

Injective Modules And Divisible Groups

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Abstract

An R -module M is injective provided that for every R -monomorphism g from R -modules A to B , any R -homomorphism f from A to M can be extended to an R -homomorphism h from B to M such that $hg = f$. That is one of several equivalent statements of injective modules that we will be discussing, including concepts dealing with ideals of rings, homomorphism modules, short exact sequences, and splitting sequences. A divisible group G is defined when for every element x of G and every nonzero integer n , there exists y in G such that $x = ny$. We will see how these two ideas (injectivity and divisibility) compare with each other in general rings, as well as special ones such as Noetherian, Dedekind, and Semi-simple. Since this thesis is a synopsis, the research gathered is scattered throughout the paper (Head, 1974), (Hungerford, 1974), (Lam, 1999), (Rotman, 1995), (Rotman, 1979), and (Sharpe and Vamos, 1972).

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Chapter 1

Preliminaries

Definition 1.0.1. An additive Abelian group M is said to be a **left module of a ring R** (or a **left R -module**) provided that there exists a function $R \times M \rightarrow M$ such that for every $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$,

- (i) $(r_1 + r_2)m = r_1m + r_2m$,
- (ii) $r(m_1 + m_2) = rm_1 + rm_2$,
- (iii) $r_1(r_2m) = (r_1r_2)m$.

If R has an identity 1_R , then a left R -module M is said to be a **unitary left R -module** provided that for every $m \in M$,

- (iv) $1_R m = m$.

If R is a division ring, then a unitary left R -module is called a left **vector space**.

Remark 1.0.1. A unitary right R -module is defined similarly for a function $M \times R \rightarrow M$ that satisfies the analogues of (i) – (iv).

From now on, unless otherwise specified, “ R -module” means “unitary left R -module.” As usual, we will denote the zero module $\{0\}$ by 0 and notate $\mathbb{Z}$ and $\mathbb{Q}$ as the rings of integers and rational numbers, respectively.

Example 1.0.1.

(1) R itself is an R -module, with the usual addition and rt being the ordinary product in R for every $r, t \in R$.

- (2) Any left ideal I of ring R is an R -module, with the usual addition and ra also being the ordinary product in R for every $r \in R$ and $a \in I$.
- (3) Every Abelian group is a $\mathbb{Z}$ -module, defined by $0g = 0$, $ng = g + \cdots + g$ (n times), and $(-n)g = n(-g)$ for $g \in G$ and for every positive integer n .

Definition 1.0.2. Let A be a subset of an R -module M . Then A is an **R -submodule** of M provided that A is an additive Abelian subgroup of M such that $ra \in A$ for every $r \in R$ and $a \in A$.

There is an interesting fact about submodules.

Proposition 1.0.1. Let A , B , and C be submodules of an R -module M , where $A \subseteq B$. Then $A + (B \cap C) = B \cap (A + C)$.

Proof. Let $x \in A + (B \cap C)$, so $x = a + y$, where $a \in A$ and $y \in B \cap C$. Since $y \in C$, $x \in A + C$, and since $y \in B$ and $a \in A \subseteq B$, $x \in B$. Thus, $x \in B \cap (A + C)$, and therefore, $A + (B \cap C) \subseteq B \cap (A + C)$.

Now, let $x \in B \cap (A + C)$, so $x = a + c$, where $a \in A$ and $c \in C$. Since $x = a + c \in B$ and $a \in A \subseteq B$, $c \in B$. Then $c \in B \cap C$. Thus, $x \in A + (B \cap C)$, and therefore, $B \cap (A + C) \subseteq A + (B \cap C)$.

Hence, $A + (B \cap C) = B \cap (A + C)$. □

Remark 1.0.2. Note that if A is a submodule of an R -module M , then the set $M/A = \{m + A \mid m \in M\}$ is also an R -module via $r(m + A) = (rm) + A$ for every $r \in R$ and $m + A \in M/A$. We call this set the **quotient R -module**.

There is a special function that connects R -modules.

Definition 1.0.3. Let $f : A \rightarrow B$ be a function between two R -modules. We say that f is an **R -homomorphism** provided that

$$f(a_1 + a_2) = f(a_1) + f(a_2) \quad \text{and} \quad f(ra) = rf(a)$$

for every $r \in R$ and $a, a_1, a_2 \in A$.

Remark 1.0.3. Note that an R -monomorphism (also called an embedding) is a one-to-one R -homomorphism and an R -epimorphism is an onto R -homomorphism. Similarly, an R -isomorphism is a one-to-one and onto R -homomorphism.

There are special types of R -homomorphisms that connect R -modules with their products.

Definition 1.0.4. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of R -modules. For every $\lambda \in \Lambda$, we will describe the following mappings:

$$\pi_\lambda : \prod_{\lambda \in \Lambda} M_\lambda \rightarrow M_\lambda \quad \text{and} \quad \iota_\lambda : M_\lambda \rightarrow \prod_{\lambda \in \Lambda} M_\lambda.$$

Let $m_\lambda \in M_\lambda$ and $\{m_\gamma\}_{\gamma \in \Lambda} \in \prod_{\lambda \in \Lambda} M_\lambda$, where each $m_\gamma \in M_\gamma$. We define each function by $\pi_\lambda(\{m_\gamma\}_{\gamma \in \Lambda}) = m_\lambda$ and $\iota_\lambda(m_\lambda) = \{\tilde{m}_\gamma\}_{\gamma \in \Lambda}$, where $\tilde{m}_\gamma = 0$ if $\gamma \neq \lambda$ and $\tilde{m}_\gamma = m_\lambda$ if $\gamma = \lambda$. We call these functions the **projection mapping** and the **injection mapping**, respectively. Note that the projection and injection mappings are both R -homomorphisms.

Lemma 1.0.1. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of R -modules. Then for every $\lambda \in \Lambda$, $\pi_\lambda \iota_\lambda : M_\lambda \rightarrow M_\lambda$ is the identity function on M_λ .

Proof. Let $\lambda \in \Lambda$. Note that $\pi_\lambda \iota_\lambda(m_\lambda) = \pi_\lambda(\{\tilde{m}_\gamma\}_{\gamma \in \Lambda})$, where $\tilde{m}_\gamma = 0$ if $\gamma \neq \lambda$ and $\tilde{m}_\gamma = m_\lambda$ if $\gamma = \lambda$. Since $\pi_\lambda(\{\tilde{m}_\gamma\}_{\gamma \in \Lambda}) = m_\lambda$, $\pi_\lambda \iota_\lambda$ is the identity on M_λ . $\square$

We will now consider a special type of R -module [Hungerford \(1974\)](#).

Definition 1.0.5. An R -module F is called a **free R -module** provided that one of the following equivalent conditions is satisfied:

- (i) F is isomorphic to a direct sum of copies of R ;
- (ii) there exists a nonempty set X and a function $\theta : X \rightarrow F$ such that for every R -module A and function $f : X \rightarrow A$, there is a unique R -homomorphism $g : F \rightarrow A$ such that $g\theta = f$.

Proposition 1.0.2. Every R -module is an R -homomorphic image of a free R -module.

Proof. Let M be an R -module, let G be a generating set for M , and let $\theta : F \rightarrow \bigoplus_{g \in G} R$ be an R -isomorphism between a free R -module and a direct sum of copies of R . Define $\gamma : \bigoplus_{g \in G} R \rightarrow M$ by $\gamma(\{r_g\}) = \sum_{g \in G} r_g g$. Note that $r_g = 0$ for all but a finite number of g . Observe that for every $\{r_g\}, \{s_g\} \in F$ and $t \in R$,

$$\begin{aligned} \gamma(\{r_g\} + \{s_g\}) &= \gamma(\{r_g + s_g\}) \\ &= \sum (r_g + s_g)g \\ &= \sum r_g g + \sum s_g g \\ &= \gamma(\{r_g\}) + \gamma(\{s_g\}) \end{aligned}$$

and

$$\begin{aligned} \gamma(t\{r_g\}) &= \gamma(\{tr_g\}) \\ &= \sum tr_g g \\ &= t \sum r_g g \\ &= t\gamma(\{r_g\}), \end{aligned}$$

so γ is an R -homomorphism. Since G is a generating set for M , γ is onto. Hence, M is an R -homomorphic image of F because $\gamma\theta(F) = M$. $\square$

Remark 1.0.4. Note that if the R -module is finitely generated, we can choose the free R -module $F \approx \bigoplus_{g \in S} R$, where S is a finite set.

Using these concepts of R -homomorphisms, we can create induced mappings between quotient R -modules.

Theorem 1.1. Let $f : M \rightarrow N$ be an R -homomorphism, let A be a submodule of M , let B be a submodule of N , and let $f(A) \subseteq B$. Define the function $\rho : M/A \rightarrow N/B$ by $\rho(m + A) = f(m) + B$. We say that ρ is the R -homomorphism induced by f . Then ρ is an R -epimorphism if f is an R -epimorphism, and ρ is a R -monomorphism if $A = f^{-1}(B)$.

Proof. Note that ρ is an R -homomorphism since f is an R -homomorphism. Suppose that f is an R -epimorphism, and let $n + B \in N/B$. Since $n \in N$ and f is onto, there exists $m \in M$ such that $f(m) = n$. Then $\rho(m+A) = f(m)+B = n+B$. Thus, ρ is an R -epimorphism. Now, suppose that $A = f^{-1}(B)$, and let $m + A \in \text{Ker}(\rho)$. Then $0 + B = \rho(m + A) = f(m) + B$, so $f(m) \in B$. Since $A = f^{-1}(B)$, $m \in A$. Thus, $m + A = 0 + A$, and ρ is an R -monomorphism. $\square$

Corollary 1.1.1. *Let $f : M \rightarrow N$ be an R -epimorphism. Then $M/\text{Ker}(f) \approx N$.*

Proof. Note that $\text{Ker}(f)$ is a submodule of M , 0 is a submodule of N , $f(\text{Ker}(f)) = 0$, and $\text{Ker}(f) = f^{-1}(0)$. So, by Theorem 1.1, $M/\text{Ker}(f) \approx N/0$. Since $N/0 \approx N$, $M/\text{Ker}(f) \approx N$. $\square$

Corollary 1.1.2. *Let A, B be submodules of an R -module M . Then $B/(A \cap B) \approx (A+B)/A$.*

Proof. Let $f : B \rightarrow A+B$ be the inclusion function, so f is an R -homomorphism. Note that $A \cap B$ is a submodule of B , A is a submodule of $A+B$, $f(A \cap B) \subseteq A$, and $A \cap B = f^{-1}(A)$. By Theorem 1.1, there is an R -monomorphism $\rho : B/(A \cap B) \rightarrow (A+B)/A$. We will now show that ρ is onto. Let $n + A \in (A+B)/A$. Then $n = a + b$ for some $a \in A$ and $b \in B$. So,

$$n + A = (a + b) + A = b + A = f(b) + A = f(b + [A \cap B]).$$

Thus, ρ is surjective, and hence, $B/(A \cap B) \approx (A+B)/A$. $\square$

An important proposition shows that we can extend any embedding function to an R -isomorphism.

Proposition 1.1.1. *Let $f : A \rightarrow B$ be an embedding between R -modules. Then there exists an extension R -module C of A and an R -isomorphism $g : C \rightarrow B$ such that $g(a) = f(a)$ for all $a \in A$.*

Proof. Firstly, consider a set X such that $|X| = |B \setminus f(A)|$ and $X \cap A = \emptyset$. Then there is a bijection $h : X \rightarrow B \setminus f(A)$. Choose $C = A \cup X$, so it is an extension of A . Define $g : C \rightarrow B$

by $g(c) = f(c)$ if $c \in A$ or $g(c) = h(c)$ if $c \in X$. It is clear that g is one-to-one since f and h are. To show that g is onto, let $y \in B$. Observe that it is either the case that $y \in f(A)$ or $y \notin f(A)$. If $y \in f(A)$, then there exists an $x \in A \subseteq C$ such that $g(x) = f(x) = y$ and we are done. So, consider $y \notin f(A)$ so that $y \in B \setminus f(A)$. Since h is onto, there exists an $x \in X \subseteq C$ such that $g(x) = h(x) = y$. Thus, g is onto, and therefore, g is bijective.

We will now show that C is an R -module under the operations $c_1 + c_2 = g^{-1}(g(c_1) + g(c_2))$ and $rc = g^{-1}(rg(c))$ for every $c, c_1, c_2 \in C$ and $r \in R$. Note that for every $r, r_1, r_2 \in R$ and $c, c_1, c_2 \in C$,

$$\begin{aligned}
 (i) \quad r_1c + r_2c &= g^{-1}(r_1g(c)) + g^{-1}(r_2g(c)) \\
 &= g^{-1}(g(g^{-1}(r_1g(c))) + g(g^{-1}(r_2g(c)))) \\
 &= g^{-1}(r_1g(c) + r_2g(c)) \\
 &= g^{-1}([r_1 + r_2]g(c)) \text{ since } g(c) \in B \text{ and } B \text{ is an } R\text{-module} \\
 &= (r_1 + r_2)c.
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad r(c_1 + c_2) &= r[g^{-1}(g(c_1) + g(c_2))] \\
 &= g^{-1}(r[g(c_1) + g(c_2)]) \\
 &= g^{-1}(rg(c_1) + rg(c_2)) \\
 &= g^{-1}(g[g^{-1}(rg(c_1))] + g[g^{-1}(rg(c_2))]) \\
 &= g^{-1}(rg(c_1)) + g^{-1}(rg(c_2)) \\
 &= rc_1 + rc_2.
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad r_1(r_2c) &= r_1g^{-1}(r_2g(c)) \\
 &= g^{-1}(r_1[r_2g(c)]) \\
 &= g^{-1}([r_1r_2]g(c)) \text{ since } g(c) \in B \text{ and } B \text{ is an } R\text{-module} \\
 &= (r_1r_2)c.
 \end{aligned}$$

$$\begin{aligned}
 (iv) \quad 1_Rc &= g^{-1}(1_Rg(c)) \\
 &= g^{-1}(g(c)) \text{ since } g(c) \in B \text{ and } B \text{ is an } R\text{-module} \\
 &= c, \text{ where } 1_R \text{ is the identity of } R.
 \end{aligned}$$

Thus, C is an R -module. Finally, to show that g is an R -homomorphism, observe that for every $c, c_1, c_2 \in C$ and $r \in R$,

$$g(c_1 + c_2) = g(g^{-1}(g(c_1) + g(c_2))) = g(c_1) + g(c_2)$$

and

$$g(rc) = g(g^{-1}(rg(c))) = rg(c),$$

so g is an R -homomorphism. Hence, there exists an extension R -module C of A and an R -isomorphism $g : C \rightarrow B$ such that $g(a) = f(a)$ for all $a \in A$. $\square$

There are special sequences of R -homomorphisms and R -modules that we will now discuss.

Definition 1.1.1. Let $\{M_i\}_{i \in \mathbb{Z}}$ be a family of R -modules, and let $\{f_i\}_{i \in \mathbb{Z}}$ be a family of R -homomorphisms such that $M_{i-1} \xrightarrow{f_i} M_i$ for every $i \in \mathbb{Z}$. Then the sequence

$$\dots \xrightarrow{f_{-1}} M_{-1} \xrightarrow{f_0} M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \dots$$

is said to be **exact** provided that $\text{Im}(f_{i-1}) = \text{Ker}(f_i)$ for every $i \in \mathbb{Z}$. Note that $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is exact if and only if f is an R -monomorphism, g is an R -epimorphism, and $\text{Im}(f) = \text{Ker}(g)$. This type of sequence is called **short exact**.

Example 1.1.1.

(1) For every R -module A and B , $0 \rightarrow A \xrightarrow{\iota_A} A \oplus B \xrightarrow{\pi_B} B \rightarrow 0$ and $0 \rightarrow B \xrightarrow{\iota_B} A \oplus B \xrightarrow{\pi_A} A \rightarrow 0$ are short exact sequences, where ι_A and ι_B are injection mappings and π_A and π_B are projection mappings.

(2) For every submodule C of an R -module D , $0 \rightarrow C \xrightarrow{\iota} D \xrightarrow{\pi} D/C \rightarrow 0$ is a short exact sequence, where ι is the inclusion mapping and π is the canonical R -epimorphism.

(3) For every R -homomorphism $f : A \rightarrow B$ between R -modules, $0 \rightarrow \text{Ker}(f) \rightarrow A \rightarrow A/\text{Ker}(f) \rightarrow 0$, $0 \rightarrow \text{Im}(f) \rightarrow B \rightarrow B/\text{Im}(f) \rightarrow 0$, and $0 \rightarrow \text{Ker}(f) \rightarrow A \xrightarrow{f} B \rightarrow B/\text{Im}(f) \rightarrow 0$ are all exact sequences, where the unlabeled maps are obvious inclusions and projections.

We see that there is a relation between short exact sequences.

Definition 1.1.2. The two short exact sequences $0 \rightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \rightarrow 0$ and $0 \rightarrow B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} B_3 \rightarrow 0$ are said to be **isomorphic** provided that there are R -isomorphisms

$h_1 : A_1 \rightarrow B_1$, $h_2 : A_2 \rightarrow B_2$ and $h_3 : A_3 \rightarrow B_3$ such that $h_2 f_1 = g_1 h_1$ and $h_3 f_2 = g_2 h_2$; i.e., the following diagram commutes:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 \longrightarrow 0 \\
& & \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 \\
0 & \longrightarrow & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 \longrightarrow 0
\end{array}$$

Figure 1.1: Diagram of isomorphic short exact sequences.

Note that isomorphism of short exact sequences is an equivalence relation, so it is symmetric.

Now, we not only can extend embedding functions, but also short exact sequences.

Proposition 1.1.2. *Every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of R -modules is isomorphic to the short exact sequence $0 \rightarrow A \rightarrow D \rightarrow D/A \rightarrow 0$, where D is an extension of A .*

Proof. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence of R -modules. Since f is an R -monomorphism, there exists an R -isomorphism $\theta : D \rightarrow B$, where D is an extension of A and $\theta(a) = f(a)$ for every $a \in A$ by Proposition 1.1.1. Then for every $a \in A$, $\theta\iota(a) = \theta(a) = f(a)$, so $\theta\iota = f$. We will find an R -isomorphism ψ that makes the following diagram commute:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{\iota} & D & \xrightarrow{\pi} & D/A \longrightarrow 0 \\
& & \downarrow 1_A & & \downarrow \theta & & \downarrow \psi \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0
\end{array}$$

Figure 1.2: Diagram to show isomorphism of specific short exact sequences.

