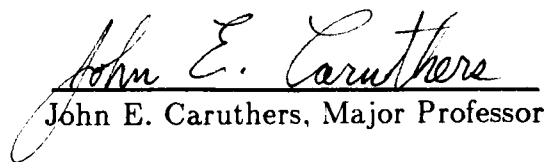
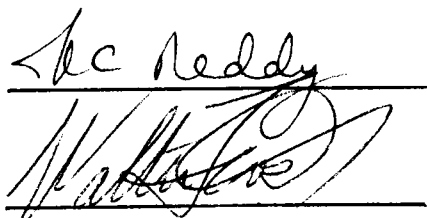
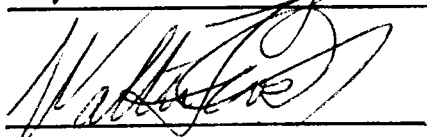


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**A FINITE-DIFFERENCE SCHEME FOR THE ANALYSIS OF
HIGH FREQUENCY SOUND RADIATION**

**A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville**

G. K. Raviprakash

March 1988

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ABSTRACT

A finite-difference formulation is developed for sound propagation in 2-D straight and cylindrical hard walled ducts with uniform flow. The procedure that has been developed is able to treat very high frequency problems without having to use a correspondingly high resolution grid. The idea is to solve for the real amplitude and phase angle of a complex potential instead of solving for its real and imaginary parts. The governing equations are recast in terms of amplitude and phase angle which do not show oscillatory behavior and hence are not difficult to solve when compared to solving for real and imaginary parts of the complex potential. Also the solution procedure is stable at very high reduced frequencies. Time marching, which utilizes the harmonic time factorization is used to solve these equations. This method has been programmed and sample calculations are presented for 2-D and cylindrical straight hard walled ducts which show excellent agreement with the analytical solutions.

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CHAPTER 1

INTRODUCTION

The noise created by aircraft during take-off, landing and ground operations has been a serious problem for aircraft passengers and residential and commercial localities near the airport. Reducing the noise pollution caused by the aircraft engines is a major concern for aircraft industries. One of the major sources of noise generation is the noise due to rotating turbomachinery. To predict the noise propagation from a compressor inlet, sound propagation inside the inlet cowl and radiation from the inlet to the far-field has to be analyzed. As a first step, to predict the noise radiation from inlets, a procedure has been developed to solve these high frequency problems in 2-D and cylindrical straight hard walled ducts.

1.1 Typical Duct Problem

Consider a 1-D straight duct problem without flow. The noise propagation inside the duct satisfies the wave equation

$$\frac{\partial^2 \tilde{\phi}}{\partial x^2} = \frac{\partial^2 \tilde{\phi}}{\partial t^2} \quad (1.1)$$

where

$$x = x'/L$$

$$t = t'c/L$$

L = length of the duct

c = speed of sound

t' and x' are dimensional

The analytical solution for the above problem with noise source $\tilde{\phi}|_1 = Ae^{ikt}$ at

(1) and radiating boundary condition at (2) as shown in Figure 1.1 is given by

$$\tilde{\phi} = Ae^{-ikx}e^{ikt} \quad (1.2)$$

where $k = \text{Reduced frequency} = \frac{\omega L}{c}$

The real part of $(\tilde{\phi}/e^{ikt})$ is plotted in Figure 1.2. To solve the same problem numerically, a considerably higher number of grid points are needed as the reduced frequency is increased due to the increasingly oscillatory behaviour shown.

1.2 Previous Work

There has been much work to date in duct acoustics. Baumeister, K. J. and Rice, E. J. [2], Baumeister, K. J. and Bittner, E. C. [3] have developed a finite difference technique to solve the noise propagation in soft walled ducts. They have solved the steady state equations obtained by solving the set of simultaneous equations obtained by writing a difference equation at each grid point. They suggest for the general 2-D case $n > 12\eta(\frac{L}{H})$ where n is the number of grids in the axial direction.

