The Effects of Small-Group Discourse in Collective Argumentation on Secondary Students' Mathematical Arguments and Mathematical Identities

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DEDICATION

This dissertation is dedicated to all my students, who pushed me to be a better teacher and better person in every possible way. Your different strengths, weaknesses, passions, and views on life stretched me to see how I could become better to instruct you, assess you, and guide you as you developed from young children into intelligent and mature adults ready to take on the world and become influential citizens in our global society. Teaching mathematics and French and earning this doctorate go far beyond changing the letters at the end of my name because of you, and I am thankful for each of you for inspiring me to reach this milestone in my life. Thank you for inspiring me to always achieve something greater and humbling me to remember what is most important.

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ABSTRACT

The purpose of this study was to examine how middle grades students collaborate and communicate in development of mathematical arguments through collective arguments. Using a cognitive situate learning perspective, this study explored how and in what ways middle grade students engaged in collective argumentation to advance their mathematical arguments, to discuss mathematics collaboratively with one another, and to view themselves as doers of mathematics. In particular, the collective small-group arguments, audio recordings of the small-group discussions, and the pre- and post-surveys were examined. The participants in the study included approximate 45 eighth-grade students enrolled in one of four different Pre-Algebra classes taught by the same eighth grade math teacher at the same private school in the Southeast United States during the 2018-2019 and 2019-2020 school years. Data was collected via written collective arguments of the small groups, audio recordings of the small group discussions, and surveys before and after the instructional series. As for data analysis, a process of coding the arguments based on a hierarchy of arguments from the well-accepted work of Harel and Sowder (1998) was used for the quantitative analysis, and a coding system heavily based on work from Saldaña (2013) was used for the qualitative analysis. This inductive approach better explains the data from the participants themselves as they grew in mathematical argumentation and mathematical identity. Students advanced in their knowledge of and abilities to develop mathematical arguments, recognized the value of specific elements of argumentation according to Toulmin's model (Brockriede & Ehninger, 1960), and recognized developments in their identities as doers of mathematics. As a result of this study, middle grades students developed in their creations of mathematical arguments when collaborating in small groups, and they grew in different aspects of personal identity as doers of mathematics.

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CHAPTER 1

INTRODUCTION

Student collaboration has long been considered an important teaching strategy in secondary mathematics education. The National Council for Teachers of Mathematics (2000) and the Common Core State Standards for Mathematics (2010) recommended principles and standards for grades preK-12, which include the ability to communicate mathematical thinking coherently and clearly to peers, teachers, and others. Particularly, in the context of developing mathematics arguments and proofs, NCTM (2000) recommends students test their ideas publicly through a classroom discussion. In these public discussions, the teacher can monitor student learning (Lampert, 1990). However, this recommended consideration of argumentation is not regularly well-practiced in mathematics classrooms. In the context of K-12 mathematics, mathematical argumentation is largely accepted as students participating in discursive communication to develop and make sense of mathematics as they inquire about the reasoning of others and justify their own (Ball & Bass, 2003; Rojas-Drummond & Zapata, 2004). Students often are weak in their argumentation skills in this context, as they struggle with reasoning and justification due to a lack of opportunities to engage in problem-solving activities that encourage development of reasoning skills that enhance mathematical arguments (Mueller & Yankelewitz 2014). Although current mathematical teaching strategies often include discourse among teachers and students, this practice manifests itself in the teacher as the giver of information to the receiving students (MacDougall, 2013). When teachers do utilize student group work and collaboration, it is important to note that effective collaboration does not occur spontaneously, and that all collaboration does not necessarily lead to better learning outcomes (Dillenbourg,

2002). The use of collaboration as a regular teaching strategy, then, must be carefully designed to promote successful student learning in mathematics.

Teachers often use collaboration as a regular tool in the K-12 classroom, and the standards obviously support the idea of argumentation, but the use of collaboration for true argumentation is an underutilized tool in promoting student problem solving and mathematical reasoning via argumentation. Further, it still remains a challenge to effectively utilize collaboration in any K-12 setting because of the dynamics of working with various student opinions and mathematical abilities, but collaboration can be a tool in K-12 education to unlock students' abilities to deeply analyze mathematics. The idea of using collaboration to develop problem-solving skills is a research-based, student-centered approach to allow for mathematical argumentation development in the classroom, because it allows students to build knowledge, develop metacognition, and develop higher-order thinking skills (Cáceres, Nussbaum, Marroquín, Glesiner, & Marquínez, 2018). Fostering meaningful and purposeful collaboration that supports student growth in these skills requires certain criteria for successful lessons in K-12 mathematics.

In addition to empowering students to engage in meaningful collaboration discussions that support student growth in mathematics learning, the literature affirms the notion that students' open discussions about their mathematical discussions "closes the gaps" between students with different understandings of the math, thus addressing the issue of equity in a mathematics classroom for students of different mathematics, socioeconomic, and racial backgrounds, to name a few (Boaler, 2018). By allowing students to have a voice in their mathematical learning, students feel included and invested when working in small groups in mathematics class.

Although the power of student discourse is recognized and the importance of mathematical argumentation is emphasized in both national standards and recommended practices of practitioners and researchers alike across several national mathematics education organizations, there still remains extensive research to be completed to support the role that discourse plays in promoting the advancement of K-12 student mathematical arguments (Brown, 2017; Byrne, 2013; Yee, Boyle, Ko, & Bleiler-Baxter, 2017). This research will analyze the roles of collaboration and discourse in K-12 students' development of mathematical arguments, the impact and conceptualization of discourse on students' development of mathematical arguments, and the identity that students adopt as doers of mathematics via their production of mathematical arguments in collaboration with one another. Specifically, the project will study the act of argumentation as a social practice within the situation of learning mathematics collectively in collaborative small groups. It recognizes the specifics of collaborative discussion that impact students to advance in their abilities to create their own mathematical arguments, the elements of their discussion that contribute to advanced mathematical arguments, and the impact that collaborative interpersonal discourse has on their identities as doers of mathematics. Finally, this study will conclude with recommendations for future research and practice to support K-12 teacher understanding of how best to encourage meaningful discussion among their students in the form of interpersonal discourse and discursive activities to promote mathematical argumentation.

The concept of student collaboration through discursive activities has been a regular practice in K-12 mathematics classrooms. As students work with one another, they are given the chance to share their thinking with one another, which provides them the opportunity to individually and collectively approach problem-solving in a way that both challenges and

supports all types of students and different learning styles (Hodge & Walther, 2017). To emphasize student thinking and improvement of their own mathematical processes and thinking, collaborative efforts among students emphasize their discourse with one another and the impact that student conversations have on student learning and progression of mathematical thinking. According to both researchers and practitioners in K-12 mathematics, mathematical discourse lies at the intersection of social, emotional, and mathematical intelligence (Bertolone-Smith & Gillette-Koyen, 2019). Thus, viewing mathematical discourse, as a social practice highlights the value of student discourse in the promotion of national standards and mathematical practices, particularly the creation of mathematical arguments. A primary focus in K-12 mathematics classrooms by national mathematics education organizations such as the National Council of Teachers of Mathematics and the Common Core State Standards, both NCTM (1989) and CCSS (2010) recommend that reasoning and proof should regularly be incorporated in K-12 classrooms. Although argumentation has been established as a recommendation for K-12 mathematics, classroom teachers still find it difficult to promote student discourse so that students successfully engage in argumentation through justification and generalization (Melhuish, Thanheiser, & Fagan, 2019). Thus, this review of literature delineates the role of student collaboration through the lens of student discourse and the current state of mathematics education in utilizing discourse for argumentation in K-12 mathematics, while also providing implications for the K-12 mathematics education community and recommendations for future practice and research.

Statement of the Problem

Student development of their own mathematical arguments at the K-12 level remains a priority in mathematics education. To fortify one's own position of knowledge in mathematics

education, scholars agree that developing mathematical arguments ensures true understanding when one can convince oneself and others of one's mathematical explanation (Ellis, 2007; Cáceres, Nussbaum, Marroquín, Gleisner, & Marquínez, 2017, Stylianou, 2013). Additionally, the eight mathematical practice standards in which students should consistently engage include the development of mathematical arguments and the critique of others' arguments (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). A major factor in creating that space for students to develop and critique mathematical arguments with one another is the dynamic of social interaction in the mathematics classroom. Student participation in a mathematical learning community in a classroom is dependent on the culture. Civil and Hunter (2015) found it was necessary to have an open atmosphere that allows social talk and humor, so that students feel comfortable to share, make mistakes, and engage in dialogue about the mathematical learning. To further the mathematical learning, Rojas-Drummond and Zapata (2004) found that this opened the door for students to engage in exploratory talk, where they felt free to examine their own opinions, observations, and explanations of the mathematics. This interpersonal discourse allowed students to engage in healthy and open dialogue focused on the mathematics.

The connections between student interpersonal dialogue and both mathematical understanding and mathematical identity remain the areas that research has yet to fully explore. The mathematics education community seems to agree upon and have substantial research on the importance of mathematical argument development at the K-12 level (Brown, 2017; Byrne, 2013; Yee, Boyle, Ko, & Bleiler-Baxter, 2017), as well as the utilization of student discourse and mathematical discussions as an effective teaching strategy (Cáceres, Nussbaum, Marroquín, Glesiner, & Marquínez, 2018). However, a dearth of research about the interpersonal dialogue

and its relationship to the development of mathematical arguments and the development of student identity at the K-12 level remains. This dissertation hopes to analyze student discourse and the impact it has, particularly within small-group collaboration emphasizing argumentation, on the students' arguments themselves, the types and impacts of discussions they have within small group settings, and the students' personal identities of doers of mathematics and constructors of their own mathematical arguments. Further, the project will provide insight into the elements of and reasoning for social discourse in student interactions with mathematics as well as pedagogical approaches that K-12 mathematics teachers can take towards implementation into regular practices in their classrooms. Further, as a result of collaborative discourse, students will begin to grow in their autonomous thoughts about the mathematics and thus develop their own veracious mathematical arguments.

Purpose of the Study

The purpose of this study is to understand the elements and design of small-group collaborative discourse in collective argumentation as students participate in creating veracious mathematical arguments in their classroom learning community. The underlying premise is that students engage in interpersonal discourse to begin to deeply understand their own analytical thoughts in developing a mathematical proof or argument. By understanding the process of mathematical argumentation development, the hope is to provide researchers and practitioners with actionable insight into how best to support student collaboration and discourse in meaningful argumentation development. Moreover, in studying the discourse from a small-group perspective through a cognitive situated learning lens, as students are in the context of creating mathematical arguments in small group discussions, this study hopes to pursue the impact of discourse as a major factor in all mathematical learning settings on both student arguments and

student identity as doers of mathematics and constructors of mathematical arguments. The following research questions will be used to address the purpose of my study:

1. In what ways do K-12 mathematics students use interpersonal discourse in small-group class discussions to create and advance mathematical arguments and proofs?
2. How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?
3. How does student collaboration through interpersonal discourse in collective argumentation impact students' identity as doers of mathematics?

Previous studies have found student collaborative work and mathematical arguments as beneficial teaching strategies to promoting student mathematics (Yee, Boyle, Ko, & Bleiler-Baxter, 2017). Therefore, this study's implementation of interpersonal discourse small-group discussions via collective argumentation has the potential to positively influence K-12 mathematics students' understanding of the importance of collaborative interpersonal discourse in developing more advanced mathematical arguments and overall become better doers of mathematics.

Need for the Study

This study is needed as a practical perspective for pre-service and in-service teachers, as well as from the theoretical perspective of mathematics education researchers. From a perspective of practice, this study has the potential to impact future mathematics educators in several ways. Firstly, understanding how to better promote mathematical arguments through interpersonal discourse specifically within small-group settings will impact future educators in the design of their lesson plans and implementation of their lesson plans. As a result, findings

from this study can influence future professional development for current and future mathematics teachers to unlock the power of student voice. Because research has largely emphasized the impact of the teachers as the authority on students' learning of mathematics (Lampert, 1990), this study opens doors of opportunity for teachers to learn how to utilize the students as equally powerful tools for mathematics understanding. Lastly, all students begin to develop their own personal identities as doers of mathematics as they progress throughout their formal K-12 education, as they begin to see themselves in relation to mathematics and the value of mathematics as it is realized in the classroom (Cobb, Gresalfi, & Hodge, 2009). This study will reflect student understanding of mathematics and the value of mathematics through participation in these small group arguments, and it can impact how students begin to see themselves in relation to mathematics. Thus, the results of this study may be useful for future designs of small-group mathematics lessons for teachers and students as they develop in their abilities to create collective mathematical arguments.

Another contribution of this study is the emphasis of a situated discursive framework for future mathematics education research. By particularly looking at the situation of mathematical argument development in K-12 classrooms, this theoretical framework can then highlight the discursive elements that students utilize in mathematics classrooms to identify particular patterns of discussion that students use to create, modify, or enhance their justifications and of their mathematical thought processes. Thus, the study will allow researchers to gain better insight into the preparation of future mathematics educators as well as the reinforcement of teaching strategies and skills in current practicing K-12 mathematics teachers. There is a gap in the literature concerning a discourse approach to researching the situation of mathematics small-group arguments. Thus, the instructional sequence in this study will positively impact both the

teaching practices for practicing K-12 mathematics teachers and the theoretical perspectives of mathematics education research today.

Limitations

As is true with most studies with design in qualitative work by nature, this study is limited by its small sample size. The sample was drawn from one eighth grade teacher's classroom at a private middle school in the Southeastern United States, which did not allow for a diverse or large collection of data, limiting the generalizability of the study's results. Particularly, this study has the potential to be limited to schools of similar status, size, and/or location, as well as to similar middle grades level. However, the results can be used to inform future research related to discourse and argumentation and study identity as doers of mathematics within this context. Furthermore, as emphasized by the situated learning lens at the foundation of the theoretical framework, this study's findings can provide understanding of what is successful and not successful in the specific situation of small-group collective argumentation, which can guide future designs and implementations of tasks centered around this design.

As the researcher, my relationship with participants may also be a limitation of the study. I will be the facilitator for the instructional sequence, which may influence participants to view me as a figure of authority. Furthermore, my beliefs related to student discourse in collective argumentation as an effective teaching practices will likely be expressed from my role as the facilitator, consequently sending specific messages to participants that may affect their responses on data collections, particularly the surveys that address their identity as doers of mathematics. However, I am not the main classroom teacher, and I will not make any decisions on their performance in the course such as assigning grades. By not having a pre-existing relationship with the participants, the risks of influencing their responses are likely reduced.

Delimitations

This section provides details of the delimitations that may impact the study's findings. These delimitations are characteristics of the research design chosen by the researcher to set boundaries for the study. First, this study is delimited by the collection of self-report data in the surveys to analyze participants' identity in participation in the small-group collective argument instructional sequence. Self-report data raise an important concern because participants may not answer honestly to items. Thus, their responses may not reflect their actual thinking, feelings, or behaviors (Christenson, 2012). Students may be influenced by social demands to answer in a certain direction, especially under some conditions such as responding without anonymity. Although there is the risk of dishonest responses, survey reflections allow the collection of data on participants' subjective perceptions. This type of data is necessary to learn about participants' experiences participating in the small-group collective arguments, including their perceptions of what aspects of the collaborative discussions were beneficial and not beneficial.

This study's findings are also delimited by the short period of time that the instructional sequence takes place. This short time span provides a very specific situation, which is the purpose of the qualitative aspects of this study. These boundaries are appropriate given this study's focus on interpersonal discourse and small-group collective argumentation in mathematics. In addition, these boundaries were set for manageability of the study, as well as accessibility to the classroom.

Assumptions

A few assumptions are considered for this study. First, I believe students learn mathematics powerfully when given the chance to engage in interpersonal discourse; therefore, I assume that the use of collective argumentation, particularly in the small group setting, is an

effective teaching practice. Additionally, I believe students' utilization of their voices in their mathematical arguments impacts their views of themselves and of mathematics, and ultimately, their identity as doers of mathematics. I am taking these beliefs into the study. However, the design of the study, including the tasks and facilitation of the instructional sequence of discourse-oriented collective argumentation, as well as the questions asked on the surveys, are based on research evidence, as found in the review of the literature to follow. Lastly, I assume students will respond honestly on all reflections on the survey questions. Student identifiers will be replaced with pseudonyms to ensure confidentiality. With these efforts to keep participants' information confidential, it is assumed students will feel comfortable to respond honestly.

Definition of Terms

This section provides definitions for key terms regarding how they are used and applied in this study.

Collaboration – Collaboration within the context of mathematics education is the idea of grouping students purposefully to "increase their productivity, accountability, engagement, and performance levels, while still providing them with the level of support they need to complete the assigned task" (McGlynn & Kozlowski, 2016).

Collective Argumentation – Using a sociocultural approach, collective argumentation (CA) is based on interactive principles necessary for coordinating student engagement in the discourse of the mathematics classroom (Brown, 2017). The definition of collective argumentation is "multiple people working together to establish a claim" (Conner, Singletary, Smith, Wagner, & Francisco, 2014).

Interpersonal Discourse – For this study, interpersonal discourse is defined from a linguistic perspective, as students engage in oral discourse with one another during the process of

collective argumentation. Interpersonal discourse is defined as students' engagement in oral communication about a particular topic (Adair-Hauck, Glisan, Koda, Swender, & Sandrock, 2006).

Mathematical Argument – As one of the eight standards for mathematical practice published by the National Council of Teachers of Mathematics (2010), mathematical arguments are defined as problem-solving as students move through the process of defending a claim by backing it with evidence, not relying on other truths or other authorities, such as other people. It also involves the process of critiquing one or another's reasoning.

Mathematical Discussion – Mathematical discourse lies at the intersection of social, emotional, and mathematical intelligence. Leading a mathematical discussion requires providing a space that is socially safe for students to engage in mathematical discourse so that the platform lets them use their voice to share mathematical ideas (Bertolone-Smith & Gillette-Koyen, 2019).

Student Personal Identity – From the study of normative and personal identity by Cobb, Gresalfi, and Hodge (2009), personal identity concerns the "extent to which students identify with their classroom obligations, merely cooperate with the teacher, or resist in engaging in classroom activities."

Organization of the Study

As with other work of this nature, this dissertation is organized into five chapters. The current chapter introduces the study with general information including the statement of the problem, the purpose of the study, the need for the study, limitation, delimitations, assumptions, and definition of terms. The remaining chapters are organized in the following manner. The second chapter presents the theoretical framework for this study, as well as the review of literature that is relevant to student discourse, collaboration (both generally in mathematics

classrooms and in specific small-group settings), mathematical argumentation and justification, and mathematical identity. The third chapter describes the research methodology, including the research design, participant selection, data collection, and data analysis procedures for the three distinct papers in this dissertation study. The fourth chapter details the results and analyses from data collection and how the data helps answers the research, divided into three papers of analyses each addressing different aspects of the studies that respond to the research questions in the first chapter. The final chapter explicates the conclusions and implications based on the findings from this study.

CHAPTER 2

LITERATURE REVIEW

The ability for students to develop their own mathematical arguments at the K-12 level remains a priority in mathematics education. Scholars agree that developing mathematical arguments ensures true understanding when one can convince oneself and others of one's mathematical explanation (Ellis, 2007; Cáceres, Nussbaum, Marroquín, Gleisner, & Marquínez, 2017). Additionally, as evidenced by the National Council of Teachers of Mathematics principles and standards for school mathematics (2000) and the Common Core State Standards Initiative (2010), the eight mathematical practice standards in which students should consistently engage include the development of mathematical arguments and the critique of others' arguments. According to the literature, a major factor in creating the space for students to develop and critique mathematical arguments with one another is the dynamic of social interaction in the mathematics classroom. Student participation in a mathematical learning community in a classroom is dependent on the culture. Civil and Hunter (2015) found it was necessary to have an open atmosphere that allows social talk and humor, so that students feel comfortable to share, make mistakes, and engage in dialogue about the mathematical learning. To further the mathematical learning, Rojas-Drummond and Zapata (2004) state that this opened the door for students to engage in exploratory talk, where they felt free to examine their own opinions, observations, and explanations of the mathematics. In emphasizing the use of students' interpersonal discourse with one another, researchers have begun to study how students engage in healthy and open dialogue focused on the mathematics to create opportunities that advance student understanding and thinking towards mathematics through the lens of mathematical argumentation.

There remain connections, then, between student and educator understanding of a mathematical argument and the ways in which classrooms are positioned to support student development of these arguments. The mathematics education community seems to agree upon and have substantial research on the importance of mathematical argument development at the K-12 level (Brown, 2017; Byrne, 2013; Yee, Boyle, Ko, & Bleiler-Baxter, 2017). However, a dearth of research remains concerning the means in which students support the building of their knowledge of mathematical arguments through interpersonal communication. This study analyzes both the conceptualization of mathematical arguments and the discussions of K-12 students as they use their own discussions to advance in critical thinking and deeper cognitive mathematical arguments. Further, the project will not only discover research-based evidence for the discursive elements of K-12 argumentation, particularly for mathematics classrooms, but also the pedagogical approaches that K-12 mathematics teachers can take towards regular implementation of argumentation in their classrooms. Finally, as a result of collaborative discourse, students will begin to grow in their autonomous thoughts about the mathematics and thus develop their own veracious mathematical arguments.

Theoretical Framework

The learning theory of situated cognition, specifically in the context of mathematics education, arose as a result of the cognitive revolution. In this cognitive revolution that began in the 1960s, educational psychologists recognized a shift beyond a singular belief in B.F. Skinner's behaviorism, recognizing the influence of Jerry Bruner's social psychology on an internal cognitive process (DeCorte et al., 1996; Miller, 2003). Thus, cognitive learning theory was officially established in the world of educational psychology, as a symposium organized by the "Special Interest Group in Information Theory" in 1956 at the Massachusetts Institute of

Technology began to recognize the working of behaviors in the brain and the connections to the mind (Miller, 2003). However, psychologists continued to contend for the power of the interactions one has with the physical world around them, the ideology in which behaviorism is originally grounded. The rise of social constructivism by Leo Vygotsky became prominent, with the idea that learning occurs as a result of the interactions an individual has with others around him or her (Jaramillo, 1996). Psychologists and educational psychologists at this point grounded themselves in these two seemingly opposing views of learning, cognition vs. social constructivism.

This led to a combination of these theories in Jean Lave's idea of situated learning, which was an explanation of the idea of transfer of information through a sociocultural lens (Billett, 1996). Because of the focus on the idea of transfer of information, particularly information learned from educational institutions to situations and circumstances outside those institutions (i.e. when college students enter the workforce), learning became more focused on the social interactions of people and its influence on the learning from an internal, cognitive standpoint. From this, then, the idea that learning is still an internal process but appropriated within its specific external situations developed the learning theory of situated cognition, explicated clearly by Brown, Collins, and Duguid in the *Handbook of Educational Psychology* in 1989. Within mathematics education, educational researchers tend to fall within the middle of the scale of "situatedness" of cognition and sociocultural constructivism, believing in the marriage of the original two theories (Anderson, Reder, & Simon, 1996). Due to a varying degree of agreement on social interactions vs. internal cognitive processes, different learning theories in the middle of the scale of cognition vs. sociocultural constructivism have developed.

To clarify a distinction between cognitive theory and situated learning theory, cognitive theory, as explicated by Cobb and Bowers, specifically identifies the idea of transfer of knowledge from being able to take learning from one physical situation and applying the same knowledge to another physical situation (1999). Jean Lave, the pioneer of situated learning, sees the transfer of knowledge more from a sociocultural point of view, where a learner is able to transfer knowledge between different situations of differing worldviews, not necessarily physical location (Cobb & Bowers, 1999). Because of the differences in focus on physical location vs. sociocultural situations, Cobb & Bowers (1999) developed a graphic to represent the influence of physical location on these learning theories (see Figure 2.1).

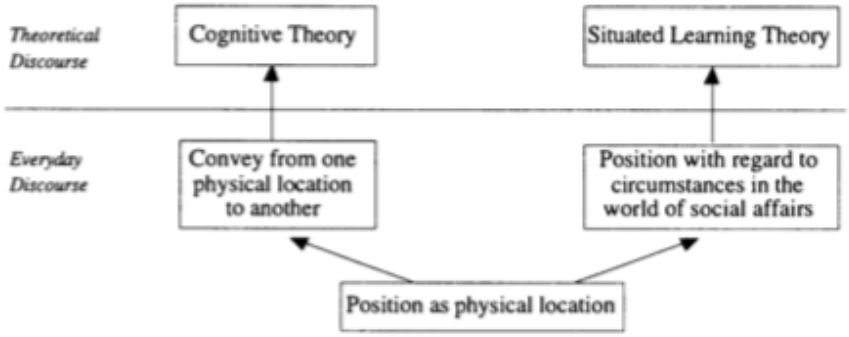


Figure 2.1. Metaphorical Underpinnings of Cognitive Theory and Situated Learning Theory

This graphic explains that variations in the idea of position are a major factor in defining situatedness. Further, Lave's situated learning does not identify the importance of cognition as heavily as does Brown, Collins, and Duguid's situated cognition theory, which recognizes the influence of the social interactions in a situation on the cognitive learning of information, such as in mathematics education. Situated cognition theory, then, serves as a blending of Lave's situated learning, which moves further away from the constructivism extreme, and cognitive learning theory (see Figure 2.2). With an overlap between situated learning and cognitivism, situated

cognition accounts for the ability to transfer knowledge between different social contexts as well as different physical locations. These ideas related to a situated perspective (Lave & Wenger, 1991; Putnam & Borko, 2000) provide the lens from which to examine the learning and the development of mathematical arguments of the students in this study. Next, I will present relevant literature regarding students' discourse, particularly in small groups, and mathematical argumentation in K-12 mathematics.

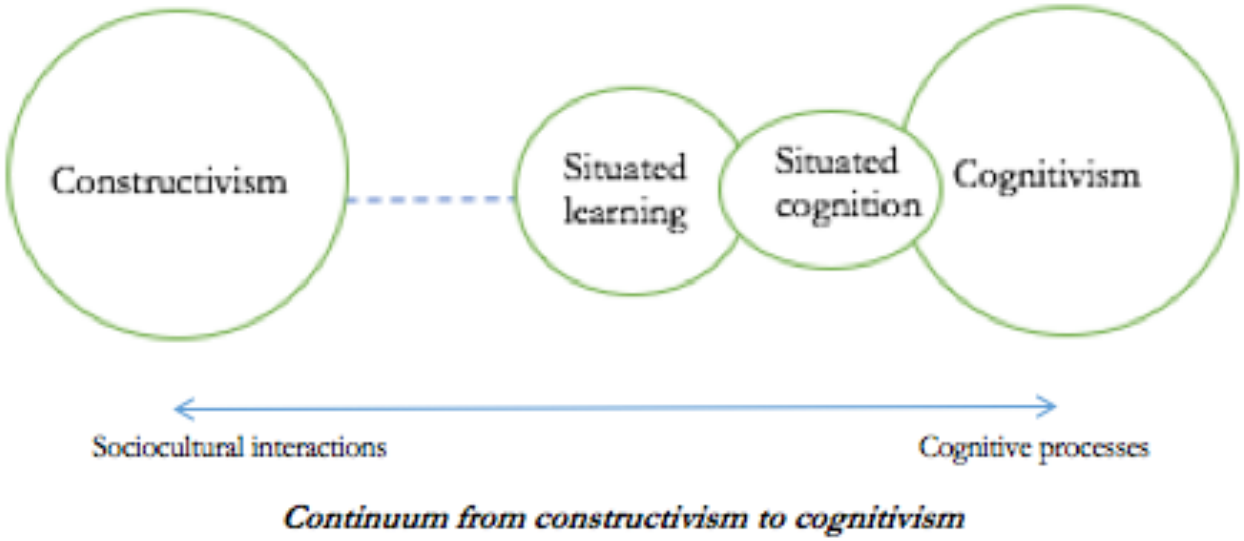


Figure 2.2. Situated Cognition along the Continuum of Constructivism and Cognitivism

Finally, because equity has become a central focus of mathematics education and mathematics education research, it is important to note that this situated cognition framework balances the social and cognitive aspects of mathematical learning to make mathematics learning more equitable for all students. Because students engage in social learning with one another, they can contribute whatever they bring to the study so that all students can advance cognitively in their level of mathematical argumentation.

Theories and Frameworks in Student Argumentation in Mathematics

Current international curricula and frameworks for mathematics education have advocated for an emphasis on proof and argumentation in K-12 mathematics. Taking a systematic approach to reviewing current research about conceptions, content, and supportive actions for proof and argumentation in K-12 mathematics, this literature review highlights how current mathematical education researchers view argumentation in classes today, particularly in regard to conceptions and implementation of argumentation techniques in K-12 mathematic classrooms. In looking closely at the empirical research that has been most recently published since the emphasis of argumentation via the National Council of Teachers of Mathematics publication of their national standards, firstly in 1989 then again in 2000, this paper highlights both major theories and frameworks that mathematics education researchers have adopted to deeply study the impact of argumentation on K-12 mathematics in the past two decades. This research remains important and relevant in the mathematics education field because the implementation of argumentation in K-12 mathematics is becoming a priority not only in the United States according to published national standards, but also internationally amongst policymakers across the globe (Nardi & Knuth, 2017). Further supported by the Common Core State Standards for Mathematics in 2010, the practice of justification and argumentation in K-12 mathematics warranted deeper empirical research that have transformed and improved argumentation classroom techniques in the classroom. By reviewing the literature systematically, this paper fulfills the purpose of understanding various theories and frameworks that underline current educational researchers' understanding of K-12 mathematical argumentation, proof, and justification, as well as highlights future considerations of argumentation through a lens that

supports all stakeholders in the investment of K-12 student advancement of mathematical understanding and learning.

The Concept of Argumentation

In taking a deep dive into the current published research on justification and argumentation in K-2 mathematics, it became evident that practitioners, researchers, and other stakeholders in the mathematics education community consider the concept of argumentation and the implementation of teaching practices that support argumentation in different manners and at different levels of priority (Campbell, Boyle, & King, 2019). Whereas argumentation has taken a priority in mathematics education in the past twenty decades, several different researchers emphasize various aspects of the process of argumentation, whether it be the social aspect via the discourse that students engage in (Berland & McNeill, 2010; Evagorou & Osborne; 2013, Knight & McNeill, 2015) or the advancement of the level of mathematical arguments (Harel & Sowder, 1998; Stylianides & Stylianides, 2009), as mere examples. No matter the emphasis of the various research studies that have previously been conducted concerning K-12 mathematics argumentation, a concentration of frameworks and theories have regularly served as the foundations of these research studies. As a result of this analysis of research studies, this paper will report on the current state of mathematics education research concerning the practice of teacher implementation of proof, justification, and argumentation techniques for K-12 mathematics. Further, it will expound on future directions that this research can take, carefully considering past research and areas that can be further explored and the theoretical and framework foundations supporting them. Finally, as a result of this research, research can begin to take shape to inspire best practices to most effectively impact practices for current mathematics teachers and practitioners in the classroom.

Context of the Review

As one member of a team of three mathematics education researchers, I situate this review in the context of two recent literature reviews performed by Stylianides, Stylianides, and Weber (2017) and one working group session by Staples, Newton, Kosko, Cirillo, Weber, Vieda, and Conner (2016). Stylianides et al. (2017) reviewed proceedings of papers for the annual conference of the Psychology of Mathematics Education (PME) written between 2005-2015 related to proof and argumentation. They chose to focus on proof and argumentation because argumentation and proof are closely related, and the consideration of both argumentation and proof together emphasizes a wider range of mathematical processes than if considered individually. Aligning with this sentiment, our review similarly considers articles related to proof and argumentation. In their review, these authors noted themes related to student conceptions, classroom-based research, and teacher knowledge. Following this review, Stylianides, Stylianides, and Weber reviewed literature related to proof from three differing perspectives. The three perspectives included proving as a form of problem-solving, proving as convincing, and proving as a socially-embedded activity. Staples et al. (2016) formed a working group at the annual conference for the North American Chapter of the Psychology of Mathematics Education (PME-NA) to determine the perspectives of what the field calls argumentation, justification, and proof. Through their interactions, they noted there was some variability in how the general field of mathematics education understood argumentation and proof at the time of their review of the literature. In noting the different conceptualizations of argumentation and proof in K-12 mathematics, this study reveals the various frameworks for research study in the field of mathematical proofs and argumentation in mathematics education today, delineated in the results of the review that are following.

Methodology of the Review

In this review of literature, we used a systematic review methodology (Cooper, 2017; Hannes & Claes, 2007) to find and analyze research studies on K-12 proof and argumentation. The protocol for a systematic literature review entails creating a list of search terms and choosing databases in which the search will be run, running a search and gathering all articles which contain the search terms, and using a screening process to choose articles based on a predetermined set of criteria. After consulting with an education research librarian, we ran an initial search in three databases: ERIC EBSCOhost, PsycINFO, and Education Full Text (H.W. Wilson). Due to the release of the NCTM recommendations for reasoning and proof in 2000, we considered articles written after the year 2000. Therefore, the following limiters were applied to the initial search: published between January 1, 2000 and May 16, 2018 and peer reviewed. We created a bank of search terms using words commonly found in nationally accepted curriculum documents (NCTM, 2000; CCSSI, 2010; Department of Education, 2013, literature reviews (Stylianides, 2016; Stylianides et al, 2017), working groups (Staples et al, 2016), or research studies which claim to examine proof and argumentation in mathematics. While it is likely impossible to consider all terms related to proof and argumentation, the chosen bank of terms is found often and well-represented in the literature, and the terms are used in conjunction with the words *proof* and *argumentation*. We retrieved articles that included the search terms appearing in any location of the article.

