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To the Graduate Council:

I am submitting herewith a thesis written by Philip B. Burt entitled Part I. "Quantum Kinetic Equations for Plasmas and Radiation; Part II. Cyclotron Instabilities in a Bounded Plasma." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

Edward D Harris
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PART I: QUANTUM KINETIC EQUATIONS FOR PLASMAS AND RADIATION

PART II: CYCLOTRON INSTABILITIES IN A BOUNDED PLASMA

A Thesis

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Philip B. Burt

June 1961

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PART I

QUANTUM KINETIC EQUATIONS FOR PLASMAS AND RADIATION

CHAPTER I

INTRODUCTION AND REVIEW OF THE LITERATURE

One method of examining many-body problems is to utilize a statistical approach, replacing the system in question by a collection of representative ensembles. This method has led to some successes in determining the properties of plasmas in which quantum mechanical effects are significant, but, as yet, no complete, consistent exposition of the theory, based on the quantum analogue of the Liouville equation for the system, has been given. This investigation is concerned primarily with such an exposition. The entire non-relativistic Hamiltonian is retained, in order that transverse electromagnetic interactions may be studied.

In Chapter II the density matrix formalism is reviewed briefly and shown to be most useful in investigating the problem of quantum plasmas. The formalism is extended to include the degrees of freedom of the radiation field, in order that both fields and particles may be treated statistically. Inclusion of the transverse interactions is effected by replacing the electromagnetic fields by an infinite set of radiation oscillators and then defining a density matrix for this system. Finally, an alternate approach--that of quantum mechanical distribution functions--is discussed and the disadvantages of such a formulation indicated.

A preliminary problem is considered in Chapter III. Here, the potential energy of a particle is assumed to be derivable in a self-consistent manner. That is, the electrostatic potential is calculated from Poisson's equation with the sources derived from the particle

distributions. Transverse electromagnetic interactions are ignored. The dispersion relation obtained from the N-particle equations is found to be the same as that from the one-particle equations. Thus, particle correlations are not included.

In Chapter IV a hierarchy of equations for the "internal" or reduced density matrices for particles and oscillators is obtained by taking partial traces of the Liouville equation for the entire system. One integrates the equation over the coordinates of all but a small number of particles and oscillators. Due to interactions, the equation for the m-particle, t-oscillator density matrix contains the density matrices for $m+1$ particles and $t+1$ oscillators, etc. However, these interaction terms follow in a completely consistent manner, so that one no longer needs to insert ad hoc forms for exchange integrals, etc. Then, a perturbation theory is developed which enables one to close the chain of equations. The limits of validity of this treatment appropriate for a plasma are discussed. Finally, the quantum analogues of the classical Vlasov and Fokker-Planck equations are exhibited.

In Chapter V the theory is used to calculate dispersion relations for the frequencies of small disturbances in the plasma. The effects of particle correlation are demonstrated for the case of longitudinal Coulomb interactions and are found to be due to exchange. In addition, the dispersion relation for transverse interactions is also derived and some rather unusual features of this relation are compared with the classical case.

The investigation is summarized in Chapter VI and suggestions for

further studies are given.

I. BASIS FOR THE INVESTIGATION

The procedure for obtaining "kinetic" equations for internal distribution functions of a system was first developed by Bogoliubov* (1) in his study of the properties of un-ionized gases. Bogoliubov also indicated some of the problems which would be encountered in a similar development for systems interacting through long-range Coulomb forces. Born and Green (3), Kirkwood and collaborators (13, 25) and Yvon (37) also studied classical and quantum systems, using techniques similar to those developed by Bogoliubov. However, they too were primarily interested in un-ionized gases and liquids. Recently, Rosenbluth and Rostoker (26) derived kinetic equations for a classical plasma, assuming only Coulomb interactions. Simon and Harris (30) extended the theory to include transverse electromagnetic interactions.

Most of the investigations of quantum plasmas have employed techniques differing somewhat from those used here. Several texts have been devoted to the methods appropriate for various many-body problems, but some of them most often employed in plasma studies will be indicated here. Perhaps the best known treatment is due to Bohm and Pines (2). Here, "collective" variables replace the usual coordinates of the system, facilitating the solution of problems in which the individual particle nature is not as important as the gross features of the system. In par-

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*References are listed alphabetically and numerically in the bibliography

ticular, Bohm and Pines obtained a dispersion relation for the frequencies of collective oscillations of a quantum plasma. This same relation has been obtained by several other authors (8, 15, 29, 39) in different ways and will also be derived in this investigation. Of especial interest is the work of Klimontovich and Selin (7), in which kinetic equations for the quantum plasma were obtained and applied to several problems, including the small-amplitude Coulomb disturbances. Ehrenreich and Cohen (8) have also studied this problem, obtaining the quantum dispersion relation by means of the one-particle Liouville equation and the self-consistent field approximation for the Coulomb potential. Finally, von Roos (36), formulating the problem in terms of a quantum mechanical distribution function similar to that used first by Wigner (37), obtained the dispersion relation mentioned above and showed how exchange affects the relation.

None of the above treatments have included a development of kinetic equations for particles and the electromagnetic field, although Osborn and Klevans (24) initiated an investigation of this problem at about the same time that the present study was begun. However, the direction of these authors' work seems to be somewhat different from this dissertation.

CHAPTER II

THE DENSITY MATRIX

In order to develop a system of kinetic equations for a quantum mechanical plasma, it is necessary to introduce a distribution function containing the statistical information pertinent to the system under consideration. In this chapter, it is shown that the density matrix of Dirac (6) and von Neumann (35) can be employed, and the generalization of the resulting formalism to radiation fields is given. In the final section, an alternative formalism is discussed briefly and compared with the one utilized in this work.

I. PARTICLE DENSITY MATRICES

In ordinary non-relativistic quantum mechanics the state of a system is described by a wave function ψ, a function of the particle coordinates and time, which obeys the Schrodinger equation

$$H\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t} \tag{1}$$

where H is the Hamiltonian operator for the system. With this wave function or probability amplitude one calculates expectation values of operators for the system.

Instead of using the wave function of the system, an equivalent formalism can be developed using the density matrix. The density matrix is usually defined as

$$R(x;x') = \psi(x)\,\psi^{\dagger}(x'). \tag{2}$$

If we consider ψ to be a column vector in Hilbert space, then $\psi^{\dagger}$, the Hermetian conjugate of ψ, will be a row vector and the direct product,

represented by Eq. (2) will be an infinite matrix.

It is a simple matter to obtain an equation for the time dependence of R, in fact

$$\dot{R} = \dot{\psi}\psi^{+} + \psi\dot{\psi}^{+} = -\frac{i}{\hbar} H\psi\psi^{+} + \frac{i}{\hbar}\psi\psi^{+}H^{+} = -\frac{i}{\hbar}[H, R], \tag{3}$$

since H is Hermetian. It is also easy to see that the expectation value of an operator A(x) is given by

$$\langle A\rangle = \int \psi^{+}(x)\, a(x)\,\psi(x)\,dx = \int dx\,dx'\, a(x)\,\delta(x-x')\,R(x;x) \tag{4}$$
$$= \mathrm{Trace}\,(aR).$$

In the above, the xx' "matrix element" of the operator is written

$$a(x,x') = a(x)\,\delta(x-x'). \tag{5}$$

Thus the analogy to a matrix formalism is preserved. This particular notation is not essential to the development of the theory, but is less cumbersome than others commonly used (e.g., Lowdin (16), McWeeny (17)).

In general, the wave function ψ carries indices or quantum numbers which label the operators of which ψ is an eigenfunction. The density matrix for a state k is denoted by

$$R^{kk}(x;x') = \psi_k(x)\,\psi_k^{+}(x'). \tag{6}$$

The diagonal element of this matrix, $R^{kk}(x,x)$, gives the probability that a system in the state k (k may be a composite quantum number) is located at the point x. In addition to these one may define transition matrices

$$R^{kl}(x;x') = \psi_k\,\psi_l^{+}(x') \tag{7}$$

The diagonal elements of these matrices are not physical observables. However, if A is an operator which causes a transition in a system, then

$$\langle l|A|k\rangle = \int \psi_l^{+}(x)\, a\, \psi_k(x)\,dx = \mathrm{Tr}\, R^{kl} a. \tag{8}$$

This is simply the matrix element or transition element of the operator A. From this element we can calculate the transition probability.

The density matrix formulation of quantum theory has found many applications, in particular in the study of atomic and molecular systems. Recently, Lowdin (16) has developed a generalized Hartree-Fock approximation using the density matrix, and has applied it to many-electron and many-atom systems. Lowdin's work has the advantage of being more amenable to numerical treatment than previous theories. U. Fano (9), in an excellent review article, has discussed the non-statistical applications of the density matrix, with particular emphasis on nuclear physics and scattering theory.

The original development of the density matrix theory was intended to introduce a formalism analogeous to classical statistical mechanics. It is this particular approach which will be emphasized below. In presenting the ensemble theory, the treatment given by Lowdin will be followed. Since the theory can be found in any standard text in statistical mechanics (e.g., (33), (34)), only the main features will be presented here.

We assume that the system under consideration is, by reason of complexity in a physical sense, incapable of exact treatment using the Schrodinger equation. In other words, the values of a complete set of constants of motion or eigenvalues necessary to specify the state of the systems are not available. One then can consider representative systems, each of which has the same number of particles (canonical ensemble) and obeys the same Schrodinger equation, for which we can specify exactly the state by giving the eigenvalues. Then the expectation value

of an operator for the system is given by

$$\langle A \rangle = \sum_K P_K \langle \psi_K | A | \psi_K \rangle. \tag{9}$$

The P_k are the "weights" or probabilities of finding the system in the state described by the wave function ψ_k. Note that ψ_k is not necessarily a stationary state, although for most purposes it will be. The only requirement placed on the ψ_K is that they must be orthogonal to prevent mixing. The P_k are time independent quantities determined by the initial conditions. They are sometimes referred to as Boltzmann factors.

In Eq. (9), two distinct averages have been taken, the first being the usual quantum mechanical average or expectation value of the operator A, while the second is a statistical average over all possible states available for the ensemble. It is apparent from the above that a convenient definition of the density matrix of the ensemble is

$$R = \sum_K p_K \psi_K \psi_K^\dagger = \sum_K p_k R^{KK}, \tag{10}$$

and the average value of the operator A can be written as before

$$\langle A \rangle = Tr RA \tag{11}$$

Since the P_k are time-independent quantities, it is apparent that the new density matrix will also obey a Schrodinger equation, or rather its equivalent, Eq. (4). Thus, this treatment differs from a time dependent perturbation theory, where one writes

$$\psi = \sum_n a_n(t) \psi_n, \tag{12}$$

and the ψ_n are stationary wave functions. In fact, in some situations exactly such a procedure may be followed, giving for the density matrix (see McWeeny (17))

$$R = \sum_k p_k \left\{ \sum_{\lambda\lambda'} a_\lambda(t) a_{\lambda'}(t) \psi_\lambda \psi_{\lambda'}^\dagger \right\}. \tag{13}$$

Now, where one writes Eq. (4) for the ensemble density matrix, one is actually imparting a different kind of information than previously. This is, in fact now a statement of conservation of probability in two senses, the quantum mechanical and the statistical. In the latter sense, it is analogous to the Liouville equation of classical statistical mechanics, a continuity equation in the $6N$-dimensional phase space of the classical system. The importance of this two-fold statistical nature has been emphasized by Tolman (34).

With the density matrix for an ensemble defined, it is possible to develop a quantum statistical mechanics quite similar to classical statistical mechanics. In particular, the concept of the micro-canonical, canonical, and grand ensembles (see, e.g., ter Haar (33)) are all retained, and many other features of classical statistical mechanics can be adopted.

II. REDUCED DENSITY MATRICES

Since the complete density matrix for a system of N particles is a function of the coordinates of all these particles, it is seldom convenient to work with this quantity, disregarding completely the fact that the mathematical problem of calculating it is practically insurmountable. For this reason Husimi (12) introduced the "reduced" density matrices. These are "correlation" functions for $m(\leq N)$ particles of the N-particle system. They are defined by*

---

*This definition differs from the standard one in the introduction of the factor V^m. For the purposes of this investigation it is more convenient and only changes the normalization. Henceforth, this new normalization will also be used for R, the full density matrix for N particles.

$$R^{(m)}(x_1, x_2 \ldots x_m; x_1' x_2' \ldots x_m') = V^m \int \psi \psi^+ dx_{m+1} \ldots dx_N, \tag{14}$$

where V is the volume of the system. Thus, the original elements of $R^{(m)}$ multiplied by $Vd^{-m}x_1 \ldots dx_m$, gives the probability that particle 1 can be found in the volume dx_1 centered at x_1, particle 2 in dx_2 centered at x_2, etc., with the remainder of the N-m particles unspecified. The generalization of Eq. (14) to ensembles is obvious, and the elements have similar interpretations.

Lowdin (16) and McWeeny (17) have discussed the reduced density matrices quite completely, but some of the more important considerations will be included here. Since the most interesting and useful applications occur for identical particles, the discussion will be restricted to a system of N identical fermions. According to the Pauli exclusion principle, the wave function of this system must be completely antisymmetric under interchange of any two particles. Thus, for the two-particle reduced density matrix

$$R^{(2)}(x_1 x_2; x_1' x_2') = -R^{(2)}(x_1 x_2; x_2' x_1') = R^{(2)}(x_2 x_1; x_2' x_1'), \tag{15}$$

and for the diagonal elements,

$$R^{(2)}(x_1 x_2; x_1 x_2) = R^{(2)}(x_2 x_1; x_2 x_1) \tag{16}$$

and finally,

$$R^{(2)}(x_1 = x_2; x_1' = x_2') = 0. \tag{17}$$

The last is the result of the anti-symmetry of the wave function. The important point to note here is that the diagonal elements, used to calculate expectation values, are symmetric. This is simply an expression of the fact that the particles are indistinguishable.

Now, suppose that the Hamiltonian of the system contains a two-

particle operator, e.g.,

$$H^{(2)} = \sum_{i>j} U(x_i, x_j). \tag{18}$$

Then, the expectation value of this operator is

$$\langle H^{(2)} \rangle = \sum_{i>j} \int U(x_i, x_j)\, \psi\psi^{\dagger} dx = \frac{1}{V^2} \sum_{i>j} \int U(x_i, x_j) R^{(2)}(x_i x_j; x_i x_j) dx_i dx_j \tag{19}$$

$$= \frac{N(N-1)}{2V^2} \int U(x_1, x_2) R^{(2)}(x_1 x_2; x_1 x_2) = \frac{N(N-1)}{2V^2} \mathrm{Tr}^{(1,2)} U(x_1, x_2).$$

Hence, in order to calculate the expectation values of two-body operators one need know only $R^{(2)}$. While the previous discussion has been restricted to fermions, it is clear that similar considerations apply to bosons. The situations for mixed systems is only slightly more complicated and need not be considered.

Another interesting relation is obtained from the definition of the reduced density matrix.

$$R^{(m)} = V^m \int \psi\psi^{\dagger} dx_{m+1} \ldots dx_N = V^{-1} \int dx_{m+1} \left[V^{m+1} \int \psi\psi^{\dagger} dx_{m+2} \ldots dx_N \right]$$

$$= V^{-1} \int dx_{m+1} R^{(m+1)}(x_1 x_2 \ldots x_{m+1}; x_1' x_2' \ldots x_m', x_{m+1} \ldots) \tag{20}$$

$$= V^{-1} \mathrm{Tr}^{(m+1)} R^{(m+1)},$$

where $\mathrm{Tr}^{(m+1)}$ denotes the partial trace, i.e., integration over the coordinates of particle $(m+1)$. Thus, if the interactions among particles are confined to m-body forces, then it is only necessary to calculate $R^{(m)}$ to be able to determine all expectation values. One can then use the recurrence relation given by Eq. (20) for calculations of expectation values of $j(\leq m)$-body forces.

For many physical systems (possible exceptions being nuclei and molecules) interactions are only of the two-body type. For this reason, as well as mathematical complexity, most of the applications of the reduced density matrices have been confined to orders one and two. Husimi

and Nishiyama (13, 21) have given detailed discussions of the algebraic properties of the reduced density matrices for the canonical ensemble. In particular, recurrence relations for $R_N^{(m)}$ in terms of $R_{N-1}^{(m)}$, $R_N^{(m-1)}$, etc., have been found.* These relations are much more useful than Eq. (20) since it is clear that $R^{(m)}$ is, at best, obtained by solving an m-body problem. Thus, for $m > 2$, Eq. (20) is of academic interest only except in very special circumstances.

III. THE DENSITY MATRIX FOR RADIATION FIELDS

As is well known, Bohr originally demonstrated that if the uncertainty principle is to be universally valid, electromagnetic fields interacting with quantized systems must also be quantized. Quantization of the electromagnetic field essentially consists of requiring certain field components to obey commutation relations, i.e., the fields are no longer "c" numbers, but operators. In this section the density matrix theory will be extended to include the radiation fields.

Originally, fields were quantized by defining generalized "coordinate" fields and their conjugate "momentum" fields and then requiring the two classes to satisfy commutation relations similar to those obeyed by particle coordinates and moments. Although this procedure is not necessary, it has the advantage of simplicity and clarity. For this reason the canonical formalism given by Heitler (10) will be given here.

In order to define a Hamiltonian for the radiation field, one expands the vector potential in a series of orthogonal functions

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*The subscript refers to the number of particles in the system.

$$\vec{A} = \sum_\lambda q_\lambda \vec{A}_\lambda + q_\lambda^+ A_\lambda^+ . \tag{21}$$

The A_λ satisfy the equation

$$\left(\nabla^2 - \frac{\omega_\lambda^2}{c^2}\right) \vec{A}_\lambda = 0, \tag{22}$$

with $$q_\lambda(t) = |q_\lambda| e^{i\omega_\lambda t}. \tag{23}$$

Further, for most purposes, one employs solenoidal gauge, i.e.,

$$\vec{\nabla} \cdot A_\lambda = 0. \tag{24}$$

The generalized coordinates and momenta of the field are defined respectively as

$$Q_\lambda = q_\lambda + q_\lambda^+ \tag{25}$$

$$P_\lambda = -i\omega_\lambda (q_\lambda - q_\lambda^+). \tag{26}$$

Then, with these definitions, the Hamiltonian becomes

$$H = \sum_\lambda H_\lambda = \tfrac{1}{2} \sum_\lambda \left\{ P_\lambda^2 + Q_\lambda^2 \omega_\lambda^2 \right\} = 2 \sum_\lambda \omega_\lambda^2 q_\lambda^+ q_\lambda . \tag{27}$$

Quantization is attained by applying the commutation relations

$$[P_\lambda, Q_\lambda] = P_\lambda Q_\lambda - Q_\lambda P_\lambda = \hbar/i, \tag{28}$$

$$[P_\mu, P_\nu] = [Q_\lambda, Q_\mu] = 0. \tag{29}$$

The Q_λ and P_λ are now time-independent operators. The evolution of the system is determined by a Schrodinger equation

$$H \psi = \frac{\hbar}{i} \frac{\partial \psi}{\partial t} \tag{30}$$

ψ is the state vector of the system, depending on the generalized coordinates and time. The stationary state solutions for H_λ are

$$H_\lambda \psi_\lambda = E_\lambda \psi_\lambda = n \hbar \omega_\lambda \psi_\lambda \tag{31}$$

and, in general, any state vector can be written

$$\psi = \sum_{a_1 a_2 \cdots a_\lambda} C_{a_1 a_2 \cdots a_\lambda} \psi_1 \psi_2 \cdots \psi_\lambda . \tag{32}$$

Thus, by employing the amplitudes as generalized coordinates and moments, one can quantize the radiation field by imposing commutation relations in the usual manner. Furthermore, from Eq. (27), it is apparent

that the Hamiltonian is a sum of Hamiltonians of harmonic oscillators. Thus, the radiation field can be replaced by a system of oscillators, each with a different natural frequency ω_λ. Now, instead of discussing the photons of the electromagnetic field, one may alternatively consider the interactions of the radiation oscillators. This is, in fact, a consistent formulation of the original ideas of Planck. In the problems to be investigated subsequently this viewpoint will offer many advantages.

The simplicity resulting from the oscillator "picture" is illustrated in Eq. (32), which is significantly different from the corresponding expression for identical particles. Since oscillators of different natural frequencies are distinguishable, no symmetrization of the state vector is necessary. Each oscillator represents a separate degree of freedom for the fields: consequently, the state vector for a given configuration is simply the product of state vectors for each oscillator if there are no interactions. However, since photons are bosons, the state vector of a system of these particles must explicitly include their statistical correlation even in the absence of interactions.

Since the foregoing formalism so closely resembles the particle theory, an obvious definition of the density matrix is

$$R = \psi \psi^{\dagger}. \tag{33}$$

That R will satisfy an equation of motion like Eq. (4) needs no demonstration. The expectation values of operators follow in a simple manner. As an example, consider the expectation value of the number operator $\hat{n}_\lambda = q_\lambda^* q_\lambda$, using Eq. (32) in the number representation,

$$\mathrm{Tr}\, q_\lambda^* q_\lambda R = \sum_{n_1 n_2 \ldots n_1' n_2' \ldots} \langle n_1 n_2 \ldots | q_\lambda^{\dagger} q_\lambda | n_1' n_2' \ldots \rangle \langle n_1' n_2' \ldots | R | n_1 n_2 \ldots \rangle$$

$$= \sum_{NN'} \langle N|q^* q|N'\rangle \sum_{N''N'''} \langle N''|N'''\rangle C_{N''} C^*_{N'''} \langle N'''|N\rangle$$
$$= \sum_{NN'} C^*_N C_{N'} \langle N|q_\lambda^* q_{\lambda'}|N'\rangle = \langle \psi|q_\lambda^* q_{\lambda'}|\psi\rangle. \tag{34}$$

In order to define reduced density matrices for the oscillators, some comment is necessary about the state vector in the quantum theory of radiation. Generally, the emphasis is on the operators in this theory, and the state vector is only a formal concept. For this reason, one finds that most authors are not explicit in defining the variables upon which the state vector depends (e.g., see Heitler (10)). In the ensuing calculations the variables will, in most instances, be indicated by $(Q_1, Q_2, \ldots, Q)$. In the amplitude or coordinate representation, the eigenfunctions of H_λ are Hermite polynomials and are, in fact, explicit functions of the Q_λ. However, one should not infer that the representation is being used unless noted. With this exception in mind, the reduced density matrices can be defined as

$$R^{(s)}(Q_1 \ldots Q_s; Q_1' \ldots Q_s') = \int dQ_{s+1} \ldots dQ_\infty R(Q_1 \ldots Q_s, Q_{s+1} \ldots; Q_1' \ldots Q_s', Q_{s+1} \ldots). \tag{35}$$

This definition differs formally from the corresponding particle reduced density matrix only in the absence of the factor V^t.

IV. QUANTUM MECHANICAL DISTRIBUTION FUNCTIONS

Before preceding with the development of the kinetic equations for a plasma, it will be useful to consider an alternate approach to quantum statistical problems introduced by E. Wigner (37).

Instead of employing the density matrix in calculations, Wigner suggested the use of a quantum mechanical "distribution function" which, being a function of both coordinates and momentum, would be more similar

to the classical distribution function than the density matrix. His definition of the distribution function is given by the equation

$$f(\vec{x}_1,\vec{x}_2\ldots\vec{x}_N;\vec{p}_1,\vec{p}_2\ldots\vec{p}_N) = (h\pi)^{-N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d\vec{y}_1^{\,3}\,d\vec{y}_2^{\,3}\cdots d\vec{y}_N^{\,3}\;\psi^{\dagger}(\vec{x}_1+\vec{y}_1\ldots\vec{x}_N+\vec{y}_N)\psi(\vec{x}_1-\vec{y}_1\ldots\vec{x}_N-\vec{y}_N)\,e^{\frac{2i\vec{p}\cdot\vec{y}}{\pi}}, \tag{36}$$

with

$$\vec{p}\cdot\vec{y}=\sum_i \vec{p}_i\cdot\vec{y}_i, \tag{37}$$

and $\vec{p}_i$ is the Fourier wave vector

$$\vec{p}_i = p_{ix}\hat{e}_x + p_{iy}\hat{e}_y + p_{iz}\hat{e}_z . \tag{38}$$

This "probability" distribution function is easily seen to have the properties

$$\int\cdots\int f(\vec{x}_1\ldots\vec{x}_N;\vec{p}_1\ldots\vec{p}_N)\,d\vec{p}_1\,d\vec{p}_2^{\,3}\ldots d\vec{p}_N^{\,3} = |\psi(x_1\ldots x_N)|^2 \tag{39}$$

and

$$\int\cdots\int f(x_1\ldots x_N;p_1\ldots p_N)\,dx_1^3\ldots dx_N^3 = |\Phi(p_1\ldots p_N)|^2, \tag{40}$$

where $\phi(p_1\ldots p_N)$ is the momentum space representation of ψ. Eqs. (39) and (40) are simply the probabilities of finding particles 1...N at $\vec{x}_1\ldots\vec{x}_N$ and particles 1...N at $\vec{p}_1\ldots\vec{p}_N$ respectively.

Wigner showed that the expectation values of a certain class of operators could be calculated by direct integration of f with the operator over coordinates and moments. Later, Irving and Zwanzig (14) indicated that the expectation values of all operators can be calculated with f, provided that one obtains the quantum mechanical operators from their classical counterparts by the prescription given by Weyl (14).

The equation of change satisfied by the distribution function f is:

$$\frac{\partial f}{\partial t} = -\sum_{k=1}^{N}\frac{\vec{p}_k}{m_k}\cdot\frac{\partial f}{\partial \vec{x}_k} + \tag{41}$$

$$+\sum_{\lambda_i}\left[\frac{\partial^{\lambda_1+\lambda_2+\cdots+\lambda_N}}{\partial\vec{x}_1^{\lambda_1}\partial\vec{x}_2^{\lambda_2}\cdots\partial\vec{x}_N^{\lambda_N}}V(\vec{x}_1,\vec{x}_2,\cdots\vec{x}_N)\right]\left(\frac{\hbar}{2i}\right)^{\left(\sum_i\lambda_i-1\right)}\frac{\partial^{\sum\lambda_i}}{\Pi_i\lambda_i!\,\partial p^{\lambda_1}\cdots\partial p_N^{\lambda_N}}f, \quad \text{(cont.)}$$

where $V(x_1, x_2 \cdots x_N)$ is the potential energy of the system, N is the number of particles in the system and the Hamiltonian is assumed to consist of kinetic energies and scalar potential only. In the sum over λ_i, the λ_i are subject to the restriction

$$\sum_i\lambda_i = \text{odd integer} \tag{42}$$

It is apparent that as $\hbar \to 0$, this equation reduces to the classical Liouville equation. Also, one should observe that quantum mechanical corrections to the Liouville equation are second order in $\hbar$. This rather interesting point has also arisen in some recent work by von Roos (36) on quantum corrections to plasma dispersion relations.

The Wigner distribution function has had many applications. Wigner employed it to calculate lowest order quantum corrections to the classical Boltzmann function (37). It has also been applied extensively by Kirkwood and collaborators (14, 25) in deriving transport equations for low density gases and to formulate a hydrodynamics of quantum fluids. A. W. Saenz (27) also derived a transport equation for a dilute, non-degenerate, spinless gas using this function. Recently, von Roos (36), by defining the quantum distribution function as

$$f = \psi^*(x)\,e^{i\frac{p\cdot x}{\hbar}}\int\psi(x)\,e^{-\frac{ip\cdot x}{\hbar}}\,dx, \tag{43}$$

obtained an apparently less complicated quantum transport equation and with this calculated the lowest order quantum correction to the plasma dispersion relation.