$$\eta = \frac{k}{2\pi} = \frac{\omega H}{2\pi}$$

and $(\frac{L}{H})$ is length to height ratio

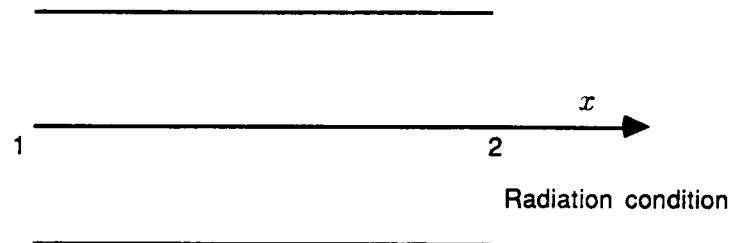


Figure 1.1 1-D sound propagation

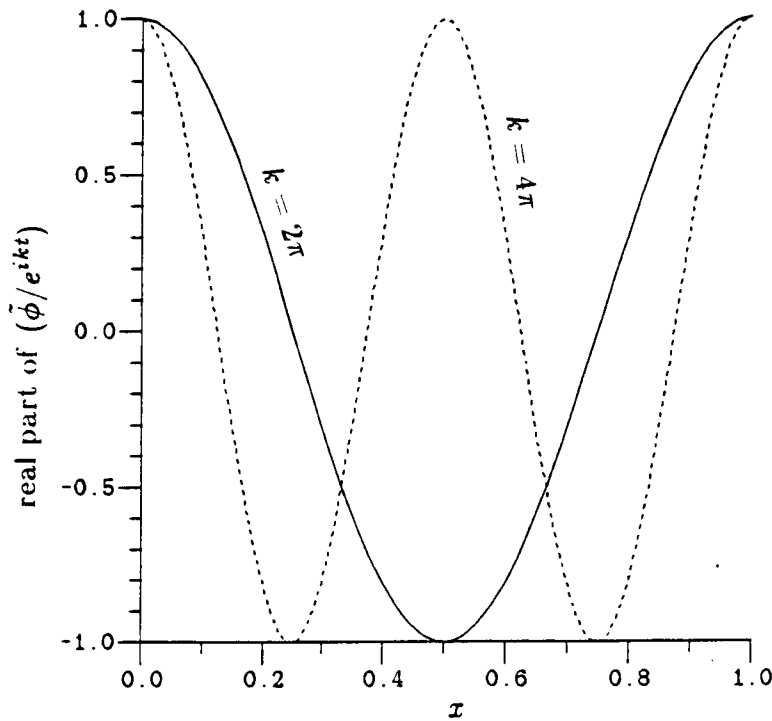


Figure 1.2 Analytical solution for 1-D sound propagation with no reflection

The above relation suggests that for a reduced frequency of $k = 10\pi$ and $(\frac{L}{H}) = 1$, the number of points needed in the axial direction is greater than 60 (which is considerably high.) The wave envelope technique was developed by Baumeister, K. J. [4] to reduce the required number of grid points in the axial direction. But here the behavior of the solution in the axial direction is assumed. The solution for duct problems varies as $e^{-ik_x x}$ in the axial direction. Here k_x is defined using the relation $\lambda_x = \frac{2\pi L}{k_x}$, where λ_x is wavelength in the axial direction and L is the characteristic length of the duct. For a typical 1-D problem considered here, $k_x = k$. But in higher dimensions, for eg. in 2-D problem, $k_x = f(k, k_y)$, where k_y is defined similar to k_x in the transverse direction. Factoring out $e^{-ik'_x x}$ reduces the oscillatory behavior

to $e^{-i(k_x - k'_x)x}$. This depends on how close k'_x is to k_x . The value of k'_x is assumed in the above method. In (5), he uses a time-marching technique. Quinn, D. W. [6] has used conformal mapping to solve the non-uniform duct problem.

Finite element techniques have been developed by Sigman, R. K., Majjigi, R. K., Zinn, B. T. [7], Horowitz, S. J., Sigman, R. K., Zinn B. T. [8], for predicting sound radiation from turbofan inlets. Parrett, A. V. and Eversman W. [9] have used finite elements in conjunction with the wave envelope element approximation method. A status report by Baumeister, K. J. [10] lists different methods and relative merits.

1.3 Present Work

The work presented herein was conducted to treat high frequency problems. The method solves for real amplitude and phase instead of real and imaginary parts of the complex potential. The governing equation in $\tilde{\phi}$ is transformed using $\frac{\tilde{\phi}}{e^{ikt}} = \phi_{amp} e^{i\beta_{phase}}$. This transformation, even though it yields non-linear equations in ϕ_{amp} and β_{phase} , is not very difficult to solve numerically.