After retrieving an initial 8,094 articles, we transported each article from the research databases to Refworks, a computer tool to create bibliographies. Then, we removed all duplicates, leaving a remaining 5,263 articles. After deleting duplicates, we transported the citations to Google Sheets for purposes of screening. We screened the title and abstract of all

5,263 articles using the following inclusion criteria: content (the focus of the study must be on proof or argumentation), participants (research subjects must be K-12 students), empirical (the study must only report empirical finding), publication type (the publication must be a journal article), and language (the article must be written in English). Screening involved testing for overall inclusion through three phases: the article title met the inclusion criteria, the abstract of the article met the inclusion criteria, and the article met the inclusion criteria after a full scan of the article. After the first phase, 902 articles remained to go through the abstract screen. After the second phase, 119 articles remained to go through a full scan. Upon completion of the third phase, 73 articles remained for inclusion in the systematic review. We double coded 10% of the articles at phase 1 and phase 2 of the screening process to calculate inter-rater reliability (as calculated by dividing the number of agreements by the sum of agreements and disagreements and multiplying by 100) with a cutoff set for 85% agreement. For phase 1, the inter-rater reliability was 90.7%, and for phase 2 the inter-rater reliability was 91.1%. After screening articles for inclusion, we followed the same screening process to evaluate references of the 73 articles. The ancestral search yielded 3 more articles for a total of 76 articles included in this review.

Results of the Review

In completing this systematic review, I recognized that mathematics argumentation was generally categorized into three different theoretical perspectives. Within each of these perspectives, various research has found different frameworks to guide their research. As a result of this review, the following results summarize main theories behind these views of mathematics argumentation and the historical perspectives of these findings.

Argumentation for Problem Solving. Much literature is devoted to understanding K-12 mathematics students' proving and argumentation knowledge and capabilities (e.g. Harel & Sowder, 1998; Healy & Hoyles, 2000; Flores, 2006; Lin, et al., 2004; Liu & Manouchehri, 2013; Martin & Harel, 1989; Weber, 2001). Scholars claim that students often rely on either the authority of an expert in the field or previously proven or previously accepted mathematical concepts (Flores, 2006; Healy & Hoyles, 2000; Lin, et al., 2004; Martin & Harel, 1989). As students learn to engage in argumentation that does not rely on other truths or other people, students are engaging in the act of problem-solving as they move through the process of solving a problem by defending a claim by backing it with evidence and critiquing the reasoning of others.

To take a deeper look at how students choose to create an argument and defend their thoughts with others, several seminal mathematics education research studies about proof and argumentation are widely cited for confirming deficiencies in K-16 students' proving practices (Healy & Hoyles, 2000; Martin, 1989; Weber, 2001). Research is beginning to uncover that students and teachers alike often rely on their written final submissions as proof of student understanding through argumentation, whereas student oral argumentation is just as powerful at allowing students to solve problems through the process of argumentation in their discourse out loud with one another or in explanation to a teacher or other mathematical authority. Students' written understandings are not always reflective of their cognitive knowledge (Evens & Houssart, 2004). Thus, the framework of argumentation as problem-solving gives credibility and importance to future studies of student engagement in oral argumentation, which also provides foundation to the second view of argumentation, which is argumentation as the act of convincing.

Argumentation for Convincing. The view of argumentation for the sake of convincing has led to several well-established and well-accepted frameworks for measuring K-12 mathematics arguments tangibly on a hierarchy of student understanding. These frameworks largely reflect on accuracy of the mathematical concept as well as what is socially and academically appropriate for the students at those levels, which guides many of the frameworks (Balacheff, 1988) used in mathematics education today.

One example of a commonly accepted mathematical argumentation framework is the proof scheme of Harel and Sowder (1998), who developed a holistic view of students' proof and argumentation schemes by observing the proving practices of mostly college students, along with a case study of one junior high school student. In their analysis, they concluded that students' proof schemes fell within three categories: external conviction, empirical, or analytical. The external conviction proof scheme occurs when a student relies on authority such as a teacher, a parent, or a textbook to create a mathematical argument. As an example, Harel and Sowder (1998) found that students often believed proofs should contain complicated language or symbols because they often saw sophisticated arguments in textbooks or other authoritative domains. They also found that students used authority figures as warrants for claims by using language such as 'because my teacher said so'. Other scholars similarly found authority to be a proving scheme exhibited by students of all ages (Flores, 2006; Fried & Amit, 2008; Sen & Guler, 2015). The empirical proof scheme refers to student usage of examples, experiences, or perceptions to justify a claim (Harel & Sowder, 1998). Students use this proof scheme when they attempt to show that a conjecture is true by verifying a finite number of cases. Often a first step observed in some research studies (King & Campbell, 2019), some students rely on an empirical proof scheme to begin their argumentation reasoning processes before being able to advance into

a more advanced proof scheme. Lastly, the analytical proof scheme is defined as arriving at a conclusion by using a sequence of logical deductions (Harel & Sowder, 1998). In current K-12 mathematics, particularly at the 9-12 level, students prove that a statement is true in mathematics by using axioms, definitions, and previously proved theorems to create a logical argument. According to the standards published by NCTM (2000), students should develop analytical proof schemes by the time they reach high school, although it is regularly encouraged in these standards across all levels of K-12 mathematics.

Though Harel and Sowder (1998) warned against treating their taxonomy as hierarchical structure, they acknowledge that "there is often at least a partially hierarchical nature implicit in the categories" (p. 277). In fact, several scholars have used Harel and Sowder's (1998) taxonomy as a hierarchical rubric for determining how students' arguments change over time (e.g. Ellis, 2007; Flores, 2006; Lee, Chen, & Chang, 2014; Liu & Manouchehri, 2013; Stylianou, 2013). The framework established by Harel and Sowder (1998) provide different types of arguments that are lower in the hierarchy as ones that are based on external proofs, where what convinces the student of the validity of the argument and what it used in their argument is something found outside of their own reasoning, including arguments that put the authority in the truth of a textbook, the teacher, or someone else. Other external proofs include ritual proof schemes, where the validity is based on the process of the proving and argumentation and not necessarily the content, and symbolic proof schemes, where students place faith in representing mathematics abstractly with symbols. The next level of arguments is an empirical proof scheme, where students rely on examples, before reaching the highest level of an argument, which is an analytic proof scheme, or true mathematical proofs. Figure 2.3 below summarizes these arguments and proof schemes.

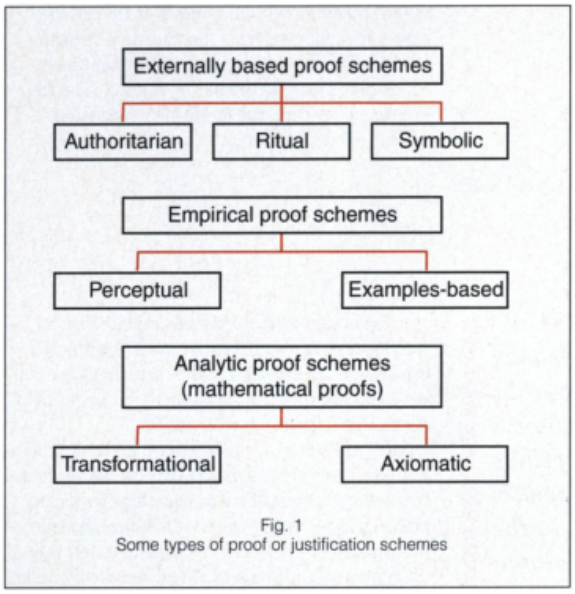


Figure 2.3: Harel and Sowder (1998) Hierarchy of Mathematical Proofs and Arguments

Because the act of convincing has become a widely accepted goal of argumentation in K-12 mathematics, it is important to note that convincing refers back to the content of mathematics, as the engagement of students in discourse with one another for reasoning, critique, and justification leads students to a deeper understanding of the mathematics so that they can gain authority of their own learning of the mathematics (Stein, Engle, Smith, & Hughes, 2008). Students often take ownership or authority of their own mathematical learning when they take on the role of the one who does the convincing, either of himself or of another student.

Of course, convincing oneself or another of the validity of a mathematical claim in argumentation does requires students to engage with the content and often with one another according to the standard of mathematical practice set by NCTM (2000), where students not only create viable mathematical arguments, but they also critique the reasoning of others. Thus, argumentation as convincing does emphasize student ownership of the learning but also

recognizes the need to do this in the social practice of argumentation with one another as students engage in discourse with one another.

Argumentation as a Socially Embedded Practice. Most recently in mathematics education research, scholars have recognized argumentation as a socially embedded practice, as research is highlighting the necessity of studying student discourse with one another. Primarily, this is in the rise of mathematics education research because of the recent emphasis on student voice because of the reformed mathematical practices that place the authority of learning on the student (Stein et al., 2008). Argumentation as a socially embedded practice looks at both the role of students individually as they engage in discussion as well as collectively as they make up a mathematical learning community.

Creating an environment that encourages student collaboration and communication, particularly in the context of developing mathematics arguments and proofs, opens up spaces for students to test their ideas publicly through a classroom discussion as they work towards developing a true mathematical argument, as recognized by the broader mathematical community. An effective mathematics learning environment successfully implements multiple or all aspects of argumentation as problem solving, convincing, and a socially embedded practice as recognized and previously delineated in the literature. When done carefully, Lampert (1990) argues that teachers can monitor student learning in the context of their discussion. However, this recommended consideration of argumentation is not regularly practiced in mathematics classrooms, as students struggle with reasoning and justification due to a lack of opportunities to engage in problem-solving activities that encourage development of reasoning skills that enhance mathematical arguments (Mueller & Yankelewitz, 2014). The idea of using collaboration to develop mathematical arguments and solve mathematical problems is a research-based student-

centered approach to allow for mathematical argumentation development in the classroom, because it allows students to build knowledge, develop metacognition, and develop higher-order thinking skills (Cáceres, Nussbaum, Marroquín, Glesiner, & Marquínez, 2018). Fostering meaningful and purposeful collaboration that supports student growth in argumentation is a social practice that fosters student learning in the context of mathematical argumentation development. One primary example of student discussion as a social practice is the idea of using the classroom as a learning community, which can go deeper in the context of mathematical argumentation as they develop their own communal criteria by which they can judge arguments and proofs.

One way to balance authority in the classroom is to allow students to engage in the process of deciding what counts as proof in a specific setting. Mathematicians rely on the broader mathematical community to decide what does and does not count as proof (Harel & Sowder, 2007), and the criteria for what counts as proof is socially negotiated among the field of mathematics education. However, students rarely understand the criteria for what counts as proof in a given context (Harel & Sowder, 2007), so recently, scholars suggested that students should negotiate communal criteria for proof to engage them in practices similar to those of mathematicians (Boyle, Bleiler-Baxter, Yee, & Ko, 2015; Stylianides & Stylianides, 2009; Yee, Boyle, Ko, & Bleiler-Baxter, 2018).

The concept of communal criteria is still relatively new to mathematics education, but some studies have been done in other contexts outside of K-12 mathematics classrooms. Stylianides & Stylianides (2009) first implemented the idea of communal criteria with prospective elementary teachers in a mathematics education course on proof. The class socially agreed upon three criteria for what constitutes a proof, but it is important to note that the teacher

acted as the representative of the mathematical community by helping to shape the criteria and making sure it met certain standards for proving. A major finding from this study is that teachers relied less on empirical arguments over time as a result of explicating the criteria for proof. Additionally, preservice mathematics teachers that used empirical arguments were aware of their limitations.

Looking at an example from collegiate mathematics instruction, Yee et al. (2018) designed an instructional sequence in undergraduate mathematics classes wherein the classroom community developed communal criteria and judged other classmates' arguments based on the criteria. Similar to Stylianides and Stylianides (2009), they found that explicitly stating the criteria for proof aided students in creating increasingly sophisticated arguments over time. In another study, Boyle et al. (2015) found that the undergraduate mathematics students in his study changed their perceptions of proof as a result of engaging with communal criteria. Students learned that proof is socially negotiated, and they gained further clarity for what counted as proof.

Although much research has been completed on the theories and frameworks for argumentation in K-12 mathematics and the usage of mathematical arguments to support student learning, there are still more implications for future research and practice that remain. As seen in the previous examples, further research can examine the impact of communal criteria on K-12 mathematical argument development instead of pre-service teacher or college student mathematical argument development. Recognizing argumentation as both a pedagogical practice, a student learning process, and an end product and goal for K-12 mathematics, future research can be broken down into different studies that highlight the student participation as well as their

advancement of mathematical thinking through tools and teaching strategies developed as a result of current mathematics education research.

As a relevant example evident in the social approach to mathematics argumentation, the use of student discourse with one another can appropriately take on a more discursive theoretical approach in a social theoretical foundation to better study and understand the aspects of dialogue that students have with one another to better support student advancement of mathematical arguments. Even further, research can then utilize a framework that has been previously established, or one created with the inspiration and research-based evidence of other argumentation frameworks for mathematics education, to study how the social practice of student discourse can impact student understanding as arguments develop and increase in level through these argumentation frameworks, emphasizing argumentation more for problem-solving and for convincing. A hierarchical framework such as Healy and Hoyles (2000) helps emphasize how students go through the process of problem-solving to engage in convincing that truly defines mathematical understanding for the sake of student understanding of mathematics at the K-12 level.

Thus, overall argumentation by nature has historically taken on several different frameworks for research study, and different theories have served as foundations for these studies on mathematics argumentation. Because of the broad nature and the recent emphasis on argumentation in the context of K-12 mathematics, mathematics education research has viewed argumentation in several different lights, requiring various theories and frameworks as foundations to the recent research. More research can move forward with these theoretical and framework underpinnings to uncover more practical approaches to K-12 mathematics argumentation.

Supporting K-12 Student Development of Mathematical Argumentation

The establishment of an environment that encourages student collaboration is an important practice in secondary mathematics learning. The National Council for Teachers of Mathematics (2000) and the Common Core State Standards for Mathematics (2010) recommended principles and standards for grades preK-12, which include the ability to communicate mathematical thinking coherently and clearly to peers, teachers, and others. Particularly, in the context of developing mathematics arguments and proofs, NCTM (2000) recommends students test their ideas publicly through a classroom discussion. In these public discussions, the teacher can monitor student learning (Lampert, 1990). However, this recommended consideration of argumentation is not regularly well-practiced in mathematics classrooms. In the context of K-12 mathematics, mathematical argumentation is largely accepted as students participating in discursive communication to develop and make sense of mathematics as they inquire about the reasoning of others and justify their own (Ball & Bass, 2003; Rojas-Drummond & Zapata, 2004). Students often are weak in their argumentation skills in this context, as they struggle with reasoning and justification due to a lack of opportunities to engage in problem-solving activities that encourage development of reasoning skills that enhance mathematical arguments (Mueller & Yankelewitz 2014). Teachers often use collaboration as a regular tool in the K-12 classroom, and the standards obviously support the idea of argumentation, but the use of collaboration for true argumentation is an underutilized tool in promoting student problem solving and mathematical reasoning via argumentation. Further, it still remains a challenge to effectively utilize collaboration in any K-12 setting because of the dynamics of working with various student opinions and mathematical abilities, but collaboration can be a tool in K-12 education to unlock students' abilities to deeply analyze the mathematics.

The idea of using collaboration to develop problem-solving skills is a research-based, student-centered approach to allow for mathematical argumentation development in the classroom, because it allows students to build knowledge, develop metacognition, and develop higher-order thinking skills (Cáceres, Nussbaum, Marroquín, Glesiner, & Marquínez 2018). Fostering meaningful and purposeful collaboration that supports student growth in these skills requires certain criteria for successful lessons at the secondary level.

Although the power of student discourse is recognized and the importance of mathematical argumentation is emphasized in both national standards and recommended practices of practitioners and researchers alike across several national mathematics education organizations, there still remains extensive research to be completed to support the role that discourse plays in promoting the advancement of K-12 student mathematical arguments (Brown, 2017; Byrne, 2013; Yee, Boyle, Ko, & Bleiler-Baxter, 2017). This study analyzes the roles of collaboration and discourse K-12 mathematics classrooms, and the impact and conceptualization of discourse on students' development of mathematical arguments. Further, the project will study the act of argumentation as a social practice and recognize the specifics of collaborative discussion that impact students to advance in their abilities to create their own mathematical arguments. Finally, this study will conclude with recommendations for future research and practice to support K-12 teacher understanding of how best to encourage meaningful discussion among their students in the form of interpersonal discourse to promote mathematical argumentation.

Student Collaboration

The implementation of student collaboration through discursive activities has been a regular practice in K-12 mathematics classrooms. As students work with one another, they are

given the chance to share their thinking with one another, which provides them the opportunity to individually and collectively approach problem-solving in a way that both challenges and supports all types of students and different learning styles (Hodge & Walther, 2017). To emphasize student thinking and improvement of their own mathematical processes and thinking, collaborative efforts among students emphasize their discourse with one another and the impact that student conversations have on student learning and progression of mathematical thinking. According to both researchers and practitioners in K-12 mathematics, mathematical discourse lies at the intersection of social, emotional, and mathematical intelligence (Bertolone-Smith & Gillette-Koyen, 2019). Thus, viewing mathematical discourse, as a social practice highlights the value of student discourse in the promotion of national standards and mathematical practices, particularly the creation of mathematical arguments. A primary focus in K-12 mathematics classrooms by national mathematics education organizations such as the National Council of Teachers of Mathematics and the Common Core State Standards, both NCTM (1989) and CCSS (2010) recommend that reasoning and proof should regularly be incorporated in K-12 classrooms. Although argumentation has been established as a recommendation for K-12 mathematics, classroom teachers still find it difficult to promote student discourse so that students successfully engage in argumentation through justification and generalization (Melhuish, Thanheiser, & Fagan, 2019). Thus, this review of literature delineates the role of student collaboration through the lens of student discourse and the current state of mathematics education in utilizing discourse for argumentation in K-12 mathematics, while also providing implications for the K-12 mathematics education community and recommendations for future practice and research.

The concept of student collaboration establishes classrooms as communities of inquiry where students are able to build knowledge through collaborative group work and the dialogue with one another (Kendrick, 2010). In fact, a classroom that successfully facilitates student collaboration maintains that the teacher facilitates student thinking through orchestrating student-to-student interactions in a climate that promotes such collaboration (MacDougall, 2013). Within the mathematics education community, it has become a common teaching practice to facilitate classrooms that promote an environment that not only encourages students to discuss with one another in collaborative efforts, but in fact requires it through the design of the classroom activities or experiences. In this design, students share their ideas about the content concepts with each other, particularly in STEM fields such as science and mathematics (MacDougall, 2013), especially as students are required to justify their thinking and reach a consensus.

The idea that student collaboration leads to justification via evidence and the necessity of critiquing the reasoning of others' arguments or ideas directs back to the standards of mathematical practice encouraged and published by the National Council of Teachers of Mathematics (NCTM, 2000) and the Common Core State Standards for Mathematics (CCSSM, 2010) that establish the standards and widely-accepted practices of K-12 classrooms today. As students work with one another to justify their own arguments and critique each other's ideas in alignment with these recommended practices, K-12 STEM teachers highlight the necessity of empowering students to be able to use their own strengths via student collaboration by appropriately implementing these practices that keep students accountable and hold them to the expectations not only of behavior in the classroom, but also of high standards for academic learning in the context of the content of the classroom lesson (McGlynn & Kozlowski, 2016). Because both creating and critiquing an argument in a K-12 STEM classroom is both

academically challenging and socially engaging, standards for accountability for the classroom task hold students accountable to their individual work and to the group as they work together towards this mathematical practice of argumentation. McGlynn and Kozlowski (2016) recognize that collaboration among students in their groupwork with one another requires students to hold each other accountable to be involved, to have a valid role or job with a set standard of completion, to be invested in the task or learning goal, and to maintain accountability both individually and collectively. In the context of mathematical argumentation, the creation of an individual argument and the critique of another's argument provides students the role with a set standard of completion (both creation of an argument and a critique to challenge the justification of one's rationale) while also holding students accountable to themselves and to each other as they create and critique each other. By the design of this task, students must be invested in the task because the claim of a mathematical argument sets the mathematical standard and the learning objective for the mathematics lesson so that students can find an entry point into the lesson for the day. Overall, collaboration is seen as an action that promotes students to work with one another and maintains high standards of investment and academic work from all students both individually and collectively as they work in groups to create and justify their arguments.

The Concept of Student Discourse

The concept of student discourse is prioritized by the National Council for Teachers of Mathematics (2000) in the establishment of the communication standard that states: "Instructional programs from pre-kindergarten through grade 12 should enable all students to organize and consolidate their mathematical thinking through communication, and communicate their mathematical thinking coherently clearly to peers, teachers, and others." As researchers have studied the power of student discourse in mathematical learning, they have come to realize

that the social activity of student collaboration and communication creates mathematical understanding from mathematical context. More specifically, the power in student discourse lies in the transformation of language of/for mathematics into knowledge of mathematics that is gained socially, known as "sociomathematical knowledge," through the classroom activity that is implemented by the classroom teacher (Ochs, 1998). This concept of transformation of mathematics language into mathematical understanding through carefully facilitated instruction by the K-12 teacher has guided several specific practices that emphasize a greater understanding and an end goal of acquisition of mathematical content.

However, research is more recently emphasizing the importance of student discourse as the end product of mathematics-rich student conversations in themselves. The recognition of discourse as a social practice ties together the social aspect of student discourse with the mathematical content, which remains the goal in K-12 mathematics education research. A closer look at student discourse as students work in small groups with one another unpacks the role that discourse has on student development of mathematical understanding according to the national standards and mathematical practices that are regularly expected of K-12 mathematics classrooms.

In looking at student discourse as a vehicle to the development of mathematical arguments, Kuhn and Udell (2003) suggest that argumentation is a skill that is not only essential to the critical thinking required in formal education, but also as a means to developing thinking skills in general. The basis in this concept is that the ability to propose, critique, and defend a position is universal across all domains of knowledge, mathematics-related or not. Thus, it is necessary that students learn to engage in topics of discussion with one another where they are required to not only pose their own thoughts, but also then critique the reasoning of others, as

recommended by the NCTM Standards (200) and the CCSS Standards (2010). The call for collaborative student discourse for the sake of argumentation is recognized and heavily emphasized in current mathematics education.

To expound on the necessity of mathematics collaboration for argumentation, discourse analysis reveals that students are capable of using, learning, and expounding their thoughts about mathematical concepts in a variety of different ways, particularly as they discuss mathematics with one another in classroom activities (Huang, Normandia, & Greer, 2005). More specifically, students are able to describe, sequence, and choose their thoughts about mathematical processes as they engage in discourse with one another initially. As teachers intervene with guided practices to promote advancement in mathematical thought and argument development, students are able to reference other truths about mathematical content, explain a method, and/or justify a decision made for a method or solution in their mathematics (Huang et al, 2005). Thus, appropriate understanding and conceptualization of discourse for the development of argumentation to promote student mathematical learning continues to be an area of research that can impact K-12 mathematics instruction.

Argumentation in the Development of Student Mathematical Learning

Though argumentation has consistently been a recommended practice of K-12 mathematics, students are generally unable to produce valid arguments (Stylianou, Blanton, & Knuth, 2010), and current methods of teaching proofs and arguments are largely inaccessible to K-12 students (e.g. Karunakaran, Freeburn, Konuk, & Arbaugh, 2014). In several K-12 mathematics classrooms, students learn to create arguments simply by watching the teacher's approach and attempting to recreate the same process. This type of instruction leads students to believe that the teacher is the supreme authority of an argument in terms of what is acceptable

and what is not. Thus, students develop an authoritarian proof scheme (Harel & Sowder, 1998), wherein the process of developing a proof becomes a computational exercise to find the specific solution for which the teacher is searching.

In contrast to the traditional teaching approach to teaching mathematical arguments, encouraging collaboration and allowing students to engage in the process of proving can improve students' proof development (e.g. Brown, 2017; Byrne, 2013; Yee, Boyle, Ko, & Bleiler-Baxter, 2017). Collective argumentation, as one example, allows students to engage in a process where they collaborate to create arguments and come to agreement on the arguments that can be accepted within a community. Students create arguments within a group and present their work to the class, which gives the students a chance to gain ownership and authority (Stein, Engle, Smith, & Hughes, 2008) in their work and participate in an authentic mathematical learning community.

For effective implementation of collective argumentation within the mathematical learning community, student collaboration must occur via student discourse. Research has shown that mathematical communication within a classroom community is crucial for the development of students' reasoning and mathematical understanding (Alrø & Skovsmose, 2003; Forman, 2003). Lampert and Cobb (2003) argue that providing students the opportunity to discuss their ideas with others can develop their mathematic reasoning more readily. According to Howe and her colleagues, the most successful instances of collaboration occur when students propose and defend their ideas and when they explain their reasoning to each other (Howe et al, 2007). Further, Howe et al. (2007) discovered that collaboration was most productive when the teacher offered little intervention and allowed students to exercise their own authority in solving the mathematical tasks proposed to them.

Teacher Use of Mathematical Arguments

To discover more about the role of the teacher in fostering a community that encourage collaboration to develop student learning via mathematical arguments, Mercer (2008) builds upon the research of Howe et al. to explain that the teacher's role should be one where he or she guides the students in creating mathematical arguments. In this capacity, the teacher assists the students as they learn to collaborate effectively and utilize "exploratory talk" as a cultural and psychological tool to contribute to their development of reasoning (Mercer, 2008; Mercer, Wegerif, Dawes, & Sams, 2004). Exploratory talk is the idea where partners engage critically but constructively with each other's ideas. Their statements and suggestions are considered jointly, as they challenge and counterchallenge, requiring justification and alternative hypotheses (Mercer et al., 2004). Exploratory talk holds students accountable to reasoning.

To create an atmosphere where exploratory talk is commonplace in the mathematics classroom, Brown (2017) encourages teacher participation in a way that guides and pushes student thinking, as they listen to and observe the activities of students in their small groups, as in the aforementioned collective argumentation model. The observation of activities can inspire students by then challenging them to engage in different types of representations, explanations, and justifications (Brown, 2017) than what they had previously created. Brown (2017) continues that the teacher can do this by asking questions about representations, adding to the representations, or even by providing his or her own personal representation. The active role of the teacher can create an environment where students are not only accountable to developing viable mathematical arguments, but they also are inspired to actively engage in them as they are challenged to create new representations of the mathematics.

Discourse in Collaboration of Mathematical Arguments in a Mathematical Learning Community

Discourse in the classroom is dependent upon the social setting of the classroom and can have multiple meanings, involving more than language (Gee, 1996; Moschkovich, 2007). Moschkovic (2007) contends that discourse also involves representations and behaviors, which involves collaboration about arguments and proofs in the context of mathematics classrooms. The discourse of a mathematics classroom is important to note, then, because the language, representations, and behaviors of the teacher and the students may differ in such a way that individuals will have varying interpretations, meanings, and foci of attention in the context of the particular mathematics tasks being studied.

During the act of collaboration in a mathematics class, collective mathematical understanding may take place when students work together on one mathematical task (Martin, Towers, & Pirie, 2006). This collective understanding requires the social context of the learning environment, and it cannot be described by looking at the actions of the individual learners. Through the process of working jointly on a problem, problem-solving leads students to share ideas and their ways of solving, so individual understanding becomes shared. Teachers should establish a classroom environment that encourages this type of discourse where students will jointly partake in a discourse that transforms individual student thinking about mathematics due to the collective understanding that takes place via student language, patterns, behavior, and interactions with the mathematics as well as each other. This study pursues the nature of the small group discussions and the class-wide, whole-group consensus in collective argumentation, hoping to clarify the elements of these discussions that best encourage student learning through the development of mathematical arguments.

As represented in previous research, the power of student discourse in promoting mathematical argumentation is recognized and encouraged in various classroom practices across all levels of K-12 mathematics. However, an important factor to note for the success of implementation of social practices to enhance mathematical argumentation is the assessment of student engagement and understanding in social discourse by the teacher. Thus, it is necessary to understand how teachers conceptualize and implement discussion as a social practice to enhance mathematical argumentation.

As a first step to promote student justification and generalization in mathematical argumentation, Melhuish et al. (2019) establish that teachers should be able to recognize when students are truly engaging in the activities that teachers use to implement successful discourse for mathematical argumentation. Even deeper, Melhuish et al. (2019) recognize that as a mathematics teacher community, educators must commonly agree on the definition of argumentation through justification and generalization, which they define as showing why an idea/solution is true, refusing the validity of an idea, and giving mathematical defense of an idea that was challenged.

Some research has combined with practical steps and recommendations for K-12 mathematics teachers to enhance purposeful student discourse in developing mathematical arguments. As one example that emphasizes the classroom as a community that engages in the social practice of student discourse, Ko, Yee, Bleiler-Baxter, and Boyle (2016) have recognized that a before-during-after instructional sequence would best give students the chance to use each other as a resource as a classroom community as they work via their social discourse towards a common goal of developing advanced mathematical arguments. In this framework (Ko et al., 2016), before engaging wholeheartedly in group discussion with one another, students

individually create their own argument to a mathematical claim that is the center of the mathematical learning activity. This can often be done as homework or in a previous class. Then, during the activity, students evaluate the validity of arguments based on classroom criteria in small and whole groups, emphasizing discourse with one another, and then after the class they critique reasoning by creating a rubric and using that rubric to critique each other's arguments (Ko et al., 2016). By engaging students in the process of creating a rubric for critique of arguments, students now have a standard by which they can make generalizations and justifications for their arguments, and they know how to properly engage in discourse with one another that is focused on the appropriate K-12 mathematics activity.

Melhuish et al. (2019) recently developed the SDOT, which stands for Student Discourse Observation Tool, as an effective measure to promote teacher noticing in student engagement in mathematical practices that advance their thinking and, particularly, their argumentation skills in K-12 mathematics. This tool serves as just one example of a practical support to effectively implement mathematical discourse to promote K-12 student development of argumentation through justification and generalization. As research pushes forward in mathematical discourse and argumentation, more tools and instruments for practical implication of mathematical arguments for practitioners will be developed to enhance student discussions and social practices in K-12 mathematics classrooms.

Argumentation

Student creation of mathematical arguments has been established as a primary focus in K-12 mathematics classrooms by national organizations and the recent creation of national standards for mathematical practices. The National Council of Teachers of Mathematics (1989) recommended that reasoning and proof should regularly be incorporated in K-12 classrooms,

which was emphasized again in 2010 by the Common Core State Standards (CCSS) publication of eight mathematical practice standards in which students should consistently engage (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The Common Core State Standards specifically define an argument as the ability to make both claims and counterclaims, develop them, and link them to reasons and evidence in a cohesive format that ends with a logical conclusion (Shivers, Levenson, & Tan, 2017). Although argumentation has been established as a recommendation for K-12 mathematics, because several mathematics curricula and teaching practices have associated the concept of proof and argumentation with upper levels of schooling (Stylianides & Stylianides, 2009), classroom teachers still lack in skills necessary to foster student development of veracious mathematical arguments in a way that satisfies these standards. The review of the literature considers the definition of argumentation and its application at the K-12 level, including the components of an argument and an analysis of the purpose of arguments in K-12 education. Specifically, the review emphasizes mathematics classrooms and the environments that support student learning and understanding of mathematics through proof and argumentation.

Argumentation in K-12 Education

The ability for students to develop arguments in general has consistently become a regular mode of learning for K-12 classrooms. The ability to create, defend, and challenge others' arguments points to the idea of discovering the truth (Joshi, 2016), which lies foundational to academic learning for K-12 students. Joshi (2016) expounds on this idea of truth as the cornerstone of education in arguments by stating that beliefs come from arguments, and the students' belief directly influences student knowledge and their learning. Further, the act of argumentation develops student conceptual understanding and critical thinking, both skills that

are critical in all academic realms, but particularly STEM fields such as mathematics education (Nussbaum, 2011). In looking at how argumentation has developed throughout various disciplines of K-12 education, Nussbaum (2011) points generally to the idea that argumentation, psychologically, is more focused on the process and the social interaction in which students engage to both construct and critique arguments. The emphasis on a social structure and the social interaction of students with one another lays the foundation to the value of student discussion with one another to develop arguments for learning at the K-12 level. To go deeper into this idea, O'Keefe (1982) distinguishes between two different senses of the word "argument," where one considers the idea of producing a final decision or statement as an argument and the other refers to the social processes in which students (at minimum of two) must engage in a dialogue to both construct and critique their arguments. Considering both of these definitions of the word "argument," mathematics education researchers and scholars recognize the value of the process of the formation of an argument through the dialogue students have to construct and critique their final decisions or considerations about mathematical truths. Different types of arguments have been defined and determined throughout educational psychological research, but the importance of recognizing the role that dialogue plays in the process of development an argument is crucial to K-12 education and mathematics and STEM education.

Much argumentation research refers back to the purpose of argumentation at the K-12 level. Since one major purpose of argumentation is analytical understanding, one of the most widely accepted models of argumentation is Toulmin's model, which breaks down arguments according to an analytical framework while also helping judge the strength of an argument and describing the claims and explanations that qualify the strength of an argument (Nussbaum, 2011). The philosopher Stephen Toulmin first introduced his model in 1953 in his book *The Uses*

of Argument, where he provides the six major components of an argument: claim, grounds, warrants (linking grounds to claims), backing for warrants, rebuttals, and modal qualifiers (Nussbaum, 2011). These components of Toulmin's model, which Toulmin re-emphasized in an updated version of his book *The Uses of Arguments* in 2003 to apply to arguments in several different purposes, are the driving forces behind future research in mathematics education's implementation of argumentation in several different manners, as the research emphasizes students socially interacting with one another in dialogue with one another to create and critique arguments according to these six major components of an argument of Toulmin's model.