Wigner's intention in introducing the distribution function, aside

from considerations of mathematical simplicity, was that quantum statistics might embody more of the concepts of classical statistics than previously. In particular, the very useful phase space of coordinates and momenta could be adopted. Thus, large portions of the classical formalism could be maintained without change, even though the fundamental postulates of the theory must, of course, differ. In addition, the transition from quantum to classical treatments is simplified. However, there are several annoying features which one encounters, aside from the complexity of Eq. (21). As was mentioned earlier, the calculation of operator averages is not straightforward. Of more importance, due to the uncertainty principle, a function giving a simultaneous distribution of position and momentum is obviously unobservable, and inferences from relation between such quantities must be made with great caution. Finally, there is the embarassing feature that none of the "probability" functions are positive definite. However, it cannot be denied that the striking similarities in appearance of the quantum distribution functions and their classical counterparts are extremely useful for parallel developments of the statistical theories. Certainly one useful feature of all these distribution functions, irrespective of their apparent differences, is that in the classical limit they all reduce to the classical distribution function. Of course, the density matrix also has this property, but the correspondence is not so clear.

CHAPTER III

THE SELF-CONSISTENT FIELD APPROXIMATION

In this chapter the dispersion relation for a quantum plasma is calculated using the N-particle self-consistent field (SCF) approximation. It is shown that the statistics enter the dispersion relation in the same manner as obtained previously by other authors in one-particle treatments.

From Chapter II, one knows that the density matrix R for the N-particle system satisfies the Liouville equation

$$i\hbar \frac{\partial R}{\partial t} = [H, R]. \tag{1}$$

Now, in the SCF approximation, the Hamiltonian is simply

$$H = \sum_i -\frac{\hbar^2}{2m_i} \nabla_i^2 + \sum_i q_i \phi(x_i) = H_0 + V, \tag{2}$$

where q_i is the charge on the ith particle.

Here, no transverse electromagnetic terms have been included. The potential ϕ is calculated from the equation

$$\nabla^2 \phi = -4\pi \sum_i q_i \, Tr\left(R\,\delta(\vec{x}-\vec{x}_i)\right). \tag{3}$$

Now, we consider a system of electrons slightly disturbed from equilibrium in the presence of a "smeared" positive background. The density matrix can then be written

$$R = \rho^0 + \rho, \tag{4}$$

where ρ_0 is the density matrix for the canonical ensemble and ρ is a small perturbation. Eq. (2) then becomes

$$\nabla^2 \phi = e \sum_i Tr\left(\rho\,\delta(x-x_i)\right) \tag{5}$$

since the uniform positive background cancels the zero-order negative charge density. Then, Eq. (1) becomes

$$i\hbar \frac{\partial \rho}{\partial t} = [H_0, \rho] + [V, \rho^0] \tag{6}$$

since ρ_0 commutes with H_0.

Now, since ρ_0 is diagonal in the energy representation, one can take matrix elements of Eq. (6) with free particle eigenfunctions

$$\psi_{\vec{K}} = \sum_P \frac{(-)^P P_K \psi_{k_1} \psi_{k_2} \cdots \psi_{k_N} \alpha_1 \cdots \alpha_N}{\sqrt{N! V^N}}, \tag{7}$$

where

$$\psi_{k_i} = e^{i \vec{k} \cdot \vec{r}_i} \tag{8}$$

P_k is the permutation operator for N objects and α_m is a spin function. The sum is taken over all N! permutations. It is a simple matter to calculate the matrix elements (see, e.g., Condon and Shortley (5)). One has

$$i\hbar \langle \xi | \frac{\partial \rho}{\partial t} | \zeta \rangle = (E_\xi - E_\zeta) \langle \xi | \rho | \zeta \rangle + (f^0(E_\zeta) - f_0(E_\xi)) \langle \xi | V | \zeta \rangle, \tag{9}$$

where

$$\Sigma_\xi = \sum_i^{(\xi)} \frac{\hbar^2}{2m_i} k_i^2 \tag{10}$$

and

$$f_0(E_\xi) = \langle \xi | \rho_0 | \xi \rangle. \tag{11}$$

The restricted sum in Eq. (10) indicates that the $\vec{k}_i$ must be chosen from the set $\xi \equiv (\vec{k}_1, \vec{k}_2 \cdots \vec{k}_N)$.

Furthermore, one can write

$$\phi(\vec{r}_i, t) = \sum_{\vec{q}} \phi(\vec{q}, t) e^{i \vec{q} \cdot \vec{r}_i} \tag{12}$$

and

$$V = \sum_i \sum_q e \phi(\vec{q}, t) e^{-i \vec{q} \cdot \vec{r}_i}. \tag{13}$$

Now, consider a matrix element of V. It is easy to see (5) that if the sets ξ and ζ differ by more than one quantum number

$$\langle \xi | V | \zeta \rangle = 0. \tag{14}$$

Thus,

$$\langle \xi | V | \zeta \rangle = \langle \xi | \sum_i e\phi_i | \zeta \rangle$$
$$= \pm \prod_{j \neq n} \delta_{\vec{k}_j, \vec{k}_j'} \delta_{SS'} \langle \vec{k}_n | \phi | \vec{k}_n' \rangle, \tag{15a}$$

with

$$\delta_{SS'} = \prod_{j=1}^{N} \delta_{\vec{s}_j, \vec{s}_j'} \tag{15b}$$

and

$$S = (\vec{s}_1, \vec{s}_2 \ldots \vec{s}_N) \tag{15c}$$

δ_{ij} is the Kronecker delta. Also, $\vec{k}_n$ and $\vec{k}_n'$ are the two differing k vectors in the sets ξ and ζ. Thus, $\langle \vec{k}_n | \phi | \vec{k}_n' \rangle$ is a single-particle matrix element. The sign depends on whether an even (+) or odd (-) number of permutations is necessary to put $\vec{k}_n$ and $\vec{k}_n'$ in the same position in the sets ξ and ζ. But

$$\langle \vec{k}_n | \phi | \vec{k}_n' \rangle = \frac{1}{V} \int e^{i(\vec{k}_n - \vec{k}_n') \cdot \vec{r}} \sum_q \phi(\vec{q}, t) e^{i\vec{q} \cdot \vec{r}} d^3 r$$
$$= \phi(\vec{q}, t) \, \delta_{\vec{k}_n' - \vec{k}_n, \vec{q}}, \tag{16}$$

hence, Eq. (6) becomes

$$i\hbar \langle \xi | \frac{\partial \rho}{\partial t} | \zeta \rangle = (E_\xi - E_\zeta) \langle \xi | \rho | \zeta \rangle$$
$$\pm [f_0(E_\zeta) - f_0(E_\xi)] \, \phi(\vec{q}, t) \prod_{j \neq n} \delta_{\vec{k}_j, \vec{k}_j'} \delta_{SS'} \delta_{\vec{k}_n' - \vec{k}_n, \vec{q}}. \tag{17}$$

In addition,

$$\sum_i \langle \xi | \delta(\vec{r} - \vec{r}_i) | \zeta \rangle = \pm \prod_{j \neq n} \delta_{\vec{k}_j, \vec{k}_j'} \delta_{S,S'} \langle \vec{k}_n | \delta(\vec{r} - \vec{r}_n) | \vec{k}_n' \rangle$$
$$= \pm \prod_{j \neq n} \delta_{\vec{k}_j, \vec{k}_j'} \delta_{S,S'} \frac{e}{V} e^{i\vec{q} \cdot \vec{r}} \delta_{\vec{k}_n' - \vec{k}_n, \vec{q}}, \tag{18}$$

Hence

$$Tr\left(\sum_i \delta(\vec{r} - \vec{r}_i) \rho\right) = \sum_{(\xi)(\zeta)} \langle \xi, S | \sum \delta(\vec{r} - \vec{r}_i) | \zeta, S' \rangle \langle \zeta, S' | \rho | \xi, S \rangle =$$
$$\sum_{\substack{(\xi)(\zeta) \\ \vec{k}_1 \ldots \vec{k}_N, S \\ \vec{k}_1' \ldots \vec{k}_N', S'}} \langle \vec{k}_1 \ldots \vec{k}_N; S | \sum \delta(\vec{r} - \vec{r}_i) | \vec{k}_1' \ldots \vec{k}_N'; S' \rangle \langle \vec{k}_1' \ldots \vec{k}_N' | \rho | \vec{k}_1 \ldots \vec{k}_N; S \rangle = \tag{19}$$

$$= \sum_{\xi', i, \vec{k}_i, s} (\pm) \frac{e}{V} e^{-i\vec{q}\cdot\vec{r}_i} \langle \xi', \vec{k}_i; s | \rho | \xi', \vec{k}_i + \vec{q}; s \rangle, \qquad \text{(cont.)}$$

where now ξ' is the set ξ with $\vec{k}_1$ omitted. The sign is determined as before. However, changing the order of the differing indices also introduces a sign change (4), so that one may perform the sum over i (= 1...N), giving

$$Tr\left(\Sigma_i \delta(\vec{r} - \vec{r}_i)\rho\right) = \sum_{\vec{k}_1, \xi', s} N \frac{e}{V} e^{-i\vec{q}\cdot\vec{r}} \langle \vec{k}_1, \xi', s | \rho | \vec{k}_1 + \vec{q}, \xi', s \rangle, \qquad (20)$$

and now we have re-named the differing indices $\vec{k}_1$ and $\vec{k}_1'$. Thus Eq. (5) can now be written, using Eqs. (12) and (20),

$$\phi(\vec{q}, t) = \frac{4\pi N e}{q^2 V} \sum_{\vec{k}_1, \xi', s} \langle \vec{k}_1, \xi', s | \rho | \vec{k}_1 + \vec{q}, \xi', s \rangle . \qquad (21)$$

Now, since the Hamiltonian is independent of spin, the initial arbitrary spin orientation will be preserved. Furthermore, the spinless density matrix is given by (16)

$$\langle \xi | \rho | \xi' \rangle = \sum_{s_1 s_2 \ldots} \langle \xi; s_1 \ldots s_N | \rho | \xi'; s_1 \ldots s_N \rangle, \qquad (22)$$

so, one has finally

$$\phi(\vec{q}, t) = \frac{4\pi N e}{q^2 V} \sum_{\vec{k}_1, \xi'} \langle \vec{k}_1, \xi' | \rho | \vec{k}_1 + \vec{q}, \xi' \rangle. \qquad (23)$$

Now, if one assumes that $\phi(\vec{q},t)$ and ρ (t) are proportional to $e^{-i\omega t}$, Eq. (17) becomes

$$\left[-\hbar\omega + E_{\xi, \vec{k}_1'} - E_{\xi', \vec{k}_1}\right] \langle \xi', \vec{k}_1'; s | \rho | \xi', \vec{k}_1'; s' \rangle =$$

$$\frac{4\pi N e^2}{q^2 V} \left[f_0(E_{\xi'}, \vec{k}_1) - f_0(E_{\xi'}, \vec{k}_1') \right] \prod_{j \neq i} \delta_{\vec{k}_j, \vec{k}_j'} \delta_{ss'} \delta_{\vec{k}_1' - \vec{k}_1, \vec{q}} \, \phi(\vec{q}, t). \qquad (24)$$

Dividing both sides by the term in ω and letting

$$\xi' = \xi' , \qquad (25a)$$

$$\vec{k}_1' = \vec{k}_1 + \vec{q}, \qquad (25b)$$

$$s = s' \qquad (25c)$$

and summing both sides, one obtains

$$\sum_{\xi',\vec{k}_1,s} \langle \xi',\vec{k}_1;s|\rho|\xi',\vec{k}_1+\vec{q};s\rangle =$$
$$\sum_{\xi',\vec{k}_1,s} \frac{4\pi Ne^2}{q^2 V} \frac{[f_0(E_{\xi',\vec{k}_1+\vec{q}}) - f_0(E_{\xi',\vec{k}_1})]}{E_{\xi',\vec{k}_1+\vec{q}} - E_{\xi',\vec{k}_1} - \hbar\omega} \sum_{\zeta,\vec{\ell}} \langle \zeta,\vec{\ell}|\rho|\zeta,\vec{\ell}+\vec{q}\rangle . \tag{26}$$

Now, one can use Eq. (22) and cancel the sums over $\langle \xi',\vec{k}_1|\rho|\xi',\vec{k}_1+\vec{q}\rangle$ since there are only dummy indices to consider. This gives

$$1 = \sum_{\xi',\vec{k}_1,s} \frac{4\pi Ne^2}{q^2 V} \frac{[f_0(E_{\xi',\vec{k}_1+\vec{q}}) - f_0(E_{\xi',\vec{k}_1})]}{E_{\xi',\vec{k}_1+\vec{q}} - E_{\xi',\vec{k}_1} - \hbar\omega} . \tag{27}$$

But Eq. (22) holds for ρ_0 also. Furthermore,

$$E_{\xi,\vec{k}_1+\vec{q}} - E_{\xi,\vec{k}_1} = \sum_{i>1}^{(\xi)} \frac{\hbar^2 k_i^2}{2m} + \frac{\hbar^2}{2m}(\vec{k}_1+\vec{q})^2 - \sum_{i>1}^{(\xi)} \frac{\hbar^2 k_i^2}{2m} - \frac{\hbar^2 k_1^2}{2m} =$$
$$\frac{\hbar^2}{2m}\left[(\vec{k}_1+\vec{q})^2 - k_1^2\right]. \tag{28}$$

Finally, since the denominator depends only on $\vec{k}_1'$, one can perform the sum over all $\vec{k}_i$ ($i > 1$). From Chapter II, this gives the reduced density matrix for one particle (partial traces are independent of the representation) in k space. But this is simply (3k)

$$f_0^{(1)}(\vec{k}) = \sum_{i>1}^{(K)} \langle k_1 \ldots k_N|\rho^0|k_1 \ldots k_N\rangle$$
$$= \left[e^{-\frac{E_k - E_f}{\kappa T}} + 1\right]^{-1}, \tag{29}$$

where E_f is the Fermi energy. Thus, the final dispersion relation becomes

$$1 = \frac{4\pi Ne^2}{q^2 V} \sum_{\vec{k}} \frac{f_0^{(1)}(E_{\vec{k}+\vec{q}}) - f_0^{(1)}(E_{\vec{k}})}{\frac{\hbar^2}{2m}[(\vec{k}+\vec{q})^2 - k^2] - \hbar\omega} \tag{30}$$

where the subscript on k has been dropped.

This is precisely the dispersion relation obtained by Bohm and Pines (2) and others (8, 15, 36, 40) by several different methods. Generally, one takes the volume to be infinite, keeping N/V finite. Then,

the sum over $\vec{k}$ can be replaced by an integral, giving

$$I = \frac{4\pi Ne^2}{Vq^2} \int dk \, \frac{[f_0^{(1)}(\varepsilon_{k+q}) - f_0^{(1)}(E_k)]}{\frac{\hbar^2}{2m}[(k+q)^2 - k^2] - \hbar\omega} \tag{31}$$

Unfortunately, this expression can be integrated in closed form only at zero temperature (15). Furthermore, the integral in series gives a doubly-infinite series of transcendental functions, the arguments of which contain ω . Needless to say, these functions cannot be inverted to give $\omega(k)$. However, there is one point to which attention should be called: in this treatment, the statistics of the particles enters only through $f_0^1(E_k)$, which is the same as $f_0(E_k)$ for the one-particle system (33). Thus, nothing new is obtained in the N-particle SCF treatment. In order to include exchange effects it is necessary to go to a more exact treatment and include explicitly the interactions which will lead to correlation effects. This requires a more complete development of kinetic equations for single-particle reduced density matrices and will be considered in the next chapters.

CHAPTER IV

DEVELOPMENT OF THE QUANTUM KINETIC EQUATIONS

In this chapter, new equations for a system of charged particles will be developed in order that a more detailed study than the self-consistent field approximation can be made. In the first section, the Liouville equation for a system of particles and radiation oscillators is reduced to a less complex set of equations for the reduced density matrices of the system by integrating over groups of particle and oscillator coordinates. In the second section, the two simplest sets of kinetic equations--the analogues of the classical Vlasov and Fokker-Planck equations--are displayed and some of their interesting properties compared with the classical theory of Simon and Harris (30).

I. THE HIERARCHY OF KINETIC EQUATIONS

As was indicated earlier, the procedure employed to obtain dynamical equations for the particle distributions is the following: the N-particle, ∞ -oscillator Liouville equations will be integrated over the coordinates of all but a few particles and oscillators. This will yield equations for the internal or reduced density matrices.

Physically, the plasma state consists of an equal number of positively and negatively charged particles in unbound states. Although there may be other types of forces besides those electromagnetic in origin which are important in some plasmas, for our purposes we shall consider only electromagnetic fields. Furthermore, as an additional simplification, this development will apply to a system consisting of N electrons

and a "smeared" background of positive charge. In addition, for the present, external electromagnetic fields will be omitted, since they can be added at any time with no difficulties. For the system considered, the Hamiltonian is (10)

$$H = \sum_j \Big\{ -\frac{\hbar^2}{2m}\nabla_j^2 - \frac{e}{mc}\sum_{\lambda,\mu}(q_\lambda a_\lambda + q_\lambda^* a_\lambda^+)\hat{\epsilon}_{\lambda\mu}\cdot\frac{1}{i}\vec{\nabla}_j$$
$$+\sum_{\substack{\lambda,\rho\\ \mu,\nu}}\frac{e^2}{2mc}(q_\lambda a_\lambda + q_\lambda^+ a_\lambda^+)(q_\rho A_\rho + q_\rho^* a_\rho^+)\hat{\epsilon}_{\lambda\mu}\cdot\hat{\epsilon}_{\rho\nu} \qquad (1)$$
$$-\sum_{\lambda,\mu}\frac{e\hbar}{mc}\vec{\sigma}_j\cdot\vec{\nabla}\times\hat{\epsilon}_{\lambda\mu}(q_\lambda a_\lambda + q_\lambda^+ a_\lambda^+)$$
$$+\sum_{i\neq j}\frac{e^2}{|\vec{r}_i - \vec{r}_j|} + 2\sum_\lambda \omega_\lambda^2 q_\lambda^+ q_\lambda .$$

Here, the A_λ are the functions discussed in Chapter II, $\hat{\epsilon}_{\lambda\mu}$ ($\mu = 1,2$) is the polarization vector in the μth direction and $\vec{\sigma}_i$ is the spin operator for the ith particle. Now, as was mentioned previously, it will be convenient to use a "coordinate space" representation of the above operators. For simplicity, all variables will be treated as if they were continuous. The coordinate "matrix elements" of the above operator are then, taking the kinetic energy of the ith particle as an example,

$$\nabla_j^2(x_1 x_2 \ldots x_N; x_1' \ldots x_N' \,|\, Q_1 \ldots Q_\sigma; Q_1' \ldots Q_\sigma')$$
$$= \nabla_j^2 \prod_{k=1}^{N}\delta(x_k - x_k')\prod_{\tau=1}^{\sigma}\delta(Q_\tau - Q_\tau') \qquad (2)$$

where now

$$x_j = (\vec{r}_j, s_j; m_{sj}) \tag{3}$$

and the Q_λ are the "coordinates" of the oscillators discussed in Chapter II and include the polarization coordinates. This is simply an extension of the Weyl representation (14) to include spin and oscillator coordinates. Also, one has similar expressions for the other operators with the delta functions multiplying the usual operators on the right. Physically, these are all "local" operators for any given particle. (The Coulomb potential depends on the coordinates of <u>two</u> particles, but is "local" in each of these coordinates.)

Now it is possible to consider the Liouville equation as a matrix equation; thus, e.g.,

$$\begin{aligned}
&(\nabla_j^2 R)(x_1 \cdots x_N; x_1' \cdots x_N' \mid Q_1 \cdots Q_\sigma; Q_1' \cdots Q_{\sigma'}) = \\
&\int dx_1'' \cdots dx_N'' \, dQ_1'' \cdots dQ_\omega'' \{ \nabla_j^2 (x_1 \cdots x_N; x_1' \cdots x_N' \mid Q_1 \cdots Q_\omega; Q_1' \cdots Q_{\sigma'}) \\
&\quad \times R(x_1'' \cdots x_N''; x_1' \cdots x_N' \mid Q_1'' \cdots Q_\sigma''; Q_1' \cdots Q_{\sigma'}) \} \\
&= \int dx_j'' \, \nabla_j^2 \delta(x_j - x_j'') \, R(x_1 \cdots x_j''; x_1 \cdots x_j' \mid Q_1 \cdots; Q_1' \cdots) \\
&= \nabla_j^2 R(x_1 \cdots x_j \cdots; x_1' \cdots x_j' \cdots \mid Q_1 \cdots; Q_1' \cdots),
\end{aligned} \tag{4}$$

with similar expressions for the other terms in the Hamiltonian. While this notation is somewhat cumbersome, it is nonetheless simpler than other forms used.

Before performing the calculations, a further word about notation is necessary. In order to reduce the complexity of the arguments of the various functions, we shall write:

$$x_{[m]} = (x_1, x_2 \cdots x_m), \tag{5}$$

$$Q_{[\lambda]} = (Q_1 \cdots Q_\lambda) \tag{6}$$

and

$$x_{[\overline{m}]} = (x_{m+1}, \ldots x_N), \tag{7}$$

$$Q_{[\overline{\lambda}]} = (Q_{\lambda+1}, \ldots Q_\sigma) \tag{8}$$

The same notation will be used for differentials, i.e.

$$dx_{[\overline{m}]} = dx_{m+1} \ldots dx_N, \tag{9}$$

$$dQ_{[\overline{\lambda}]} = dQ_{\lambda+1}, \ldots dQ_\sigma. \tag{10}$$

Then, for example, the reduced density matrix for m particles and μ oscillators will be:

$$R^{m;\mu}(x_1 \ldots x_m; x_1' \ldots x_m' \mid Q_1, \ldots Q_\mu; Q_1' \ldots Q_\mu') = R^{m;\mu}(x_{[m]}; x_{[m]}' \mid Q_{[\mu]}, Q_{[\mu]}') \tag{11}$$

$$= V^m \int dx_{[\overline{m}]}\, dQ_{[\overline{\mu}]}\, R(x_{[m]}, x_{[\overline{m}]}; x_{[m]}', x_{[\overline{m}]} \mid Q_{[\mu]}, Q_{[\mu]}'; Q_{[\overline{\mu}]}, Q_{[\overline{\mu}]}).$$

With these preliminary remarks, let us consider the form of the Hamiltonian in Eq. (1). This can be written

$$H = H^{1;0} + H^{2;0} + H^{0;1} + H^{1;1} + H^{1;2} \tag{12}$$

where $H^{1;0}$ is a sum of one-particle operators, $H^{2;0}$ is a sum of two-particle operators, $H^{0;1}$ is a sum of one-oscillator operators, $H^{1;1}$ contains one-particle, one-oscillator operators only and $H^{1;2}$ is a sum of one-particle, two oscillator operators. In the integration over all but m particles and τ oscillators, some of the terms in the Hamiltonian will vanish. For the sake of brevity, we will consider examples of the various operators as classified in Eq. (12).

First, let us examine terms of the type found in $H^{1;0}$, i.e., single-particle operators. One such term will be

$$\int dx_{[\overline{m}]}\, dQ_{[\overline{\tau}]} [R, U^2] = \int dx_{[\overline{m}]}\, dx''_{[\overline{m}]}\, dx''_{[\overline{m}]} \{ R^{m;\tau}(x_{[m]}, x_{[\overline{m}]}; x''_{[m]}, x''_{[\overline{m}]}) \tag{13}$$

$$\times \nabla_j''^2 \delta(x_{[m]}'' - x_{[m]}') \delta(x_{[\bar{m}]}'' - x_{[\bar{m}]}) - \nabla_j^2 \delta(x_{[\bar{m}]} - x_{[\bar{m}]}'') R^{N;\tau}(x_{[\bar{m}]}'', x_{[m]}', x_{[\bar{m}]}). \qquad \text{(cont.)}$$

The integrations on Q_τ do not affect the operator, so they have been performed and the arguments suppressed. Now, suppose that j is in the set $[\bar{m}]$. One can perform the integrations over all $x''[\bar{m}]$ and $x[\bar{m}]$ except for the jth coordinate and also over all $x''[\bar{m}]$ to get

$$\frac{1}{V^{m+1}} \int dx_j\, dx_j'' \Big\{ R^{m+1;\tau}(x_{[m]}, x_j; x_{[m]}', x_j'') \nabla_j''^2 \delta(x_j'' - x_j) -$$

$$\nabla_j^2 \delta(x_j - x_j'') R^{m+1;\tau}(x_{[m]}, x_j''; x_{[m]}', x_j) \Big\} =$$

$$\frac{1}{V^{m+1}} \int dx_j\, dx_j'' \Big\{ \nabla_j''^2 R^{m+1}(x_{[m]}, x_j; x_{[m]}', x_j'') \delta(x_j - x_j'') \qquad (14)$$

$$- \nabla_j^2 R^{m+1;\tau}(x_{[m]}, x_j''; x_{[m]}', x_j) \delta(x_j - x_j'') \Big\} = 0.$$

The result follows since in each term, the operator acts on the right coordinate of the density matrix. Another way of seeing the same result is to interchange double-primed and unprimed variables of particle j in the second term. This is possible sime they are integration variables. However, if j had been in the set $[m]$, the right coordinate of the jth particle would have been primed, and there would be only one integral over x_j'' to consider. Then the operator would have been operating on different variables in R, i.e., the right in the first term and the left in the second. Then, one would obtain

$$\int dx_{[\bar{m}]} [R^{m;\tau}, \nabla_j^2] = \frac{1}{V} m [R^{m;\tau}, \nabla_j^2]. \qquad (15)$$

Now, let us consider a typical term appearing in $H^{2:0}$. If both particles are in the set $[\bar{m}]$, considerations similar to those of the previous example hold, and the commutator vanishes. If only one particle is in the set $[\bar{m}]$, one has

$$\int dX_{[\bar{m}]}\left[R^{N:\tau},\frac{1}{|\vec{r}_i-\vec{r}_j|}\right]=\frac{1}{V^{m+1}}\int dx_j dx_j''\left[R^{m+1:\tau}_{(x_1\cdots x_j;x_1'\cdots x_j'')}\frac{\delta(x_j''-x_j)}{|\vec{r}_i-\vec{r}_j|}-\right.$$

$$\left.\frac{\delta(x_j-x_j'')}{|\vec{r}_i-\vec{r}_j|}R^{m+1:\tau}_{(x_1\cdots x_j'';x_1'\cdots x_j)}\right]=\int\frac{dx_j}{V^{m+1}}R^{m+1:\tau}_{(x_1\cdots x_j;x_1'\cdots x_j)}\left[\frac{1}{|\vec{r}_i-\vec{r}_j|}-\frac{1}{|\vec{r}_i-\vec{r}_j|}\right] \tag{16}$$

$$\neq 0.$$

Finally, if neither particle is in the set $[m]$, one has

$$\int dx_{[\bar{m}]}\left[R^{N:\tau},\frac{1}{|\vec{r}_i-\vec{r}_j|}\right]=\frac{1}{V^m}\left[R^{m:\tau},\frac{1}{|\vec{r}_i-\vec{r}_j|}\right] \tag{17}$$

Next, we consider a typical term in $H^{1:1}$. One such term is $q_\lambda a_\lambda \vec{\epsilon}_{\lambda\mu}\cdot\vec{v}_j$. If λ is in the set $[\bar{\tau}]$ and j is in the set $[\bar{m}]$, one will again get no contribution. However, if j is not in $[m]$, one has

$$\frac{1}{V^m}\int dx_j''\,dQ_\lambda\,dQ_\lambda''\left\{R^{m:\tau+1}_{(x_j x_j''|Q_\lambda Q_\lambda'')}q_\lambda''\vec{A}_\lambda\delta(Q_\lambda''-Q_\lambda)\frac{1}{i}\vec{v}''\delta(x_j''-x_j)\right.$$

$$\left.-q_\lambda\vec{A}_\lambda\delta(Q_\lambda-Q_\lambda'')\frac{1}{i}\vec{v}\delta(x_j-x_j'')R^{m:\tau+1}_{(x_j'';x_j|Q_\lambda'';Q_\lambda)}\right\}= \tag{18}$$

$$\frac{1}{V^m}\int dQ_\lambda\left[R^{m:\tau+1},q_\lambda\vec{A}_\lambda\cdot\frac{1}{i}\vec{v}\right]\neq 0,$$

where $$\vec{A}_\lambda=\sum_\mu\hat{\epsilon}_{\lambda\mu}a_\lambda \tag{18a}$$

and $$\vec{A}_\lambda^{\dagger}=\sum_\mu\hat{\epsilon}_{\lambda\mu}a_\lambda^{+}. \tag{18b}$$

That this term does not vanish can be seen by examining the arguments of

the operators in Eq. (18). Although q_λ acts on the same coordinates in each term, i.e., the right coordinate, $\vec{v}_j$ acts on different ones. Hence the term must be retained. Finally, if neither λ nor j is included in the integration, one has

$$\int dx_{[\bar{m}]} dQ_{[\bar{\tau}]} \left[R, q_\lambda \vec{A}_\lambda \cdot \frac{1}{i} \vec{v} \right] = \frac{1}{V^m} \left[R^{m;\tau}, q_\lambda \vec{A}_\lambda \cdot \frac{1}{i} \vec{v}_j \right] \tag{19}$$

The integrals of terms in $H^{1;2}$ are similar to those in $H^{1;1}$ except one will also have integrals over $R^{m+1;\tau+1}$ and $R^{m;\tau+2}$ in addition to the types considered above.