For the typical 1-D wave equation considered earlier, the analytical solution is given by Equation (1.2). Now the solution in terms of ϕ_{amp} and β_{phase} is very simple

$$\phi_{amp} = A \quad (1.3)$$

$$\beta_{phase} = -kx \quad (1.4)$$

The real and imaginary parts of the solution for $\tilde{\phi}$ oscillate, whereas the amplitude is constant and the phase is varying linearly for the typical 1-D problem considered here. This is beneficial in the numerical procedure since a lesser number of grids is sufficient to resolve high frequency propagation. A time marching

procedure with Ni's [1] harmonic factorization in time is used to solve the problem. Relaxation in time and averaging in space are used to stabilize the solution.

This method does not assume any behavior of the solution in the axial direction. The problem being formulated in terms of the amplitude like term ϕ_{amp} and phase like term β_{phase} , does not need to assume k_x to eliminate the oscillatory behavior of the solution.

CHAPTER 2

PROBLEM FORMULATION

2.1 Governing Equations

The continuity, momentum and energy equations governing the adiabatic unsteady flow of an inviscid gas are given by

1) Continuity Equation

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{V}) = 0 \quad (2.1)$$

2) Momentum Equation

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P \quad (2.2)$$

3) Energy Equation

$$\frac{Ds}{Dt} = 0 \quad (2.3)$$

where

ρ = density of the gas

P = Pressure of the gas

s = specific entropy of the gas

$\vec{V}$ = Velocity vector

2.1.1 Assumptions

The following assumptions are made concerning the flow:

- 1) The flow is subsonic, inviscid and adiabatic;
- 2) In the absence of noise, the flow is steady;
- 3) All unsteady disturbances (noise) are small. We consider sound disturbance to otherwise steady flow.

Assuming the flow to be initially irrotational, the flow will remain irrotational, so that the governing equations can be formulated in terms of a potential Φ . We assume that

$$\Phi = \Phi_s + \tilde{\phi} \quad (2.4)$$

where

Φ_s is the steady part

$\tilde{\phi}$ is the perturbation potential.

It can be shown (Goldstein [11]) that an equation for the perturbation potential $\tilde{\phi}$ can be obtained from the Equations (2-1) to (2-4) of the form

$$\frac{1}{\rho_o} \nabla \cdot (\rho_o \nabla \tilde{\phi}) - \frac{D_o}{Dt} \left(\frac{D_o \tilde{\phi}}{Dt} / C_o^2 \right) = 0 \quad (2.5)$$

where

ρ_o = steady mean flow density

C_o = steady mean flow speed of sound

and

$$\frac{D_o}{Dt} = \left(\frac{\partial}{\partial t} + \vec{V}_o \cdot \nabla \right)$$

For uniform steady flow, Equation (2.5) reduces to

$$(1 - M^2) \frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} = \frac{\partial^2 \tilde{\phi}}{\partial t^2} + 2M \frac{\partial^2 \tilde{\phi}}{\partial x \partial t} \quad (2.6)$$

in 2-D rectangular Cartesian coordinates

where

$$x = x'/H$$

$$y = y'/H$$

$$t = \frac{t' C_o}{H}$$

x', y', t' are dimensional quantities

H = height of the duct

and to

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{\phi}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \tilde{\phi}}{\partial \theta^2} + (1 - M^2) \frac{\partial^2 \tilde{\phi}}{\partial x^2} = \frac{\partial^2 \tilde{\phi}}{\partial t^2} + 2M \frac{\partial^2 \tilde{\phi}}{\partial x \partial t} \quad (2.7)$$

in cylindrical coordinates

where

$$r = r'/R$$

$$x = x'/R$$

$$t = \frac{t' C_{\infty}}{R}$$

r', x', t' are dimensional quantities

R = radius of the duct

Using Harmonic decomposition in θ

$$\tilde{\phi} = \sum_{-\infty}^{\infty} \tilde{\phi}_m e^{im\theta} \quad (2.8)$$

Equation (2.7) reduces to (2.9) for each spinning mode m

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{\phi}_m}{\partial r} \right) - \frac{m^2}{r^2} \tilde{\phi}_m + (1 - M^2) \frac{\partial^2 \tilde{\phi}_m}{\partial x^2} = \frac{\partial^2 \tilde{\phi}_m}{\partial t^2} + 2M \frac{\partial^2 \tilde{\phi}_m}{\partial x \partial t} \quad (2.9)$$

For a time marching scheme, Ni's harmonic factorization in time [1] is used.