Let $\psi : D/A \rightarrow C$ be given by $\psi(d + A) = g\theta(d)$. Observe that ψ is well-defined because $\psi(0 + A) = g\theta(0) = 0$.

We will now show that ψ is an R -homomorphism. Notice that for every $d + A \in D/A$ and $r \in R$,

$$\begin{aligned}
\psi(r[d + A]) &= \psi([rd] + A) \\
&= g\theta(rd) \\
&= rg\theta(d).
\end{aligned}$$

Since $r\psi(d + A) = rg\theta(d)$, $\psi(r[d + A]) = r\psi(d + A)$. Now, for every $d_1 + A, d_2 + A \in D/A$,

$$\begin{aligned}
\psi([d_1 + A] + [d_2 + A]) &= \psi([d_1 + d_2] + A) \\
&= g\theta(d_1 + d_2) \\
&= g\theta(d_1) + g\theta(d_2).
\end{aligned}$$

Since $\psi(d_1 + A) + \psi(d_2 + A) = g\theta(d_1) + g\theta(d_2)$, $\psi([d_1 + A] + [d_2 + A]) = \psi(d_1 + A) + \psi(d_2 + A)$.

Therefore, ψ is an R -homomorphism.

To show that ψ is one-to-one, let $x + A \in \text{Ker}(\psi)$, so $\psi(x + A) = 0$. Then $g\theta(x) = \psi(x + A) = 0$. Since $\theta(x) \in \text{Ker}(g) = \text{Im}(f)$, there exists $a \in A$ such that $f(a) = \theta(x)$. Thus, $\theta(x) = f(a) = \theta\iota(a) = \theta(a)$. Since θ is one-to-one, $x = a \in A$. Therefore, $x + A = 0 + A$, so ψ is one-to-one.

Finally, to see that ψ is onto, let $c \in C$. Since $g\theta$ is an R -epimorphism, there exists $d \in D$ such that $g\theta(d) = c$. Since $d + A \in D/A$, $\psi(d + A) = g\theta(d) = c$. Thus, ψ is onto.

Therefore, ψ is an R -isomorphism such that $g\theta(d) = \psi(d + A) = \psi\pi(d)$ for every $d \in D$. Hence, $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is isomorphic to $0 \rightarrow A \rightarrow D \rightarrow D/A \rightarrow 0$. $\square$

There is a specific type of short exact sequences that is important for future proofs.

Definition 1.1.3. Let $0 \rightarrow A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \rightarrow 0$ be a short exact sequence of R -homomorphisms. We say that the sequence is **split** provided that it is isomorphic to the short exact sequence $0 \rightarrow A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \rightarrow 0$, where the identity map 1_{A_1} is the

R -isomorphism between A_1 and A_1 , and 1_{A_2} is the R -isomorphism between A_2 and A_2 . This is shown by the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 \longrightarrow 0 \\
& & \downarrow 1_{A_1} & & \downarrow \ell & & \downarrow 1_{A_2} \\
0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 \longrightarrow 0
\end{array}$$

Figure 1.3: Diagram of isomorphism representing split sequences.

In particular, $B \approx A_1 \oplus A_2$.

Proposition 1.1.3. Let $0 \rightarrow A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \rightarrow 0$ be a short exact sequence of R -homomorphisms. Then the following statements are equivalent:

- (i) The sequence is split;
- (ii) There is an R -homomorphism $h : A_2 \rightarrow B$ with $gh = 1_{A_2}$;
- (iii) There is an R -homomorphism $k : B \rightarrow A_1$ with $kf = 1_{A_1}$;

Proof. (i) $\Rightarrow$ (ii) Suppose that the sequence is split, with the R -isomorphism $\ell : B \rightarrow A_1 \oplus A_2$. Then $\iota_1 = \ell f$ and $g = \pi_2 \ell$. Define an R -homomorphism $h : A_2 \rightarrow B$ by $\ell^{-1} \iota_2$. So, $gh = g\ell^{-1} \iota_2 = \pi_2 \ell \ell^{-1} \iota_2 = \pi_2 \iota_2 = 1_{A_2}$. Thus, h exists.

(ii) $\Rightarrow$ (i) Conversely, suppose that there is an R -homomorphism $h : A_2 \rightarrow B$ with $gh = 1_{A_2}$. We will show that there exists an R -isomorphism $m : A_1 \oplus A_2 \rightarrow B$ that makes the following diagram commute:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 \longrightarrow 0 \\
& & \downarrow 1_{A_1} & & \downarrow m & & \downarrow 1_{A_2} \\
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 \longrightarrow 0
\end{array}$$

Figure 1.4: Diagram to show sequence splits from an R -homomorphism h .

i.e., $f = m\iota_1$ and $\pi_2 = gm$. Define $m : A_1 \oplus A_2 \rightarrow B$ by $m((a_1, a_2)) = f(a_1) + h(a_2)$. Then m is an R -homomorphism since f and h are. Observe that for every $a_1 \in A_1$, $m\iota_1(a_1) =$

$m((a_1, 0)) = f(a_1) + h(0) = f(a_1)$. Thus, $m\iota_1 = f$. Also, notice that for every $(a_1, a_2) \in A_1 \oplus A_2$,

$$\begin{aligned} gm((a_1, a_2)) &= g(f(a_1) + h(a_2)) \\ &= gf(a_1) + gh(a_2) \\ &= 0 + 1_{A_2}(a_2) \\ &= a_2 \\ &= \pi_2((a_1, a_2)). \end{aligned}$$

Thus, $gm = \pi_2$. We will now show that m is a bijection. Firstly, let $(a_1, a_2) \in \text{Ker}(m)$. Then $m((a_1, a_2)) = f(a_1) + h(a_2) = 0$ so that $h(a_2) = -f(a_1)$. So, $a_2 = 1_{A_2}(a_2) = gh(a_2) = -gf(a_1) = 0$. Then $f(a_1) = -h(a_2) = 0$, implying that $a_1 = 0$ since $a_1 \in \text{Ker}(f)$ and f is one-to-one. Thus, $(a_1, a_2) = (0, 0)$ and m is one-to-one. Now, to show that m is onto, let $b \in B$ so that $g(b) \in A_2$. Since π_2 is onto, there exists $(a_1, a_2) \in A_1 \oplus A_2$ such that $\pi_2((a_1, a_2)) = g(b)$. Then $g(b) = \pi_2((a_1, a_2)) = gm((a_1, a_2))$, implying that $g(m((a_1, a_2)) - b) = 0$. This means that $m((a_1, a_2)) - b \in \text{Ker}(g) = \text{Im}(f)$ so that there exists an $x \in A_1$ such that $f(x) = m((a_1, a_2)) - b$. Consider $(a_1, a_2) - \iota_1(x) \in A_1 \oplus A_2$. Then

$$\begin{aligned} m((a_1, a_2) - \iota_1(x)) &= m((a_1, a_2)) - m\iota_1(x) \\ &= m((a_1, a_2)) - f(x) \\ &= m((a_1, a_2)) - m((a_1, a_2)) + b \\ &= b. \end{aligned}$$

Thus, m is onto, which means that m is an R -isomorphism. Therefore, the sequence is split.

(i) $\Rightarrow$ (iii) Now, suppose that the sequence is split, with the R -isomorphism $\ell : B \rightarrow A_1 \oplus A_2$. Then $\iota_1 = \ell f$ and $g = \pi_2 \ell$. Let $k : B \rightarrow A_1$ be the R -homomorphism defined by $\pi_1 \ell$. Then $kf = \pi_1 \ell f = \pi_1 \iota_1 = 1_{A_1}$. Thus, k exists.

(iii) $\Rightarrow$ (i) Conversely, suppose that there is an R -homomorphism $k : B \rightarrow A_1$ with $kf = 1_{A_1}$. We will show that there exists an R -isomorphism $\ell : A_1 \oplus A_2 \rightarrow B$ that makes the following diagram commute:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 \longrightarrow 0 \\
 & & \downarrow 1_{A_1} & & \downarrow \ell & & \downarrow 1_{A_2} \\
 0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 \longrightarrow 0
 \end{array}$$

Figure 1.5: Diagram to show sequence splits from an R -homomorphism k .

i.e., $\iota_1 = \ell f$ and $g = \pi_2 \ell$. Define $\ell : B \rightarrow A_1 \oplus A_2$ by $\ell(b) = (k(b), g(b))$. Then ℓ is an R -homomorphism since k and g are. Note that for every $a_1 \in A_1$,

$$\ell f(a_1) = (kf(a_1), gf(a_1)) = (1_{A_1}(a_1), 0) = (a_1, 0) = \iota_1(a_1),$$

so $\ell f = \iota_1$. Also, observe that for every $b \in B$, $\pi_2 \ell(b) = \pi_2((k(b), g(b))) = g(b)$, so $\pi_2 \ell = g$. We will now show that ℓ is a bijection. Firstly, let $b \in \text{Ker}(\ell)$. Then $\ell(b) = (k(b), g(b)) = (0, 0)$, so $g(b) = 0$. Since $b \in \text{Ker}(g) = \text{Im}(f)$, there exists an $a_1 \in A_1$ such that $f(a_1) = b$. So, $(a_1, 0) = \iota_1(a_1) = \ell f(a_1) = \ell(b) = (0, 0)$, which means that $a_1 = 0$ and $b = f(a_1) = f(0) = 0$. Thus, $b = 0$ and ℓ is one-to-one. Now, to show that ℓ is onto, let $(a_1, a_2) \in A_1 \oplus A_2$, so $\pi_2((a_1, a_2)) = a_2$. Since g is onto, there exists $b \in B$ such that $g(b) = a_2$. Then $\pi_2 \ell(b) = g(b) = a_2 = \pi_2((a_1, a_2))$, implying that $\pi_2(\ell(b) - (a_1, a_2)) = (0, 0)$. This means that $\ell(b) - (a_1, a_2) \in \text{Ker}(\pi_2) = \text{Im}(\iota_1)$, so there exists an $\tilde{a}_1 \in A_1$ such that $\iota_1(\tilde{a}_1) = \ell(b) - (a_1, a_2)$. Consider $b - f(\tilde{a}_1) \in B$. Then

$$\begin{aligned}
 \ell(b - f(\tilde{a}_1)) &= \ell(b) - \ell f(\tilde{a}_1) \\
 &= \ell(b) - \iota_1(\tilde{a}_1) \\
 &= \ell(b) - \ell(b) + (a_1, a_2) \\
 &= (a_1, a_2).
 \end{aligned}$$

Thus, ℓ is onto, which means that ℓ is an R -isomorphism. Therefore, the sequence is split. $\square$

Chapter 2

Injective Modules

Definition 2.0.4. An R -module M is **injective** provided that for every R -monomorphism $g : A \rightarrow B$ between R -modules, any R -homomorphism $f : A \rightarrow M$ can be extended to an R -homomorphism $h : B \rightarrow M$ such that $hg = f$; i.e., the following diagram commutes

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B \\ & & \downarrow f & \nearrow h & \\ & & M & & \end{array}$$

Figure 2.1: Diagram of an injective R -module using modules.

There are multiple equivalent statements to the definition of injective modules. We start by discussing the ideal one.

Theorem 2.1 (Baer's Criterion). An R -module M is injective if and only if every R -homomorphism $f : I \rightarrow M$, where I is a left ideal of R , can be extended to an R -homomorphism $g : R \rightarrow M$ so that $g(a) = f(a)$ for every $a \in I$; i.e., $g|_I = f$.

Proof. Suppose that M is an injective R -module. Let $f : I \rightarrow M$ be an R -homomorphism, where I is a left ideal of R . Note that I and R are both R -modules, where the inclusion mapping $\iota : I \rightarrow R$ is an R -monomorphism. Since M is injective, there exists an R -homomorphism $g : R \rightarrow M$ such that $f = g\iota$. Thus, $g(a) = g\iota(a) = f(a)$ for every $a \in I$.

Conversely, suppose that every R -homomorphism $f : I \rightarrow M$, where I is a left ideal of R , can be extended to an R -homomorphism $g : R \rightarrow M$. Let $g : A \rightarrow B$ be an R -monomorphism between R -modules, and let $f : A \rightarrow M$ be an R -homomorphism. Since g is an embedding, we know by Proposition 1.1.1 that there exists an extension R -module C of A that is an isomorphic extension $h : C \rightarrow B$ of g . Consider the inclusion function where A is an R -submodule of C . Let Ω be the collection of tuples (D, γ) , where D is an R -module such that $A \subseteq D \subseteq C$ and $\gamma : D \rightarrow M$ is an R -homomorphism such that $\gamma|_A = f$. Note that $(A, f) \in \Omega$, so Ω is nonempty. Let us order Ω by $(D_1, \gamma_1) \leq (D_2, \gamma_2)$ if and only if $D_1 \subseteq D_2$ and $\gamma_2|_{D_1} = \gamma_1$. Let $\{(D_\lambda, \gamma_\lambda)\}_{\lambda \in \Lambda}$ be a totally ordered subset of Ω . Consider $(\tilde{D}, \tilde{\gamma}) \in \Omega$, where $\tilde{D} = \bigcup_{\lambda \in \Lambda} D_\lambda$ and $\tilde{\gamma} : \tilde{D} \rightarrow M$ is an R -homomorphism defined by $\tilde{\gamma}(\tilde{d}) = \gamma_\lambda(\tilde{d})$ if $\tilde{d} \in D_\lambda$. Then $(\tilde{D}, \tilde{\gamma})$ is an upper bound of $\{(D_\lambda, \gamma_\lambda)\}_{\lambda \in \Lambda}$. Thus, by Zorn's Lemma, Ω has a maximal member (D_0, γ_0) .

We will now show that $D_0 = C$. Assume by way of contradiction that $D_0 \neq C$. Then there exists $x \in C$ such that $x \notin D_0$. Let $I = \{r \in R \mid rx \in D_0\}$, so I a left ideal of R . Define a well-defined R -homomorphism $\alpha : I \rightarrow M$ by $\alpha(r) = \gamma_0(rx)$. By assumption, there exists $\beta : R \rightarrow M$ such that $\beta|_I = \alpha$.

Now, define $\rho : D_0 + Rx \rightarrow M$ by $\rho(d_0 + rx) = \gamma_0(d_0) + \beta(r)$. To show that ρ is well-defined, observe that if $d_0 + rx = 0$, then $rx = -d_0 \in D_0$, so $r \in I$. Thus,

$$\begin{aligned} \rho(0) &= \rho(d_0 + rx) \\ &= \gamma_0(d_0) + \beta(r) \\ &= -\gamma_0(rx) + \alpha(r) \\ &= -\alpha(r) + \alpha(r) \\ &= 0. \end{aligned}$$

Since γ_0 and β are R -homomorphisms, ρ is a well-defined R -homomorphism. Thus, $(D_0 + Rx, \rho) \in \Omega$. Note that for every $d_0 \in D_0$, $\rho(d_0) = \gamma_0(d_0)$, so $(D_0, \gamma_0) < (D_0 + Rx, \rho)$. However, this contradicts the fact that (D_0, γ_0) is the maximal member of Ω . Therefore, $D_0 = C$.

Lastly, we will show that $\gamma_0 h^{-1} : B \rightarrow M$ is an R -homomorphism such that $\gamma_0 h^{-1} g = f$. Note that for every $a \in A$,

$$\gamma_0 h^{-1} g(a) = \gamma_0 h^{-1} h(a) = \gamma_0(a) = f(a).$$

Hence, M is injective. □

Example 2.1.1.

(1) If R is a division ring, then every R -module is injective by Theorem 2.1 since the only ideals of R are 0 and R .

(2) Note that $\mathbb{Q}$ is an injective $\mathbb{Z}$ -module by Theorem 2.1 since for every $\mathbb{Z}$ -homomorphism $f : n\mathbb{Z} \rightarrow \mathbb{Q}$, where $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$ for $0 \neq n \in \mathbb{Z}$, there exists a $\mathbb{Z}$ -homomorphism $g : \mathbb{Z} \rightarrow \mathbb{Q}$ defined by $g(z) = \frac{zf(n)}{n}$, so $g(nz) = \frac{(nz)f(n)}{n} = zf(n) = f(nz)$ for every $nz \in n\mathbb{Z}$.

(3) Note that $\mathbb{Z}$ is **not** an injective $\mathbb{Z}$ -module since using the $\mathbb{Z}$ -homomorphism $f : 2\mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(2z) = z$, there is no $\mathbb{Z}$ -homomorphism $g : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $g(2z) = f(2z)$ for every $2z \in 2\mathbb{Z}$. Otherwise, $1 = f(2) = g(2) = 2g(1)$, implying that $g(1) = \frac{1}{2}$. However, since $g(1) \in \mathbb{Z}$, this is impossible.

The next equivalent injective statement comes from short exact sequences and the homomorphism group.

Definition 2.1.1. Let M and N be R -modules. Define $\text{Hom}_R(M, N)$ to be the Abelian group of R -homomorphisms $f : M \rightarrow N$, given by $(f + g)(m) = f(m) + g(m)$ for every $m \in M$ and $f, g \in \text{Hom}_R(M, N)$.

Theorem 2.2. Let M be an R -module. Then M is injective if and only if for every short exact sequence $0 \rightarrow A \xrightarrow{\theta} B \xrightarrow{\psi} C \rightarrow 0$ of R -modules, $0 \rightarrow \text{Hom}_R(C, M) \xrightarrow{\Psi} \text{Hom}_R(B, M) \xrightarrow{\Theta} \text{Hom}_R(A, M) \rightarrow 0$ is also a short exact sequence, where $\Psi(f) = f\psi$ and $\Theta(f) = f\theta$.

Proof. Suppose that M is injective.

We will first show that Ψ is an R -monomorphism. Suppose that $\Psi(f) = \Psi(g)$ for some $f, g \in \text{Hom}_R(C, M)$. Then for every $b \in B$, $f(\psi(b)) = g(\psi(b))$. We want to see that $f = g$. Note that for every $c \in C$, there exists $b \in B$ such that $\psi(b) = c$ since ψ is an R -epimorphism. Then for every $c \in C$, $f(c) = f(\psi(b)) = g(\psi(b)) = g(c)$. Thus, $f = g$ and Ψ is an R -monomorphism.

Secondly, we will see that Θ is an R -epimorphism. Let $g \in \text{Hom}_R(A, M)$. Since M is injective, there exists an $f \in \text{Hom}_R(B, M)$ such that $g = f\theta = \Theta(f)$. Thus, Θ is an R -epimorphism.