Finally, the terms in $H^{0;1}$ will be similar to those in $H^{1;0}$. If λ is in $[\bar{\tau}]$, the term vanishes; otherwise it must be retained.

With this final consideration, it is now possible to write the kinetic equation for the m-particle, τ -oscillator reduced density matrix. This equation is (multiplying each term by V^m)

$$\frac{\hbar}{i} \frac{\partial R^{m;\tau}}{\partial t} + \left[R^{m;\tau}, H^{m;\tau} \right] =$$

$$\sum_{j=1}^{m} \sum_{\chi=1}^{\infty} \frac{e\hbar}{mc} \int dQ_\chi \left[R^{m;\tau+1}_{(m,\tau;\lambda)}, (q_\lambda \vec{A}_\lambda + q_\lambda^+ A_\lambda^+) \cdot \frac{1}{i} \vec{v}_j \right] \tag{20}$$

$$+ \frac{N-m}{V} \sum_{\lambda=[\tau]} \frac{e\hbar}{mc} \int dx \left[R^{m+1;\tau}, (q_\lambda \vec{A}_\lambda + q_\lambda^+ A_\lambda^+) \cdot \frac{1}{i} \vec{v}_j \right]$$

$$- \frac{e^2}{mc^2} \sum_{j=1}^{m} \sum_{\lambda=[\tau]} \sum_{\rho=1}^{\infty} \int dQ_\rho \left[R^{m;\tau+1}, (q_\lambda \vec{A}_\lambda + q_\lambda^+ \vec{A}_\lambda^+) (q_\rho \vec{A}_\rho + q_\rho^+ A_\rho^+) \right]$$

$$-\frac{e^2}{2mc^2}\sum_{j=1}^{m}\sum_{\lambda,\rho=1}^{\infty}{}^{[\tau]}\int dQ_\rho\, dQ_\lambda\left[R^{m:\tau+2}_{(m:c;j\lambda\rho)},(g_\lambda\vec{A}_\lambda+g_\lambda^+\vec{A}_\lambda^+)\cdot(g_\rho\vec{A}_\rho+g_\rho^+A_\rho^+)\right]$$

$$-\frac{N-m}{V}\frac{e^2}{2mc^2}\sum_{\lambda,\rho=[\tau]}\int dx\left[R^{m+1:\tau},(g_\lambda\vec{A}_\lambda+g_\lambda^+\vec{A}_\lambda^+)\cdot(g_\rho\vec{A}_\rho+g_\rho^+\vec{A}_\rho^+)\right]$$

$$-\frac{e^2(N-m)}{2mc^2V}\sum_{\rho=1[\tau]}\sum_{\lambda=1}^{\infty}\int dQ_\lambda\, dx\left[R^{m+1:\tau+1},(g_\lambda\vec{A}_\lambda+g_\lambda^+\vec{A}_\lambda^+),(g_\rho\vec{A}_\rho+g_\rho^+A_\rho^+)\right] \quad \text{(cont.)}$$

$$+\sum_{j=1}^{m}\sum_{\lambda=1}^{\infty}{}^{[\tau]}\frac{e\hbar}{mc}\int dQ_\lambda\left[R^{m:\tau+1},\vec{\sigma}_j\cdot\vec{\nabla}\times(g_\lambda\vec{A}_\lambda+g_\lambda^+A_\lambda^+)\right]$$

$$+\frac{N-m}{V}\frac{e\hbar}{mc}\sum_{\lambda=[\tau]}\int dx\left[R^{m+1:\tau},\vec{\sigma}\cdot\vec{\nabla}\times(g_\lambda\vec{A}_\lambda+g_\lambda^+\vec{A}_\lambda^+)\right]$$

$$+\frac{N-m}{V}\sum_{j=1}^{m}\int dx\left[R^{m+1},\frac{e^2}{|\vec{r}_i-\vec{r}_j|}\right],$$

where

$$H^{m:\tau}=\sum_{j=1}^{m}\left\{-\frac{\hbar^2}{2m}\nabla_j^2-\sum_{\lambda=[\tau]}\frac{e\hbar}{mc}(g_\lambda\vec{A}_\lambda+g_\lambda^+A_\lambda^+)\cdot\frac{1}{i}\vec{\nabla}_j\right.$$

$$+\sum_{\lambda,\rho=[\tau]}\frac{e^2}{2mc^2}(g_\lambda\vec{A}_\lambda+g_\lambda^+\vec{A}_\lambda^+)\cdot(g_\rho\vec{A}_\rho+g_\rho^+A_\rho^+)$$

$$\left.+\sum_{\lambda=[\tau]}\frac{e\hbar}{mc}\vec{\sigma}_j\cdot\vec{\nabla}\times(g_\lambda A_\lambda+g_\lambda^+A_\lambda^+)\right\}+\frac{1}{2}\sum_{i\neq j}\frac{e^2}{|\vec{r}_i-\vec{r}_j|}$$

$$+\sum_{\lambda=[\tau]}\omega_\lambda^2 g_\lambda^+ g_\lambda. \qquad (21)$$

In Eq. (21), the last term follows since the particles are identical.

As a reminder, each term on the right side of Eq. (20) has arguments of the form

$$\int dQ_\lambda\left[R^{m:\tau+1}_{(\lambda)},(g_\lambda\vec{A}_\lambda+g_\lambda^+A_\lambda^+)\cdot\frac{1}{i}\vec{\nabla}_j\right]= \qquad (22)$$

$$\int dQ_\lambda \left[R^{m+1,\tau+1}, (g_\lambda \vec{A}_\lambda + g_\lambda^+ \vec{A}_\lambda^+) \cdot \frac{1}{i}\vec{\nabla}_1 \right] (Q_{[\tau]}, Q_\lambda; Q'_{[\tau]}, Q_\lambda \,|\, X_{[m]}, X'_{[m]}) . \qquad \text{(cont.)}$$

Since every term on the right side of Eq. (20) contains higher correlation functions, it actually represents an infinite set of coupled equations for the reduced density matrices of the system. The solution to this set would eventually necessitate solving the full Liouville equation, so that at this point there is not apparent advantage to the development. However, thus far the entire treatment has been exact. It will be seen in the following section that an expansion of the reduced density matrices is possible, which will enable one to close this set of equations. This, in turn, will make it possible (in principle) to calculate all reduced density matrices in terms of the several lowest orders.

II. THE QUANTUM VLASOV EQUATION

In this section, the simplest of the kinetic equations will be given and it will be shown that by assuming a special form for the reduced density matrices the infinite chain of kinetic equations represented by Eq. (20) can be closed.

The equation for the single-particle reduced density matrix can be obtained from Eq. (20) by setting m = 1 and τ = 0. Then, one has

$$\frac{\hbar}{i}\frac{\partial R^{1,0}}{\partial t} + \left[R^{1,0}, \frac{-\hbar^2}{2m}\nabla^2 \right] - \frac{e\hbar}{mc}\sum_{\lambda=1}^{\infty} \int dQ_\lambda \left[R^{1,0}, (g_\lambda \vec{A}_\lambda + g_\lambda^+ \vec{A}_\lambda^+)\cdot\frac{1}{i}\vec{\nabla} \right]$$

$$+ \frac{e^2}{2mc^2}\sum_{\lambda,\rho=1}^{\infty} \int dQ_\lambda\, dQ_\rho \left[R^{1,1,2}_{(1,\lambda,\rho)}, (g_\lambda \vec{A}_\lambda + g_\lambda^+ A_\lambda^+)\cdot(g_\rho \vec{A}_\rho + g_\rho^+ A_\rho^+) \right] \qquad (23)$$

$$- \sum_{\lambda=1}^{\infty} \frac{e\hbar}{mc} \int dQ_\lambda \left[R^{1,1}, \vec{\sigma}\cdot\vec{\nabla}\times(g_\lambda \vec{A}_\lambda + g_\lambda^+ A_\lambda^+) \right] +$$

$$+\frac{N-1}{V}\int d\chi_2\left[R^{2;0}_{(1,2)},\frac{e^2}{|\vec{r}_1-\vec{r}_2|}\right]=0. \qquad \text{(cont.)}$$

Similarly, the single-oscillator equation can be obtained from Eq. (20) by setting m = 0 and z = 1. This is

$$\frac{\hbar}{i}\frac{\partial R^{0;1}_{(\rho)}}{\partial t}+2\omega_\rho^2\left[R^{0;1}_{(\rho)},q_\rho^\dagger q_\rho\right]-\frac{Ne\hbar}{mc}\int d\chi\left[R^{1;1}_{(\rho)},(q_\rho\vec{A}_\rho+q_\rho^\dagger A_\rho^\dagger)\cdot\frac{1}{i}\vec{\nabla}\right]$$

$$+\frac{Ne^2}{mc^2V}\sum_{\lambda=1}^{\infty}\int d\chi\, dQ_\lambda\left[R^{1;2}_{(\lambda,\rho)},(q_\lambda\vec{A}_\lambda+q_\lambda^\dagger\vec{A}_\lambda^\dagger)\cdot(q_\rho\vec{A}_\rho+q_\rho^\dagger A_\rho^\dagger)\right] \qquad (24)$$

$$-\frac{Ne\hbar}{mcV}\int d\chi\left[R^{1;1}_{(\rho)},\vec{\sigma}\cdot\vec{v}\times(q_\rho\vec{A}_\rho+q_\rho^\dagger\vec{A}_\rho^\dagger)\right]=0.$$

Now, if one assumes that $R^{2;0}$, $R^{1;1}$ and $R^{1;2}$ can be written as products of single-particle and single-oscillator reduced density matrices, i.e.,

$$R^{2;0}_{(1,2)}=PR^{1;0}_{(1)}R^{1;0}_{(2)}, \qquad (25)$$

$$R^{1;1}_{(1,\lambda)}=R^{1;0}_{(1)}R^{0;1}_{(\lambda)}, \qquad (26)$$

$$R^{1;2}_{(1,\lambda,\rho)}=R^{1;0}_{(1)}R^{0;1}_{(\lambda)}R^{0;1}_{(\rho)}, \qquad (27)$$

then Eqs. (23) and (24) are sufficient to describe the system, since higher order reduced density matrices can also be written as products of the single-particle and single-oscillator reduced density matrices. In Eq. (25), the operator P symmetrizes the product of single-particle functions. These are the quantum analogues of the Vlasov equations for a plasma and radiation (30).

The assumption of Eqs. (25)-(27) is essentially equivalent to an expansion of the density matrix in terms of interaction parameters characteristic of the system as has been done in the classical case by Rosenbluth and Rostoker (26) and Simon and Harris (30). Here, only the zero-order terms have been kept. Thus, Eqs. (25)-(27) are "zero interaction" form of the reduced density matrices.

While a detailed examination of the perturbation treatment for a quantum plasma will not be given, some of the qualitative results should be discussed. Before doing this, it is necessary to examine the various operators in the Hamiltonian to determine whether they contain parameters characteristic of the system.

First, the normalization of the $\vec{A}_\lambda$ is such that (Heitler (10), p. 39, Eq. 6)

$$\frac{1}{V}\int \vec{A}_\lambda \cdot \vec{A}_\lambda \, d^3x = 4\pi c^2, \tag{28}$$

so a factor $(4\pi C^2/V)^{\frac{1}{2}}$ must multiply each $\vec{A}_\lambda$ used in this treatment. Also, the commutation relation for q_λ and q_λ^* is (10)

$$q_\lambda q_\lambda^\dagger - q_\lambda^\dagger q_\lambda = \frac{\hbar}{2\omega_\lambda V}, \tag{29}$$

so q_λ and q_λ^* must be multiplied by a factor $\left(\frac{\hbar}{2\omega_\lambda V}\right)^{\frac{1}{2}}$. Actually, since it will not affect the results of the problem, we will include a factor 1/V in the single-oscillator density matrix to account for the presence of 1/V in Eq. (29). This, in effect, means that we assume that

$$q_\lambda q_\lambda^\dagger - q_\lambda^\dagger q_\lambda = 1 \tag{30a}$$

and

$$\sum_\lambda \frac{1}{V}\int dQ_\lambda \, \rho(Q_\lambda', Q_\lambda) = 1, \tag{30b}$$

since the sum over wavelengths is proportional to the volume of the system.

Although we will not re-write Eq. (20) with the parameters shown explicitly, one should consider that a factor $\left(\frac{2\pi c^2\hbar}{\omega_\lambda V}\right)^{\frac{1}{2}}$ multiplies each term in $\vec{A}_\lambda$. The general procedure for the perturbation theory is then to assume that such quantities as e, 1/V, 1/N, 1/m, etc., are small, or more correctly, approach zero in such a way that certain ratios, e.g., e/m, N_e/V, etc., remain constant. Thus, some physical characteristics of the system are not affected by the perturbation theory. For example,

requiring that e/m remain constant means that cyclotron radiation, the frequency of which is proportional to e/m, will be described correctly by the perturbed equations. After determining which parameters should be varied, one can then examine the Hamiltonian to determine the "order" of each term. Thus, a term proportional to e^2/m would be first order. Finally, the density matrix is expanded in a series of the form

$$R^{m;\tau} = \sum_{\ell=0}^{\infty} R_\ell^{m;\tau} \beta^\ell \tag{31}$$

where β is a small quantity characteristic of the plasma. (In the treatment of Rosenbluth and Rostober (26), β was taken to be the reciprocal of the number of particles in a Debye sphere. Thus, when the number of particles in the Debye sphere is large, the system exhibits plasma characteristics.) The physical parameters such as e, m, etc., are regarded as being proportional to β. The next step is to equate powers of β in the kinetic equations, so that one obtains equations coupling $R_\ell^{m;\tau}$ to $R_{\ell-1}^{m;\tau}$, etc. Finally, one assumes that the reduced density matrices can be written as products of single-particle and single-oscillator density matrices and various correlation functions to a given degree of accuracy. For example, to first order one would have

$$\begin{aligned} R_1^{m;\tau} = {} & P \sum_{i \neq j} R_1^{2;0}(i,j) \prod_{k \neq i,j}^{m} R^{1;0}(k) \prod_{\lambda=1}^{[\tau]} R^{0;1}(\lambda) \\ & + P \sum_{\lambda=1}^{[\tau]} \sum_{j=1}^{2m} R_1^{1;1}(\lambda,j) \prod_{\rho \neq \lambda}^{[\tau]} R_0^{0;1}(\rho) \prod_{k \neq j}^{m} R_0^{1;0}(j), \end{aligned} \tag{32}$$

where again P is a symmetrization operator. Then, one finds that only a small number of equations are needed to obtain the functions necessary to describe the system.

The procedure outlined above has been applied many times to various physical systems. However, in the case of a plasma and radiation fields, there are several difficulties inherent in such a procedure. This is true both for the classical and the quantum cases. The primary difficulty is due to the fact that the method described above is based on a one-parameter expansion of the density matrix. However, when there are both longitudinal and transverse interactions such a treatment is not strictly correct. The reason for this is easy to see. The determination of the parameter β is not arbitrary, but depends on the type of system considered. In fact, one chooses certain "fundamental units" appropriate to the system, e.g., the Debye length, the plasma frequency, and some third quantity. When the equations are written in terms of the fundamental units, certain physically meaningful contributions appear, e.g., the reciprocal of the number of particles in the Debye sphere. However, when radiation fields are present, there are several sets of reasonable fundamental units, e.g., the plasma frequency or the time required for a light wave to cross the Debye sphere. One is not justified in making an arbitrary choice between these two. Thus, strictly speaking, the one-parameter expansion is inconsistent. In fact, in the course of this dissertation, such an expansion was made, taking λ, e, m, 1/N and1/V as the quantities varied. The result of this treatment gave Eqs. (23) and (24) as the zero-order equations, but for systems known to exhibit plasma-like or collective behavior, the parameter β became greater than unity.

Although they will not be employed in this study, it is of some interest to write down the "first order" equations corresponding to those in the treatment of Simon and Harris (30), and referred to by those

authors as the "Fokker-Planck" equations. Here, one assumes that Eq. (32) gives the proper form of the density matrix and writes the correlation functions in the following form.

$$R_1^{2:0}(1,2) = P\left[R_1^{1:0}(1)\,R_0^{1:0}(2) + R_0^{1:0}(1)\,R_1^{1:0}(2) + q(1,2)\right], \tag{33}$$

$$R_1^{1:1}(\lambda) = R_1^{1:0}(1)\,R_0^{0:1}(\lambda) + R_0^{1:0}(1)\,R_1^{0:1}(\lambda) + q(1,\lambda), \tag{34}$$

$$R_1^{0:2}(\lambda,\rho) = R^{0:1}(\lambda)\,R_1^{0:1}(\rho) + R_1^{0:1}(\lambda)\,R_0^{0:1}(\rho), \tag{35}$$

where q(1,2) and q(1,λ) are correlation functions which vanish in the limit of no interactions. One finds that four equations are sufficient to describe the plasma. These are obtained by setting m = 1, τ = 0; m = 0, τ = 1; m = 1, τ = 1; and m = 0, τ = 2. Since we will not use them here, the derivation will not be given. The equations are

$$m=0,\ \tau=1;$$

$$\begin{aligned}
&\frac{\hbar}{i}\frac{\partial R_1^{1:0}}{\partial t} + \left[R_1^{1:0}, -\frac{\hbar^2}{2m}\nabla_1^2\right] - \sum_{\lambda=1}^{\infty}\frac{e\hbar}{mc}\int dQ_\lambda\left[R_1^{1:1}, (g_\lambda\vec{A}_\lambda + g_\lambda^+\vec{A}_\lambda^+)\cdot\frac{1}{i}\vec{\nabla}\right] \\
&-\frac{e\hbar}{mc}\sum_{\lambda=1}^{\infty}\int dQ_\lambda\left[R_1^{1:1}, \vec{\sigma}\cdot\vec{\nabla}\times(g_\lambda\vec{A}_\lambda + g_\lambda^+\vec{A}_\lambda^+)\right] \\
&+\frac{e^2}{2mc^2}\sum_{\lambda,\rho=1}^{\infty}\int dQ_\lambda\, dQ_\rho\left[R_1^{1:2}(1,\lambda,\rho), (g_\lambda\vec{A}_\lambda + g_\lambda^+\vec{A}_\lambda^+)\cdot(g_\rho\vec{A}_\rho + g_\rho^+\vec{A}_\rho^+)\right] \\
&+\frac{Ne^2}{V}\int dx_2\left[R_1^{2:0}(1,2), \frac{1}{|\vec{r}_1-\vec{r}_2|}\right] = 0;
\end{aligned} \tag{36}$$

$n=0, \tau=1;$

$$\frac{\hbar}{i}\frac{\partial R_1^{0;1}}{\partial t} + 2\omega_\lambda^2\left[R_1^{0;1}, g_\lambda^+ g_\lambda\right] - \frac{Ne\hbar}{mVc}\int dx\left[R_1^{1;1}, (g_\lambda \vec{A}_\lambda + g_\lambda^+ \vec{A}_\lambda^+)\cdot\frac{1}{i}\vec{\nabla}\right]$$

$$+\frac{Ne^2}{mVc^2}\sum_\rho \int dQ_\rho\, dx\left[R_1^{1;2}(1,\lambda,\rho), (g_\lambda \vec{A}_\lambda + g_\lambda^+ \vec{A}_\lambda^+)\cdot(g_\rho \vec{A}_\rho + g_\rho^+ \vec{A}_\rho^+)\right] \qquad (37)$$

$$-\frac{Ne\hbar}{mVc}\int dx\left[R_1^{1;1}, \vec{\sigma}\cdot\vec{v}\times(g_\lambda \vec{A}_\lambda + g_\lambda^+ A_\lambda^+)\right] = 0;$$

$n=1, \tau=1;$

$$\frac{\hbar}{i}\frac{\partial R^{1;1}}{\partial t} + \left[R_1^{1;1}, -\frac{\hbar^2}{2m}\nabla_1^2\right] + 2\omega_\lambda^2\left[R_1^{1;1}, g_\lambda^+ g_\lambda\right]$$

$$-\sum_{\rho\neq\lambda}^{\infty}\frac{e\hbar}{mc}\int dQ_\rho\left[R^{1;2}(1,\lambda,\rho), (g_\rho \vec{A}_\rho + g_\rho^+ \vec{A}_\rho^+)\cdot\frac{1}{i}\vec{\nabla}\right]$$

$$-\sum_{\rho\neq\lambda}^{\infty}\frac{e\hbar}{mc}\int dQ_\rho\left[R^{1;2}(1,\lambda,\rho), \vec{\sigma}\cdot\vec{v}\times(g_\rho \vec{A}_\rho + g_\rho^+ A_\rho^+)\right]$$

$$+\frac{Ne^2}{mVc^2}\sum_{\rho\neq\lambda}\int dQ_\rho\, dx_2\left[R_1^{2;2}(1,2;\lambda,\rho), (g_\lambda \vec{A}_\lambda + g_\lambda^+ \vec{A}_\lambda^+)\cdot(g_\rho \vec{A}_\rho + g_\rho^+ A_\rho^+)\right] \qquad (38)$$

$$+\frac{Ne\hbar}{mcV}\int dx_2\left[R_1^{2;1}(1,2,\lambda), \vec{\sigma}_2\cdot\vec{\nabla}_2\times(g_\lambda \vec{A}_\lambda + g_\lambda^+ \vec{A}_\lambda^+)\right]$$

$$+\frac{Ne^2}{V}\int dx_2\left[R_1^{2;1}, \frac{1}{|\vec{r}_1-\vec{r}_2|}\right]$$

$$= 0;$$

$m = 0, \tau = 2;$

$$\frac{k}{i}\frac{\partial R^{0:2}}{\partial t} + \sum_{\kappa=\lambda,\rho}\left[R^{0:2}_{1(\lambda,\rho)}, 2\omega_\kappa^2 q^*_\kappa q_\kappa\right]$$

$$-\frac{Nek}{mVc}\sum_{\kappa=\lambda,\rho}\int dx\left[R^{1:2}_{1(1,\lambda,\rho)}, (q_\kappa \vec{A}_\kappa + q^*_\kappa A^*_\kappa)\cdot\frac{1}{i}\vec{\nabla}\right]$$

$$+\frac{Ne^2}{mVc^2}\sum_{\kappa=\lambda,\rho}\sum_{\nu\neq\lambda,\rho}^{\infty}\int dQ_\nu\, dx\left[R_1^{1:3}, (q_\kappa\vec{A}_\kappa + q^*_\kappa A^*_\kappa)\cdot(q_\nu\vec{A}_\nu + q^*_\nu A^*_\nu)\right] \tag{39}$$

$$-\frac{Nek}{mVc}\sum_{\kappa=\lambda,\rho}\int dx\left[R_1^{1:2}, \vec{\sigma}\cdot\vec{\nabla}\times(q_\lambda\vec{A}_\lambda + q^*_\lambda\vec{A}^*_\lambda)\right]$$

$$+\frac{Ne^2}{2mVc^2}\sum_{\kappa=\lambda,\rho}\sum_{\tau=\lambda,\rho}\int dx\left[R^{1:2}, (q_\kappa\vec{A}_\kappa + q^*_\kappa\vec{A}^*_\kappa)\cdot(q_\tau\vec{A}_\tau + q^*_\tau A^*_\tau)\right]$$

$$= 0\,.$$

Although it appears that there are more variables than equations, $R_0^{1:0}$ and $R_0^{0:1}$ are found from Eqs. (23) and (24). In addition, using Eq. (32), one can find expressions for $R_1^{1:2}$, $R_1^{1:3}$, $R_1^{2:2}$ and $R_1^{2:1}$. It seems unnecessary to give them here.

There are several differences between the zero- and first-order equations and their classical counterparts. The most important of these is a result of the fact that $R_0^{2:0}$ must be properly symmetrized. In fact, one has (17)

$$R_0^{2:0}(x_1,x_2;x_1',x_2') = R^{1:0}(x_1,x_1')R^{1:0}(x_2,x_2') - R^{1:0}(x_1,x_2')R^{1:0}(x_2,x_1'), \tag{40}$$

where the sign is chosen for fermions. This means, for example, that the Coulomb integral has a term of the form

$$\int dx_2\, dx_1''\, R_0^{1:0}(x,x_1)\, \frac{R_0^{1:0}(x_2,x_1'')}{|\vec{r}_1-\vec{r}_2|}\, \delta(x_1''-x_2) = \int \frac{dx_2\, R_0^{1:0}(x_1,x_2)\, R_0^{1:0}(x_2,x_1')}{|\vec{r}_1-\vec{r}_2|}. \tag{41}$$

Thus, this term cannot be written in the form

$$\int dx_2\, R^{2:0}(x_1,x_1';x_2,x_2)\, \frac{1}{|\vec{r}_1-\vec{r}_2|} \neq (\phi(x_1))\, R(x_1,x_1') \tag{42}$$

where ϕ is a multiplication operator. This exchange correlation means that, even in the zero-order approximations, the quantum equations retain a particle aspect, whereas in the classical treaty these equations correspond to a "fluid" limit (26, 30). This rather important effect will be discussed in more detail in the next chapter. An additional difference due to exchange is that partial reduction of Eqs. (36)-(39), possible in the classical case (30), is no longer possible here.

In the next chapter, applications of the "Vlasov" equations, Eqs. (23) and (24), to the calculation of dispersion relations will be considered. However, before considering the applications, it is useful to examine the coupled Vlasov equations for particles and oscillators in a slightly different manner. This will be done in the following section.