This preserves the unsteady nature of the equations. Equation (2.6) is transformed to Equation (2.11) using

$$\tilde{\phi}(x, y, t) = \hat{\phi}(x, y, t) e^{ikt} \quad (2.10)$$

where

k is the reduced frequency = $\frac{\omega H}{C_{\infty}}$

ω is the frequency .

$$\begin{aligned} (1 - M^2) \frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} &= \frac{\partial^2 \hat{\phi}}{\partial t^2} + 2ik \frac{\partial \hat{\phi}}{\partial t} - k^2 \hat{\phi} \\ &+ 2M \left[\frac{\partial^2 \hat{\phi}}{\partial x \partial t} + ik \frac{\partial \hat{\phi}}{\partial x} \right] \end{aligned} \quad (2.11)$$

Similarly using

$$\tilde{\phi}_m(x, r, t) = \hat{\phi}_m(x, r, t) e^{ikt} \quad (2.12)$$

in cylindrical co-ordinates, the Equation (2.9) is transformed to

$$\begin{aligned} & \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{\phi}_m}{\partial r} \right) - \frac{m^2}{r^2} \hat{\phi}_m + (1 - M^2) \frac{\partial^2 \hat{\phi}_m}{\partial x^2} \\ &= \frac{\partial^2 \hat{\phi}_m}{\partial t^2} + 2ik \frac{\partial \hat{\phi}_m}{\partial t} - k^2 \hat{\phi}_m + 2M \left\{ \frac{\partial^2 \hat{\phi}_m}{\partial x \partial t} + ik \frac{\partial \hat{\phi}_m}{\partial x} \right\} \end{aligned} \quad (2.13)$$

where now $k = \omega R/C_o$

2.2 Boundary Conditions

The boundary conditions for a typical duct problem have to be prescribed at

- 1) Entrance of the duct
- 2) Exit of the duct
- 3) Hard walls

The boundary conditions for a typical 1-D problem, 2-D straight hard walled duct and cylindrical hard walled duct are explained in detail in later paragraphs. Usually a noise source is prescribed at the entrance of the duct. For the inlet problems, posing the exit boundary condition is not easy. For the exit radiation condition, the condition has been prescribed in terms of exit impedance for duct problems [4]. However, here the exact solutions have been calculated for particular modes and are used to prescribe the boundary conditions at the entrance and exit of the duct. Farfield radiation conditions may be developed for open and flanged ducts, by assuming behavior as a monopole in the farfield. The boundary conditions are explained using Figures 2.1, 2.2 and 2.3.

2.2.1 1-D Problem

At station (1) the potential is prescribed by $\tilde{\phi}_1 = Ae^{ikt}$ where A is a constant. At (2) for radiation, we can use the characteristic boundary condition or prescribe the potential for an outgoing wave at station (2). If a characteristic boundary