Finally, we will show that $\text{Ker}(\Theta) = \text{Im}(\Psi)$. Let $f \in \text{Im}(\Psi)$, so there exists $g \in \text{Hom}_R(C, M)$ such that $\Theta(g) = g\theta = f$. Let $a \in A$. So, $\theta(a) \in \text{Im}(\theta) = \text{Ker}(\psi)$. Then $\psi(\theta(a)) = 0$, so $g\psi(\theta(a)) = g(0) = 0$. Thus, for every $a \in A$, $f(\theta(a)) = 0$. Therefore, $f \in \text{Ker}(\Theta)$ so that $\text{Im}(\Psi) \subseteq \text{Ker}(\Theta)$. Let $f \in \text{Ker}(\Theta)$ be arbitrary. Then $\Theta(f) = f\theta = 0$ so that $0 = f(\text{Im}\theta) = f(\text{Ker}\psi)$. Since $\text{Ker}(\psi) \subseteq B$ and $f(\text{Ker}\psi) \subseteq 0$, there is an induced R -homomorphism $\rho : B/\text{Ker}(\psi) \rightarrow M$ given by $\rho(m + \text{Ker}\psi) = f(m)$ by Theorem 1.1. Also, since $\text{Ker}(\psi) \subseteq B$ and $\psi(\text{Ker}\psi) \subseteq 0$, there is an induced R -isomorphism $\varsigma : B/\text{Ker}\psi \rightarrow C$ given by $\varsigma(b + \text{Ker}\psi) = \psi(b)$ by Corollary 1.1.1. Consider $\rho\varsigma^{-1} : C \rightarrow M$. Notice that $\rho\varsigma^{-1}$ is an R -homomorphism since ρ and ς^{-1} are. Since $\phi(b + \text{Ker}\psi) = \psi(b)$ implies that $b + \text{Ker}\psi = \varsigma^{-1}\psi(b)$, for every $b \in B$, $\rho\varsigma^{-1}\psi(b) = \rho(b + \text{Ker}\psi) = f(b)$, so $f = \rho\varsigma^{-1}\psi = \Psi(\rho\varsigma^{-1})$. Thus, $f \in \text{Im}(\Psi)$ and $\text{Ker}(\Theta) \subseteq \text{Im}(\Psi)$. Therefore, $\text{Ker}(\Theta) = \text{Im}(\Psi)$, and hence, $0 \rightarrow \text{Hom}_R(C, M) \xrightarrow{\Psi} \text{Hom}_R(B, M) \xrightarrow{\Theta} \text{Hom}_R(A, M) \rightarrow 0$ is a short exact sequence.

Conversely, suppose that whenever $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence between R -modules, $0 \rightarrow \text{Hom}_R(C, M) \rightarrow \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(A, M) \rightarrow 0$ is also exact. Let I be a left ideal of a ring R , so I is a submodule of an R -module R , where $0 \rightarrow I \xrightarrow{\iota} R$ is a sequence with the inclusion map ι and $f : I \rightarrow M$ is an R -homomorphism. Consider the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & I & \xrightarrow{\iota} & R & \xrightarrow{\pi} & R/I \longrightarrow 0 \\
& & \downarrow f & \swarrow g & & & \\
& & M & & & &
\end{array}$$

Figure 2.2: Diagram to show that M is injective by using ideals.

where $\pi : R \rightarrow R/I$ is the projection R -homomorphism. Since the inclusion map ι is an R -monomorphism, the projection map π is an R -epimorphism, and $\text{Ker}(\pi) = \text{Im}(\iota)$, the top row is exact. Then $0 \rightarrow \text{Hom}_R(R/I, M) \xrightarrow{\Pi} \text{Hom}_R(R, M) \xrightarrow{I} \text{Hom}_R(I, M) \rightarrow 0$ is also exact. Since $f \in \text{Hom}_R(I, M)$ and I is an R -epimorphism, there exists $g \in \text{Hom}_R(R, M)$ such that $I(g) = g\iota = g|_I = f$. Thus, M is injective. $\square$

We now see an equivalent injective statement using split exact sequences.

Theorem 2.3. *An R -module M is injective if and only if every short exact sequence $0 \rightarrow M \rightarrow B \rightarrow C \rightarrow 0$ splits.*

Proof. Suppose that M is an injective R -modules. Let $0 \rightarrow M \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an exact sequence of R -modules. Consider the R -homomorphism $1_M : M \rightarrow M$, the identity function of M . Since M is injective, there exists an R -homomorphism $k : B \rightarrow M$ such that $kf = 1_M$. By Proposition 1.1.3, the sequence splits.

Conversely, suppose that every exact sequence $0 \rightarrow M \rightarrow B \rightarrow C \rightarrow 0$ of R -modules is split. Let $g : A \rightarrow B$ be an R -monomorphism between R -modules, and let $f : A \rightarrow M$ be an R -homomorphism. Then there exist functions $f' : B \rightarrow D$ and $g' : M \rightarrow D$ given by $f'(b) = (0, b) + W$ and $g'(m) = (m, 0) + W$, where $W = \{(f(a), -g(a)) \mid a \in A\}$ is a submodule of $M \oplus B$ and $D = (M \oplus B)/W$; this is shown by the following diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & A & \xrightarrow{g} & B \\
& & \downarrow f & & \downarrow f' \\
& & M & \xrightarrow{g'} & D
\end{array}$$

Figure 2.3: Pushout diagram to show that M is injective by using modules.

Then f' and g' are R -homomorphisms. Note that for every $a \in A$, $(f(a), -g(a)) \in W$, so $g'f(a) = (f(a), 0) + W = (0, g(a)) + W = f'g(a)$. Thus, $g'f = f'g$. We will now show that g' is one-to-one. Let $x \in \text{Ker}(g')$. Then $(x, 0) + W = g'(x) = (0, 0) + W$. Thus, $(x, 0) \in W$, meaning that there exists $a \in A$ such that $f(a) = x$ and $-g(a) = 0$. Since $a \in \text{Ker}(g)$ and g is one-to-one, $a = 0$, so $0 = f(0) = f(a) = x$. Then, $x = 0$ and g' is an R -monomorphism. So, we can extend g' to an exact sequence $0 \rightarrow M \xrightarrow{g'} D \xrightarrow{\phi} C \rightarrow 0$ for some R -module C . By assumption, there exists an R -homomorphism $k : D \rightarrow M$ such that $kg' = 1_M$. Define $h : B \rightarrow M$ by $h = kf'$, so h is an R -homomorphism. Note that for every $a \in A$, $hg(a) = kf'g(a) = kg'f(a) = 1_M f(a) = f(a)$, showing that $hg = f$. Hence, M is injective. $\square$

We will now illustrate the relationship between injective R -modules and their product and direct sum.

Theorem 2.4. *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of R -modules. Then $\prod M_\lambda$ is injective if and only if M_λ is injective for every $\lambda \in \Lambda$.*

Proof. Suppose that $\prod M_\lambda$ is injective. Fix $\lambda \in \Lambda$. Let $f : I \rightarrow M_\lambda$ be an R -homomorphism, where I is a left ideal of R . Consider the following diagram:

$$\begin{array}{ccccc}
0 & \longrightarrow & I & \xrightarrow{\iota} & R \\
& & \downarrow f & & \nearrow h \\
& & M_\lambda & & \\
& & \uparrow \phi_\lambda & \nearrow \pi_\lambda & \\
& & \prod_{\lambda \in \Lambda} M_\lambda & &
\end{array}$$

Figure 2.4: Diagram to show that each module in an injective product is injective.

where ι is the inclusion map, ϕ_λ is the injection mapping, and π_λ is the projection mapping. Since $\phi_\lambda f$ is an R -homomorphism and $\prod M_\lambda$ is injective, there exists an R -homomorphism $h : R \rightarrow \prod M_\lambda$ such that $\phi_\lambda f = h|_I$. Then $\pi_\lambda \phi_\lambda f = \pi_\lambda h|_I$. Since $\pi_\lambda \phi_\lambda$ is the identity function of M_λ by Lemma 1.0.1, $f = \pi_\lambda h|_I$. Thus, M_λ is injective for every $\lambda \in \Lambda$.

Conversely, suppose that M_λ is injective for every $\lambda \in \Lambda$. Let $f : I \rightarrow \prod M_\lambda$ be an R -homomorphism, where I is a left ideal of R . Consider the following diagram:

$$\begin{array}{ccccc}
0 & \longrightarrow & I & \xrightarrow{\iota} & R \\
& & \downarrow f & \swarrow g & \nearrow \\
& & \prod M_\lambda & & \\
& & \downarrow \pi_\lambda & \swarrow g_\lambda & \nearrow \\
& & M_\lambda & &
\end{array}$$

Figure 2.5: Diagram to show that product of injective modules is injective.

where ι is the inclusion map and π_λ is the projection mapping. Since $\pi_\lambda f$ is an R -homomorphism and M_λ is injective for every $\lambda \in \Lambda$, there exists an R -homomorphism $g_\lambda : R \rightarrow M_\lambda$ such that $\pi_\lambda f = g_\lambda|_I$. Define $g : R \rightarrow \prod M_\lambda$ by $g(r) = \{g_\lambda(r)\}$. Then g is a well-defined R -homomorphism such that for every $a \in I$, $g(a) = \{g_\lambda(a)\} = \{\pi_\lambda f(a)\} = f(a)$. Thus, $\prod M_\lambda$ is injective. $\square$

Corollary 2.4.1. *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of R -modules. Then*

- (1) *If $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is injective, then M_λ is injective for every $\lambda \in \Lambda$.*
- (2) *If Λ is finite and M_λ is injective for every $\lambda \in \Lambda$, then $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is also injective.*

In general, the direct sum of injective modules is not necessarily injective. We will discuss a case that is true in Chapter 5 with Noetherian rings.

The dual notion of injective is as follows.

Definition 2.4.1. An R -module P is **projective** provided that for every R -epimorphism $g : A \rightarrow B$ between R -modules and R -homomorphism $f : P \rightarrow B$, there exists an R -homomorphism $h : P \rightarrow A$ such that $gh = f$; i.e., the following diagram commutes

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow h & \downarrow f & & \\
 A & \xrightarrow{g} & B & \longrightarrow & 0
 \end{array}$$

Figure 2.6: Diagram of projective R -module using modules.

Proposition 2.4.1. Every free R -module is projective.

Proof. Suppose that F is a free R -module on a nonempty set X with a function $\theta : X \rightarrow F$. Let $g : A \rightarrow B$ be an R -epimorphism between R -modules, and let $f : F \rightarrow B$ be an R -homomorphism, so $f\theta(x) \in B$ for every $x \in X$. Since g is onto, there exists $a_x \in A$ such that $g(a_x) = f\theta(x)$. Define $\varsigma : X \rightarrow A$ by $\varsigma(x) = a_x$. Since F is free, there exists a unique R -homomorphism $h : F \rightarrow A$ such that $h\theta(x) = \varsigma(x) = a_x$. Then for every $x \in X$, $gh\theta(x) = g(a_x) = f\theta(x)$. Since h is unique, $gh = f$. Hence, F is projective. $\square$

Remark 2.4.1. Note that projective modules have dual notions for most of our injective concepts.

(1) An R -module P is projective if and only if for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of R -modules, $0 \rightarrow \text{Hom}_R(P, A) \rightarrow \text{Hom}_R(P, B) \rightarrow \text{Hom}_R(P, C) \rightarrow 0$ is also a short exact sequence.

- (2) *An R -module P is projective if and only if every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$ splits.*
- (3) *A direct sum of projective modules is projective.*

Chapter 3

Divisible Groups

Definition 3.0.2. Let M be an R -module. An element $m \in M$ is **divisible** provided that for any $r \in R$ that is not a right zero-divisor, there exists an $x \in M$ such that $m = rx$. We also say that M is a **divisible module** provided that every element of M is divisible. Note that a divisible group is a divisible $\mathbb{Z}$ -module.

Remark 3.0.2. Recall that an element $a \in R$ is a right zero-divisor provided that there exists $0 \neq b \in R$ such that $ba = 0$.

Example 3.0.1.

(1) Note that $\mathbb{Q}$ is a divisible $\mathbb{Z}$ -module since for every $q \in \mathbb{Q}$, where $q = \frac{a}{b}$ for integers $a, b \in \mathbb{Z}$ with $b \neq 0$, and for every $0 \neq z \in \mathbb{Z}$, there exists $x \in \mathbb{Q}$ such that $x = \frac{a}{zb}$ so that $q = zx$.

(2) Note that $\mathbb{Z}$ is not a divisible $\mathbb{Z}$ -module since there is no $x \in \mathbb{Z}$ with $3 = 2x$.

There are multiple equivalent statements to the definition of divisible modules.

Theorem 3.1. Let M be an R -module. Then M is divisible if and only if $M = rM$ for every $r \in R$ which is not a right zero-divisor.

Proof. Suppose that M is divisible. Since M is an R -module, $rM \subseteq M$ for every $r \in R$ which is not a right zero-divisor. Fix $r \in R$ such that r is not a right zero-divisor. Let

$m \in M$. Since M is divisible, there exists an $x \in M$ such that $m = rx$, so $m \in rM$. Thus, $M \subseteq rM$, showing $M = rM$ for every $r \in R$ which is not a right zero-divisor.

Conversely, suppose that $M = rM$ for every $r \in R$ which is not a right zero-divisor. Let $m \in M$ and let $r \in R$ be such that r is not a right zero-divisor. Since $M = rM$, $m \in rM$, so $m = rx$ for some $x \in M$. Thus, M is divisible. $\square$

We see how divisibility affects quotient modules.

Lemma 3.1.1. *Let A be a submodule of a divisible R -module M . Then the quotient R -module M/A is also divisible.*

Proof. Let $m+A \in M/A$ and let $r \in R$ be such that r is not a right zero-divisor. Since $m \in M$ and M is divisible, there exists an $x \in M$ such that $m = rx$. So, $m+A = (rx)+A = r(x+A)$ for some $x+A \in M/A$. Hence, M/A is divisible. $\square$

We will now illustrate the relationship between divisible R -modules and their product and direct sum.

Theorem 3.2. *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of R -modules. Then*

- (i) $\prod M_\lambda$ is divisible if and only if M_λ is divisible for every $\lambda \in \Lambda$
- (ii) $\bigoplus M_\lambda$ is divisible if and only if M_λ is divisible for every $\lambda \in \Lambda$.

Proof. (i) Suppose that $\prod M_\lambda$ is divisible. For every $\lambda \in \Lambda$, let $m_\lambda \in M_\lambda$ and $r \in R$ be such that r is not a right zero-divisor. Since $\{m_\lambda\} \in \prod M_\lambda$, there exists $\{x_\lambda\} \in \prod M_\lambda$ such that $\{m_\lambda\} = r\{x_\lambda\} = \{rx_\lambda\}$. Then for every $\lambda \in \Lambda$, $m_\lambda = rx_\lambda$. Thus, M_λ is divisible for every $\lambda \in \Lambda$.

Conversely, suppose that M_λ is divisible for every $\lambda \in \Lambda$. Let $\{m_\lambda\} \in \prod M_\lambda$ and $r \in R$ be such that r is not a right zero-divisor. For every $\lambda \in \Lambda$, since M_λ is divisible, there exist $x_\lambda \in M_\lambda$ such that $m_\lambda = rx_\lambda$. So, $\{m_\lambda\} = \{rx_\lambda\} = r\{x_\lambda\}$, where $\{x_\lambda\} \in \prod M_\lambda$. Hence, $\prod M_\lambda$ is divisible.

(ii) Using a similar argument to part (i), $\bigoplus M_\lambda$ is divisible if and only if M_λ is divisible for every $\lambda \in \Lambda$. $\square$

3.1 L-Divisibility

There are many disagreements with the definition of divisible modules over arbitrary rings. T. Y. Lam (1999) uses a different divisible definition based on the annihilator ideal of elements.

Definition 3.2.1. *Let M be an R -module and let $m \in M$. Then the **annihilator** $\text{Ann}(m)$ of m is the left ideal $\{r \in R \mid rm = 0\}$ of R .*

Definition 3.2.2. *Let M be an R -module. Then M is L -divisible provided that for every $m \in M$ and every $r \in R$ that satisfy $\text{Ann}(r) \subseteq \text{Ann}(m)$, there exists an $x \in M$ such that $m = rx$.*

There is a connection between these two definitions.

Theorem 3.3. *Let M be an R -module. If M is L -divisible, then M is divisible.*

Proof. Suppose that M is L -divisible. Let $m \in M$ and $r \in R$ be such that r is not a right zero-divisor. Then $\text{Ann}(r) = 0$ and $0 \in \text{Ann}(m)$, so $\text{Ann}(r) \subseteq \text{Ann}(m)$. Thus, there exists an $x \in M$ such that $m = rx$. Therefore, M is divisible. $\square$

However, the converse may not be true. Of course, they are equivalent statements for integral domains.

Example 3.3.1. *Consider the subring $S = \{0, 2\}$ of $\mathbb{Z}_4$. Since S contains only right-zero divisors, it is certainly true that S is a divisible S -module. However, $\text{Ann}(2) \subseteq \text{Ann}(2)$ and there is no $x \in S$ such that $2 = 2x$, so S is not an L -divisible S -module.*

3.2 Injectivity vs. Divisibility

We now see the relationship between divisibility and injectivity.

Theorem 3.4. *Every injective R -module is divisible.*

Proof. Suppose that M be an injective R -module. Let $m \in M$ and let $r \in R$ be such that r is not a right zero-divisor. Note that Rr is a left ideal in R . Define the well-defined function $f : Rr \rightarrow M$ by $f(tr) = tm$ since r is not a right zero-divisor. Note that for every $s, t \in R$

$$f(tr + sr) = f((t + s)r) = (t + s)m = tm + sm = f(tr) + f(sr)$$

and

$$f(t(sr)) = f((ts)r) = (ts)m = t(sm) = tf(sm),$$

so f is an R -homomorphism. Since M is injective, there is an R -homomorphism $g : R \rightarrow M$ such that $g|_{Rr} = f$. Thus, $m = 1_R m = f(1_R r) = g(1_R r) = rg(1_R)$, where $g(1_R) \in M$. Hence, M is divisible. $\square$

However, not all divisible modules are injective. There are some cases when this is true, see more in Dedekind domains in Chapter 5.

Proposition 3.4.1. *Let R be a principle ideal domain. Then an R -module M is injective if and only if it is divisible.*

Proof. Suppose that M is an injective R -module. Then by Theorem 3.4, M is divisible.