III. AVERAGE POTENTIALS AND THE CLASSICAL LIMIT

In this section it will be shown that the oscillator equation, Eq. (24), can be eliminated and instead, Maxwell's equations for "average" fields can be substituted if one neglects spin terms.

We begin by noting that in the particle equation, Eq. (23), one has for the $A \cdot \vec{\nabla}$ term,

$$\sum_\lambda \int dQ_\lambda \left[R^{1:0}_{(i)} R^{0:1}_{(\lambda)}, (g_\lambda \vec{A}_\lambda + g_\lambda^+ \vec{A}_\lambda^+) \cdot \frac{1}{i} \vec{\nabla} \right] =$$

$$\int dx_1'' R^{1:0}_{(x,x_1'')} \left\{ \left\{ \sum_\lambda \int dQ_\lambda dQ_\lambda' R^{0:1}(Q_\lambda, Q_\lambda') (g_\lambda' \vec{A}_\lambda + g_\lambda'^* \vec{A}_\lambda^*) \delta(Q_\lambda' - Q_\lambda) \right\} \frac{1}{i} \cdot \vec{\nabla}'' \delta(x''-x') \right.$$

$$\left. - \left\{ \sum_\lambda \int dQ_\lambda dQ_\lambda' (g_\lambda \vec{A}_\lambda + g_\lambda^+ A_\lambda^+) \delta(Q_\lambda - Q_\lambda') R^{0:1}(Q_\lambda', Q_\lambda) \right\} \cdot \frac{1}{i} \vec{\nabla} \delta(x-x'') \right] R^{1:0}(x'', x') \tag{42}$$

$$= \int dx_1'' \left\{ R^{1:0}_{(x,x_1'')} \vec{A}_{ov}(x_1'') \cdot \vec{\nabla}'' \delta(x''-x') - \vec{A}_{ov} \cdot \vec{\nabla} \delta(x-x'') R^{1:0}(x'', x') \right.$$

$$= \left[R^{1:0}, \vec{A}_{ov} \cdot \frac{1}{i} \vec{\nabla} \right],$$

where

$$\vec{A}_{ov} = \sum_\lambda \int dQ_\lambda dQ_\lambda' R^{0:1}(Q_\lambda, Q_\lambda') (g_\lambda' \vec{A}_\lambda + g_\lambda'^* \vec{A}_\lambda^*) \delta(Q_\lambda' - Q_\lambda) = \tag{43}$$

$$\sum_\lambda T_n^{(\lambda)} \left\{ R^{0:1}(\lambda) (g_\lambda \vec{A}_\lambda + g_\lambda^* \vec{A}_\lambda^*) \right\}.$$

Similar considerations hold for the term in A^2. In fact, it becomes

$$\sum_{\lambda,\rho} \int dQ_\lambda dQ_\rho \left[R^{1:0}_{(i)} R^{0:1}_{(\lambda)} R^{0:1}_{(\rho)}, (g_\lambda \vec{A}_\lambda + g_\lambda^+ \vec{A}_\lambda^+)(g_\rho \vec{A}_\rho + g_\rho^+ A_\rho^+) \right] = \left[R^{1:0}_{(i)}, \vec{A}_{ov}^2 \right]. \tag{44}$$

Finally, using Eq. (39), one can write the scalar potential term as

$$\int dx_2 \left[R^{1:0}_{(1)} R^{1:0}_{(2)}, \frac{1}{|\vec{n}_1 - \vec{n}_2|} \right] + \left[R^{2:0}, \phi_{ex} \right] =$$

$$\int dn_2 dn_2' dn_1'' \left\{ R^{1:0}_{(n_1, n_1'')} R^{1:0}_{(n_2, n_2'')} \frac{1}{|\vec{n}_1'' - \vec{n}_2''|} \delta(n_1'' - n_1') \delta(n_2'' - n_2) \right. \tag{45}$$

$$\left. - \frac{1}{|\vec{n}_1' - \vec{n}_2|} \delta(n_1 - n_1'') \delta(n_2 - n_2'') R^{1:0}_{(n_1'', n_1')} R^{1:0}_{(n_2'', n_2)} \right\} + \left[R^{2:0}, \phi_{ex} \right] =$$

$$= \left[R^{1:0}, \phi_{av}\right] + \left[R^{2:0}, \phi_{ex}\right], \qquad \text{(cont.)}$$

with

$$\phi_{av} = \int d\mathbf{1}_2 \, \frac{R^{1:0}(\mathbf{1}_2, \mathbf{1}_1)}{|\vec{r}_1 - \vec{r}_2|} = T_r^{(2)}\left(R^{1:0}_{(2)} \frac{1}{|\vec{r}_1 - \vec{r}_2|}\right), \tag{46}$$

where now we must remember that ϕ_{ex} is an integral operator. In fact, one has

$$\left(\phi_{ex} R^{2:0}_{(1,2)}\right) = \int d\mathbf{1}_2 \, R^{1:0}(\mathbf{1}, \mathbf{1}_2) \frac{R^{1:0}(\mathbf{1}_2, \mathbf{1}_1')}{|\vec{r}_1 - \vec{r}_2|} \,. \tag{47}$$

Thus, one can write the particle equation in the following form,

$$\frac{\hbar}{i}\frac{\partial R^{1:0}}{\partial t} + \left[R^{1:0}, \frac{\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}_{av}\right)^2}{2m}\right] + \left[R^{1:0}, \phi_{av}\right] + \left[R^{1:0}, \phi_{ex}\right] = 0, \tag{48}$$

since

$$\vec{\nabla} \cdot \vec{A}_{av} = 0. \tag{49}$$

Now, let us take the classical limit of the above operators. We know that

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}\left[A, B\right] = \left\{A, B\right\}, \tag{50}$$

where the expression in curly brackets is the Poisson bracket. Then, taking the limit of Eq. (48), one has

$$\lim_{\hbar \to 0} R^{1:0}_{(1,1')} = f(\vec{q}, \vec{p}), \tag{51}$$

where $\vec{q}$ and $\vec{p}$ are the canonical coordinates and momenta and f is the classical distribution function. Now, let

$$H' = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}_{ov}\right)^2 + \Phi_{ov} \tag{52}$$

The exchange term will not contribute in the classical limit since particles are distinguishable. One has

$$\lim_{\hbar\to 0} \frac{1}{i\hbar}\left[R^{1:0}, H'\right] = \left\{f(\vec{q},\vec{p}), H'\right\}$$

$$= \frac{\partial H'}{\partial \vec{p}}\cdot\frac{\partial f}{\partial \vec{q}} - \frac{\partial H'}{\partial \vec{q}}\cdot\frac{\partial f}{\partial \vec{p}} = \dot{\vec{p}}\cdot\frac{\partial f}{\partial \vec{p}} + \vec{v}\cdot\frac{\partial f}{\partial \vec{q}} = -\frac{\partial f}{\partial t}, \tag{53}$$

where $\vec{v}$ is the velocity,

$$\vec{v} = \frac{\vec{p} - \frac{e}{c}\vec{A}}{m}. \tag{53a}$$

This is the result obtained by Harris (11a) in the classical treatment.

The oscillator equation, Eq. (24), can now be written

$$\frac{\hbar}{i}\frac{\partial R^{0:1}}{\partial t} + 2\omega_\lambda^2\left[R^{0:1}, q_\lambda^\dagger q_\lambda\right] - \frac{Ne}{mc} Tr^{(1)}\left[R^{1:0}_{(1)} R^{0:1}_{(\lambda)}, (q_\lambda \vec{A}_\lambda + q_\lambda^\dagger \vec{A}_\lambda^\dagger)\cdot\frac{1}{i}\vec{\nabla}\right]$$

$$+ \frac{Ne^2}{mc^2 V}\sum_\rho Tr^{(1,\rho)}\left[R^{1:0}_{(1)} R^{0:1}_{(\lambda)} R^{0:1}_{(\rho)}, (q_\lambda \vec{A}_\lambda + q_\lambda^\dagger \vec{A}_\lambda^\dagger)\cdot(q_\rho \vec{A}_\rho + q_\rho^\dagger A_\rho^\dagger)\right]$$

$$= \frac{\hbar}{i}\frac{\partial R^{0:1}}{\partial t} + 2\omega_\lambda^2\left[R^{0:1}, q_\lambda^\dagger q_\lambda\right] - \frac{Ne}{mcV} Tr^{(1)}\left[R^{1:0}_{(1)} R^{0:1}_{(\lambda)}, (q_\lambda A_\lambda + q_\lambda^\dagger A_\lambda^\dagger)\cdot\frac{1}{i}\vec{\nabla}\right] \tag{54}$$

$$+ \frac{Ne^2}{mcV} Tr^{(1)}\left[R^{1:0} R^{0:1}_{(\lambda)}, (q_\lambda \vec{A}_\lambda + q_\lambda^\dagger \vec{A}_\lambda^\dagger)\cdot\vec{A}_{ov}\right] = 0.$$

Now, we can re-write Eq. (54) in terms of the canonical variables, Q_λ, P_λ, and for convenience, change to a representation in which the fields are real (10); then we get

$$\frac{\hbar}{i}\frac{\partial R^{011}_{(\lambda)}}{\partial t} + \left[R^{011}_{(\lambda)}, \frac{1}{2}(P_\lambda^2 + \omega_\lambda^2 Q_\lambda^2)\right] - \frac{Ne}{Vc}\left[R^{011}_{(\lambda)}, Q_\lambda T_n^{(1)}(R''^{0}\vec{A}_\lambda \cdot \vec{v})\right] \tag{55}$$

$$= \frac{\hbar}{i}\frac{\partial R^{011}_{(\lambda)}}{\partial t} + \left[R^{011}_{(\lambda)}, H^\lambda\right],$$

where

$$\vec{v} = \frac{1}{m}\left(\frac{\vec{\nabla}}{i} - \frac{e}{c}\vec{A}_{av}\right). \tag{56}$$

Again, we take the classical limit. One has

$$\lim_{\hbar\to 0}\frac{1}{i\hbar}\left[R^{011}, H^\lambda\right] = \left\{f^\lambda, H^\lambda\right\}$$

$$= -Q_\lambda\omega_\lambda^2\frac{\partial f}{\partial P_\lambda} + \frac{\partial f}{\partial Q_\lambda}P_\lambda + \frac{Ne}{cV}\frac{\partial f}{\partial P_\lambda}\int \vec{A}_\lambda\cdot\vec{v}\, f(\vec{p},\vec{q})\,d\vec{p}\,d\vec{q}. \tag{57}$$

which again agrees with Harris' result (11a).

The next step is to define the average electric and magnetic fields. We take

$$\vec{E}_{av} = -\frac{1}{c}\frac{\partial \vec{A}_{av}}{\partial t} - \vec{\nabla}\phi, \tag{58}$$

and

$$\vec{H}_{av} = \vec{\nabla}\times\vec{A}_{av} \tag{59}$$

It is simple to show that three of Maxwell's equations are satisfied. First, we take the divergence of $\vec{E}$, giving,

$$\vec{\nabla}\cdot\vec{E}_{av} = -\frac{1}{c}\vec{\nabla}\cdot\vec{A}_{av} - \nabla^2\phi_{av}$$

$$= 0 - \nabla^2\frac{Ne}{V}\int R(1_2,1_2)\frac{1}{|1_1-1_2|}\,d1_2 \tag{60}$$

$$= 4\pi\frac{Ne}{V}\,T_n^{(2)}\left[R(2)\,\delta(1_1-1_2)\right].$$

Then, taking the divergence of $\vec{H}$, one has

$$\vec{\nabla} \cdot \vec{H}_{av} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}_{av}) = 0. \tag{61}$$

Taking the curl of $\vec{\mathcal{E}}_{av}$, one has

$$\vec{\nabla} \times \vec{\mathcal{E}}_{av} = -\vec{\nabla} \times \frac{1}{c} \frac{\partial \vec{A}_{av}}{\partial t} = -\frac{1}{c} \frac{\partial \vec{H}_{av}}{\partial t}. \tag{62}$$

The final equation, giving the curl of $\vec{H}$, is somewhat more involved. One must calculate $\dot{\vec{A}}_{av}$ and $\ddot{\vec{A}}_{av}$ from the oscillator equations. Thus, one has (using the Q_λ instead of q_λ and q_λ^* for simplicity)

$$\begin{aligned} \vec{\nabla} \times \vec{H} &= \vec{\nabla} \times (\vec{\nabla} \times \vec{A}_{av}) = -\nabla^2 A_{av} \\ &= \sum_\lambda k_\lambda^2 \vec{A}_\lambda T_2^{(\lambda)}(Q_\lambda, R_{(\lambda)}^{0;11}) \\ &= \sum_\lambda \frac{\omega_\lambda^2}{c^2} \vec{A}_\lambda T_2^{(\lambda)}(Q_\lambda R_{(\lambda)}^{0;11}) \end{aligned} \tag{63}$$

and

$$\frac{1}{c} \frac{\partial \vec{\mathcal{E}}_{av}}{\partial t} = -\frac{1}{c^2} \frac{\partial^2 \vec{A}_{av}}{\partial t^2} - \frac{1}{c} \vec{\nabla} \frac{\partial \phi_{av}}{\partial t} \tag{64}$$

Consider first $\frac{1}{c} \frac{\partial \phi}{\partial t}$ av. We get, using Eq. (46),

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial t} \frac{Ne}{V} \int \frac{R^{11;0}(r_1, r_1)}{|\vec{r}_1 - \vec{r}_1|} dr_1 = \int \frac{\partial R^{11;0}(r_2, r_1, t)}{\partial t} \frac{dr_2}{|\vec{r}_1 - \vec{r}_2|}. \tag{65}$$

But $\frac{\partial R^{11;0}}{\partial t}$ is given by Eq. (48). Taking a typical term, one has

$$\begin{aligned} \int \frac{dr_2}{|r_1 - r_1|} \left(\frac{Ne}{V}\right) \left[R^{11;0} \left(\vec{p} - \frac{e}{c} \vec{A}_{av}\right)^2 \right](r_2, r_2) = \frac{1}{2m} \frac{Ne}{V} \int \frac{dr_2 dr_1'}{|r_1 - r_1|} \Big\{ (p - \tfrac{e}{c} \vec{A}_{av})^2(r_2') \delta(r_2' - r_2) R^{11;0}(r_2, r_1') \\ - (\vec{p} - \tfrac{e}{c} \vec{A}_{av})^2(r_2) \delta(r_2 - r_1') R(r_2', r_2) \Big\} = 0, \end{aligned} \tag{66}$$

since in each case, the operator acts on the right coordinate. The other terms vanish for similar reasons.

Next, consider the form of $\frac{\partial^2 \vec{A}_{av}}{\partial t^2}$. We get

$$\frac{\partial^2}{\partial t^2}\vec{A}_{\alpha\nu} = \frac{\partial}{\partial t}\sum_\lambda \vec{A}_\lambda T_n^{(\lambda)}\left(Q_\lambda, \frac{\partial R_\lambda^{0;1}}{\partial t}\right) =$$

$$-\frac{\partial}{\partial t}\frac{i}{\hbar} T_n^{(\lambda)}\left\{Q_\lambda\left(\left[R_\lambda^{0;1}, \tfrac{1}{2}(P_\lambda^2+\omega_\lambda^2 Q_\lambda^2)\right] - \frac{e}{c}\left[Q_\lambda, R_\lambda^{0;1} T_n^{(1)} R^{1;0}\vec{A}_\lambda\cdot\vec{v}\right]\right)\right\}. \tag{67}$$

The first term gives (10)

$$T_n^{(\lambda)}\left\{Q_\lambda\left[R_{(\lambda)}^{0;1}, \tfrac{1}{2}(P_\lambda^2+\omega_\lambda^2 Q_\lambda^2)\right]\right\} = \tfrac{1}{2} T_n^{(\lambda)}\left(Q_\lambda R_\lambda^{0;1}\{P_\lambda^2+\omega_\lambda^2 Q_\lambda^2\} - Q_\lambda\{P_\lambda^2+\omega_\lambda^2 Q_\lambda^2\}R_{(\lambda)}^{0;1}\right)$$

$$= \frac{T_n^{(\lambda)}}{2}\left(R_\lambda\left[Q_\lambda P_\lambda^2 - P_\lambda^2 Q_\lambda\right]\right) = -\frac{\hbar}{i} T_n^{(\lambda)}\left(P_\lambda R_\lambda^{0;1}\right). \tag{68}$$

The second term gives

$$T_n^{(\lambda)}\left\{Q_\lambda\left[Q_\lambda, R_\lambda^{0;1}\left(T_n^{(1)} R^{1;0}\vec{A}_\lambda\cdot\vec{v}\right)\right]\right\} = T_n^{(\lambda)}\left\{T_n^{(0)}\left(R^{1;0}\vec{A}_\lambda\cdot\vec{v}\right)R_{(\lambda)}^{0;1}\left[Q_\lambda, Q_\lambda\right]\right\} \tag{69}$$

$$= 0.$$

Then, one has from Eq. (67)

$$+\sum_\lambda T_n^{(\lambda)}\vec{A}_\lambda\left(P_\lambda \frac{\partial R_{(\lambda)}^{0;1}}{\partial t}\right) = -\frac{i}{\hbar}\sum_\lambda \vec{A}_\lambda T_n^{(\lambda)} P_\lambda\left\{\left[R, \tfrac{1}{2}(P_\lambda^2+\omega_\lambda^2 Q_\lambda^2)\right] - \frac{e}{c}\left[Q_\lambda, R_{(\lambda)}^{0;1} T_n^{(1)} R^{1;0}\vec{A}_\lambda\cdot\vec{v}\right]\right\}$$

$$= +\sum_\lambda \omega_\lambda^2 T_n\left(R^{0;1}, Q_\lambda\right) + \frac{e}{c}\frac{i}{\hbar} T_n^{(\lambda)}\left\{R^{0;1}\left(P_\lambda Q_\lambda - Q_\lambda P_\lambda\right) T_n^{(1)} R^{1;0}\vec{A}_\lambda\cdot\vec{v}\right\} \tag{70}$$

$$= +\sum_\lambda \omega_\lambda^2 T_n^{(\lambda)} R_{(\lambda)}^{0;1} Q_\lambda + \frac{e}{c}\sum_\lambda \vec{A}_\lambda\left(T_n^{(1)} R^{1;0} A_\lambda\cdot\vec{v}\right),$$

Thus, we obtain finally

$$\vec{\nabla}\times\vec{H}_{\alpha\nu} = \frac{1}{c}\frac{\partial\vec{E}_{\alpha\nu}}{\partial t} + \frac{e}{c}\sum_\lambda \vec{A}_\lambda T_n^{(1)}\left(R^{1;0}\vec{A}_\lambda\cdot\vec{v}\right), \tag{71}$$

which can be simplified somewhat since A λ can be written (10)

$$\vec{A}_\lambda \sum_\mu \hat{\epsilon}_{\lambda\mu}\cos\vec{k}_\lambda\cdot\vec{n}\,/\left(\frac{4\pi c^2}{v}\right)^{1/2}. \tag{72}$$

Then, one has (10)

$$\sum_\lambda \vec{A}_\lambda(\vec{n}) T_n^{(1)}\left\{R^{1;0}(n_1)\vec{A}_\lambda(\vec{n}_1)\vec{v}(n_1)\right\} \tag{73}$$

$$= T_n^{(1)}\left\{R^{1;0}(n_1)\vec{v}(\vec{n}_1)\sum_\lambda A_\lambda(\vec{n}_1)A_\lambda(\vec{n}) =\right.$$

$$T_{\alpha}^{(1)}\left\{R''^{(0)}_{(1)}\,\vec{v}_{(1)}\,\delta(\vec{r}-\vec{r}_{(1)})\cdot\overleftrightarrow{T}\,4\pi c\right\} = 4\pi c^{2}\,T_{\alpha}^{(1)}\left(R'^{1;0}_{(1)}\,\vec{v}_{(1)}\,\delta(\vec{r}-\vec{r}_{(1)})\right), \qquad \text{(cont.)}$$

where $\overleftrightarrow{T}$ is the unit dyad. Thus, the average fields obey Maxwell's equations, just as in the classical treatment (11a). This result will be employed in the next chapter to calculate the dispersion relation for transverse waves.

CHAPTER V

DISPERSION RELATIONS FOR A QUANTUM PLASMA

In this chapter, applications of the zero-order equations to a quantum plasma are considered. By means of a perturbation treatment, dispersion relations for an electron plasma in which the equilibrium state is the canonical ensemble are derived. The similarities between these relations and their classical counterparts are discussed.

I. DISPERSION RELATION FOR LONGITUDINAL WAVES

In this section, the zero-order equation derived in the previous chapter will be employed to calculate a dispersion relation for the frequencies of small amplitude oscillations in an electron plasma when the transverse modes are of no significance.

The starting point for this calculation is Eq. (23) of Chapter IV, with the transverse interactions omitted. One has

$$\frac{\hbar}{i}\frac{\partial R^{11\,0}}{\partial t} + \left[R^{11\,0}, -\frac{\hbar^2}{2m}\nabla_1^2\right] + \frac{Ne^2}{V}\int dx_2\left[R^{2\,1\,0}_{(1,2)}, \psi(|\vec{r}_1-\vec{r}_2|)\right] = 0. \quad (1)$$

Here, in the potential energy we have written $\psi(|\vec{r}_1-\vec{r}_2|)$ in place of $1/|\vec{r}_1-\vec{r}_2|$ since it will prove necessary to introduce a "screened" potential function later. Now, it is useful to re-write Eq. (1) in the momentum representation. This is

$$\frac{\hbar}{i}\frac{\partial R^{11\,0}}{\partial t}(k_1,k_1') + \left[R^{11\,0}, -\frac{\hbar^2}{2m}\nabla_1^2\right](k_1,k_1') + \quad (2)$$

$$+\frac{Ne^2}{V}\int dk_2\left[R^{2;0}_{(1,2)},\ \psi(|\vec{r}_1-\vec{r}_2|)\right](k_1,k_2;k_1',k_2') = 0 \qquad \text{(cont.)}$$

Now, consider the form of $\psi(\vec{k}_1, \vec{k}_2; \vec{k}_1", \vec{k}_2")$. Since $\psi(|\vec{r}_1-\vec{r}_2|)$ is a function only of $|\vec{r}_1-\vec{r}_2|$, one can define two new variables

$$\vec{R} = \frac{\vec{r}_1+\vec{r}_2}{2}, \tag{3}$$

$$\vec{r} = \vec{r}_1 - \frac{\vec{r}_2}{2} \tag{4}$$

and find the matrix elements in terms of these. Then, one has for

$$\psi(\vec{k}_1,\vec{k}_2;\vec{k}_1'',\vec{k}_2'') = \int d^3r_1 d^3r_2\, e^{i(\vec{k}_1''-\vec{k}_1)\cdot\vec{r}_1 + i(\vec{k}_1''-\vec{k}_2)\cdot\vec{r}}\,\psi(|\vec{r}_1-\vec{r}_2|)$$

$$= \frac{1}{8}\int d^3R\, d^3r\, \psi(r)\, e^{i(\vec{k}_1''-\vec{k}_1)\cdot(\vec{R}+\vec{r}) + i(\vec{k}_2''-\vec{k}_2)\cdot(\vec{R}-\vec{r})} \tag{5}$$

$$= 4\pi\,\delta(\vec{k}_1''-\vec{k}_1+\vec{k}_2''-\vec{k}_2)\,\Psi\left(\tfrac{1}{2}|\vec{k}_1''-\vec{k}_1-\vec{k}_1''+\vec{k}_2|\right),$$

and the factor 4π comes from the integration over angles in r space and the factor 1/8 has been included in Ψ.

Now, one assumes that the density matrix for the system can be written

$$R^{1;0} = F + \rho \tag{6}$$

where F is diagonal in kinetic energies (or momenta) and ρ is a small perturbation of the equilibrium density matrix F, due to the presence of the Coulomb forces. We also assume that the terms of the form ρ^2 can be neglected. This is a linearization or first-order perturbation theory

similar to that of Chapter III. With these assumptions, Eq. (2) becomes

$$\frac{\hbar}{i}\frac{\partial \rho}{\partial t}(\vec{k}_1,\vec{k}_1') + \left[\rho, -\frac{\hbar^2}{2m}\nabla_1^2\right](\vec{k}_1,\vec{k}_1') + \frac{Ne^2}{V}\int dk_2 dk_2'' dk_1'' \Big\{ R^{2:0}(\vec{k}_1,\vec{k}_2;\vec{k}_1'',\vec{k}_1'')\,\psi(\vec{k}_1''\vec{k}_2'';\vec{k}_1'\vec{k}_2) - \psi(\vec{k}_1,\vec{k}_2;\vec{k}_1''\vec{k}_2'')\,R^{2:0}(\vec{k}_1''\vec{k}_2'';\vec{k}_1'\vec{k}_2)\Big\} = 0. \tag{7}$$

Now, one assumes that initially, the spins are randomly oriented, and that this orientation is not changed by the perturbation. Then, since there are no spin terms in Eq. (1), one can sum over spins. The spinless correlation function for two electrons, assuming random spin orientation, is (17, 36a)

$$R_0^{2:0}(\vec{k}_1,\vec{k}_2;\vec{k}_1',\vec{k}_2') = R^{1:0}(\vec{k}_1,\vec{k}_1')R^{1:0}(\vec{k}_2,\vec{k}_2') - \tfrac{1}{2}R^{1:0}(\vec{k}_1,\vec{k}_2')R^{1:0}(\vec{k}_2,\vec{k}_1') \tag{8}$$

Taking the form of $R_0^{1:0}$ assumed in Eq. (6), Eq. (8) becomes

$$R^{2:0} = F(\vec{k}_1,\vec{k}_1')F(\vec{k}_2,\vec{k}_2') - \tfrac{1}{2}F(\vec{k}_1,\vec{k}_2')F(\vec{k}_2,\vec{k}_1') + F(\vec{k}_1,\vec{k}_1')\rho(\vec{k}_2,\vec{k}_2') + \rho(\vec{k}_1,\vec{k}_1')F(\vec{k}_2,\vec{k}_2') - \tfrac{1}{2}F(\vec{k}_1,\vec{k}_2')\rho(\vec{k}_2,\vec{k}_1') - \tfrac{1}{2}\rho(\vec{k}_1,\vec{k}_2')F(\vec{k}_2,\vec{k}_1') + 0^2. \tag{9}$$

Now, the next step is to consider the commutator of the potential energy with each of these terms; that is, the last term in Eq. (7). Taking each term in Eq. (9) in order and using Eq. (5) one has

$$\int dk_2 dk_2'' dk_1'' F(\vec{k}_1,\vec{k}_1'')F(\vec{k}_2,\vec{k}_2'')\,\psi(\vec{k}_1''\vec{k}_2'';\vec{k}_1'\vec{k}_2) - \tag{10}$$

$$- \int dk_2\, dk_1''\, dk_1''\; \psi(\vec{k}_1, \vec{k}_2; \vec{k}_1'', \vec{k}_2'')\, F(\vec{k}_2'', \vec{k}_2)\, F(\vec{k}_1'', \vec{k}_1') =$$

$$\int dk_2\, dk_2''\, dk_1'' \Big\{ F(\vec{k}_1)\, \delta(\vec{k}_1'' - \vec{k}_1')\, F(\vec{k}_2)\, \delta(\vec{k}_2 - \vec{k}_2'')\, \psi(\vec{k}_1'', \vec{k}_2''; \vec{k}_1' \vec{k}_2) - \psi(\vec{k}_1, \vec{k}_2; \vec{k}_1'' \vec{k}_2'')\, F(\vec{k}_1')\, F(\vec{k}_2)\, \delta(\vec{k}_2'' - \vec{k}_1')\, \delta(\vec{k}_2 - \vec{k}_2'') \Big\}$$

$$= \int dk_2 \Big\{ F(\vec{k}_1)\, F(\vec{k}_2)\, \psi(\vec{k}_1, \vec{k}_2; \vec{k}_1', \vec{k}_2) - \psi(\vec{k}_1, \vec{k}_2; \vec{k}_1', \vec{k}_2)\, F(\vec{k}_1')\, F(\vec{k}_2) \Big\} =$$

$$4\pi \int dk_2 \Big\{ F(\vec{k}_1)\, F(\vec{k}_2)\, \delta(\vec{k}_1' - \vec{k}_1)\, \underline{\Psi}(|\vec{k}_1' - \vec{k}_1|) - \qquad \text{(cont.)}$$

$$\underline{\Psi}(|\vec{k}_1' - \vec{k}_1|)\, \delta(\vec{k}_1' - \vec{k}_1)\, F(\vec{k}_1)\, F(\vec{k}_2) \Big\} =$$

$$4\pi\, \delta(\vec{k}_1' - \vec{k}_1)\, \psi(|\vec{k}_1' - \vec{k}_1|) \Big(F(\vec{k}_1) - F(\vec{k}_1') \Big) \int F(\vec{k}_2)\, dk_2$$

Now, because the delta function is in the above equation, it will contribute only for $\vec{k}_1' = \vec{k}_1$. However, for all potentials such that (0) is finite, this diagonal term will vanish. Now, the matrix element $\psi(\vec{k}_1 \vec{k}_1')$ of the Coulomb potential is

$$\psi_c(k_1, k_1') = \frac{4\pi}{|\vec{k}_1' - \vec{k}_1|^2} \,. \tag{11}$$

This clearly diverges for $\vec{k}_1 = \vec{k}_1'$. This is the reason that we have preferred to leave the potential unspecified so that a screened potential can be employed. Henceforth, we shall assume that the potential is screened so that $\psi_k(0)$ remains finite. This will enable one to omit the term in Eq. (10). Similar considerations apply for the second term of Eq. (9).