To elaborate on Toulmin's model, Brockriede and Ehninger (1960) identify the major components of Toulmin's model and the role that they play in advancing student critical thinking and thus understanding. Firstly, Brockriede and Ehninger (1960) recognize that Toulmin defines an argument as movement from accepted data to a claim through a warrant. The necessity of believing accepted data and having to make a claim through evidence gained in the warrant encourages students to consider the truth as they certify assumptions and approach their understanding from the viewpoint of having to be able to justify and defend what is known as truth. At the heart of this process, students are beginning to learn how they must think on their own, carefully analyzing various claims of truth and making connections between different types of information to create a logical connection from data to a claim. Particularly concerning mathematics education, students approach making mathematical proofs as arguments as they consider the truths that believe and know in geometry, number skills, and algebra, to name a few (Metaxas, Potari, & Zachariades, 2016). By developing the skills of justifying, challenging, counterchallenging, and conceding, students are able to improve the quality of an argument in that they are offering their own warrants and backings by engaging in the process of

argumentation (Metaxas et al, 2016). The idea of using counterarguments or rebuttals, which is the fifth of the six components of Toulmin's model of arguments, challenges students to find exceptions to what they have already considered and the data that they have accepted and utilized for their claims, requiring students to more deeply analyze the truths and data that they have considered and other ways to conceptualize this information. The different elements of Toulmin's model directly incorporate into the critical thinking that lies at the heart of mathematics education.

Further, research in terms of winning oral arguments in dialogue and speech has elaborated on the credibility of Toulmin's model for argumentation. Brockriede and Ehninger (1960) make specific claims to the value of Toulmin's model because the components require backing of arguments and rationale, which are the major components that students engage in to develop the critical thinking and the conceptual understanding that lay at the heart of K-12 education, particularly in mathematics education. As students must back their arguments with logic and consider other opinions and viewpoints through proposed rebuttals, students must engage in conversation with another to hear another opinion. Nussbaum (2011) states that Toulmin's model was first used in the formation of arguments in speech and communication courses, but the same principles of development of claims and then defense of rebuttals applies in mathematics classrooms.

In looking at other disciplines as they approach an appropriate model for argumentation in K-12 education, the main tenets of teaching an argument still lie in a student's interaction with both the content material and his or her classmates. The Perception, Interpretation, Expression model, or PIE model, reflects how students' change their minds as they develop, critique, and modify an argument after the intervention of working with one another (Shivers et al, 2017).

This process recognizes that students initially conceive and develop an argument in phases. In the first phase, students perceive and describe what is presented before them (Shivers et al., 2017), which in mathematics argumentation, is the known and accepted data or facts. The second phase is when the students build the argument based on their own perceptions and descriptions, which is reflected in Toulmin's necessity of warrant to back the arguments that students develop in mathematics classes (Shivers et al., 2017). Finally, the final phase occurs when students critique, revise, and structure their arguments differently as a result of their creative thinking and interactions with the content and with one another, particularly as they justify their ideas so that their argument is considered successful, or grounded in clear reason and supported claims (Shivers et al., 2017). This model recognizes that argument do not all have to look the same, even if students do progress through this model of perception, interpretation, and expression equitably. This is important to note in the field of mathematics education because a broader understanding of the phases of argumentation that allows students to meet the criteria of Toulmin's model creates the opportunity for mathematics education to occur at all levels in K-12 education.

Underlying Toulmin's model and the PIE model of argument development is student collaboration, as they utilize their own discourse to perceive information, interpret different accepted information and data in mathematics, and express their own thoughts and understanding via mathematical arguments. Collaboration, then, remains a necessary process that allows students to view their own beliefs and actions differently and, according to Toulmin's model, modify their arguments as necessary as they develop a deeper understanding via critical thinking and collaborative work. Combining Toulmin's model with the PIE model, this model of argumentation was used for the purposes of visual critiques in artistic expression through art and

digital media classes, but the social process of argumentation remained vital to the student development of critical thinking and deeper conceptual understanding of the content material.

As a final characteristic of teaching argumentation in K-12 schooling in general throughout all disciplines, Shivers and colleagues (2017) recognize the power of argumentation development across different modes of student expression to best benefit the students' understanding and learning. Firstly, Shivers recognizes that writing is one mode that helps students develop argumentation skills, but it reiterates that writing should point to the overall goal of student thinking, which is what the student collaborative oral arguments does to support their writing. Secondly, Shivers (2017) explains that argumentation through art education required students to persuade others of their points of view and their knowledge of artistic qualities, which is similar to mathematics education in that persuasion lies at the heart of the definition of argumentation in the context of mathematics education. Finally, looking at the more practical approach of how to incorporate argumentation skills in the K-12 setting, Shivers (2017) agrees that a more open-ended task without a finite correct single answer for which the teacher is looking is best to promote student discovery and the dialogue necessary for rich dialogue among K-12 students. These tenets of argumentation of art education ring true for STEM education, as all education at this level aims to produce critical thinkers and deep conceptual understanding of their discipline and content material.

Arguments in the Mathematics Classroom

With a foundation in recognizing what are the values and the teaching strategies appropriate for K-12 argumentation, research has identified certain characteristics or aspects of arguments specifically for mathematics classrooms that drive pedagogical practices and student learning in today's K-12 mathematics classrooms. In regard to mathematics arguments, many

researchers, scholars, and teachers agree that math arguments largely also lie in the field of mathematics proofs. Proof is one of the core aspects of mathematical activity, which is partially why math argumentation and proof has become such a priority in national standards and expectations today (Ball, Hoyles, Jahnke, & Moshovitz-Hadar, 2002; National Governors Association/Council of Chielf State School Officers, 2010; Harel & Sowder, 1998). As an example, argumentation schemes have been studied with the use of the components of Toulmin's model to study how students have moved from a simple example in a proof to a more rigorous, procedural algorithm to make a more advanced proof or mathematical argument (Metaxas, Potari, & Zachariades, 2016). Even further, Harel and Sowder (1998) developed a framework for proofs to help qualify the cognitive level of student mathematical proofs, which reflects the characteristics of an advanced mathematics argument according to Toulmin's model. Stylianides & Stylianides (2009) explain the analytical framework for reasoning-and-proving that supports student understanding of the development of a viable mathematical argument through the process of reasoning through a mathematical problem and proving its validity via various components of a mathematical argument. The value of this framework is the clear distinction of the qualities of a good mathematical argument that meets the criteria of an argument according to Toulmin's model within the context of K-12 mathematical learning.

In this framework, Harel and Sowder (1998) recognizes that students will often rely on specific examples as an entry point into thinking about the field of mathematics and the content proposed in this claim. Known as an empirical argument, Stylianides & Stylianides (2009) acknowledges the weakness in a solely empirical argument as it does not meet the criteria of an argument being true for all cases mathematically, which elicits the necessity of backing of an argument in Toulmin's model. They continue in acknowledging that K-12 mathematics teachers,

particularly at the elementary level, must also consider how best to teach students to move beyond an empirical argument. A secondary type of argument that lacks in the evidence required in a strong mathematical argument is one that appeals to authority, where students rely on authority of someone who is considered higher or more knowledgeable in the field of mathematics or upon accepted data itself as the warrant or evidence behind the claims in their arguments (Harel & Sowder, 1998). Although the students are seeking to make a conclusive statement about the mathematical claim, the lack of strong evidence prevents the argument from fully satisfying the components of a strong mathematical argument according to Toulmin's model.

Ascending the hierarchy of mathematical arguments, Harel & Sowder recognize other types of arguments that more robustly satisfy the components of a veracious mathematical argument, but they are lacking only in some of the validity of the evidence. In this next level, students either inaccurately use accepted mathematical information to create a generalized mathematical argument or proof, or they use information or data that is largely inaccessible to the appropriate mathematics for their content and level, such as relying on calculations via technology or graphical representations that do not prove the algebraic reasoning or thinking appropriate for the class (Harel & Sowder, 1998). Arguments that use appropriate mathematics are considered arguments but not "proofs," which defines the highest level of an argument according to Harel and Sowder (1998). The most advanced level of an argument is one that correctly and accurately uses the evidence that is appropriate to the class and level of the student (Harel & Sowder, 1998). These types of arguments described in this framework have been hierarchized in recent research (King & Campbell, 2019), and is used to inspire the hierarchy

used in this research. Descriptions of the levels and types of arguments are seen in Table 2.1 below.

Table 2.1: King and Campbell (2019) Hierarchy of Mathematical Arguments

Level 1		**Level 2**		**Level 3**
Empirical	Appeal to Authority	Unsuccessful Valid Argument	Valid Argument, Not a Proof	Proof

Further research has been completed to better understand the students' conceptual understandings and misunderstandings so that practitioners can address these understandings to better pinpoint appropriate pedagogical tools to teach these skills, as well as to pinpoint misunderstandings that students can address (Healy & Hoyles, 2000). The study by Healy and Hoyles is unique but powerful in its application to mathematics education to better address how to support both K-12 educators and students in the development of mathematical arguments.

Because Healy and Hoyles (2000) researched student perceptions of mathematical arguments, they were able to dive deeper into the reasoning for and power behind different types of arguments that make argumentation more accessible to the K-12 student population. They recognized formal, algebraic, and narrative proofs (Healy & Hoyles, 2000) all as valid forms of mathematical representation of student learning when developing their own arguments, helping underline the necessity of both evidence according to Toulmin's model and student social interaction as they engage in narrative dialogue to develop their own thoughts to provide warrant and evidence to their arguments. With a broader perspective of appropriate mathematical arguments among diverse populations of students across K-12 students, Healy and Hoyles (2000)

promote various modalities and representations of arguments in collective argumentation. As evidenced by this research and the concept of mathematical arguments, the social practice of student discourse is necessary for K-12 mathematics argumentation.

Argumentation and Identity

Much literature has evolved in the past twenty years concerning students' mathematical identities as they develop within the mathematics classroom. At the heart of this research lies the belief that one's identity in academic contexts, such as the mathematics classroom, is found in the interaction of one's participation in their surroundings and the community around them. Particularly, Nasir (2002) recognizes that new practices in the mathematics classroom can lead to students' evolving identities in the classroom setting, as she observes that as students "experience changing (more engaged) identities, they come to learn new skills and bodies of knowledge, facilitating new ways of participating, which in turn, helps to create new identities relative to their community." Within mathematics classrooms, students begin to evolve their identities into those who have belief that they can do mathematics as a result of the skills and knowledge developed by participation and engagement within the activities of the mathematics class. Cobb and Hodge (2002), then, emphasize the power that this identity construction within a community context can have on a student's identity when these identities can be utilized as instructional resources, so long as the "focus is on potential continuities with the mathematical practices that the teacher intends will become established in the mathematics classroom." By utilizing students' evolving constructed identities through a situated perspective of community and engagement, mathematics teachers can untap the potential to involve students in a way that their mathematical identities can contribute to the enhancement of their mathematical learning.

Student Identity in the Classroom

Identity, as defined by Hand and Gresalfi (2015), describes "how one thinks about oneself and others while acting in the world." Although several different views currently exist in educational research concerning how students develop their identity within the context of their learning environments such as the classroom, all of them highlight the value of realization of the context in which learning takes place (i.e. the classroom), engagement in the community, and then production of an evolved identity formation. Thus, from an educational standpoint, identity becomes more refined into a definition about "one's participation in and across activities and the sense one makes of oneself in relation to these activities" (Hand & Gresalfi, 2015). Particularly, student collaboration and student discourse have been individually studied as activities that students actively participate in that can impact student learning, so from the perspective of identity formation, they can also be viewed as activities that impact change on student identity as well, particularly within an academic context such as the mathematics classroom. From a situated learning perspective, literature from several previous mathematics education studies have studied discourse in student collaboration to understand how students begin to make sense of oneself in relation to the process of mathematical collaboration.

It is important to take a deeper look into the setting where the practice of mathematical collaboration takes place in the context of student identity development. Looking deeper at the establishment of classrooms as social systems that are organized through regularities of shared practice (Gresalfi, Martin, Hand, & Greeno, 2009), literature also reveals the importance of the purpose of the classroom as a place to develop student competence, or "what students need to know or do in order to be successful" (Gresalfi, Martin, Hand, & Greeno, 2009). This idea of competence helps visualize the roles that students play in relationship to one another as they

engage in the activities that contribute to students' evolutions of mathematical identities. As students work with one another collaboratively in mathematical activities, they are able to ask others for help or check with other students to ensure that their own answers are correct (Gresalfi, Martin, Hand, & Greeno, 2009). Students' interactions with one another in this manner through the lens of competence reveals patterns of students' collaboration that contribute to students' development of mathematical identity.

Looking specifically at the interactions that students have in their actions in the environment of the mathematics classroom, Cobb, Gresalfi, and Hodge (2009) defined two specific constructs of identity that begin to transform as a result of the practices that occur with the intent of mathematical learning in the classroom: normative identity, or the identity of students as doers of mathematics that is established in the classroom, and personal identity, which is what students individually develop about themselves as a result of participation in the classroom activities. More specifically, Hodge and Harris (2015) clarify that personal identity within the mathematics classroom focuses attention on "understanding both how students regard mathematics and what they consider mathematics to be." Through this lens, research places value in students' perspectives as well as the classroom practices that create opportunities to learn. Additionally, mathematical personal identity celebrates individual student success stories (Hodge & Harris, 2015) and individual classroom experiences that different individuals experience in mathematics class. This perspective of personal identity reveals deeper insights into students' transformations of identity as a result of the activities built into the design of the mathematics classroom.

Classroom Design for Student Identity

As seen previously, the establishment of classrooms as social systems impact student identity development because of shared practices in the classroom (Gresalfi, Martin, Hand, & Greeno, 2009). Further, current research verifies the belief that classroom communities emerge interactionally and naturally through social activity of the students with respect to "curricular materials, pedagogical approaches, physical space and resources, and, importantly, interactional routines" (Gresalfi & Hand, 2019). By defining classroom community as a space that utilizes materials, pedagogy, physical space and resources, and the interactions between students within classroom routines, research considers the design of classrooms so that practically, teachers, researchers, and designers can best set up students in a community and setting where students can become learners and doers of mathematics.

The classroom design and the effect on student identity, then, opens up the door to understanding specific pedagogical approaches to mathematical learning. Previous research has shown how students, when engaged in mathematical activity specific to the classroom setting, take on certain views of authority and agency because of the classroom design and the interactions with other students socially and the mathematics cognitively. Students begin to view themselves as doers of mathematics differently when classroom design contributes to their transformation in these aspects of normative identity, but also the teacher may capitalize on student identity as a way to intervene and use classroom design as a resource to engage students in better mathematical learning.

Concerning argumentation and identity, a dearth or research remains to be uncovered concerning student identity development as doers of mathematics as a result of engagement in collective argumentation and development of mathematical arguments. The aforementioned study by Cobb, Gresalfi, and Hodge (2009) will inspire the study of student identity in this

research, as students will begin to develop both understandings of their normative identity as a result of the classroom social structure in collective argumentation and their personal identities as doers of mathematics. Due to the novelty of this research, more potential publications are forthcoming concerning how students view themselves within the social context of collective argumentation and their personal identities as doers of mathematics as a result of their collaborative argumentation.

Summary

Although much research has been completed on the frameworks for argumentation in K-12 mathematics and the usage of mathematical arguments to support student learning, there are still more implications for future research and practice that remain. Because argumentation has only been widely accepted and nationally encouraged recently in the past few decades, the mathematics education research realm still seeks answers to the questions about what causes students to develop deeper conceptual understanding and critical thinking via argumentation. Nardi and Knuth (2017) only recently questioned how to change classroom cultures to support mathematical proofs and arguments that produce long-lasting effects in student understanding. For the purposes of future research, longitudinal studies and studies of student motivation and engagement to create and critique viable mathematical arguments could contribute to practitioners' and researchers' understanding of what makes mathematical argumentation successful in K-12 classrooms.

Future research that emphasizes the use of student discourse with one another also stands to narrow in what aspects of the dialogue that students have with one another support student advancement of mathematical arguments. In referring to Toulmin's model's components of argumentation, future research could emphasize the actions, interactions, and reactions that

students have with one another in oral dialogue as they initially create, listen to and critique, and modify arguments in open-ended mathematical problems that push students to critically think and analyze various mathematical concepts across all K-12 levels. It is important to note that argumentation can be encouraged and use through all levels of K-12 mathematics when scaffolded correctly. As an example, elementary school mathematics teachers and researchers such as Rumsey and Langrall (2016) have promoted mathematical argumentation even at the elementary level with language supports, specified conditions, and manipulation of familiar content to the unfamiliar, to name a few concrete examples of teaching practices. Continued research studies on the insights of teacher practices to support K-12 student development of mathematical arguments can focus on the student learning, the elements of the discourse in the arguments, the elements of the arguments, and the teacher practices to promote student learning through mathematical argumentation.

CHAPTER 3

METHODOLOGY

This mixed methods study uses a situated learning perspective to examine how middle grades students utilize small group collaboration to develop mathematical arguments and, in turn, their own identities as doers of mathematics in the context of a classroom learning situation. Because this study situates students into small groups for the sake of mathematical argument development, I was able to analyze the data in a way that supports group collaboration and interpersonal discourse. The situated perspective, then, addresses the research questions in a way that best analyzes the data through the situated learning perspective to uncover findings and discoveries about collective argumentation and student discourse supported by the data. Particular points of interest are the advancement of the levels of mathematical arguments, the elements of discourse within small group discussions that support mathematical argumentation, and the impact of small-group argumentation on students' mathematical identities. With this intention, I will apply methods from ethnography as supported by Wolcott (2010) to develop themes from data provided by the middle grades students about their arguments and their developing mathematical identities. Further, this mixed methods approach with ethnographic methods accounts for the participation of all students in the study so that equity and access to the mathematical learning are considered for this study, as Cobb, Hodge, and Gravemeijer (2011) state that equity should be considered to some extent in all research studies in mathematics education. This chapter provides descriptions of the mixed methods study that draws on ethnographic methods, the study participants, instructional sequence, and the procedures I will follow for data collection, data analysis, and ensuring validity and reliability of my research findings.

Mixed Methods Study

This study is a mixed methods study, utilizing a quantitative analysis to understand the advancement of mathematical arguments and qualitative analyses in analyzing the small-group discussions and the students' identities. Specifically, the mixed methods study utilizes ethnographic methods, as ethnography is the "study of the culture of a group," but particularly as that culture is revealed "through the course of ongoing events" (Preissle & Grant, 2004, p. 164). Ethnography emphasizes an insider's view of a specific group of people (Preissle & Grant, 2004), thus is often becomes known more generally as fieldwork. Fieldwork, as characterized by Michael Crotty (1998), outlines studies that range on the continuum from realism to idealism, with constructionism falling in the middle of the two. Because fieldwork largely represent social studies fields, scholars at the realism end of the objective believe in collecting systematic empirical research to provide a coherent view of the social setting and the culture of the people group of interest, but at the same time Crotty (1998) posits that social conditions are fluid and able to change, ascribing to the idealism position that reality is a creation of the human mind. This explains how constructionism, in the center of the continuum, very often describes ethnographic studies as knowledge becomes known as a result of "meanings developed in social contexts" (Preissle & Grant, 2004, p. 174). Ethnography as fieldwork, then, becomes more prevalent in classroom observation studies, as the researcher often has invested interest and experience within the classroom and other educational settings. This makes ethnographic methods appropriate for this particular aspect of the study because of my background as a mathematics classroom teacher for a decade, which will influence my perspective as the researcher (Jones, Holmes, Macrae, & Maclure, 2010). Particularly, my focus in this study is the impact of the interpersonal discourse that students have with one another in mathematics class on

both their mathematical arguments and identities. By utilizing this mixed methods study with ethnographic methods to better understand how students develop knowledge within the social context of the small groups in mathematics class, this theoretical framework ensures that equity is considered so that all students feel that they can contribute to the discussions centered around the mathematics content in developming mathematical arguments.

Thus, mixed methods with ethnographic methods remains an appropriate methodological approach to the analyses of the data collected in this study. The scoring of the improvement of mathematical arguments according to a hierarchy of mathematical arguments is the quantitative analysis of the mixed methods study, whereas the ethnographic methods are embedded into the coding of the themes of the small group discussions and surveys in the qualitative analyses. As stated before, this approach assumes the researcher is part of the research process, so the researcher attempts to examine data by observation of the participants "to develop a holistic understanding of the phenomena under study that is as objective as possible" (Dewalt & Dewalt, 2002, p. 92). By using fieldwork such as student work, audio recordings of student discourse, and survey results, this study uses ethnographic methods in the qualitative aspect of the mixed methods study because of the attempt to best understand how the social setting within the classroom of small-group discussion impacts the arguments and identities of mathematics students. This approach can provide opportunities to gain insight into how students use small-group discourse in argumentation to develop their mathematical arguments and their mathematical identities.

Research Context

Conducting this mixed methods research with ethnographic methods requires a look into the social context of the study (Becker & Geer, 1957). In the following paragraphs, I describe

both the school and my pre-existing relationship with the school, along with the data collection procedures via the planned instructional series and the data analysis.

Participants

Participants in the first iteration of the study include the approximate 40 eighth-grade students enrolled in one of four different Pre-Algebra classes taught by the same eighth grade math teacher at a private school in the Southeast United States during the 2018-2019 school year. The second iteration include the approximate 35 eight-grade students enrolled in one of four different Pre-Algebra classes taught by the same eight grade math teacher at the same private school in the Southeast United States during the 2019-2020 school year. The students in the first iteration consisted of 22 boys and the remaining participants were girls, and the second iteration reflected a similar breakdown of participants by gender (18 boys and the remaining participants were girls). All students were white in both iterations of the study. These courses with this one regular classroom teacher, henceforth known as Mrs. K, were chosen simply by an existing working relationship I have with Mrs. K at this K-12 private school. It is important to note that Mrs. K is a veteran teacher who practices much student engagement and active learning, with regular use of small groups in her classroom learning. Her practices, although very relevant to the standards of mathematical practice published by NCTM (2000), do not regularly involve students in developing mathematical arguments, so this study and the practices that made up the collective argumentation were novel to the participants in the study. This did not impact the study, though, as I did not teach or know any of the participants in the middle school setting because of my affiliation solely with the high school in the K-12 private school. The participants, although not necessarily recruited, have consented to the study, as well as provided parent consent for the study (Appendix A). I was the facilitator for the instructional sequence on

mathematical argumentation, but I was not the instructor of the course. All the students were taught by the same teacher, and they were accustomed to collaborating with one another on mathematical tasks, but not with an emphasis on developing mathematical arguments. This middle school class made sense for the context for this study because it focused on student discourse with one another, but the study's emphasis on argumentation revealed new cultural patterns about how students interact with one another in discourse in the context of mathematical argumentation in small groups. Understanding how students utilize discourse with one another in small groups can unlock great potential for mathematical argumentation (Rojas-Drummond & Zapata, 2004). Therefore, collective argumentation is highly relevant and an important study to contribute to the body of research on mathematical collaboration and argumentation. All participants engaged in the argumentation instructional series to develop mathematical arguments. In the second replication of the study only, students also completed surveys about their experiences in the small-group argumentation context.

The Instructional Series

For the purposes of my study, I facilitated each of the four class periods for three consecutive days of instruction (50-minute class periods), which made up the entire teaching series for the first iteration. On each day of instruction, students worked on one task (Appendix B), where they created a collective mathematical argument. To begin each class, students discussed the criteria to create a valid mathematical argument, and the criteria were written on a whiteboard to ensure that it was visible to each member of the class. This was the first five to ten minutes of class, depending on the amount of time allotted for class that day. Students were then presented with a mathematical claim, then they worked individually for three minutes, then in small groups for ten to twenty minutes, again modified for the amount of time in class that day.

Finally, as a whole class, each small group presented their collective argument to the whole class for a final round of validation so that students can practice not only creating, but also critiquing each other's mathematical arguments. This lasted anywhere between three to fifteen minutes in class, depending on the time remaining in class.

Within each class, students were placed into groups of three or four, and I implemented the instructional series as mentioned in above. The student desks were already arranged in small groups of three to five, facing each other for maximum communication with each other. The students sat in these small groups for the entire class period, but they were told to only work with each other during the small-group portion in the middle of the daily activities in the instructional series. The students were loosely guided through a PowerPoint presentation just to guide students in the discussions of the communal criteria each day, as well as to help students see the original mathematical claim. I also used the white board to help provide any examples and to remind students of the communal criteria, and the white board was also used in the final round of validation when the small groups presented their arguments to the entire class. As previously stated, students spent three minutes creating an argument on their own, then up to twenty minutes creating a collective argument within their groups. After creating collective arguments, groups shared their arguments with the class to be validated based on the criteria from the beginning of the teaching series.

The tasks that the students engaged in were similar across the two iterations of the study, with one exception in the second iteration. The first iteration largely focused on students' understanding of numeracy, utilizing facts and characteristics about sets of numbers that students learned in the beginning of a Pre-Algebra classroom. As an example, a mathematical fact or claim to prove true that is similar to the first two claims of both iterations of the instructional

series is “The product of three odd integers is even. Write an argument for why or why not. Explain.” In the first iteration, the third and final task was largely focused on geometrical knowledge, but it proved to be insufficient data for this study because of the students’ lack of precedent knowledge necessary for success and useful data in this iteration of the study. In the second iteration of the study, the last two (of four) tasks in the iteration asked students about the nature of solutions in a system of equations, as this was relevant to the current learning segment of the students with the regular classroom teacher Mrs. P; this was done at the request of Mrs. P. An example of a mathematical claim for this content is: “Two parallel lines have no solution. Write an argument for why or why not. Explain.” These tasks are examples similar to the content and the rigor of the tasks in both iterations of this study.

The second iteration of the study follows the exact same sequence, except that I was able to facilitate four consecutive days, but each class beyond the first class was only 30 minutes due to a modified assembly schedule for the rest of the week. This second iteration of the instructional series differs from the first in the differing amounts of time per class, as well as the survey given before and after the facilitation of the entire iteration of this study.

Data Collection Procedures

Data were collected to examine both the advancement of middle grade students’ mathematical arguments as well as the discourse that they engaged in to find what types of discourse contributed to the development of their mathematical arguments. Further, when this research revealed new possibilities to study, and I repeated the study for the purpose of analyzing student identity, surveys were given before and after the instructional series. The purpose of the survey was to better understand student mathematical identity and the possible shift in identity towards being a doer of mathematics as a result of the argumentation instructional series. To

understand the students' mathematical arguments and experiences, student work (including the mathematical arguments themselves), audio recordings of their small group discussions, and surveys were the primary data sources. Firstly, all student work data were collected. Next, the small group discussions were audio recorded and transcribed. Finally, in the second iteration of the study, surveys were given at the beginning and at the end of the instructional series for the sake of insight into student perceptions and identities before and after the implementation of the instructional series. The upcoming paragraphs provide details for each data source, including student work, audio recordings, and surveys.

Student Work

The activities that were completed during the instructional series include a collective argumentation survey before the instructional series in the second iteration of the study (Appendix B), student work of the arguments they create individually and collectively, audio recordings of their small-group discussions, and a collective argumentation survey after the instructional series in the second iteration of the study (Appendix D). Spradley (1980) states that an ethnographic record includes fieldnotes, tape recordings, artifacts, and anything else that documents the social setting of the study. Collecting student work, both individual work and group work, and reflections via surveys before and after the instructional series is an appropriate method to provide insight into the students' argumentation abilities and mathematical identities. These methods allow theories to be grounded in data provided by the students where they give their thoughts about the argumentation instruction. These data were collected using paper for the mathematical tasks and for the pre-surveys, and online surveys for the post-surveys.

In addition, the small group discussions were audio recorded. Each group had their small group discussions audio recorded to allow me to examine what types of discourse the small

group discussions utilized to develop more advanced mathematical arguments, as well as what could possibly impact student mathematical identity as a result of the small-group mathematical argumentation.

Data Analysis Procedures

The focus of data analysis is threefold: to understand how student mathematical arguments advanced as a result of collective argumentation, to analyze what types of discourse students engaged in to develop veracious mathematical arguments, and to discover the impact of mathematical argumentation in collective small groups on mathematical identity. As a mixed methods study, this study uses data analysis that consists of quantitative methods to consider advancement of mathematical arguments and qualitative methods with ethnographic methods to examine the discourse that inspired mathematical arguments as well as the surveys that reveal more about student identity. Creswell (2014) states that a mixed methods design such as this one can use the research question to better understand the content of the study. In this situation, the argument advancement explains how student discourse supports mathematical argument development among middle school mathematics students. In the following paragraphs, I describe the analysis for each research question within this study.

First, quantitative analysis of mathematical arguments and qualitative analysis of the instructional series were utilized to study how student arguments both increased as a result of collective argumentation and as a result of engaging in collective argumentation over time. Student collective arguments were coded for the first task, then collective arguments for the second task, and finally for the third task. To determine how students' arguments developed as a result of the discourse in collective argumentation, a coding system was developed. Students' collective arguments as evidenced by the audio recordings and written work were coded

according to Stylianides and Stylianides (2009) framework. Their framework consists of five hierarchical levels for judging the sophistication of a mathematical argument. Students create *non-genuine arguments* (Level 1) when they commit little effort. *Empirical arguments* (Level 2) are arguments that rely on examples as warrants for a mathematical claim. An *unsuccessful attempt at a veracious general argument* (Level 3) is an argument that uses general warrants but contains flaws. A *veracious general argument but not a proof* (Level 4) is an argument that uses a deductive chain to argue a claim but uses warrants that are not accepted by the mathematical community. A *proof* (Level 5) is a deductive argument that uses warrants accepted by the mathematical community. This coding system, as seen in the review of the literature, has been well-accepted in the proof and argumentation mathematics education learning community.

Using the above hierarchy of mathematical argument, I ran a dependent samples t-test of the scores of the small-group collective arguments on Task 1 and Task 2 to see if there was a statistically significant difference in the arguments as students had time to engage in collectively argumentation again after Task 2. Secondarily, I ran another dependent samples t-test for a statistically significant difference in the students' oral arguments on the two tasks, to study the impact of the oral discussions that students had in their small-group collaborations.

Secondly, I listened to the audio recordings and transcribed and coded them in two stages to better understand how students use discourse in developing mathematical arguments. Firstly, I utilized open coding to initially understand the mathematical concepts that students discussed within the context of the specific mathematical claim presented to them. As an example, several students discussed the mathematical fact that "the sum of two odds is an even" to identify specific concepts relevant to the mathematics for each mathematical task. I coded this into "odd + odd = even," to identify the major theme of this major mathematical fact. Because coding is

more an “interpretive act” instead of an “exact science,” (Saldaña, 2013, p. 4), open coding was particularly useful in the first phase of coding of the audio recorded small group discussions. In the second stage of coding, I utilized in vivo coding to pinpoint the mathematical concepts and the themes in the mathematical thoughts that the students engaged in according to Toulmin’s model. The codes in the vivo coding for this stage of the analysis are the six elements of Toulmin’s model, to better understand what elements of an argument the students discuss when working in small groups as they develop a collective mathematical argument. To build on the previous example, the math fact about the sum of two adds equaling an even became coded as a “backing” in Toulmin’s model of argumentation because this mathematical fact is the justification for the warrant of many students’ mathematical arguments. This two-stage process of coding identified the themes in the discourse that students engaged in during collective argumentation. Upon understanding the themes of the student discourse, the second phase of coding utilized in vivo coding, where I, as the researcher, began to more deeply understand how students appeal to different parts of an argument, using Toulmin’s model as the guide to the codes in the in vivo coding, to reveal deeper insights into the specific types of discourse according to Toulmin’s model that the students engaged in to best develop a mathematical argument. As I coded according to the in vivo codes, I wrote specific quotes from the transcriptions of the small group discussions into the right side of the table to correspond to the element of Toulmin’s model in the left side of the table. Table 3.1 shows examples of one group’s supporting quotes for understanding aspects of Toulmin’s model that students referred to in their argumentation.

Table 3.1 Examples of Supporting Quotes for Different Aspects of Toulmin's Model in Arguments

Conjecture: *The sum of two odd numbers is even*. Is this conjecture true? Write an argument for why or why not.	
Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	
Data (the facts from which the claim was based)	Oh, yeah, but if we did two different number, odd numbers, it's still even, but why?
Warrants (the justification for using the data to establish the claim)	It is true because I worked the problems out on the calculator to prove this, and it is true. More than that. Why? Because we've worked it out, and it's clearly that it will be an even number. OK, so we'll write some examples.
Rebuttals (statements that call the warrant into question)	Except if it's 3 plus 3. 3 plus 3 is an even number. 3 plus 3 is 6. Two – Oh wait. Oh. Two different or the same ones? Uh oh, that confused me. What, 3 plus 3? Yeah. 'Cause it's 9. But, 7 plus 7 is 14, 11 plus 11 is 22. No, that's not true because 10 plus 3 is 13.
Qualifiers (utterances that indicate the certainty of a statement)	So, in some cases with even numbers it's odd. But with odd numbers it's always even.
Backings (supports for the warrant that are usually unstated and related to the academic field)	Everything plus itself is even.

In all three analyses, selected quotes were shown to reveal the themes and subthemes that address each individual research question. By utilizing the themes discovered through open and in vivo coding, I was able to identify specific quotes that support each theme to illustrate the findings from the analyses. When providing the quotes throughout this dissertation, students were either identified as members of a specific group, where each group was labeled with a letter, or by a pseudonym.