Conversely, suppose that M is a divisible R -module. Let $f : I \rightarrow M$ be an R -homomorphism, where I is a left ideal of R . Since R is a principle ideal ring, $I = Rt$ for some $t \in R$. Since M is divisible, there exists an $m \in M$ such that $f(t) = tm$. Define $g : R \rightarrow M$ by $g(r) = rm$. Then g is an R -homomorphism. Note that for every $st \in Rt$, $g(st) = (st)m = s(tm) = sf(t) = f(st)$. Thus, M is injective. $\square$

Corollary 3.4.1. *An Abelian group is injective if and only if it is divisible.*

Up to this point, we have discussed divisible modules over general rings. We will now explore a more specific type of module - the divisible $\mathbb{Z}$ -module, a.k.a. divisible group.

3.3 Divisible Groups

Remark 3.4.1. We see that for any element x of a divisible group G and for every $0 \neq n \in \mathbb{Z}$, there exists $y \in G$ such that $x = ny$.

There is a special subgroup for us to consider.

Definition 3.4.1. Let G be an Abelian group. Then dG is the subgroup of G generated by all divisible subgroups of G ; i.e., $dG = \sum \{D_\alpha \mid D_\alpha \leq G \text{ is divisible}\}$. We say that G is **reduced** provided that $dG = 0$.

There are important facts to be discussed about the subgroup dG of G .

Theorem 3.5. Let G be an Abelian group. Then

- (i) dG is divisible and contains every divisible subgroup of G .
- (ii) $G = dG \oplus R$ for some reduced subgroup R of G .

Proof. (i) Let $\{D_\lambda\}_{\lambda \in \Lambda}$ be the family of all divisible subgroups of G , so $dG = \sum D_\lambda$. To show that dG is divisible, let $x \in dG$, and let $0 \neq n \in \mathbb{Z}$. Then $x \in \sum D_\lambda$, so $x = \sum d_\lambda$, where for every $\lambda \in \Lambda$, $d_\lambda \in D_\lambda$. So, for all $\lambda \in \Lambda$, since D_λ is divisible, there exists $y_\lambda \in D_\lambda$ such that $d_\lambda = ny_\lambda$. Choose $y = \sum y_\lambda$, so $y \in dG$. Thus,

$$x = \sum d_\lambda = \sum ny_\lambda = n \sum y_\lambda = ny,$$

and therefore, dG is divisible. Clearly, for any divisible subgroup D_{λ_0} of G and $d_{\lambda_0} \in D_{\lambda_0}$, $d_{\lambda_0} = \sum x_\lambda \in dG$, where $x_\lambda = d_{\lambda_0}$ if $\lambda = \lambda_0$ and $x_\lambda = 0$ if $\lambda \neq \lambda_0$. Therefore, dG contains every divisible subgroup of G .

(ii) Since we showed that dG is divisible in part (i), dG is injective by Corollary 3.4.1. Consider the sequence $0 \rightarrow dG \xrightarrow{\iota} G \xrightarrow{\pi} G/dG \rightarrow 0$, where ι is the inclusion mapping and π is the projection mapping. Since ι is a monomorphism, π is an epimorphism, and $\text{Im}(\iota) = \text{Ker}(\pi)$, the sequence is exact. Then because dG is injective, it splits, so there exists a homomorphism $h : G/dG \rightarrow G$ such that $\pi h = 1_{G/dG}$ by Proposition 1.1.3. We will show that G is the direct sum of dG and $h(G/dG)$. Note that $dG + h(G/dG) \subseteq G$, so let

$g \in G$. Then $g + dG = \pi h(g + dG) = h(g + dG) + dG$, which implies that $g - h(g + dG) \in dG$. So, $g = x + h(g + dG)$ for some $x \in dG$. Thus, $G = dG + h(G/dG)$. Now, let $x \in dG \cap h(G/dG)$, so there exists $g + dG \in G/dG$ such that $h(g + dG) = x$. Then $g + dG = \pi h(g + dG) = \pi(x) = x + dG = 0 + dG$, which means that $h(0 + dG) = x$. Therefore, $x = 0$, and $dG \cap h(G/dG) = 0$. Hence, $G = dG \oplus h(G/dG)$.

We will now show that $h(G/dG)$ is reduced. Assume by way of contradiction that S is a nonzero divisible subgroup of $h(G/dG)$. We will show that $h^{-1}(S)$ is a divisible subgroup of $h(G/dG)$. Let $z \in h^{-1}(S)$ and let $n \in \mathbb{Z}$ be nonzero, so there exists $x \in S$ such that $h(z) = x$. Since S is divisible, there exists $y \in S$ such that $x = ny$. So, there exists $w \in h^{-1}(S)$ such that $h(w) = y$ because $y \in h(G/dG)$. Thus, $h(z) = x = ny = nh(w) = h(nw)$. Note that h is one-to-one since for every $a \in \text{Ker}(h)$, $a + dG = \pi h(a + dG) = \pi(0) = 0 + dG$. Therefore, $z = nw$ for some $w \in h^{-1}(S)$. However, this means that G/dG has a divisible subgroup, contradicting that it is reduced. Hence, $h(G/dG)$ is reduced. $\square$

We see that not only is dG a divisible subgroup, but it is the largest divisible subgroup of G .

Remark 3.5.1. *Note that $G \approx H$ if and only if $dG \approx dH$ and $G/dG \approx H/dH$.*

We will consider specific types of isomorphic groups.

Definition 3.5.1. *Let M be an Abelian group. An element $m \in M$ is called a **torsion element** of M provided that there exists $0 \neq r \in R$ such that $rm = 0$. We say that M is **torsion-free** provided that its only torsion element is 0.*

Theorem 3.6. *A torsion-free divisible group G is a vector space over $\mathbb{Q}$, and thus is isomorphic to a direct sum of copies of $\mathbb{Q}$.*

Proof. Let G be a torsion-free divisible group. Note that for every $x \in G$ and $0 \neq n \in \mathbb{Z}$, there exists $y \in G$ such that $x = ny$. Define the function $\mathbb{Q} \times G \rightarrow G$ by $\frac{m}{n}x = my$ for every $\frac{m}{n} \in \mathbb{Q}$ and $x \in G$. We will see that the function is well-defined. Since for any $0 \neq n \in \mathbb{Z}$,

$0 = ny$ for some $y \in G$, y is torsion, and thus, $y = 0$. So, for every $\frac{m}{n} \in \mathbb{Q}$, $\frac{m}{n}0 = m0 = 0$. Therefore, the function is well-defined. A routine check will show that this makes G a vector space over $\mathbb{Q}$. Since every vector space is free, G is isomorphic to a direct sum of copies of $\mathbb{Q}$.

□

Chapter 4

More Injective Modules

An important result shows that every R -module has an injective extension, and we will begin working towards it.

4.1 Embedding Modules

Lemma 4.0.1. *Every Abelian group can be embedded in a divisible Abelian group.*

Proof. Let G be an Abelian group, and choose $F = \bigoplus_{g \in G} \mathbb{Z}$ and $E = \bigoplus_{g \in G} \mathbb{Q}$, where $\mathbb{Q}$ is the additive group of rational numbers. Define the well-defined function $\theta : F \rightarrow G$ by $\theta(\{n_g\}) = \sum n_g g$, where $n_g = 0$ for all but a finite number of $g \in G$. We will first show that θ is a $\mathbb{Z}$ -epimorphism. Note that for every $\{n_g\}, \{m_g\} \in F$ and $z \in \mathbb{Z}$,

$$\begin{aligned}
\theta(\{n_g\} + \{m_g\}) &= \theta(\{n_g + m_g\}) \\
&= \sum_{g \in G} (n_g + m_g)g \\
&= \sum_{g \in G} n_g g + \sum_{g \in G} m_g g \\
&= \sum_{g \in G} n_g g + \sum_{g \in G} m_g g \\
&= \theta(\{n_g\}) + \theta(\{m_g\})
\end{aligned}$$

and

$$\begin{aligned}
\theta(z\{n_g\}) &= \theta(\{zn_g\}) \\
&= \sum_{g \in G} zn_g g \\
&= z \sum_{g \in G} n_g g \\
&= z\theta(\{n_g\}),
\end{aligned}$$

so θ is an $\mathbb{Z}$ -homomorphism. Let $g_0 \in G$, and choose $\{n_g\} \in F$ so that $n_g = 0$ if $g \neq g_0$ and $n_{g_0} = 1$ if $g = g_0$. Thus, $\theta(\{n_g\}) = \sum n_g g = 1g_0 = g_0$, showing that θ is a $\mathbb{Z}$ -epimorphism. Then by Corollary 1.1.1, $\mathbb{Z}$ -isomorphism $G \approx F/Ker(\theta)$.

Note that $F/Ker(\theta)$ is a subgroup of $E/Ker(\theta)$. Now, since $\mathbb{Q}$ is divisible, so is $E = \bigoplus_{g \in G} \mathbb{Q}$ by Theorem 3.2. Thus, $E/Ker(\theta)$ is a divisible Abelian group by Lemma 3.1.1. Because $G \approx F/Ker(\theta) \xrightarrow{\iota} E/Ker(\theta)$, where ι is the inclusion mapping, G can be embedded in a divisible Abelian group. $\square$

Lemma 4.0.2. *Let D be a divisible abelian group. Then $\text{Hom}_{\mathbb{Z}}(R, D)$ is an injective R -module.*

Proof. Recall that $\text{Hom}_{\mathbb{Z}}(R, D)$ is an R -module via $(f + g)(x) = f(x) + g(x)$ and $(rf)(x) = f(xr)$ for every $r, x \in R$ and $f, g \in \text{Hom}_{\mathbb{Z}}(R, D)$. Let $f : I \rightarrow \text{Hom}_{\mathbb{Z}}(R, D)$ be an R -homomorphism, where I is a left ideal of R . Define the well-defined function $h : I \rightarrow D$ by $h(a) = [f(a)](1_R)$. Also, note that for every $a, b \in I$ and $z \in \mathbb{Z}$,

$$\begin{aligned} h(a + b) &= [f(a + b)](1_R) \\ &= [f(a)](1_R) + [f(b)](1_R) \\ &= h(a) + h(b), \end{aligned}$$

so h is an $\mathbb{Z}$ -homomorphism, a.k.a. a group homomorphism. Since D is an injective $\mathbb{Z}$ -module by Corollary 3.4.1, there exists a $\mathbb{Z}$ -homomorphism $\ell : R \rightarrow D$, so $\ell|_I = h$. Now, let $g : R \rightarrow \text{Hom}_{\mathbb{Z}}(R, D)$ be the well-defined function given by $[g(r)](x) = \ell(xr)$ for every $x \in R$, where $g(r) \in \text{Hom}_{\mathbb{Z}}(R, D)$. Note that for every $a, b \in R$, $g(a + b) = g(a) + g(b)$ since for every $x \in R$,

$$\begin{aligned} [g(a + b)](x) &= \ell(x[a + b]) \\ &= \ell(xa + xb) \\ &= \ell(xa) + \ell(xb) \\ &= [g(a)](x) + [g(b)](x) \end{aligned}$$

and $g(ab) = ag(b)$ since for every $x \in R$,

$$\begin{aligned} [g(ab)](x) &= \ell(x[ab]) \\ &= \ell([xa]b) \\ &= [g(b)](xa) \\ &= a[g(b)](x). \end{aligned}$$

Thus, g is an R -homomorphism. Lastly, observe that $g(r) = f(r)$ for every $r \in I$ since for every $x \in R$,

$$[g(r)](x) = \ell(xr) = h(xr) = [f(xr)](1_R) = x[f(r)](1_R) = [f(r)](x).$$

Hence, $\text{Hom}_{\mathbb{Z}}(R, D)$ is an injective R -module. □

Corollary 4.0.1. *Let M be an injective Abelian group. Then $\text{Hom}_{\mathbb{Z}}(R, M)$ is an injective R -module.*

Proposition 4.0.1. *Every R -module can be embedded in an injective R -module.*

Proof. Let M be an R -module. Since M is an abelian group, there exists a $\mathbb{Z}$ -monomorphism $f : M \rightarrow D$ for some divisible abelian group D by Lemma 4.0.1. Define the well-defined function $g : M \rightarrow \text{Hom}_{\mathbb{Z}}(R, D)$ by $[g(m)](r) = f(rm)$, where $g(m) \in \text{Hom}_{\mathbb{Z}}(R, D)$. Note that for every $a, b \in M$, $g(a + b) = g(a) + g(b)$ since for every $x \in R$,

$$\begin{aligned} [g(a + b)](x) &= f(x(a + b)) \\ &= f(xa + xb) \\ &= f(xa) + f(xb) \\ &= [g(a)](x) + [g(b)](x) \\ &= [g(a) + g(b)](x). \end{aligned}$$

Also, observe that for every $m \in M$ and $r \in R$, $g(rm) = rg(m)$ since for every $x \in R$,

$$\begin{aligned} [g(rm)](x) &= f(x[rm]) \\ &= f([xr]m) \\ &= [g(m)](xr) \\ &= r[g(m)](x). \end{aligned}$$

Thus, g is an R -homomorphism. Finally, to show that g is one-to-one, let $m \in \text{Ker}(g)$. In particular, $0 = 0(1_R) = [g(m)](1_R) = f(1_R m) = f(m)$, so $m \in \text{Ker}(f)$. Since f is a $\mathbb{Z}$ -monomorphism, $m = 0$. Therefore, g is an R -monomorphism. Hence, because $\text{Hom}_{\mathbb{Z}}(R, D)$ is injective by Lemma 4.0.1, M can be embedded in an injective R -module. $\square$

Theorem 4.1. *Every R -module has an injective extension.*

Proof. Let M be an R -module. Then by Proposition 4.0.1, there is an injective R -module J such that a function $f : M \rightarrow J$ is an R -monomorphism. By Proposition 1.1.1, there is an R -module extension B of M that is R -isomorphic to J . Thus, B is an injective extension of M . $\square$

We will now revisit Theorem 2.3 and prove it again using injective extensions.

Theorem 4.2. *Let M be an R -module. Then the following statements are equivalent:*

- (i) M is injective.
- (ii) Every exact sequence $0 \rightarrow M \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is split; in particular, $B \approx M \oplus C$.
- (iii) M is a direct summand of every extension of itself.

Proof. Let M be an R -module.

(i) $\Rightarrow$ (ii) Suppose that M is injective. Let $0 \rightarrow M \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an exact sequence of R -modules. We will show that it is split. Consider the R -homomorphism $1_M : M \rightarrow M$, the identity function of M . Since M is injective, there exists an R -homomorphism $k : B \rightarrow M$ so that $kf = 1_M$. By Lemma 1.1.3, the sequence is split.

(ii) $\Rightarrow$ (iii) Secondly, assume that every exact sequence $0 \rightarrow M \rightarrow B \rightarrow C \rightarrow 0$ is split, so $B \approx M \oplus C$. Let D be an extension of M . We will show that M is a direct summand of D . Consider the sequence $0 \rightarrow M \xrightarrow{\iota} D \xrightarrow{\pi} D/M \rightarrow 0$, where ι is the inclusion mapping and π is the projection mapping. Since ι is an R -monomorphism, π is an R -epimorphism, and $\text{Im}(\iota) = \text{Ker}(\pi)$, the sequence is exact. Then, by assumption, it splits, so there exists an R -homomorphism $h : D/M \rightarrow D$ such that $\pi h = 1_{D/M}$ by Proposition 1.1.3. We will show that D is the direct sum of M and $h(D/M)$. Note that $M + h(D/M) \subseteq D$, so let $d \in D$. Then $d + M = \pi h(d + M) = h(d + M) + M$, which implies that $d - h(d + M) \in M$. So, $d = m + h(d + M)$ for some $m \in M$. Thus, $D = M + h(D/M)$. Now, let $x \in M \cap h(D/M)$, so there exists $d + M \in D/M$ such that $h(d + M) = x$. Then $d + M = \pi h(d + M) = \pi(x) = x + M = 0 + M$, which means that $h(0 + M) = x$. Therefore, $x = 0$, and $M \cap h(D/M) = 0$. Hence, M is a direct summand of every extension of itself.

(iii) $\Rightarrow$ (i) Finally, suppose that M is a direct summand of every extension. By Theorem 4.1, M has an injective extension Q . Then by assumption, Q is the direct sum of M and some submodule X of M . Thus, M is injective by Corollary 2.4.1. $\square$

4.2 Injective Envelopes

Definition 4.2.1. Let M be a submodule of an R -module B . We say that B is an **essential extension** of M provided that for every nonzero submodule S of B , $M \cap S \neq 0$.

We will now see an equivalent statement to essential extensions.

Proposition 4.2.1. Let M be a submodule of an R -module B . Then B is an essential extension of M if and only if for every $0 \neq b \in B$, there exists an $r \in R$ such that $0 \neq rb \in M$.

Proof. Suppose that B is an essential extension of M . Let $0 \neq b \in B$. Then the left principle ideal $(b) = \{rb \mid r \in R\}$ is a submodule of B , so $(b) \cap M \neq 0$. Thus, there exists an $r \in R$ such that $0 \neq rb \in M$.

Conversely, suppose that for every $0 \neq b \in B$, there exists an $r \in R$ such that $0 \neq rb \in M$. Let S be a nonzero submodule of B . Then there exists $0 \neq b \in S \subseteq B$. So, there is an $r \in R$

such that $0 \neq rb \in M$. Since S is an R -module, $0 \neq rb \in S$. Thus, $M \cap S \neq 0$. Therefore, B is an essential extension of M . $\square$

Example 4.2.1.

- (1) Every R -module M is an essential extension of itself since for every nonzero submodule S of M , $M \cap S = S \neq 0$.
- (2) Note that the $\mathbb{Z}$ -module $\mathbb{Q}$ is an essential extension of $\mathbb{Z}$ since for every $0 \neq q \in \mathbb{Q}$, where $q = \frac{a}{b}$ for some $0 \neq a, b \in \mathbb{Z}$, $0 \neq bq = a \in \mathbb{Z}$.
- (3) Note that the $\mathbb{Z}$ -module $\mathbb{R}$ is not an essential extension of $\mathbb{Q}$ because there is no $z \in \mathbb{Z}$ with $0 \neq z\sqrt{2} \in \mathbb{Q}$.

There is an important fact to know about an essential extension of an R -module with an intermediate R -module.