In order to avoid an undue amount of algebra, let us consider the form of two typical terms in Eq. (9), the third and the last. Then the

other terms can be written in analogy with these. The third term, taking each part of the commutator separately, gives

$$I_{3+} = \int dk_2\, dk_1''\, dk_1''\, F(\vec{k}_1, \vec{k}_1'')\, \rho(\vec{k}_2, \vec{k}_2'')\, \Psi(\vec{k}_1'', \vec{k}_2''; \vec{k}_1', \vec{k}_2) =$$

$$4\pi \int dk_2\, dk_1''\, dk_1''\, F(\vec{k}_1)\, \delta(\vec{k}_1 - \vec{k}_1'')\, \Psi(\tfrac{1}{2}|\vec{k}_1' - \vec{k}_1'' - \vec{k}_2 + \vec{k}_2''|)\, \delta(\vec{k}_1' - \vec{k}_1'' + \vec{k}_2 - \vec{k}_2'') =$$

$$4\pi \int dk_2\, dk_2''\, F(\vec{k}_1)\, \rho(\vec{k}_2, \vec{k}_2'')\, \Psi(\tfrac{1}{2}|\vec{k}_1' - \vec{k}_1 - \vec{k}_2 + \vec{k}_2''|)\, \delta(\vec{k}_1' - \vec{k}_1 + \vec{k}_2 - \vec{k}_2'') = \tag{12}$$

$$4\pi\, F(\vec{k}_1)\, \Psi(|\vec{k}_1 - \vec{k}_1'|) \int dk_2\, \rho(\vec{k}_2, \vec{k}_2 + \vec{k}_1' - \vec{k}_1),$$

and

$$I_{3-} = -\int dk_2\, dk_2''\, dk_1''\, \Psi(\vec{k}_1, \vec{k}_2; \vec{k}_1'', \vec{k}_2'')\, F(\vec{k}_1'', \vec{k}_1')\, \rho(\vec{k}_2'', \vec{k}_2)\, \delta(\vec{k}_1' - \vec{k}_1)$$

$$= -\int dk_2\, dk_1''\, \Psi(\vec{k}_1, \vec{k}_2; \vec{k}_1', \vec{k}_2'')\, F(\vec{k}_1')\, \rho(\vec{k}_2'', \vec{k}_2)$$

$$= -4\pi \int dk_2\, dk_2''\, \Psi(\tfrac{1}{2}|\vec{k}_1' - \vec{k}_1 - \vec{k}_2'' + \vec{k}_2|)\, \delta(\vec{k}_1' - \vec{k}_1 + \vec{k}_2'' - \vec{k}_2)\, F(\vec{k}_1')\, \rho(\vec{k}_2'', \vec{k}_2) \tag{13}$$

$$= -4\pi\, \Psi(|\vec{k}_1' - \vec{k}_1|)\, F(\vec{k}_1') \int \rho(\vec{k}_2, \vec{k}_2 + \vec{k}_1' - \vec{k}_1)\, dk_2,$$

where $\vec{k}_2''$ and $\vec{k}_2$ have been interchanged in Eq. (13) since they are integration variables. Adding Eqs. (12) and (13) one gets

$$I_{3+} + I_{3-} = 4\pi\, \Psi(|\vec{k}_1' - \vec{k}_1|) \left\{ F(\vec{k}_1) - F(\vec{k}_1') \right\} \int \rho(\vec{k}_2, \vec{k}_2 + \vec{k}_1' - \vec{k}_1)\, dk_2 \tag{14}$$

Now, for the last term in Eq. (9), again considering each part of the commutator separately, one has

$$I_{4+} = -\tfrac{1}{2} \int dk_2\, dk_1''\, dk_1''\, \rho(\vec{k}_1, \vec{k}_1'')\, F(\vec{k}_2, \vec{k}_1'')\, \Psi(\vec{k}_1'', \vec{k}_1''; \vec{k}_1', \vec{k}_2)$$

$$= -\tfrac{1}{2} \int dk_2\, dk_2''\, dk_1''\, \rho(\vec{k}_1, \vec{k}_2'')\, F(\vec{k}_2)\, \delta(\vec{k}_2 - \vec{k}_1'')\, \Psi(\vec{k}_1'', \vec{k}_2''; \vec{k}_1', \vec{k}_2) \tag{15}$$

$$= -2\pi \int dk_2\, dk_2''\, \rho(\vec{k}_1, \vec{k}_2'')\, F(\vec{k}_2)\, \Psi(\tfrac{1}{2}|\vec{k}_1' - 2\vec{k}_2 + \vec{k}_2''|)\, \delta(\vec{k}_1' - \vec{k}_2'')$$

$$= -2\pi \int dk_2\, \rho(\vec{k}_1, \vec{k}_1')\, F(\vec{k}_2)\, \Psi(|\vec{k}_1' - \vec{k}_2|), \qquad \text{(cont.)}$$

and similarly, the second term is

$$I_{6-} = \tfrac{1}{2} \int dk_2\, dk_1''\, dk_2''\, \Psi(k_1 k_2; k_1'' k_2'')\, \rho(\vec{k}_1'', \vec{k}_2)\, F(\vec{k}_2'', \vec{k}_1')$$

$$= \tfrac{1}{2} F(k_1') \int dk_2\, dk_1''\, 4\pi \underline{\Psi}(|k_1'' - k_1|)\, \rho(\vec{k}_1'', \vec{k}_1'' + \vec{k}_1' - \vec{k}_1). \qquad (16)$$

Hence, adding Eqs. (15) and (16), one gets

$$I_6 = -\tfrac{1}{2} \rho(\vec{k}_1, \vec{k}_1') \int dk_2\, F(\vec{k}_2)\, 2\pi\, \underline{\Psi}(|\vec{k}_1' - \vec{k}_2|)$$
$$+ \tfrac{1}{2} F(k_1') \int dk_1''\, 2\pi\, \underline{\Psi}(|\vec{k}_1'' - \vec{k}_1|)\, \rho(\vec{k}_1'', \vec{k}_1'' + \vec{k}_1' - \vec{k}_1) \qquad (17)$$

Without considering each term in detail, one can now write for the other expressions:

$$I_4 = 0 \qquad (18)$$

$$I_5 = -2\pi F(\vec{k}_1') \int dk_2\, \rho(k_2, \vec{k}_2 + \vec{k}_1' - \vec{k}_1)\, \underline{\Psi}(|\vec{k}_2 - \vec{k}_1|)$$
$$+ 2\pi \rho(\vec{k}_1, \vec{k}_1') \int dk_2\, F(\vec{k}_2)\, \underline{\Psi}(|\vec{k}_2 - \vec{k}_1|). \qquad (19)$$

Now, evaluating the kinetic energy commutator and using Eqs. (10), (14), (17)-(19) in Eq. (7), one has

$$\frac{\hbar}{i} \frac{\partial \rho}{\partial t} - \frac{\hbar^2}{2m} (k_1'^2 - k_1^2) \rho(\vec{k}_1, \vec{k}_1') =$$
$$\frac{4\pi N e^2}{V} \{ F(\vec{k}_1') - F(\vec{k}_1) \}\, \underline{\Psi}(|\vec{k}_1' - \vec{k}_1|) \int dk_2\, \rho(\vec{k}_2, \vec{k}_2 + \vec{k}_1' - \vec{k}_1)$$
$$+ \frac{2\pi N e^2}{V} \{ \rho(\vec{k}_1, \vec{k}_1') \int F(\vec{k}_2) \{ \underline{\Psi}(|\vec{k}_2 - \vec{k}_1''|) - \Psi(|\vec{k}_2 - \vec{k}_1|) \} \qquad (20)$$

$$+\frac{2\pi Ne^2}{V}\left\{F(\vec{k}_1)-F(\vec{k}_1')\right\}\int dk_2\,\Psi(|\vec{k}_2-\vec{k}_1|)\,\rho(\vec{k}_2,\vec{k}_2+\vec{k}_1'-\vec{k}_1). \qquad \text{(cont.)}$$

One now assumes that $\rho \propto e^{i\omega t}$, whence Eq. (20) becomes

$$\rho(\vec{k}_1,\vec{k}_1') = \frac{4\pi Ne^2}{\hbar V}\frac{1}{(\omega-[k_1'^2-k_1^2]\frac{\hbar}{2m})}\left\{\left[F(\vec{k}_1')-F(\vec{k}_1)\right]\Psi(|\vec{k}_1'-\vec{k}_1|)\int dk_2\,\rho(\vec{k}_2,\vec{k}_2+\vec{k}_1'-\vec{k}_1)\right.$$

$$+\tfrac{1}{2}\rho(\vec{k}_1,\vec{k}_1')\int dk_2\,F(\vec{k}_2)\left\{\Psi(|\vec{k}_2-\vec{k}_1'|)-\Psi(|\vec{k}_2-\vec{k}_1|)\right. \qquad (21)$$

$$\left.\left.+\tfrac{1}{2}\left\{F(\vec{k}_1)-F(\vec{k}_1')\right\}\int dk_2\,\Psi(|\vec{k}_2-\vec{k}_1|)\,\rho(\vec{k}_2,\vec{k}_2+\vec{k}_1'-\vec{k}_1)\right\}.\right.$$

Unfortunately, the solution of this integral equation cannot be effected. However, if one assumes that the contribution of exchange to the allowed frequencies is small, Eq. (21) can be simplified somewhat. Let

$$\omega = \omega_0 + \omega_1, \qquad (22)$$

$$\rho = \rho_0(\omega_0) + \rho_1(\omega_1), \qquad (23)$$

where ω_1 is the correction due to exchange to the usual frequency. Then, if $\omega_1 \ll \omega_0$ and $\rho_1 \ll \rho_0$, one can obtain two equations for these quantities. These are

$$\left[\omega_0 - \frac{\hbar}{2m}(k_1'^2-k_1^2)\right]\rho(\vec{k}_1,\vec{k}_1') = \frac{4\pi Ne^2}{\hbar V}\left[F(\vec{k}_1')-F(\vec{k}_1)\right]\Psi(|\vec{k}_1-\vec{k}_1'|)\int dk_2\,\rho(\vec{k}_2,\vec{k}_2+\vec{k}_1'-\vec{k}_1), \qquad (24)$$

and

$$\omega_1\rho_0(\vec{k}_1,\vec{k}_1') + \left[\omega_0 - \frac{\hbar}{2m}(k_1'^2-k_1^2)\right]\rho(\vec{k}_1,\vec{k}_1') = \qquad (25)$$

$$= \frac{4\pi Ne^2}{\kappa V}\left[\left\{F(\vec{k}_1') - F(\vec{k}_1)\right\}\Psi(|\vec{k}_1'-\vec{k}_1|)\int dk_2\,\rho_1(\vec{k}_2,\vec{k}_2+\vec{k}_1'-\vec{k}_1)\right.$$

$$+\tfrac{1}{2}\rho_0(\vec{k}_1,\vec{k}_1')\int dk_2\,F(\vec{k}_2)\left\{\Psi(|\vec{k}_2-\vec{k}_1'|) - \Psi(|\vec{k}_2-\vec{k}_1|)\right\} \qquad \text{(cont.)}$$

$$\left.-\tfrac{1}{2}\left\{F(\vec{k}_1) - F(\vec{k}_1')\right\}\int dk_2\,\Psi(|\vec{k}_2-\vec{k}_1|)\,\rho_0(\vec{k}_2,\vec{k}_2+\vec{k}_1'-\vec{k}_1)\right].$$

If one writes

$$\vec{k}_1' = \vec{k}_1 + \vec{\xi}, \tag{26}$$

and integrates Eq. (24) over $\vec{k}_1$, one obtains

$$\int \rho(\vec{k}_2,\vec{k}_2+\vec{q})\,dk_2\left[1 - \frac{4\pi Ne^2}{\kappa V}\Psi(q)\int \frac{F(\vec{k}_1+\vec{q}) - F(\vec{k}_1)}{\omega_0 - \frac{\hbar}{2m}\left[(\vec{k}_1+\vec{q})^2 - k_1^2\right]}\right], \tag{27}$$

This is the same relation as found previously in the self-consistent field treatment (Eq. (31), Chapter III). Now, using Eq. (26) and integrating Eq. (25) over $\vec{k}_1$, taking into account Eq. (27), one has an equation for ω_1, i.e.,

$$\omega_1\int \frac{\rho_0(\vec{k}_1,\vec{k}_1+\vec{q})\,dk_1}{\omega_0 - \frac{\hbar}{2m}\left[(\vec{k}_1+\vec{q})^2 - k_1^2\right]} =$$

$$\frac{2\pi Ne^2}{\kappa V}\left[\int \frac{\rho_0(\vec{k}_1,\vec{k}_1+\vec{q})\,dk_1}{\omega_0 - \frac{\hbar}{2m}\left[(\vec{k}_1+\vec{q})^2 - k_1^2\right]}\int dk_2\,F(\vec{k}_2)\left\{\Psi(|\vec{k}_2-\vec{k}_1-\vec{q}|) - \Psi(|\vec{k}_2-\vec{k}_1|)\right\}\right. \tag{28}$$

$$\left. + \int dk_1\left[\frac{F(\vec{k}_1+\vec{q}) - F(\vec{k}_1)}{\omega_0 - \frac{\hbar}{2m}\left[(\vec{k}_1+\vec{q})^2 - k_1^2\right]}\right]\int dk_2\,\Psi(|\vec{k}_2-\vec{k}_1|)\,\rho(\vec{k}_2,\vec{k}_2+\vec{q})\right].$$

This is the same equation found recently by von Roos and Zmuidzinas (36a). It can be solved at zero temperature for $|q| \ll |k_F|$. Since the solution has already been given by the aforementioned authors, there is little point in repeating the calculations.

II. TRANSVERSE OSCILLATIONS OF A QUANTIZED PLASMA

In this section an equation which, in the absence of exchange terms, leads to the dispersion relation for transverse waves will be derived. The problem is formulated in terms of the average fields discussed in the previous chapter. Now, however, one assumes that the time dependence of the fields is arbitrary and Maxwell's equations may be substituted for the oscillator equation. We employ a perturbation treatment similar to that of the last section, letting

$$R^{1:0} = F + \rho \tag{29}$$

where F is the equilibrium density matrix of the previous section. We also assume that the sources of the fields are due to the departure from equilibrium, i.e., one inserts ρ in the traces. Thus, there must be a positive background of change to cancel the contribution from F. Finally, it will be convenient to change gauges so that

$$\vec{\nabla} \cdot \vec{A}_{ov} + \frac{1}{c} \frac{\partial \phi_{ov}}{\partial t} = 0, \tag{30}$$

i.e., the Lorentz condition holds.

With these assumptions, the basic equations for the particles (neglecting spin) is, from Eq. (48) of Chapter IV,

$$\frac{\hbar}{i} \frac{\partial \rho}{\partial t} + \left[\rho, \frac{-\hbar^2}{2m} \nabla^2\right] + e\left[F, \phi_{ov}\right] + \tag{31}$$

$$-\frac{e}{mc}\left[F,\vec{A}\cdot\frac{\hbar}{i}\vec{\nabla}+\frac{\hbar}{i}\vec{\nabla}\cdot\vec{A}\right]+\left[R^{2\,1\,0},\phi_{ex}\right]=0, \qquad \text{(cont.)}$$

since $\vec{A}$ is no longer solenoidal. (We drop the subscript av for convenience.) Also, A^2 is second order, hence negligible. For the fields, one has

$$\vec{\nabla}\cdot\vec{H}=0, \tag{32}$$

$$\vec{\nabla}\cdot\vec{\mathcal{E}}=-\frac{1}{c}\vec{\nabla}\cdot\dot{\vec{A}}-\nabla^2\phi=\frac{4\pi Ne}{V}T_n^{(1)}\rho\,\delta(\vec{r}-\vec{r}_1), \tag{33}$$

$$\nabla\times\mathcal{E}=-\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \tag{34}$$

and

$$\vec{\nabla}\times\vec{H}=\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}=-\frac{1}{c}\frac{\partial\dot{\vec{A}}}{\partial t}-\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\phi+\frac{4\pi Ne}{V}T_n^{(1)}\rho\,\delta(\vec{r}-\vec{r}_1)\vec{v} \tag{35}$$

Now, as usual, one can use the Lorentz condition in Eqs. (33) and (34), adding and subtracting $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ in Eq. (34) to give

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi=-\frac{4\pi Ne}{V}T_n^{(1)}\rho\,\delta(\vec{r}-\vec{r}_1) \tag{36}$$

and for Eq. (35) one has

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}=-\frac{4\pi Ne}{V}T_n^{(1)}\rho\vec{v}\,\delta(\vec{r}-\vec{r}_1) \tag{37}$$

Now, one can Fourier transform Eqs. (36) and (37) in space and time and evaluate the traces, recalling that

$$\vec{v}=\frac{1}{m}\left(\vec{p}-\frac{e}{c}\vec{A}\right). \tag{38}$$

One obtains

$$\left(q^2-\frac{\omega^2}{c^2}\right)\phi(\vec{q},\omega)=\frac{4\pi Ne}{V}\int dk\,\rho(\vec{k},\vec{k}+\vec{q}) \tag{39}$$

and

$$\left(q^2-\omega^2/c^2\right)\vec{A}(\vec{q},\omega)=\frac{4\pi Ne}{V}\frac{\hbar}{mc}\int\vec{k}\,\rho(\vec{k},\vec{k}+\vec{q})+\frac{4\pi Ne^2}{Vmc^2}\vec{A}(\vec{q},\omega). \tag{40}$$

(The traces are evaluated as in Chapter III, but now we have assumed continuous variables.)

Next, we shall write Eq. (31) in the momentum representation. Since the procedure is quite similar to that in the preceding section, only one typical term will be derived. Consider, for example,

$$\left(F\vec{A}\cdot\frac{\hbar}{i}\vec{\nabla}\right)(\vec{k},\vec{k}+\vec{q}) = \int dk''\,dk'''\,F(\vec{k},\vec{k}'')\,A(\vec{k}'',\vec{k}''')\,\frac{\hbar}{i}\left(\vec{\nabla}'\delta(\vec{k}'-\vec{r})\right)(\vec{k}''',\vec{k}'), \tag{41}$$

But, one has

$$\left[\vec{\nabla}'\delta(\vec{k}'-\vec{r})\right](\vec{k}''',\vec{k}') = \int \vec{\nabla}'\delta(\vec{r}'-\vec{r})\,e^{-i\vec{k}'\cdot\vec{r}'-\vec{k}'''\cdot\vec{r}'}\,dr\,dr'$$

$$= i\vec{k}'''\,\delta(\vec{k}'-\vec{k}'''), \tag{42}$$

where an integration by parts was performed. Also, F is diagonal in $\vec{k}$, so Eq. (41) becomes

$$\int dk''\,dk'''\,F(\vec{k})\,\delta(\vec{k}-\vec{k}'')\,A(\vec{k}'',\vec{k}''')\cdot\hbar\vec{k}'''\,\delta(\vec{k}'-\vec{k}''')$$

$$= F(\vec{k})\,\vec{A}(\vec{k},\vec{k}')\cdot\hbar\vec{k}'. \tag{43}$$

The other terms may be obtained in a similar manner. Finally, Eq. (31) becomes, dividing each side by $\hbar$,

$$\left(\omega - \frac{\hbar}{2m}\left([\vec{k}+\vec{q}]^2 - k^2\right)\right)\rho(\vec{k},\vec{k}+\vec{q}) =$$

$$\frac{e}{\hbar}\,\phi(\vec{k},\vec{k}+\vec{q})\,\Delta F + \frac{e}{mc}\,\vec{A}(\vec{k},\vec{k}+\vec{q})\,\Delta(\vec{k}F) \tag{44}$$

$$-\frac{e}{2mc}\,\vec{q}\cdot\vec{A}(\vec{k},\vec{k}+\vec{q})\,\Delta F + \phi_{ex},$$

where

$$\Delta F = F(\vec{k}+\vec{q}) - F(\vec{k}), \tag{45}$$

$$\Delta(\vec{k}F) = \vec{k}F(\vec{k}+\vec{q}) - (\vec{k}+\vec{q})F(\vec{k}) \tag{46}$$

and

$$k' = \vec{k}+\vec{q}, \tag{47}$$

with ϕ_{ex} given by the last two terms of Eq. (21). Now, we note from Eqs. (39) and (40) that $A(\vec{k}, \vec{k}_1+\vec{q})$ and $\phi(\vec{k}, \vec{k}_1+\vec{q})$ can only be functions of the difference of their arguments, i.e., of $\vec{q}$.

Let us now drop the exchange potential for a moment. Then, dividing each side of Eq. (44) by the term in ω, one has

$$\rho(\vec{k},\vec{k}+\vec{q}) = \frac{e}{\hbar}\phi(\vec{q})\frac{\Delta F}{D} + \frac{e}{mc}\vec{A}(\vec{q})\cdot\frac{\Delta(\vec{k}F)}{D} - \frac{e}{2mc}\vec{q}\cdot\vec{A}(\vec{q})\frac{\Delta F}{D} \tag{48}$$

with

$$D = \omega - \frac{\hbar}{2m}\left[(\vec{k}+\vec{q})^2 - k^2\right]. \tag{48a}$$

We can multiply Eq. (48) by $(4\pi N e/V)$ and integrate over $\vec{k}$, using Eq. (39), to get

$$\left(q^2 - \frac{\omega^2}{c^2}\right)\phi(\vec{q}) = \omega_{pe}^2\frac{m}{\hbar}\phi\int\frac{\Delta F}{D}dk + \frac{\omega_{pe}^2}{c}\vec{A}\cdot\int\frac{\Delta(\vec{k}F)}{D}dk - \frac{\omega_{pe}^2}{c}\vec{q}\cdot\vec{A}(\vec{q})\int\Delta F/D\,dk, \tag{49}$$

where

$$\omega_{pe}^2 = 4\pi N e^2/mV. \tag{50}$$

Similarly, multiplying Eq. (48) by $\left(4\pi\frac{Ne}{mcV}\frac{\hbar}{}\vec{k}\right)$ and integrating over $\vec{k}$, one has, from Eq. (40),

$$\left[q^2 - \left(\frac{\omega^2+\omega_{pe}^2}{c^2}\right)\right]\vec{A} = \frac{\omega_{pe}^2}{mc^2}\int\vec{k}\frac{\Delta F}{D}dk + \frac{\omega_{pe}^2\hbar}{mc^2}\vec{A}\cdot\int\frac{\Delta(\vec{k}F)}{D}\vec{k}\,dk \tag{51}$$

$$- \omega n_e^2 \frac{t}{2mc^2} \vec{q} \cdot \vec{A} \int \vec{k} \frac{\Delta F}{D} dk . \qquad \text{(cont.)}$$

Now, Eqs. (50) and (51) can be written

$$A_i = \sum C_{ij} A_j , \qquad (i,j = 1-4) \tag{52}$$

with

$$A_4 = \phi \tag{53}$$

Thus, one has a set of homogeneous equations. The condition that there be a solution is that the determinant of the coefficients vanish. This will lead to the dispersion relation.

Now, let us examine the C_{ij} explicitly. Without any loss in generality, one can take $\vec{q}$ to lie along the Z direction. Then, some of the C_{ij} will vanish because of the parity of the integrands. (The function $F(\vec{k})$, as discussed previously, is a function only of k^2.) Explicitly, the C_{ij} are

$$C_{\mu 4} = \frac{\omega n_e^2}{c} \int k_\mu \frac{\Delta F}{D} dk , \qquad (\mu, \nu = 1-3) \tag{54}$$

$$C_{\mu\nu} = \omega n_e^2 \frac{t}{mc^2} \int \frac{\Delta (k_\nu F) k_\mu dk}{D} - \frac{\omega n_e^2 t}{2 mc^2} q \int \frac{\Delta F k_\mu dk}{D} \delta_{\nu 3} - \left[q^2 - \left(\frac{\omega^2 + \omega n^2}{c^2} \right) \right] \delta_{\mu\nu} , \tag{55}$$

$$C_{4\mu} = \frac{\omega n_e^2}{c} \int \frac{\Delta (k_\mu) F}{D} dk \tag{56}$$

(1, 2, 3 corresponds to x, y, z.)

and

$$C_{44} = \omega \pi e^2 \frac{m}{\kappa} \int \frac{\Delta F dk}{D} - (g^2 - \frac{\omega^2}{c^2}). \tag{57}$$

Now, let us examine the function D. One has

$$D = \frac{1}{\omega - \frac{\kappa}{2m}[(\vec{k}+\vec{g})^2 - k^2]} = \frac{1}{\omega - \frac{\kappa}{m} k_3 g - \frac{\kappa}{2m} g^2}, \tag{58}$$

Thus, D is only a function of k_3. Taking the first integral in C_{44}, one has

$$C_{44} = \frac{\omega \pi e^2}{c} \int_{-\infty}^{\infty} \frac{F(\vec{k}+\vec{g}) k_1 dk}{D(k_3, g, \omega)}$$

$$= \frac{\omega \pi e^2}{c} \int_{-\infty}^{\infty} \frac{dk_3}{D} \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} k_1 dk_1 F(k_1^2, k_2^2, (k_3+g)^2), \tag{59}$$

But, since F is an even function of k_1 and D does not depend on k_1, one has for the k_1 integration an odd integrand integrated between symmetric limits. Thus, the integral vanishes. Similar considerations apply to the other expressions in C_{44} and, in fact, we get

$$C_{\mu\nu} = 0 \quad , \mu \neq \nu \quad , \tag{60}$$

$$C_{14} = C_{24} = C_{41} = C_{42} = 0. \tag{61}$$

Thus the determinant of the coefficients is of the form

$$\begin{vmatrix} C_{11} & 0 & 0 & 0 \\ 0 & C_{22} & 0 & 0 \\ 0 & 0 & C_{33} & C_{34} \\ 0 & 0 & C_{43} & C_{44} \end{vmatrix} = 0, \tag{62}$$

for which

$$C_{11} = 0, \tag{63}$$

or $$C_{22} = 0 \tag{64}$$

are solutions for transverse waves. Although it appears that the dispersion relation is still coupled, one can use the Lorentz condition to eliminate ϕ (or A_3), i.e.,

$$q A_3 = -\frac{\omega}{c}\phi . \tag{65}$$

Thus, the dispersion relation for transverse waves (neglecting exchange), using Eqs. (55) and (63), is

$$1 = \frac{\omega_{pe}^2 \hbar}{mc^2}\left[q^2-\left(\frac{\omega^2+\omega_{pe}^2}{c^2}\right)\right]\int \frac{\Delta(k,F)\,k_\perp\,dk}{D} . \tag{66}$$

This expression can be integrated at zero temperature. The result is

$$q^2-\left(\frac{\omega^2+\omega_{pe}^2}{c^2}\right) = -\frac{\pi\omega_{pe}^2\frac{\hbar^2}{m^2}q^2\left\{\frac{B^2(v^2k_f^2-B^2)}{v^5|A-B|}\ln\left|\frac{vk_f+B}{vk_f-B}\right|\right.}{\left(4\pi\frac{k_f^3}{3}\right)}$$

$$\left.-\frac{A^2(v^2k_f^2-A^2)}{v^5|A-B|}\ln\left|\frac{vk_f+A}{vk_f-A}\right| - \frac{(2A^2+AB)k_f}{v^4}\right. \tag{67a}$$

$$\left.+\frac{k_f^4}{8(A+B)v}\left[\ln\left|\frac{vk_f-B}{vk_f+B}\right| - \ln\left|\frac{vk_f-A}{vk_f+A}\right|\right] - \frac{A^3}{2v^5}\ln\left|\frac{vk_f-A}{vk_f+A}\right| + \frac{B^3}{2v^5}\ln\left|\frac{vk_f-B}{vk_f+B}\right|\right\},$$

with

$$v = \hbar q/m, \tag{67b}$$

$$A = \omega + \hbar q^2/2m \tag{67c}$$

$$B = \omega - \hbar q^2/2m \tag{67d}$$

and k_F is the Fermi momentum (33). It hardly seems necessary to state

that one cannot solve this expression for ω.

