_{i+1,j} + \frac{\dfrac{1}{(\Delta y)^2 G_j} - \dfrac{R\Delta y}{2(\Delta y)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}} v^{n}_{i,j+1} + \frac{\dfrac{1}{(\Delta y)^2 G_j} + \dfrac{R\Delta y}{2(\Delta y)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}}$$

$$v^{n+1}_{i,j-1} + \frac{1}{\frac{2}{(\Delta x)^2 G_i} + \frac{2}{(\Delta y)^2 G_j}} \left[- \frac{p_{i,j+1} - p_{i,j-1}}{2\,\Delta y \cdot \mu} \right] \tag{9}$$

where $R\Delta x = \frac{u \cdot \Delta x}{\nu}$, $R\Delta y = \frac{v \cdot \Delta y}{\nu}$, G_i and G_j are based upon $v^n_{i,j}$ and the most recent value of $u_{i,j}$ and finally for the p-calculation, there is:

$$\overline{p^{n+1}_{i,j}} = \frac{\frac{1}{(\Delta x)^2}}{\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}} \; p^{n+1}_{i-1,j} + \frac{\frac{1}{(\Delta x)^2}}{\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}}$$

$$p^n_{i+1,j} + \frac{\frac{1}{(\Delta y)^2}}{\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}} \; p^n_{i,j+1} + \frac{\frac{1}{(\Delta y)^2}}{\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}}$$

$$p^{n+1}_{i,j-1} + \frac{1}{\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}} \left[-(s_p)_{i,j} \right] \tag{10}$$

$$p^{n+1}_{i,j} = p^n_{i,j} + \alpha(\overline{p^{n+1}_{i,j}} - p^n_{i,j}) \tag{11}$$

where α is a relaxation parameter.

Two points must necessarily be presented about the pressure calculation scheme. First, the pressure is solved by using a central-difference approximation which has second-order accuracy. The central difference equations are recovered by setting $G_i = G_j = 1$ in equation (7). Thus, the central-difference formulation is a special case imbedded within the decay scheme. Second, the pressure field was solved by means of an SOR (Successive-Overrelaxation) method, using equations (10) and (11).

In general, the determination of an optimal relaxation parameter, α, is obtained from computational experiments. Present study did not use the optimal value. Here the optimal relaxation parameter is selected to give the fastest convergence for a given matrix. Generally, this optimal value is dependent upon the boundary condition, the domain and the mesh size for elliptic equations, but does not depend upon the source term. Overrelaxation occurs when $\alpha>1$, while $\alpha<1$ produces underrelaxation. A Liebman method (or Gauss-Seidel iteration) is recovered by placing $\alpha=1$.

The present study used a block tridiagonal matrix in the rectangular domain (with constant mesh size and Neumann boundary condition), and the optimal value of the relaxation parameter was obtained from a numerical experiment. However, this optimal value, $\alpha>1$, was not used for the present study since the system diverges due to a large step or change of pressure iteration level. Thus, underrelaxation is used, with the value $\alpha=0.5$, which was chosen arbitrarily. The iteration methods for solving the pressure equation, such as Jacobian iteration (or Richardson iteration), Liebman iteration (or Gauss-Seidel iteration), SOR, ADI (Alternate-Direction-Implicit) method, etc., all need, in general, a numerical experiment for the determination of the relaxation parameter. Direct methods do not need to establish any such parameter. More detail on iteration methods and direct methods can be found in refs. [3], [11]. Amsden and Harlow [15] provide a detailed discussion of the SOR method.

Calculation Procedure

The governing equations:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u \tag{2}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v \tag{3}$$

$$\nabla^2 p = s_p \tag{4}$$

where $s_p = - 2\rho \ [(\frac{\partial u}{\partial y}) (\frac{\partial v}{\partial x}) - (\frac{\partial u}{\partial x}) (\frac{\partial v}{\partial y})]$

were solved in the present study. These make a set of nonlinear partial differential equations in terms of u, v and p. However, if v and p are known, then equation (2) is a quasilinear elliptic equation in u if the value of u from the previous iteration is used in the convection term.

Similarly, if u and p are known, then equation (3) can be solved, while equation (4) is linear and can be solved if u and v are known.

The locally linear or quasilinear equations suggest that an iterative algorithm must be adopted with assumed initial values. The selection of proper initial values will help prompt convergence. For the present study, the initial values were all zero except for conditions on the boundary. This enables the convergence history to be watched. For a given convergence criterion a stable solution may be obtained provided that there is a unique solution for the problem as posed.

A discretization error is expected during the transformation from the continuum equations to the discrete system of finite-difference equations. An inherent round-off error in the computer is also inevitable. There may be an optimal grid size for the problem, because the decreased grid size reduces the discretization error but increases the round-off error. The optimum value may be found from computational experiments. Ames[16] provides more detail.

Besides the uniqueness and existence problem in the solution of the Navier-Stokes equation, there is also the problem of finding optimum parameters such as the grid size and relaxation parameter, etc. The convergence criteria for the present study is decreased $\mathrm{Max}_{i,j} | F^{n+1}_{i,j} - F^{n}_{i,j} |$ in a given outer iteration number. For the inner iteration, $\sum_{i,j} | F^{n+1}_{i,j} - F^{n}_{i,j} | < \varepsilon$ is the parameter used for convergence. Here, F represents the flow properties, such as u, v and p. For u and v, 10^{-3} is used for ε, and 10^{-5} for p. The convergence criteria was implemented in such a way that the calculation stopped to save computing time when $\mathrm{Max}_{i,j} | F^{n+1}_{i,j} - F^{n}_{i,j} |$ increased, or when the maximum iteration number was reached. The calculation procedure is shown by Figure 2.

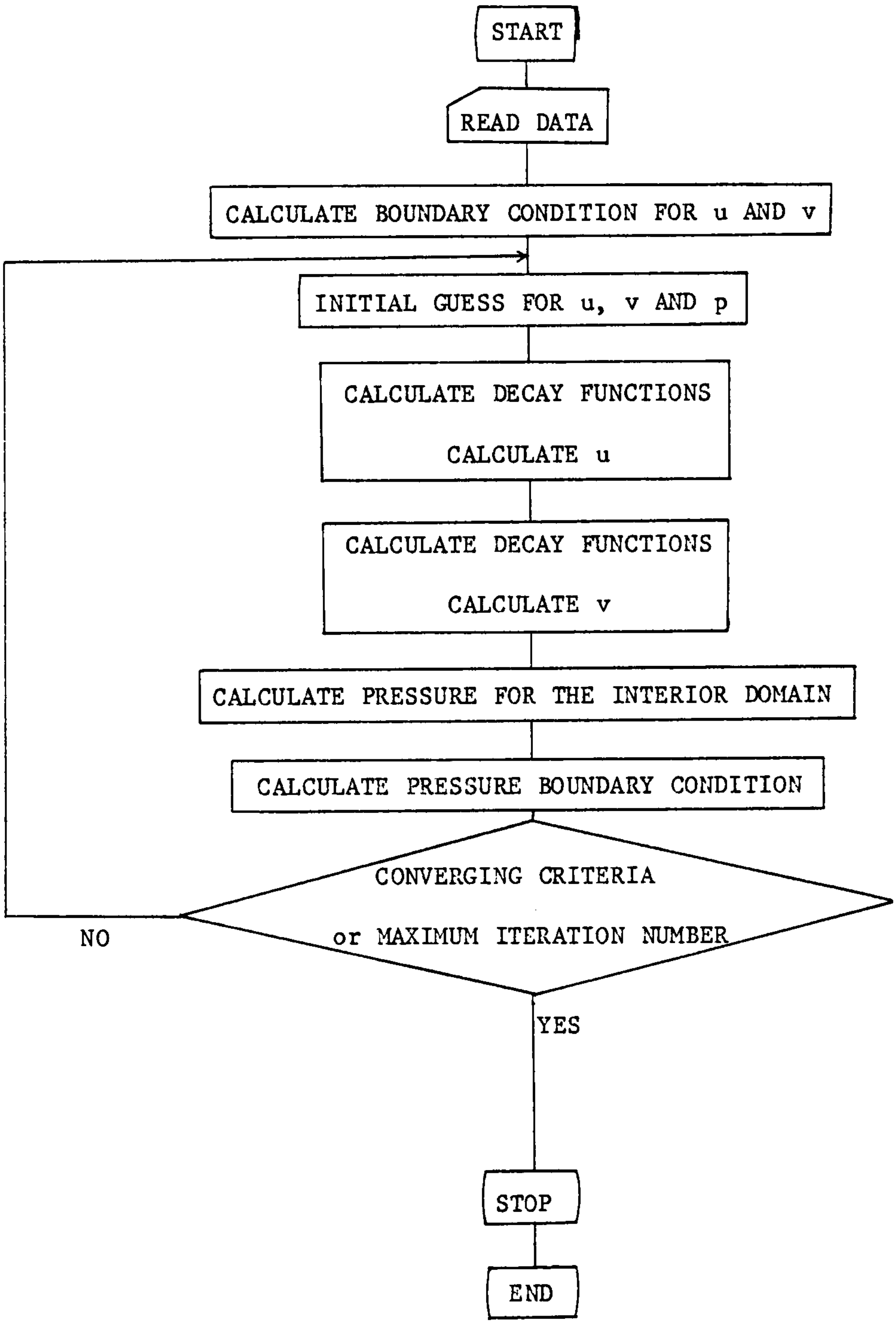

Figure 2. Flow chart of the problem.

CHAPTER III

RESULTS AND DISCUSSION

In this section the results of the cavity problem with Re=10 are compared with the published results. The results of the furnace study are also presented, analyzed and evaluated.

The flow in a cavity with a moving wall has been solved for Re=10. Here, the characteristic length was the width or depth (i.e., width/depth = 1), 1 ft, and the density was 0.00234 slugs/ft^3. The characteristic velocity was that of the moving wall, 1 ft/sec. The absolute viscosity was a fictitious value, taken to be: 0.000234 lbf-sec/ft^2. It is noted that air at 1 atmosphere pressure has the density of 0.00234 slugs/ft^3 and an absolute viscosity of 3.8×10^{-7} lbf-sec/ft^2, when the temperature is 68°F.[1]

The cavity problem was chosen as the test case because a successful calculation of this flow requires consideration of the full Navier-Stokes equation (rather than the boundary layer equation) and the flow field contains vortices. The value, Re=10, was selected to allow a coarse mesh spacing, i.e., 11 x 11 or 15 x 15. Comparison is made with the calculations of Hirsh[7]. The velocity vector field, the velocities through the grid point closest to the vortex center, the horizontal velocity at y=1/14 from the moving wall, and the wall

[1]These values were taken from the American Society of Civil Engineers, Manual 25, 1942.

pressures are plotted in Figures 3 through 8. The present results were obtained after only 20 iterations. The pressure map is produced as part of the solution.

The flow vector field shows that the vortex center for the present study is around 1/5 from the moving wall on Figure 4, while the calculation of Hirsh[7] gives a value around 4/15 from the wall, as on Figure 3. The wall pressure Re=10 (Figure 7) could not be compared with others since no result for wall pressure with Re=10 was found. The general behavior of the flow is similar to the higher Reynolds number cases obtained by other authors[4], [7]. Here, Cp' was defined as $\frac{p - p_{ref}}{1/2\ \rho\ u^2}$, and p_{ref} was the pressure of a reference point, u the velocity of the moving wall. Wall pressure for Re=100 (Figure 8) was compared, although the present result is not fully converged. The pressure near A began to show large differences-compared to Ghia's [4] 57 x 57 grid point case. The reason was thought to be due to improper convergence and the pressure boundary condition of A on Figure 8. Probably, a reduction in mesh size would improve the comparison. Most of the results produced by the present program are thought to be reasonable, because they have produced good comparison with other published results. Moreover, only 20 iterations were used and the pressure was obtained as part of the solution. Although the program of Greenspan[17] gives the converged solution with less iterations, his formulation was based upon the steady state vorticity-stream function equations. Time dependent procedures normally require larger

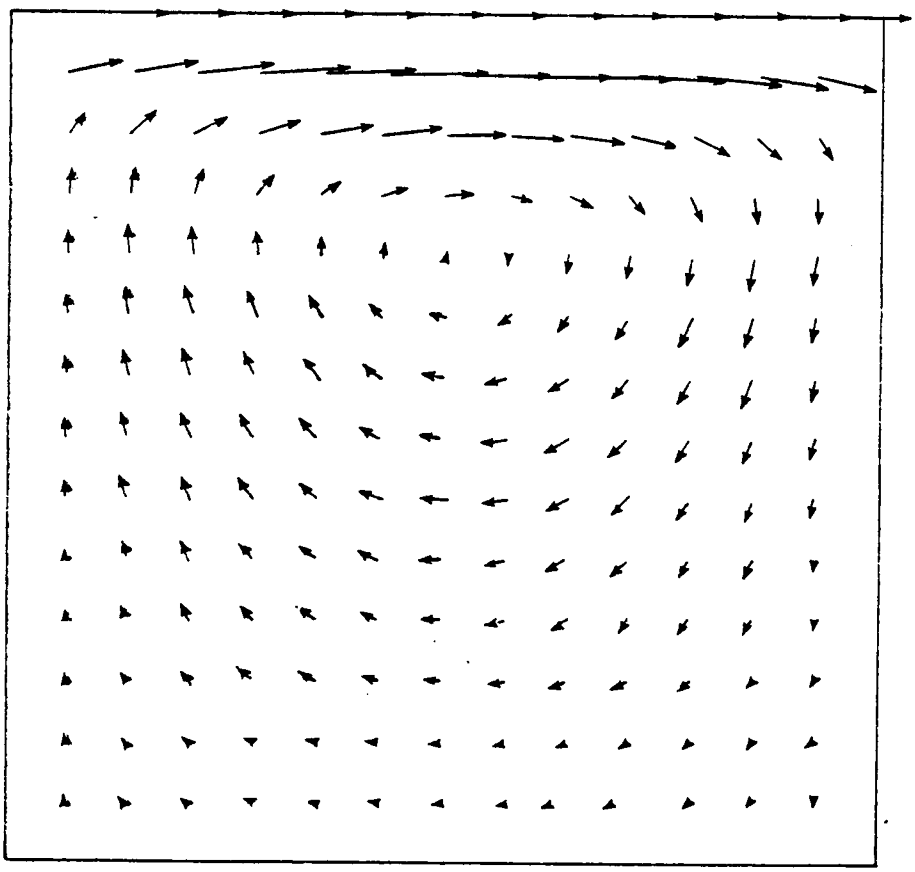

Figure 3. Plot of velocity vectors of the cavity problem Re=10; 15 x 15 grid by Hirsh.