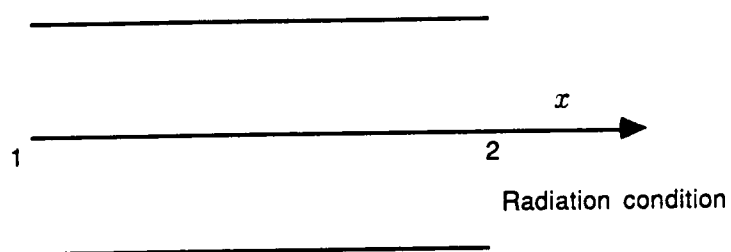


Figure 2.1 1-D sound propagation

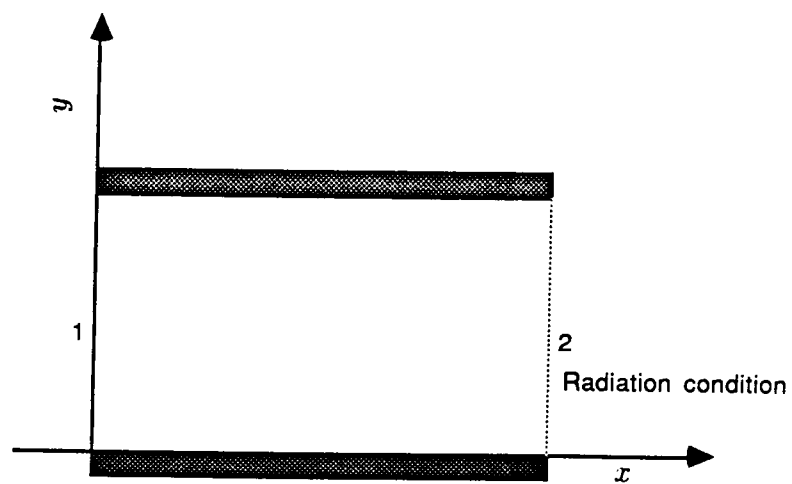


Figure 2.2 Sound propagation in a 2-D straight duct

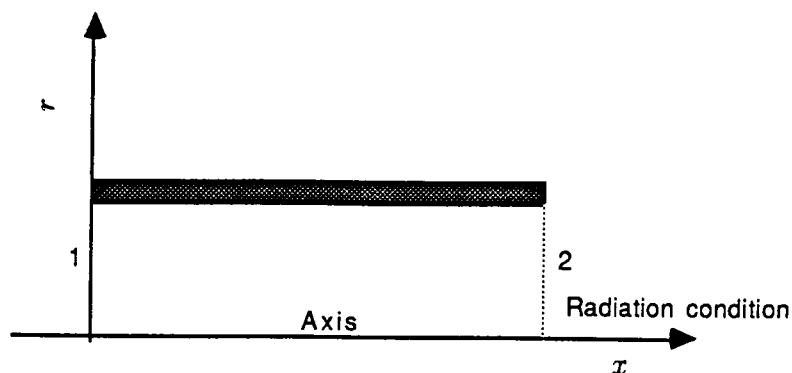


Figure 2.3 Sound propagation in a cylindrical duct

condition is used, the potential at (2) is calculated from the grid point just adjacent to (2) by interpolating at the grid intersection with the forward characteristic which has a slope of $1/(1 + M)$.

2.2.2 2-D Straight Duct

The potential is prescribed along the entrance plane (1) and exit plane (2), for radiating boundary conditions at the exit, using the analytical solution

$$\tilde{\phi} = \cos(k_y y) e^{ikt} e^{-ik_r x} \quad (2.14a)$$

where

$$x = \frac{z'}{H}$$

$$t = \frac{t' C_u}{H}$$

H = Height of the duct

$k_y = n\pi$, $n = 0, 1, 2$. integer values

$k_x = \sqrt{k^2 - k_y^2}$ without flow

and

$k_x = \frac{-Mk + \sqrt{k_y^2(M^2 - 1) + k^2}}{(1 - M^2)}$ with flow

So the boundary conditions take the form as follows:

1. Along the entrance and the exit of the duct

$$\tilde{\phi}|_1 = \cos(k_y y) e^{ikt} \quad (2.14b)$$

$$\tilde{\phi}|_2 = \cos(k_y y) e^{ikt} e^{-ik_x x_2} \quad (2.14c)$$

where

$$x_2 = \frac{L}{H}$$

L = Length of the duct

$$\hat{\phi}|_1 = \cos(k_y y) \quad (2.14d)$$

$$\hat{\phi}|_2 = \cos(k_y y) e^{-ik_x x_2} \quad (2.14e)$$

2. Along the hard walls

$$\frac{\partial \hat{\phi}}{\partial y} = 0 \quad (2.15)$$

2.2.3 Cylindrical Duct

The solution is prescribed along the entrance and exit of the duct using the analytical solution for the radiating boundary condition at the exit

$$\tilde{\phi}_m = J_m(k_{mn} r) e^{-i\mu x} e^{ikt} \quad (2.16a)$$

$$\hat{\phi}_m = J_m(k_{mn} r) e^{-i\mu x} \quad (2.16b)$$

where $\mu = \sqrt{k^2 - k_{mn}^2}$ without flow

and $\mu = \frac{-Mk + \sqrt{k_{mn}^2(M^2 - 1) + k^2}}{(1 - M^2)}$ with flow

and k_{mn} is obtained from satisfying

$$\frac{\partial}{\partial r} [J_m(k_{mn}r)] \big|_{r=1} = 0$$

So the boundary conditions take the form as follows:

1. Along the entrance and the exit of the duct

$$\hat{\phi}_m \big|_1 = J_m(k_{mn}r) \quad (2.16c)$$

$$\hat{\phi}_m \big|_2 = J_m(k_{mn}r) e^{-i\mu x_2} \quad (2.16d)$$

where

$$x_2 = \frac{L}{R}$$

L = Length of the duct

R = Radius of the duct

2. Along the hard walls

$$\frac{\partial \hat{\phi}_m}{\partial r} = 0 \quad (2.17)$$

Two cases have been considered

- a) $m = 0$ (axisymmetric mode, no spinning mode)
- b) $m = 1$ (1st spinning mode)

Since the duct is assumed to be axisymmetric, the problem can be solved for half the portion of the duct by prescribing the conditions along the axis of the duct.