Finally, one can examine this expression in the classical limit. To do so, one should note that

$$\lim_{k\to 0} \int d^3k F \to \int d^3p f(p) \tag{68}$$

where f is the Maxwell-Boltzmann distribution function (33). Also, one has

$$\hbar \vec{k} = \vec{p}\,, \tag{69}$$

$$\begin{aligned} \Delta(k_1 F) &= \frac{\hbar k_1 F(\hbar k_1 + \hbar q) - \hbar(k_1) F(k_1)}{\hbar} \\ &= p_1 \frac{\left(F(p) + \hbar q \cdot \frac{\partial F}{\partial p_3}\right) - p_1 F}{\hbar} \\ &= p_1 q \frac{\partial f}{\partial p_3}\,, \end{aligned} \tag{70}$$

since q is an arbitrary wave vector. Thus, Eq. (66) becomes

$$1 = \frac{\omega_{pe}^2}{nc^2} \frac{1}{q^2 - \frac{(\omega + \omega_{pe})^2}{c^2}} \int \frac{p_1^2 q \frac{\partial f}{\partial p_3}}{D'} d^3p \tag{71}$$

where

$$\lim_{k\to 0} D = \lim_{k\to 0} \frac{1}{\omega_0 - \frac{\hbar k q}{m} - \hbar q^2/m} = \frac{1}{\omega - p_3 q/m} = D' \tag{72}$$

This is the result obtained in the classical treatment (11a). Now, all the previous considerations depend on the fact that one can neglect exchange terms. A glance at Eq. (21) indicates that one cannot obtain dispersion equations when exchange terms are included. Moreover, the simple perturbation procedure employed in the first section is no longer useful, since the resulting equations are still coupled.

However, one expects exchange effects to be relatively small in most cases and the dispersion relations should be quite accurate. While it is inaccurate to say that the exchange effects "couple" transverse and longitudinal dispersion relations, since these cannot be obtained when exchange is included, it is not unreasonable physically. The exchange terms arise from particle-particle (longitudinal) interactions, but give rise to "currents" which couple the longitudinal and transverse modes.

CHAPTER VI

SUMMARY

In the preceding chapters, the problem of formulating a quantum kinetic theory of plasmas has been examined in detail. It was shown first in Chapter III that the N-particle self-consistent field solutions give no new results and offer no advantages over the single-particle self-consistent field approach. In Chapter IV, the derivation of kinetic equations for reduced particle-oscillator density matrices was given. The hierarchy of coupled kinetic equations was decoupled by an expansion of the density matrices. Zero- and first-order equations were displayed, and it was shown that by taking several of the lowest order kinetic equations, the density matrices needed to specify the system could, in principle, be found. In Chapter V, the zero-order equations were applied to the calculation of dispersion relations for collective oscillators of the plasma. It was found that, in contrast to the corresponding classical case, separate dispersion relations for longitudinal and transverse waves were not obtained. This result is due to the presence of particle exchange terms in the electrostatic potential. A calculation assuming distinguishable particles indicates that the coupling no longer exists.

In view of the fact that the equations obtained in this treatment reduce to the proper form in the classical limit, one can hope that, at least for the zero-order or "Vlasov" equations, other collective properties of quantum systems can be studied within the formalism. Although there is no experiment with which to compare the results obtained here, they are apparently substantiated to some extent by the correspondence

principle arguments. Since the entire treatment up to the expansion of the density matrices in Chapter IV is exact and is based on the Liouville equation, any inaccuracies must enter in or after the expansion. However, the expansion is essentially equivalent to a Hartree-Fock treatment, which is known to be valid for weakly interacting systems.

Perhaps the most critical inaccuracy in the zero-order equations arises from the electrostatic potential. The fact that this term diverges at short distances, or in the momentum representation, at long wavelengths, makes it necessary to assume a screened potential. This must be done on a phenomenalogical basis and is consequently not rigorous. A second difficulty is that when radiation fields are represented, there is no rigorous justification for the expansion of the density matrix. However, it is reasonable to suppose that this is possible, just as it is when only Coulomb forces are considered.

Although the previous considerations indicating the weak points in this thesis are quite pertinent, it is nonetheless appropriate to indicate future directions of research. Foremost among these is a detailed examination of a perturbation treatment applicable to a system in which both longitudinal and transverse electromagnetic interactions are significant. This study would have meaning both in the classical and quantum mechanical realms. Since, as was mentioned earlier, situations exist in which quantum systems display collective behavior, the arguments for such an investigation are quite cogent.

Another interesting problem would be to include the spin variables in the calculations in order to determine their effects on the dispersion relation. One would expect these terms to contribute to the transverse

fields.

Again, in relation to the first suggestion, first-order kinetic equations would contain information about particle correlations. Derivation of a true, irreversible Fokker-Planck equation from these could be expected to yield information concerning energy losses, etc., in a quantized plasma. Along these same lines, the determination of the particle correlation function for a plasma is a quite useful objective. However, both of these last two problems depend on the proper derivation of first-order equations. Consequently, this seems to be the most appropriate step in extending the theory.

PART II

CYCLOTRON INSTABILITIES IN A BOUNDED PLASMA

CHAPTER I

INTRODUCTION

Since the pioneering work of Langmuir and Tonks* (16) in 1929, the theory of plasma oscillations has received increasing attention, particularly in the last ten years. One of the major obstacles to the evaluation of the theory has been the extreme difficulty encountered in the interpretation of experimental data. However, with the improvement in plasma diagnostics, it has become evident that oscillatory phenomena play an important role in the interactions of the particles composing a plasma. Of particular interest are the unstable modes of oscillation, in which a small disturbance grows in time or space, eventually disrupting the confined system.

In Chapter II of this part of the dissertation, the oscillations of a cylindrical shell of plasma in a uniform axial magnetic field are considered. The formulation of the problem in terms of the two-fluid magnetohydrodynamic equations is shown to lead to a dispersion relation for the frequencies of oscillation of the plasma. This dispersion relation is solved in the limit of short wave length axial disturbances and under certain circumstances growing modes are predicted. Finally, a comparison of the results with experimental data is given.

I. REVIEW OF THE THEORY

After the classic papers of Langmuir and Tonks (16), plasma oscil-

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*References are listed as in Part I.

lations were considered in a slightly different manner by A. A. Vlasov (18), who treated the system of charged particles by means of a modified Boltzmann equation for each species of particle. Vlasov omitted the collision term in the Boltzmann equation and calculated the average or self-consistent electromagnetic field by using the particle distribution functions as sources in Maxwell's equations. After linearization of this system of equations, Vlasov was able to obtain a dispersion relation for the oscillation frequencies.

Later, Landau (12), employing the Vlasov equations, showed that damped and growing modes of oscillation could exist in a plasma for certain initial configurations. In addition, Landau showed that, in the most rigorous sense, a proper dispersion relation for a plasma does not exist. Van Kampen (17) also demonstrated that for a given wave number of a disturbance, a continuous range of frequencies is possible.

In recent years plasma oscillations have been scrutinized carefully by theorists in an effort to find possible unstable or growing modes of oscillation in a plasma, since, under certain circumstances, a small disturbance will propagate and grow either in space or time, preventing containment. All of these studies, as well as those previously discussed, are based upon linearized versions of the statistical or magneto-hydrodynamic equations governing a plasma, and their predictions are contingent upon the validity of the linearization. A recent paper by Bernstein and Trehan (4) contains a complete bibliography of linearized treatments as well as the few attempts to study non-linear properties of a plasma. However, the usefulness of the linear theories, in the absence of any interesting non-linear investigations, is considerable. In par-

ticular, Harris (8-10) has shown that unstable longitudinal oscillations may exist in a plasma in a uniform magnetic field when the initial ion or electron velocity distributions are sufficiently anisotropic and when the density of the plasma exceeds a certain critical value. Harris showed that when the ion cyclotron frequency ω_{ci} became less than the electron plasma frequency ω_{pe}, instabilities would develop. This has particular significance for thermonuclear devices such as DCX. Drummond, Rosenbluth and Johnson (6) have determined lower limits for instability for both ion and electron longitudinal oscillations in terms of the degree of anisotropy of the initial velocity distributions. A similar result has been obtained by Post (13) for unstable transverse hydromagnetic waves in a magnetic mirror machine.

A limitation on the applicability of Harris' treatment of longitudinal oscillations arises because this work, like most other previous studies, considers oscillations in an unbounded plasma. As yet, only a few authors have attempted to study finite systems. Thus, one has no idea how the finite boundaries will affect plasma oscillation frequencies. For this reason it is of considerable interest to examine the problem of the finite, cylindrical shell of plasma in a uniform magnetic field. In Chapter II one form of this problem which approximates the situation existing in the DCX machine of Project Sherwood (2) is examined.

Specifically, the problem consists of an examination of the oscillations of a cylindrical shell of plasma of infinite length and limiting radii r_1 and r_2. Initially, the ions are assumed to more in Larmor orbits in a uniform external magnetic field and the electrons are assumed sta-

tionary. The equations governing the system are linearized and the dispersion relation for longitudinal (electrostatic) waves is derived. See Fig. 1.

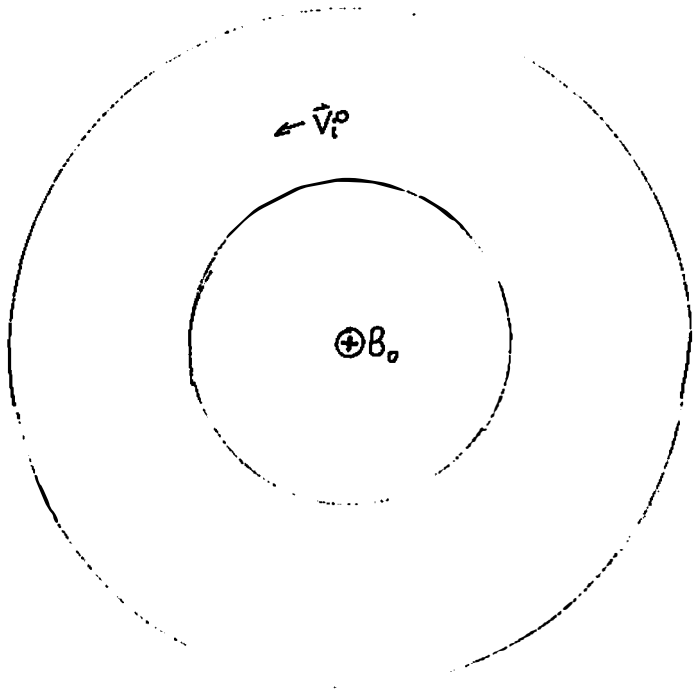

FIGURE 1

THE INITIAL PLASMA CONFIGURATION

CHAPTER II

SOLUTION OF THE PROBLEM

In this chapter the basic equations governing the plasma are derived from the Vlasov equations and the range of validity of the theory is discussed. A perturbation treatment which consists of linearizing the equations is used to obtain the dispersion relation for longitudinal or electrostatic waves in a plasma. This dispersion relation is solved for the oscillation frequencies in the limit of short wavelength axial disturbances. Criteria for instability are derived and discussed.

I. THE VLASOV EQUATIONS

As was mentioned previously, the basis for many theoretical investigations of plasmas is the Vlasov equations. These are a set of coupled equations for the distribution functions of each species of the plasma and Maxwell's equations with the distributions used as sources, i.e.,

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\vec{\nabla} f_s + \left[\frac{q_s}{m_s}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) + \frac{\vec{F}_e}{m_s}\right]\cdot\frac{\partial f_s}{\partial \vec{v}} = 0, \tag{1}$$

and

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t}, \tag{2a}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\sum_s q_s\int f_s\,\vec{v}\,d^3v, \tag{2b}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\sum_s q_s\int f_s\,d^3v, \tag{2c}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{2d}$$

where f_s is the distribution function for species 's', $\vec{v}$ is the velocity and e_s and m_s the charge and mass respectively. In the term $\vec{F}_e$ are included all external forces imposed upon the system, while $\vec{E}$ and $\vec{B}$ are

the average internal electric and magnetic fields arising from the charges and currents within the system.

Eq. (1) is quite similar to the Boltzmann equation, except that the collision term is neglected. This is a quite common practice in plasma physics, although it necessarily places a certain limitation on the validity of the treatments. However, for relatively low density systems in which collisions are infrequent, or for times short compared to the collision time, it is a valid approximation. The use of the "average" or self-consistent electromagnetic field in Eq. (1) is, of course, subject to the same restriction since the distribution functions are sources.

Unfortunately, the complexity of the Vlasov equations limits their usefulness. While some exact solutions have been found, most treatments employ a linearized form of the above equations in which the distributions and fields are assumed to differ slightly from "equilibrium" values. Most discussions of plasma oscillations are based on this procedure.

The Vlasov equations yeild information about the change in space and the change in velocity space of the distributions. In this investigation, a simplification is possible which will reduce the number of variables necessary to specify the distributions. Here, one considers a plasma of sufficiently low density to make it possible to assume that there are no random or thermal velocities. That is, the distribution function f is written

$$f_s = N^s(\vec{r},t)\,\delta(\vec{v}-\vec{V}(\vec{r},t)). \tag{3}$$

Now, it is fairly simple to show that this distribution function satisfies the Vlasov equation provided certain restrictions are placed

upon N^s, the number of particles of type s at a point in space, and $\vec{V}^s$, the velocity of a particle of species s at a point in space. These restrictions will be seen to comprise the two-fluid magnetohydrodynamic equations for electrons and ions.

Now, let us consider each term of Eq. (1) with the distribution function of Eq. (3). One has

$$\frac{\partial}{\partial t}\left[N^s(\vec{r},t)\,\delta(\vec{v}-\vec{V}(\vec{r},t))\right] = \frac{\partial N^s}{\partial t}\delta(\vec{v}-\vec{V}(\vec{r},t)) + N^s\frac{\partial}{\partial \vec{V}}\delta(\vec{v}-\vec{V}^s)\cdot\frac{\partial \vec{V}^s}{\partial t}, \tag{4}$$

$$\vec{v}\cdot\frac{\partial}{\partial \vec{r}}\left[N^s\delta(\vec{v}-\vec{V}^s)\right] = \vec{v}\cdot\frac{\partial N^s}{\partial \vec{r}}\delta(\vec{v}-\vec{V}^s) + N^s\vec{v}\cdot\frac{\partial \vec{V}^s}{\partial \vec{r}}\cdot\frac{\partial}{\partial \vec{V}^s}\delta(\vec{v}-\vec{V}^s) \tag{5}$$

and $$\frac{\vec{F}_s}{m_s}\cdot\frac{\partial}{\partial \vec{v}}\left[N^s\delta(\vec{v}-\vec{V}^s)\right] = \frac{\vec{F}_s}{m_s}N^s\cdot\frac{\partial}{\partial \vec{v}}\delta(\vec{v}-\vec{V}^s). \tag{6}$$

Now, we note that from the properties of the delta function,

$$(v_i - V_k^s)\frac{\partial}{\partial V_k}\delta(\vec{v}-\vec{V}^s) = 0, \quad i \neq k, \tag{7}$$

$$(v_i - V_k^s)\frac{\partial}{\partial V_k}\delta(\vec{v}-\vec{V}^s) = -\delta(\vec{v}-\vec{V}^s), \quad i = k. \tag{8}$$

Thus, one can add and subtract $N\frac{\partial V_i^s}{\partial x_i}\delta(\vec{v}-\vec{V})$ to Eqs. (4) and (5), and add Eqs. (4)-(6) to get:

$$\begin{aligned}&\left[\frac{\partial N^s}{\partial t} + \vec{v}\cdot\vec{\nabla}N^s + N^s\vec{\nabla}\cdot\vec{V}^s\right]\delta(\vec{v}-\vec{V}^s) - N^s\vec{\nabla}\cdot\vec{V}^s\,\delta(\vec{v}-\vec{V}^s)\\ &+ N^s\vec{v}\cdot\frac{\partial \vec{V}^s}{\partial \vec{r}}\cdot\frac{\partial}{\partial \vec{V}^s}\delta(\vec{v}-\vec{V}^s) - \frac{\vec{F}_s}{m_s}N^s\frac{\partial}{\partial \vec{V}}\delta(\vec{v}-\vec{V}^s) + N^s\frac{\partial \vec{V}^s}{\partial t}\cdot\frac{\partial}{\partial \vec{V}}\delta(\vec{v}-\vec{V}^s)\\ &= \left[\frac{\partial N^s}{\partial t} + \vec{\nabla}\cdot N^s\vec{V}^s\right]\delta(\vec{v}-\vec{V}^s) + \left[\frac{\partial \vec{V}^s}{\partial t} + \vec{V}^s\cdot\vec{\nabla}\vec{V}^s - \frac{\vec{F}_s}{m_s}\right]\cdot\frac{\partial}{\partial \vec{V}^s}\delta(\vec{v}-\vec{V}^s),\end{aligned} \tag{9}$$

since $$\vec{v}\,\delta(\vec{v}-\vec{V}_s) = \vec{V}^s\delta(\vec{v}-\vec{V}^s), \tag{10}$$

$$\delta(\vec{v}-\vec{V}^s)\left[\vec{v}\cdot\vec{\nabla}N^s + N^s\vec{\nabla}\cdot\vec{V}^s\right] = \delta(\vec{v}-\vec{V}^s)\left[\vec{V}^s\cdot\vec{\nabla}N^s + N^s\vec{\nabla}\cdot\vec{V}^s\right] = \delta(\vec{v}-\vec{V}^s)\vec{\nabla}\cdot N^s\vec{V}^s, \tag{11}$$

and Eqs. (7) and (8) have been used. Thus, the distribution function given by Eq. (3) satisfies the Vlasov equation if

$$\frac{\partial N^s}{\partial t} + \vec{\nabla}\cdot N^s\vec{V}^s = 0, \tag{12}$$

$$\frac{\partial \vec{V}^s}{\partial t} + \vec{V}^s\cdot\vec{\nabla}\vec{V}^s = \frac{\vec{F}_s}{m_s} = \frac{q_s}{m_s}\left[\vec{E} + \frac{\vec{V}_s}{c}\times\vec{B}\right]. \tag{13}$$

Eq. (4) and Eq. (13) represent "continuity" equations and "equations of motion" for the particles. They must be used in conjunction with Maxwell's equations to describe the plasma.

The justification for the use of the Vlasov equations or, equivalently, Eqs. (4), (13), and Eqs. (2a)-(2d) is simple. Since the most interesting cases will occur for plasmas of densities $< 10^{14}$ particles $cm.^{-3}$, it is easy to see that the collision integral will not be needed. Following the arguments of Simon (22), one can assume that the initial distribution has been randomized when a particle has suffered a deflection of 90°. In a plasma, the significant interaction for low energies (non-relativistic) is the Coulomb interaction. It can be shown (22) that small angle deflections make a more significant contribution than large angle scatters, since the Coulomb forces are active at long ranges. The "effective" cross-section for a series of random encounters leading to a 90° deflection is

$$\sigma_{eff} = 80\pi\left(\frac{e^2}{mv^2}\right)^2 \tag{14}$$

where v is the average velocity of the particles. For 300 kev. protons, this is

$$\sigma_{eff} = 80\pi\left(\frac{23\cdot10^{-20}}{600\cdot10^{3}\cdot1.6\cdot10^{-12}}\right)^2 \simeq 40\cdot10^{-24}\ cm.^2 \tag{15}$$

Hence, for a particle density of 10^{14} $cm.^{-3}$, the average time necessary to randomize the distribution (assuming that the magnitude of the initial velocity is not significantly changed during collisions) is

$$\tau_{coll.} = \frac{1}{N\sigma_{eff}v} \simeq \frac{1}{10^{14}\cdot40\cdot10^{-24}\cdot10^{9}} \sim .25\ sec. \tag{16}$$

However, the times of observation will be determined by the frequencies of oscillation of the disturbances, which one expects to be of the order of ω_p, where $\omega_p^2 = 4\pi Ne^2(1/m_i + 1/m_e)$ is the plasma frequency. For a

plasma of electrons and protons, this is approximately:

$$\omega_p^{-1} \sim \left(\frac{4\pi N e^2}{m}\right)^{-\frac{1}{2}} \sim \left(\frac{4\pi 10^{14.23\cdot 10^{-10}}}{9.10^{-27}}\right)^{-\frac{1}{2}} \sim 10^{-9}\ \text{sec.} \qquad (17)$$

Hence, one sees that for a relatively long time, the Vlasov equations are quite useful. In addition, the approximation becomes better for lower densities since $\omega_p^{-1} \sim 1/N^{\frac{1}{2}}$ while $t_{coll} \sim 1/N$. Furthermore, for DCX, where the proton energy is about 300 kev and the electron energy about 1/40 ev, the ratio of electron velocity to proton velocity is

$$\frac{V_e}{V_i} = \sqrt{\frac{M_p}{M_e}\frac{E_e}{E_p}} \sim 10^{-2}, \qquad (18)$$

so it is quite reasonable to assume initially that $\vec{V}_e = 0$. Thus, the assumptions of the problem as stated in Chapter I are seen to be justifiable for low densities and in the energy range in which we are interested. Finally, one finds experimentally in DCX that the average time for an ion to be lost via charge exchange is of the order of seconds--again much longer than the period of one plasma oscillation.

One final approximation remains to be discussed. As indicated, in this investigation only longitudinal or electrostatic interactions are considered. This means that in the low frequency approximation, the displacement current and the perturbed magnetic field are neglected. In effect, one assumes that $\omega/k \ll c$. This type of approximation has been discussed in some detail by Bernstein and Trehan (4) and Bernstein and Kulsrud (3). When the equations governing the plasma are linearized and Fourier analyzed (exp $i[\omega t + \ell\phi + kz$), one finds that the transverse fields are multiplied by factors $\frac{\omega_{ci}}{k}$, $\frac{\omega_{ce}}{k}$ ($\omega_{ci,e}$ is the cyclotron frequency for ions or electrons), $\frac{\omega_{pi,e}}{k}$ ($\omega_{pi,e}$ is the plasma frequency for ions or electrons) and V^0/c. Hence, if one restricts the treatment to longitudinal (electrostatic) disturbances, one is assuming that the foregoing quantities are small, i.e.,

$\omega_{ci}/k, \omega_{ce}/k, \omega_{pi}/k, \omega_{pe}/k \ll c.$ (18a)

Thus the phase velocities of the plasma waves and the particle velocities must be small compared to the velocity of light.

In view of the fact that this investigation is primarily concerned with the state of a plasma such as occurs in DCX, some of these requirements can be seen to be valid already. For example, the densities considered will range between 10^6-10^8 particles/cm^3. For such a system, the plasma frequency is 10^8-10^9 cycles/sec. Consequently, for $\frac{\omega_p}{k} \ll c$, the treatment must be restricted to $k < 1 cm^{-1}$, i.e., to short wavelengths.

For 300 kev protons and electrons at room temperature, it has already been seen that the initial particle velocities are $< 10^9$ cm/sec., so $V^o/c \sim 0$ is a fairly good approximation for the ions and very good for the electrons.

Finally, the magnetic field in DCX is about 10^4 gauss. For such a field, the ion cyclotron frequency is about 10^8 cycles per second. Thus, $\frac{\omega_{ci}}{k}$ will be essentially zero for $k < 1$ cm^{-1}, again in the short wavelength region. However, the electron cyclotron frequency is approximately three orders of magnitude greater than ω_{ci}. Consequently, one must go to very large $k (> 100)$ before $\frac{\omega_{ce}}{k}$ can be neglected compared with c. The restriction on ω, the frequency of oscillation of the disturbance in the plasma, must be justified a posteriori, since one does not know what values will assume until the dispersion relation is solved.

All of the foregoing is simply an expression of the fact (3,4) that Coulomb disturbances are most responsible for plasma instabilities.