Finally, open coding similar to that used in the second analysis was used in the surveys that students completed before and after the instructional series about mathematical argumentation in the second iteration of the study. By listening to and transcribing all the audio recordings, I was able to code the transcriptions to discover the themes and subthemes discussed in the findings of the analyses of this study. By identifying how students first felt as doers of mathematics, then following up with the students in a survey after the completion of the argumentation, these codes reveal themes into identity, including their valuations of their experiences with small groups and argumentation and how they regarded the key practices of argumentation as a result of the activities in the study. As a mixed methods study, the analyses have both quantitative and qualitative results. The results of the quantitative scores on the mathematical arguments show an increase in the arguments, and as a result, the idea that students increased in their mathematical learning. Further, the students improved their rationales and backings to the warrants of their arguments as a result of better argumentation processes of qualifying and providing rebuttals to others' arguments. This provides support for the claim of student learning through improved mathematical argumentation. I maintain interrater reliability by engaging in peer debriefing with a group of qualitative researchers, including two doctoral students familiar with this area of research. These qualitative researchers reviewed and asked

questions about the qualitative study throughout the data analysis process to ensure similar findings in all codes, themes, and subthemes.

Validity and Reliability Procedures

To validate and ensure reliability of the proposed study's findings, Creswell and Creswell (2018) recommended taking multiple steps. To check for validity in a qualitative procedure, Creswell (2014) recommended checking "for the accuracy of the findings by employing certain procedures" (p. 201), based on determining whether the findings were accurate from the standpoint of the researcher, the participants, or the readers (Creswell & Miller, 2000). By employing validity strategies, the researcher bolsters the trustworthiness, authenticity, and credibility of the study's findings. Creswell (2014) recommended using multiple validity procedures to assess accuracy of findings. I used four strategies he references to address validity. First, I engaged in peer debriefing with a group of qualitative researchers, including two doctoral students familiar with mathematics education. These qualitative researchers reviewed and asked questions about the qualitative study throughout the data analysis process. Second, for the quantitative aspect of the mathematical arguments, I had another doctoral student familiar with this area of research also analyze each mathematical argument and conduct a quantitative analysis for the interrater reliability of the argument levels. This form of member checking was conducted through follow-up interviews. Third, I provided "rich, thick description to convey findings" (Creswell, 2014, p. 202) with detailed descriptions of the study's context and examples of codes and themes to add validity to the findings. Lastly, I clarified the bias I brought to the study and how my interpretations of the findings are impacted by memo writing. This procedure allowed me to document how my interpretation of the findings was shaped by my background (Creswell, 2014) during the analysis process.

As for reliability, I used procedures suggested by Gibbs (2007) to check the consistency of my data analysis approach. Firstly, I ensured transcripts were accurate and did not contain mistakes made during transcription. Then, throughout the analysis process, I continually compared data with codes to make sure there were no inconsistencies in the definitions of the codes. Lastly, I cross checked codes with other qualitative researchers, consisting of the two doctoral students familiar with mathematics education, for intercoder agreement (Guest, MacQueen, & Namey, 2012). The same group of qualitative researchers who engaged in peer debriefing also cross-checked codes that we developed independently to determine the consistency of coding.

Summary

In using a mixed methods approach with ethnographic methods, I was able to examine the culture of student development of mathematical arguments in small group discussions, as well as the surveys that give insight into mathematical identity of the students. Firstly, a pre-survey analyzed student identity as mathematics doers before the argumentation instructional series. This instrument, along with all the others, are provided in the appendices. Student work, including written arguments and audio recordings of discussions in small groups, were completed by participants during the instructional series on argumentation and collected for analysis. After completion of the instructional series, post-surveys were distributed for analysis on identity similar to that of the pre-survey. Small group transcriptions and student work were analyzed using ethnographic methods. To better understand how the collected data reflects each individual research question, as well as the type of analysis used to understand the information in the data, refer to Table 3.2 on this page and the following page. This table also references more specifically the data sources by providing the appendices where the exact instruments are

located. The data analysis for each data source explains both the timelines during which the data was analyzed and a reference to the types of coding used for the qualitative analyses.

Table 3.2: Summary of Data Source and Analysis per Research Question

Research Question	Data Source	Data Analysis
In what ways do K-12 mathematics students use interpersonal discourse in small-group class discussions to create mathematical arguments and proofs?	Audio recordings and written arguments from small-group collaborative discussions (Appendix C)	Analyzed as collected prior to post-surveys as part of initial coding
How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	Audio recordings from (Appendix C) Pre-Survey data (Appendix B) *Questions pertaining to working in groups, making arguments, and talking with other students (Questions 2, 3, 4, and 5). Student work artifacts will be used for reference and guidance.*	Analyzed as collected prior to post-surveys as part of initial coding Analyzed prior to post-surveys as part of initial coding
How does student collaboration through interpersonal discourse in collective argumentation impact students' identity as doers of mathematics?	Pre-Survey data (Appendix B) *Questions pertaining to working in groups, making arguments, and talking with other students (Questions 1, 6, 7, and 8). Student work artifacts will be used for reference and guidance.* Post-Survey data (Appendix D) *Questions pertaining to working in groups, making arguments, and talking with other students (Questions 1, 6, 7, and 8). Student work artifacts will be used for reference and guidance.*	Analyzed prior to post-surveys as part of initial coding Analyzed as collected

CHAPTER 4.1

ANALYSIS AND FINDINGS

The purpose of this study is to understand the elements and design of small-group collaborative discourse in collective argumentation as students create and develop mathematical arguments in their classroom learning community. In looking at both the development of the mathematical arguments through discourse and the development of student identity, this study considered the following research questions: 1) In what ways do K-12 mathematics students use interpersonal discourse in small-group class discussions to create and advance mathematical arguments and proofs? 2) How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments? 3) How does student collaboration through interpersonal discourse in collective argumentation impact students' identity as doers of mathematics? This chapter details the themes and major findings for each research question. The first research question elicited three common themes for the first research question: advancement of a true mathematical argument, understanding of a true mathematical argument by the communal criteria, and student use of collaborative discourse to develop mathematical arguments.

The Instructional Context: Advancement of a Mathematical Argument

Before delving deeper into these themes that respond to the first research question regarding advancement of mathematical arguments, it is important to discuss the context of learning and the students' previous understanding of a mathematical argument that guided the analysis of the themes to answer this first question. According to the communal discussion concerning mathematical arguments at the beginning of the first day of each class, the students

had an elementary understanding of what defines a true mathematical argument, meaning that the students understood the purpose of a mathematical proof or argument was to provide evidence for the truth behind something. Beyond this understanding, however, the students did not know more about how to create a veracious mathematical argument beyond providing evidence. The purpose of the communal criteria is for students to better understand what constitutes an appropriate proof within the specific social learning context of their own learning classroom, and to recognize that negotiation with each other develops their abilities to make sense of the mathematics (Ko, Yee, Bleiler-Baxter, & Boyle, 2016). According to previous literature, I determined four major criteria for mathematical proofs or arguments appropriate for the middle grades level in alignment with the definition of a true argument amongst the broader mathematical community. This literature that emphasizes mathematical arguments (Boyle, Bleiler-Baxter, Yee, & Ko, 2015) should (a) utilize mathematical facts that are accepted by the classroom learning community, (b) be understandable by the classroom community, (c) follow a logical sequence, and (d) show that a statement is true for all cases.

As I introduced the instructional series on the first day, I explained to the students that the tasks require students to develop their own mathematical arguments, thus it was necessary that they have the understanding of what defines a mathematical argument. As a way to have the students comprehend the definition of a mathematical argument, I asked leading questions based on an example of winning an argument in general. Specifically, I asked them if they were to get into an argument with a sibling, what types of information would they present to their parents in their argument to ensure that they win the argument. I allowed the students to openly share their opinions, and after considering the input from several students, I explained that arguments and proving in mathematics means to convince oneself and others of the truth of a mathematical

claim, based on the definition of mathematical proof from Harel and Sowder (2007). After this explanation, with the preceding context of winning an argument with a sibling, in discussing what defines a mathematical argument, most every class stated the idea that they must be able to present some sort of facts or examples to support their reasoning, as opposed to personal opinion, in a mathematical argument. Similarly, this brought up discussion in some other classes about how they had to be able to prove that the other party is incorrect. As a class, we negotiated this criterion into the mathematics classroom context, where they stated that a mathematical argument must have facts or evidence to support their answer.

Other criteria that many of the classes created aligned well with the pre-determined criteria that I had devised before the discussion. With these student-generated ideas, I slightly altered the specific language that they had used to make the criteria more consistent with what is accepted in the broader mathematical community for a veracious mathematical argument. These other criteria include following a logical sequence and being understandable in terms of language for all members of the classroom community.

Beyond these three pre-determined criteria that the students had already developed, one incorrect criterion came up consistently that opened up the classroom to a deeper understanding of mathematical arguments as a whole. Every class explained that using specific examples to back up their argument are necessary for a veracious mathematical argument, but as the facilitator, I discussed with the students, using their examples, how one example is not enough evidence to support an entire claim in all cases. This may be because of the hypothetical situation I originally provided about the sibling argument to encourage students to think about how they support an argument, but this particular criterion of providing examples proved to be insufficient for a mathematical argument. The students were understanding and well-receptive of the idea

that examples are not sufficient for a mathematical argument, particularly when given an example of not being able to provide enough examples to prove a claim wrong if there are infinite numbers. I encouraged the students to use examples as a great beginning point to consider the validity of a mathematical claim, but then they must understand how to develop the arguments into a more general argument true for all cases. Finally, by all appearances from the whole-group discussions, the students all generally agreed, with some careful facilitation by me, that the criterion should, in fact, be that the argument must be true for all cases.

After this discussion, the students completed the rest of the instructional series, where they developed individual arguments, worked in small-groups via collective argumentation to develop a group-wide argument, then validated and justified their arguments through whole-class discussions. The students repeated this process each day of the instructional series, where the communal criteria conversation was a more succinct discussion reminding students of the criteria at the beginning of class after the initial day of development of the criteria.

In analyzing if the students developed a greater understanding and greater development of a mathematical argument, it is important to note the methodology of the research and the data that was studied. In this particular study, the complexity of the third analysis mandated an increased amount of scaffolding from myself as the facilitator, thus the arguments that the students developed did not provide meaningful data in regard to the students' collaborative discussions. Thus, only Tasks 1 and 2 will be reported for this part of the analysis. Task 1 asks students to write an argument for the truth or lack of truth to the claim that the sum of any three consecutive integers is divisible by 3. Task 2 asks students to write an argument for the truth or lack of truth to the claim that the sum of two odd numbers is even.

Upon completing the tasks, the final collaborative written arguments that the small groups wrote were analyzed. In responding to the research question, it is necessary to remember the hierarchy by which the student arguments were scored, as detailed in Chapter III in the methodology. The student arguments were scored a level 1 if the argument was not genuine, because it was either not interpreted correctly by the students or little effort was given to define an argument according to the classroom criteria. Level 2 arguments were empirical, or reliant on examples, and level 3 were unsuccessful attempts at a veracious general argument. Level 4 was a veracious general argument but not a proof, distinguished by acceptance by the general mathematical community, whereas level 5 is a genuine proof. Examples of each level of argument are provided in Table 4.1 below.

Table 4.1 Examples of Each Argument Levels 1 through 5

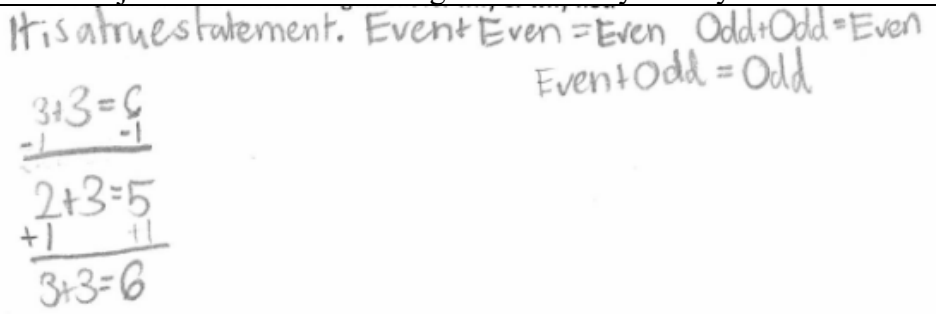
	Task 2: Conjecture: *The sum of two odd numbers is even.* Is this conjecture true? Write an argument for why or why not.
Level 1	

Table 4.1 Continued

Level 2	.

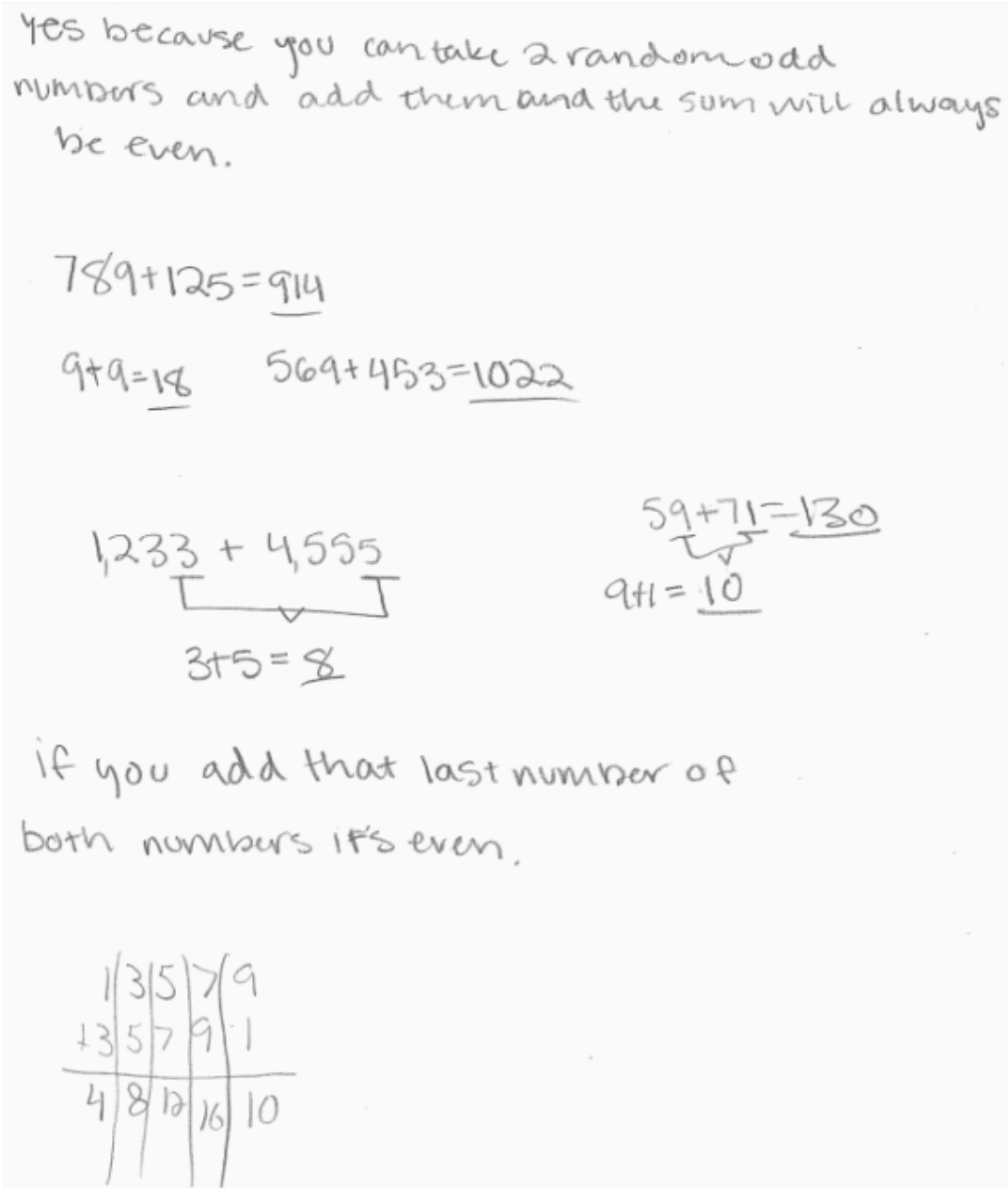


Table 4.1 Continued

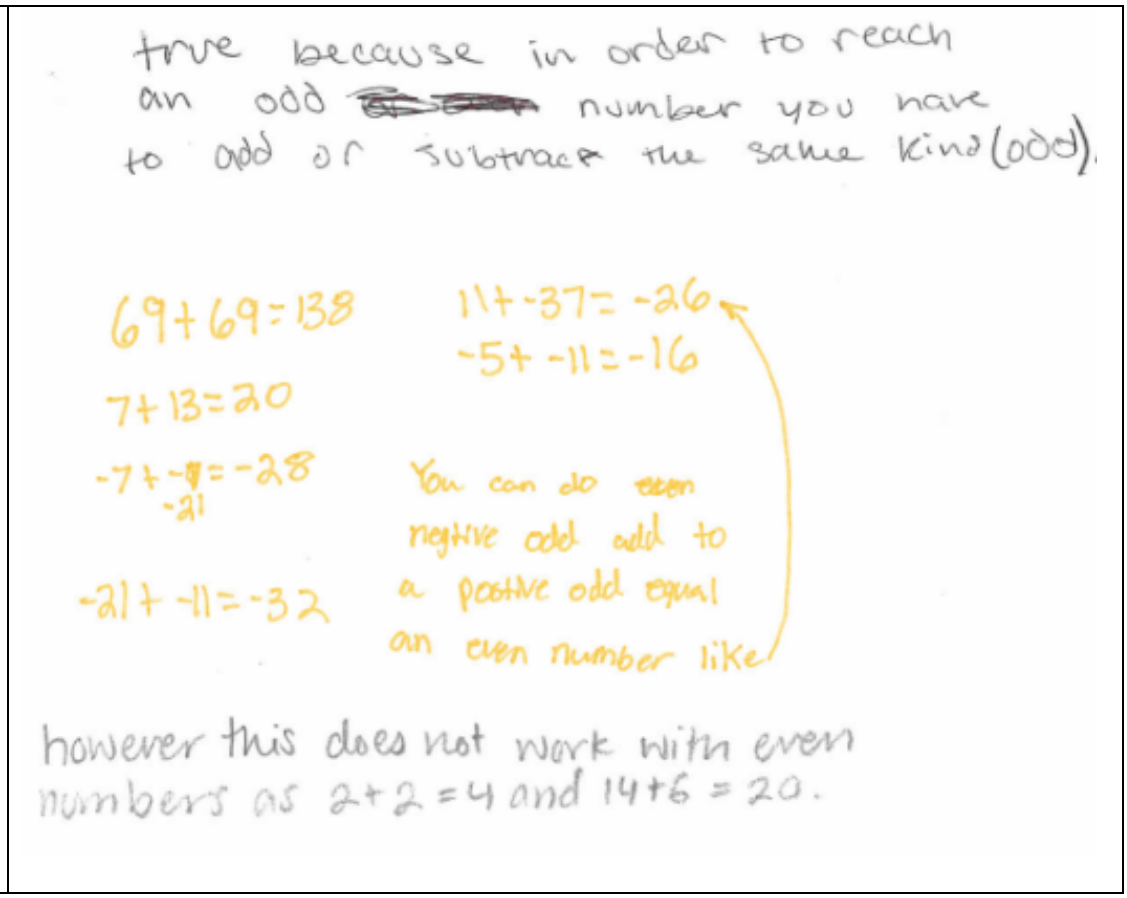
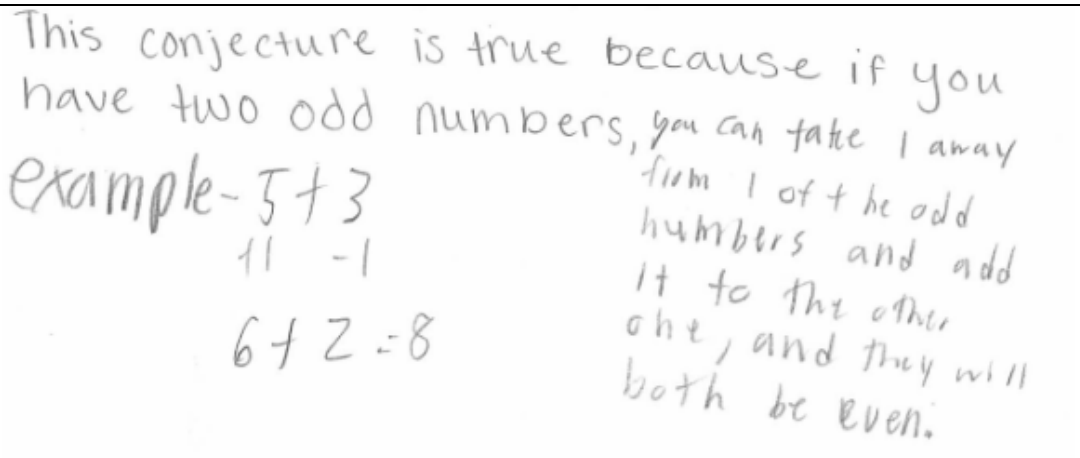
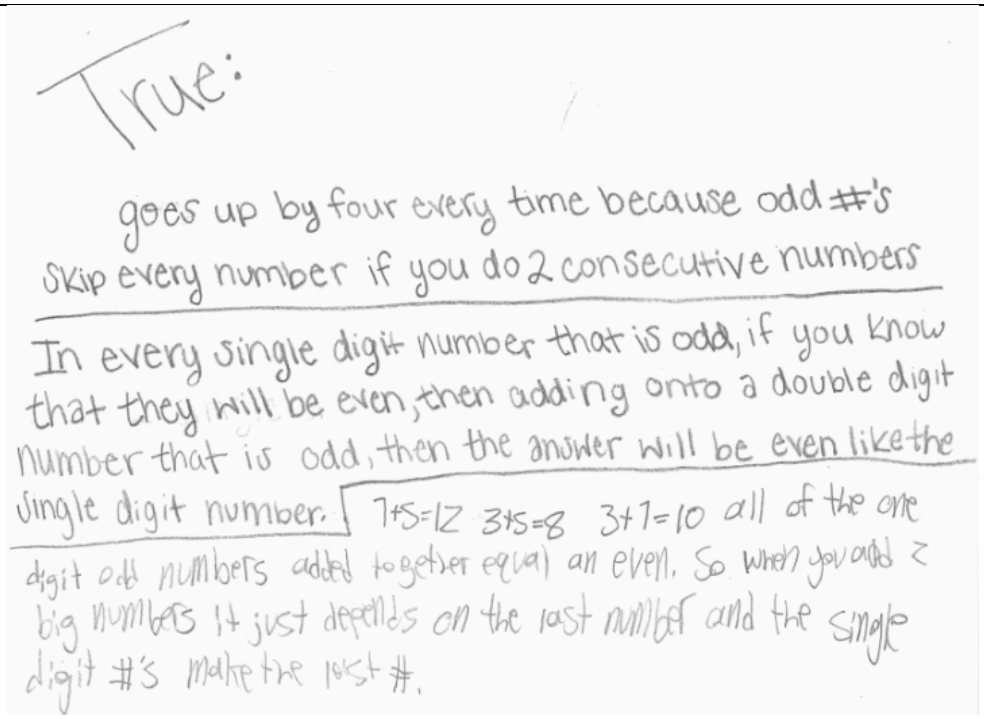
Level 3	

Table 4.1 Continued

Level 4	
Level 5	

Of all the written arguments by the 15 groups from whom data was collected, 28 were scored and presented in Table 4.1 below, as the data from two of the arguments did not get collected because of student error (i.e. the students disposed of or submitted the wrong data). The arguments were coded by me and by another doctoral candidate whose research interest lies in the same field, reaching 100% agreement on the arguments. The interrater reliability indicated a high level of consistency amongst the raters. Of these 28 arguments, 13 groups were able to complete the written arguments for both Task 1 and Task 2. The scores of the collective arguments for all the groups for the first two tasks are shown in Table 4.2 below.

Table 4.2 Distribution of Scored Arguments in Task 1 and Task 2

	Task 1	Task 2
Group A	2	*
Group B	1	2
Group C	2	2
Group D	3	3
Group E	3	5
Group F	1	1
Group G	3	2
Group H	2	3
Group I	2	2
Group J	3	3
Group K	2	4
Group L	2	3
Group M	2	2
Group N	3	3
Group O	*	3
Average Code for Groups B-N	2.23	2.69

* Indicates missing data

The research question, then, seeks to answer the question if students were able to improve their mathematical arguments after engaging in more opportunities to collaborate and collectively argue.

To test the hypothesis that the coded scores of the collective arguments after the first task ($M = 2.23, SD = 0.73$) and after the second task ($M = 2.69, SD = 1.03$), a dependent-samples t-test at the alpha level $\alpha = 0.05$ was used to examine the difference in the eighth-grade students' collective arguments from the first task to the second task. The sample consists of 13 subjects of small groups of eighth-grade Pre-Algebra students in groups of 3-4 students (Groups B – N) as matched pairs data because the same subjects were measured for their collective arguments, once after Task 1 and once after Task 2. Because the research question seeks to understand how the students improved in their abilities to create veracious mathematical arguments via collective argumentation, the hypotheses for this test are $H_0: \mu_1 = \mu_2$ vs. $H_a: \mu_1 < \mu_2$, where $\mu_1 =$ the score on the small group collective arguments after the first task and $\mu_2 =$ the score on the small group collective arguments after the second task. The null hypothesis of equal argument levels was rejected, $t(12)=1.90$, $p = 0.041$, since the calculated p-value of 0.041 is less than the alpha level 0.05. Thus, there is evidence to conclude that there is a significant difference in the argument levels after Task 2 vs. after Task 1; more specifically, that the students improved their collective arguments after Task 2 after Task 1.

The quantitative analysis indicates an improvement in students' abilities to create and develop their own mathematical arguments. However, further qualitative analysis of the collective mathematical arguments breaks down even deeper understanding of students' understanding of what constitutes a mathematical argument, as well as the necessity of

interpersonal discourse to develop and advance their mathematical arguments within a small-group context.

First Theme: Understanding of a True Mathematical Argument

The first major theme that became evident in this data is student understanding of a true mathematical argument, particularly in reference to the communal criteria. In looking at the collective arguments of the groups, several groups indicated greater understanding of the criteria that make up a veracious mathematical argument as decided upon during the communal criteria at the beginning of the discussion, or they even were able to apply the criteria to a more advanced mathematical argument in the second task after having engaged in more practice with collective argumentation in comparison to the first task. Of the 13 groups, only one group decreased in the mathematical argument from Task 1 to Task 2, and there are several factors that could have impacted this decrease, including a better sophisticated argument in their oral collaborative discussions in comparison to what they submitted as a finished written argument, which will be addressed in the next analysis. Five group arguments showed significant improvement in their arguments as a result of better understanding of the communal criteria, and the remaining arguments stayed at the same level for both arguments. The following seven examples from the 13 groups that have written collective arguments for both Tasks 1 and 2 indicate improved understanding of a veracious mathematical argument by increased reference to the communal criteria or improved mathematical arguments themselves, as different groups of students validate their thoughts with some or all of the communal criteria that the class agreed upon for a true mathematical argument (supported by evidence, logical sequence, understandable, and true for all cases). The first two examples did now have improved

arguments, but improved understanding of a true mathematical argument, while the last five examples are improved arguments from Task 1 to Task 2.

Focusing in on Two Group Tasks: Improved Understanding of a Mathematical Argument

Although not every argument did increase, some of the groups did show at least a better understanding of what constitutes a veracious mathematical argument because of their attempt at creating one, addressing the communal criteria that were required for a veracious argument by the classroom community at the beginning of the instructional series. Groups H and I are two groups whose arguments are examples of improved student understanding of an argument.

Attention to Argument Criteria: Group H Arguments. Group H, although unsuccessful in producing a mathematical argument attempt beyond examples, did show improvement in at least the acknowledgement of the necessity of communal criteria in collective argumentation. Their first argument, coded at a Level 2, relies heavily on the use of examples, with little other explanation of the mathematical argument, referencing none of the communal criteria discussed as a class. See Figure 4.1 below for the first task argument of Group H.

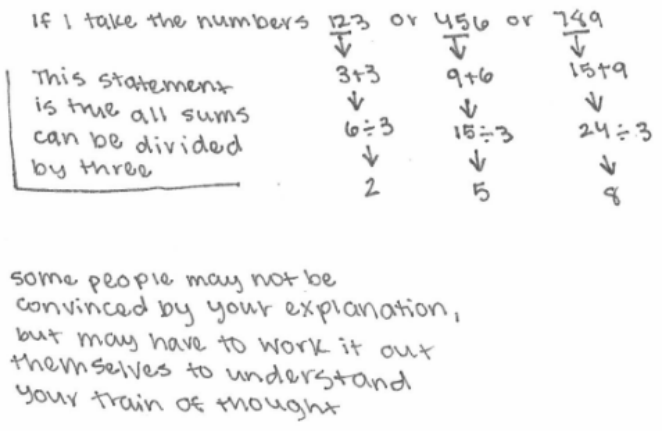


Figure 4.1 Group H Task 1

Similar to their argument in Task 1, Group H relied heavily on examples to portray their reasoning for their argument, which is the reason for another argument coded at Level 2. This is evident in the way that they listed multiple examples of sums of two odd integers at the top of the argument, as well as by the second row of multiple examples of sums of odd integers below their address of the multiple communal criteria. Further, this group attempted to validate more about the reasoning in their argument by organically bringing up the scientific method, hoping to validate more of their reasoning by appealing to the authority of a process of criteria like they did with the communal criteria. Refering to the communal criteria, the group intentionally addressed three of the four communal criteria, showing an acute awareness for the necessity of an argument to meet these criteria (even if they were unsuccessful in being able to do so). Although their address of the criteria are simply restatements of the three criteria, this group distinguishes itself from the previous group because they intentionally attempted to incorporate them as reasonings for validation of the collective argument that they created as a group. See Figure 4.2 below for the second task argument of Group H.

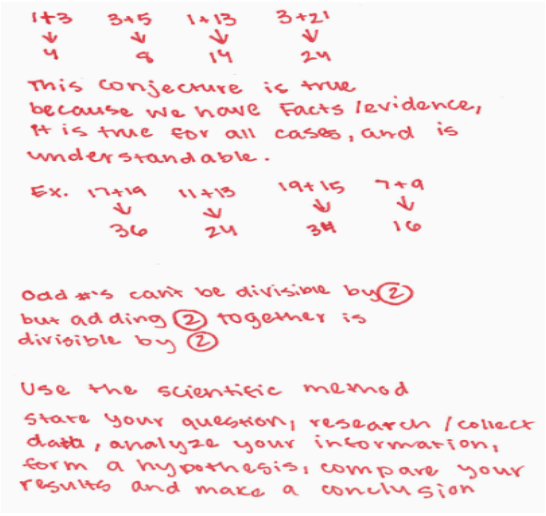


Figure 4.2 Group H Task 2

Necessity of Evidence: Group I Arguments. Similar to Group H, Group I showed an equivalent understanding of referencing to the communal criteria by developing similar attempts at arguments, although both were coded at level 3, or unsuccessful attempts at a veracious mathematical argument. Seen in Figure 4.3, the group attempted to provide some reasoning as to why any three consecutive integers were divisible by 3 by referring to all cases in numbers 0-9, but the attempt is unsuccessful without truly showing how all numbers that end with 0-9 makes the sum divisible by 3. Additionally, the group attempted to provide justification for their reasoning by confirming the veracity of their claims with an authority that they trusted, in this case an online search engine. At the end of the argument, the students wrote or reiterated that their argument met certain criteria in the communal criteria, such as using evidence and being true for all cases, indicating a recognition of the need for the criteria, but they did not successfully incorporate them into their argument. See Figure 4.3 below for the first task argument of Group I.

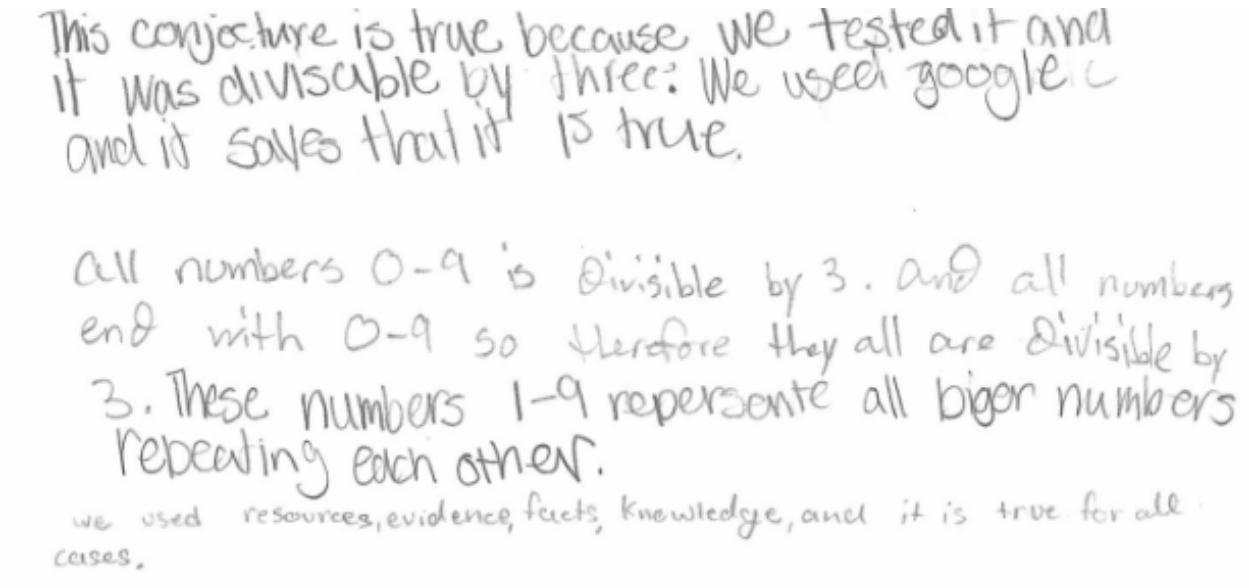


Figure 4.3 Group I Task 1

Although the second written argument for Group I did not improve in the coded score from the first task, this example shows a better understanding of providing evidence to support their argument, although it again was unsuccessful in its attempt. In this task, it is evident to see how the students provided a better sequence of logic to make their argument true for all cases, citing several difference examples of an odd number added to another odd number to result in an even sum. Because, as the review of the literature discussed, examples are not a veracious proof or mathematical argument, Group I's argument score did not receive above a 3. Additionally, the students showed a secondary response that was another attempt at an argument by reversing the equation, stating the argument that an odd number subtracted from any even number is an odd result, which is an intelligent observation. However, no veracoius argument supported this evidence. Again, the group referenced the communal criteria, particularly being true for all cases, indicating a true understanding of a veracious mathematical argument. See Figure 4.4 below for the second task argument of Group I.