3. Along the axis

$$\frac{\partial \hat{\phi}_m}{\partial r} = 0 \text{ (for } m = 0) \quad (2.18a)$$

$$\hat{\phi}_m = 0 \text{ (for } m \neq 0) \quad (2.18b.)$$

Because of the absence of spinning modes for $m = 0$ case, $\frac{\partial \hat{\phi}_m}{\partial r} = 0$. For $m \neq 0$ the contribution due to the term $\frac{-m^2}{r^2} \hat{\phi}_m$ in the Equation (2.13) is considered. As $r \rightarrow 0$, for $m \neq 0$, $\frac{m^2}{r^2}$ becomes very large and hence necessitates that ϕ_m approach zero for $\frac{m^2}{r^2} \hat{\phi}_m$ to be finite on the axis [9].

CHAPTER 3

SOLUTION METHOD

The governing equations, which are formulated in terms of $\hat{\phi}$ are transformed using the relation

$$\hat{\phi} = \phi e^{i\beta} \quad (3.1)$$

where ϕ is amplitude like term and is real and β is the phase like term.

The above transformation gives two equations by separating the real and imaginary terms in the transformed equation. Separating real parts give a second order equation in ϕ . Collecting the imaginary terms gives a second order equation in β . The above equations in ϕ and β are nonlinear. These equations can be solved numerically by a time-marching explicit method.

3.1 2-D Straight Duct

The wave equation in 2-D with Ni 's [1] decomposition as developed in Section 2 is of the form with $M = 0$ in (2.11).

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = \frac{\partial^2 \hat{\phi}}{\partial t^2} + 2ik \frac{\partial \hat{\phi}}{\partial t} - k^2 \hat{\phi} \quad (3.2)$$

Using (3.1), the terms in the above expression take the form

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} = \left\{ \frac{\partial^2 \phi}{\partial x^2} - \phi \left(\frac{\partial \beta}{\partial x} \right)^2 + i \left(\phi \frac{\partial^2 \beta}{\partial x^2} + 2 \frac{\partial \phi}{\partial x} \frac{\partial \beta}{\partial x} \right) \right\} e^{i\beta} \quad (3.3a)$$

$$\frac{\partial^2 \hat{\phi}}{\partial y^2} = \left\{ \frac{\partial^2 \phi}{\partial y^2} - \phi \left(\frac{\partial \beta}{\partial y} \right)^2 + i \left(\phi \frac{\partial^2 \beta}{\partial y^2} + 2 \frac{\partial \phi}{\partial y} \frac{\partial \beta}{\partial y} \right) \right\} e^{i\beta} \quad (3.3b)$$

$$\frac{\partial^2 \hat{\phi}}{\partial t^2} = \left\{ \frac{\partial^2 \phi}{\partial t^2} - \phi \left(\frac{\partial \beta}{\partial t} \right)^2 + i \left(\phi \frac{\partial^2 \beta}{\partial t^2} + 2 \frac{\partial \phi}{\partial t} \frac{\partial \beta}{\partial t} \right) \right\} e^{i\beta} \quad (3.3c)$$

$$2ik \frac{\partial \hat{\phi}}{\partial t} = 2ik \left(\frac{\partial \phi}{\partial t} + i \phi \frac{\partial \beta}{\partial t} \right) e^{i\beta} \quad (3.3d)$$

$$-k^2 \hat{\phi} = -k^2 \phi e^{i\beta} \quad (3.3e)$$

Using (3.3a), (3.3b), (3.3c), (3.3d), (3.3e), in (3.2) and separating real and imaginary parts, we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \phi \left(\frac{\partial \beta}{\partial x} \right)^2 - \phi \left(\frac{\partial \beta}{\partial y} \right)^2 + k^2 \phi + 2k\phi \frac{\partial \beta}{\partial t} = \frac{\partial^2 \phi}{\partial t^2} - \phi \left(\frac{\partial \beta}{\partial t} \right)^2 \quad (3.4)$$

and

$$\phi \frac{\partial^2 \beta}{\partial x^2} + \phi \frac{\partial^2 \beta}{\partial y^2} + 2 \frac{\partial \phi}{\partial x} \frac{\partial \beta}{\partial x} + 2 \frac{\partial \phi}{\partial y} \frac{\partial \beta}{\partial y} - 2k \frac{\partial \phi}{\partial t} = \phi \frac{\partial^2 \beta}{\partial t^2} + 2 \frac{\partial \phi}{\partial t} \frac{\partial \beta}{\partial t} \quad (3.5)$$

The Equations (3.4) and (3.5) are used to solve for ϕ and β .