II. THE DISPERSION RELATION

In the linear approximation one assumes that each quantity can be

written in the form $G^0 + g$, where G^0 is the initial value and g is the perturbed quantity. Powers and products greater than one of the perturbed quantities are neglected. If the initial quantities are time independent, Eqs. (4) and (13) become

$$\frac{\partial N_s}{\partial t} + \vec{\nabla}\cdot(N_s^0 \vec{V}^s + N_s^0 \vec{v}_s + n_s \vec{V}_s^0) = 0, \tag{19}$$

$$\frac{\partial \vec{V}_s}{\partial t} + \vec{V}_s^0 \cdot \vec{\nabla}(\vec{V}_s^0 + \vec{v}_s) + \vec{v}_s \cdot \vec{\nabla}\vec{V}_s^0 = \frac{q_s}{m_s}\left[\vec{E} + \frac{\vec{V}_s^0 + \vec{v}_s}{c} \times \vec{B}_0\right]. \tag{20}$$

Now, in keeping with the assumptions previous discussed, one has (subscript i for ions, e for electrons):

$$\vec{V}_i = \vec{V}_i^0 + \vec{v}_i, \tag{21}$$

$$\vec{V}_i^0 = r\omega_{ci}\hat{\phi}, \tag{22}$$

$$\vec{V}_e = \vec{v}_e, \tag{23}$$

$$N_i = N_i^0 + n_i, \tag{24}$$

$$N_e = N_e^0 + n_e, \tag{25}$$

$$N_i^0 = N_e^0 = N^0 \epsilon(r), \tag{26}$$

$$\epsilon(r) = \begin{cases} 0, & r < r_1, \; r > r_2 \\ 1, & r_1 < r < r_2, \end{cases} \tag{27}$$

$$\vec{B} = -B_0 \hat{z}, \tag{28}$$

$$\vec{E} = \vec{\mathcal{E}}. \tag{29}$$

In the above, $\hat{\phi}$ is a unit vector in the (cylindrical) aximuthal direction, $\hat{z}$ is a unit vector in the z direction, $\omega_{ci} = eB_0/m_i c$ and the perturbed electric field is written in script rather than lower case. The resulting equations for the plasma and fields are

$$\frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot\left[N_0 \epsilon \vec{v}_i + n_i r\omega_{ci}\hat{\phi}\right] = 0, \tag{30}$$

$$\frac{\partial \vec{v}_i}{\partial t} + r\omega_{ci}\hat{\phi}\cdot\vec{\nabla}\left[r\omega_{ci}\hat{\phi} + \vec{v}_i\right] + \vec{v}_i \cdot \vec{\nabla}(r\omega_{ci}\hat{\phi}) = \frac{e}{M_i}\left[\vec{\mathcal{E}} - \frac{n\omega_{ci}\hat{\phi} + \vec{v}_i}{c} \times B_0 \hat{z}\right], \tag{31}$$

$$\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(N_0 \in \vec{v}_e) = 0, \tag{32}$$

$$\frac{\partial \vec{v}_e}{\partial t} = \frac{-e}{m_e}\vec{E} + \frac{e}{m_e c}\vec{v}_e \times B_0 \hat{z}, \tag{33}$$

and

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2\phi = 4\pi e(n_i - n_e). \tag{34}$$

Here, since the perturbed magnetic field and the displacement current have been neglected, $\mathcal{E} = -\vec{\nabla}\phi$, where ϕ is the scalar potential.

The problem now consists of solving Eqs. (30)-(34), subject to the appropriate boundary conditions. The first step in effecting the solution is to assume that all quantities can be written

$$g = g(r)\, e^{i(\omega t + \ell\phi + kz)} \tag{35}$$

Changing notation, with $\vec{v}_i = \vec{v}$, $\vec{v}_e = \vec{u}$, $M_i = M$ and $M_e = m$, one can obtain expressions for the components of the velocities in terms of the components of the fields, i.e.,

$$v_r = \frac{e}{M}\,\frac{i\Omega_\ell \mathcal{E}_r + \omega_{ci}\mathcal{E}_\phi}{\omega_{ci}^2 - \Omega_\ell^2}, \tag{36}$$

$$v_\phi = \frac{e}{M}\,\frac{i\Omega_\ell \mathcal{E}_\phi - \omega_{ci}\mathcal{E}_r}{\omega_{ci}^2 - \Omega_\ell^2}, \tag{37}$$

$$v_z = \frac{e}{M}\,\frac{\mathcal{E}_z}{i\Omega_\ell}, \tag{38}$$

with

$$\Omega_\ell = \omega + \ell\omega_{ci} \tag{39}$$

and

$$u_r = \frac{-e}{m}\,\frac{i\omega\mathcal{E}_r + \omega_{ce}\mathcal{E}_\phi}{\omega_{ce}^2 - \omega^2}, \tag{40}$$

$$u_\phi = -\frac{e}{m}\,\frac{i\omega E_\phi - \omega_{ce} E_r}{\omega_{ce}^2 - \omega^2}, \tag{41}$$

$$u_z = -\frac{e}{m}\,\frac{E_z}{i\omega}, \tag{42}$$

where

$$\omega_{ce} = \frac{e B_0}{mc}. \tag{43}$$

At this point it is necessary to emphasize a significant point with regard to Eqs. (36)-(38) and (40)-(42). Although $\vec{E}$ can be written as the gradient of a scalar, this is only an approximate relation, subject to the assumptions on the frequencies. But the rate at which work is done on a particle by the field (or vice versa) is simply $\vec{E} \cdot v\vec{e}$ and the average of this quantity over a period T is

$$\langle e\vec{E}\cdot\vec{v}\rangle = \frac{1}{T}\oint e\vec{E}\cdot\vec{v}\,dt. \tag{44}$$

It is clear that if $\vec{v}$ and $\vec{E}$ were $\pi/2$ out of phase, this quantity would be zero. However, the presence of the uniform magnetic field serves essentially to mix the phases of $\vec{v}$ and $\vec{E}$ with the result that they are no longer $\pi/2$ out of phase and consequently, the time average of the power is non-zero. This important fact will be useful later in interpreting the results of the problem. At this point it is sufficient to say that it is the mechanism for energy gain by the particles (fields) at the expense of the fields (particles).

Continuing with the solution of the problem, on the basis of Eq. (35), Eqs. (30) and (32) become

$$n_i = -N_0 \left\{ \Delta(n_1, n_2)\, v_2 - \epsilon\, \vec{v}\cdot\vec{v} \right\} / i\Omega_\ell, \tag{45}$$

$$n_e = -N_0 \{\Delta(r_1, r_2) u_r - \epsilon \vec{\nabla} \cdot \vec{u}\}/i\omega, \tag{46}$$

where

$$\Delta(r_1, r_2) = \frac{d}{dr} \epsilon(r) = \delta(r_1 - r) - \delta(r_2 - r). \tag{47}$$

Insertion of the expressions for $\vec{v}$ and $\vec{u}$ from Eqs. (36)-(38) and (40)-(42) give a relation for n_i and n_e in terms of $\vec{\mathcal{E}}$. Then, this relation can be used in Poisson's equation to obtain a final equation for $\vec{\mathcal{E}}$ or actually, $\phi(r)$. This equation is

$$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\phi + (\lambda'^2 - \frac{\ell^2}{r^2})\phi = -\Delta(r_1, r_2)\left\{\frac{W'^2}{1+W'^2}\frac{d\phi}{dr} + \frac{\phi}{r} P'\right\}, \tag{48}$$

where

$$\lambda'^2 = -k^2\left\{\frac{1 - \epsilon\left[\omega_{pe}^2/\omega^2 + \omega_{pi}^2/\Omega_e^2\right]}{1+W'^2}\right\}, \tag{49}$$

$$W'^2 = \epsilon\left\{\frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \Omega_e^2}\right\}, \tag{50}$$

$$P' = \frac{\ell}{\omega}\left\{\frac{\omega_{pe}^2 \omega_{ce}}{\omega_{ce}^2 - \omega^2} + \frac{\omega_{pi}^2 \omega_{ci}}{\omega_{ci}^2 - \Omega_e^2}\right\}, \tag{51}$$

and

$$\omega_{pi}^2 = 4\pi N e^2/M, \tag{52}$$

$$\omega_{pe}^2 = 4\pi N e^2/m. \tag{53}$$

The boundary conditions on this equation are continuity of the potential (and regularity at r = 0 and r = ∞)

$$\phi(r_1+) = \phi(r_1-) \tag{54}$$

$$\phi(r_2+) = \phi(r_2-) \tag{55}$$

and a second set of conditions which may be obtained by integrating Eq. (48) over the discontinuities at r_1 and r_2. These are:

$$\left.\frac{d\phi}{dr}\right|_{r_1+} + \left.\frac{P'\phi}{r_1}\right|_{r_1+} = \left.\frac{d\phi}{dr}\right|_{r_1-}, \tag{56}$$

$$\left.\frac{d\phi}{dr}\right|_{r_2-} + \left.\frac{P'\phi}{r}\right|_{r_2-} = \left.\frac{d\phi}{dr}\right|_{r_2+}. \tag{57}$$

Note that the latter set of conditions are the discontinuity conditions placed on the normal component of $\vec{\mathcal{E}}$, due to space charges piling up, i.e.,

$$\Delta\mathcal{E}_n = 4\pi\sigma, \tag{58}$$

where $\Delta\mathcal{E}_n$ is the change in the normal component of $\vec{\mathcal{E}}$ in crossing the boundary from vacuum to plasma and σ is the surface charge density.

The solutions of Eq. (48) are combinations of Bessel functions of imaginary argument and Hankel functions. Specifically, $\phi(r)$ is (the notation is that of Watson (19))

$$\phi(r) = \begin{cases} A_1 I_\ell(kr), & r < r_1, \quad (59) \\ A_2 H_\ell^{(1)}(\lambda r) + A_3 H_\ell^{(2)}(\lambda r), & r_1 < r < r_2, \quad (60) \\ A_4 K_\ell(kr), & r_2 < r, \quad (61) \end{cases}$$

where the $A_i (i = 1 \ldots 4)$ are arbitrary constants. In the above, λ is the expression given in Eq. (49) with $\epsilon(r)$ replaced by unity. Insertion of these functions in the boundary conditions leads to a set of homogeneous algebraic equations for the constants A_i, i.e.,

$$A_1 I_\ell(kr_1) - A_2 H_\ell^{(1)}(\lambda r_1) - A_3 H_\ell^{(2)}(\lambda r_1) = 0, \tag{62a}$$

$$- A_2 H_\ell^{(1)}(\lambda r_2) - A_3 H_\ell^{(2)}(\lambda r_2) + A_4 K_\ell(kr_2) = 0, \tag{62b}$$

$$-A_1 \frac{dI_\ell}{dr}(kr_1) + A_2\left[\frac{dH_\ell^{(1)}}{dr}(\lambda r_1) + \frac{P}{r_1} H_\ell^{(1)}(\lambda r_1)\right] + A_3\left[\frac{dH_\ell^{(2)}}{dr}(\lambda r_1) + \frac{P}{r_1} H_\ell^{(2)}(\lambda r_1)\right] = 0, \tag{62c}$$

$$-A_2\left[\frac{dH_\ell^{(1)}}{dr}(\lambda r_2) + \frac{P}{r_2} H_\ell^{(1)}(\lambda r_2)\right] - A_3\left[\frac{dH_\ell^{(2)}}{dr}(\lambda r_2) + \frac{P}{r_2} H_\ell^{(2)}(\lambda r_2)\right] + A_4 \frac{dK_\ell}{dr}(kr_2) = 0. \tag{62d}$$

The condition that this set of equations has a solution is that the determinant of the coefficients of A_i vanish, i.e.,

$$\begin{vmatrix} I_\ell(kr_1) & -H_\ell^{(1)}(\lambda r_1) & -H_\ell^{(2)}(\lambda r_1) & 0 \\ 0 & -H_\ell^{(1)}(\lambda r_2) & -H_\ell^{(2)}(\lambda r_2) & K_\ell(kr_2) \\ -\frac{dI_\ell}{dr}(kr_1) & \left[\frac{dH_\ell^{(1)}}{dr}(\lambda r_1) + \frac{P}{r_1} H_\ell^{(1)}(\lambda r_1)\right] & \left[\frac{dH_\ell^{(2)}}{dr}(\lambda r_1) + \frac{P}{r_1} H_\ell^{(2)}(\lambda r_1)\right] & 0 \\ 0 & -\left[\frac{dH_\ell^{(1)}}{dr}(\lambda r_2) + \frac{P}{r_2} H_\ell^{(1)}(\lambda r_2)\right] & -\left[\frac{dH_\ell^{(2)}}{dr}(\lambda r_2) + \frac{P}{r_2} H_\ell^{(2)}(\lambda r_2)\right] & \frac{dK_\ell}{dr}(kr_2) \end{vmatrix} = 0. \tag{63}$$

In the above, P is the function defined by Eqs. (50) and (51) with $\epsilon(r)$ replaced by unity.

Eq. (63) is the desired dispersion relation. In principle, one can solve the equation for the allowed frequencies ω. Of course, it is apparent that no such solution can be effected due to the complexity of the relations. For this reason, it is necessary to make some sort of approximations with the hope of obtaining a tractable expression. This will be considered in the next section.

III. THE SHORT WAVELENGTH LIMIT

Fortunately, as will be seen later, short wavelength oscillations are the most unstable. For this reason, one can simplify Eq. (63) considerably by replacing the cylinder functions by their asymptotic forms. This requires kr_1, kr_2, λr_1, $r_2 \gg 1$. Hence, any roots ω of the dispersion relation must be consistent with these conditions. The asymptotic forms of the cylinder functions are:

$$I_\ell(kr) \sim e^{kr}/\sqrt{kr}, \tag{64a}$$

$$K_\ell(kr) \sim e^{-kr}/\sqrt{kr}, \tag{64b}$$

$$H_\ell^{(1)}(\lambda r) \sim e^{i\lambda r}/\sqrt{\lambda r}, \tag{64c}$$

$$H_\ell^{(2)}(\lambda r) \sim e^{-i\lambda r}/\sqrt{\lambda r} \tag{64d}$$

Furthermore, the derivatives of these functions are, taking I (kr) as an example,

$$\frac{dI_\ell(kr)}{dr} \sim \frac{d}{dr}\frac{e^{kr}}{\sqrt{kr}} \sim \frac{ke^{kr}}{\sqrt{kr}}\left(1-\frac{1}{2kr}\right) \sim \frac{ke^{kr}}{\sqrt{kr}}, \tag{65}$$

with similar expressions for the other functions. Now, using Eqs. (64a)-(64d) and Eq. (65) in Eq. (63) gives

$$\begin{vmatrix} \frac{e^{hr_1}}{\sqrt{hr_1}} & -\frac{e^{i\lambda r_1}}{\sqrt{\lambda r_1}} & -\frac{e^{-i\lambda r_1}}{\sqrt{\lambda r_1}} & 0 \\ 0 & -\frac{e^{i\lambda r_2}}{\sqrt{\lambda r_2}} & -\frac{e^{-i\lambda r_2}}{\sqrt{\lambda r_2}} & 0 \\ -k\frac{e^{hr_1}}{\sqrt{hr_1}} & \left[i\lambda+\frac{p}{r_1}\right]\frac{e^{i\lambda r_1}}{\sqrt{\lambda r_1}} & \left[-i\lambda+\frac{p}{r_1}\right]\frac{e^{-i\lambda r_1}}{\sqrt{\lambda r_1}} & 0 \\ 0 & \left[i\lambda+\frac{p}{r_2}\right]\frac{e^{i\lambda r_2}}{\sqrt{\lambda r_2}} & \left[-i\lambda+\frac{p}{r_2}\right]\frac{e^{-i\lambda r_2}}{\sqrt{\lambda r_2}} & 0 \end{vmatrix} = 0. \quad (66)$$

This expression can be further simplified. Multiplication of the first row by k and addition to row three and multiplication of the second row by (-k) and addition to row four gives

$$\begin{vmatrix} \frac{e^{hr_1}}{\sqrt{hr_1}} & -\frac{e^{i\lambda r_1}}{\sqrt{\lambda r_1}} & -\frac{e^{-i\lambda r_1}}{\sqrt{\lambda r_1}} & 0 \\ 0 & -\frac{e^{i\lambda r_2}}{\sqrt{\lambda r_2}} & -\frac{e^{-i\lambda r_2}}{\sqrt{\lambda r_2}} & \frac{e^{-hr_2}}{\sqrt{hr_2}} \\ 0 & \left[-k+i\lambda+\frac{p}{r_1}\right]\frac{e^{i\lambda r_1}}{\sqrt{\lambda r_1}} & \left[-k-i\lambda+\frac{p}{r_1}\right] & 0 \\ 0 & \left[k+i\lambda+\frac{p}{r_2}\right]\frac{e^{i\lambda r_2}}{\sqrt{\lambda r_2}} & \left[-i\lambda+k+\frac{p}{r_2}\right]\frac{e^{-i\lambda r_2}}{\sqrt{hr_2}} & 0 \end{vmatrix} = 0. \quad (67)$$

Expanding along the first column gives a 3x3 determinant which in turn may be expanded along its third column, yielding

$$\frac{e^{k(r_1-r_2)}}{\sqrt{k^2 r_1 r_2}} \begin{vmatrix} \left[-k+i\lambda+P/r_1\right] e^{i\lambda r_1}/\sqrt{\lambda r_1} & \left[-k-i\lambda+P/r_1\right] e^{-i\lambda r_1}/\sqrt{\lambda r_1} \\ \left[k+i\lambda+P/r_2\right] e^{i\lambda r_2}/\sqrt{\lambda r_2} & \left[-i\lambda+k+P/r_2\right] e^{-i\lambda r_2}/\sqrt{\lambda r_2} \end{vmatrix} = 0, \tag{68}$$

But, since $e^{k(r_1-r_2)}$ has only imaginary roots and k is real, this factor may be thrown away.

This determinant when expanded leads to the equation

$$\left[-ik+k+P/r_2\right]\left[-k+i\lambda+P/r_1\right]e^{-i\lambda a}-\left[-k-i\lambda+P/r_1\right]\left[k+i\lambda+P/r_2\right]e^{i\lambda a}=0, \tag{69}$$

where $a = r_2-r_1$ (70)

and a factor $(\lambda^2 r_1 r_2)^{-1}$ has been thrown away. Performing the individual multiplications yields:

$$\begin{aligned} &\left[P^2-k^2 r_1 r_2+\lambda^2 r_1 r_2-i\lambda Pa+kPa+2ik\lambda r_1 r_2\right]e^{-i\lambda a} \\ &-\left[P^2-2ik\lambda r_1 r_2-k^2 r_1 r_2+kPa+\lambda^2 r_1 r_2+i\lambda Pa\right]e^{i\lambda a}=0; \end{aligned} \tag{71}$$

re-arranging terms one has

$$\left[P^2+\lambda^2 r_1 r_2-k^2 r_1 r_2-kPa\right]\left(e^{-i\lambda a}-e^{i\lambda a}\right)=i\left[\lambda Pa-2\lambda r_1 r_2 k\right]\left[e^{i\lambda a}+e^{-i\lambda a}\right], \tag{72}$$

or,

$$\left[-P^2-\lambda^2+k^2 r_1 r_2+kPa\right]\tan\lambda a=\lambda Pa-2k\lambda r_1 r_2, \tag{73}$$

and finally,

$$\tan\lambda a=\frac{\lambda Pa-2k\lambda r_1 r_2}{k^2 r_1 r_2-P^2-\lambda^2 r_1 r_2+kPa} \tag{74}$$

Unfortunately, this equation is still somewhat too complicated even to find approximate solutions. However, if one considers the limit $k \to \infty$, it is possible to simplify Eq. (74) further. Note that the terms in P are due to the discontinuity in $d\phi/dr$ at the boundaries. But the boundary conditions, Eqs. (54)-(57) can be re-written,

$$\frac{d\phi}{dr}/\phi\Big|_{r_1-} = \frac{d\phi}{dr}\Big|_{r_1+} + P/r_1, \tag{75}$$

$$\frac{d\phi}{dr}/\phi\Big|_{r_2+} = \frac{d\phi}{dr}/\phi\Big|_{r_2-} + P/r_2 . \tag{76}$$

From Fig. 2, it can be seen that as $k \to \infty$, $\phi(r_1)$, $\phi(r_2) \to 0$ and the derivatives become continuous. That is,

$$\frac{d\phi}{dr}/\phi\Big|_{r_1-} \approx \frac{A_1 k e^{kr_1}/\sqrt{kr_1}\left(1-\frac{1}{2kr_1}\right)}{A_1 e^{kr_1}/\sqrt{kr_1}} \approx k \to \infty, \tag{77}$$

where the asymptotic form for $I_{\ell}(kr)$, Eq. (64a), has been used. But $k \to \infty$ implies from Eq. (77) that $\phi \to 0$, since $d\phi/dr \to \infty$ would imply physically unrealistic infinite fields. However, if $\phi \to 0$, one is justified in neglecting terms in P. This argument can be substantiated in another manner. One can write Eq. (74):

$$\tan \beta ka = \frac{\beta Pa/kr_1r_2 - 2\beta}{1 - P^2/k^2r_1r_2 - \beta^2 + Pa/kr_1r_2}, \tag{78}$$

with

$$\beta = \lambda/k. \tag{79}$$

Now, the terms in P contain factors k, r_1, r_2 in the denominator and, from the short wavelength conditions, can be neglected. Thus, Eq. (78)

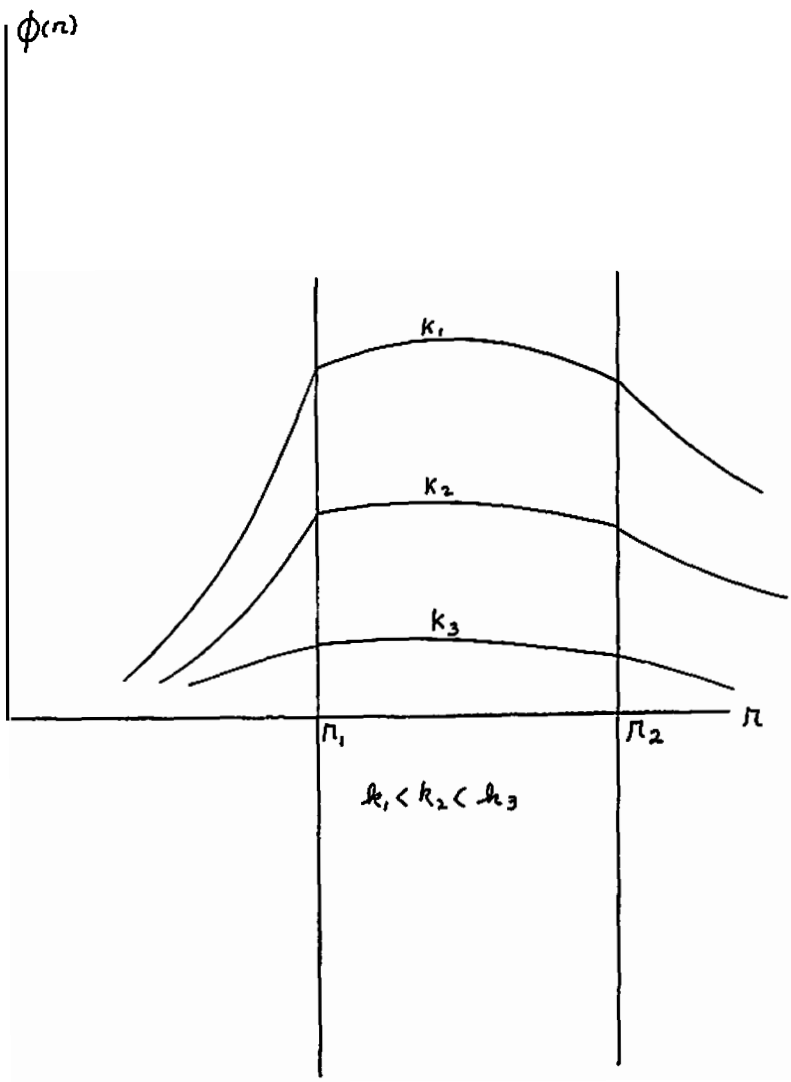

FIGURE 2

$\phi(r)$ FOR SEVERAL VALUES OF k

becomes

$$\tan \beta ka = \frac{-2\beta}{1-\beta^2} . \tag{80}$$

The roots of this equation can be found only by numerical methods. However, Fig. 3 indicates the general form of the functions in Eq. (80). It is clear that the solutions β_n of Eq. (80) can be written

$$\beta_n = \frac{(2n+1)}{2ka}\pi + \delta_n , \tag{81}$$

where n is an integer. Of course, one cannot find δ_n exactly, so Eq. (81) is not very informative. However, this equation is useful in one respect, since the dependence of β_n on ka is indicated crudely. In fact, as ka becomes very large, δ_n approaches zero for all n.

With the knowledge that the roots β_n can, in principle, be found, one can now find approximate solutions for the allowed frequencies of a disturbance. From Eqs. (49)-(50), one has

$$1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\Omega_i^2} + \beta_n^2 \left\{ 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \Omega_i^2} \right\} =$$

$$1 + \beta_n^2 - f(\omega) = 0. \tag{82}$$

From Fig. 4 one sees that of the eight roots of Eq. (82), three real roots always occur at values of $|\omega| \gg l\omega_{ci}$. However, in the region $\omega \sim -l\omega_{ci}$, pairs of complex roots may occur. From the knowledge of Harris' work (9-10), these are the interesting roots. Consequently, one can simplify the above expression if one confines it to the region $\omega \sim -l\omega_{ci}$.