Lemma 4.2.1. *Let A, B, C be R -modules such that $A \subseteq B \subseteq C$. Then C is an essential extension of A if and only if C is an essential extension of B and B is an essential extension of A .*

Proof. Suppose that C is an essential extension of A . Firstly, let $0 \neq c \in C$. Then there exists an $r \in R$ such that $0 \neq rc \in A \subseteq B$. Thus, C is an essential extension of B by Proposition 4.2.1. Secondly, let $0 \neq b \in B \subseteq C$. Then there exists an $s \in R$ such that $0 \neq sb \in A$. Thus, B is an essential extension of A by Proposition 4.2.1.

Conversely, suppose that C is an essential extension of B and B is an essential extension of A . Let $0 \neq c \in C$. Then there exists an $s \in R$ such that $0 \neq sc \in B$. So, there exists a $t \in R$ such that $0 \neq t(sc) \in A$. Choose $r = ts$. Then there exists an $r \in R$ such that $0 \neq rc \in A$. Thus, C is an essential extension of A by Proposition 4.2.1. $\square$

We will now see how R -monomorphisms preserve essential extensions.

Lemma 4.2.2. *Let B be an essential extension of an R -module M , and let $h : B \rightarrow C$ be an R -monomorphism between R -modules. Then $h(B)$ is an essential extension of $h(M)$.*

Proof. Let $0 \neq h(b) \in h(B)$. Since h is well-defined, $0 \neq b \in B$. Because B is an essential extension of M , there exists an $r \in R$ such that $0 \neq rb \in M$. Then $0 \neq h(rb) \in h(M)$ since h is a R -monomorphism. Thus, there exists $r \in R$ such that $0 \neq rh(b) \in h(M)$. Hence, $h(B)$ is an essential extension of $h(M)$. $\square$

There is a connection between injectivity and essential extensions.

Theorem 4.3. *Let M be an R -module. Then M is injective if and only if M has no proper essential extension.*

Proof. Suppose that M is injective. Let B be a proper extension of M . Then by Theorem 4.2, $B = M \oplus X$, where $M \cap X = 0$ for some submodule $X \subseteq B$. Thus, M has no proper essential extension.

Conversely, suppose that M has no proper essential extension. Let B be an extension of M . If $B = M$, then M is a direct summand of B so that M is injective. Then, assume that B is a proper extension of M . Since B is not an essential extension of M , there exists a submodule X of B , so $X \cap M = 0$. Define $\Omega = \{0 \neq X \subseteq B \mid X \cap M = 0\}$. Note that Ω is nonempty. Also, observe that for every subset $\{X_\lambda\}_{\lambda \in \Lambda}$ of Ω , $\bigcup X_\lambda$ contains X_λ for every $\lambda \in \Lambda$, and $\bigcup X_\lambda \cap M = \bigcup (X_\lambda \cap M) = 0$. So, $\bigcup X_\lambda \in \Omega$ is an upper bound of $\{X_\lambda\}_{\lambda \in \Lambda}$. Thus, Ω has a maximal member X_0 by Zorn's Lemma. Since $M \cap X_0 = 0$, we now must show that $B = M + X_0$. Assume by way of contradiction that $B \neq M + X_0$, so $M + X_0 \subset B$. Note that by Lemma 4.2.1, B is not an essential extension of $M + X_0$ since it is not an essential extension of M . Then B/X_0 is not an essential extension of $(M + X_0)/X_0$. So, there exists $A \subseteq B$ such that $X_0 \subset A$ and $A/X_0 \cap (M + X_0)/X_0 = 0$. Thus, $A \cap (M + X_0) = X_0$, showing that $A \cap M \subseteq A \cap (M + X_0) = X_0$. Since $A \cap M = M \cap (A \cap M) \subseteq M \cap X_0 = 0$, $A \cap M = 0$ and $X_0 \subset A \in \Omega$. However, this contradicts X_0 being the maximal element of Ω . Therefore, $B = M \oplus X_0$ and hence, M is injective by Theorem 4.2. $\square$

Once again, we will illustrate the connection between essential extensions of R -modules and their respective products and direct sums.

Theorem 4.4. *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of R -modules. Assume that B_λ is an essential extension of M_λ for every $\lambda \in \Lambda$. Then $\bigoplus_{\lambda \in \Lambda} B_\lambda$ is an essential extension of $\bigoplus_{\lambda \in \Lambda} M_\lambda$.*

Proof. Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ and $B = \bigoplus_{\lambda \in \Lambda} B_\lambda$. Let $0 \neq b \in B$. There exists a finite number $1 \leq i \leq n$ such that $0 \neq b_i \in B_i$. Consider $0 \neq b_1 \in B_1$. Since B_1 is an essential extension of M_1 , there is $r_1 \in R$ such that $0 \neq r_1 b_1 \in M_1$. Then $0 \neq r_1 b \in B$. If $r_1 b$ is the only nonzero component of $r_1 b$, then $0 \neq r_1 b \in M$ and we are done. Suppose that this is not the case. Then there exists $m \leq n$ such that for each b_i with $2 \leq i \leq n$, we renumber them b_j with $1 \leq j \leq m$ where $0 \neq r_1 b_j$. Choose $0 \neq r_1 b_x \in B_x$ for some $1 \leq x \leq m$. Since B_x is an essential extension of M_x , there exists $r_x \in R$ such that $0 \neq r_x r_1 b_x \in M_x$. Then $0 \neq r_x r_1 b \in B$. Again, if $r_x r_1 b$ is the only nonzero component of $r_x r_1 b$, then $0 \neq r_x r_1 b \in M$. Otherwise, we will continue our argument process. Observe that in the end, we will have an $r \in R$ such that $0 \neq rb \in M$. Hence, B is an essential extension of M . $\square$

There is a special type of essential extension.

Definition 4.4.1. *An R -module B is an **injective envelope** for M provided that it is an injective essential extension of M ; we typically denote an injective essential extension of M by $E(M)$.*

Example 4.4.1.

- (1) *Note that the $\mathbb{Z}$ -module $\mathbb{Q}$ is an injective envelope of $\mathbb{Z}$ because it is injective and is an essential extension of $\mathbb{Z}$.*
- (2) *Note that the $\mathbb{Z}$ -module $\mathbb{R}$ is not an injective envelope of $\mathbb{Q}$ because it is not an essential extension of $\mathbb{Q}$.*

There are two major theorems connected to injective envelopes.

Theorem 4.5. *Let M be an R -module, and let B be an essential extension of M . Then B is an injective envelope for M if and only if C is not an essential extension of M for every proper extension C of B .*

Proof. Suppose that B is an injective envelope for M . Let C be an arbitrary proper extension of B . Since B is injective, there exists $0 \neq X \subseteq C$ such that $C = B \oplus X$ where $B \cap X = 0$ by Theorem 4.2. Then

$$X \cap M = X \cap (B \cap M) = (X \cap B) \cap M = 0 \cap M = 0.$$

Thus, for every proper extension C of B , C is not an essential extension of M .

Conversely, suppose that for every proper extension C of B , C is not an essential extension of M . Since every C is not an essential extension of M and B is an essential extension of M , it must be the case that C is not an essential extension of B by Lemma 4.2.1. So, B has no proper essential extension. Thus, B is injective by Theorem 4.3 and therefore is an injective envelope for M . $\square$

Remark 4.5.1. *We refer to this result as a maximal essential extension.*

This lemma is instrumental in the second major theorem.

Lemma 4.5.1. *Let B be an injective extension of the R -module M . Then there exists a submodule X of B such that X is an injective envelope of M .*

Proof. Let $\Omega = \{C \subseteq B \mid C \text{ is an essential extension of } M\}$. Note that Ω is nonempty since $M \in \Omega$. Using inclusion as partial order on Ω , there exists a maximal member X by Zorn's Lemma such that $M \subseteq X \subseteq B$. If $X = B$, then X is an injective envelope of M and we are done. Otherwise, $M \subseteq X \subset B$. Let D be an essential extension of X . Then since B is injective, there exists an R -homomorphism $\theta : D \rightarrow B$ such that $\theta(x) = x$ for every $x \in X$. By the way θ is defined, $X \cap \text{Ker}(\theta) = 0$. Because D is an essential extension of X , it must be the case that $\text{Ker}(\theta) = 0$, so θ is an R -monomorphism. Then, by Lemma 4.2.2, $\theta(D)$ is an essential extension of $\theta(X) = X$. Thus, $\theta(D)$ is an essential extension of M by Lemma 4.2.1, so $X \subseteq \theta(D) \in \Omega$. Since X is the maximal member of Ω , $\theta(D) = X = \theta(X)$, so $D = X$. Therefore, for every proper extension C of X , C is not an essential extension of M . Hence, by Theorem 4.5, there exists a submodule X of B such that X is an injective envelope of M . $\square$

Theorem 4.6. *Let M be a submodule of an injective R -module B . Then B is an injective envelope for M if and only if N is not injective for every proper submodule N of B containing M .*

Proof. Suppose that B is an injective envelope for M . Let N be a proper submodule of B such that $M \subseteq N$. Since B is an essential extension of M , B is an essential extension of N by Lemma 4.2.1. Thus, N has a proper essential extension, and therefore, N is not injective by Theorem 4.3.

Conversely, suppose that for every proper submodule N of B containing M , N is not injective. Since B is an injective extension of M , there exists a submodule X of B such that X is an injective envelope of M by Lemma 4.5.1. However, by the assumption, this means that $X = B$. Hence, B is an injective envelope for M . $\square$

Remark 4.6.1. *We refer to this result as a minimal injective extension.*

We will see that all injective envelopes of an R -module are R -isomorphic.

Lemma 4.6.1. *Let B_1, B_2 be injective envelopes of an R -module M . Then there exists an R -isomorphism $\theta : B_1 \rightarrow B_2$ such that $\theta(m) = m$ for every $m \in M$.*

Proof. Since B_2 is injective, there exists an R -homomorphism $\theta : B_1 \rightarrow B_2$ such that $\theta(m) = m$ for every $m \in M$. We will show that θ is one-to-one. Notice that $\text{Ker}(\theta)$ is a submodule of B_1 . Let $q \in M \cap \text{Ker}(\theta)$. So, $\theta(q) = q$ since $q \in M$. Of course, because $q \in \text{Ker}(\theta)$, $\theta(q) = 0$, showing that $q = 0$. Thus, $M \cap \text{Ker}(\theta) = 0$. Since B_1 is an essential extension of M , this is only possible if $\text{Ker}(\theta) = 0$. To show that θ is onto, assume by way of contradiction that there exists $y \notin \text{Im}(\theta)$, so $\text{Im}(\theta)$ is a proper submodule of B_2 . By Theorem 4.6, $\text{Im}(\theta)$ is not injective. However, this is a contradiction since B_1 is injective and $B_1 \approx \text{Im}(\theta)$. Hence, θ is an R -isomorphism such that $\theta(m) = m$ for every $m \in M$. $\square$

We can now expand upon Theorem 4.1.

Proposition 4.6.1. *Every R -module has an injective envelope.*

Proof. Let M be an R -module. By Proposition 4.1, M has an injective extension B . Then by Lemma 4.5.1, there exists a submodule X of B such that X is an injective envelope of M . Hence, M has an injective envelope. $\square$

We will show that module can be its own injective envelope.

Proposition 4.6.2. *Let M be an R -module. Then M is injective if and only if M is its own injective envelope; i.e., $E(M) = M$.*

Proof. By Proposition 4.6.1, M has an injective envelope $E(M)$.

Suppose that M is injective. If M is a proper submodule of $E(M)$, then $M \subsetneq M \subset E(M)$. So, M is not injective by Theorem 4.6. However, because M is injective, M is not a proper submodule of $E(M)$. Thus, $E(M) = M$.

Conversely, suppose that $E(M) = M$. Since $E(M)$ is injective, M is injective. $\square$

There is a connection between injective envelopes and other extensions of modules.

Proposition 4.6.3. *Let B be an extension of an R -module M with an injective envelope $E(B)$. Then*

- (i) *M has an injective envelope $E(M)$ which is a direct summand of $E(B)$.*
- (ii) *If B is an essential extension of M , then $E(M)$ is an injective envelope of B .*

Proof.

(i) Since $E(B)$ is an injective extension of M , there exists an injective envelope $E(M)$ of M which is a submodule of $E(B)$ by Lemma 4.5.1. Because $E(M)$ is injective, $E(M)$ is a direct summand of X by Theorem 4.2.

(ii) Assume that B is an essential extension of M . Then $E(B)$ is an essential extension of M by Lemma 4.2.1. Since $E(B)$ is injective, $E(B) = E(M)$. Hence, we have proven that $E(M)$ is an injective envelope of B . $\square$

There is a connection between simple modules and other modules. Recall that a simple module is defined as having only two submodules, namely 0 and itself.

Proposition 4.6.4. *Let M be an R -module and let $0 \neq m \in M$. Then there is a simple module S and an R -homomorphism $\psi : M \rightarrow E(S)$ such that $\psi(m) \neq 0$.*

Proof. Consider the left ideal $X = \{r \in R \mid rm = 0\}$ of R . Since $m \neq 0$, X is a proper left ideal of R so that it is contained in a maximal left ideal N of R . Define the well-defined function $\theta : Rm \rightarrow R/N$ by $\theta(rm) = r + N$. Then θ is an R -homomorphism. Since R/N is a simple R -module, let $S = R/N$. Observe that $Rm \rightarrow M$ is the inclusion mapping and that $Rm \xrightarrow{\theta} S \rightarrow E(S)$ is an R -homomorphism. So, because $E(S)$ is injective, there exists an R -homomorphism $\psi : M \rightarrow E(S)$ such that $\psi(rm) = \theta(rm)$. Thus, $\psi(m) = \theta(m) = 1 + N \neq 0 + N$. $\square$

We now will discuss the relationship between injective envelopes of R -modules and their respective products and direct sums.

Theorem 4.7. *Let $\{M_i\}_{i=1}^n$ be a finite family of R -modules. Then $\bigoplus_{i=1}^n E(M_i)$ is an injective envelope of $\bigoplus_{i=1}^n M_i$.*

Proof. Note that $\bigoplus_{i=1}^n E(M_i)$ is injective by Corollary 4.2 since $E(M_i)$ is injective for every $1 \leq i \leq n$. Also, $\bigoplus_{i=1}^n E(M_i)$ is an essential extension of $\bigoplus_{i=1}^n M_i$ by Theorem 4.4. Thus, $\bigoplus_{i=1}^n E(M_i)$ is an injective envelope of $\bigoplus_{i=1}^n M_i$. $\square$

4.3 Indecomposable Injective Modules

We will now look at two new concepts and how they relate to injectivity.

Definition 4.7.1. *An R -module M is said to be **indecomposable** provided that*

- (i) $M \neq 0$;
- (ii) *the only direct summands of M are 0 and M .*

Example 4.7.1.

- (1) *Every simple module is indecomposable.*
- (2) *Note that $\mathbb{Q}$ is an indecomposable $\mathbb{Z}$ -module.*

Definition 4.7.2. Let A be a submodule of an R -module M . We say that A is an **irreducible submodule** of M provided that

- (i) $A \neq M$;
- (ii) there do not exist submodules A_1, A_2 of M with a proper submodule A such that $A_1 \cap A_2 = A$.

Example 4.7.2.

- (1) Note that for any positive integer n and prime p , $p^n\mathbb{Z}$ is an irreducible submodule of the $\mathbb{Z}$ -module $\mathbb{Z}$.
- (2) Observe that $n\mathbb{Z}$ is not an irreducible submodule of the $\mathbb{Z}$ -module $\mathbb{Z}$ for any positive integer n , where n is the product of at least 2 distinct primes since if $n = \ell k$ where $\ell = p_1^{d_1} \cdots p_r^{d_r}$ and $k = q_1^{e_1} \cdots q_t^{e_t}$ for distinct primes $p_1, \dots, p_r, q_1, \dots, q_t$, then $\ell\mathbb{Z}$ and $k\mathbb{Z}$ are submodules of $\mathbb{Z}$ properly containing $n\mathbb{Z}$ such that $\ell\mathbb{Z} \cap k\mathbb{Z} = n\mathbb{Z}$.

There is a connection between indecomposable and irreducible modules.

Proposition 4.7.1. Let M be an injective R -module. Then the following statements are equivalent:

- (i) M is indecomposable;
- (ii) $M \neq 0$ and is an injective envelope of every nonzero submodule of itself;
- (iii) 0 is an irreducible submodule of M .

Proof. (i) $\Rightarrow$ (ii) Suppose that M is indecomposable. Then $M \neq 0$. Let A be a nonzero submodule of M . Since M is injective, there exists a submodule X of M such that X is an injective envelope of A by Lemma 4.5.1. Since M is an extension of an injective X , X is a direct summand of M by Theorem 4.2. However, since M is indecomposable and $A \neq 0$, X must be equal to M , so M is an injective envelope of A . Thus, $M \neq 0$ and is an injective envelope of every nonzero submodule of itself.

(ii) $\Rightarrow$ (iii) Now, suppose that $M \neq 0$ and is an injective envelope of every non-zero submodule of itself. Let M_1, M_2 be nonzero submodules of M . Since M is an essential extension of M_1 , $M_1 \cap M_2 \neq 0$. Thus, 0 is an irreducible submodule of M .

(iii) $\Rightarrow$ (i) Lastly, suppose that 0 is an irreducible submodule of M . Then $M \neq 0$. Let M_1, M_2 be nonzero submodules of M such that $M = M_1 \oplus M_2$, where $M_1 \cap M_2 = 0$. Since 0 is irreducible in M , this is only possible if $M_1 = 0$ or $M_2 = 0$. So, the only direct summands of M are 0 and M itself. Thus, M is indecomposable. $\square$

There are several corollaries that can be derived from this proposition.

Corollary 4.7.1. *Let M be an R -module. Then*

(i) *$E(M)$ is indecomposable if and only if 0 is irreducible in M .*

(ii) *a submodule A of M is irreducible if and only if $E(M/A)$ is indecomposable.*

Proof. (i) Suppose that $E(M)$ is indecomposable. Since $E(M)$ is injective, 0 is an irreducible submodule of $E(M)$ by Proposition 4.7.1. Note that $M \neq 0$ since $E(M) \neq 0$. Let M_1, M_2 be nonzero submodules of M . Then M_1, M_2 are submodules of $E(M)$ and $E(M)$ is an essential extension of M_1 , so $M_1 \cap M_2 \neq 0$. Thus, 0 is irreducible in M .