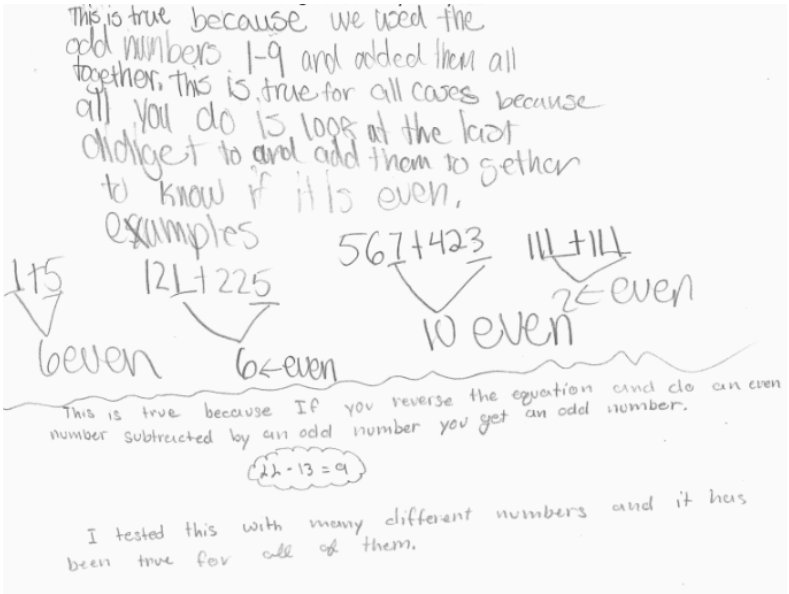


Figure 4.4 Group I Task 2

Second Theme: Improved Mathematical Argument

The next five examples elaborate not only on how the arguments of the groups increased levels from Task 1 to Task 2, but also how the groups came to develop more advanced mathematical arguments as a result of the communal criteria that they collaborate about during the collaborative discussions during the collective argumentation. The major difference between these five examples and the previous two is that the groups maintained a similar understanding of what constitutes a veracious mathematical argument across the two tasks, but instead the arguments themselves developed from Task 1 to Task 2. This is seen not only in the increased argument for each of these five groups, but also in the explanations that these groups provided for Task 2 in comparison to their explanations for Task 1. The discussion of each example provides insight into how the groups chose to incorporate their knowledge of communal criteria into their reasoning that they used in their more advanced argument for Task 2 in comparison to Task 1. Some of the arguments even explicitly stated the communal criteria as a way to validate their argument, which is direct evidence of their knowledge of the need for a mathematical argument to meet the criteria necessary for a veracious argument. The following five examples are indicative of these students' improved mathematical arguments.

Increase in Evidence: Group A Arguments. The arguments for Group A were weaker than most all other groups, but they did show improvement between Task 1 and Task 2. As indicated by the first argument, Group A did not put any real effort into the first task, or perhaps showed much confusion about how to make an argument, as their argument was basically a re-statement of the claim and an attempt at one example. This resulted in the lowest level argument, a Level 1 Argument. See Figure 4.5 for their first argument below.

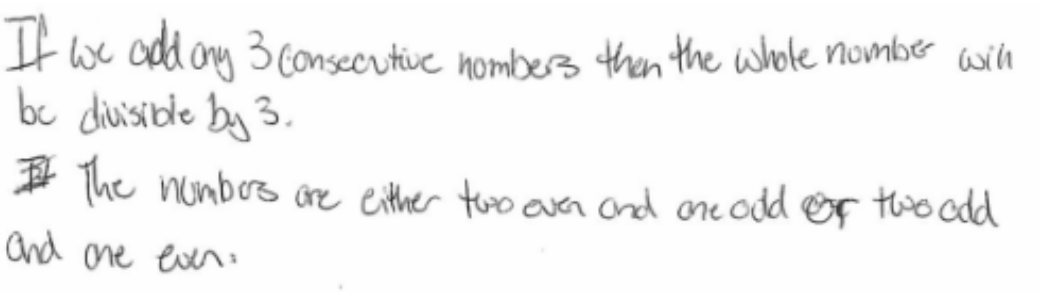


Figure 4.5 Group A Task 1

Although their second argument only coded as a level 2, this is improvement from a Level 2 largely because they began to look for more evidence by showing examples. Although examples alone were defined as a Level 2 argument, Group A took the examples and attempted to use them to explain their reasonings, hoping to cite evidence to their argument, appealing to more of the communal criteria that they learned about the beginning of the class. Also, Group A felt the necessity to express that they could support their argument with facts from the calculator, which indicates some knowledge of the necessity to provide some support, unlike their first argument. Although the use of the calculator places authority in outside sources beyond the mathematics that they were able to come up with, the students did show great effort in validating their rationale, which indicates much improvement from the first argument that they create dfor Task 1. See Figure 4.6 for their second argument below.

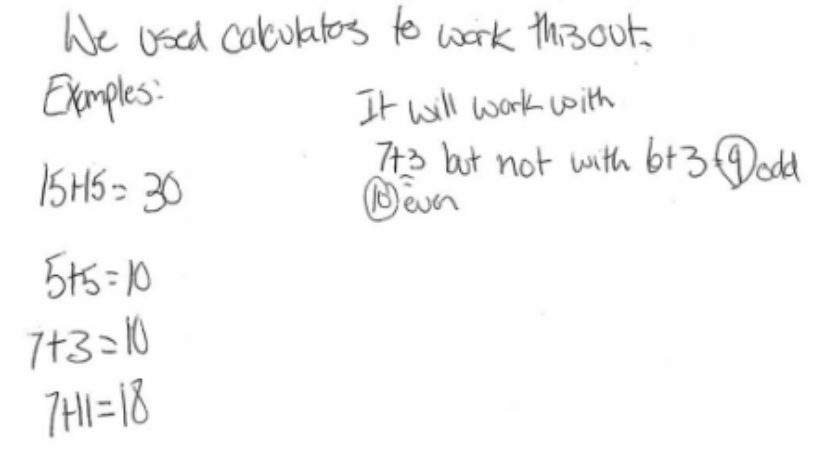


Figure 4.6 Group A Task 2

Increase in Multiple Criteria: Group G Arguments. The argument for Group G did not indicate any reference to any of the communal criteria in the final written argument after Task 1, as it was solely one example and a restatement of the claim. The students, although providing an example, did not show any rationale or indication of understanding any of the communal criteria. By providing at least one example, though, this group provided what they believed their reasoning for the truth of the mathematical claim via an empirical argument, moving their argument from a Level 1 to a Level 2. However, there is no real attempt to make a general argument for the mathematical claim, resulting in their argument being coded at a Level 2. See Figure 4.7 below for the first task argument of Group G.

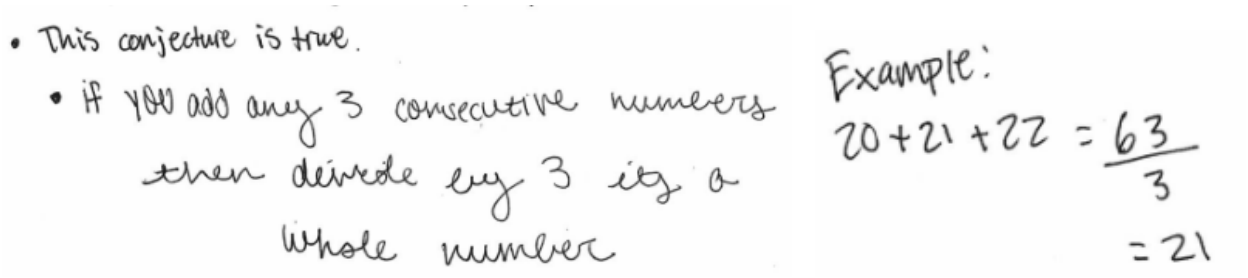


Figure 4.7 Group G Task 1

However, the Task 2 argument explicitly stated and even attempted to make an argument for multiple of the communal criteria, including a logical sequence and being true for all cases. The figure shows the written argument that portrays a logical sequence for why the sum of two odd numbers is even, as well as a direct conclusion that their argument is true for all cases (although not necessarily being entirely supported). The logical sequence is based on one example and building from there to show that any two odd numbers sum to an even number, and their progression of this logical sequence leads them to conclude that they have met the criteria of their mathematical argument begin true for all cases of the sum of any two odd numbers being even. Although the argument was not validated well because of their lack of explanation about the even sums, Group G did improve this task by purposefully addressing the communal criteria, resulting in an unsuccessful attempt at a mathematical argument, or a Level 3 argument. See 4.8 below for the second task argument for Group G.

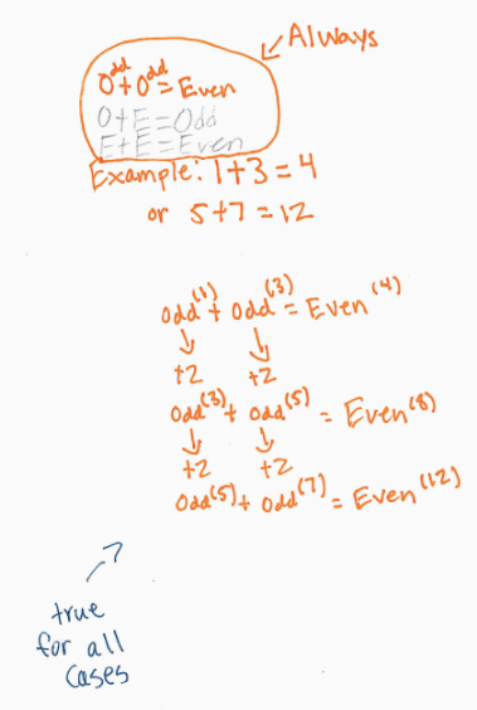


Figure 4.8 Group G Task 2

Increase in Multiple Criteria: Group K Arguments. The next example of arguments for Task 1 and Task 2 are arguments from Group K, who improved from a level 2 to a level 3 argument. In their first argument, it is evident that these students relied heavily on one example as their rationale for their argument, failing to move into anything more concrete for cases other than this one example. The second argument also largely relies on examples as their main evidence for the argument, but the explanation they provide helps show their attempt at a more general argument beyond an empirical argument as they move into a Level 3 argument. This is the reasoning for the Task 1 argument for Group K being coded at Level 2. See Figure 4.9 below for the first task argument by Group K.

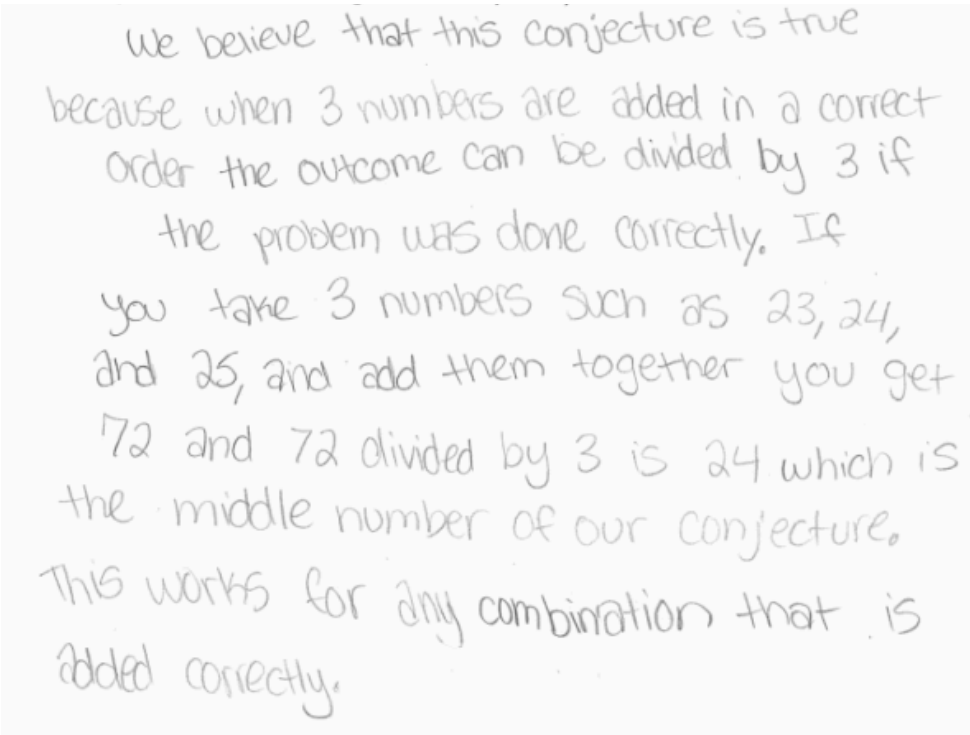
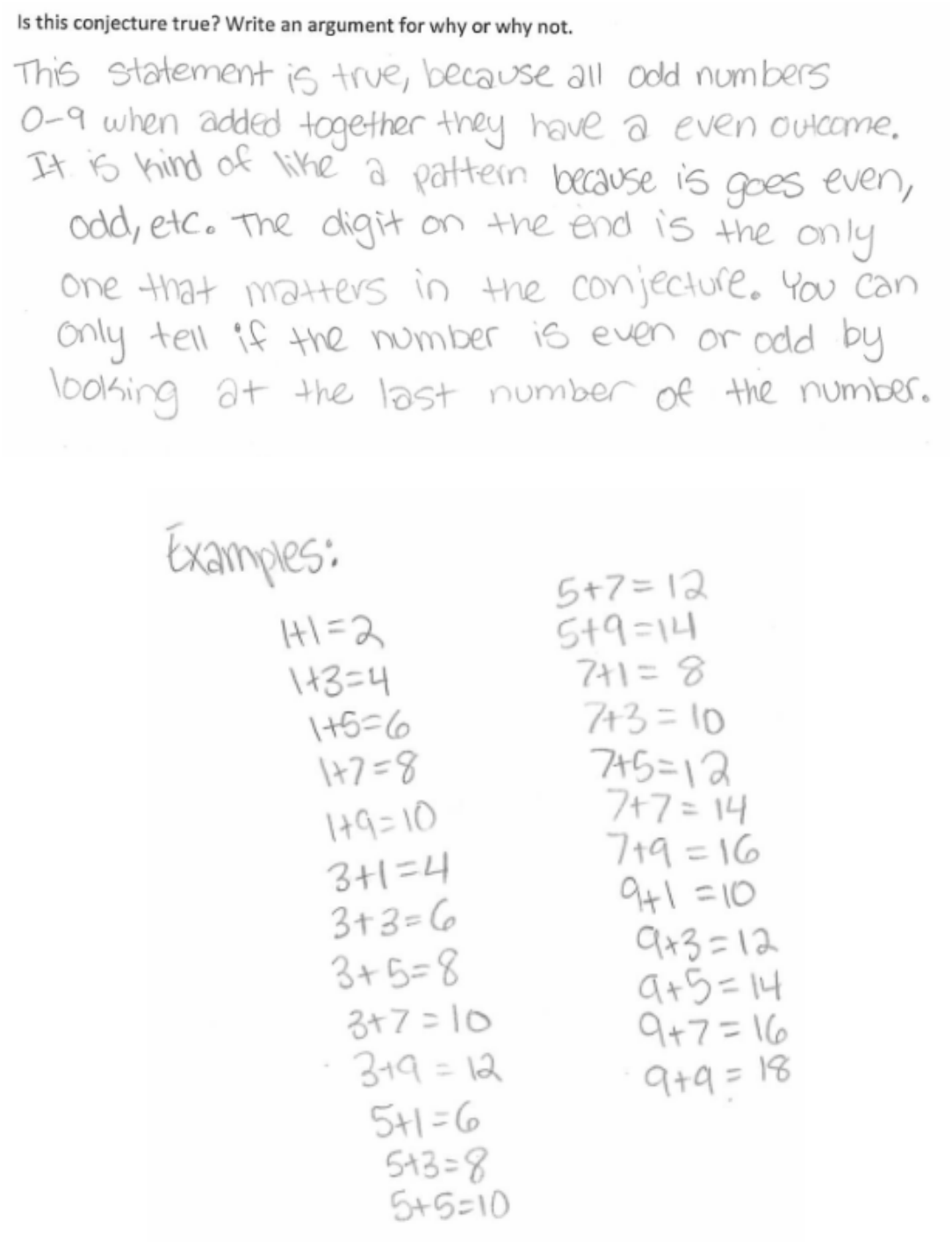


Figure 4.9 Group K Task 1

The improvement for Group K comes with their attempt to better explain their reasoning with a logical sequence, addressing two of the communal criteria with their explanation of the sequence as a pattern and a thorough logical sequence of their reasoning. As a result of addressing these communal criteria, the argument moves into an unsuccessful attempt at a veracious argument because of their reasoning that provides an argument that extends beyond one example, or a Level 3 argument. See Figure 4.10 below for the second task argument by Group K.



Is this conjecture true? Write an argument for why or why not.

This statement is true, because all odd numbers 0-9 when added together they have a even outcome. It is kind of like a pattern because is goes even, odd, etc. The digit on the end is the only one that matters in the conjecture. You can only tell if the number is even or odd by looking at the last number of the number.

Examples:

1+1=2
1+3=4
1+5=6
1+7=8
1+9=10
3+1=4
3+3=6
3+5=8
3+7=10
3+9=12
5+1=6
5+3=8
5+5=10
5+7=12
5+9=14
7+1=8
7+3=10
7+5=12
7+7=14
7+9=16
9+1=10
9+3=12
9+5=14
9+7=16
9+9=18

Figure 4.10 Group K Task 2

Proving the Criteria True: Group J Arguments. Group J showed great improvement because of the way that the students began to address the communal criteria directly in their attempts to prove at least one of the communal criteria to be true for the second task after not addressing the criteria at all in the first task. The arguments for Group J were weaker than most all other groups, but they did show improvement because of this attempt to improve the argument by using the communal criteria. In the first task, Group J provided the example that one plus two plus three equals six, which they then state as a number that is divisible by 3. The first argument, then, was coded at a level 2, or empirical argument, for reliance on an example as the reasoning for the argument. Beyond providing the example, the group states that the conjecture is a true statement, then they restate the conjecture within the context of the example to validate the truth to their mathematical argument. See Figure 4.11 below for the first task argument by Group J.

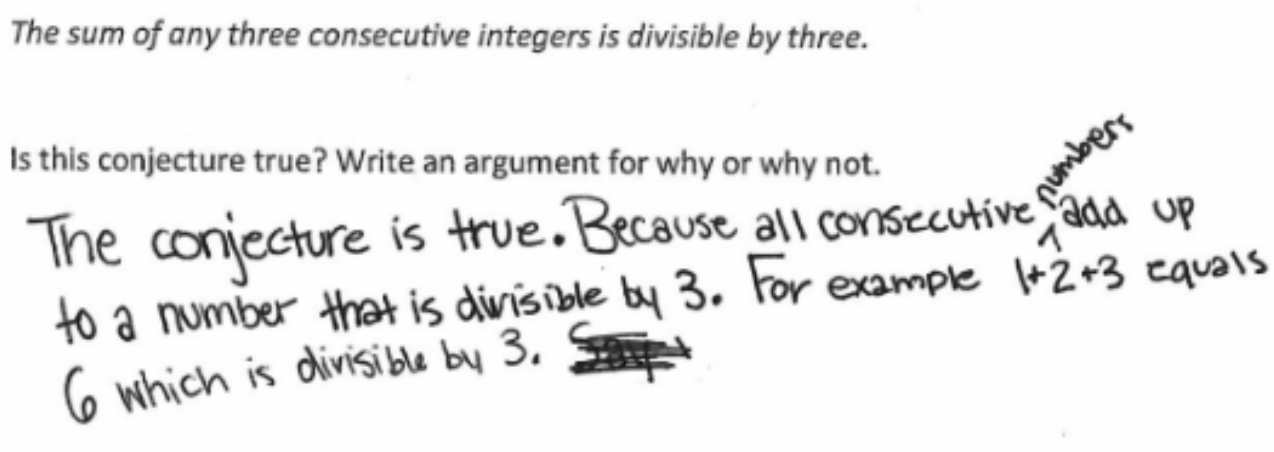


Figure 4.11 Group J Task 1

As evidenced by the following figure, the argument for Task 2 improved in that the group attempted to show evidence and explain their reasoning clearly about the sum of two odd numbers being even by simply taking 1 from one of the odd numbers and adding it to the other odd number, resulting in two even numbers. The argument is successful because their work shows a logical reasoning for this statement, and the nature of this statement proves the argument true for all cases. Due to their lack of explanation (particularly in showing how their two odd numbers come to a result of an even sum and not just two even numbers, as well as their lack of explanation of two even numbers summing to an even number), the argument did not code as a full veracious proof (level 5). This was coded as a level 4 argument. See Figure 4.12 below for the second task argument by Group J.

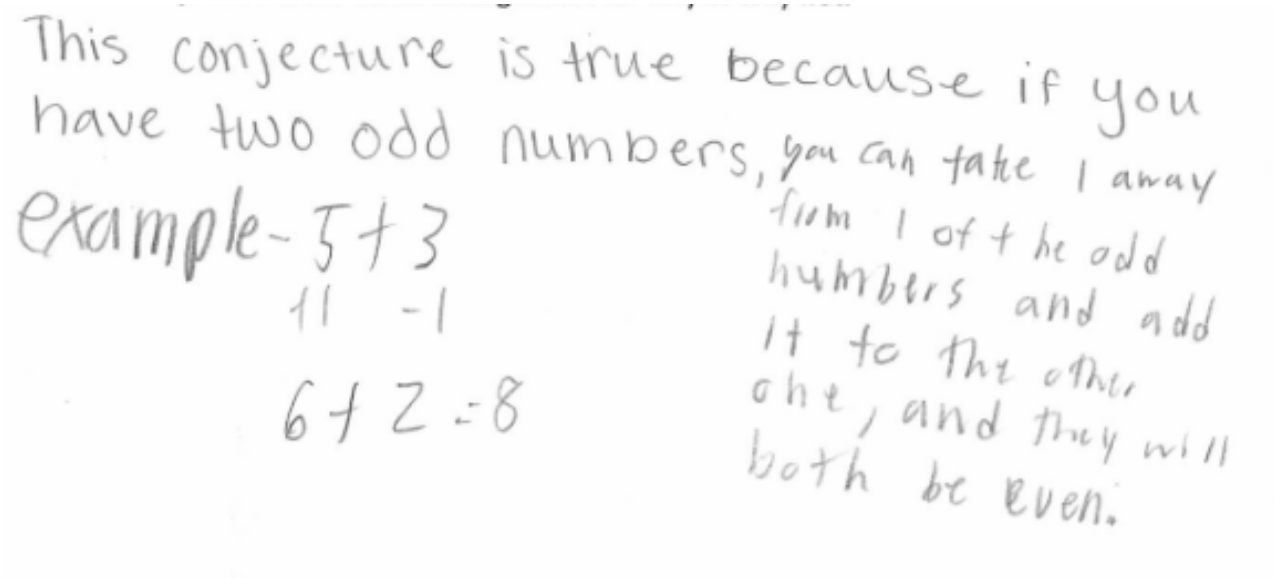


Figure 4.12 Group J Task 2

Addressing All Criteria: Group D Arguments. The best example of showing an improved understanding of a mathematical argument is Group D, whose argument level increased from a level 3 – an unsuccessful attempt at a veracious argument – to the highest level 5, a genuine proof or mathematical argument. As evident in the figure below, this group attempted a veracious argument by giving an example of three consecutive integers and finding its sum, then recognizing that this result was divisible by 3. The written argument was not clear in showing the divisibility by 3, though, which invalidates the argument. The argument portrayed a genuine attempt at proving the argument is true for all cases by indicating that adding 1 to each number in this first example, showing an increase in each of the three numbers so that the resulting three numbers are still consecutive, also resulted in an sum that was divisible by 3. This process was repeated several times. This reasoning, although valid, shows an invalid attempt at a proof because the reasoning indicates understanding of this being true for all cases, referring to one of the communal criteria. Additionally, the sequence was logical in adding 1 to each individual number to indicate that this sum result occurs for any three consecutive integers. However, the argument was not written out entirely, especially because there was no evidence of any of the sums being divisible by three. See Figure 4.13 below for the first task argument by Group D.

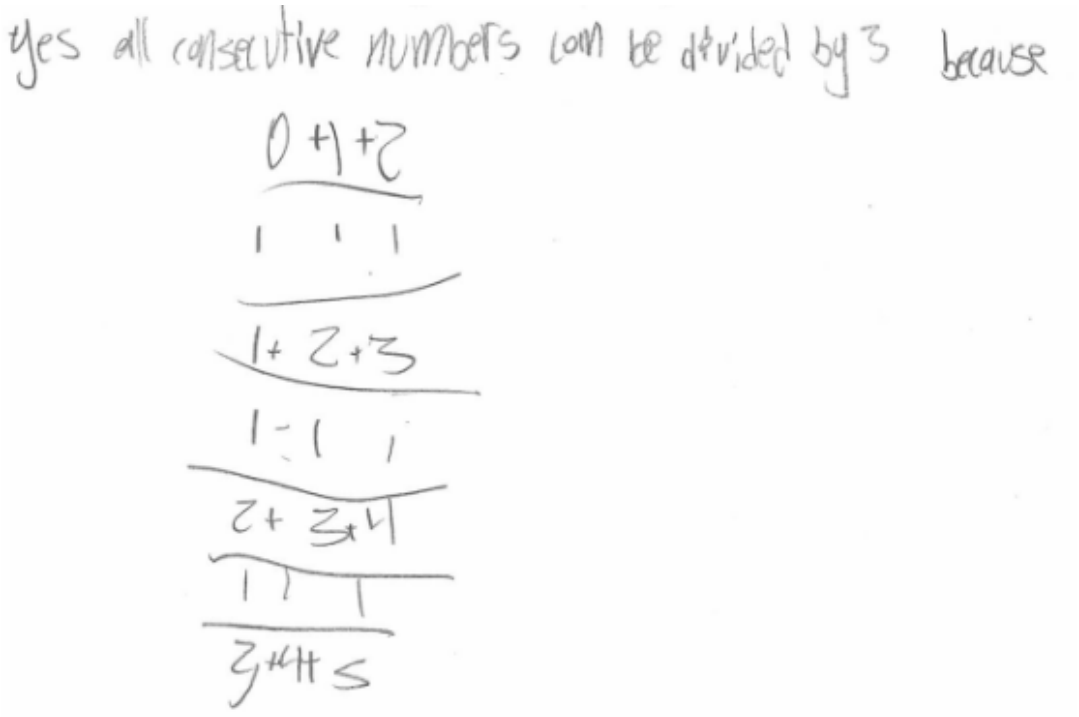


Figure 4.13 Group D Task 1

The argument for the second task in Group D is the one most evident of a true mathematical proof or argument. The task, asking for an argument proving that the sum of two odd numbers is even, firstly provides evidence by referring to the fact that the sum of any two one-digit odd numbers is even, a mathematical fact that they have previously learned. They even support this mathematical fact by stating, within their own words, how odd numbers work and the patterns that they recognized with counting odd numbers. This also makes the argument understandable in the mathematical learning community, as these facts and patterns are observations and pieces of information that should be learned before entering this class. They state that any numbers beyond one-digit (two digits or beyond) do not change the sum of being or odd. This is logical to reason because they state that adding the double-digit numbers does not

change the even or odd characteristics of the sum since being odd or even is only dependent on the last digit of any number anyway. Although the group does not provide full reasoning as to why this is logical to follow, the students' logical reasoning makes it clear that the claim is true for all cases. In this thorough explanation, then, the students very purposefully appealed to all four major communal criteria that were discussed at the beginning of the instructional series, and again reviewed at the beginning of the second task on the second day of this research study. So, by stating the argument with all their reasoning and appeals to each of the communal criteria, Group D was able to confirm their understanding of a true mathematical argument by meeting all the communal criteria for this mathematical task. See Figure 4.14 below for the written argument.

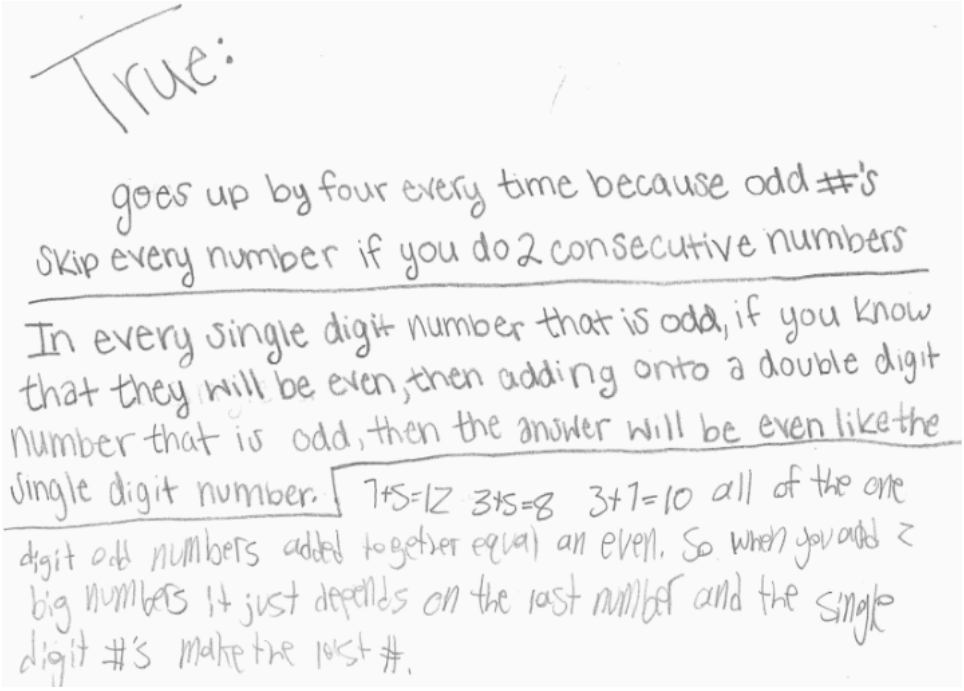


Figure 4.14 Group D Task 2

This qualitative analysis indicates a clear improvement in most students' understanding of what constitutes a veracious mathematical argument across several of the groups in this analysis, as well as their abilities to develop a more advanced mathematical argument, particularly in relation to their knowledge of an argument in accordance to the communal criteria that they developed as a mathematical learning community in the eighth grade Pre-Algebra classroom. To look more specifically at the role that interpersonal discourse plays in these advanced mathematical arguments, more qualitative analysis of the audio recordings of the collaborative discussions in which the small groups engaged reveals the ways in which students used small-group discussions to develop and advance their mathematical arguments.

Collaborative Discourse to Develop Mathematical Arguments

The previous discussions of the results of the data emphasized students' understanding and improvement of a veracious mathematical argument through analyses of the written arguments at the completion of the first two tasks. However, the first research question also equally emphasizes the ways that students engage in interpersonal discourse with one another to develop these mathematical arguments. Because of this, as indicated in the methodology of the study, the students' small-group collaborative discussions were audio recorded. Similar to the dependent samples t-test that was run on the written arguments for the students, it is necessary to remember the hierarchy by which the student arguments were scored, as detailed in Chapter III in the methodology. These argument levels – ranging from Level 1 (an ingenuine argument due to little or no effort or understanding of the mathematics) to Level 5 (a veracious proof or mathematical argument) – are the same levels used to measure the level of argument that students were able to reach orally during the development stage of their argument as they worked

collaboratively in their small-group discussions through the modality of verbal discourse with one another.

Of all the oral arguments by the 15 groups from whom data was collected, 27 were scored and presented in Table 4.2 below, as the data from three of the arguments did not get collected because of equipment error (i.e. the audio recorder did not work or lost power). The arguments were coded by me and by the same doctoral candidate whose research interest lies in the same field as mentioned in the first analysis, reaching 93% agreement on the arguments. The interrater reliability indicated a high level of consistency amongst the raters. Of these 27 arguments, 12 groups were able to have audio recordings for both Task 1 and Task 2. The scores of the oral arguments for all the groups for the first two tasks are shown in Table 4.2 below.

Table 4.3 Distribution of Scored Oral Arguments in Task 1 and Task 2

	Task 1	Task 2
Group A	3	*
Group B	2	2
Group C	2	*
Group D	5	3
Group E	5	5
Group F	3	4
Group G	3	3
Group H	3	3
Group I	2	2
Group J	3	5
Group K	3	5
Group L	4	5
Group M	3	2
Group N	3	4
Group O	*	3
Average Code for Groups B, D-N	3.25	3.58

*Indicates missing data

The research question, then, seeks to answer the question if students were able to both improve their mathematical arguments after engaging in more opportunities to collaborate and collectively argue as well as find value in their interpersonal discourse via collaborative discussions in the development of their mathematical arguments.