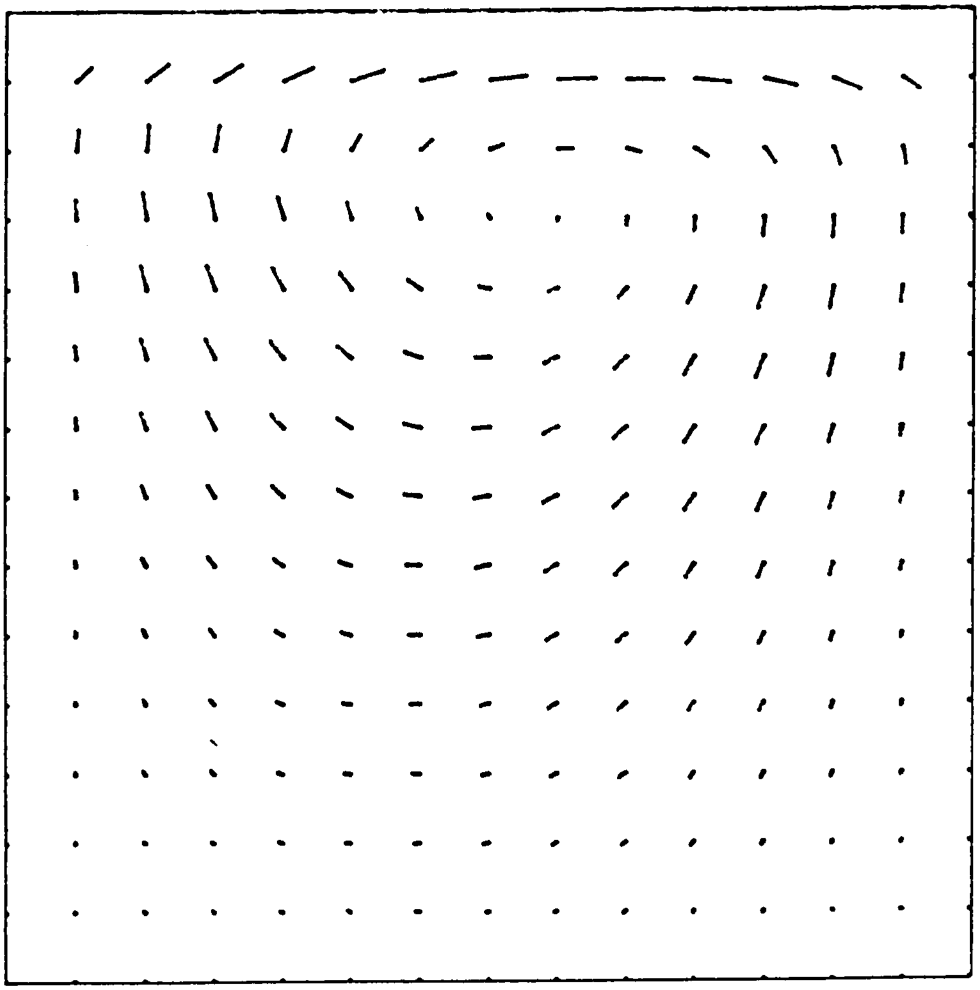

Figure 4. Plot of velocity vectors of the cavity problem Re=10; 15 x 15.

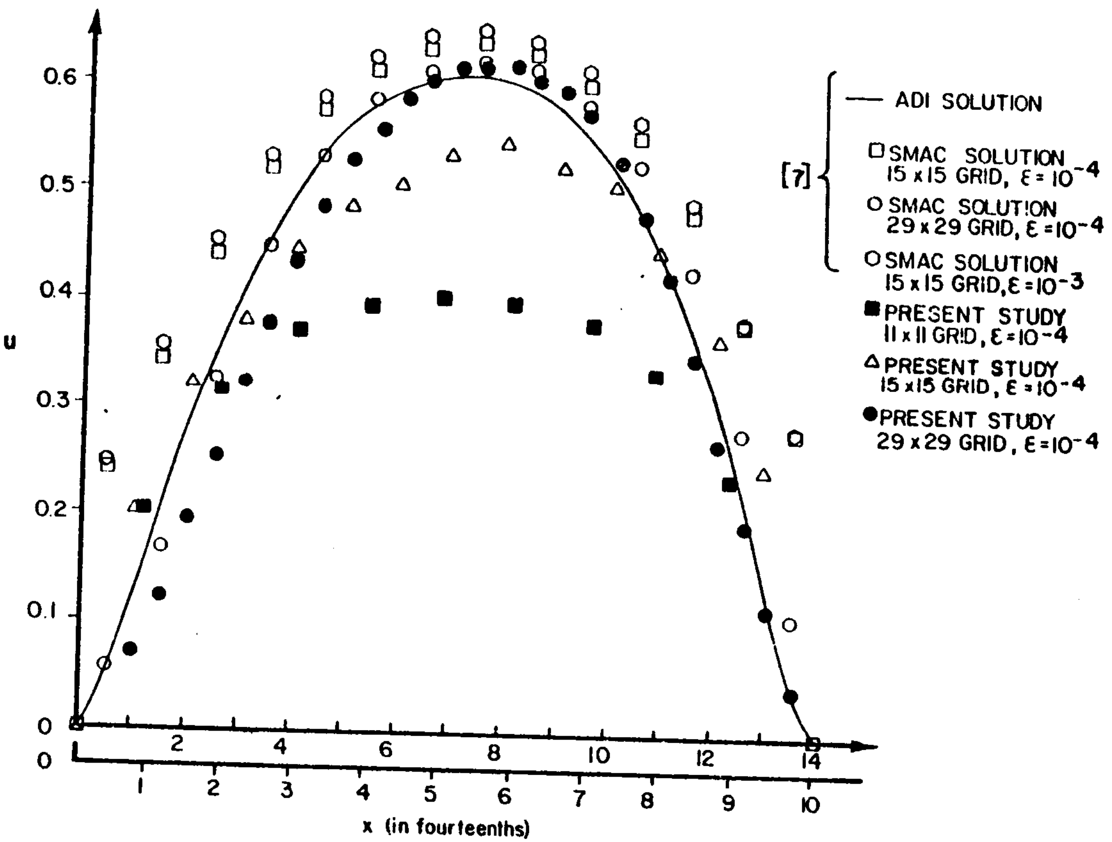

Figure 5. Plot of velocities through closest grid point to vortex center of the cavity problem.

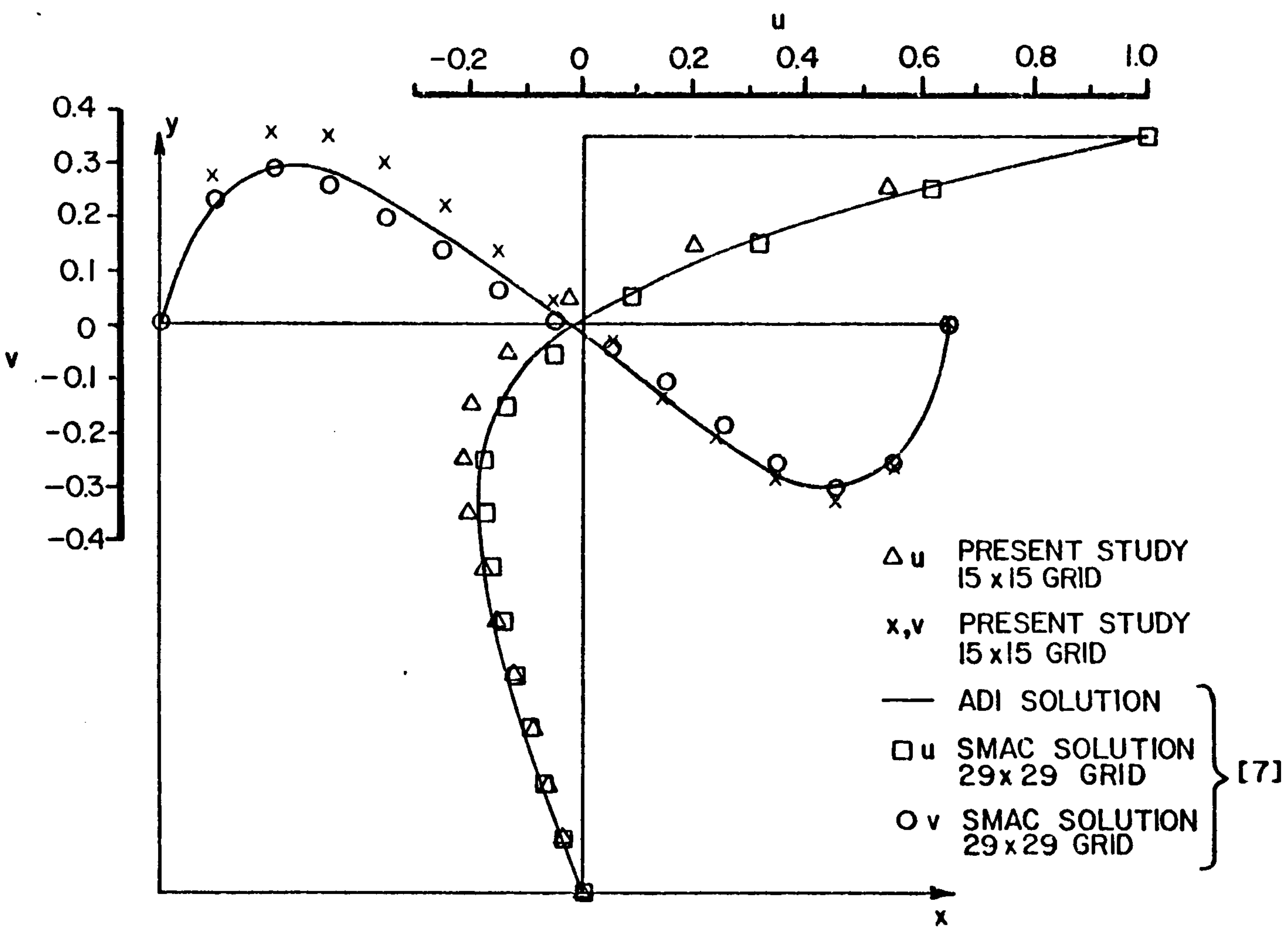

Figure 6. Plot of velocity u at y = 1/14 from moving wall.