Conversely, suppose that 0 is irreducible in M . We will show that 0 is irreducible in $E(M)$. Note that $E(M) \neq 0$ since $M \neq 0$. Now, let M_1, M_2 be nonzero submodules of $E(M)$. Assume by way of contradiction that $M_1 \cap M_2 = 0$. Since $E(M)$ is an essential extension of M , $M_1 \cap M \neq 0$ and $M_2 \cap M \neq 0$. Since $M_1 \cap M, M_2 \cap M$ are submodules of M , $(M_1 \cap M) \cap (M_2 \cap M) \neq 0$ because 0 is irreducible in M . However, since $M_1 \cap M_2 = 0$ by assumption,

$$(M_1 \cap M) \cap (M_2 \cap M) = (M_1 \cap M_2) \cap M = 0 \cap M = 0,$$

resulting in a contradiction. Thus, $M_1 \cap M_2 \neq 0$ so that 0 is irreducible in $E(M)$. Since $E(M)$ is injective, $E(M)$ is indecomposable by Proposition 4.7.1.

(ii) We will first show that 0 is irreducible in M/A if and only if A is irreducible in M . Note that $M \neq A$ if and only if $M/A \neq 0$. Then M_1, M_2 are submodules of M properly containing A if and only if $M_1/A, M_2/A$ are nonzero submodules of M/A . Since $M_1 \cap M_2 \neq A$ if and only if $(M_1/A) \cap (M_2/A) \neq 0$, 0 is irreducible in M/A if and only if A is irreducible in M .

Now, observe by part (i) that a submodule A of M is irreducible if and only if 0 is irreducible in M/A , if and only if $E(M/A)$ is indecomposable. $\square$

Corollary 4.7.2. *Let S be a simple R -module. Then $E(S)$ is indecomposable.*

Proof. Since 0 is an irreducible submodule of S , $E(S)$ is indecomposable by Corollary 4.7.1. $\square$

Corollary 4.7.3. *Let M be an indecomposable injective R -module, and let $0 \neq m \in M$. Then $M \approx E(R/\text{Ann}(m))$ and $\text{Ann}(m)$ is an irreducible left ideal of R .*

Proof. Note that $Rm \approx R/\text{Ann}(m)$ is an R -isomorphism defined by $rm \mapsto r + \text{Ann}(m)$. Since Rm is a nonzero submodule of M , $M = E(Rm) \approx E(R/\text{Ann}(m))$ by Proposition 4.7.1. Then $E(R/\text{Ann}(m))$ is indecomposable because M is. Thus, by Corollary 4.7.1, $\text{Ann}(m)$ is an irreducible left ideal of R . $\square$

Corollary 4.7.4. *Let M be an R -module. Then M is an indecomposable injective R -module if and only if there exists an irreducible left ideal I of R such that $M \approx E(R/I)$.*

Proof. Suppose that M is an indecomposable injective R -module. Then by Corollary 4.7.3, $\text{Ann}(m)$ is an irreducible left ideal of R such that $M \approx E(R/\text{Ann}(m))$.

Conversely, suppose that there is an irreducible left ideal I of R such that $M \approx E(R/I)$. Since I is an irreducible submodule of the R -module R , $E(R/I)$ is an indecomposable R -module by Corollary 4.7.1. Thus, M is an indecomposable injective R -module. $\square$

4.4 Prime Ideals

We will now consider a special type of ideal.

Definition 4.7.3. *Let P be an ideal of a commutative ring R . We say that P is a **prime ideal** of R provided that*

- (i) $P \neq R$;
- (ii) for every $\alpha, \beta \in R$, if $\alpha\beta \in P$, then either $\alpha \in P$ or $\beta \in P$.

Remark 4.7.1.

(1) Let P be a prime ideal of a commutative ring R , and let $r_1, \dots, r_n \in R$ for some $1 \leq n \in \mathbb{Z}$. If $r_1 \cdots r_n \in P$, then $r_i \in P$ for some $1 \leq i \leq n$. In particular, for any $r \in R$, if $r^m \in P$ for some $1 \leq m \in \mathbb{Z}$, then $r \in P$.

(2) Let P be a prime ideal of a commutative ring R , and let I, J be ideals of R such that $IJ \subseteq P$. Then either $I \subseteq P$ or $J \subseteq P$.

There is a connection between prime ideals and indecomposable modules.

Proposition 4.7.2. *Let P be a prime ideal of a commutative ring R . Then P is an irreducible submodule of R , and $E(R/P)$ is indecomposable.*

Proof. To show that P is irreducible in R , note that $P \neq R$. Now, let I_1, I_2 be submodules of the R -module R that contain P and $I_1 \cap I_2 = P$. Since $I_1 I_2 \subseteq I_1 \cap I_2 = P$, either $I_1 \subseteq P$ or $I_2 \subseteq P$. Then either $I_1 = P$ or $I_2 = P$. Thus, P is irreducible in R . By Corollary 4.7.1, this means that $E(R/P)$ is indecomposable. $\square$

We will now see the relationship between prime ideals and annihilator modules.

Lemma 4.7.1. *Let P be a prime ideal of a commutative ring R . Then the collection of all annihilators of nonzero elements of the R -module $E(R/P)$ has a unique maximal member, namely P itself.*

Proof. Let Ω be the collection of all $\text{Ann}(m + P)$, where $m + P \in E(R/P)$ is nonzero. Note that since P is a proper ideal of R , there exists a nonzero $x + P \in R/P \subseteq E(R/P)$, so Ω is nonempty. With an order of subsets, Ω has a maximal member $\text{Ann}(m_0)$ by Zorn's Lemma. Since $E(R/P)$ is an essential extension of R/P , there exists $r \in R$ such that $r(m_0 + P) = (rm_0) + P \in R/P$ is nonzero by Proposition 4.2.1.

We claim that $\text{Ann}(rm_0 + P) = P$. Let $x \in \text{Ann}(rm_0 + P)$, so $x(rm_0 + P) = xrm_0 + P = 0 + P$. Then $x(rm_0) \in P$. Since P is prime and $rm_0 \notin P$, $x \in P$, so $\text{Ann}(rm_0 + P) \subseteq P$. Now, let $x \in P$, so $xrm_0 \in P$. Then $xrm_0 + P = x(rm_0) + P = 0 + P$. So, $x \in \text{Ann}(rm_0 + P)$, and $P \subseteq \text{Ann}(rm_0 + P)$. Thus, $\text{Ann}(rm_0 + P) = P$.

Since $\text{Ann}(m_0 + P) \subseteq \text{Ann}(rm_0 + P) = P$ and $\text{Ann}(m_0 + P)$ is maximal, $\text{Ann}(m_0 + P) = P$. Thus, P is the unique maximal member. $\square$

Corollary 4.7.5. *Let P_1 and P_2 be prime ideals of a commutative ring R , where $E(R/P_1) \approx E(R/P_2)$. Then $P_1 = P_2$.*

Chapter 5

Injective Modules Over Various Rings

We will now discuss how different rings affect injectivity.

5.1 Noetherian

Definition 5.0.4. *Let M be an R -module. We say that M is a **Noetherian module** provided that every ascending chain of submodules $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ terminates, i.e., there exists a positive integer n such that $K_n = K_m$ for every $m \geq n$. We also say that R is a **Noetherian ring** provided that it is Noetherian as an R -module.*

There are equivalent statements for Noetherian modules.

Proposition 5.0.3. *Suppose that M is an R -module. Then M is Noetherian if and only if every nonempty collection Ω of differing submodules of M possesses a maximal member, i.e., there is a submodule $K_0 \in \Omega$ such that for every $K \in \Omega$, $K_0 \not\subset K$.*

Proof. Suppose that M is Noetherian. Assume by way of contradiction that there exists a nonempty collection Ω of differing submodules of M which doesn't possess a maximal member. Then any member $K_1 \in \Omega$ is not the maximal member, so there exists $K_2 \in \Omega$ such that $K_1 \subset K_2$. Since K_2 is not the maximal member, there exists $K_3 \in \Omega$ such that $K_2 \subset K_3$. Continuing this way, we get an ascending chain of submodules $K_1 \subset K_2 \subset K_3 \subset \cdots$.

Because this ascending chain doesn't terminate, M cannot be Noetherian; a contradiction. Thus, every nonempty collection Ω of submodules of M possesses a maximal member.

Conversely, suppose that every nonempty collection Ω of differing submodules of M possesses a maximal member. Assume by way of contradiction that there exists an infinite strictly ascending chain $K_1 \subset K_2 \subset K_3 \subset \cdots$ of submodules of M . Define $\Omega = \{K_1, K_2, K_3, \dots\}$. By assumption, there is a maximal member $K_n \in \Omega$ for some positive integer n such that for every $K_m \in \Omega$, $K_n \not\subset K_m$. However, since $n+1 \neq n$ and $K_n \subset K_{n+1}$, this creates a contradiction. Thus, M is Noetherian. $\square$

Here is a second equivalent statement based on finitely generated submodules.

Proposition 5.0.4. *Let M be an R -module. Then M is Noetherian if and only if every submodule of M is finitely generated.*

Proof. Suppose that M is Noetherian. Let S be a submodule of M . Define Ω to be the collection of finitely generated submodules of S . Since $0 \in \Omega$, Ω is nonempty. Since M is Noetherian and each submodule of S is a submodule of M , there exists a maximal member $K_0 \in \Omega$ such that for every $K \in \Omega$, $K_0 \not\subset K$. Since K_0 is finitely generated, let $K_0 = \sum_{i=1}^n Rk_i$ for some positive integer n . Assume by way of contradiction that $K_0 \neq S$. So, there exists $k \in S$ such that $k \notin K_0$. Then $K_0 + Rk = Rk_1 + \cdots + Rk_n + Rk$ is a finitely generated submodule of S , so $K_0 + Rk \in \Omega$. However since $K_0 \subset K_0 + Rk$, this creates a contradiction with K_0 being the maximal member of Ω . Thus, $S = K_0$. Therefore, every submodule of M is finitely generated.

Conversely, every submodule of M is finitely generated. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be an ascending chain of submodules of M . Define $K = \bigcup_{i=1}^{\infty} K_i$. Then K is a submodule of M , so K is finitely generated. Thus, $K = Rk_1 + \cdots + Rk_r$ for some positive integer r . Since $k_j \in K_i$ for some $1 \leq j \leq r$ and positive integer i , there exists an n such that $k_1, \dots, k_r \in K_n$. Then, $K = K_n$ so that $K_m = K_n$ for every $m \geq n$. Therefore, M is Noetherian. $\square$

There are several simple corollaries that come from this proposition.

Corollary 5.0.6. *Every simple R -module is Noetherian.*

Proof. Let S be a simple R -module. Since 0 and S are the only submodules of S and they are finitely generated, S is Noetherian by Proposition 5.0.4. $\square$

Corollary 5.0.7. *Let I be an ideal of a Noetherian ring R . Then R/I is also a Noetherian ring.*

Proof. Let J/I be an ideal of R/I . Then J is an ideal of R that properly contains I . Since R is Noetherian, J is finitely generated by Proposition 5.0.4. Thus, J/I is also finitely generated, and hence, R/I is a Noetherian ring by Proposition 5.0.4. $\square$

Corollary 5.0.8. *Let A be a submodule of a Noetherian R -module M . Then A is also Noetherian.*

Proof. Let X be a submodule of A . Then X is a submodule of M , so X is finitely generated by Proposition 5.0.4 because M is Noetherian. Since every submodule of A is finitely generated, A is Noetherian by Proposition 5.0.4. $\square$

Here are some examples and counterexamples of Noetherian rings.

Example 5.0.3.

- (1) *Note that any principle ideal ring is Noetherian since every left ideal I of R is finitely generated by a set $\{r\}$ for some $r \in R$.*
- (2) *Any field is a Noetherian ring since the only ideals of a field is 0 and itself.*
- (3) *Note $\mathbb{Q}$ is not Noetherian since the ascending chain of submodules $\frac{1}{2}\mathbb{Z} \subset \frac{1}{4}\mathbb{Z} \subset \cdots \subset \frac{1}{2^i}\mathbb{Z} \subset \cdots$ does not terminate.*

We will now see how short exact sequences are affected by Noetherian modules.

Proposition 5.0.5. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of R -modules. Then B is Noetherian if and only if A and C are Noetherian.*

Proof. Note that $0 \rightarrow A \rightarrow B \rightarrow C$ is isomorphic to a short exact sequence $0 \rightarrow A \rightarrow D \rightarrow D/A \rightarrow 0$ of R -modules, where D is an extension of A by Proposition 1.1.2.

Suppose that B is Noetherian, so D is Noetherian. We will first show that A is Noetherian. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be an ascending chain of submodules of A . Then they

are submodules of D . Since D is Noetherian, the chain terminates. Thus, A is Noetherian. We will now show that D/A is Noetherian. Let $L_1/A \subseteq L_2/A \subseteq L_3/A \subseteq \cdots$ be an ascending chain of submodules of D/A . Then $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ is an ascending chain of submodules of D containing A . So, there exists a positive integer n such that $L_n = L_m$ for every $m \geq n$. Thus, $L_n/A = L_m/A$ for every $m \geq n$, showing that D/A is Noetherian. Therefore, C is Noetherian since $C \approx D/A$.

Conversely, suppose that A and C are Noetherian, so D/A is Noetherian. Let $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ be an ascending chain of submodules of D . Then $(J_1 \cap A) \subseteq (J_2 \cap A) \subseteq (J_3 \cap A) \subseteq \cdots$ is a chain of submodules of A , and $(J_1 + A)/A \subseteq (J_2 + A)/A \subseteq (J_3 + A)/A \subseteq \cdots$ is a chain of submodules of D/A . So, both chains terminate at an integer $n \geq 1$, meaning that $J_n \cap A = J_m \cap A$ and $J_n + A = J_m + A$ for every $m \geq n$. Then for every $m \geq n$,

$$\begin{aligned} J_n &= J_n \cap (J_n + A) \\ &= J_n \cap (J_m + A) \\ &= J_m + (J_n \cap A) \text{ by Proposition 1.0.1} \\ &= J_m + (J_m \cap A) \\ &= J_m. \end{aligned}$$

Therefore, since the chain of submodules of D terminates, D is Noetherian, and hence, B is Noetherian. $\square$

There are relationships between Noetherian modules and their products and direct sums.

Theorem 5.1. *Let $\{M_i\}_{i=1}^n$ be a finite family of R -modules. Then $\bigoplus_{i=1}^n M_i$ is Noetherian if and only if M_i is Noetherian for every $1 \leq i \leq n$.*

Proof. We will show this proof by induction. Since the base case of $n = 1$ is trivial, assume that the statement is true with all $n \geq k$. Considering $k + 1$, $0 \rightarrow \bigoplus_{i=1}^k M_i \rightarrow \bigoplus_{i=1}^{k+1} M_i \rightarrow M_{k+1} \rightarrow 0$ is an exact sequence.

If $\bigoplus_{i=1}^{k+1} M_i$ is Noetherian, then by Proposition 5.0.5, $\bigoplus_{i=1}^k M_i$ and M_{k+1} are Noetherian. By induction, M_i is Noetherian for every $1 \leq i \leq k + 1$.

Conversely, if M_i is Noetherian for every $1 \leq i \leq k+1$, then $\bigoplus_{i=1}^k M_i$ and M_{k+1} are both Noetherian. Thus, again by Proposition 5.0.5, $\bigoplus_{i=1}^{k+1} M_i$ is Noetherian.

Hence, $\bigoplus_{i=1}^n M_i$ is Noetherian if and only if M_i is Noetherian for every $1 \leq i \leq n$. $\square$

Since finite direct sums and products are the same, the previous theorem also applies with finite products.

Corollary 5.1.1. *Let R be a ring. Then R is Noetherian if and only if every finitely generated R -module is Noetherian.*

Proof. Suppose that R is a Noetherian ring. Let M be a finitely generated R -module. By Proposition 1.0.2, there exists an R -epimorphism from F to M such that F is a free R -module. So, F is R -isomorphic to $\bigoplus_{i=1}^n R$ for some positive integer n because M is finitely generated. Let $g : \bigoplus_{i=1}^n R \rightarrow M$ be this R -epimorphism. Then $0 \rightarrow \text{Ker}(g) \rightarrow \bigoplus_{i=1}^n R \rightarrow M \rightarrow 0$ is a short exact sequence. Since $\bigoplus_{i=1}^n R$ is Noetherian by Theorem 5.1, M is Noetherian by Proposition 5.0.5.

Conversely, suppose that every finitely generated R -module is Noetherian. Since the R -module R is finitely generated, R is Noetherian. $\square$

We now see how Noetherian rings relate to injective modules.

Theorem 5.2. *The following statements are equivalent:*

- (i) R is a Noetherian ring;
- (ii) Every direct sum of injective R -modules is injective;
- (iii) Every direct sum of a countably infinite family of injective envelopes of simple R -modules is injective.

Proof. (i) $\Rightarrow$ (ii) Suppose that R is a Noetherian ring. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of injective R -modules. Define $f : \bigoplus M_\lambda \rightarrow \bigoplus M_\lambda$ to be an R -homomorphism, where I is a left ideal of R . Then, I is finitely generated by Proposition 5.0.4. So, $f(I) \subseteq \bigoplus_{i=1}^n M_{\lambda_i} \subseteq \bigoplus M_\lambda$ for some positive integer n . Since $\bigoplus_{i=1}^n M_{\lambda_i}$ is injective by Corollary 2.4.1, there exists an R -homomorphism $h : R \rightarrow M$, where $M = \bigoplus_{i=1}^n M_{\lambda_i}$ such that $h|_I = f$. Define the R -homomorphism

$g : R \rightarrow \bigoplus M_\lambda$ by $g = \iota_M h$, where $\iota_M : M \rightarrow M$ is the identity function of M . Note that for every $a \in I$, $g(a) = \iota_M h(a) = \iota_M f(a) = f(a)$. Thus, $\bigoplus M_\lambda$ is injective.

(ii) $\Rightarrow$ (iii) Now, assume that every direct sum of injective R -modules is injective. Let S be a simple R -module, and let $\{E(S)_i\}_{i \in \mathbb{N}}$ be a countably infinite family of injective envelopes of S . Since $\{E(S)_i\}_{i \in \mathbb{N}}$ is a family of injective R -modules, $\bigoplus E(S)_i$ is injective.