To test the hypothesis that the coded scores of the oral arguments after the first task ($M = 3.25, SD = 0.97$) and after the second task ($M = 3.54, SD = 1.24$), a dependent-samples t-test at the alpha level $\alpha = 0.05$ was used to examine the difference in the eighth-grade students' oral arguments from the first task to the second task. The sample consists of 12 subjects of small groups of eighth-grade Pre-Algebra students in groups of 3-4 students (Groups B, D – N) as matched pairs data because the same subjects were measured for their oral arguments, once after Task 1 and once after Task 2. Because the research question seeks to understand how the students improved in their abilities to create veracious mathematical arguments via collective argumentation, the hypotheses for this test are $H_0: \mu_1 = \mu_2$ vs. $H_a: \mu_1 < \mu_2$, where $\mu_1 =$ the score on the small group oral arguments after the first task and $\mu_2 =$ the score on the small group oral arguments after the second task. The null hypothesis of equal argument levels failed to be rejected, $t(11)=1$, $p = 0.17$, since the calculated p-value of 0.17 is greater than the alpha level 0.05. Thus, there is not sufficient evidence to conclude that there is a significant difference in the argument levels after Task 2 vs. after Task 1; more specifically, that the students improved their oral arguments after Task 2 after Task 1.

Although the dependent samples t-test failed, there are several takeaways from the comparison between the oral arguments from Task 1 to Task 2, as well as different conditions that could have contributed to the failure of the dependent samples t-test. The table above shows that there is missing data in three of the fifteen groups, whose data could have contributed to the

statistical significance of the test if the equipment had not failed. Further, the table reveals that, although the test was not statistically significant, the average score of the argument level did improve from Task 1 to Task 2. Even more notably, multiple groups of students were able to create a veracious mathematical argument or proof, at the highest level of argumentation on the scale, Level 5. Finally, it is important to note that only two groups decreased in argument level from Task 1 to Task 2, and this could be due to other factors outside of the argumentation design of the instructional series, including a lack of background knowledge in the content of the second task or ability to orally state what they recognize to be true. With all of these variables impacting this statistical significance, it is still noteworthy of the increased means of the oral arguments from Task 1 to Task 2, so a qualitative analysis followed.

Beyond the statistical test for significance, analysis of the students' collaborative discussions provides a deeper understanding into the value of the discourse that students engaged in while developing a collective argument for the task as an entire group. To look into the role that the discourse played, it is important to consider the following examples of the discussions that students engaged in to advance their mathematical arguments along the hierarchy of arguments utilized in this study.

Convincing within the Small Group Discussion: Group D Oral Arguments

This first example, Group D, is an example of the tool of oral discussion in being able to convince one another of a mathematical argument (Harel & Sowder, 1998). In Group D's oral argument for Task 1, one group member clearly began the collaborative task with his reasoning for his argument about the sum of any three consecutive integers being divisible by 3 via an inductive argument, seen in his quote below:

> So, you start with 0,1,2 as your base one. And then, what you do to bring it up to 1,2,3, is you add one to each of the integers. So, when you're adding one to each of the integers, you're really just adding three. So, that's why it's divisible by three. And then, that's why it's divisible by three every single time.

This group member explains that, via induction, his base case of 0 + 1 + 2 is divisible by 3, and then adding one to each integer produces the next case. He explains that this case is also divisible by 3 because, by adding 1 to each integer, this only adds 3 to the original sum, making this sum also divisible by 3, using the property that adding 3 to a number already divisible by 3 creates a new number that is also divisible by 3. In bringing this to the group, another group member asks this member to support this claim with facts and evidence, appealing to the communal criteria of the need for evidence:

> Group D Member 2: But, I do feel like in some instance it won't be right.
>
> Group D Member 1: No, it will be right every time because you're just adding three every time.
>
> Group D Member 2: But, why? We need facts and evidence.
>
> Group D Member 1: Because if you have three consecutive numbers, if you go up one for each of them, you just add three. And if the base one does—is divisible by three—that means all the other ones will be divisible by three. And in this instance the base one is divisible by three.

Upon receiving clarification of this discussion, Group D accepted this idea and pursued this argument as the collective argument. However, recall from the previous analysis that their final written argument was coded at a Level 3 because the written argument did not provide enough rationale and understanding for why all the sums in their argument were divisible by 3, even

though this was evident in their oral discussions. These quotations by members of Group D indicate the value and understanding of the students' mathematical arguments when they orally communicated in interpersonal small-group discourse, which was not evident in their final written arguments.

Sliding Up the Scale: Group E Oral Arguments

Group E is one example of a small-group discussion that showed the effectiveness of students' discussions with one another, in productive struggle, to eventually move beyond the current level of argument into a more advanced level as a result of their collaborative reasoning. In Group E's written arguments, there was not enough evidence or explanation to code either of the arguments above a Level 1 for both Tasks 1 and 2. However, their oral discussions prove that the group had a much higher understanding of the mathematics, and they even developed sound mathematical arguments in their discussions. To begin Task 1, this group collaborated to try one example of three consecutive numbers to determine if they are divisible by 3 on a calculator:

> Group E Member 1: If you add them by three, yeah it can.
>
> Group E Member 2: Any three numbers?
>
> Group E Member 3: Like 42, 43, 44. If you add them together—I need to think. I need to do the math in my head. Hold on, I'm just going to do my calculator.
>
> Group E Member 1: OK, so… yes.
>
> Group E Member 3: So, each time…
>
> Group E Member 1: Why do we believe it? Because….
>
> Group E Member 3: I just did the math in my head, and each time I got—it was divided by three.

When I prompted them to explain more about their argument, particularly because I recognized that their foundational argument at this moment was an empirical argument, coded at Level 2, Member 3 of Group E attempted to make a general argument for their reasoning.

> Me: OK, that's good thoughts, yeah. Anything else? Something different? Same thing?
>
> Group E Member 3: Well, I just said I, um, three consecutive integers until I got to number 30, and then I figured that they could—if 30 is able to be divided by three, then you could add 30s and 30s…

This reasoning did not make any explicit comments about Member 3's argument, but continued frustration in their discussions revealed that they then tried to advance the empirical argument to a general argument, although unsuccessful, moving the oral argument from a Level 2 to a Level 3 argument:

> Group E Member 1: I don't know—like, facts. Like, without examples, I don't know how to do it.
>
> Group E Member 3: Each time I add three consecutive numbers together, it's able to be divided by three. And this can go up to the millions.
>
> Group E Member 1: I think it depends on like the integers that are given cause if you just use random ones, or if it just starts out with the first ones, I think it would.
>
> Group E Member 2: 1, 2, yeah.
>
> Group E Member 1: And then, you just add three more, and then you divide it by three, and you get the base.
>
> Group E Member 3: The, every—you add this, it would be divided by three.

Member 3's comment "then, you just add three more, and then you divide it by three, and you get the base," although invalid, shows her attempt to appeal to a more general argument, which is

a clear improvement to a Level 3 argument, which is an unsuccessful attempt at a general argument.

These two examples are just two explicit examples of quotes from different groups' oral discussions that show the power of engaging in interpersonal discourse with one another. These groups recognized the necessity to convince one another of the veracity of their arguments by referring back to communal criteria, particularly as they ask each other for evidence and to make their arguments true for all cases, or that they simply had to engage in some deep discussion with one another to struggle with the conceptual ideas at the foundation of the mathematics to move along the scale to a more advanced mathematical argument. With these two examples providing some clear insight into the value of oral discussion and collaborative mathematical argumentation, this analysis supports the notion of student voice as a tool for developing better student collaboration and mathematical argumentation.

In summary, the analyses of the written and oral arguments provide evidence for many students' abilities to improve in their abilities to understand what constitutes a true mathematical argument as they appeal to communal criteria for a mathematical argument with one another via interpersonal discourse within small group discussions. Further, according to the data, the research shows that several students developed a more advanced mathematical argument when given the opportunity to engage in collaborative discussion with one another, especially in small-group settings in the collective argumentation process. Finally, the analysis of this paper related to the first research question shows that students' collaborative discussions proved just as important as their formal written arguments in revealing students' true abilities to develop a mathematical argument.

CHAPTER 4.2

ANALYSIS AND FINDINGS

This study has so far established the importance of collective argumentation in developing students' understanding of a true mathematical argument and the ability to create a more advanced mathematical argument. Further, different examples in the data distinctly portrayed the value in students' collaborative discussions with one another as they begin to wrestle with one another's ideas and attempt to progress their ideas to more advanced mathematical thoughts. As a reminder, this study considered the following research questions: 1) In what ways do K-12 mathematics students use interpersonal discourse in small-group class discussions to create and advance mathematical arguments and proofs? 2) How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments? 3) How does student collaboration through interpersonal discourse in collective argumentation impact students' identity as doers of mathematics? Looking intently at the students' development of their mathematical arguments through discourse in small-group discussions, this analysis study considered the second research question: 2) How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments? This second analysis in this chapter will provide literature and context to understand Toulmin's Model in mathematics argumentation, and then detail the two major themes that this analysis discovered concerning students' discourse when developing collective arguments. Each theme will discuss two major subthemes that the data uncovers to better understand how students interact with each other within the context of developing an argument as they appeal to various aspects of an argument in Toulmin's model of argumentation.

This chapter details the themes and major findings for this research question, using Toulmin's model to qualify students' reasoning during the process of their discourse in collective argumentation. This second research question elicited two common themes for the second research question: an emphasis on warrants as justification, and progressive meaningful backings to explain justifications.

Background: Toulmin's Model in Mathematics Argumentation

Before delving deeper into these themes that respond to the first research question regarding advancement of mathematical arguments, it is important to remember each component of Toulmin's model within the context of collective argumentation for mathematics learning. Generally, collective argumentation in a mathematics classroom is characterized by "a group of students and the teacher working collaboratively to establish the veracity of a claim" (Wagner, Smith, Conner, Francisco, & Singletary, 2013). According to Toulmin, as summarized by Wagner et al. (2013), an argument consists of claims (a statement, the truth of which is being established), data (the facts from which the claim was based), warrants (the justification for using the data to establish the claim), rebuttals (statements that call the warrant into question), qualifiers (utterances that indicate the certainty of a statement), and backings (supports for the warrant that are usually unstated and related to the academic field). Toulmin conceptualized these components in the structure seen in Figure 4.15 below.

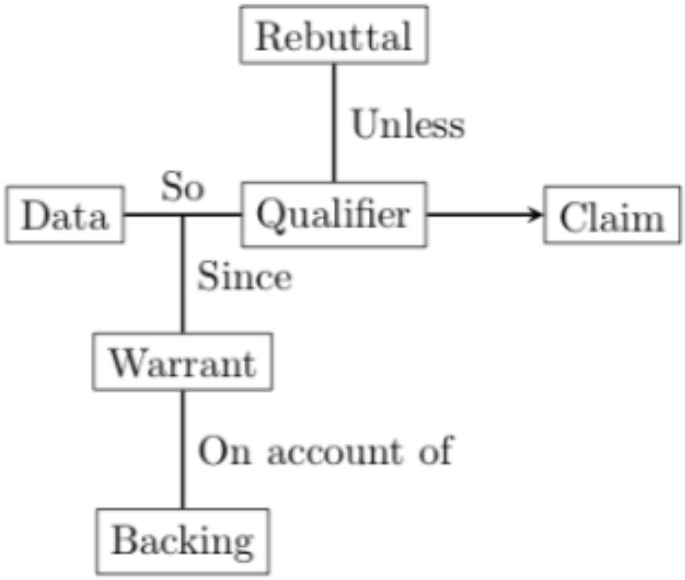


Figure 4.15 Toulmin's (1958/2003) Model of Argumentation

Other mathematics education research, particularly by Krummheuer (1995), has taken Toulmin's model (1958/2003) and adapted it to account for the social nature of the interpersonal discourse that occurs during collective argumentation, as he recognized that components of an argument are contributed and interpreted by a group of people. Other research has taken into account a model to consider which member of the classroom learning community (i.e. student, teacher, or both) has contributed various components of a mathematical argument (e.g. Conner, 2008) and the ways that teachers support each component of Toulmin's model (e.g. Conner, Gleason, Singletary, Smith, & Wagner, 2011). However, this research emphasizes students' natural utterances for each component of Toulmin's models as they work in small-group collaborative

discussions centered around the process of collective argument, with a discussion of communal criteria at the beginning of the instructional series to frame the argumentation tasks.

Before analyzing individual quotes from the small-group collaborative discussions, it is important to remember that both open coding and in vivo coding to better understand how students appeal to different aspects of an argument within the framework of Toulmin's model. As stated in the previous analysis addressing research question one, only data from the first two Tasks were analyzed because of the difficulty of the content in Task 3, yielding inaccurate data for this analysis. The following selected quotations highlight the aforementioned themes of students' collective arguments within the framework of Toulmin's model.

First Theme: Emphasis on Warrants through Justification

Before delving deeper into the themes of the students' collaboration discussions, it is important to review the students' experiences with argumentation and small-group work, as well as their discussions concerning communal criteria immediately preceding the collaborative discussions. As stated in the initial design of the study, the students in this class regularly engaged in small-group discussions, as this information was provided by the regular classroom teacher. However, the students and teacher equally admitted that the small-group discussions centered more around the ability to solve problems that develop procedural fluency and some conceptual understanding, and less around the students' abilities to develop a mathematical argument. Because of this, the students' argumentation was largely influenced by the communal criteria discussion that occurred before the small group discussions each day.

Upon initial analysis of the transcriptions of each small-group collaborative discussion for Tasks 1 and 2, several different themes emerged in the ways that students engaged in discourse with one another during the collective argumentation process. Because of the emphasis

on communal criteria, the students gravitated towards conversation that provided evidence of their reasoning and justifications for their arguments, alluding to the need for evidence that makes an argument understandable, two of the four communal criteria decided upon at the beginning of each class. However, it is notably apparent that students depended upon their warrants as the focal points of their arguments.

Where Students Began: Empirical Arguments as Initial Utterances

As the students engaged in collective argumentation for these tasks, particularly Task 1, they were, for the most part, willing to jump to some evidence or noticing that they could bring to the table concerning the claim that they were tasked with arguing as true or false. Thus, the students chose to skip over a statement of the claim and right into either some data from within the claim that restates or clarifies a fact within the claim or a warrant that justifies the data in the claim. As an example, several students immediately began their collective arguments by bringing an example to the conversation as a fact or piece of evidence towards the veracity of the claim. The following quotes about the first claim, which asks students to write an argument for or against the sum of any three consecutive integers being divisible by three, show students' initial excitement to use examples as an initial argument:

> Group B member: "This is true because if you add up all your three – underline three times consecutive numbers and divide by three, you will always get a whole number, but, only consecutive numbers… I said it's true because if I use my calculator to add the three consecutive numbers, then they will be divisible by three."

This group member restated the claim as truth, but only because she was using her calculator to prove to her group that several different examples were true, as she summed multiple groups of three consecutive integers then divided the sums by 3 to show a divisibility by 3 each time. The

next quote shows a different group stating that proving *every* example to be true should be a valid enough argument, despite the conversation about the communal criteria.

> Group I member: "You think it is true because if you look at it at all – just look at any example. It works, coach. You can do anything – 1 billion, 3 billion, 7 billion. It's true."

This is interesting because the student seems to believe that only one example is not good enough as a veracious mathematical argument, but instead a statement that every example is true moves beyond an empirical argument. Although this is a good thought, he continues just by stating different examples of various quantities. He unfortunately is not able to advance the argument because of a lack of rationale.

The next quote explains a student's thoughts that enough examples can prove a pattern, which is also a movement towards a more advanced mathematical argument.

> Group G member: "Yeah, because – well, I haven't tested all of the numbers yet, but the numbers that I did test were 1, 2, 3, 4, 5, 6 and 7, 8, 9. And if you divided them by 3, all of them turned out evenly."

Similarly, this group depended upon starting the problem by showing several different examples, then attempted to make a more advanced argument out of it by appealing to a rule about this being possible for all values of single-digit numbers. Although this may be true, and the conversation could have turned towards a more advanced mathematical argument – even a Level 4 or Leve 5 argument – the initial decisions by several groups were to begin with an example to get their initial thoughts out of their heads and out of their mouths, eventually onto the paper as a place to begin their collaboration discussion to come up with a more advanced mathematical argument. In this case, the several different examples came to be the final collective mathematical argument for this group.

Next Steps: Uncertain or Unauthoritative Justifications

Beyond many of the students' initial empirical arguments, the data also showed that, particularly as these students engaged in their first critiques of each other's work in small-group collaborative discussions, that students were either not certain with their justifications or falsely placed authority in the foundation of the veracity of their reasoning in an authority outside of acceptable mathematics. Particularly, some students did not want to explain with certainty out of fear of incorrectness. Several students chose to question their own judgment or ask for another example as they were pushed to explain reasoning for their arguments. On the other end of the spectrum, other students chose to believe in their argument because they found authority in an outside source to affirm the veracity of their argument. These seemingly opposing stances on belief in their mathematical arguments affirm that students believed that the primary goal in developing a mathematical argument is a viable justification, or a warrant in Toulmin's model, but they were not entirely certain or were falsely confident in presenting their warrants to the small group. The following quotes are examples of students choosing to ask questions for confirmation from one another rather than declaratively stating their reasonings as true. This first quote is from Group N's discussion about Task 1:

> Group N member: "You like just go through and take a few examples I guess, and all the ones I use worked."

Although this member of Group N goes on to explain her thoughts that a few examples would work, which reverts to the empirical argument, she shows uncertainty with the "I guess" statement. By qualifying her empirical arguments – which was discussed in the communal criteria as a weaker argument – with a statement of uncertainty, this member of Group M confirms a lack of knowledge or confidence in proving her argument's value. The following

quote is from Group F about Task 2, which asks statements to write an argument for or against the claim that the sum of two odd numbers is even.

> Group F member: “Two odd numbers, and if you do 10 - like the number divided by 2 because there's two odd numbers. If it equals a whole number, it's probably - it will be - it's probably. I don't know what I'm trying to say. It equals a whole number, so that means that's probably an even number? I really don't know.

Although this quote comes in the second iteration of collective argumentation, it comes near the beginning of Task 2, showing the student’s uncertainty by his statement of “I really don’t know.” Outright stating that he is uncertain of what he just said, this member of Group F is willing to present his ideas to the group, or perhaps is using the oral discussions as a time to state his thoughts and gather some feedback. This is by no means a poor learning strategy. Instead, it may be quite the opposite. However, the quote does provide affirmation of students’ unwillingness to be certain with their statements when working collectively in a collaborative discussion towards the common goal of a collective argument.

Beyond signs of doubt or uncertainty in their own reasonings, some students reasoned that their rationale was sound because of their belief that outside authorities could validate their mathematical claims. Because these students placed authority elsewhere, the justifications that they provided themselves proved to lack any authority in their own beliefs or thoughts. The next quotes are examples of students placing their trust in authorities outside of the truth of the mathematics that is accessible to them and appropriate for their learning at the middle school level. These quotes may reveal several different considerations about students’ willingness to resign at a mathematical task due to difficulty or inability to consider how to move forward

because of lack of knowledge of a mathematical argument, content knowledge, or other factors. These quotes come from discussions about Task 2:

> Group B member: "This is true because if you add up all your three – underline three times consecutive numbers and divide by three, you will always get a whole number, but, all capitalized--only consecutive numbers… I said it's true because if I use my calculator to add the three consecutive numbers, then they will be divisible by three."

Other members of Group M continued with the following quotes:

> Other Group B member: "I said it's true because if I use my calculator to add the three consecutive numbers, then they will be divisible by three."
>
> Other Group B member: "We have been shown by teachers. And calculators."

Beyond this group, several other groups were willing to state that their proofs were true mathematical arguments because the math that they checked with their calculators proved their calculations to be true, so their empirical arguments were veracious.

Overall, this initial coding revealed some deeper insights into students' approaches towards developing an argument once tasked with creation and justification with one another in small-group settings. The collaboration revealed that the students were quick to develop warrants as their main focus of an argument, utilizing empirical arguments particularly as either their initial focus or their only access to a mathematical argument. Additionally, the small-group interactions proved that students were either not entirely confident in their own justifications, or they were willing to be unconfident in themselves because of the small-group setting. As the students continued the instructional series and gained more experience with the collective argumentation process, the data revealed more insights into how students engaged in collaborative discourse with one another about their arguments.

Second Theme: Meaningful Backings to Explain Justification Based on Evidence

Because the instructional series repeated the same process of collective argumentation each day, it is also interesting to note how the students improved in their abilities to develop mathematical arguments over time. As previously stated, only Tasks 1 and 2 were coded for this analysis because of the lack of access to the content – and thus, meaningful arguments – in Task 3 prevented the data from being significant. As students began to see what a true mathematical argument is, and what an advanced mathematical argument upon finishing one entire iteration of the collective argumentation process, students began to speak differently about the mathematical task. Students recognized at the end of this iteration of collective argumentation that they were required to discuss with only each other, and the role of myself as facilitator was only to question and help as necessary. This gave the students room to explore, which developed some themes into the ways the students chose to explain their justifications with more robust knowledge in their backings. Rojas-Drummond and Zapata (2004) explain that exploratory talk might be a way for students to feel that they could express their own opinions, observations, and explanations of the mathematics in the task at hand. As a result of this exploratory talk, a majority of the groups increased in their willingness and ability to engage in meaningful conversation with one another to elicit more evidence in their rationale, which impressed major differences in their patterns of discourse with one another. Thus, many students provided more meaningful evidence in both their backings of their warrants and the justifications in their warrants.

Using Each Other: Collaborative Discovery of Justifications

Because of the nature of collective argumentation, collaborative work towards a more advanced cognitive mathematical thought was to be expected in this instructional series.

However, seeing the progression of collaborative work, specifically in the warrants of the students' arguments, is notable in comparison to their uncertain or unauthoritative justifications seen in the first task. Students recognized that they were not convinced of the truth of their explanations until they were able to converse with one another, or they didn't recognize the validity of a claim until they were able to converse with one another. The following are some quotes that show interaction between two groups as they come to the discovery of the veracity, in their minds, of a justification.

> Group F member #1: "And if you do the same thing for--yeah, if you do the same thing for odd plus even."
>
> Group F member #2: "OK."
>
> Group F member #1: "If you take away one, it would be either odd plus odd or even plus even."

Although this is just a simple exchange, the words and intonation show that the first group member is using this time to explain his or her thoughts out loud while writing, and the second group member is acting as confirmation, as they are both equally working towards full belief in the veracity of this mathematical thought about taking one away from odd plus even equaling either odd plus odd or even plus even. Although this thought is not entirely correct, and is not fully developed, it shows how the student progresses towards a basic mathematical thought about the math fact that the sum of two even integers is even. This shows up later in a backing for the warrant, with the following exchange that also takes place in collaboration with one another.

> Group F member #1: "Does all even plus even numbers equal even?"
>
> Group F member #2: "Yes."

Even though this is a short exchange, it is an example that shows that the group is now willing to not just provide a justification, but provide a *backing* for their justification, even if in a collaborative group where they are questioning one another for confirmation of truth in their thoughts. A similar exchange s held between two members of Group H:

> Group H member #1: If there's an odd and an even...
>
> Group H member #2: No, if you add an even and an odd, it will always be odd.
>
> Group H member #1: And an even and an even will always be...
>
> Group H member #2: Even.

Again, this is a very simple exchange. However, the finishing of each other's sentences and wiliness to interrupt show that students are comfortable with the argumentation process, and they are using each other's thoughts to confirm their math facts about the sums of odds and evens. In this case, the second member was actually correcting the first member's original thought (not shown), so this exchange is actually classified as a rebuttal, which is used to revert back to the truth of the warrant. In this situation, the rebuttal acts in a way that the backing does, except it is asking to question the justification for the warrant rather than support it for the sake of seeking the truth in their statement. Whether students are working through thoughts together or using each other for confirmation of truth, students learned that they could use one another as authorities to move towards justifications when they collaborated in small-group discussions.

A Critical Shift: Movement towards Assured Utterances

As students began to gain more confidence by working with one another, their statements moved away from questioning to more authoritative as they gained confidence in their ability to make sound mathematical judgments. As a result, students began to discuss more mathematical

facts as evidence of their claims, addressing one of the communal criteria that they realized was necessary for a mathematical argument, particularly as they had the opportunity to attempt another full cycle of argumentation on Task 2. The students learned more precisely what makes a more veracious mathematical argument after the class-wide critique and justifications at the end of class on the first day with Task 1, and they were able to appeal to their own authority and utilize mathematical facts in using more backings to support their justifications in the warrants of their arguments. The following are a few examples of groups who used sound mathematical facts in their backings to show more advanced assuredness in their discussions with one another. This first example shows a student explaining his thoughts to another, utilizing two different mathematical facts that they have learned in previous math courses as justifications for his reasoning on the second task:

> Group E member #1: "So, it all goes up by 4 each time. Figured that out."
>
> Group E member #2: "It all goes up by 4?"
>
> Group E member #1: "That's because an odd is every other number…"
>
> Group E member #2: "So, that's interesting."
>
> Group E member #1: "…and every other number twice is 4."
>
> Group E member #1: "Um, if you go back down to the basics, 7 plus 5 um, is 12. Yeah, it is - OK. And 7 plus 3 is 10. And all the basic, tiny numbers, the one-digit numbers. They all equal evens, so that means, uh, because it just depends on the last number in the number, uh, to make it an even. So, since all of the one-digit numbers are even, it just comes down to the one-digit numbers in the big number."

In this exchange, the first group member is moving beyond simply stating that the sum of the next pair of odd integers if 4 greater than the previous pair of odd integers. He qualifies that an

odd number is every other number, implying that the next odd number adds 2 to the previous number, and not just 1. Although this is not stated as explicitly as possible, choosing to explain how the sum increased by 4 because odd numbers are every other number is a movement towards using supporting information, or a backing, for the warrants in their justifications.

This next example shows a student excitedly exclaiming their discoveries because they are working with one another and willing, or even wanting, to share their thoughts with the other group members. Because of the willingness to engage in mathematical discussion (Cáceres, Nussbaum, Marroquín, Glesiner, & Marquínez, 2018), the students' thoughts become those of more advanced quality when they realize that they must provide backings to their warrants. The students often refer back to the communal criteria of needing evidence and making their arguments true for all cases, which prompts them to provide a warrant to make the argument sophisticated enough that it is valid for their mathematical learning community. The following is an example of the students' excited engagement as their thoughts promote a higher cognitive thought:

> Group J member #1: "So, I tried subtraction. Like, if you do an even number minus an odd number, you get an odd number. So, like 22 minus 9 is 13. Or, like 14 minus 3..."
>
> Group J member #2: "Wait, wait. Why would you do minus if it's the sum?"
>
> Group J member #1: "I'm like reversing it."
>
> Group J member #2: "Oh! To see..."
>
> Group J member #1: "Yeah. So, like 14 minus 3 is 11."

Although this thought was entirely developed, because both group members seemed to understand each other's thoughts, it's clear to see that the first member of group J was trying to find a mathematical fact that proves the sum of two odd numbers as even by stating it in a

different manner, or the difference of an even and an odd number is always odd. By the willingness to question the first group member's thoughts and the excitement in understanding what he or she was doing, it's clear to see that the students are now more willing and even more engaged by discussing their thoughts with one another.

In summary, many students showed great progress in their willingness to engage in and understanding of a true mathematical argument when grouped together to discuss in collaborative small groups. Based on the analysis of this research, overall, students were more comfortable to begin with what they knew and recognized as accessible, particularly empirical arguments. Although a minority of groups did not show much increase in diversity in the types of discourse in which they engaged, particularly in relation to various elements of Toulmin's model of argumentation, most groups began to move into patterns of discussion that appealed to more advanced mathematical reasoning because of their willingness and desire to qualify their statements, provide rebuttals for critical analysis, and justify their arguments with deeper warrants and backings of their arguments. When asked to advance to a higher mathematical argument, the students were reliant on other authority or questioned their own thoughts. Then, as students recognized the process and the value in the process of collective argumentation, they were willing to use their voices with one another to collaboratively discover mathematical truths, and they even gained more assuredness in their natural utterances as they casually discussed the validity of their mathematical claims. All these thoughts put into perspective the need for students to recognize argumentation as a valuable learning tool in the classroom, and collective argumentation very clearly meets all three conceptualizations of argumentations seen in literature today. Firstly, the communal criteria discussion pushed students to find more evidence in their thoughts, and thus provide better backing and warrants to their arguments, creating an

atmosphere where argumentation is seen as convincing. Secondarily, the design of the instructional series requires students to discuss with one another and no other authority, including the teacher, textbook, or any other resource provided by the teacher. This, by nature, makes the process of argumentation a socially-embedded practice. Finally, at the heart of each task is a problem to me solved in mathematics, which requires students to engage with one another in discussion and thought to arrive at a viable explanation for the potential problem at hand. Recall that, at the heart of argumentation, one is searching for truth (Joshi, 2016), and collective argumentation seeks to allow students to find truth in mathematics through their collaborative discussions in collective argumentation. As Civil and Hunter (2015) stated, the social atmosphere in a mathematics classroom that allows students to participate lets them be comfortable to share what they want and to make mistakes, which was evident in this analysis. More research into the students' insights would reveal more about how teachers can choose to pedagogically implement mathematics lessons where students engage in the social classroom while confidently believing in their thoughts to justify veracious mathematical arguments.

Finally, to address the third research question, it is beneficial to take a look at how students have transformed in their own identity as doers of mathematics as a result of collective argumentation. Firstly, students began to recognize a change in the normative identity of the class because they see that they are not only allowed, but encouraged, to engage in mathematical discussion with one another in collective argumentation (Cobb, Gresalfi, & Hodge, 2009). The shift in voice and authority from teacher-directed activities to student-oriented discussions changed the students' patterns of discourse in the mathematics classrooms, as recognized by their increased engagement with discussions with one another. Secondarily, it is important to note that students' personal identities appeared to have grown with their confidence in the collective

argument tasks. Student utterances of “We’re the smart group!” and “I came up with another way to prove it” show student personal beliefs in themselves as able to do mathematics, both individually and collectively within the small groups. Future research on student reflections after collective argumentation would provide more information on student identity in collective argumentation, but this analysis showed a genuine impact on student ability to create mathematical arguments through interpersonal discourse in the collaborative discussions that students had in collective argumentation.

CHAPTER 4.3

ANALYSIS AND FINDINGS

Beyond analyzing both the argument improvement of students via collective argumentation and the elements of an argument via Toulmin's model in small-group discourse, this analysis seeks to understand what aspects of the small-group discourse and argumentation impacted change in student identity in the mathematics classroom. Mathematics education research has established the importance of recognizing the classroom as a community in which students socially interact with one another as well as the activities and the content to serve an overall purpose of improvement in mathematics learning (Nasir, 2002). In looking specifically at the classroom community, activities that became regular practices (Cobb & Hodge, 2002) and classroom design (Gresalfi, Martin, Hand, & Greeno, 2009) have centered the analysis of this study on collective argumentation as a practice that designs the classroom as a time and space to engage students in mathematical learning. Further, a recognition of student personal success stories frames this analysis in a way that highlights the comments that students made in response to the survey questions before and after the instructional series to elicit themes about student transformation of mathematical identity, through both constructs of normative and personal identity.

Understanding Normative Identity

To delve deeper into the idea of normative identity, Hodge and Harris (2015) state that the valuation of norms from the participant's perspective leads to informing the design of the mathematics classroom learning experiences. This view of identity helps inform students' vision of what constitutes mathematics via the classroom actions and interactions, as it clarifies what

types of thoughts students engage in to reveal their identity in mathematics because the students' views center around what they believe it means to do math. Additionally, normative identity takes into account how students choose to consider themselves competent in mathematics and their decision to persist in the process of learning mathematics, as well as their value in the experience of engaging in competence and persistence in the mathematics classroom.

Understanding Personal Identity

Personal identity, then, concerns the extent to which students identify with, cooperate with, or resist engagement with the classroom activities (Cobb, Gresalfi, & Hodge, 2009), as a result of their engagement and interactions in the activities in the class which have the intent of promoting mathematical learning. Moving away from students' views of what is mathematics and their decision to persist in mathematics to become competent, personal identity looks at students' valuation of the obligatory activities that they engage in during mathematics class, their valuation of these obligations, and their reasonings for assigning these values to these classroom activities (Cobb, Gresalfi, & Hodge, 2009). With this in mind, research pays close attention to the ways in which students assess their own capabilities in mathematics, as well as the capabilities of others, particularly as students engage in the classroom activities and the extent to which they choose to contribute to the mathematics classroom.

Further, it is important to note that, due to the dynamic nature of activity in a learning environment, normative identity and personal identity are consistently being reformed and transformed as students respond to the various elements of the activities in a mathematics classroom, including but not limited to the content, other students, and the setting and materials used in the mathematics classroom. Keeping this in mind, the data collected and analyzed for this

study reveal students' initial identity before the instructional series and their transformations as a result of the instructional series.

The data collected were a qualitative pre- and post-survey that reflected the same questions concerning students' beliefs about doing mathematics and about doing mathematics in small groups, as well as their abilities to create mathematical arguments. Through open coding for themes in the students' responses to the questions, I discovered some patterns that were prevalent in much of the student responses concerning students' beliefs about themselves and their mathematical identities, both before they began the study and their transformed identity as a result of the instructional series. Coding according to different themes about normative identity and personal identity, I discovered different subthemes for each construct of identity in the students' responses.

Normative Identity

The norms that define the class or instructional setting define normative identity. In this study, the norms are changed based on my view as the researcher and the facilitator of the instructional series that implements collective argumentation. It is important to note that the setting of this instruction is a unique situation to the participants in this study, thus the sociomathematical norms will be very different from what the students are accustomed to. The students in this study do not regularly engage in small-group discourse that encourages mathematical argumentation. By recognizing how students' normative identity will change because of the different expectations placed upon them by myself as the facilitator of the instructional series, I can delineate the sociomathematical norms that will likely impact students' personal identities as doers of mathematics, as well as the transformations in some students' mathematical personal identities as a result of this instructional series.