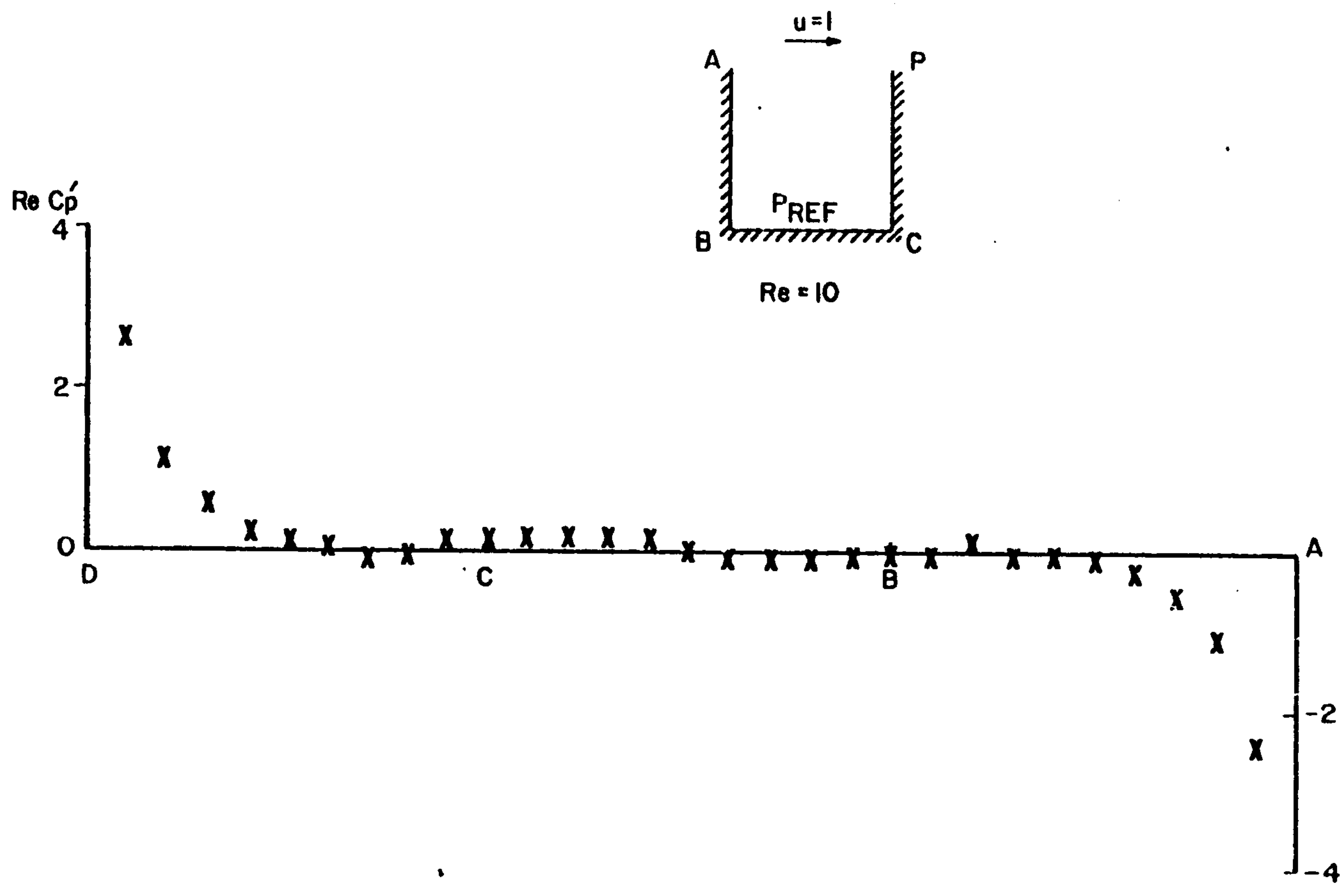

Figure 7. Plot of wall pressure of the cavity problem Re=10.

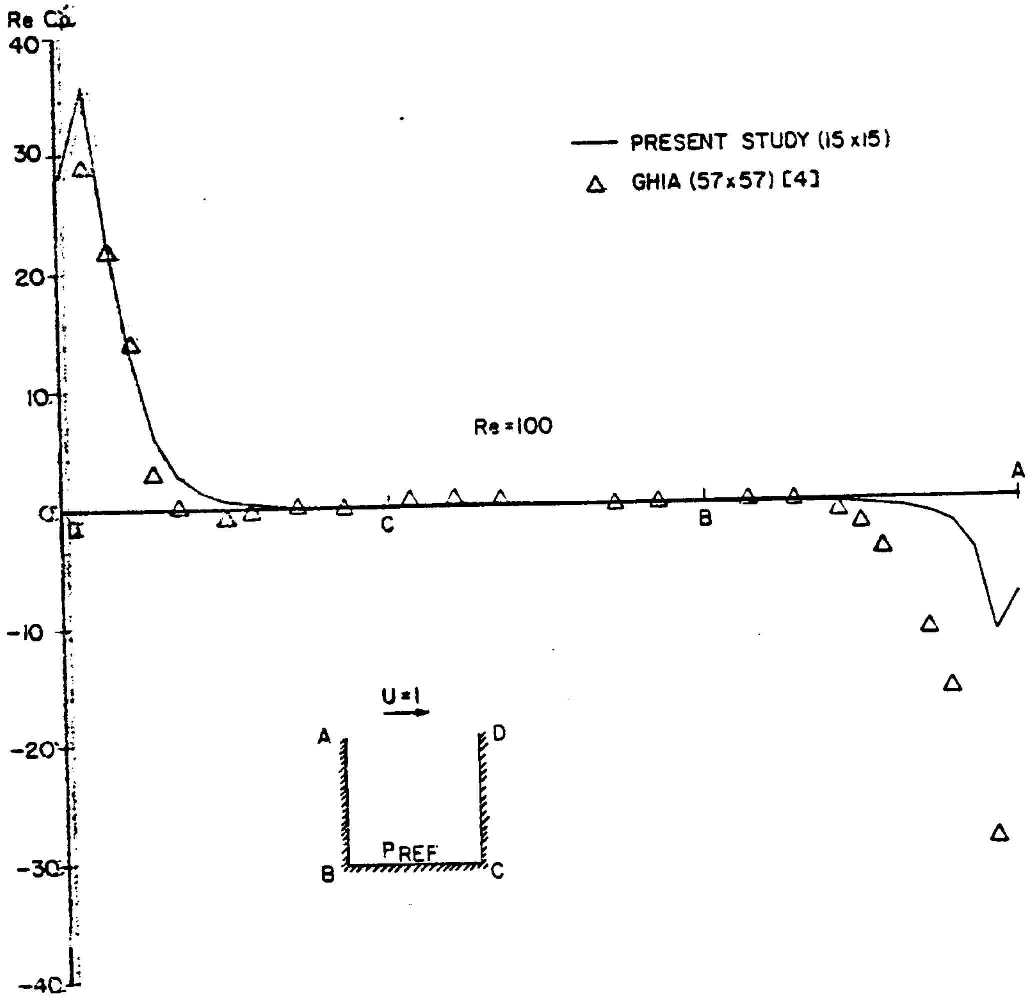

Figure 8. Plot of wall pressure of the cavity problem Re=100.

computing time (i.e., when the explicit method is used). Higher Reynolds number cases were tried but these failed to converge with the same 15 x 15 grid and this was thought to be mainly due to the high velocity gradients near the wall.

The furnace problem was also attacked for Re=10. Here, the characteristic length for defining the Reynolds number was the entry vertical length, 0.15 ft, and the fluid density was taken to be 0.00234 slugs/ft^3. The characteristic velocity was that of the inlet, 1 ft/sec. The absolute viscosity was also a fictitious value: 0.0000351 lbf-sec/ft^2. It is noted that the present method did not use non-dimensional equations and the ratio of the model to the prototype was 3/40 in two-dimensional geometry. The present study is confined to a cold-flow model, while the prototype of the radiant furnace has a temperature of more than 2000°F (i.e., the design temperature of the inlet is 3960°F). Due to its high temperature the radiant furnace is a radiation-dominant heat-exchanger. Better simulation of the radiant furnace would be obtained through including the energy equation and some radiation modeling. More detail for a radiant furnace can be found in ref. [18]. The geometry is shown on Figure 9.

The velocity map and convergence history during the calculation are plotted in Figures 10 and 11. To the author's knowledge, results for the same geometry have not been reported, and no comparison could be made. To evaluate the result in some way, the dilation, i.e., $D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, the fourth central-differences of velocities, i.e.,

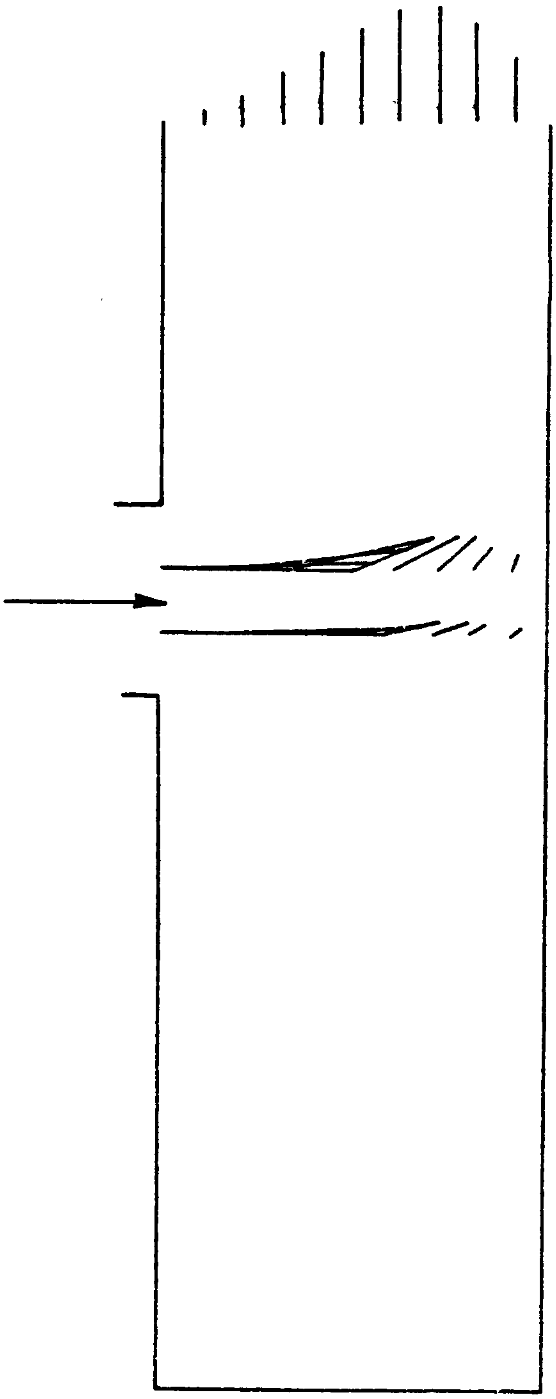

Figure 9. Geometry of furnace problem.

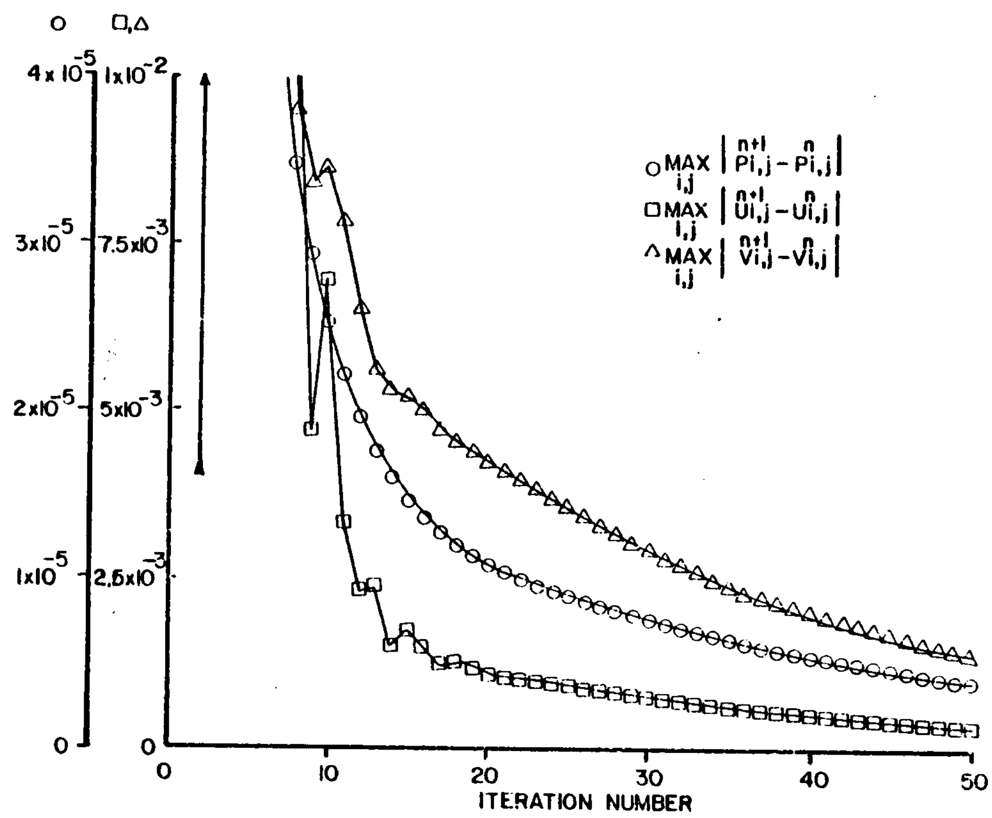

Figure 10. Plot of converging history of the furnace.