(iii) $\Rightarrow$ (i) Lastly, suppose that every direct sum of a countably infinite family of injective envelopes of simple R -modules is injective. Assume by way of contradiction that $I_1 \subset I_2 \subset I_3 \subset \cdots$ is a strictly infinite increasing sequence of left ideals of R . Consider the left ideal $I = \bigcup_{k=1}^{\infty} I_k$ of R , so $I/I_k \neq 0$ for every positive integer k . For every positive integer k , there exists a nonzero R -homomorphism $\theta_k : I/I_k \rightarrow E(S_k)$ for some simple R -module S_k by Proposition 4.6.4. Define the well-defined R -homomorphism $\psi : I \rightarrow \bigoplus_{k=1}^{\infty} E(S_k)$ by $\psi(r) = \{\psi_k(r)\}_{k=1}^{\infty}$, where $\psi_k : I \rightarrow E(S_k)$ is the nonzero composition of the natural mapping $I \rightarrow I_k$ with θ_k . Since $\bigoplus_{k=1}^{\infty} E(S_k)$ is injective by assumption, there exists an R -homomorphism $\alpha : R \rightarrow \bigoplus_{k=1}^{\infty} E(S_k)$ such that $\alpha(a) = \psi(a)$ for every $a \in I$. Observe that the R -module R is finitely generated by the set $\{1_R\}$. So, there exists a positive integer n such that $\alpha(R) \subseteq \bigoplus_{k=1}^n E(S_k)$. Then ψ_k is the zero map for all $k > n$, causing a contradiction. Therefore, there is no strictly increasing sequence of left ideals of R . Hence, R is Noetherian. $\square$

We will now see how Noetherian modules relate to prime ideals. For the rest of this section, consider the ring R to be commutative.

Proposition 5.2.1. *Let M be a nonzero Noetherian R -module. Then there exists $m \in M$ such that $\text{Ann}(m)$ is a prime ideal of R .*

Proof. Define $\Omega = \{\text{Ann}(x) \mid 0 \neq x \in M\}$. Since $M \neq 0$, Ω is nonempty. To show that Ω has a maximal member, assume the contrary. Then there exists an infinite strictly ascending chain $\text{Ann}(m_1) \subseteq \text{Ann}(m_2) \subseteq \text{Ann}(m_3) \subseteq \cdots$ of elements of Ω , so $0 \subseteq \text{Ann}(m_2)/\text{Ann}(m_1) \subseteq \text{Ann}(m_3)/\text{Ann}(m_1) \subseteq \cdots$ is also an infinite strictly ascending chain of submodules of the R -module $R/\text{Ann}(m_1)$. Note that $Rm_1 \approx R/\text{Ann}(m_1)$ is R -isomorphic by $rm_1 \mapsto r + \text{Ann}(m_1)$, where Rm_1 is a submodule of M . By Corollary 5.0.8, $R/\text{Ann}(m_1)$ is Noetherian. However,

this means that there is no infinite strictly ascending chain of submodules of $R/\text{Ann}(m_1)$. Thus, Ω has a maximal member, say $P = \text{Ann}(m)$. Since P is maximal, $P \neq R$. Let $\alpha, \beta \in R$ be such that $\alpha\beta \in P$. Assume that $\beta \notin P$. Since $\text{Ann}(m) \subseteq \text{Ann}(\beta m)$ for every $\beta \in R$ and $\text{Ann}(\beta m) \subseteq \text{Ann}(m)$ since $\text{Ann}(m)$ is maximal, $\text{Ann}(\beta m) = P$. Thus, $\alpha \in P$ because $\alpha\beta m = 0$. $\square$

There are a couple of corollaries that come from this proposition

Corollary 5.2.1. *Let M be a nonzero Noetherian R -module. Then there is a prime ideal of P such that M has a submodule that is R -isomorphic to R/P .*

Proof. By Proposition 5.2.1, there exists $m \in M$ such that $\text{Ann}(m)$ is a prime ideal of R . Since Rm is a submodule of M , M has a submodule that is R -isomorphic to $R/\text{Ann}(m)$ by $rm \mapsto r + \text{Ann}(m)$. $\square$

Corollary 5.2.2. *Every maximal ideal of R is prime.*

Proof. Let N be a maximal ideal of R , so R/N is simple. Since R/N is Noetherian by Corollary 5.0.6, there is a prime ideal P of R such that $R/N \approx R/P$ by Corollary 5.2.1. Thus, $N = P$, so N is prime. $\square$

We now have new special prime ideals and injective modules specifically for Noetherian rings.

Definition 5.2.1. *Let P be an ideal of R . Then P is said to be **N -prime** provided that*

- (i) P is prime;
- (ii) R/P is a Noetherian ring.

Definition 5.2.2. *Let M be an R -module. Then M is said to be **N -injective** provided that*

- (i) M is injective;
- (ii) if $M \neq 0$, then M has a nonzero Noetherian submodule.

Remark 5.2.1. *If R is a Noetherian ring, then there is no difference between its prime and N -prime ideals. In such a case, there is also no difference between injective and N -injective R -modules.*

We will see an interesting theorem that connects these two concepts.

Theorem 5.3.

- (i) If P is an N -prime ideal of R , then $E(R/P)$ is an indecomposable N -injective R -module.
- (ii) If M is a nonzero indecomposable N -injective R -module, then there exists an N -prime ideal P of R such that $M \approx E(R/P)$.

Proof. (i) Let P be an N -prime ideal of R . Since P is prime, $E(R/P)$ is indecomposable by Proposition 4.7.2. Note that $E(R/P)$ is an injective R -module that has a nonzero Noetherian submodule R/P . Thus, $E(R/P)$ is an indecomposable N -injective R -module.

(ii) Let M be a nonzero indecomposable N -injective R -module. Then M has a nonzero Noetherian submodule N . By Corollary 5.2.1, there exists a prime ideal P of R such that N has a submodule A , where $A \approx R/P$ is an R -isomorphism. Then A is Noetherian by Corollary 5.0.8, so R/P is Noetherian. Thus, P is an N -prime ideal. Since M is indecomposable and A is a submodule of M by Proposition 4.7.1, $M = E(A) \approx E(R/P)$. $\square$

Corollary 5.3.1. *Let R be a commutative Noetherian ring, and let M be an R -module. Then M is an indecomposable injective R -module if and only if there exists a prime ideal P of R such that $M \approx E(R/P)$.*

Proof. Suppose that M is an indecomposable injective R -module.

Conversely, suppose that there exists a prime ideal P of R such that $M \approx E(R/P)$. Since R is Noetherian, then so is R/P by Corollary 5.0.7. Thus, P is an N -prime ideal of R . By Theorem 5.3, $E(R/P)$ is an indecomposable N -injective R -module. Thus, M is an indecomposable injective R -module. $\square$

We will now discuss the special Abelian group $\mathbb{Z}(p^\infty)$ for some prime p .

Remark 5.3.1. *Note that for some prime p , define $\mathbb{Z}(p^\infty) = \{g + \mathbb{Z} : |g + \mathbb{Z}| = p^n \text{ for some nonnegative integer } n\}$.*

Lemma 5.3.1. *The Abelian groups $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$, where p is prime, are non-isomorphic, indecomposable, and injective.*

Proof. Note that $\mathbb{Q}$ is divisible, so $\mathbb{Q}$ is injective by Corollary 3.4.1. Also, $\mathbb{Z}(p^\infty) = \mathbb{Q}/\mathbb{Z}_p$ is divisible by Lemma 3.1.1, so $\mathbb{Z}(p^\infty)$ is injective.

We claim that $\mathbb{Q}$ is a torsion-free. Assume by way of contradiction that $\mathbb{Q}$ has a nonzero torsion element q , where $q = \frac{a}{b}$ for some nonzero integers a and b . Then there exists a nonzero $z \in \mathbb{Z}$ such that $zq = \frac{za}{b} = 0$, so $za = 0$. However, since z and a are nonzero, this is a contradiction. Thus, $\mathbb{Q}$ is torsion-free.

We now will show that $\mathbb{Z}(p^\infty)$ is not torsion-free. Consider $\frac{1}{p} \in \mathbb{Z}(p^\infty)$. Since $p \in \mathbb{Z}$ such that $p \left(\frac{1}{p} + \mathbb{Z} \right) = 0 + \mathbb{Z}$, $\mathbb{Z}(p^\infty)$ is not torsion-free.

Hence, $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ are non-isomorphic, indecomposable, and injective. $\square$

Theorem 5.4. *Every indecomposable injective Abelian group is isomorphic to $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$ for some prime p .*

Proof. Let M be an indecomposable injective module over a commutative Noetherian ring R , and suppose that M is not isomorphic to $\mathbb{Z}(p^\infty)$ for any p . Since $\mathbb{Z}$ is a commutative Noetherian ring, $M \approx E(\mathbb{Z}/P)$ for some prime ideal P by Corollary 5.3.1. This is only possible if $P = 0$, so $M \approx \mathbb{Q}$. $\square$

5.2 Dedekind Domains

In this section, let R be an integral domain with a field of fractions K , unless otherwise stated.

Definition 5.4.1. *A submodule F of the R -module K is called a **fractional ideal** of R provided that there exists $0 \neq r \in R$ such that $rF \subseteq R$. Ideals of R are called **integral ideals** of R .*

Remark 5.4.1.

- (1) *Note that every integral ideal is a fractional ideal.*
- (2) *If $I = rF$, then I is an integral ideal of R and $F = (1/r)I$.*

Definition 5.4.2. Given a fractional ideal F of R , if there is a fractional ideal $F' = \{x \in K \mid xF \subseteq R\}$ of R such that $FF' = R$, then F' is unique and called the **inverse** of F ; in which case F is said to be **invertible**.

Remark 5.4.2. Note that for every fractional ideal F and G of R , $FG = \{\sum_{i=1}^n f_i g_i \mid f_i \in F, g_i \in G\}$.

There is an important new ring for us to consider.

Definition 5.4.3. A **Dedekind domain** is an integral domain such that every nonzero fractional ideal is invertible.

We see that Dedekind domains also can be defined with integral ideals.

Lemma 5.4.1. Let R be an integral domain such that every nonzero integral ideal is invertible. Then R is a Dedekind domain.

Proof. Let F be a nonzero fractional ideal of R . Then there exists an $0 \neq r \in R$ such that $rF \subseteq R$. Since rF is a nonzero integral ideal of R , there exists a fractional ideal F' of R , so

$$\begin{aligned} R &= (rF)F' \\ &= \{\sum r f_i f'_i \mid f_i \in F, f'_i \in F'\} \\ &= \{\sum f_i r f'_i \mid f_i \in F, f'_i \in F'\} \\ &= F(rF'). \end{aligned}$$

Thus, F has an inverse fractional ideal rF' of R so that F is invertible. Hence, R is a Dedekind domain. □

Lemma 5.4.2. Invertible ideals in Dedekind domains are finitely generated.

Proof. Let I be an invertible ideal of R , where R is a Dedekind domain. Then there exists a fractional ideal F of R such that $IF = R$. Note that there exists $a_1, \dots, a_n \in I$ and

$b_1, \dots, b_n \in F$ such that $\sum_{i=1}^n a_i b_i = 1_R$ for some positive integer n . We claim that $I = Ra_1 + \dots + Ra_n$. It is clear that $Ra_1 + \dots + Ra_n \subseteq I$. Let $x \in I$. Then $xF \subseteq IF = R$, and

$$x = x \sum_{i=1}^n a_i b_i = \sum_{i=1}^n x b_i a_i \in \sum_{i=1}^n x F a_i \subseteq \sum_{i=1}^n R a_i.$$

Thus, $I \subseteq Ra_1 + \dots + Ra_n$, so $I = Ra_1 + \dots + Ra_n$. Hence, I is finitely generated. $\square$

We will see how Dedekind domains relate to different rings.

Theorem 5.5. *Every Dedekind domain is Noetherian.*

Proof. Let I be an integral ideal of a Dedekind domain R . Since I is a fractional ideal of R , I is an invertible ideal. Thus, by Lemma 5.4.2, I is finitely generated. Hence, by Proposition 5.0.4, R is Noetherian. $\square$

However, the converse is not necessarily true.

Example 5.5.1. *Let K be a field and let the ring R be defined by $R = K[X, Y]$. Consider $I = (X, Y)$. Then $I^{-1} = R$, so $II^{-1} = I$. Thus, R is Noetherian, but not Dedekind.*

Proposition 5.5.1. *Every principal ideal domain is a Dedekind domain.*

Proof. Suppose that R is a principal ideal domain with a field of fractions K . Let I be a nonzero integral ideal of R . Then $I = Rr$ for some $0 \neq r \in R$. Note that $R(1/r)$ is a fractional ideal of R since $r(R(1/r)) \subseteq R$. Since $RrR(1/r) = R$, I is invertible. Thus, R is a Dedekind domain by Lemma 5.4.1. $\square$

We see how Dedekind domains affect injectivity.

Theorem 5.6. *In a Dedekind domain, an R -module M is injective if and only if it is divisible.*

Proof. Let R be a Dedekind domain. Suppose that M is injective, so M is divisible by Theorem 3.4.

Conversely, suppose that M is divisible. Let $f : I \rightarrow M$ be an R -homomorphism, where I be an integral ideal of R . We will show that there exists an R -homomorphism $g : R \rightarrow M$ that is an extension of f . Since I is a fractional ideal, there exists a unique fractional ideal F of R such that $IF = R$. So there are $a_1, \dots, a_n \in I$ and $b_1, \dots, b_n \in F$ such that $\sum_{i=1}^n a_i b_i = 1_R$ for some positive integer n . Since M is divisible, for every $1 \leq i \leq n$, there exists $x_i \in M$, so $f(a_i) = a_i x_i$. Consider $t \in I$. Since $tb_i \in IF = R$, for every $1 \leq i \leq n$,

$$f(t) = f(t1_R) = f\left(t \sum_{i=1}^n a_i b_i\right) = \sum_{i=1}^n tb_i f(a_i) = t \sum_{i=1}^n b_i a_i x_i.$$

Then $b_i a_i \in FI = R$, so $\sum_{i=1}^n b_i a_i x_i \in M$. Define $g : R \rightarrow M$ by $g(r) = r \sum_{i=1}^n b_i a_i x_i$. Thus, g is an R -homomorphism such that $g|_I = f$. Hence, M is injective. $\square$

To prove the next theorem, we need the following lemma concerning projective modules.

Lemma 5.6.1. *An R -module P is projective if and only if for every R -epimorphism $g : Q \rightarrow D$ between R -modules, where Q is injective, and for every R -homomorphism $f : P \rightarrow D$, there exists an R -homomorphism $h : P \rightarrow Q$ such that $gh = f$.*

Proof. Suppose that P is a projective R -module. Let $g : Q \rightarrow D$ be an R -epimorphism between R -modules, where Q is injective, and let $f : P \rightarrow D$ be an R -homomorphism. Since P is projective, there exists an R -homomorphism $h : P \rightarrow Q$ such that $gh = f$.

Conversely, let $g : A \rightarrow B$ be an R -epimorphism between R -modules, and let $f : P \rightarrow B$ be an R -homomorphism. Note that A has an injective extension Q by Theorem 4.1, so $\sigma : A \rightarrow Q$ and $\iota : \text{Ker}(g) \rightarrow A$ are R -monomorphisms and $\pi : Q \rightarrow Q/\text{Im}(\sigma\iota)$ is an R -epimorphism. We will find an R -monomorphism ψ that makes the following diagram commute:

$$\begin{array}{ccccccc}
& & & & P & & \\
& & & & \downarrow f & & \\
0 & \longrightarrow & \text{Ker}(g) & \xrightarrow{\iota} & A & \xrightarrow{g} & B \longrightarrow 0 \\
& & \downarrow 1 & & \downarrow \sigma & & \downarrow \psi \\
0 & \longrightarrow & \text{Ker}(g) & \xrightarrow{\sigma\iota} & Q & \xrightarrow{\pi} & Q/\text{Im}(\sigma\iota) \longrightarrow 0
\end{array}$$

Figure 5.1: Diagram to show that P is projective.

Note that for every $b \in B$, there exists $a \in A$ such that $g(a) = b$ since g is an R -epimorphism. Define $\psi : B \rightarrow Q/A$ by $\psi(b) = \sigma(a) + Q/\text{Im}(\sigma\iota)$. Note that ψ is well-defined because $\psi(0) = \sigma(x) + \text{Im}(\sigma\iota)$ for some $x \in \text{Ker}(g)$, meaning that $\iota(x) = x$ and $\psi(0) = \sigma\iota(x) + \text{Im}(\sigma\iota) = 0 + \text{Im}(\sigma\iota)$. We will now show that ψ is an R -homomorphism. Firstly, consider $b \in B$ and $r \in R$, so $r\psi(b) = r(\sigma(a) + \text{Im}(\sigma\iota)) = (r\sigma(a)) + \text{Im}(\sigma\iota) = \sigma(ra) + \text{Im}(\sigma\iota)$ for some $a \in A$ such that $g(a) = b$. Since $g(ra) = rg(a) = rb$, $\psi(rb) = \sigma(ra) + \text{Im}(\sigma\iota)$. Thus, $r\psi(b) = \psi(rb)$. Secondly, consider $b_1, b_2 \in B$, so $\psi(b_1) + \psi(b_2) = (\sigma(a_1) + \text{Im}(\sigma\iota)) + (\sigma(a_2) + \text{Im}(\sigma\iota)) = (\sigma(a_1) + \sigma(a_2)) + \text{Im}(\sigma\iota) = \sigma(a_1 + a_2) + \text{Im}(\sigma\iota)$ for some $a_1, a_2 \in A$ such that $g(a_1) = b_1$ and $g(a_2) = b_2$. Since $g(a_1 + a_2) = g(a_1) + g(a_2) = b_1 + b_2$, $\psi(b_1 + b_2) = \sigma(a_1 + a_2) + \text{Im}(\sigma\iota)$. Thus, $\psi(b_1) + \psi(b_2) = \psi(b_1 + b_2)$. Therefore, ψ is an R -homomorphism. To show that ψ is one-to-one, let $x \in \text{Ker}(\psi)$, so $\psi(x) = 0 + \text{Im}(\sigma\iota) = \sigma(a) + \text{Im}(\sigma\iota)$ for some $a \in A$ such that $g(a) = x$. Then $\sigma(a) \in \text{Im}(\sigma\iota)$, which implies that there exists $z \in \text{Ker}(g)$ such that $\sigma\iota(z) = \sigma(z) = \sigma(a)$. Since σ is an R -monomorphism, $z = a$, so $x = g(a) = g(z) = 0$. Thus, ψ is an R -monomorphism.