Working in Small Groups

The students will be arranged in a way that the will work within their small groups on each task to not only create a veracious mathematical argument, but also to engage in discussion to arrive at their final mathematical collective argument. The norm here is different from regular classroom routine, as students now assume the authority of their learning, and discussion between each other is encouraged and not discouraged since the teacher is not at the center of the discussion or leading the lesson in the mathematics classroom today. Further, this implies that studies should be respectful and open to each other as they work towards a collective argument. This also applies as the small groups work with each other to present their arguments to the whole class for validation at the end of the collective argumentation in the instructional series.

Encouraging Mathematical Discourse

Further, students will be encouraged to engage in discussion not just to answer a specific question, but to question each other, create arguments, and even disagree with one another to develop a deeper mathematical argument. The expectation of regular discourse will push students to continually discuss with one another but also to push them to think critically deeper about the mathematics content presented to them because of their task to create a mathematical argument in accordance with the communal criteria that they created at the beginning of the instructional series.

These expectations and changes in norms in the design of collective argumentation impact a new normative identity for the design of the mathematics classroom, and they will inspire different views in students' personal identities and transformations of students'

mathematical personal identities as they engage in discourse, work in small groups, and create mathematical arguments.

Personal Identity

Beliefs about Creating Mathematical Arguments

Perhaps due to the relative inexperience the students had in creating mathematical arguments before the instructional series, several students stated that they had little belief in their ability to create a mathematical argument. The third survey question asked students "Do you think you can make a mathematical argument or proof? Explain." The following are just a few examples of statements that students made in response to this question, indicating their doubt about their abilities to create mathematical arguments. As a reminder, all names below are pseudonyms.

Darius: "No, because I don't know what I am talking about."

Linus: "Maybe, but I'm not very experienced in that field."

Kelly: "Probably not because I don't know that many math terms."

Lola: "No, because I don't know a whole lot about math to make a good statement."

Several of the responses reflected something of this nature, indicating that they do not know what the nature of a mathematical argument is, or indicating that it just is too much time or effort to gain the experience to successfully create a mathematical argument. Only about one out of four students in this study responded with anything positive to say about being able to create a mathematical argument. Four students responded with utmost confidence and a reason, varying between ideas about mathematics and ideas about their own abilities. Their responses are shown below.

Patricia: “Yes I can (most of the time) articulate what I did in a problem.”

Ralph: “I can make [a] proof for a problem having an answer and how to solve it.”

Sam: “Yes because I can use words.”

Jeremiah: “Yes, I’ve done it before. I can do it again.”

These responses are the only few that show confidence in ability to do a proof, even if the reasoning is lacking in explanation or rationale. The remaining seven responses, although positive in their initial responses, reveal some sort of conditions or doubt to their abilities to create a mathematical argument. Several responses show that students feel that they must have some other information or authority to gain some ability to engage with the problem (such as knowing the concept or having an answer

Manny: “Depends on how well I know the formula.”

Daniel: “I think I can if I understand the problem I’m given.”

Phoebe: “Yes, if I really think.”

Amy: “Yes, but I can’t guarantee it will be any good.”

Felicia: “I think I can if I understand the concept.”

AJ: “Yes if I have proof.”

These responses reflect the idea that there must be some condition, pre-existing knowledge, or time or effort to dedicate to creating a mathematical argument or proof. This lack of complete confidence, according to the answers provided to the survey in this data, suggests one of two things. In context of mathematical argument, these responses suggest that there is room for improvement in the abilities to create a mathematical argument. Students show indication of doubt or uncertainty because they know that they can improve in what they can currently do in developing a mathematical argument. Secondarily, these responses also reflect the students’

potential views of their abilities and competencies in mathematics in general. That being said, the students in this study showed some strong beliefs in their lack of mathematical knowledge or affinity towards mathematics and mathematical argument development.

Beliefs about Mathematics in Small Groups

The survey also questioned students on their beliefs about being able to do mathematics in small groups. These answers varied based on how students approached working with each other. Some students saw the social benefits to working with others, while others saw the cognitive benefits in working towards a more correct answer, while others focused more on the process of getting to an answer or a correct answer in the mathematics. The following subthemes break down students' various responses to beliefs about working on mathematics in small groups.

Helping Others. Several students made comments about helping others, including some of the following quotes:

Ronnie: "I like that I can help others."

Ashley: "I also like being able to get help from peers by working together."

Kevin: "I like being able to ask for help or walking through a problem."

Lola: "I like working with others to see what they came up with and helping them with the problem."

Manny: "I like that I can get help or check to make sure we did it the same way."

Daniel: "I like it because I can help others."

Samantha: "I like being able to ask for help when I need it."

Several students discussed being able to ask for help, or that they enjoyed giving help to others, which also means that students on various confidence levels of mathematical abilities

commented on the idea of giving and receiving help in different manners. Some of the students explicated that this works best for them because they can walk through the process of a problem, as well as compare to someone else's thoughts or ideas. In alignment with the standard of mathematical practice published by NCTM (2000), students see that importance of critiquing each other's reasoning when they are given the opportunity to work in small group and ask for help.

Being Able to Talk Through the Mathematics. Other students went into further detail about small group work contributing to their ability to talk through the mathematics. They expressed that the process of discussion with one another was beneficial to their growth in mathematical understanding. The following are a few of these examples.

Ashley: "I can expand my thinking by seeing how other people think."

Darius: "I like it because we can all talk about our mistakes and learn to fix them."

Sadie: "I like getting other people's opinion or having them explain to me."

All these comments indicate many of these students' preference for group collaboration because someone else is able to help them learn something that they could not or did not understand beforehand without the help of the groupmate. This is important to note because it reflects the students' overall positive views of mathematical learning in the context of collaboration through student discussion within a small group. In reference to the discussion about normative identity, this is valuable for teachers as facilitators of small-group work so that, when classroom lessons are designed appropriately to maximize students helping one another in small groups, students can begin to develop a better sense of the mathematics content at hand and cognitively understand the mathematics at a deeper level.

Beyond being able to receive or give help or discuss the mathematics with each other, students also began to show changes in their attitudes and beliefs about themselves, impacting their personal identity within the secondary mathematics classroom. As literature has previously established, a student's personal identity is a reflection of how students regard mathematics, as well as their own individual view of mathematics. Through the survey responses, students revealed how they initially viewed math people and themselves as math people before the instructional series.

Time. A few students made comments with negative connotations towards working in groups because of the time necessary to complete a task in small groups, generally indicating that groupwork requires more time than if they had worked individually. The following are quotes from some of the students discussing time as a negative factor to completing their mathematics work:

> Gerry: "I do not like group work in math. I'm a bit of a lone wolf in this subject. I'm always way ahead of everyone else because [of] my fast pace. I have to hang back, wait, and answer questions."
>
> Manny: "I don't like it 'cause sometimes they (partner group) move [too] fast."
>
> Daniel: "Sometimes I hate it because people are slower than me."
>
> Ralph: "Sometimes people disagree, and it takes longer."
>
> Phoebe: "I don't like how some people are slow and fast."
>
> Amy: "It is easier, so I don't get super frustrated. When I'm mad, I can just ask my group. It is also quicker."

This is an interesting finding because it shows that students value time when working to complete a mathematical task. This may due to students' perceptions of themselves based on

comparison to others, or it may be just a desire to complete a task on time for the sake of following directions. No matter the reason, students put much value in the time necessary to complete a mathematical task. Further, one can see that this affects students who both consider themselves fast and slow at mathematics, so groupwork should be carefully considered for students of all abilities in mathematics classes.

Negative Feelings about Personal Intelligence. Still another comment that students made were comments about either being the person who just didn't "get it" or felt "stupid" about the mathematics. On the opposite end, one student made a comment about having to be the only person who understands the mathematics, thus he or she has to take time to explain the math to everyone else in the group for not understanding. The following are quotes from some of the students expressing this concern about group work in mathematics:

Laura: "I like helping people, but I don't like people telling me I'm dumb for doing problems differently."

Linus: "I don't like being paired with someone who doesn't understand."

Negative Feelings about Working Independently in the Group. Finally, two students made comments about how sometimes small group degenerates into just one person doing all the work, thus they find the small group work unnecessary and unproductive. These are the two students' quotes indicating this negative affinity towards group work in mathematics class:

Al: "I don't like it being just one person working."

Felicia: "I like being able to work together and compare answers, but sometimes one person does all the work."

Students expressed several different feelings about working in small groups in mathematics class, both positive and negative. It is worth understanding how students identified themselves as doers of mathematics with these comments taken into consideration.

Beliefs about Being a "Math Person"

Being "Good" at Mathematics. Students defined being a "math person" in a few different ways, but several comments narrow down to a more dichotomous view of doing mathematics. Students either believed that being "good" at math defines a math person, or that that being good or even "best" at math comparative to other subjects defines a math person. When asked if students are a math person, the students responded with responses that ascribe to this dichotomous belief of being good math or not good at math. The following are the best examples of comments that reveal the insights into these students about being "good" at mathematics. The following are student responses in the pre-survey to the question that asks: "Do you consider yourself a math person? Explain why or why not."

Phoebe: "Yes because I'm not bad at math."

Joe: "Yes because it's what I'm best at."

Felicia: "Yes, I enjoy math because I understand it."

Samantha: "Yes I am good at math."

Daniel: "Yes I do because I understand it and I'm good at it."

Only one student mentioned that being a "math" person is defined as finding it fun, or some response to the doing of mathematics instead of their self-perceptions of being able to do mathematics. The quote by this student follows here:

Lola: "Yes, because I believe it's fun, yet challenging."

Being "Bad" at Mathematics. Several more students, however, made comments about being bad at mathematics, believing in a similar dichotomous view of a math person. Their reasons are similar to students above – they either find it confusing or have difficulty with understanding the math, or they do not enjoy the subject or engaging in doing the mathematics itself. The question asked directly about students' general views of mathematics, so the answers may vary in students' beliefs about the ability to change their level of mathematics, as some may believe their mathematics ability is simply inherent. Either way, several students made a comment that reflected this dichotomous view of mathematics. The following are quotes in response to the same question where students consider themselves bad at math:

Stella: "No because I'm not good at it."

Alex: "No, I do not understand everything about it all the time, and it can be confusing."

Ashley: "No, because I just don't enjoy it."

Kaya: "No because I don't really see math in my future and it's not my favorite."

Darius: "No I have never been good at math or liked it."

Laura: "No, I struggle to understand/memorize formulas."

Ralph: "No I am not good at figuring math problems out."

Some of the students mentioned that the process of doing mathematics is not fun or enjoyable, hence why they believed themselves to not be math people. Their responses to the same question follow here:

Ashley: "No, because I just don't enjoy it."

Al: "No I don't like doing math."

These responses vary slightly from the previous ones because they refer to the process of doing mathematics as their belief in mathematical identities, and not their beliefs in their own abilities to do mathematics as their definition of a math person.

Thinking Critically. Only one student commented on being a math person as being someone who thinks critically or is able to cognitively retain or think through mathematics problems. The following response to the same question that reveals the theme of critical thinking is listed here:

Jeremiah: "Yes, I love thinking critically and challenging myself."

This student revealed his belief that math people are those who are able to think critically, retain and understand the information to solve a math problem, and to enjoy the challenge of mathematics. Still other students believed that outside factors defined the characteristics of a math person.

External Factors to Mathematical Ability. Interestingly enough, a few students mentioned that their ability to do mathematics quickly or to persevere when mathematics is not understood quickly or easily defines their self-beliefs about their mathematical abilities and thus their mathematical identities (or the lack of these qualities informs their negative mathematical identities or keeps them from seeing themselves as math people). Their responses to the same question above are listed here:

Manny: "No because I give up too easily on math problems."

Alicia: "Yes because I'm good at learning it quickly."

Finally, it is clear to see that a few people find that math people are those who either persist in their attempts to do mathematics, implying that there is struggle initially in understanding mathematics, but they are willing to push through, or seem to understand the mathematics

immediately. Manny, particularly, notes that persistence in mathematics is a quality that makes a person a math problem, as he does not consider himself a math person because he gives up too easily.

Excitement Towards Mathematics

Although several of these comments that the students made were on a wide spectrum about what makes a math person, some comments to one question on the survey reveal what makes them excited to do mathematics. These more personal comments reflect what students already believe about what makes them excited for mathematics before the instructional series, as the survey question states: "What makes you excited when you do math? Explain." These answers reflect different sentiments from the question about being a math person, as it personalizes the question to each student concerning their excitement level towards mathematics. The following responses are chosen to reflect various pedagogical tools, and the explanations about each will follow. The first quotes show students' excitement when they are able to do math correctly.

Peyton: "When I don't fail the class or do a difficult problem correctly."

Lola: "When I answer a problem correctly, it boosts my confidence a little and I like that."

Felix: "Figuring out a solution."

Daniel: "Understanding it, being challenged by it, and getting the problems right."

These students place excitement in math learning in their ability to do it correctly, while others look at more external factors in the learning process, such as being able to display to others their knowledge, working hard, playing games, or having food present. All these responses are listed below.

Linus: “This may sound a little egotistical, but I like showing off my intelligence.”

Ralph: “When I work hard and get good grades.”

Amy: “Playing games to help learn it can be fun.”

Quincy: “When it’s simple or when food is involved. It makes it better to listen.”

These first two responses, although more unique than the others, indicate that students have a general interest in being able to express their intelligence and see the fruits of their labor. Additionally, other students also found excitement in mathematics learning in class when the instructional design is purposefully altered to something more unique than the norm, such as having food or playing a game. The students provided different reasons for this excitement, such as better engagement or finding more fun in the learning, but they were important responses to consider.

No matter the views of what makes math exciting, students showed great interest in doing mathematics correctly, which directly impacted their personal identities. The final section of this analysis reveals how students generally evolved in their identities as a result of engaging in the instructional series that emphasized student interpersonal discourse in small-group collaboration for argumentation.

Transformations of Identity

The study sought to see the impact that small-group collective argumentation would have on students’ transformed identities. Upon completion of the instructional series, the same survey was given to students that was given to them before the instructional series to better reveal any changes in students’ normative or personal identities in the mathematics classroom. The survey revealed three unique transformations in students’ mathematical identities. The first two themes – creating mathematical arguments and working in small groups – reveal a general shift in

students' views of their abilities to do mathematics by creating mathematical arguments and by working in small groups. Some small examples of quotes chosen here reveal comparative comments before and after the instructional series.

Creating Mathematical Arguments

As noted in the surveys administered before the instructional series, several students did not feel that they could create a mathematical argument. The overall belief in the ability to create a mathematical argument changed as a result of this instructional series, with most responses to the survey question revealing a better understanding of a mathematical argument and a better confidence in being able to create a mathematical argument. These responses to the third question in the post-survey, which states "Do you think you can make a mathematical argument or proof? Explain," reveals students' transformations as a result of better understanding a mathematical argument.

> Daniel: "Yes I can! Now I understand how to show proof of how a math problem is solved and why it is solved a certain way."
>
> Ralph: "Yes because if you critically think through a math problem you can prove it."
>
> Linus: "Yes I can, by analyzing what I have to prove or disprove and find special examples."
>
> Phoebe: "Yes if I have evidence and facts."

These examples show that these students seem to better understand what makes an argument or how to approach making an argument (such as using examples as a starting point), or have a better confidence in being able to create a mathematical argument as long as they critically think through it or because they know the criteria it takes to make a mathematical argument. It should be noted, though, that perhaps some students now simply had a better understanding of what

defines a mathematical argument, so they recognize upon completing this instructional series that they indeed can create a mathematical argument, while before the instructional series they could have created a mathematical argument without recognizing that they could according to the definition of mathematical argument for their particular classroom learning community. Even considering this observation, these examples provide a great insight into recognizing how students can improve in creating mathematical arguments just by better understanding what defines a mathematical argument and what makes up the elements of a mathematical argument.

About half of the students, though, still stated that they cannot make a mathematical argument. These comments reflect similar ones in the pre-survey, but the number of students with this negative response has decreased, and the responses center more around the difficulty of making a mathematical argument and not about their inability to understand the nature of a veracious mathematical argument. Only a few responses stated that they still could not make a mathematical argument because they did not understand what it is. This shows an overall increase in the data of student understanding what is a mathematical argument as a result of the instructional series on collective argumentation.

Working in Small Groups

In response to the question about the small groups, some students said similar comments about being able to get help, collaborating with others to check work, and being able to talk through the mathematics. However, a few responses reflected a more robust view of small-group work in mathematics. They reveal themes about how students' personal identities transformed as a result of the collective argumentation process. The following quotes show an overall more positive outlook about small-group collaboration because of the discussion in the small-group collective argumentation.

Quincy: “We get to have fun and be more loose. Also, we have ideas if we don’t understand, others can help.”

Phoebe: “It’s fun to [work] together and see how we all figure it out together.”

Amy: “If we don’t understand, we can ask each other.”

Kelly: “I like that I can ask others and see different ways to do it.”

Although these quotes were largely positive responses to group work, the instructional series of collective argumentation did reveal more negative responses to group work because of the way that the group dynamics worked. The following are responses that had more negative views about working in small groups.

Time. A similar number of students made comments with negative connotations towards working in groups because of the time necessary to complete a task in small groups, generally indicating that groupwork holds them back from working at a quicker or more efficient pace in comparison to working individually. Gathering all the responses from every participant in this study, it is important to note that only one student repeated this same sentiment before and after the instructional series (Gerry). The other students are new students realizing this notion. Their responses are given below.

Gerry: “I don’t like working in groups. I usually find myself ahead and don’t want to wait on others. You could say I don’t play well with others.”

Link: “I like being in a group with friends, because it is fun to work it out together. I don’t like it because I work quicker by myself.”

Sadie: “I like that if we don’t get something, somebody else can teach, but sometimes if I really get it and I am on a roll, I just like to go at my own pace.”

Kevin: “I like working in groups because we can share ideas and help each other out, but I don’t like it sometimes because I can’t work as fast.”

Garrett: “I don’t like it because it takes me longer. And, when I get an answer on a problem and I know it is right. And some people have different answers, I get confused.”

The number of students who mentioned time were very similar before and after the instructional series, but the fact that almost all the students who made this comment before did not mention it after the instructional series implies a positive response to collective argumentation as group work, but similarly the fact that four new students mentioned time after the instructional series implies a negative response to collective argumentation as group work. Thus, the opposite sides of this finding are important to note that different students responded differently to the collective argumentation lesson design in terms of their perception of time and its impact on their view on mathematical group work.

Negative Feelings about the Design of Group Work. Perhaps the most novel discovery about collective argumentation and these students’ views of small group work is their negative view on the design of collective argumentation itself. Students made comments about not being able to be heard by the group, or frustration with disagreements, or just not being able to reach their maximum potential that they could when working individually. The following are quotes from some of the students expressing this concern about group work in mathematics. The first three examples are about students preferring to work individually to meet their maximum potential in mathematics:

Laura: “I like having someone check my work. I don’t like when people call me stupid or doubt me when I’m clearly right.”

> Kaya: “I like having the group for support or if I need help, but if I know how to do it I’d rather do it myself.”
>
> Daniel: “I am okay with doing math in groups but I wouldn’t prefer it. I feel like sometimes being in a group at least in math holds me back from solving problems the best I can.”

These last two quotes indicate that the design of collective argumentation affects overall the behavior of students in engaging in the mathematical activity, as sometimes others just do not listen or do not focus on the task at hand:

> Lola: “I like being able to work with people and hearing their ideas. I don’t like it because I sometimes can’t get my ideas through to others (in other words they don’t listen to me.)”
>
> Amanda: “I don’t like it because some people don’t actually do math and are careless, and it’s hard to focus.”

The reasons why students varied in their negative views of small group varied from distraction to personality to speed and efficiency in finishing the mathematics, but all the answers reveal a necessary implication for practicing small-group work in secondary mathematics classrooms. When small-group work like collective argumentation emphasizes student communication heavily, like the creation of mathematical arguments and critiquing each other’s’ reasoning as encouraged by NCTM (2000), teachers should focus on how to socially situate the students in a way that distractions, personality conflicts, and time and efficiency problems are minimized so that all students can engage in and find value in their discourse with one another as they produce mathematical arguments in collective argumentation.

Finally, students' excitement levels about mathematics and their views of doing mathematics changed dramatically as a result of the small-group discourse in collective argumentation. Several students began to evolve their view of mathematics beyond a dichotomous belief in being "good" at mathematics, the external factors that make math enjoyable, or finding excitement in doing mathematics. The following quotes reveal how students' identities as doers of mathematics changed as a result of collective argumentation

Being a Doer of Mathematics

Revisiting Being "Good" at Mathematics. An interesting result of this post-survey is that more students ascribed to the dichotomous view of a math person, with several more responses repeating similar sentiments found in the pre-survey about being a math person. More students utilized statements about being "good" or "bad" at math, quoting similar reasons such as not being able to quickly solve the problems, not getting past confusion, and just stating that mathematics is something that has always been hard for them or something that they've always excelled at or enjoyed. An interesting finding to this study is this increased dichotomous view of being "good" at mathematics, implying some necessary practitioner recommendations for further implementation of collective argumentation and student discourse in the future.

Revisiting Excitement towards Mathematics. Sometimes, students see doing mathematics as a characteristic about the excitement that they discover in solving mathematics. The following are quotes from the students' post-survey that reveal their view of being excited about mathematics.

> Gerry: "Whenever I find a hard problem, I feel like it taunts and challenges me to complete it. Like an itch that makes me want to complete it."

Daniel: “Just learning something new and catching onto what the teacher is talking about and understanding.”

Linus: “I like the feeling whenever I’m on fire going problem after problem.”

Laura: “When I do something well, easily, or when something clicks.”

Kaya: “Being able to know how to do the math makes me excited or when something clicks in my brain and I understand it a lot.”

These quotes reveal that they began to see a math person or a doer of math person not as someone who is smart at math, but as someone who sees when the math “clicks,” or persists and suddenly has a breakthrough in thinking. This is a powerful tool to capitalize on when instructing students to have a growth mindset so that they may become better students in mathematics.

But beyond the students’ discoveries of mathematics, a few of them still resonated with the idea that excitement for mathematics is in the classroom design. The following quotes are examples of such sentiments, in response to the same question as above.

Amy: “When we play a game or sing a song.”

Bethany: “I like when you have an immersive class. For instance, when we do scavenger hunt those are fun.”

Even if these answers are similar to above, it highlights the idea that collective argumentation, or whatever pedagogical tool a mathematics teacher uses, can be an engaging tool that changes a student’s ability to create mathematical arguments, as well as his or her identity and view of learning mathematics. This provides great insight into the implications for the design of the lesson by the teacher to involve students in a way that they fully understand collective argumentation and how to utilize each other as resources for learning when they engage in collective argumentation.

Summary

The findings in this study reveal several different findings concerning students' beliefs about their abilities and competencies to create mathematical arguments, about working in small groups, and about their own views of themselves as doers of mathematics. It is interesting to note that, in general, most students in this study had a better understanding of and better confidence in development of veracious mathematical arguments. When students are given criteria and time and space to practice the creation of a mathematical argument, they generally, according to this data, improved in both knowledge of and ability to create veracious mathematical arguments in their small-group collective argumentation.

Beyond argument development, students had mixed views about working in small groups in general, both before and after the instructional series. Several similar responses were found in the answers to the pre-survey and post-survey, but the data clearly shows that more students saw the value in helping each other and being able to talk through the mathematics, but many students highlighted the issues with small-group work when not utilized well, including lack of focus, lack of working efficiently, and lack of being able to participate with the small group, to name the main examples. This provides important feedback to consider when teachers design a small-group lesson such as one that revolves around student discourse like collective argumentation.

Finally, students' identities for the most part began to show more positive views of themselves as doers of mathematics as a result of collective argumentation. More responses to the post-survey ascribed to the dichotomous view of being either "good or math" at mathematics, but more students were able to identify as a math person by seeing how they could contribute to the math learning and be "good at mathematics." Other findings showed more general

excitement for mathematics when given the chance to work in small groups and work through a difficult problem such as the creation of a mathematical argument or proof. Interestingly enough, the post-survey also revealed a few students having a better affinity towards the process of doing mathematics instead of just being "good" at mathematics because of their more in-depth responses to the questions. They recognized that "doing" mathematics can make someone a "math person," and not just being considered good at it, which is a great result of collective argumentation for teachers to be able to capitalize on when engaging students in discourse or in small-group work.

Overall, students seemed to have a better understanding of what defines a mathematical argument and a more open idea to what a small group can both contribute to and detract from students' learning of mathematics. Additionally, further results have revealed that students did reveal more negative dispositions towards small group work as a result of collective argumentation, as well as an increased belief in the dichotomous good vs. bad mathematical abilities that define a doer of mathematics. Finally, this data analysis revealed a number of students seeing an excitement in the process of doing mathematics that helped shape their view of one's personal identity as a doer of mathematics.

These findings provide necessary but important implications for what can happen in the future of mathematics education. For teachers, lesson designs can be framed around argumentation so that students engage in productive discussion with one another that pushes students to improve in their abilities to create veracious mathematical arguments that inspires a better view of themselves of doers of mathematics. For researchers, there are more aspects of collective argumentation to consider so that math educators and math education researchers can understand how individual students engage in discourse, and what elements of discourse truly

contribute to students' advanced mathematical learning through collective argumentation. Thus, I will complete this dissertation with necessary and important implications for future practice and research involving student collaboration, student small-group discussions, and student argumentation.

CHAPTER 5

DISCUSSION AND IMPLICATIONS

Student collaboration has been established as an important learning tool for K-12 mathematics classroom, as supported by the recommended principles for mathematical practice by both the National Council for Teachers of Mathematics (2000) and the Common Core State Standards for Mathematics (2010). Specifically, the idea behind the mathematical practice of constructing viable arguments and critiquing the reasoning of others allows students to see that creating a mathematical argument successfully is relevant to problem-solving (Harel, 2017), which is an expectation of mathematics class on a regular basis. Beyond the standards, though, student collaboration centers around the idea of students engaging in discourse with one another. To emphasize the development of a mathematical argument, the process of collective argumentation has proven successful in opening up spaces for students to have a voice while the teacher still advances mathematical learning and other important standards of mathematical practice. As literature shows, classroom discussions allow the teacher to monitor student learning (Lampert, 1990), and collective argumentation is one example in mathematics education where students are able to engage in focused but open discussions with one another. In K-12 mathematics, mathematical argumentation is defined as students participating in discursive communication to develop and make sense of mathematics as they inquire about the reasoning of others and justify their own (Ball & Bass, 2003; Rojas-Drummond & Zapata, 2004). However, many students' competencies to successfully engage in argumentation are still developing, as they struggle with reasoning and justification. Typically, because students do not have enough opportunities to engage in problem-solving activities that encourage development of reasoning

skills that enhance mathematical arguments (Mueller & Yankelewitz 2014). The purpose of this study was to examine how middle-school students use collective argumentation to advance mathematical arguments and the ways that the interpersonal discussion in their small-groups impact students' mathematical arguments and identities. Applying these insights to K-12 mathematics classrooms can inform researchers, educators, and teacher educators of approaches to develop students' mathematical argumentation abilities, collaborative discussions in mathematics, and mathematics identity. In this chapter, I detail this study's relevance to today's mathematics classroom, its practical and theoretical implications, and my growth and learning from conducting this research.

Application to the 21st Century Mathematics Education Classroom

The relevance of collective argumentation to today's mathematics classroom is best understood in the context of students in mathematics classrooms today. The expectations of today's mathematics classrooms are to teach rigorous content that "engages students in cognitively demanding performance-based tasks with direct connections to students' lives" (Nava, Park, Dockterman, Kawasaki, Schweig, Quartz, & Martinez, 2019). Further, the skills of constructing, justifying, and explaining a viable argument that are developed in collective argumentation and encouraged by the standards of mathematical promoted by NCTM (2000) are important to developing active, participatory citizenship. By giving students the skills to the right to participate actively in mathematics education, and education in general, collective argumentation develops students who see the value in community and discourse with one another and the opportunity to make math and science literacy accessible to all students (Moses & Cobb, 2002). In alignment with the theoretical framework, collective argumentation advances mathematical thoughts in a way that students are asked to engage with high cognitive activity in

the mathematical tasks while making the material more personalized to them because of their social engagement. This sociomathematical approach, additionally, meets the students in their needs for developing mathematical proficiency. Mathematical proficiency, as defined by Kilpatrick, Swafford, and Findell (2001), covers five domains: proficiency in procedural knowledge, conceptual knowledge, strategic competence, adaptive reasoning, and mathematical dispositions. Collective argumentation fulfills the standards of each of these domains in its instructional design, which prepares the students of today with the critical thinking skills and mathematical proficiency necessary for today's classrooms.

To expound on these various domains of mathematical proficiency, note the design of collective argumentation that lends itself towards student self-discovery of these domains. As students engage in conversation with one another, their conceptual knowledge is built because they must learn how to justify each other's reasoning in a logical sequence, requiring strategic competence in their progression of thoughts. By referring back to the communal criteria, students must upon their own mathematical knowledge to develop arguments that focus on problem-solving and understanding why their process of argumentation works (Singletary & Conner, 2015). This type of critical thinking is the type of skills that students are expected to have in today's STEM-oriented society. Because mathematics is a key component of all STEM content areas (Rittmayer & Beier, 2009) and education emphasizes a social ability to contribute to the community, it is relevant for math educators to consider ways to foster student abilities to engage in argumentation in K-12 mathematics through collaborative small-group discussions.

Practical Implications

This study's results provide practical implications for both teacher education and K-12 mathematics classrooms. Firstly, results about students' abilities to improve argumentation

through collective argumentation informs pre-service and in-service teachers of the pedagogical benefits as well as the learning opportunities that interpersonal discourse in collective argumentation provides. Secondly, this study can contribute to mathematics teacher educator programs by providing awareness of the impact that this teaching strategy can have on mathematics classrooms, especially in reference to the standards for mathematical practice. The themes found for both math teachers and math teacher educators' understandings of mathematical argumentation and their perceived benefiting and limiting aspects will be revisited in this section to discuss the implications offered by this study.

Understandings of Mathematical Arguments

As noted in the previous chapter, the students in this study grew not only in their knowledge of mathematical arguments, but also in their competencies and abilities to develop mathematics arguments. The themes for K-12 students' mathematical argument understanding include the following: 1) development of conceptualization of mathematical arguments, 2) student development of mathematical arguments, and 3) argumentation development through discourse.

Development of conceptualization of mathematical arguments. As noted in the description of the beginning of this study and in the literature, many K-12 students are continuing to develop in the competency to correctly engage in cognitively challenging mathematical argumentation due to a lack of access or opportunities to engage in argumentation. Specifically, several students recognized that they had to appeal to certain criteria that make a veracious argument, and the study showed they really understood how to do so once challenged in the facilitated instructional series and by their peers in the collaborative discussions. As these students learned how to engage productively in argumentation through a collective

argumentation instructional series, students began to realize that their discourse with one another allowed them to appeal to the criteria for a mathematical argument and, in turn, develop more advanced arguments as they took advantage of each other within the learning community and developed the competencies to create viable mathematical arguments. The students benefitted from an instructional design that promoted their voices in collectively developing an argument to an open-ended mathematical claim, as well as gave them the opportunity to critique each other and the other small groups' arguments. According to the data, many students learned how to provide support or evidence for their arguments, and they particularly learned of the necessity of an argument to be true for all cases. Because of the emphasis on communal criteria, most students begin to negotiate their understanding of the mathematics with one another, as the utilize each other to gain authority of their own mathematical understanding (Boyle et al., 2015). As students grew in their conceptualizations of mathematical arguments, they then were able to create and validate their own mathematical arguments within their small groups, utilizing the power of interpersonal discourse with their group members in collective argumentation.

Student development of mathematical arguments. Additionally, as students in this study learned more about what makes a veracious argument through the discussion of and appeals to communal criteria, many of them began to recognize *how* to validate their thoughts with justifications and warrants that make their arguments logical, understanding, and true for all cases. Again, by appealing to communal criteria in the collective argumentation process, students are able to cognitively concretize ideas of what makes a more advanced cognitive thought to justify their reasonings in their final arguments.

Argumentation development through discourse. Finally, as the qualitative analysis of this study uncovered, students' felt more and more comfortable to engage in exploratory talk

(Rojas-Drummond & Zapata, 2004) with one another so that they could reason out their own thoughts, ask for clarification or confirmation of a mathematical thought, or justify their reasoning to someone else in the group with a mathematical fact as the backing to their warrants and justifications. In giving the students the opportunity to not just engage in discourse, but also utilize it as a mathematical tool for promoting their own thoughts and contributions to mathematical arguments, collective argumentation promotes students' understanding of and competencies in developing more advanced proofs and mathematical arguments.

In terms of implications for practicing teachers, teachers can understand how better to implement small-group discussions and argumentation tasks to develop students' abilities to justify their reasoning in mathematical arguments. The results show that teachers should utilize communal criteria well so that students know how to create a mathematical argument, but they should also consider how to prepare students so that all students can feel comfortable to participate to maintain equity for mathematical learning. This also means that teachers should reinforce that all students understand the vocabulary in a mathematical argument lesson design to ensure all students can engage in discussion required in the instructional series. Further, teachers should pay attention to timing to allow all students to engage so that they do not feel like they are working too quickly or slowly. Finally, teachers should hold students accountable with the tangible products they must create in the small-group work so that students can all feel that they engaged in and found purpose in the mathematical small-group work.