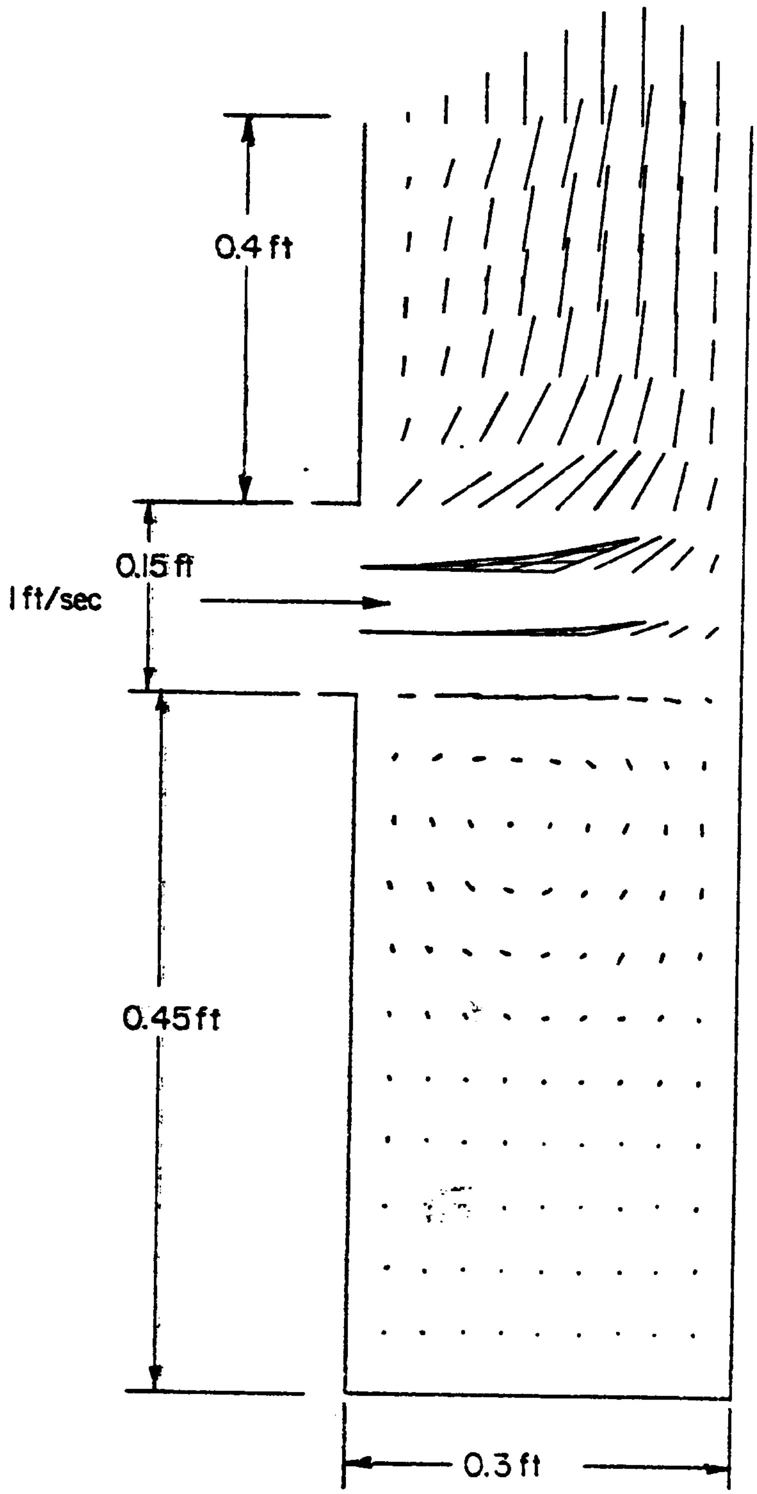

Figure 11. Plot of velocity vectors of the furnace.

$\delta^4 f_o$, and stream function values were evaluated. The results are tabulated in Table I through Table IV.

The dilation was set to zero when the source term, s_p, was written during the formulation of the pressure equation (4). As a result, the continuity concept is not included explicitly in the numerical discretization, and suggests that the nonconservative form of the equations has a shortcoming in that the continuity requirement is not explicit. The dilation term gives a source idea as to where the maximum error occurs in satisfying the continuity requirement. In the furnace study this area is near the upper inlet corner (i.e., I=2, J=15), because the coarse grid size did not produce recirculating flow above the main stream. If a recirculating flow near the wall occurs, the dilation is expected to have smaller values at the expense of more grid points.

Table IV also shows that the continuity error exists along the inlet stream because of the large u-gradients in the x-direction. The first approximation for the continuity is used for the former part of Table IV, and the second approximation is used for the latter part, i.e., $\frac{\partial f_o}{\partial x} = \frac{1}{12\Delta x}(- f_2 + 8f_1 - 8f_{-1} + f_{-2})$. The first and second approximation is explained in Appendix B. Appendix B is mainly based upon the work of Crawford[19] as well as Ames[11] and Hildebrand[20].

The fourth central-differences of velocity, i.e., $\delta^4 f_o = f_2 - 4f_1 + 6f_o - 4f_{-1} + f_{-2}$, are calculated to see how much the truncation

Table I. u-velocity, v-velocity and pressure.

U-VELOCITY RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I = 10	I = 11
J = 21	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J = 20	0.00000	0.01408	0.04087	0.06490	0.07896	0.08087	0.07172	0.05420	0.03164	0.01034	0.00000
J = 19	0.00000	0.00904	0.02644	0.04248	0.05234	0.05378	0.04693	0.03391	0.01848	0.00571	0.00000
J = 18	0.00000	0.00516	0.01626	0.02893	0.03808	0.04021	0.03441	0.02282	0.01026	0.00260	0.00000
J = 17	0.00000	0.00911	0.02834	0.04961	0.06405	0.06667	0.05666	0.03770	0.01736	0.00453	0.00000
J = 16	0.00000	0.03046	0.08594	0.13050	0.15260	0.14990	0.12500	0.08557	0.04352	0.01264	0.00000
J = 15	0.00000	0.10320	0.26240	0.32700	0.33370	0.30000	0.23820	0.16120	0.08451	0.02527	0.00000
J = 14	1.00000	0.94840	0.84010	0.72180	0.59890	0.47290	0.34600	0.22380	0.11570	0.03417	0.00000
J = 13	1.00000	0.92580	0.78350	0.65360	0.53010	0.41040	0.29510	0.18800	0.09592	0.02795	0.00000
J = 12	0.00000	0.07360	0.18430	0.22190	0.21730	0.18680	0.14180	0.09170	0.04583	0.01309	0.00000
J = 11	0.00000	0.01233	0.03440	0.05133	0.05706	0.05202	0.03939	0.02366	0.00978	0.00221	0.00000
J = 10	0.00000	-0.00160	-0.00363	-0.00332	-0.00310	-0.00401	-0.00564	-0.00678	-0.00597	-0.00244	0.00000
J = 9	0.00000	-0.00366	-0.01020	-0.01516	-0.01805	-0.01889	-0.01768	-0.01439	-0.00928	-0.00325	0.00000
J = 8	0.00000	-0.00290	-0.00841	-0.01338	-0.01658	-0.01748	-0.01602	-0.01247	-0.00756	-0.00254	0.00000
J = 7	0.00000	-0.00180	-0.00537	-0.00885	-0.01122	-0.01192	-0.01083	-0.00825	-0.00482	-0.00157	0.00000
J = 6	0.00000	-0.00099	-0.00299	-0.00503	-0.00645	-0.00686	-0.00617	-0.00458	-0.00257	-0.00081	0.00000
J = 5	0.00000	-0.00051	-0.00152	-0.00256	-0.00327	-0.00343	-0.00301	-0.00213	-0.00111	-0.00033	0.00000
J = 4	0.00000	-0.00026	-0.00076	-0.00121	-0.00147	-0.00146	-0.00118	-0.00072	-0.00028	-0.00005	0.00000
J = 3	0.00000	-0.00016	-0.00043	-0.00058	-0.00059	-0.00047	-0.00024	0.00000	0.00015	0.00009	0.00000
J = 2	0.00000	-0.00013	-0.00030	-0.00031	-0.00022	-0.00006	0.00013	0.00027	0.00030	0.00014	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

V-VELOCITY RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I = 10	I = 11
J = 21	0.00000	0.03636	0.12930	0.25450	0.38790	0.50510	0.58180	0.59390	0.51720	0.32730	0.00000
J = 20	0.00000	0.05330	0.14480	0.25860	0.37550	0.47560	0.53830	0.54310	0.46860	0.29390	0.00000
J = 19	0.00000	0.07810	0.16740	0.26330	0.35460	0.42800	0.46940	0.46320	0.39280	0.24220	0.00000
J = 18	0.00000	0.08478	0.17340	0.26120	0.34100	0.40330	0.43660	0.42690	0.35970	0.22090	0.00000
J = 17	0.00000	0.08544	0.17300	0.25520	0.32660	0.38060	0.40810	0.39740	0.33470	0.20660	0.00000
J = 16	0.00000	0.09473	0.17910	0.25030	0.30800	0.34940	0.36830	0.35530	0.29860	0.16510	0.00000
J = 15	0.00000	0.12620	0.18950	0.23440	0.26910	0.29270	0.30040	0.28470	0.23670	0.14580	0.00000
J = 14	0.00000	0.05566	0.10220	0.14040	0.16970	0.18820	0.19300	0.18060	0.14700	0.08790	0.00000
J = 13	0.00000	-0.00252	0.01717	0.04181	0.06276	0.07576	0.07886	0.07141	0.05376	0.02800	0.00000
J = 12	0.00000	-0.00077	0.00385	0.00876	0.01136	0.01039	0.00572	-0.00163	-0.00916	-0.01187	0.00000
J = 11	0.00000	0.02194	0.02143	0.01327	0.00210	-0.01020	-0.02204	-0.03125	-0.03456	-0.02690	0.00000
J = 10	0.00000	0.02283	0.02498	0.01633	0.00270	-0.01226	-0.02567	-0.03471	-0.03618	-0.02623	0.00000
J = 9	0.00000	0.01609	0.01907	0.01320	0.00244	-0.00989	-0.02089	-0.02788	-0.02828	-0.01967	0.00000
J = 8	0.00000	0.00915	0.01134	0.00788	0.00085	-0.00750	-0.01493	-0.01939	-0.01911	-0.01282	0.00000
J = 7	0.00000	0.00420	0.00526	0.00328	-0.00084	-0.00578	-0.01012	-0.01250	-0.01188	-0.00766	0.00000
J = 6	0.00000	0.00132	0.00152	0.00034	-0.00189	-0.00452	-0.00674	-0.00777	-0.00704	-0.00435	0.00000
J = 5	0.00000	-0.00007	-0.00033	-0.00106	-0.00220	-0.00347	-0.00445	-0.00475	-0.00409	-0.00242	0.00000
J = 4	0.00000	-0.00054	-0.00092	-0.00137	-0.00192	-0.00247	-0.00282	-0.00281	-0.00231	-0.00132	0.00000
J = 3	0.00000	-0.00051	-0.00081	-0.00104	-0.00126	-0.00145	-0.00155	-0.00148	-0.00119	-0.00068	0.00000
J = 2	0.00000	-0.00022	-0.00033	-0.00040	-0.00045	-0.00048	-0.00048	-0.00045	-0.00035	-0.00020	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table I. (continued)

PRESSURE X 10**3 RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	-0.35260	-0.26460	-0.34380	-0.44390	-0.55850	-0.66990	-0.77010	-0.85740	-0.92800	-0.96330	-0.80770
J =20	-0.44870	-0.41060	-0.42400	-0.46360	-0.51460	-0.56590	-0.61120	-0.64740	-0.67100	-0.67380	-0.64010
J =19	-0.44260	-0.42920	-0.42820	-0.43640	-0.45050	-0.46630	-0.48050	-0.49090	-0.49520	-0.49060	-0.46120
J =18	-0.44210	-0.42880	-0.41750	-0.40620	-0.39420	-0.38100	-0.36700	-0.35330	-0.34060	-0.32780	-0.31840
J =17	-0.45890	-0.43360	-0.40710	-0.37520	-0.33760	-0.29600	-0.25430	-0.21720	-0.18780	-0.16530	-0.14970
J =16	-0.51110	-0.44490	-0.38930	-0.32930	-0.26330	-0.19470	-0.12970	-0.07467	-0.03241	0.00037	0.02950
J =15	-0.53110	-0.41190	-0.31600	-0.22770	-0.14200	-0.06094	0.01071	0.06860	0.11300	0.15170	0.19710
J =14	0.31020	0.03515	-0.02896	-0.01172	0.03495	0.08858	0.13830	0.17970	0.21450	0.25180	0.30610
J =13	0.00000	0.01451	0.04197	0.07478	0.11060	0.14660	0.18050	0.21060	0.23910	0.27230	0.31910
J =12	-0.24660	-0.09526	-0.01094	0.04319	0.08493	0.12100	0.15310	0.18170	0.20780	0.23380	0.26200
J =11	-0.07395	-0.03443	0.00482	0.03891	0.06930	0.09721	0.12280	0.14540	0.16470	0.18020	0.19080
J =10	0.01672	0.02264	0.03638	0.05301	0.07060	0.08812	0.10460	0.11900	0.13020	0.13690	0.13730
J = 9	0.05972	0.05707	0.06041	0.06734	0.07618	0.08571	0.09484	0.10250	0.10760	0.10920	0.10590
J = 8	0.07800	0.07433	0.07417	0.07653	0.08042	0.08497	0.08936	0.09280	0.09456	0.09399	0.09051
J = 7	0.08435	0.08160	0.08066	0.08115	0.08257	0.08444	0.08623	0.08746	0.08771	0.08663	0.08410
J = 6	0.08556	0.08387	0.08308	0.08296	0.08332	0.08391	0.08446	0.08470	0.08442	0.08347	0.08197
J = 5	0.08494	0.08402	0.08356	0.08335	0.08335	0.08343	0.08347	0.08334	0.08296	0.08227	0.08154
J = 4	0.08395	0.08348	0.08331	0.08319	0.08312	0.08304	0.08292	0.08269	0.08234	0.08187	0.08160
J = 3	0.08310	0.08286	0.08291	0.08293	0.08290	0.08281	0.08264	0.08239	0.08207	0.08170	0.08168
J = 2	0.08239	0.08235	0.08262	0.08279	0.08285	0.08279	0.08261	0.08231	0.08193	0.08153	0.08160
J = 1	0.08198	0.08240	0.08290	0.08323	0.08337	0.08334	0.08313	0.08273	0.08217	0.08146	0.08111

Table II. Stream function values (S1, S2, and S3).