By assumption, there exists an R -homomorphism $\theta : P \rightarrow Q$ such that $\pi\theta = \psi f$. We will show that $\text{Im}(\theta) \subseteq \text{Im}(\sigma)$. Let $x \in P$, so

$$\theta(x) + \text{Im}(\sigma\iota) = \pi\theta(x) = \psi f(x) = \sigma(a) + \text{Im}(\sigma\iota)$$

for some $a \in A$ such that $g(a) = f(x)$. Then $\theta(x) - \sigma(a) \in \text{Im}(\sigma\iota)$, implying that there exists $z \in \text{Ker}(g)$ such that $\sigma\iota(z) = \theta(x) - \sigma(a)$. Thus, $\theta(x) = \sigma\iota(z) + \sigma(a) = \sigma(z) + \sigma(a) = \sigma(z+a)$. So, for every $x \in P$, $\theta(x) \in \text{Im}(\sigma)$, meaning that $\text{Im}(\theta) \subseteq \text{Im}(\sigma)$.

Observe that for every $x \in P$, there exists $a \in A$ such that $\theta(x) = \sigma(a)$. Define $\gamma : P \rightarrow A$ by $\gamma(x) = a$. Note that γ is well-defined because there exists $z \in A$ such that $\sigma(z) = \theta(0) = 0$. Since σ is an R -monomorphism, $z = 0$ and $\gamma(0) = 0$. We will now show that γ is an R -homomorphism. Firstly, consider $x \in P$ and $r \in R$, so $r\gamma(x) = ra$ for some $a \in A$ such that $\sigma(a) = \theta(x)$. Since $\theta(rx) = r\theta(x) = r\sigma(a) = \sigma(ra)$, $\gamma(rx) = ra$. Thus, $r\gamma(x) = \gamma(rx)$ for every $x \in P$. Secondly, consider $x_1, x_2 \in P$, so $\gamma(x_1) + \gamma(x_2) = a_1 + a_2$ for some $a_1, a_2 \in A$ such that $\sigma(a_1) = \theta(x_1)$ and $\sigma(a_2) = \theta(x_2)$. Since $\theta(x_1 + x_2) = \theta(x_1) + \theta(x_2) = \sigma(a_1) + \sigma(a_2) = \sigma(a_1 + a_2)$, $\gamma(x_1 + x_2) = a_1 + a_2$. Thus, $\gamma(x_1) + \gamma(x_2) = \gamma(x_1 + x_2)$ for every $x_1, x_2 \in P$. Therefore, γ is an R -homomorphism.

Lastly, we will show that $f = g\gamma$. For every $x \in P$, notice that $\psi f(x) = \pi\theta(x) = \pi\sigma(a) = \psi g(a)$ for some $a \in A$ such that $\sigma(a) = \theta(x)$. Since $g\gamma(x) = g(a)$, $\psi f(x) = \psi g\gamma(x)$, which means that $f = g\gamma$ because ψ is an R -monomorphism. $\square$

Theorem 5.7. *Let R be an integral domain. Then the following statements are equivalent:*

- (i) *R is a Dedekind domain.*
- (ii) *Every quotient of an injective R -module is injective.*
- (iii) *Every submodule of a projective R -module is projective.*

Proof.

(i) $\Rightarrow$ (ii) Suppose that R is a Dedekind domain. Assume that A is a submodule of an injective R -module M . Then M is divisible by Theorem 3.4, so M/A is divisible by Lemma 3.1.1. Thus, by Theorem 5.6, M/A is injective.

(ii) $\Rightarrow$ (iii) Secondly, suppose that every quotient of an injective R -module is injective. Let A be a submodule of a projective R -module P . Consider an R -epimorphism $g : Q \rightarrow B$ between R -module, where Q is injective, and an R -homomorphism $f : A \rightarrow B$. Then $\iota : A \rightarrow P$ is the inclusion mapping. Since $0 \rightarrow \text{Ker}(g) \xrightarrow{\iota} Q \xrightarrow{g} B \rightarrow 0$ is isomorphic to the short exact sequence $0 \rightarrow C \rightarrow Q \rightarrow Q/C \rightarrow 0$ for some submodule C of Q by Proposition

1.1.2, $B \approx Q/C$. Since B is injective by assumption, there exists an R -homomorphism $h : P \rightarrow B$ such that $hu = f$. By Lemma 5.6.1, there exists an R -homomorphism $\ell : P \rightarrow Q$ such that $g\ell = h$. Thus, $\ell u : A \rightarrow Q$ is an R -homomorphism such that $g\ell u = hu = f$. Hence, A is projective by Lemma 5.6.1.

(iii) $\Rightarrow$ (i) Thirdly, suppose that every submodule of a projective R -module is projective. Since R is a free R -module with a basis $\{x_\lambda\}_{\lambda \in \Lambda}$, R is projective by Proposition 2.4.1. Let I be a nonzero integral ideal of R , and let $g : R \rightarrow I$ be an R -epimorphism, where $a_\lambda = g(x_\lambda)$. Since I is projective by assumption and the identity function $1_I : I \rightarrow I$ of I is an R -homomorphism, there exists $f : I \rightarrow R$ such that $gf = 1_I$, where $f(a) = \sum r_\lambda x_\lambda$ for $r_\lambda \in R$. Define the R -homomorphism $f_\lambda : I \rightarrow R$ by $f_\lambda(a) = r_\lambda$. Fix $0 \neq b \in I$, and let $q_\lambda = f_\lambda(b)/b$ be an element of K . Note that for every $a \in I$, $af_\lambda(b) = bf_\lambda(a)$ since $a, b \in I \subseteq R$. Thus, $q_\lambda a = f_\lambda(a)$, so $q_\lambda I \subseteq R$. Because $f_\lambda(b) = 0$ for all but a finite number of $\lambda \in \Lambda$, also $q_\lambda = 0$ for all but a finite number of $\lambda \in \Lambda$. Therefore, for every $a \in I$,

$$\begin{aligned} a &= gf(a) \\ &= g\left(\sum r_\lambda x_\lambda\right) \\ &= \sum f_\lambda(a)g(x_\lambda) \\ &= \sum f_\lambda(a)a_\lambda \\ &= \sum q_\lambda aa_\lambda \\ &= a \sum q_\lambda a_\lambda. \end{aligned}$$

Since these calculations are in K , $1_R = \sum q_\lambda a_\lambda$. Hence, I is invertible, and by Lemma 5.4.1, R is a Dedekind domain. $\square$

5.3 Semi-simple

Definition 5.7.1. Let M be an R -module. We say that M is a **semi-simple module** provided that every submodule of M is a direct summand of M . Also, R is said to be a **semi-simple ring** provided that it is a semi-simple module.

We will now see equivalent definitions for semi-simple.

Proposition 5.7.1. Let M be an R -module. Then the following statements are equivalent:

- (i) M is semi-simple;
- (ii) M is the direct sum of a family $\{S_\lambda\}_{\lambda \in \Lambda}$ of simple submodules;
- (iii) M has a family of simple submodules whose sum is M itself.

Proof. (i) $\Rightarrow$ (ii) Suppose that M is semi-simple. We will first show that every nonzero submodule of M has a simple submodule. Let $x \in M$ be nonzero, so $\text{Ann}(x)$ is a proper left ideal of R . Then $\text{Ann}(x)$ is contained in a maximal left ideal M of R . Note that $R/\text{Ann}(x)$ is an R -isomorphism by $rx \mapsto r + \text{Ann}(x)$. So, $M/\text{Ann}(x) \approx N$ for some maximal submodule N of $R/\text{Ann}(x)$. Thus, $R/\text{Ann}(x)/N$ is a simple R -module. Since $N \subseteq R/\text{Ann}(x) \subseteq M$, there exists a submodule W of M such that $M = N \oplus W$, where $N \cap W = 0$. Then

$$Rx = Rx \cap M = Rx \cap (N \oplus W) = N \oplus (Rx \cap W),$$

where $N \cap (Rx \cap W) = N \cap W = 0$. So, $R/(Rx \cap W)$ is a simple R -module because $Rx \cap W \approx Rx/N$. Therefore, every nonzero submodule of M has a simple submodule. Now, consider the collection of all simple submodules of M . Then it is contained in a maximal collection Υ of M such that $K = \bigoplus_{S \in \Upsilon} S$ is a direct sum. Assume by way of contradiction that $M \neq K$. Then there exists a nonzero submodule A of M such that $M = K \oplus A$, where $K \cap A = 0$. So, A has a simple submodule S' such that $S' \oplus K$ is a larger direct sum. However, this contradicts the fact that Υ is maximal. Hence, $M = \bigoplus_{S \in \Upsilon} S$.

(ii) $\Rightarrow$ (iii) Since M is a direct sum of a family $\{S_\lambda\}_{\lambda \in \Lambda}$ of simple submodules, it is certainly the case that M is a sum of that family.

(iii) $\Rightarrow$ (i) Suppose that $M = \sum S_\lambda$ for some family $\{S_\lambda\}_{\lambda \in \Lambda}$ of simple submodules. Let A be a submodule of M . Note that if $A = M$, then M is semi-simple. So, suppose that $A \neq M$. By Proposition 4.6.3, $E(A)$ is a direct summand of $E(M)$, where $E(A)$ and $E(M)$ are injective envelopes of A and M , respectively. So, there exists a submodule X of $E(M)$ such that $E(M) = E(A) \oplus X$, where $E(A) \cap X = 0$. Since $E(M)$ is an essential extension of M , $X = E(M) \cap X$ is an essential extension of $M \cap X$. Because $A \subseteq E(A)$, $A \cap X = 0$, which means that $A \cap (M \cap X) = M \cap 0 = 0$. By Theorem 4.4, $E(M) = E(A) \oplus X$ is an essential extension of $A \oplus (M \cap X)$. Then $M = M \cap E(M)$ is an essential extension of $A \oplus (M \cap X)$ since A and $M \cap X$ are submodules of M . Now, observe that for every $\lambda \in \Lambda$, $S_\lambda \cap (A \oplus M \cap X) \neq 0$, which means that $S_\lambda \cap (A \oplus M \cap X) = S_\lambda$ since S_λ is simple. Because $S_\lambda \subseteq A \oplus (M \cap X)$ for every $\lambda \in \Lambda$, $M = A \oplus (M \cap X)$. Hence, M is semi-simple. $\square$

Example 5.7.1.

- (1) Note that $\mathbb{Z}_n$ is a semi-simple ring, where n is a square-free positive integer because for every submodule A of the $\mathbb{Z}_n$ -module $\mathbb{Z}_n$ such that $|A|$ divides n , there exists a submodule X of $\mathbb{Z}_n$ such that $\mathbb{Z}_n = A \oplus X$, where $A \cap X = 0$ and $|A||X| = n$.
- (2) Observe that $\mathbb{Z}_8$ is not semi-simple because its only submodules are $0, \{0, 4\}, \{0, 2, 4, 6\}$, and $\mathbb{Z}_8$; however, $\{0, 4\}$ and $\{0, 2, 4, 6\}$ are not direct summands of $\mathbb{Z}_8$.

There are interesting consequences from semi-simple modules.

Theorem 5.8. *Let M be a semi-simple R -module. Then the following statements are equivalent*

- (i) M is the direct sum of a finite family of simple submodules.
- (ii) M is Noetherian.
- (iii) M is finitely generated.

Proof. (i) $\Rightarrow$ (ii) Suppose that M is the direct sum $\bigoplus_{i=1}^n S_i$ of a finite family of simple submodules $\{S_i\}_{i=1}^n$. Since S_i is Noetherian for every $1 \leq i \leq n$ by Corollary 5.0.6, $M = \bigoplus_{i=1}^n S_i$ is also Noetherian by Theorem 5.1.

(ii) $\Rightarrow$ (i) Suppose that M is not the direct sum of a finite family of simple submodules. So, M must be the direct sum of an infinite family of simple submodules. Then M has a strictly increasing chain of submodules. Thus, M is not Noetherian.

(i) $\Rightarrow$ (iii) Suppose that M is a direct sum $\bigoplus_{i=1}^n S_i$ of a finite family of simple submodules $\{S_i\}_{i=1}^n$. Since S_i is finitely generated for every $1 \leq i \leq n$, $M = \bigoplus_{i=1}^n S_i$ is also finitely generated.

(iii) $\Rightarrow$ (i) Suppose that M is finitely generated, so $M = Rm_1 + \cdots + Rm_k$ for some $m_1, \dots, m_k \in M$. Since M is semi-simple, $M = \bigoplus S_\lambda$ is the direct sum of simple submodules S_λ by Proposition 5.7.1. Then there exists elements $\lambda_1, \dots, \lambda_r$ of Λ such that $m_i \in S_{\lambda_1} + \cdots + S_{\lambda_r}$ for every $1 \leq i \leq k$. Thus, $M = \bigoplus_{j=1}^r S_{\lambda_j}$ is the direct sum of a finite family of simple submodules $\{S_{\lambda_j}\}_{j=1}^r$. $\square$

We will see that submodules, quotients, and homomorphic images of semi-simple modules are all semi-simple.

Lemma 5.8.1. *Every submodule of a semi-simple module is semi-simple.*

Proof. Let A be a submodule of a semi-simple R -module M . Consider a submodule S of A . Since S is a submodule of M , there exists a submodule X of M such that $M = S \oplus X$, where $S \cap X = 0$. Note that by Proposition 1.0.1

$$A = A \cap M = A \cap (S + X) = S + (A \cap X)$$

and

$$S \cap (A \cap X) = (S \cap A) \cap X = S \cap X = 0.$$

Then S is a direct summand of A . Thus, A is a semi-simple. $\square$

Corollary 5.8.1. *Every quotient of a semi-simple module is semi-simple.*

Proof. Let M/A be a quotient module of a semi-simple R -module M , where A is a submodule of M . Then there exists a submodule X of M such that M is the direct sum of A and X . Define $f : M \rightarrow X$ by $f(m) = 0$ when $m \in A$ and $f(m) = m$ when $m \in X$. A routine check

shows us that f is an R -epimorphism. So, by Corollary 1.1.1, $M/A = M/\text{Ker}(f) \approx X$. Since X is semi-simple by Lemma 5.8.1, M/A is also semi-simple. $\square$

Corollary 5.8.2. *A homomorphic image of a semi-simple module is semi-simple.*

Proof. Let M be a semi-simple R -module, and let $\alpha : M \rightarrow A$ be an R -epimorphism between R -modules. By Corollary 1.1.1, $A \approx M/\text{Ker}(\alpha)$. Since $M/\text{Ker}(\alpha)$ is semi-simple by Corollary 5.8.1, A is also semi-simple. $\square$

There are relationships between semi-simple modules and their products and direct sums.

Lemma 5.8.2.

- (i) *A direct sum of semi-simple modules is semi-simple.*
- (ii) *A finite product of semi-simple modules is semi-simple.*

Proof.

(i) Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of semi-simple R -modules. Consider a submodule S of $\bigoplus M_\lambda$, so $S = \bigoplus S_\lambda$ for some submodule S_λ of M_λ . For every $\lambda \in \Lambda$, since M_λ is semi-simple, there exists a submodule X_λ of M_λ such that M_λ is the direct sum of S_λ and X_λ . Let $X = \bigoplus X_\lambda$. We claim that $\bigoplus M_\lambda$ is the direct sum of S and X . Note that $S + X = (\bigoplus S_\lambda) + (\bigoplus X_\lambda) = \bigoplus (S_\lambda + X_\lambda) = \bigoplus M_\lambda$. Also, observe that $S \cap X = (\bigoplus S_\lambda) \cap (\bigoplus X_\lambda) = \bigoplus (S_\lambda \cap X_\lambda) = 0$. Thus, S is a direct summand of $\bigoplus M_\lambda$, and hence, $\bigoplus M_\lambda$ is semi-simple.

(ii) By similar argument to part (i), a finite product of semi-simple modules is semi-simple. $\square$

We will see how semi-simple modules connect with injectivity.

Theorem 5.9. *Let R be a ring. Then the following statements are equivalent:*

- (i) *R is semi-simple;*
- (ii) *every R -module is semi-simple;*
- (iii) *every R -module is injective;*
- (iv) *every left ideal of R is injective;*
- (v) *every short exact sequence of R -modules splits;*
- (vi) *every R -module is projective.*

Proof. (i) $\Rightarrow$ (ii) Suppose that R is semi-simple. Let M be an R -module. Then there exists an R -epimorphism $\psi : F \rightarrow M$, where F is a free R -module, so $F \approx \bigoplus_{g \in G} R$ for some generating group G of M by Proposition 1.0.2. Therefore, by Lemma 5.8.2 F is semi-simple, and thus, by Lemma 5.8.2, M is semi-simple.

(ii) $\Rightarrow$ (iii) Secondly, suppose that every R -module is semi-simple. Let M be an R -module. Consider B to be an R -module extension of M , so B is semi-simple. Then M is a direct summand of B . Thus, M is injective by Theorem 4.2.

(iii) $\Rightarrow$ (iv) Thirdly, suppose that every R -module is injective. Let I be a left ideal of R . Since I is an R -module, I is injective.

(iv) $\Rightarrow$ (i) Fourthly, suppose that every left ideal of R is injective. Let I be a left ideal of R , so I is injective. Since R is an extension of I , I is a direct summand of R . Thus, R is semi-simple.

(iii) $\Rightarrow$ (v) Now, suppose that every R -module is injective. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence between R -modules. Since A is injective, the sequence splits by Theorem 4.2.

(v) $\Rightarrow$ (iii) Conversely, suppose that every short exact sequence of R -modules splits. Let M be an R -module. Consider the exact sequence $0 \rightarrow M \rightarrow B \rightarrow C \rightarrow 0$. Then the sequence splits, and thus, M is injective by Theorem 4.2.

(v) $\Rightarrow$ (vi) Suppose that every short exact sequence splits. Let P be an R -module such that $g : A \rightarrow B$ is an R -epimorphism between R -modules and $f : P \rightarrow B$ is an R -homomorphism. Then $0 \rightarrow \text{Ker}(g) \xrightarrow{\iota} A \xrightarrow{g} B \rightarrow 0$ is a short exact sequence, where ι is the inclusion mapping. By Proposition 1.1.3, there exists an R -homomorphism $h : B \rightarrow A$ such that $gh = 1_B$. Consider the R -homomorphism $hf : P \rightarrow A$. Then $g(hf) = (gh)f = 1_B f = f$. Thus, P is projective.

(vi) $\Rightarrow$ (v) Conversely, suppose that every R -module is projective. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence of R -modules. Since C is projective and the identity function $1_C : C \rightarrow C$ of C is an R -homomorphism, there exists an R -homomorphism $h : C \rightarrow B$ such that $gh = 1_C$. Thus, the sequence splits by Proposition 1.1.3. $\square$

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Vita

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