Promotion of Mathematical Identity

Beyond mathematical argumentation advancement, students began to implicitly realize themselves as better mathematics students and people who are capable of creating and developing higher cognitive mathematical thoughts. The process of collective argumentation

allowed them to engage informally with conversation that led to more interpersonal discourse, including the students' thoughts and feelings about themselves and the mathematics, and how they relate to the mathematics itself. This appeal to the relationship between the person and the mathematics encompasses the idea of one's mathematical identity. As stated by Cobb, Gresalfi, and Hodge (2009), the mathematical identity consists of one's normative identity and one's personal identity in the mathematics classroom.

Promotion of normative identity. In collective argumentation, I carefully facilitated the lesson design according to the instructional series where students engaged in collective argumentation. Students recognized that their patterns of what they practiced in mathematics classrooms were disrupted with the process of collective argumentation. But rather than distracting them or detracting them from learning, the students embraced the sociocultural change in the framework of class and, particularly by the end of the research study, excitedly engaged in collectively argumentation. They recognized that student talk with one another was not only allowed but encouraged. Students became excited to find reasons to support their answers and offer up their own thoughts as valid warrant to their mathematical argument. Students were willingly using each other as equal authorities to validate, question, or challenge their own mathematical thoughts. Collective argumentation can create that open atmosphere in a mathematics classroom that allows students to make mistakes and learn, as Civil and Hunter (2015) found was necessary for the mathematics classroom that actively engages students in participation.

Promotion of personal identity. Beyond this transformation in the norms of the class, students also grew in their own personal identity, or how they viewed themselves in relation to the mathematics (Cobb, Gresalfi, & Hodge, 2009). According to the data in the final analysis,

students began to recognize the power of their voice, as well as the ability to use each other in the learning community to create, modify, and justify each other's arguments. Several students made comments that indicated that they began to recognize the challenge of mathematical argumentation as something that is exciting, and more students began to see themselves as intelligent at mathematics by claiming that they were, or had the potential to be, good at mathematics. Students began to see themselves as intelligent in mathematics, and they found themselves making utterances proclaiming how someone is smart because of their discovery of an undisputable warrant in their mathematical argument development.

There were, however, some limiting aspects to this study that could have prevented the research from unlocking its potential or caused some difficulty in the completion of this research. Because these students had not engaged in mathematical collective argumentation before this research activity, it may have been difficult for the students to truly understand what is expected of a veracious mathematical argument after just one discussion of communal criteria before collecting the data for the first analysis. Without modeling or more support before their first full cycle of collective argumentation, students may have been limited in what they were able to accomplish in their arguments and their collaborative discussions in the first task on the first day. Additionally, the timed aspect of the instructional series provided some pressure for some students, as evidenced by the theme of time impacting students' identities as doers of mathematics, so future practitioners should consider how best to utilize collective argumentation within the timelines of their classes. Further, the results of the survey analysis of student identity study also should be compared to similar survey analysis of student identity on days when students did not engage in collective argumentation to better understand how students' normative identity shifted or transformed concerning mathematics class as a result of this study.

Another limiting aspect of this study was the emphasis on students' collective arguments. Without deep analysis of the students' individual thoughts before the collaborative portion of collective argumentation, it is difficult to note just how much the collective argumentation contributed to the advanced mathematical thoughts and how much was a result of a single student's growth in understanding. A future analysis of this data could include individual measures to truly understand how students individually and uniquely grew in their understanding, such as using a think-aloud approach (Silbey, 2012) for mathematical understanding of the construction of an individual's mathematical argument. Further, the individual contributions within the development of a collective argument could be analyzed for a diverse group of students in the class, allowing for a longer period of study of the discourse over time to see how these diverse individuals improve in their abilities to create mathematical arguments through collective argumentation. This would provide researchers better insight into what aspects of the collaboration truly contributed to student development of understanding. Moreover, it would also provide clearer guidelines for the support and the implementation of effective collective argumentation for all levels and types of learners in the K-12 mathematics classroom.

Overall, the analyses of this research study provide support for student understanding of mathematical arguments because they had the opportunity to engage in meaningful discourse through collective argumentation to better understand the criteria for a mathematical arguent and the ways in which they can discuss with one another and critique each other to develop a more advanced mathematical argument. Further, student participation in this community gave students the opportunity to value their own contributions to the learning as they became more competent in the skills of developing mathematical argument, impacting student mathematical normative

and personal identity for the better. By opening up the classroom to be a community that promotes student discourse for productive mathematical discussion in this manner of collective argumentation, K-12 mathematics teachers can reach students of diverse abilities and mathematical identities to become overall more engaged and advanced mathematical participants in the classroom as a learning community and better doers of mathematics through their discourse in collective argumentation.

Theoretical Implications and Recommendations

In addition to practical implications, the results of this study offer theoretical implications for the implementation of situated learning theory and cognition. The implications for theory are provided in the following paragraphs, which are then followed by recommendations for future research.

Theoretical Implications

The theoretical framework of this study utilized a situated learning perspective that emphasizes the cognitive growth of students, or a hybrid lens between cognition and situated learning theory. This perspective posits that learning is sociocultural, thus there is knowledge transferred because of its relation to the context in which it takes place (Brown et al., 1989; Cobb & Bowers, 1999; Lave & Wenger, 2001; Putnam & Borko, 2000; Saivyer & Greeno, 2009). Specific to this study, the belief that cognition is situated not only in the physical location but also within the social context where it takes place (Lave & Wenger, 2001; Putnam & Borko, 2000) guides the analysis of the students' discussions as well as the data that they produce. Because of the focus on interaction between students through interpersonal discourse to discover how students advance in learning, it is necessary to take a sociocultural approach to the cognitive

growth that occurs in students through collective argumentation, thus a theoretical framework that situates cognition in the context of situated learning via collective argumentation is most appropriate. By analyzing this data through this unique lens of situated learning for the advancement of cognitive mathematical understanding, the framework of this study promotes the practice of intentionally paying attention to and utilizing the resources of student voice and a collective argumentation pedagogical lesson design that uses communal criteria. This perspective takes into account the elements of this situation within the context of mathematical learning that support the overall outcomes of increased mathematical argumentation competencies and increased student abilities and mathematical identities.

This study takes place entirely in the context of the students' regular mathematics classroom, with an emphasis on their small-group collaborations. It is important to note that the heart of the task is students' collaborative discussions that took place in small groups, so the physical space and the necessity of social interaction for small-group discussions is important for the consideration of the theoretical perspective. Students, by the design of the study, were tasked with discussing the validity of three different claims in a way that they have never done before. The small discussion of communal criteria at the beginning was the students' introduction to understanding the concept of a mathematical argument itself, and then they were engaged in discussions with one another to develop an argument. This process repeated every day of the entire instructional series. At the beginning of the instructional series, most students seemed a little reluctant to share with each other their mathematical ideas, whether it be for fear of failure, fear of embarrassment, a lack of knowledge, or anything else. However, the research shows that the students found value in working with one another, as their voices expressed more detailed arguments with more advanced cognitive mathematical thoughts, and their discussions revealed a

greater engagement with each other and the mathematics as they developed more highly developed justifications in the warrants of their arguments. The instructional series, by design, promoted an atmosphere where students were the authorities of the mathematical learning, as opposed to other classroom designs that place the teacher or the textbook at the center of the learning. Further, the main goal of creating a mathematical argument based on the criteria of a veracious mathematical argument that the students developed at the beginning of the instructional series contributed to students' understanding of collective argumentation in general and thus, their overall mathematical learning. As students were able to develop their reasoning through their social discussions (Mercer, 2008; Mercer, Wegerif, Dawes, & Sams, 2004), they were able to construct more cognitively deep ideas to justify a better mathematical argument.

This insight, then, provides substantial support for K-12 practitioners to engage their students in mathematical argumentation via collaborative discourse. The encouragement of their interpersonal conversations allows students to see themselves as authorities of the mathematics instead of someone or something else (Harel & Sowder, 1998), including the teacher, a textbook, or another person. Further, the nature of the open tasks that required students to create a mathematical argument also engaged students in a way that they could participate in the learning community, discuss the content in a manner where their voices felt valued, and contribute to the overall goal of interpersonal discourse in mathematics through argumentation. As a result, according to this data analysis, many students develop more advanced mathematical arguments and increase in their cognition and mathematical learning, and the classroom becomes a welcoming atmosphere where students feel that they can have and use their voice for growth in their academic learning.

Recommendations for Future Research

The purpose of this study was to provide better understanding of how students develop their abilities to create mathematical arguments and their mathematical identities through the interpersonal discourse in which they engage through collective argumentation. The results of this study provide insight into how students grew in their understanding of a mathematical argument and their abilities to develop a mathematical argument, as well as ways in which the process of collective argumentation contributed to their identity as doers of mathematics. However, this study highlights elicits further research that can be conducted to better understand how best to support both K-12 mathematics students and teachers in the process of collective argumentation. In the following paragraphs, three recommendations are discussed to address the limitations of this study, as well as to give possible future directions for research. These recommendations include the design of the process of collective argumentation, further examination of the discourse that the students share, and possibilities to more directly study the mathematical identity that students develop as a result of the interpersonal discourse students have with one another in the collective argumentation process.

Data analysis recognizes that poor planning of content can severely limit the data that is available, and it can also detract students from fully gaining confidence in the process of collective argumentation, one of the major findings found in this study. Additionally, improper planning and implementation can detract from allowing all students to engage appropriately in collective argumentation, hindering the equity of mathematics learning for all students in the classroom community. In particular, attention to selection of the tasks for collective argumentation will appropriately address the students' content knowledge so that they may be able to engage in interpersonal discourse and the collective argumentation to their full potential.

Another important finding in the data analysis was the development of and maintenance of critical norms in the collective argumentation instructional series. Providing students expectations and practices that support this instructional design will support results in understanding how students improve in their mathematical abilities and identities via collective argumentation. Ensuring that planning well addresses both the content standards and the timeline for the curriculum will prevent mistakes like these in the future, as well as unlock the maximum potential for collective argumentation on student small groups' collaborative discussion. Overall, future research could focus on the critical aspects of teachers' planning of collective argumentation to maximize student potential of participation and engagement in the discourse necessary for collective argumentation as well as the students' improved abilities and competencies to understand and create veracious mathematical arguments and proofs.

Further, another recommendation for future research is videotaping students' interactions with one another, so that elements of the sociomathematical classroom like positioning and movement can be analyzed to even better understand classroom layout and other elements of the classroom that could impact this research for the better or for the worse. Because mathematical discourse is not merely defined just as the words spoken, but also as the gestures and facial expressions that students use as they convey their ideas, videotaping would capture a more holistic view of discourse for data collection. Additionally, visual accompaniment with the audio will allow researchers to recognize more deeply how the discussion was delivered in terms of intonation, emotion, and passion for the content and for each other. Seeing more social interaction can lead to more discoveries about how students collaborate with one another, even in the mathematics classroom setting. By engaging in students' interactions beyond simply audio,

researchers can see how to position the teacher to maximize the time and space for effective collective argumentation in any K-12 mathematics classroom.

A final recommendation for future research is a more direct instrument or measurement of student identity. The instrument in this study to analyze student identity, a pre-survey and a post-survey, helps identify major themes about students' changes in identities as a result of the completion of the entire instructional series using collective argumentation. However, more research could focus on the development of identities-in-practice, or students' development of mathematical identities as they develop throughout the entire learning process. This could be studied more intentionally with instruments such as interviews, so that more critical shifts in students' mathematical competencies in collective argumentation and mathematical identities could be better defined. Focusing in on students with follow-up questions and chances to let them reflect and explain their answers will give deeper insight into how students' mathematical identities truly change, and what they can do to become more a more motivated doer of mathematics.

My Growth and Learning

In reflecting on this entire journey towards a doctor in the field of mathematics education, I can only begin to describe how I have grown tremendously as a researcher, educator, and advocate for mathematics learning for all. I knew I had an initial desire to learn a bit more about student discourse in the mathematics classroom. My passion has always been to assist students in the learning of mathematics, and beginning the Ph.D. was the best possible way for me to maximize this dream. However, coming to learn a holistic view of mathematics education, as well as the minute buzz words and practices that come with all areas of mathematics education, I

did not think I would be able to complete a dissertation that would satisfy myself as a practitioner and researcher.

As I continued to teach mathematics while working towards the doctorate, I only began to realize the exact opposite. I was inspired to teach better every time I stepped back in the classroom, but at the same time I just grew in the number of questions I wanted to ask. And this is very much so a direct result of my doctoral program and, particularly, my dissertation work here. What do students *really* think when the light bulb comes on after discussing math with a friend for a couple minutes? What can I do to make sure this happens? How do I foster students' mindsets to become ones where they realize that they *can* take authority over their own mathematical learning? These questions led me to this research, and the design, implementation, and analysis of this dissertation.

I was actually quite inspired to complete this program with thorough efforts and good spirits, in all aspects, to the end. But designing and completing a dissertation research project proved to be much more than I had realized. However, the questions above and many more kept inspiring me to pursue these answers. Further, the students I still taught while becoming a formal university learner myself inspired me to apply what I learned so that they can learn to ask questions and pursue answers. With such inspiration to guide me, I am more than happy to present the results of this dissertation research here.

My future goals are just to keep inspiring others to love and learn mathematics more than they did before. Impacting student lives has always been a life calling, and with the influence of those before me and those guiding me through this entire doctorate process, I must say I believe in the power of my research and the impact of student discourse even more. Seeing it in my classroom, and engaging in meaningful conversation with my professors, coworkers, family,

friends, and students inspire me to continue to teach and research about student voice and development of higher advanced thoughts in mathematics for a long time. As a result of completing this dissertation research, it has become apparent and very valuable to me to recognize the steps that can be taken to frame a classroom that poises students at the center of their mathematical learning as they take on the authority as the doers of mathematics by engaging in discourse through collective argumentation. I have come to recognize that careful lesson design to promote student voice in collective argumentation is not just important to mathematics learning in the classroom, but also to mathematics learning in general. And thus, the practices and tenets of student engagement and participation in a community support overall students' rights to participating in the community around them, as they develop into citizens who can contribute to their society. It has been an honor to teach young minds that they have access to participatory citizenship in society today through the vehicle of collective argumentation in mathematics class. To close, I request only that this research not become just another thing to read or another new tool for learning. Should you have any curiosities about mathematics education, or perhaps do not fully agree or understand my thoughts on mathematics education, I simply ask that we sit down and discourse about it.

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Appendices

Appendix A

The Effects of Small-Group Discourse in Collective Argumentation on Secondary Students' Mathematical Arguments and Mathematical Identities

Parent Permission Form

Your child is invited to be part of a research study being conducted by Shande King at the University of Tennessee, Knoxville. Your child is being invited because Mr. King is conducting research to learn more about how students justify their thinking in math collectively and its impact on student identity. Being in this research study is voluntary, and you should only agree if you completely understand the study and want to volunteer to allow your child's materials to be used. This form contains information that will help you decide if you want your child to be part of this research study or not. Please take the time to read it carefully, and if there is anything you don't understand, please ask questions.

Purpose
The purpose of this study is to understand the effectiveness of collective argumentation on students' abilities to make sense of math problems and mathematical identities by creating arguments. I plan to write my dissertation, publish articles and/or books, etc., and make presentations at conferences to share the results of this research.

Participation
If you choose to allow your child to participate and your child also agrees required for age 7 and older, I will analyze their written individual and collective arguments; pre- and post-surveys they completed to tell me more about their perceptions of mathematical collaboration, argumentation, and identity; and audio recordings of their small-group collective argumentations. Because these are all things that are part of their regular activities in the 8th grade Pre-Algebra curriculum at Christian Academy of Knoxville., participation in the research will not require any additional time.

Benefit
Your child will not receive any direct benefit from allowing their written individual and collective arguments, surveys, and collective argument audio recordings to be used in the research project, but we hope to learn things that will benefit teachers and mathematics education researchers in the future.

Risks
This research is considered to be no more than minimal risk, which means there is no more expected risk to you than what your child might experience during a typical day. There is the risk of possible loss of confidentiality, as someone could find out your child was in the study or see their study information, but I believe that risk is unlikely because of the procedures we will use to protect their information.

Confidentiality
If you and your child agree to participate in the research, I will assign your child a pseudonym and use that instead of their name on all of the materials before I begin analyzing them for the research study. These materials will be stored in a secure location on the UT campus in a locked cabinet in an office at the University of Tennessee. No information which could identify your child will be shared in publications and presentations about this study or databases in which results may be stored. If I wish to include your child's name, pictures, recordings, or other information that could identify them in publications or presentations, I will ask for separate written permission for this.

Future Research
Your child's materials will not be used or shared with other researchers for future research, even if identifiers are removed.

Contact Information
If you have any questions about this research, please contact me, Shande King, at nking5@vols.utk.edu or (865) 776-0368 or my advisor, Dr. Lynn Hodge, at lhodge4@utk.edu or (865) 974-8778. If you have any questions about your rights as a research participant, please contact the Institutional Review Board (IRB) of the University of Tennessee, Knoxville, at utkirb@utk.edu or 865-974-7697. You may also contact the IRB with any problems, complaints or concerns you have about a research study.

Voluntary Participation
It is completely up to you and your child to decide to be in this research study. Even if you decide to be part of the study now, you may change your mind at any time and stop participating by informing myself and the main classroom teacher. Your child will not lose any services, benefits, or rights they would normally have if you choose not to give permission, or if you or your child change your minds and stop participating later.

If you agree that your child may participate, please print and sign the **Parent Permission** section below, and have your child sign the **Assent** section, on both copies of this form. Return one copy to Shande King and keep one copy for your records. If you do not wish for your child to participate in the research, it is not necessary to do anything, as I cannot use their materials without your permission and their assent required for age 7 and older.

Parent Permission
I have read the above information. I have received a copy of this form. I understand that my child's participation in this research study includes allowing Shande King to use my child's materials for research purposes. I agree that my child may participate in this study.

Child's Name (printed) _____

Parent's Name (printed) _____

Parent's Signature _____ Date _____

Permission for use of images
I agree that photographs/photocopies of the materials of my child from Christian Academy of Knoxville's 8th grade Pre-Algebra class. may be analyzed for research purposes.

Parent's Signature _____ Date _____

Permission for use of images
I agree that photographs/photocopies of the materials of my child from Christian Academy of Knoxville's 8th grade Pre-Algebra class. may be used in scholarly teachings, publications, and/or presentations.

Parent's Signature _____ Date _____

Child/Youth Assent
I have talked about this research with my parent(s) and I agree that name of researcher may use my materials/information/biospecimens for research purposes. If I change my mind, and decide not to participate later, I only need to insert here procedures for discontinuing participation.

Youth Name (printed) _____

Youth Signature _____ Date _____

Assent for use of images
I agree that photographs/photocopies of the materials of me from Christian Academy of Knoxville's 8th grade Pre-Algebra class may be analyzed for research purposes.

Youth Signature _____ Date _____

Assent for use of images
I agree that photographs/photocopies of the materials of me from Christian Academy of Knoxville's 8th grade Pre-Algebra class may be used in scholarly teachings, publications, and/or presentations.

Youth Signature _____ Date _____

Appendix B

Argumentation Survey

Please answer the extent to which you agree or disagree with each statement.

1. Do you like doing math? Why or why not?
2. What does it take to make a mathematical argument or proof? Explain.
3. Do you think you can make a mathematical argument or proof? Explain.
4. What do you like about doing math in groups? What did you not like? Explain.
5. How does talking about math with someone else affect your ability to do the math?
6. Do you consider yourself a math person? Explain why or why not.
7. What makes you excited when you do math? Explain.
8. Do you prefer doing math by yourself or in a group? Explain.

Appendix C

Tasks for the first iteration of the research study:

Task 1

Conjecture: *The sum of any three consecutive integers is divisible by three*. Is this conjecture true? Write an argument for why or why not.

Task 2

Conjecture: *The sum of two odd numbers is even*. Is this conjecture true? Write an argument for why or why not.

Task 3

How many triplet primes of the form p, $p+2$, $p+4$ are there? Write an argument for how you know.

Tasks for the second iteration of the research study:

Task 1

Conjecture: *The sum of any three consecutive integers is divisible by three*. Is this conjecture true? Write an argument for why or why not.

Task 2

Conjecture: *Two parallel lines have no solution*. Is this conjecture true? Write an argument for why or why not.

Task 3

Conjecture: The sum of two odd numbers is even. Is this conjecture true? Write an argument for why or why not.

Task 4

Conjecture: Two parallel lines have no solution. Is this conjecture true? Write an argument for why or why not.

Appendix D

Argumentation Survey

Please answer the extent to which you agree or disagree with each statement.

1. Do you like doing math? Why or why not?
2. What does it take to make a mathematical argument or proof? Explain.
3. Do you think you can make a mathematical argument or proof? Explain.
4. What did you like about doing the math in groups? What did you not like? Explain.
5. How does talking about math with someone else affect your ability to do the math?
6. Do you consider yourself a math person? Explain why or why not.
7. What makes you excited when you do math? Explain.
8. Do you prefer doing math by yourself or in a group? Explain.

Appendix E

Group B Data Analysis Table for Task 1:

Conjecture: *The sum of any three consecutive integers is divisible by three*. Is this conjecture true? Write an argument for why or why not.

Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	
Data (the facts from which the claim was based)	It is true because I've worked it out, and it is right. **This is true because if you add up all your big consecutive numbers and divide by 3**, you always get a whole number, but only consecutive numbers. If they're consecutive they can't all be even or odd
Warrants (the justification for using the data to establish the claim)	This is true because if you add up all your three – underline three times consecutive numbers and divide by three, you will always get a whole number, but, all capitalized--only consecutive numbers… I said it's true because if I use my calculator to add the three consecutive numbers, then they will be divisible by three. I said it's true because if I use my calculator to add the three consecutive numbers, then they will be divisible by three. We have been shown by teachers. And calculators. This is true because if you add up all your big consecutive numbers and divide by 3, **you always get a whole number, but only consecutive numbers.**
Rebuttals (statements that call the warrant into question)	

Qualifiers (utterances that indicate the certainty of a statement)	It would be the same. It's just adding a couple zeros.
Backings (supports for the warrant that are usually unstated and related to the academic field)	

Appendix F

Group I Data Analysis Table for Task 1:

Conjecture: *The sum of any three consecutive integers is divisible by three*. Is this conjecture true? Write an argument for why or why not.

Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	
Data (the facts from which the claim was based)	This is true because if you add any consecutive integers, the number is divisible by 3.
Warrants (the justification for using the data to establish the claim)	Like, 5 plus 6 plus 7 is 18 divided by 3 equals 6. You think it is true because if you look at it at all – just look at any example. It works, Coach. You can do anything – 1 billion, 3 billion, 7 billion. It's true. 11 plus 12 plus 13… And then do like 6987, 6988…just do a bunch of examples at first. Then show that the formula works, and do the formula, you know? And then it's like, it equals a plus 1 plus a plus 2 equals a whole integer.
Rebuttals (statements that call the warrant into question)	Cause if someone else is saying it's wrong, and you think it's right, you need to tell them why it's right.
Qualifiers (utterances that indicate the certainty of a statement)	So we need to make a formula or something. a plus b plus c… Oh, it's always the number in the middle. Whenever you're adding three consecutive numbers, it's always the number in the middle. Because 1 plus 2 plus 3 is 6, and then the middle one is 2. Oh, you're so smart!
Backings (supports for the warrant that are usually	

unstated and related to the academic field)	

Appendix G

Group G Data Analysis Table for Task 1:

Conjecture: *The sum of any three consecutive integers is divisible by three*. Is this conjecture true? Write an argument for why or why not.

Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	
Data (the facts from which the claim was based)	
Warrants (the justification for using the data to establish the claim)	Yeah, because – well, I haven't tested all of the numbers yet, but the numbers that I did test were 1, 2, 3, 4, 5, 6 and 7, 8, 9. And if you divided them by 3, all of them turned out evenly. It's true because I did the math. 1 plus 2 is 3, 3 plus 3 is 6. Which is divisible by 3. 6 can be divided by 3 which is 2.
Rebuttals (statements that call the warrant into question)	You can't just have like two examples. You have to have – if you're trying to prove that what you think is right, you have to have at least three.
Qualifiers (utterances that indicate the certainty of a statement)	
Backings (supports for the warrant that are usually unstated and related to the academic field)	

Appendix H

Group N Data Analysis Table for Task 1:

Conjecture: *The sum of any three consecutive integers is divisible by three*. Is this conjecture true? Write an argument for why or why not.

Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	
Data (the facts from which the claim was based)	
Warrants (the justification for using the data to establish the claim)	So, I put that the statement is true, and then I put an example. I put 27, 28, and 29, and then that equals 84, and 84 is divisible by 3. And then I noticed that it's like averaging because you add three numbers, then you divide it by three, and that's how you average numbers. Oh, how do you know if it's divisible by three. You like just go through and take a few examples I guess, and all the ones I use worked. So, I noticed that it's like averaging because you add three numbers, and then divide it by three, and that's how you average, and the answer to this will always be the middle consecutive. So, it's like averaging. So, what do we say? I put yes this is true. Any three consecutive integers added together can be divided by three. 5, 6, and 7 added together equal 18. 18 divided by 3 is 6. So, then we should put this process...
Rebuttals (statements that call the warrant into question)	Let's just try it with a bunch of different kinds of numbers. Like, negative numbers, and ones and tens, and hundreds, and not thousands I guess.

Qualifiers (utterances that indicate the certainty of a statement)	
Backings (supports for the warrant that are usually unstated and related to the academic field)	

Appendix I

Group F Data Analysis Table for Task 2:

Conjecture: *The sum of two odd numbers is even*. Is this conjecture true? Write an argument for why or why not.

Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	
Data (the facts from which the claim was based)	
Warrants (the justification for using the data to establish the claim)	It's like saying you have two different odd numbers. This was just doing - or, they could be the same. If it's like 7 plus 7, then it equals 14. So, basically you're just doing like 7 times 2 which would be divided by two which makes it a whole number. Two odd numbers, and if you do 10 - like the number divided by 2 because there's two odd numbers. If it equals a whole number, it's probably - it will be - it's probably. I don't know what I'm trying to say. It equals a whole number, so that means that's probably an even number? I really don't know. And if you do the same thing for--yeah, if you do the same thing for odd plus even. OK. If you take away one, it would be either odd plus odd or even plus even. We saw that even plus even equals even, odd plus odd equals even, and even plus odd equals odd. And then...
Rebuttals (statements that call the warrant into question)	But if I do 3 plus 4 which is 7 and do divided by 2, it is not a whole number.

Qualifiers (utterances that indicate the certainty of a statement)	
Backings (supports for the warrant that are usually unstated and related to the academic field)	Does all even plus even numbers equal even? Yes. OK, so you would take away--what I did is take away the 1 from the 5 and added it to your final answer. Teacher: Oh OK. So, you take away 1 from one of the numbers because you know that even plus odd equals odd.

Appendix J

Group B Data Analysis Table for Task 2:

Conjecture: *The sum of two odd numbers is even*. Is this conjecture true? Write an argument for why or why not.

Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	
Data (the facts from which the claim was based)	Oh, yeah, but if we did two different number, odd numbers, it's still even, but why?
Warrants (the justification for using the data to establish the claim)	It is true because I worked the problems out on the calculator to prove this, and it is true. More than that. Why? Because we've worked it out, and it's clearly that it will be an even number. OK, so we'll write some examples.
Rebuttals (statements that call the warrant into question)	Except if it's 3 plus 3. 3 plus 3 is an even number. 3 plus 3 is 6. Two – Oh wait. Oh. Two different or the same ones? Uh oh, that confused me. What, 3 plus 3? Yeah. 'Cause it's 9. But, 7 plus 7 is 14, 11 plus 11 is 22. No, that's not true because 10 plus 3 is 13.
Qualifiers (utterances that indicate the certainty of a statement)	So, in some cases with even numbers it's odd. But with odd numbers it's always even.
Backings (supports for the warrant that are usually unstated and related to the academic field)	Everything plus itself is even.

Appendix K

Group H Data Analysis Table for Task 2:

Conjecture: *The sum of two odd numbers is even*. Is this conjecture true? Write an argument for why or why not.

Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	No, it's if there is two odd numbers, it's even.
Data (the facts from which the claim was based)	
Warrants (the justification for using the data to establish the claim)	I don't really know how to make a formula for that. Did you put even plus even will always be even and an even plus odd will always be odd?
Rebuttals (statements that call the warrant into question)	If there's an odd and an even... No, if you add an even and an odd, it will always be odd. And an even and an even will always be... Even.
Qualifiers (utterances that indicate the certainty of a statement)	
Backings (supports for the warrant that are usually unstated and related to the academic field)	

Appendix L

Group E Data Analysis Table for Task 2:

Conjecture: *The sum of two odd numbers is even*. Is this conjecture true? Write an argument for why or why not.

Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	
Data (the facts from which the claim was based)	Sum of any two odd numbers. It has to be something about two because...
Warrants (the justification for using the data to establish the claim)	I tried inverting it by minus one on the end to see if odds would equal an even, but that didn't work. OK, so all single digit odd numbers. Like if you add them to each other, they're both--um, they're all even when you add them up to each other. So, if you're adding a single digit number on the end of a double, like digit number, then it's all going to be. It's all going to add up to be like that. OK, um… Because if you did like 5 plus 9, that's uh, 14. And so, and if you have like 23, and then like 3 on the end. And then like 25, that would 48. Oh, I noticed something. What? In all of these it goes down by 4. Guys, wait. So, if you do 1 plus 3, that equals 4. And then you go, uh... let's do… That's because a negative's every 2. … 3 plus 5, it goes up by 4. So, it all goes up by 4 each time. Figured that out. It all goes up by 4? That's because an odd is every other number…

	So, that's interesting. …and every other number twice is 4. Um, if you go back down to the basics, 7 plus 5 um, is 12. Yeah, it is - OK. And 7 plus 3 is 10. And all the basic, tiny numbers, the one-digit numbers. They all equal evens, so that means, uh, because it just depends on the last number in the number, uh, to make it an even. So, since all of the one-digit numbers are even, it just comes down to the one-digit numbers in the big number.
Rebuttals (statements that call the warrant into question)	
Qualifiers (utterances that indicate the certainty of a statement)	
Backings (supports for the warrant that are usually unstated and related to the academic field)	So, it all goes up by 4 each time. Figured that out. It all goes up by 4? **That's because an odd is every other number…** So, that's interesting. **…and every other number twice is 4.** Um, if you go back down to the basics, 7 plus 5 um, is 12. Yeah, it is - OK. And 7 plus 3 is 10. And all the basic, tiny numbers, the one-digit numbers. They all equal evens, so that means, uh, because **it just depends on the last number in the number, uh, to make it an even**. So, since all of the one-digit numbers are even, it just comes down to the one-digit numbers in the big number.

Appendix M

Group J Data Analysis Table for Task 2:

Conjecture: *The sum of two odd numbers is even*. Is this conjecture true? Write an argument for why or why not.

Component of Toulmin's Model	**Supporting Quotes**
RQ2: How does the interpersonal discussion in collaborative small group proof and reasoning tasks support K-12 students in developing veracious mathematical arguments?	
Claims (statement, the truth of which is being established)	I mean an odd number plus an odd number, it always ends up to be even.
Data (the facts from which the claim was based)	
Warrants (the justification for using the data to establish the claim)	I mean 1 plus 1 equals 2, which is even. 1 plus 3 equals 4. 1 plus 5 equals 6. Then 1 plus 7 equals 8. 1 plus 9 equals 10. And, you can try that for 2 plus 1 equals, um 3. Wait, I did the wrong number. 3 plus 1 equals 4, 3 plus 2 equals 5, 3 plus 6--no, like you can go on and get all the odd numbers 1-9 and add them all together, and they'll all equal an even number. 'Cause all a number has is the last number has to be even, so if the number - you could do, um… Yeah, the last number has to be even, so you just have to go through 1 through 9. So, I tried subtraction. Like, if you do an even number minus an odd number, you get an odd number. So, like 22 minus 9 is 13. Or, like 14 minus 3... Wait, wait. Why would you do minus if it's the sum? I'm like reversing it. Oh! To see... Yeah. So, like 14 minus 3 is 11.
Rebuttals (statements that call the warrant into question)	

Qualifiers (utterances that indicate the certainty of a statement)	
Backings (supports for the warrant that are usually unstated and related to the academic field)	Like, if you do an even number minus an odd number, you get an odd number

VITA

Nicholas Chin Shan-Drew King, or Shande, was born in Taipei, Taiwan, and was adopted and raised in Knoxville, Tennessee by Willie and Billie-Jo King. He attended Christian Academy of Knoxville then the University of Tennessee, where he graduated a Chancellor's Honors Scholar with a Bachelor of Science in Mathematics and a Bachelor of Arts in Honors French. Shande then taught his first year in Knox County public schools and earned his Master of Science in Teacher Education from the University of Tennessee before moving as a Fulbright scholar to Paris, France, where he taught English at a local high school, gained fluency and appreciation for French language and culture, and researched pedagogy of mathematical communication of secondary mathematics teachers. Shande then moved back to Knoxville and taught in a private K-12 school (earning an Education Specialist in Curriculum and Instruction along the way), teaching multiple levels of middle and high school mathematics and French before and while beginning the Doctor of Philosophy at the University of Tennessee. As a Ph.D. student in Mathematics Education, with cognates in World Language Education and Mathematics, Shande served as a graduate teaching associate in the Department of Theory and Practice in Teacher Education. In this role, Shande delivered professional development; taught undergraduate and graduate STEM and math education courses; evaluated K-12 language, math, and science interns; and served as coordinator of the East Tennessee STM Hub. Shande developed his interests in student discourse and argumentation, as well as other passions including but not limited to accessible STEM for all students and representations of mathematical knowledge via different modalities. Shande completed his Ph.D. from the University of Tennessee during this tenth year as a K-12 educator. Shande believes in always becoming a better version of himself, the power of learning, and actively seeking to serve others.