X-Y S1 RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	0.10000	0.10596	0.11345	0.11208	0.10311	0.08745	0.06673	0.04368	0.02209	0.00639	0.00000
J =20	0.10000	0.10561	0.11243	0.11046	0.10113	0.08543	0.06494	0.04232	0.02130	0.00613	0.00000
J =19	0.10000	0.10503	0.11074	0.10776	0.09785	0.08206	0.06197	0.04012	0.02005	0.00573	0.00000
J =18	0.10000	0.10467	0.10967	0.10599	0.09559	0.07971	0.05994	0.03870	0.01933	0.00552	0.00000
J =17	0.10000	0.10432	0.10856	0.10403	0.09304	0.07704	0.05766	0.03719	0.01864	0.00534	0.00000
J =16	0.10000	0.10333	0.10570	0.09952	0.08762	0.07162	0.05312	0.03411	0.01712	0.00491	0.00000
J =15	0.10000	0.09999	0.09699	0.08809	0.07546	0.06038	0.04404	0.02794	0.01392	0.00396	0.00000
J =14	0.07500	0.07370	0.06943	0.06187	0.05215	0.04105	0.02943	0.01831	0.00891	0.00248	0.00000
J =13	0.02500	0.02684	0.02884	0.02748	0.02392	0.01897	0.01340	0.00802	0.00362	0.00092	0.00000
J =12	0.00000	0.00186	0.00465	0.00559	0.00524	0.00404	0.00248	0.00102	0.00008	-0.00010	0.00000
J =11	0.00000	-0.00029	-0.00082	-0.00124	-0.00152	-0.00193	-0.00205	-0.00186	-0.00131	-0.00045	0.00000
J =10	0.00000	-0.00056	-0.00159	-0.00244	-0.00297	-0.00313	-0.00289	-0.00228	-0.00141	-0.00048	0.00000
J = 9	0.00000	-0.00043	-0.00124	-0.00198	-0.00244	-0.00256	-0.00231	-0.00175	-0.00103	-0.00034	0.00000
J = 8	0.00000	-0.00026	-0.00078	-0.00126	-0.00158	-0.00165	-0.00147	-0.00108	-0.00060	-0.00019	0.00000
J = 7	0.00000	-0.00015	-0.00043	-0.00071	-0.00088	-0.00091	-0.00079	-0.00056	-0.00030	-0.00009	0.00000
J = 6	0.00000	-0.00008	-0.00023	-0.00036	-0.00044	-0.00044	-0.00037	-0.00024	-0.00011	-0.00003	0.00000
J = 5	0.00000	-0.00004	-0.00011	-0.00017	-0.00020	-0.00019	-0.00014	-0.00008	-0.00002	0.00000	0.00000
J = 4	0.00000	-0.00002	-0.00006	-0.00007	-0.00008	-0.00006	-0.00004	0.00000	0.00002	0.00001	0.00000
J = 3	0.00000	-0.00001	-0.00003	-0.00003	-0.00003	-0.00001	0.00000	0.00001	0.00002	0.00001	0.00000
J = 2	0.00000	0.00000	-0.00001	-0.00001	-0.00001	0.00000	0.00000	0.00001	0.00001	0.00000	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Y-X S2 RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	0.10000	0.09945	0.09697	0.09121	0.08156	0.06818	0.05188	0.03424	0.01758	0.00491	0.00000
J =20	0.10000	0.09920	0.09623	0.09018	0.08067	0.06790	0.05269	0.03647	0.02130	0.00986	0.00545
J =19	0.10000	0.09883	0.09515	0.08869	0.07942	0.06768	0.05422	0.04023	0.02739	0.01785	0.01423
J =18	0.10000	0.09873	0.09486	0.08834	0.07930	0.06814	0.05554	0.04259	0.03079	0.02208	0.01877
J =17	0.10000	0.09872	0.09484	0.08842	0.07969	0.06908	0.05725	0.04517	0.03419	0.02607	0.02297
J =16	0.10000	0.09858	0.09447	0.08803	0.07966	0.06980	0.05903	0.04818	0.03837	0.03111	0.02834
J =15	0.10000	0.09811	0.09337	0.08701	0.07946	0.07103	0.06214	0.05336	0.04554	0.03980	0.03761
J =14	0.07500	0.07417	0.07180	0.06816	0.06351	0.05814	0.05242	0.04682	0.04190	0.03838	0.03706
J =13	0.02500	0.02504	0.02482	0.02393	0.02236	0.02029	0.01797	0.01571	0.01384	0.01261	0.01219
J =12	0.00000	0.00001	-0.00003	-0.00022	-0.00053	-0.00085	-0.00109	-0.00115	-0.00099	-0.00068	-0.00050
J =11	0.00000	-0.00033	-0.00098	-0.00150	-0.00173	-0.00161	-0.00113	-0.00033	0.00066	0.00158	0.00199
J =10	0.00000	-0.00034	-0.00106	-0.00168	-0.00196	-0.00182	-0.00125	-0.00035	0.00072	0.00165	0.00205
J = 9	0.00000	-0.00024	-0.00077	-0.00125	-0.00149	-0.00136	-0.00091	-0.00018	0.00066	0.00138	0.00167
J = 8	0.00000	-0.00014	-0.00044	-0.00073	-0.00086	-0.00076	-0.00043	0.00009	0.00060	0.00114	0.00134
J = 7	0.00000	-0.00006	-0.00021	-0.00033	-0.00037	-0.00027	-0.00003	0.00031	0.00067	0.00097	0.00108
J = 6	0.00000	-0.00002	-0.00006	-0.00009	-0.00007	0.00003	0.00020	0.00042	0.00064	0.00081	0.00087
J = 5	0.00000	0.00000	0.00001	0.00003	0.00008	0.00016	0.00028	0.00042	0.00055	0.00065	0.00068
J = 4	0.00000	0.00001	0.00003	0.00006	0.00011	0.00018	0.00026	0.00034	0.00042	0.00048	0.00050
J = 3	0.00000	0.00001	0.00003	0.00006	0.00009	0.00013	0.00018	0.00022	0.00026	0.00029	0.00030
J = 2	0.00000	0.00000	0.00001	0.00002	0.00003	0.00005	0.00006	0.00008	0.00009	0.00010	0.00010
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table II. (Continued)

S-V 53   RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	0.10000	0.09945	0.09697	0.09121	0.08158	0.06818	0.05188	0.03424	0.01758	0.00491	0.00000
J =20	0.10000	0.09890	0.09562	0.08923	0.07931	0.06606	0.05027	0.03338	0.01746	0.00524	0.00000
J =19	0.10000	0.09813	0.09370	0.08636	0.07599	0.06286	0.04773	0.03185	0.01703	0.00553	0.00000
J =18	0.10000	0.09773	0.09280	0.08505	0.07450	0.06150	0.04675	0.03140	0.01709	0.00583	0.00000
J =17	0.10000	0.09751	0.09235	0.08436	0.07366	0.06072	0.04621	0.03123	0.01724	0.00609	0.00000
J =16	0.10000	0.09667	0.09099	0.08228	0.07114	0.05817	0.04408	0.02981	0.01661	0.00602	0.00000
J =15	0.10000	0.09411	0.08523	0.07465	0.06291	0.05042	0.03768	0.02531	0.01413	0.00522	0.00000
J =14	0.07500	0.06863	0.06091	0.05223	0.04320	0.03407	0.02512	0.01672	0.00932	0.00351	0.00000
J =13	0.02500	0.02364	0.02216	0.01990	0.01694	0.01353	0.01000	0.00664	0.00372	0.00147	0.00000
J =12	0.00000	-0.00001	0.00016	0.00021	0.00009	-0.00010	-0.00026	-0.00029	-0.00017	0.00003	0.00000
J =11	0.00000	-0.00145	-0.00312	-0.00425	-0.00472	-0.00457	-0.00392	-0.00288	-0.00165	-0.00054	0.00000
J =10	0.00000	-0.00125	-0.00280	-0.00403	-0.00468	-0.00465	-0.00403	-0.00296	-0.00170	-0.00057	0.00000
J = 9	0.00000	-0.00084	-0.00151	-0.00263	-0.00337	-0.00340	-0.00297	-0.00218	-0.00123	-0.00040	0.00000
J = 8	0.00000	-0.00051	-0.00117	-0.00174	-0.00209	-0.00213	-0.00185	-0.00134	-0.00074	-0.00023	0.00000
J = 7	0.00000	-0.00030	-0.00067	-0.00099	-0.00118	-0.00119	-0.00103	-0.00073	-0.00039	-0.00011	0.00000
J = 6	0.00000	-0.00018	-0.00037	-0.00053	-0.00062	-0.00062	-0.00052	-0.00035	-0.00017	-0.00003	0.00000
J = 5	0.00000	-0.00011	-0.00021	-0.00028	-0.00031	-0.00030	-0.00024	-0.00015	-0.00006	0.00000	0.00000
J = 4	0.00000	-0.00006	-0.00011	-0.00014	-0.00015	-0.00013	-0.00010	-0.00005	-0.00001	0.00001	0.00000
J = 3	0.00000	-0.00004	-0.00006	-0.00007	-0.00007	-0.00006	-0.00004	-0.00002	0.00000	0.00001	0.00000
J = 2	0.00000	-0.00001	-0.00002	-0.00002	-0.00002	-0.00002	-0.00001	-0.00001	0.00000	0.00000	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table III. Fourth central-differences ($\delta_x^4 u$, $\delta_y^4 u$, $\delta_x^4 v$, and $\delta_y^4 v$).

DX4-U RE = 10.00 ITERATION = 50

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J =20	0.00000	0.00000	0.00826	0.00503	0.00327	0.00160	0.00064	0.00297	0.00340	0.00000	0.00000
J =19	0.00000	0.00000	0.00490	0.00258	0.00237	0.00199	0.00164	0.00131	-0.00068	0.00000	0.00000
J =18	0.00000	0.00000	-0.00072	0.00159	0.00259	0.00305	0.00268	0.00105	-0.00573	0.00000	0.00000
J =17	0.00000	0.00000	-0.00078	0.00388	0.00418	0.00449	0.00389	0.00132	-0.00811	0.00000	0.00000
J =16	0.00000	0.00000	0.02440	0.00920	0.00494	0.00507	0.00424	0.00188	-0.00672	0.00000	0.00000
J =15	0.00000	0.00000	0.18730	-0.01920	-0.00520	0.00060	0.00261	0.00163	-0.00062	0.00000	0.00000
J =14	0.00000	0.00000	-0.04130	-0.00390	0.00070	0.00340	0.00380	0.00307	0.00832	0.00000	0.00000
J =13	0.00000	0.00000	-0.08650	0.00340	0.00320	0.00320	0.00302	0.00227	0.00682	0.00000	0.00000
J =12	0.00000	0.00000	0.14110	-0.01460	-0.00490	-0.00200	-0.00007	-0.00043	-0.00238	0.00000	0.00000
J =11	0.00000	0.00000	0.00882	0.00649	0.00275	0.00131	0.00046	-0.00049	-0.00541	0.00000	0.00000
J =10	0.00000	0.00000	-0.00523	0.00144	0.00139	0.00086	0.00019	-0.00065	-0.00459	0.00000	0.00000
J = 9	0.00000	0.00000	-0.00397	-0.00051	0.00002	0.00003	-0.00029	-0.00064	-0.00280	0.00000	0.00000
J = 8	0.00000	0.00000	-0.00196	-0.00068	-0.00047	-0.00033	-0.00046	-0.00053	-0.00134	0.00000	0.00000
J = 7	0.00000	0.00000	-0.00080	-0.00049	-0.00043	-0.00042	-0.00035	-0.00039	-0.00045	0.00000	0.00000
J = 6	0.00000	0.00000	-0.00030	-0.00027	-0.00030	-0.00030	-0.00026	-0.00021	-0.00001	0.00000	0.00000
J = 5	0.00000	0.00000	-0.00015	-0.00013	-0.00018	-0.00018	-0.00015	-0.00010	0.00019	0.00000	0.00000
J = 4	0.00000	0.00000	-0.00015	-0.00006	-0.00009	-0.00010	-0.00008	-0.00004	0.00027	0.00000	0.00000
J = 3	0.00000	0.00000	-0.00020	-0.00002	-0.00004	-0.00004	-0.00003	-0.00001	0.00032	0.00000	0.00000
J = 2	0.00000	0.00000	-0.00026	0.00001	-0.00001	-0.00002	-0.00002	-0.00001	0.00032	0.00000	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