Then, since

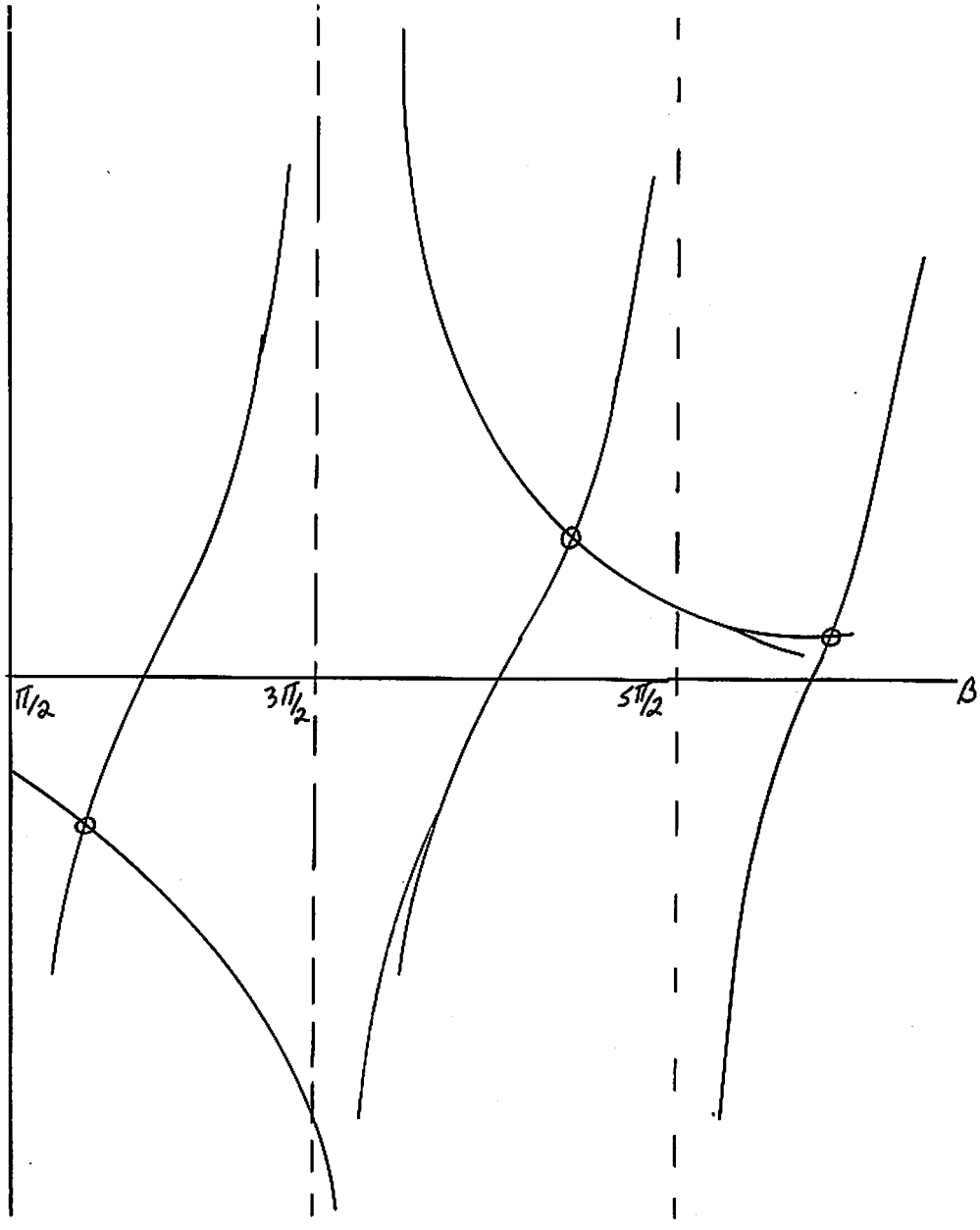

FIGURE 3

LOCATION OF THE ROOTS β_n

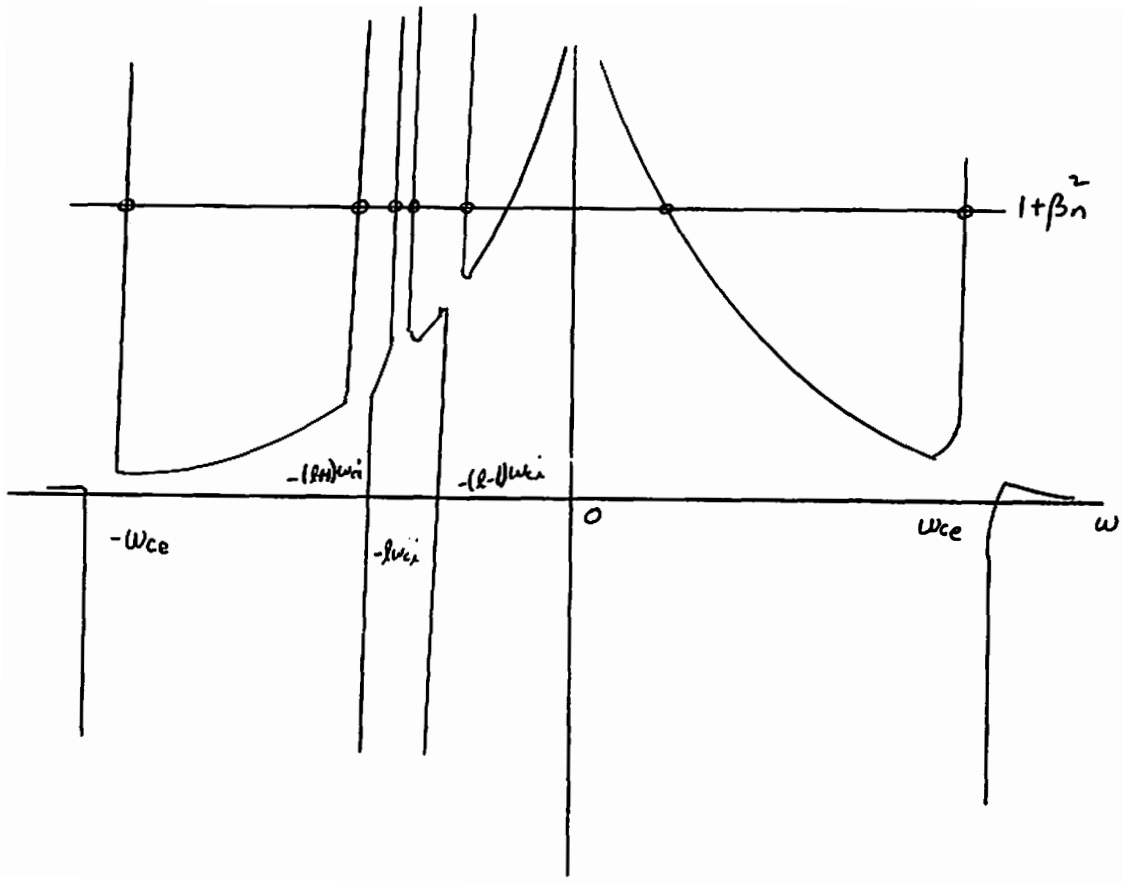

FIGURE 4

$f(\omega)$ VERSUS ω

$$\frac{\omega_{ce}}{\omega_{ci}} = \frac{M}{m} \gtrsim 2000, \tag{83}$$

one can write

$$\frac{\omega_{pe}^2}{\omega_{ce}^2-\omega^2} \sim \frac{\omega_{pe}^2}{\omega_{ce}^2} \sim \frac{4\pi N m c^2}{B_0^2} \sim 10^{-5}\frac{N}{B_0^2}. \tag{84}$$

But for the situations in which we are most interested one has,

$$B_0 \sim 10^3 - 10^4 \text{ gauss}, \tag{85a}$$

$$N \sim 10^6 - 10^8 \text{ cm}^{-3} \tag{85b}$$

and Eq. (84) gives

$$10^{-7} \leq \frac{\omega_{pe}^2}{\omega_{ce}^2} \leq 10^{-3} << 1. \tag{86}$$

Therefore, one is justified in neglecting this term for plasmas of physical interest in the region considered. Thus, one can examine the new function $g(\omega)$ defined by

$$1 - \omega_{pe}^2/\omega^2 - \omega_{pi}^2/\Omega_i^2 + \beta_n^2\left(1 + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \Omega_i^2}\right)$$

$$= 1 + \beta_n^2 - g(\omega) = 0. \tag{87}$$

The intersection of $g(\omega)$ with the line $1 + \beta_n^2$ is indicated in Fig. 5. Again, it is clear that for certain values of the parameters, complex roots may appear. In the figure, one of five possible roots are shown; the sixth, which is real, lies outside the region of interest. Beginning at line A, as the parameters are changed, one progresses to line B, where there are two pairs of equal, real roots at approximately $-(\ell+1)\omega_{ci}$. At C, the roots in the neighborhood of $-(\ell+1)\omega_{oi}$ have become complex, so that there are three real and two complex roots. At

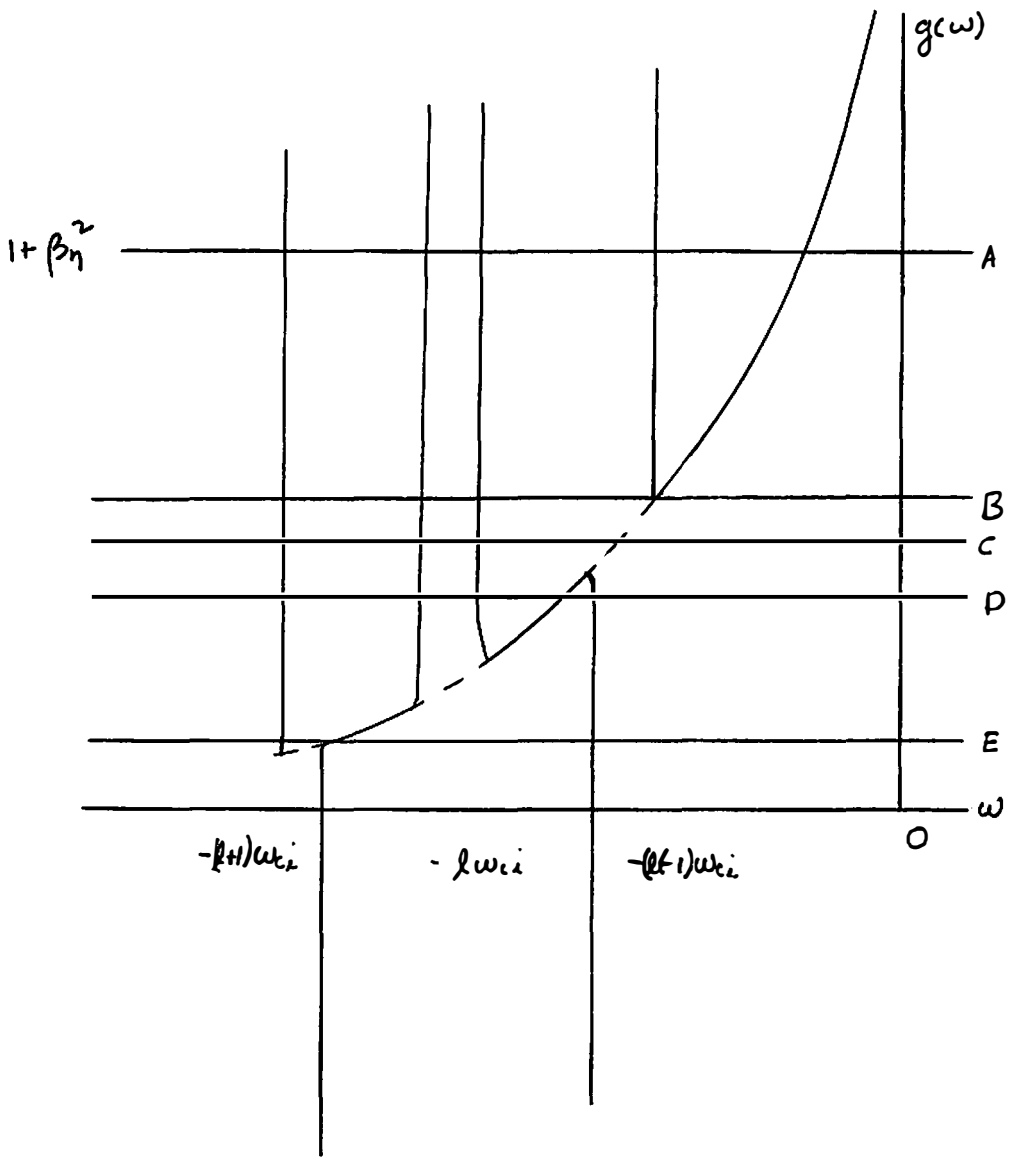

FIGURE 5

g(ω) VERSUS ω

D, one again has five real roots. Finally, at E, complex roots again appear for which the real part of ω is

$$R(\omega) \sim -\ell\omega_{ci} \tag{88}$$

In order to determine the conditions for the occurrence of complex roots as seen in E, the function $g(\omega)$ can be simplified somewhat. It will be seen later that complex roots occur only for $\beta_n \ll 1$. Furthermore, at $R(\omega) \sim -\ell\omega_{ci}$, one has

$$\beta_n^2 \frac{\omega_{pi}^2}{\omega_{ci}^2 - \ell^2} \approx \beta_n^2 \frac{\omega_{pi}^2}{\omega_{ci}^2} \ll 1. \tag{89}$$

Consequently, one is justified in examining the function $G(\omega)$ defined by

$$1 + \beta_n^2 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\Omega_e^2} = 1 + \beta_n^2 - G(\omega) = 0. \tag{90}$$

Now, two of the roots of this expression will be complex if

$$1 + \beta_n^2 < G(\omega)_{min}, \tag{91}$$

where $G(\omega)_{min}$ is the relative minimum of the function $G(\omega)$ in the region $-(\ell+1)\omega_{ci} < \omega < -(\ell-1)\omega_{ci}$. The relative minimum can be found by differentiation of $G(\omega)$; thus

$$\frac{dG}{d\omega} = 2\frac{\omega_{pe}^2}{\omega^3} + \frac{2\omega_{pi}^2}{(\omega + \ell\omega_{ci})^3} = 0, \tag{92}$$

or

$$-\omega = \left(\frac{\omega_{pi}^2}{\omega_{pe}^2}\right)^{\frac{1}{3}} (\omega + \ell\omega_{ci}), \tag{93}$$

which gives

$$\omega = -\frac{\ell\omega_{ci}}{1 + \left(\frac{m}{M}\right)^{\frac{1}{3}}}, \tag{94}$$

since

$$\frac{\omega_{pi}^2}{\omega_{pe}^2} = \frac{4\pi Ne^2/M}{4\pi Ne^2/m} = \frac{m}{M} . \tag{95}$$

Inserting the value of ω from Eq. (94), Eq. (91) becomes

$$1+\beta_n^2 < \frac{\omega_{pe}^2}{\ell^2\omega_{ci}^2}\left[1+\left(\frac{m}{M}\right)^{1/3}\right]^3, \tag{96}$$

or

$$\ell^2\omega_{ci}^2 < \frac{\omega_{pe}^2}{1+\beta_n^2}\left[1+\left(\frac{m}{M}\right)^{1/3}\right]^3 . \tag{97}$$

This is the condition for instability. One can see qualitatively the effects of the boundaries, since (r_2-r_1) occurs in the expression for β_n.

The roots of the dispersion relation now may be found approximately. Re-writing Eq. (90), one has

$$\left[(1+\beta_n^2)\omega^2 - \omega_{pe}^2\right](\omega+\ell\omega_{ci})^2 \approx \omega_{pi}^2\omega^2, \tag{98}$$

or

$$\omega = -\ell\omega_{ci} \pm \frac{\omega_{pi}\omega}{\left[(1+\beta_n^2)\omega^2-\omega_{pe}^2\right]^{1/2}} . \tag{99}$$

A sufficiently accurate result may be obtained by letting $\omega = -\ell\omega_{ci}$ on the right side of the equation, since the correction is proportional to ω_{pi} and is expected to be small compared to $-\ell\omega_{ci}$. Then, one obtains

$$\omega \approx -\ell\omega_{ci} \pm \frac{\omega_{pi}\ell\omega_{ci}}{\left[(1+\beta_n^2)\ell^2\omega_{ci}^2-\omega_{pe}^2\right]^{1/2}} . \tag{100}$$

Now, it is of some interest to determine the maximum value of the imaginary part of ω. In order to do this, one may follow Buneman (5) or Bernstein and Kulsrud (3). Let

$$X = \omega + \ell\omega_{ci} \tag{101}$$

and

$$X = |X| e^{i\theta}. \tag{102}$$

Then, Eq. (90) becomes

$$1 + \beta_n^2 - \frac{\omega_{pe}^2}{(X - \ell\omega_{ci})^2} - \frac{\omega_{pi}^2}{X^2} = 0, \tag{103}$$

or

$$(X - \ell\omega_{ci}) = \left[1 + \beta_n^2 - \frac{\omega_{pi}^2}{X^2}\right]^{-1/2} \omega_{pe}. \tag{104}$$

In the region of interest one has from Eq. (100)

$$(1+\beta_n^2)^{1/2} |X| \sim \ell\omega_{ci} >> \omega_{pi}. \tag{105}$$

Thus, the right side of Eq. (104) can be expanded, retaining only first order terms, i.e.,

$$X - \ell\omega_{ci} \simeq \frac{\omega_{pe}}{(1+\beta_n^2)^{1/2}}\left[1 + \frac{1}{2}\frac{\omega_{pi}^2}{X^2(1+\beta_n^2)} + \cdots\right]. \tag{106}$$

One can now take the imaginary part of both sides, noting that

$$\mathcal{Q}\left(\frac{1}{X^2}\right) = \mathcal{Q}\left(\frac{1}{|X|^2 e^{2i\theta}}\right) = -\frac{1}{|X|^2}\sin 2\theta = -\frac{2\sin\theta\cos\theta}{|X|^2}, \tag{107}$$

so

$$|X|\sin\theta = \frac{-\omega_{pe}}{(1+\beta_n^2)^{3/2}} \frac{\omega_{pi}^2}{|X|^2} \sin\theta\cos\theta, \tag{108}$$

or

$$|x| = -\left[\frac{\omega_{pe}}{(1+\beta_n^2)^{3/2}}\,\omega_{pi}^2\right]^{1/3}\cos^{1/3}\theta. \tag{109}$$

But one knows from Eq. (101) that

$$\mathcal{Q}(x) = \mathcal{Q}(\omega) = |x|\sin\theta = -\frac{(\omega_{pe}\omega_{pi}^2)^{1/3}}{(1+\beta_n^2)^{1/2}}\sin\theta\cos^{1/3}\theta. \tag{110}$$

Differentiating this expression with respect to θ gives

$$\cos^{4/3}\theta - \frac{1}{3}\sin^2\theta\cos^{-2/3}\theta = 0, \tag{111}$$

or

$$\tan\theta = \pm\sqrt{3}. \tag{112}$$

We choose the phase so that cos θ < 0. Then one has

$$\cos\theta = -1/2 \tag{113}$$

and

$$\sin\theta = \sqrt{3}/2\,. \tag{114}$$

Finally, for $\mathcal{Q}(\omega)_{max}$ on obtains from Eq. (110)

$$\mathcal{Q}(\omega)_{max} = \frac{(\omega_{pe}\omega_{pi}^2/2)^{1/3}\sqrt{3}}{(1+\beta_n^2)^{1/2}\,2} = \frac{\sqrt{3}}{2}\,\frac{(m/M)^{1/3}\omega_{pe}}{(1+\beta_n^2)^{1/2}}\,. \tag{115}$$

In addition, when

$$\omega_{pe}^2 >> (1+\beta_n^2)\ell^2\omega_{ci}^2) \tag{116}$$

we see from Eq. (100) that

$$\mathcal{Q}(\omega) \to \sqrt{\frac{\omega_{pi}^2}{\omega_{pe}^2}}\,\ell\omega_{ci} = \sqrt{\frac{m}{M}}\,\ell\omega_{ci}. \tag{117}$$

Fig. 6 shows one branch of ω/ω_{pe} as a function of N. One notes that as the density increases, complex roots appear. The imaginary part of ω/ω_{pe} increases to a maximum and then decreases again, approaching zero.

Finally, it is useful to consider the charge states of the plasma as represented in various modes. Fig. 7 gives several simple examples of the charge distributions. From the values of ω obtained, one knows that these configurations are rotating at approximate multiples of the ion cyclotron frequency, as is the electrostatic potential. In Fig. 8 the electric fields for such states are indicated. It is clear that a small charge separation can lead to complicated field configurations. Furthermore, as was mentioned earlier*, the particle velocities are not $\pi/2$ out of phase with these fields, so that energy transfer is possible. Physically, the ions tend to "bunch," giving rise to large electric fields within the plasma. This effect has apparently been observed in DCX.

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*See p. 12.

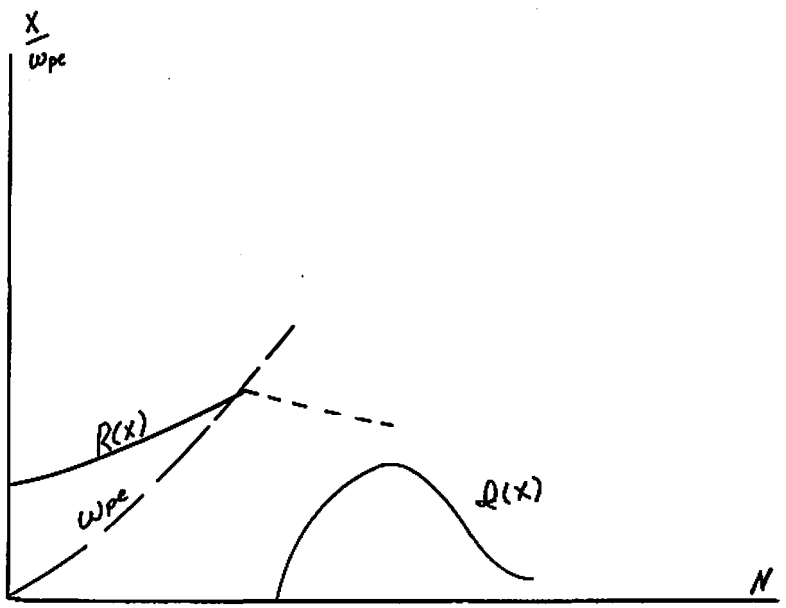

FIGURE 6

x/ω_{pe} VERSUS N

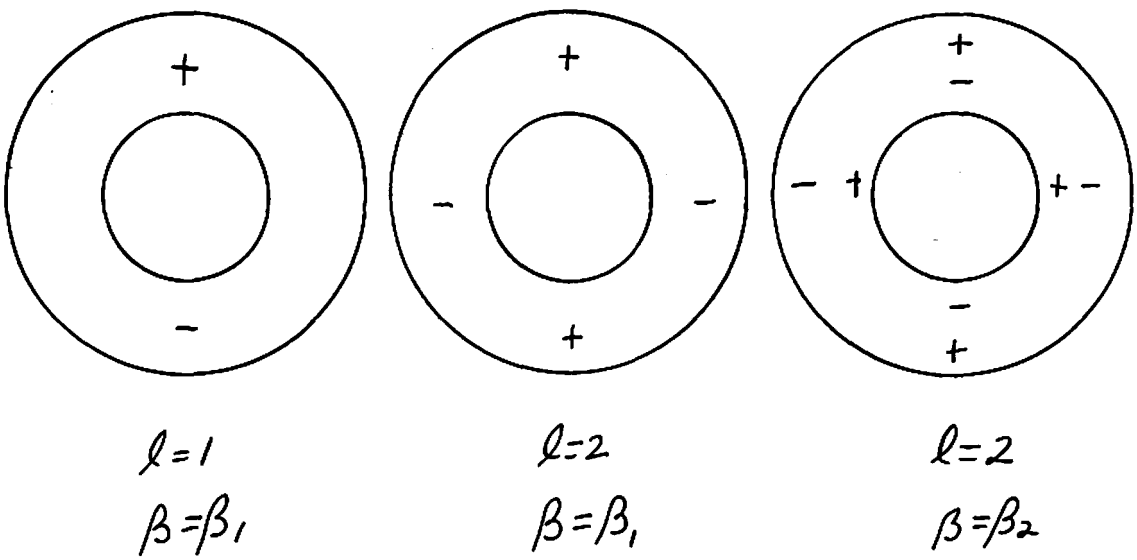

FIGURE 7

CHARGE STATES IN THE PLASMA

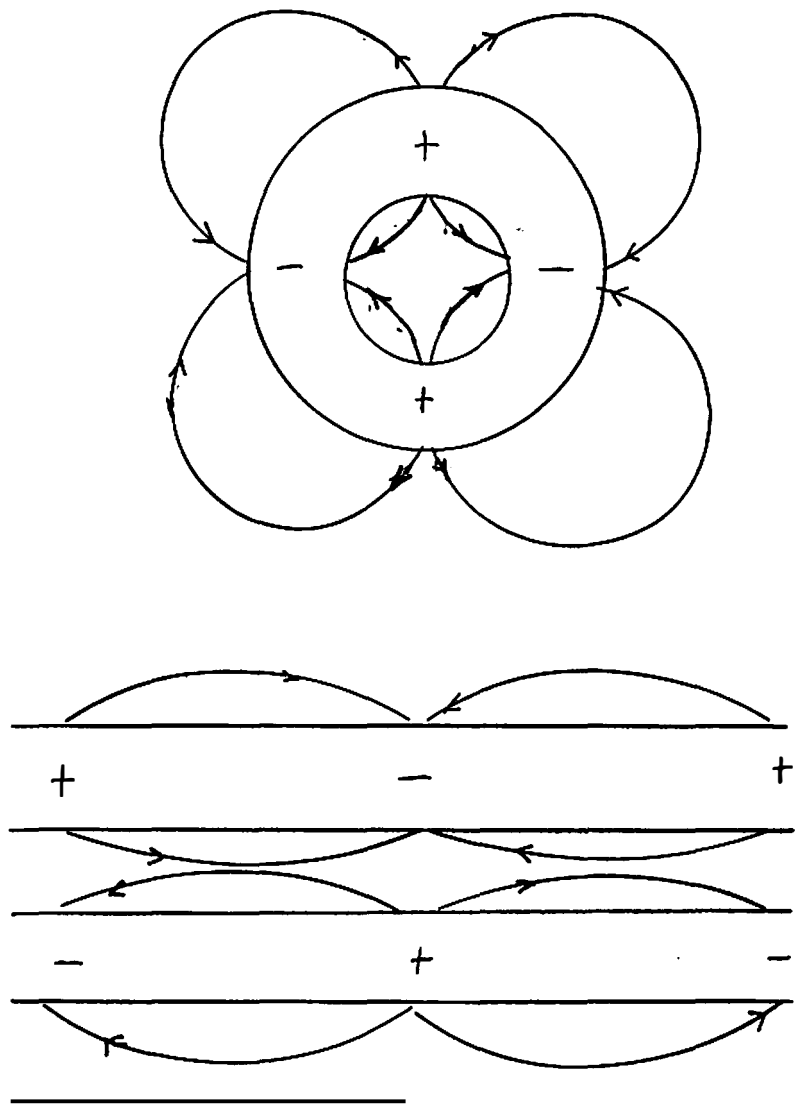

FIGURE 8

ELECTRIC FIELDS IN THE PLASMA

CHAPTER III

CONCLUSIONS

In the previous chapter it was shown that, under certain circumstances, a cylindrical shell of plasma can be expected to exhibit growing modes. The real part of the frequencies of these modes occurs at integral multiples of the ion cyclotron frequency: a result predicted earlier by Harris (9-10). In addition, the imaginary part of the frequency is dependent upon the density and the criterion for the appearence of complex frequencies is quite similar to that found by Harris.

Both of these predictions are borne out by experiment. Barnett (2) has found large amplitude disturbances in DCX apparently corresponding to superpositions of the modes predicted here. These disturbances appear after an initial time during which the plasma can be expected to have reached a density of approximately 10^6 particles/cm^3. For a proton plasma with an energy range of 200-300 kev and a magnetic field of 10^4 gauss, one has for one centimeter wavelength disturbances,

$$r_1 = V_1^0/\omega_{ci} \simeq \frac{6\cdot 10^8}{10^8} = 6\,\text{cm}, \tag{1}$$

$$r_2 = V_2^0/\omega_{ci} \simeq 8\cdot 10^8/10^8 = 8\,\text{cm}. \tag{2}$$

From Eq. (97) of Chapter II, the critical density for $\ell = 1$ and $\beta =$ /ka is

$$N_c = \frac{e^2 B_0^2}{m^2 c^2}\,\frac{m}{4\pi e^2}\,\frac{1+\beta^2}{\left(1+\left(\frac{m}{M}\right)^{1/3}\right)^3} \sim 2.5\cdot 10^6\,\text{cm}^{-3}. \tag{3}$$

Although this agrees with the experimental figure given above, one should exercise some caution in attributing quantitative significance to the

predictions of this calculation, due to the approximations made and also, due to the present uncertainties in experimental data. However, qualitatively the theory is surprisingly good. The critical density predicted is certainly correct within an order of magnitude. Furthermore, as mentioned before, the instabilities are observed after a finite time following turning on the beam. One can interpret this to mean that the plasma must build up to the critical density before becoming unstable.

Besides the aforementioned data, intense radiation, almost at integral multiples of the cyclotron frequency, has been observed in DCX. Due to the intensity of the radiation, it is believed that it must be coherent and therefore, attributable to collective oscillations of the plasma. Experiments with OGRA, the Russian counterpart of DCX, also indicate that the plasma radiates at multiples of the cyclotron frequency (7). However, it is not clear whether this is coherent or incoherent radiation.

It is apparent that from a qualitative viewpoint, the calculations presented in this study are valid. However, some weak points should be noted.

First, of course, the entire treatment is linearized, while a real plasma responds in a distinctly non-linear manner. This can be seen very simply. If, for example, the ion density were actually oscillatory, even with a growing amplitude modulation, eventually it would become negative--a physical absurdity. Unfortunately, exact, non-linear treatments for physically interesting situations are, at present, beyond one's ability.

A second objection--one which may be more easily corrected--is the restriction of the treatment to longitudinal oscillations. While it may

be quite true that the Coulomb forces are initially responsible for the appearence of unstable modes, it is apparent that there must be some coupling between these and the radiating transverse modes. Harris (11a), considering the infinite plasma, has indicated that this can occur for certain types of velocity distributions.

Finally, the announced purpose of the study, to determine the effects of the finite size of the plasma on the dispersion relation, has been accomplished to a very limited degree. Unfortunately, present experiments can give little indication of the accuracy of the present treatment in this respect. The advent of DCX-II, a larger machine with a more homogeneous magnetic field, makes it desirable to refine the calculations to determine more accurately the effect of the boundaries. However, since such a treatment will require extensive numerical calculations, one should also include the transverse fields to increase the validity of the study.

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