Similarly, using (3.1) in the Equation (2.11) for 2-D straight duct with flow gives 3.6 and 3.7.

$$\begin{aligned} (1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - (1 - M^2) \phi \left(\frac{\partial \beta}{\partial x} \right)^2 - \phi \left(\frac{\partial \beta}{\partial y} \right)^2 + k^2 \phi + 2k\phi \frac{\partial \beta}{\partial t} \\ = \frac{\partial^2 \phi}{\partial t^2} - \phi \left(\frac{\partial \beta}{\partial t} \right)^2 + 2M \left(\frac{\partial^2 \phi}{\partial x \partial t} - \phi \frac{\partial \beta}{\partial x} \frac{\partial \beta}{\partial t} - \phi k \frac{\partial \beta}{\partial x} \right) \end{aligned} \quad (3.6)$$

$$\begin{aligned} (1 - M^2) \phi \frac{\partial^2 \beta}{\partial x^2} + \phi \frac{\partial^2 \beta}{\partial y^2} + 2(1 - M^2) \frac{\partial \phi}{\partial x} \frac{\partial \beta}{\partial x} + 2 \frac{\partial \phi}{\partial y} \frac{\partial \beta}{\partial y} - 2k \frac{\partial \phi}{\partial t} \\ = \phi \frac{\partial^2 \beta}{\partial t^2} + 2 \frac{\partial \phi}{\partial t} \frac{\partial \beta}{\partial t} + 2M \left(\phi \frac{\partial^2 \beta}{\partial x \partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \beta}{\partial t} + k \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial t} \frac{\partial \beta}{\partial x} \right) \end{aligned} \quad (3.7)$$

3.1.1 Boundary Conditions

Using (3.1) the boundary conditions which are formulated in the previous section are transformed as follows

1. At both ends of the duct

$$\phi = \cos(k_y y) \quad (3.8a)$$

$$\beta = -k_x x \quad (3.8b)$$

where $k_x = \sqrt{k^2 - k_y^2}$ for no flow

$$k_x = \frac{-Mk + \sqrt{k_y^2(1-M^2) + k^2}}{(1-M^2)} \text{ with flow}$$

and $k_y = n\pi$

2. At the hard walls $\frac{\partial \tilde{\phi}}{\partial y} = 0$ is transformed as

$$\frac{\partial \phi}{\partial y} = 0 \quad (3.9a)$$

$$\frac{\partial \beta}{\partial y} = 0 \quad (3.9b)$$

3.2 Cylindrical Duct

Using the transformation

$$\hat{\phi}_m = \phi_m e^{i\beta_m} \quad (3.10)$$

in Equation (2.13) for $M = 0$, and collecting real and imaginary parts, we get

$$\begin{aligned} \frac{\partial^2 \phi_m}{\partial x^2} + \frac{\partial^2 \phi_m}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_m}{\partial r} - \frac{m^2 \phi_m}{r^2} - \phi_m \left(\frac{\partial \beta_m}{\partial x} \right)^2 - \phi_m \left(\frac{\partial \beta_m}{\partial r} \right)^2 + k^2 \phi_m + 2k \phi_m \frac{\partial \beta_m}{\partial t} \\ = \frac{\partial^2 \phi_m}{\partial t^2} - \phi_m \left(\frac{\partial \beta_m}{\partial t} \right)^2 \end{aligned} \quad (3.11)$$

$$\begin{aligned} \phi_m \frac{\partial^2 \beta_m}{\partial x^2} + \phi_m \frac{\partial^2 \beta_m}{\partial r^2} + 2 \frac{\partial \phi_m}{\partial x} \frac{\partial \beta_m}{\partial x} + 2 \frac{\partial \phi_m}{\partial r} \frac{\partial \beta_m}{\partial r} - 2k \frac{\partial \phi_m}{\partial t} + \frac{1}{r} \phi_m \frac{\partial \beta_m}{\partial r} \\ = \phi_m \frac{\partial^2 \beta_m}{\partial t^2} + 2 \frac{\partial \phi_m}{\partial t} \frac{\partial \beta_m}{\partial t} \end{aligned} \quad (3.12)$$