DY4-U RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J =20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J =19	0.00000	-0.01361	-0.04154	-0.07083	-0.09007	-0.09497	-0.08628	-0.06692	-0.03936	-0.01297	0.00000
J =18	0.00000	0.00291	0.00525	0.00062	-0.00552	-0.00977	-0.01118	-0.00975	-0.00664	-0.00238	0.00000
J =17	0.00000	0.02440	0.05008	0.02942	0.00762	-0.00664	-0.01255	-0.01225	-0.00797	-0.00280	0.00000
J =16	1.00000	0.68709	0.20904	0.02729	-0.03842	-0.05417	-0.04903	-0.03556	-0.02040	-0.00659	0.00000
J =15	-3.00000	-2.30133	-1.31792	-0.74399	-0.40965	-0.21413	-0.10304	-0.04458	-0.01654	-0.00314	0.00000
J =14	2.00000	1.67846	1.12724	0.76080	0.50810	0.33250	0.20960	0.12327	0.06183	0.01787	0.00000
J =13	2.00000	1.56233	0.90020	0.52513	0.30656	0.17562	0.09699	0.05086	0.02369	0.00614	0.00000
J =12	-3.00000	-2.36412	-1.32933	-0.76984	-0.44904	-0.25999	-0.14681	-0.07941	-0.03808	-0.01037	0.00000
J =11	1.00000	0.70813	0.25703	0.07208	-0.00237	-0.02754	-0.03086	-0.02413	-0.01413	-0.00464	0.00000
J =10	0.00000	0.02642	0.05729	0.04394	0.02605	0.01275	0.00507	0.00150	0.00045	0.00005	0.00000
J = 9	0.00000	0.00655	0.01602	0.01830	0.01628	0.01271	0.00914	0.00605	0.00340	0.00106	0.00000
J = 8	0.00000	0.00189	0.00516	0.00742	0.00805	0.00749	0.00611	0.00437	0.00249	0.00079	0.00000
J = 7	0.00000	0.00056	0.00168	0.00279	0.00347	0.00352	0.00309	0.00221	0.00123	0.00039	0.00000
J = 6	0.00000	0.00014	0.00047	0.00090	0.00121	0.00131	0.00114	0.00084	0.00044	0.00013	0.00000
J = 5	0.00000	0.00001	0.00005	0.00016	0.00026	0.00030	0.00027	0.00017	0.00007	0.00002	0.00000
J = 4	0.00000	-0.00001	-0.00003	-0.00005	-0.00006	-0.00008	-0.00010	-0.00012	-0.00010	-0.00004	0.00000
J = 3	0.00000	0.00006	0.00012	0.00004	-0.00007	-0.00018	-0.00027	-0.00032	-0.00030	-0.00013	0.00000
J = 2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table III. (Continued)

DX4-V RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	0.00000	0.00000	0.00026	-0.00034	0.00010	0.00020	-0.00010	-0.00020	0.00020	0.00000	0.00000
J =20	0.00000	0.00000	-0.00330	-0.00070	-0.00070	0.00010	-0.00090	0.00050	0.00190	0.00000	0.00000
J =19	0.00000	0.00000	-0.00660	-0.00210	-0.00080	-0.00150	-0.00100	0.00060	0.00460	0.00000	0.00000
J =18	0.00000	0.00000	-0.00252	-0.00232	-0.00200	-0.00250	-0.00050	0.00040	0.00360	0.00000	0.00000
J =17	0.00000	0.00000	0.00204	-0.00116	-0.00250	-0.00260	-0.00210	0.00040	0.00030	0.00000	0.00000
J =16	0.00000	0.00000	0.00248	-0.00247	-0.00340	-0.00320	-0.00240	-0.00130	-0.00170	0.00000	0.00000
J =15	0.00000	0.00000	-0.03630	-0.00910	-0.00390	-0.00270	-0.00140	-0.00170	-0.00140	0.00000	0.00000
J =14	0.00000	0.00000	-0.00134	-0.00134	-0.00100	-0.00060	-0.00050	-0.00030	0.00100	0.00000	0.00000
J =13	0.00000	0.00000	0.00864	0.00438	0.00231	0.00130	0.00100	0.00174	0.00378	0.00000	0.00000
J =12	0.00000	0.00000	0.00250	0.00134	0.00114	0.00114	0.00149	0.00250	0.00476	0.00000	0.00000
J =11	0.00000	0.00000	-0.01016	-0.00275	-0.00030	0.00059	0.00110	0.00180	0.00320	0.00000	0.00000
J =10	0.00000	0.00000	-0.00406	-0.00216	-0.00078	-0.00005	0.00038	0.00065	0.00101	0.00000	0.00000
J = 9	0.00000	0.00000	-0.00030	-0.00066	-0.00040	-0.00023	-0.00010	-0.00016	-0.00037	0.00000	0.00000
J = 8	0.00000	0.00000	0.00078	0.00019	-0.00004	-0.00018	-0.00028	-0.00050	-0.00075	0.00000	0.00000
J = 7	0.00000	0.00000	0.00082	0.00040	0.00012	-0.00007	-0.00031	-0.00044	-0.00075	0.00000	0.00000
J = 6	0.00000	0.00000	0.00057	0.00035	0.00014	-0.00003	-0.00021	-0.00036	-0.00051	0.00000	0.00000
J = 5	0.00000	0.00000	0.00034	0.00023	0.00012	-0.00001	-0.00012	-0.00022	-0.00030	0.00000	0.00000
J = 4	0.00000	0.00000	0.00018	0.00014	0.00008	0.00000	-0.00006	-0.00012	-0.00016	0.00000	0.00000
J = 3	0.00000	0.00000	0.00009	0.00008	0.00004	0.00001	-0.00002	-0.00005	-0.00006	0.00000	0.00000
J = 2	0.00000	0.00000	0.00003	0.00002	0.00001	0.00001	0.00000	-0.00001	-0.00001	0.00000	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

DY4-V RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J =20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J =19	0.00000	0.03808	0.03390	0.01030	-0.02390	-0.06190	-0.09330	-0.10950	-0.10450	-0.07210	0.00000
J =18	0.00000	0.00255	0.00270	0.00210	0.00470	0.01040	0.01620	0.01740	0.01540	0.00920	0.00000
J =17	0.00000	-0.00110	-0.01510	-0.01710	-0.01270	-0.00650	-0.00120	0.00350	0.00450	0.00360	0.00000
J =16	0.00000	-0.13774	-0.09980	-0.05500	-0.02410	-0.00530	0.00540	0.01090	0.01270	0.00980	0.00000
J =15	0.00000	0.23856	0.20197	0.14061	0.09316	0.06216	0.04416	0.03341	0.02626	0.01740	0.00000
J =14	0.00000	-0.06679	-0.03053	-0.00338	0.01012	0.01515	0.01498	0.01283	0.00960	0.00543	0.00000
J =13	0.00000	-0.06656	-0.11024	-0.09810	-0.07648	-0.05730	-0.04336	-0.03397	-0.02606	-0.01722	0.00000
J =12	0.00000	-0.00380	-0.00414	-0.01104	-0.01867	-0.02396	-0.02562	-0.02453	-0.02094	-0.01395	0.00000
J =11	0.00000	0.05697	0.04951	0.03428	0.02156	0.01215	0.00553	0.00139	-0.00052	-0.00067	0.00000
J =10	0.00000	-0.00677	0.00307	0.00874	0.01023	0.00969	0.00849	0.00724	0.00601	0.00421	0.00000
J = 9	0.00000	-0.00522	-0.00416	-0.00110	0.00175	0.00373	0.00490	0.00537	0.00504	0.00362	0.00000
J = 8	0.00000	-0.00215	-0.00280	-0.00196	-0.00055	0.00089	0.00205	0.00270	0.00276	0.00182	0.00000
J = 7	0.00000	-0.00062	-0.00110	-0.00107	-0.00059	0.00006	0.00062	0.00101	0.00097	0.00064	0.00000
J = 6	0.00000	-0.00002	-0.00022	-0.00032	-0.00027	-0.00011	0.00008	0.00016	0.00017	0.00005	0.00000
J = 5	0.00000	0.00016	0.00012	0.00001	-0.00006	-0.00009	-0.00011	-0.00012	-0.00015	-0.00014	0.00000
J = 4	0.00000	0.00020	0.00018	0.00009	-0.00002	-0.00011	-0.00016	-0.00018	-0.00017	-0.00011	0.00000
J = 3	0.00000	-0.00013	-0.00016	-0.00020	-0.00029	-0.00041	-0.00053	-0.00061	-0.00058	-0.00041	0.00000
J = 2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE IV. Dilation.

DILATION RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J =20	0.00000	0.26377	0.46600	0.54683	0.59917	0.65033	0.67950	0.63900	0.51300	0.32367	0.00000
J =19	0.00000	0.12567	0.27132	0.40567	0.53333	0.63283	0.68583	0.68783	0.61905	0.42200	0.00000
J =18	0.00000	0.19760	0.34017	0.44467	0.46800	0.41283	0.32317	0.25550	0.24407	0.18500	0.00000
J =17	0.00000	0.37283	0.61805	0.70417	0.61433	0.41583	0.20017	0.06100	0.05820	0.06867	0.00000
J =16	0.00000	1.02473	1.50233	1.31900	0.89833	0.41900	0.00483	-0.23100	-0.23550	-0.11733	0.00000
J =15	0.00000	4.76403	4.49900	2.28733	0.93300	0.02033	-0.56033	-0.81450	-0.74950	-0.43650	0.00000
J =14	0.00000	-1.37776	-2.05337	-2.09410	-2.08493	-2.04560	-1.93627	-1.70543	-1.33110	-0.75033	0.00000
J =13	0.00000	-3.04401	-3.55314	-2.90691	-2.46993	-2.13857	-1.83388	-1.49737	-1.10590	-0.60097	0.00000
J =12	0.00000	2.82703	2.42907	0.83540	0.02162	-0.39873	-0.57600	-0.57290	-0.42697	-0.21483	0.00000
J =11	0.00000	0.33732	0.43667	0.30195	0.09812	-0.06800	-0.15876	-0.16272	-0.06733	-0.01938	0.00000
J =10	0.00000	-0.00203	-0.00497	0.00948	-0.01499	-0.04544	-0.05762	-0.03910	0.00940	0.02718	0.00000
J = 9	0.00000	-0.03315	-0.05525	-0.04635	-0.04364	-0.04142	-0.03240	-0.01317	0.01503	0.02053	0.00000
J = 8	0.00000	-0.02135	-0.03668	-0.03692	-0.03552	-0.03175	-0.02420	-0.01277	0.00153	0.00589	0.00000
J = 7	0.00000	-0.01116	-0.01929	-0.02218	-0.02375	-0.02327	-0.02066	-0.01597	-0.00946	-0.00438	0.00000
J = 6	0.00000	-0.00707	-0.01139	-0.01429	-0.01687	-0.01844	-0.01875	-0.01759	-0.01516	-0.00958	0.00000
J = 5	0.00000	-0.00687	-0.00982	-0.01188	-0.01420	-0.01620	-0.01755	-0.01794	-0.01721	-0.01182	0.00000
J = 4	0.00000	-0.00827	-0.01104	-0.01205	-0.01354	-0.01521	-0.01669	-0.01764	-0.01779	-0.01277	0.00000
J = 3	0.00000	-0.01040	-0.01292	-0.01249	-0.01289	-0.01402	-0.01555	-0.01710	-0.01823	-0.01380	0.00000
J = 2	0.00000	-0.01007	-0.01117	-0.00907	-0.00838	-0.00877	-0.00999	-0.01187	-0.01425	-0.01189	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

DILATION RE = 10.00

	I = 1	I = 2	I = 3	I = 4	I = 5	I = 6	I = 7	I = 8	I = 9	I =10	I =11
J =21	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J =20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J =19	0.00000	0.00000	0.28920	0.41778	0.55203	0.66008	0.71900	0.72313	0.65159	0.00000	0.00000
J =18	0.00000	0.00000	0.40494	0.48169	0.46108	0.35708	0.22483	0.13213	0.13767	0.00000	0.00000
J =17	0.00000	0.00000	0.68294	0.73083	0.59794	0.36203	0.11492	-0.04356	-0.02518	0.00000	0.00000
J =16	0.00000	0.00000	1.46056	1.22556	0.80378	0.32497	-0.09656	-0.33722	-0.32126	0.00000	0.00000
J =15	0.00000	0.00000	4.81201	2.14746	0.87149	-0.02040	-0.60365	-0.85618	-0.80590	0.00000	0.00000
J =14	0.00000	0.00000	-1.91574	-1.87387	-1.90181	-1.90915	-1.84377	-1.65010	-1.32662	0.00000	0.00000
J =13	0.00000	0.00000	-3.71236	-2.81277	-2.38158	-2.06292	-1.77651	-1.46071	-1.10691	0.00000	0.00000
J =12	0.00000	0.00000	2.50644	0.59264	-0.13145	-0.50408	-0.65614	-0.64019	-0.48070	0.00000	0.00000
J =11	0.00000	0.00000	0.42956	0.24466	0.01644	-0.15655	-0.24659	-0.24407	-0.14376	0.00000	0.00000
J =10	0.00000	0.00000	0.01449	0.01787	-0.03192	-0.08080	-0.10329	-0.08608	-0.01973	0.00000	0.00000
J = 9	0.00000	0.00000	-0.05047	-0.03615	-0.04230	-0.05000	-0.04770	-0.02976	0.00873	0.00000	0.00000
J = 8	0.00000	0.00000	-0.04196	-0.03536	-0.03388	-0.03196	-0.02578	-0.01363	0.00610	0.00000	0.00000
J = 7	0.00000	0.00000	-0.02668	-0.02524	-0.02424	-0.02201	-0.01793	-0.01147	-0.00237	0.00000	0.00000
J = 6	0.00000	0.00000	-0.01772	-0.01817	-0.01626	-0.01744	-0.01559	-0.01262	-0.00935	0.00000	0.00000
J = 5	0.00000	0.00000	-0.01414	-0.01492	-0.01552	-0.01557	-0.01512	-0.01418	-0.01346	0.00000	0.00000
J = 4	0.00000	0.00000	-0.01370	-0.01398	-0.01449	-0.01491	-0.01515	-0.01522	-0.01583	0.00000	0.00000
J = 3	0.00000	0.00000	-0.01503	-0.01404	-0.01404	-0.01453	-0.01539	-0.01643	-0.01818	0.00000	0.00000
J = 2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
J = 1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

error of second derivatives is in velocity, i.e., the error due to neglecting the second leading term of second derivatives in first approximation. Table III shows the values of the fourth central-difference of u and v, which form the leading truncation error terms of second derivatives, such as $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 v}{\partial x^2}$, and $\frac{\partial^2 v}{\partial y^2}$. The result implies that the effect of second approximation, i.e., $\frac{\partial^2 u_o}{\partial y^2} \approx \frac{1}{(\Delta y)^2} [\delta^2 - 1/12\ \delta^4] u_o$ (with $\frac{\partial^2 u_o}{\partial y^2} \approx \frac{1}{(\Delta y)^2} \delta^2 u_o$ as a first approximation), would be large, particularly at the inlet area. For example, $\delta_y^2\ u_{2,15} \approx 0.77246$ and $1/12\ \delta_y^4\ u_{2,15} \approx 1/12 \times (-2.36133) \approx -0.19678$; hence the role of truncation is around 25% of the leading term. Here, δ stands for a central-difference operator, the subscripts x,y for x,y-direction difference, and the superscript for the order of the difference. More clearly, $\delta_y^2\ u_{i,j}$ denotes $(\Delta y)^2 \left.\frac{\partial^2 u}{\partial y^2}\right|_{i,j}$ or $u_{i+1,j} - 2u_{i,j} + u_{i,j-1}$.

The above results (Table III) make sense, because of the large gradient of u-velocity in y-direction and the gradient reduces along the inlet jet. The truncation error is probably improved through more grid points in the region of interest or through second approximation. Crawford[19] used this improved second approximation for the region of the expected large gradient and obtained better results.

Stream functions are calculated in different ways. Here, the S1 method represents $\psi_{i,j} - \psi_{1,1} = \int_{1,1}^{i,j} udy - \int_{1,1}^{i,1} vdx$, S2 for $\psi_{i,j} - \psi_{1,1} = \int_{1,1}^{1,i} udy - \int_{1,j}^{i,j} vdx$, and the S3 method is carried out by using $\nabla^2\psi = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ with the iteration method. The stream function is path-independent, when continuity is satisfied. However, S1, S2, and S3 are different, and S2 looks worse. S2 carries dilation error through the x-direction. The dilation error is particularly bad around the inlet, and this propagates along the x-direction, which gives and unsatisfactory result at the wall. S1 gives an unreasonable value at the exit, which, again, may be explained due to continuity and truncation error. However, S1 seems to be better than S2, because S1 follows the y-direction and gives less error propagation. S3 is believed to be more accurate because the stream-function boundary condition from known velocity boundary conditions was used. However, the large error of continuity exists around the inlet and the error affects calculation of vorticity and propagates through the stream function calculation.

Pressure result predicts the point of highest pressure, as expected, and the pressure drop can be calculated between two points of interest.

A higher Re case was tried but failed with the same grid, i.e., 15 x 21, mainly due to large gradient around the inlet portion and the walls.

CHAPTER IV

CONCLUSION

In this study the primitive variable discretization approach was applied to the steady state Navier-Stokes equation and some sample calculations were made.

This method has some advantage over the vorticity-stream function formation since the extension to three-dimensional flow is possible. Also the pressure field is obtained directly as part of the solution. This method also has some advantages over the unsteady equation approach since the computing time for a converged solution is reduced.

The decay function scheme adopted is a generalization and a hybrid of the conventional central-difference and upwind difference techniques. Thus, a wider range of Reynolds numbers may be treated. The above advantages of the decay function method were utilized to treat the flow in a cavity and to model the flow inside a furnace. The difficulty of the present method is the attendant danger that the method may converge to the wrong solution. This difficulty was encountered in the present study, i.e., when a coarse mesh was used for a high Reynolds number flow. The large velocity gradients near the walls then lead to numerical instability.

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APPENDICES

APPENDIX A

DECAY FUNCTION AND SCHEME

In this appendix, the decay theory, which Chien[13],[14] devised and applied to vorticity included formulation, is extended to the primitive variable formulation. This appendix will show that the decay functions are the same as those of the vorticity included formulation which Chien used.

The two-dimensional continuity and momentum equations in dimensional form are expressed as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

where u, v are the velocity components along the coordinates axes x, y and p is the local static pressure. ρ is the (constant) fluid density and ν is the kinematic viscosity.

Equation (2) can be rewritten as:

$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{u}{\nu} \frac{\partial u}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial y^2} - \frac{v}{\nu} \frac{\partial u}{\partial y}\right) - \frac{1}{\mu} \frac{\partial p}{\partial x} = 0 \tag{4}$$

where μ is the absolute viscosity, i.e., $\mu = \rho\nu$, or in a locally one-dimensional form as:

$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{u}{\nu} \frac{\partial u}{\partial x}\right) + A + B = 0 \tag{5}$$

where $A = \frac{\partial^2 u}{\partial y^2} - \frac{v}{\nu}\frac{\partial u}{\partial y}$ and $B = -\frac{1}{\mu}\frac{\partial p}{\partial x}$, which are treated as temporary source terms. Here, local one-dimensionality is defined as a one dimensional process existing between mesh points. A temporary source term is defined as the assumed constant term which does not vary across the next mesh point. The concept of local one-dimensionality and temporary source term is a reasonable and important assumption in obtaining the local analytic solution and establish the decay function.

The finite difference analogue of equation (5) can be derived as:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \cdot \left(\frac{1}{G_i}\right)$$

$$\frac{u}{\nu}\frac{\partial u}{\partial x} \approx \left(\frac{u}{\nu}\right)_{i,j} \quad \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}$$

$$A \approx A_{i,j}$$

$$B \approx B_{i,j}$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \cdot \frac{1}{G_i} - \left(\frac{u}{\nu}\right)_{i,j} \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} +$$

$$A_{i,j} + B_{i,j} = 0 \qquad (6)$$

After rearrangement, equation (6) becomes

$$u_{i,j} = u_{i-1,j} + (u_{i+1,j} - u_{i-1,j}) \left(1 - \frac{R\Delta x \cdot G_i}{2}\right)/2$$

$$+ (A_{i,j} + B_{i,j})\ G_i\ (\Delta x)^2/2 \qquad (7)$$

where $R\Delta x = \left(\frac{u \cdot \Delta x}{\nu}\right)_{i,j}$ is the x-direction grid Reynolds number. The determination of the x-direction decay function G_i requires an analytic solution of equation (5) subject to the boundary conditions $u = u_{i+1,j}$ at $x = (i+1)\Delta x$ and $u = u_{i-1,j}$ at $x = (i-1)\Delta x$. It is also necessary to assume that the source term, $s = A + B$, does not vary greatly across the mesh point, s can then be treated as a constant and evaluated at the nodal point, i.e., $s_{i,j} = A_{i,j} + B_{i,j}$. the analytic solution of equation (5) under this assumption is

$$u_{i,j} = u_{i-1,j} + (u_{i+1,j} - u_{i-1,j}) \left(e^{\int_{i+1}^{i}\left(\frac{u}{\nu}\right) dx} - 1\right) / \left(e^{\int_{i-1}^{i+1}\left(\frac{u}{\nu}\right) dx} - 1\right)$$

$$+ \frac{A_{i,j} + B_{i,j}}{\left(\frac{u}{\nu}\right)_{i,j}} \left[1 - 2\left(e^{\int_{i-1}^{i}\left(\frac{u}{\nu}\right) dx} - 1\right) / \left(e^{\int_{i-1}^{i+1}\left(\frac{u}{\nu}\right) dx} - 1\right)\right] \qquad (8)$$

By direct comparison between the finite difference solution (7) and its analytical counterpart (equation (8)), the decay function G_i is determined;

$$G_i = \frac{2}{Rx}\left[1 - \frac{2\left(e^{\int_{i-1}^{i}\left(\frac{u}{\nu}\right) dx} - 1\right)}{e^{\int_{i-1}^{i+1}\left(\frac{u}{\nu}\right) dx} - 1}\right] \qquad (9)$$

Similarly, G_j is obtained by using the locally one-dimensional concept in the y-direction, then

$$\left(\frac{\partial^2 u}{\partial y^2} - \frac{v}{\nu}\frac{\partial u}{\partial y}\right) + C_{i,j} + B_{i,j} = 0$$

Thus, $u_{i,j} = u_{i,j-1} + (u_{i,j+1} + u_{i,j-1})\left(1 - \frac{R\Delta y \cdot G_j}{2}\right)/2 +$

$(C_{i,j} + B_{i,j})\ G_j(\Delta y)^2/2$

$$\text{with } G_j = \frac{2}{R\Delta y}\left(1 - \frac{2\left(e^{\int_{j-1}^{j}\left(\frac{v}{\nu}\right)\,dy} - 1\right)}{e^{\int_{j-1}^{j+1}\left(\frac{v}{\nu}\right)\,dy} - 1}\right) \tag{10}$$

where $R\Delta y = \left(\frac{v \cdot \Delta y}{\nu}\right)_{i,j}$ is y-direction grid Reynolds number.

G_i and G_j can be evaluated from equation (9) and equation (10). The definite integrals can be approximated by assuming a linear variation of u and v between mesh points.

$$\int_{i-1}^{i}\left(\frac{u}{\nu}\right)\,dx \approx \Delta x\left[\left(\frac{u}{\nu}\right)_{i,j} + \left(\frac{u}{\nu}\right)_{i-1,j}\right]/2$$

$$\int_{i-1}^{i+1}\left(\frac{u}{\nu}\right)\,dx \approx \Delta x\left[\left(\frac{u}{\nu}\right)_{i+1,j} + 2\left(\frac{u}{\nu}\right)_{i,j} + \left(\frac{u}{\nu}\right)_{i-1,j}\right]/2$$

$$\int_{j-1}^{j}\left(\frac{v}{\nu}\right)\,dy \approx \Delta y\left[\left(\frac{v}{\nu}\right)_{i,j} + \left(\frac{v}{\nu}\right)_{i,j-1}\right]/2$$

$$\int_{j-1}^{j+1}\left(\frac{v}{\nu}\right)\,dy \approx \Delta y\left[\left(\frac{v}{\nu}\right)_{i,j+1} + 2\left(\frac{v}{\nu}\right)_{i,j} + \left(\frac{v}{\nu}\right)_{i,j-1}\right]/2$$

With this approximation, the decay functions become

$$G_i \approx \frac{2}{R\Delta x}\left[1 - \frac{2\left(\exp((u_{i,j} + u_{i-1,j})\Delta x/2\nu) - 1\right)}{\exp((u_{i-1,j} + 2u_{i,j} + u_{i-1,j})\Delta x/2\nu) - 1}\right]$$

$$G_j \approx \frac{2}{R\Delta y}\left[1 - \frac{2\left(\exp((v_{i,j} + v_{i-1,j})\Delta x/2\nu) - 1\right)}{\exp((v_{i-1,j} + 2v_{,j} + v_{i-1,j})\Delta x/2\nu) - 1}\right]$$

If the variations of u and v between mesh points are not very large, the integral approximations become

$$\int_{i-1}^{i}\left(\frac{u}{\nu}\right)\,dx \approx \frac{u_{i,j}\,\Delta x}{\nu} = (R\Delta x)_{i,j}$$

$$\int_{i-1}^{i+1}\left(\frac{u}{\nu}\right)\,dx \approx 2\,\frac{u_{i,j}\,\Delta x}{\nu} = 2(R\Delta x)_{i,j}$$

$$\int_{j-1}^{j}\left(\frac{v}{\nu}\right)\, dy \approx \frac{v_{i,j}\,\Delta y}{\nu} = (R\Delta y)_{i,j}$$

$$\int_{j-1}^{j+1}\left(\frac{v}{\nu}\right)\, dy \approx 2\,\frac{v_{i,j}\,\Delta y}{\nu} = 2(R\Delta y)_{i,j}$$

This simplified approximation shows that

$$G_i \approx \frac{2}{R\Delta x}\left(1 - \frac{2(e^{R\Delta x} - 1)}{e^{2R\Delta x} - 1}\right)$$

$$G_j \approx \frac{2}{R\Delta y}\left(1 - \frac{2(e^{R\Delta y} - 1)}{e^{2R\Delta y} - 1}\right) \qquad (11)$$

It is noted here that if $G_i = G_j = 1$, the conventional central-difference formula is recovered:

$$u_{i,j} = \frac{1/2\ R\Delta x\ (\Delta y)^2 + (\Delta y)^2}{2\{(\Delta x)^2 + (\Delta y)^2\}}\quad u_{i-1,j} + \frac{-1/2\ R\Delta x\ (\Delta y)^2 + (\Delta y)^2}{2\{(\Delta x)^2 + (\Delta y)^2\}}$$

$$u_{i+1,j} + \frac{-1/2\ R\Delta y\ (\Delta x)^2 + (\Delta x)^2}{2\{(\Delta x)^2 + (\Delta y)^2\}}\quad u_{i,j+1} + \frac{1/2\ R\Delta y\ (\Delta y)^2 + (\Delta x)^2}{2\{(\Delta x)^2 + (\Delta y)^2\}}$$

$$+ \frac{(\Delta x)^2\ (\Delta y)^2}{2\{(\Delta x)^2 + (\Delta y)^2\}}\ D$$

Similarly, with $G_i = \frac{2}{2 + R\Delta x}$ and $G_j = \frac{2}{2 + R\Delta y}$ the conventional upwind-difference equation is rederived:

$$\begin{aligned} u_{i,j} = \{&[u_{i-1,j} + (u_{i+1,j} - u_{i-1,j})/(2 + R\Delta x)](\Delta y)^2\ (2 + R\Delta x) + \\ &[u_{i,j-1} + (u_{i,j+1} - u_{i,j-1})/(2 + R\Delta y)](\Delta x)^2\ (2 + R\Delta y)\}/ \\ &[(\Delta y)^2\ (2 + R\Delta x) + (\Delta x)^2\ (2 + R\Delta y)] + \{(\Delta x)^2 + (\Delta y)^2\}/ \\ &[(\Delta y)^2\ (2 + R\Delta x) + (\Delta x)^2\ (2 + R\Delta y)] \end{aligned}$$

So, the complete single finite-difference equation becomes

$$\left(\frac{\partial^2 u}{\partial x^2} - \left(\frac{u}{\nu}\right)\frac{\partial u}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial y^2} - \left(\frac{v}{\nu}\right)\frac{\partial u}{\partial y}\right) - \frac{1}{\nu}\frac{\partial p}{\partial x} = 0$$

In other words,

$$X + Y + D = 0$$

where $$X = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \cdot \frac{1}{G_i} - \left(\frac{u}{\nu}\right)_{i,j} \frac{u_{i+1,j} - u_{i-1,j}}{2\,\Delta x}$$

$$Y = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} \cdot \frac{1}{G_j} - \left(\frac{v}{\nu}\right)_{i,j} \frac{u_{i,j+1} - u_{i,j-1}}{2\,\Delta y}$$

Thus, in a explicit form after arranging,

$$u_{i,j} = \frac{1/2\ R\Delta x(\Delta y)^2\ G_i G_j + (\Delta y)^2 G_j}{2\{(\Delta x)^2\ G_i + (\Delta y)^2\ G_j\}}\ u_{i-1,j} + \frac{-1/2\ R\Delta x(\Delta y)^2\ G_i G_j}{2\{(\Delta x)^2 G_i + (\Delta y)^2 G_j\}}$$

$$u_{i+1,j} + \frac{-1/2\ R\Delta y(\Delta x)^2 G_i G_j + (\Delta x)^2 G_i}{2\{(\Delta x)^2 G_i + (\Delta y)^2 G_j\}}$$

$$u_{i,j+1} + \frac{1/2\ R\Delta y(\Delta x)^2 G_i G_j + (\Delta x)^2 G_i}{2\{(\Delta x)^2 G_i + (\Delta y)^2 G_j\}} + \frac{(\Delta x)^2\ (\Delta y)^2 G_i G_j}{2\{(\Delta x)^2 G_i + (\Delta y)^2\ G_j\}}\ D$$

or, $$u_{i,j} = \frac{\dfrac{1}{(\Delta x)^2 G_i} + \dfrac{R\Delta x}{2(\Delta x)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}}\ u_{i-1,j} + \frac{\dfrac{1}{(\Delta x)^2 G_i} - \dfrac{R\Delta x}{2(\Delta x)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}}$$

$$u_{i+1,j} + \frac{\dfrac{1}{(\Delta y)^2 G_j} - \dfrac{R\Delta y}{2(\Delta y)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}}\ u_{i,j+1} + \frac{\dfrac{1}{(\Delta y)^2 G_j} + \dfrac{R\Delta y}{2(\Delta y)^2}}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}}$$

$$u_{i,j-1} + \frac{1}{\dfrac{2}{(\Delta x)^2 G_i} + \dfrac{2}{(\Delta y)^2 G_j}} D \tag{12}$$

The equation (12) is used for the determination of the u-velocity throughout the domain, with G_i and G_j calculated by equation (13).

Similar expression may be obtained for the v-velocity and the pressure may be obtained from a conventional central-difference approximation (i.e., $G_i = G_j = 1$ for the decay function) of the Poisson equation $\nabla^2 p = s_p$, where s_p is the source term.

For computational purpose, G_i and G_j can be approximated by piece-wise continuous polynomial curve-fitting based upon equation (11).

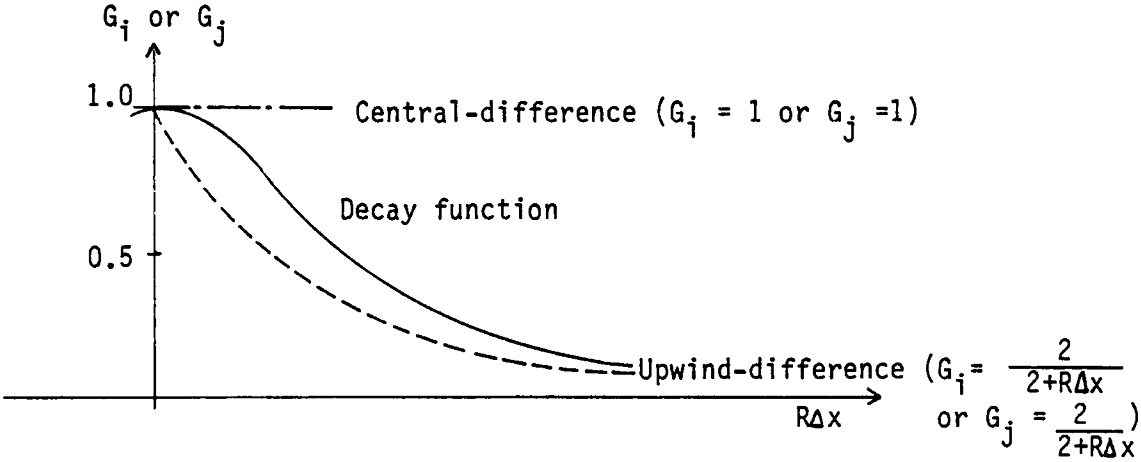
$$G_i = \begin{cases} 1 & \text{where } |R\Delta x| < 1 \\ \dfrac{2}{|R\Delta x|} - \dfrac{1}{(R\Delta x)^2} & \text{where } |R\Delta x| > 1 \end{cases}$$

$$G_j = \begin{cases} 1 & \text{where } |R\Delta y| < 1 \\ \dfrac{2}{|R\Delta y|} - \dfrac{1}{|R\Delta y|^2} & \text{where } |R\Delta y| > 1 \end{cases}$$

or,

$$G_i = \begin{cases} 1.0 - 0.0625\ (R\Delta x)^2 & \text{where } |R\Delta x| < 2 \\ \dfrac{2}{|R\Delta x|} - \dfrac{1}{(R\Delta x)^2} & \text{where } |R\Delta x| > 2 \end{cases}$$

$$G_j = \begin{cases} 1.0 - 0.0625\ (R\Delta y)^2 & \text{where } |R\Delta y| < 2 \\ \dfrac{2}{|R\Delta y|} - \dfrac{1}{(R\Delta y)^2} & \text{where } |R\Delta y| > 2 \end{cases} \tag{13}$$



It is noted here that the above graph is quoted from Chien[13],[14].

The above graph shows the character of the decay function, which is a smooth transition from the central-differencing to the upwind-differencing. Thus, it has applicability for a wider mesh Reynolds number range.

It is noted here that the known numerical stability limitation for forward-time central-difference (in space) technique is given by $R\Delta x = \frac{u\ \Delta x}{\nu} < 2$ and $s = \frac{\nu\ \delta t}{(\Delta x)^2} < 2$ for a one-dimensional equation, which for the decay scheme reduces to:

$$1 - \frac{R\Delta x\ G_i}{2} < 0 \text{ and } 1 - \frac{2\ s\ G_t}{G_i} < 0$$

Hence, the limitation is less severe than the former technique, if G_i and G_t are known. Here G_t is a time-wise decay function.

The present steady approach with primitive variables has stability, efficiency and accuracy by using the decay scheme. Probably a new definition of stability is needed to deal with the pressure-velocity iteration and the nature of the steady equation approach. The present criteria are:

a) Numerical algorithm should converge.

b) Solution must be physically correct.

APPENDIX B

APPROXIMATION BY DIFFERENCES

The derivatives may be approximated by the central difference formula as follows:

$$(\Delta x)\frac{df_o}{dx} \approx \frac{1}{2}\,(f_1 - f_{-1}) = \mu\delta f_o$$

$$(\Delta x)^2\,\frac{d^2 f_o}{dx^2} \approx f_1 - 2\,f_o + f_{-1} = \delta^2 f_o$$

$$(\Delta x)^3\,\frac{d^3 f_o}{dx^3} \approx \frac{1}{2}\,(f_2 - 2f_1 + 2f_{-1} - f_{-2}) = \mu\delta^3 f_o$$

$$(\Delta x)^4\,\frac{d^4 f_o}{dx^4} \approx f_2 - 4f_1 + 6f_o - 4f_{-1} + f_{-2} = \delta^4 f_o$$

The above equations are actually just the first approximations (first terms of a Taylor series) to the derivatives, and are the leading terms of series of central-differences of increasing orders.

In more general notations,

$$(\Delta x)\,\frac{df_o}{dx} = [\mu\,\delta - \frac{1}{6}\,\mu\,\delta^3 + \ldots.]\;f_o$$

$$(\Delta x)^2\,\frac{d^2 f_o}{dx^2} = [\delta^2 - \frac{1}{12}\,\delta^4 + \ldots.]\;f_o$$

$$(\Delta x)^3\,\frac{d^3 f_o}{dx^3} = [\mu\,\delta^3 - \frac{1}{4}\,\mu\,\delta^5 + \ldots.]\;f_o$$

$$(\Delta x)^4\,\frac{d^4 f_o}{dx^4} = [\delta^4 - \frac{1}{6}\,\delta^6 + \ldots.]\;f_o$$

where δ is a central-difference operator, e.g., $\delta f(x) = f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)$, and μ is an averaging operator, e.g. $\mu f(x) = \frac{1}{2}[f\left(x + \frac{\Delta x}{2}\right) + f\left(x - \frac{\Delta x}{2}\right)]$. The second approximation includes the next leading term to the first approximation.

It is noted here that this appendix quotes mainly the work of Crawford[19].

VITA

Bumbae Kim was born in Seoul, Korea in November, 1952. He received a B. S. (Engineering) degree from Seoul National University in 1976.

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