The Equations (3.10) and (3.11) with flow inside the cylinder take the form

$$\begin{aligned} (1-M^2) \frac{\partial^2 \phi_m}{\partial x^2} - (1-M^2) \phi_m \left(\frac{\partial \beta_m}{\partial x} \right)^2 + \frac{\partial^2 \phi_m}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_m}{\partial r} - \frac{m^2 \phi_m}{r^2} - \phi_m \left(\frac{\partial \beta_m}{\partial r} \right)^2 + k^2 \phi_m \\ + 2k \phi_m \frac{\partial \beta_m}{\partial t} = \frac{\partial^2 \phi_m}{\partial t^2} - \phi_m \left(\frac{\partial \beta_m}{\partial t} \right)^2 + 2M \left[\frac{\partial^2 \phi_m}{\partial x \partial t} - \phi_m \frac{\partial \beta_m}{\partial x} \frac{\partial \beta_m}{\partial t} - \phi_m k \frac{\partial \beta_m}{\partial x} \right] \end{aligned} \quad (3.13)$$

$$\begin{aligned}
& (1-M^2)\phi_m \frac{\partial^2 \beta_m}{\partial x^2} + \phi_m \frac{\partial^2 \beta_m}{\partial r^2} + 2(1-M^2) \frac{\partial \phi_m}{\partial x} \frac{\partial \beta_m}{\partial x} - 2k \frac{\partial \phi_m}{\partial t} + 2 \frac{\partial \phi_m}{\partial r} \frac{\partial \beta_m}{\partial r} + \frac{1}{r} \phi_m \frac{\partial \beta_m}{\partial r} \\
& = \phi_m \frac{\partial^2 \beta_m}{\partial t^2} + 2 \frac{\partial \phi_m}{\partial t} \frac{\partial \beta_m}{\partial t} + 2M \left[\phi_m \frac{\partial^2 \beta_m}{\partial x \partial t} + \frac{\partial \phi_m}{\partial x} \frac{\partial \beta_m}{\partial t} + k \frac{\partial \phi_m}{\partial x} + \frac{\partial \phi_m}{\partial t} \frac{\partial \beta_m}{\partial x} \right] \quad (3.14)
\end{aligned}$$

3.2.1 Boundary Conditions

1. Along the entrance and exit of the duct

$$\phi_m = J_m(k_{mn}r) \quad (3.14a)$$

$$\beta_m = -\mu x \quad (3.14b)$$

where $\mu = \sqrt{k^2 - k_{mn}^2}$ without flow

$\mu = \frac{-Mk + \sqrt{k_{mn}^2(M^2 - 1) + k^2}}{(1 - M^2)}$ with flow

k_{mn} is obtained from satisfying $\frac{\partial}{\partial r}(J_m(k_{mn}r))|_{r=1} = 0$

2. Along the hard walls

$$\frac{\partial \phi_m}{\partial r} = 0 \quad (3.15a)$$

$$\frac{\partial \beta_m}{\partial r} = 0 \quad (3.15b)$$

3. Along the axis

a) For $m = 0$

$$\frac{\partial \phi_m}{\partial r} = 0 \quad (3.16a)$$

$$\frac{\partial \beta_m}{\partial r} = 0 \quad (3.16b)$$

b) For $m \neq 0$

$$\phi_m = 0 \quad (3.16c)$$

The phase like term, β_m has to be defined on the axis as described in the Chapter "Computational Method."

CHAPTER 4

COMPUTATIONAL METHOD

4.1 2-D Straight Duct

The schematic of grid points used for a 2-D duct are shown in Figure 4.1.

The terms in Equation (3.6) are computed as follows:

- 1) $\frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^2 \phi}{\partial y^2}, \frac{\partial^2 \phi}{\partial t^2}$ central difference;
- 2) $\frac{\partial \beta}{\partial x}, \frac{\partial \beta}{\partial y}$ central difference;
- 3) $\frac{\partial \beta}{\partial t}$ backward difference;
- 4) $\frac{\partial^2 \phi}{\partial x \partial t}$ backward in time and central in space.

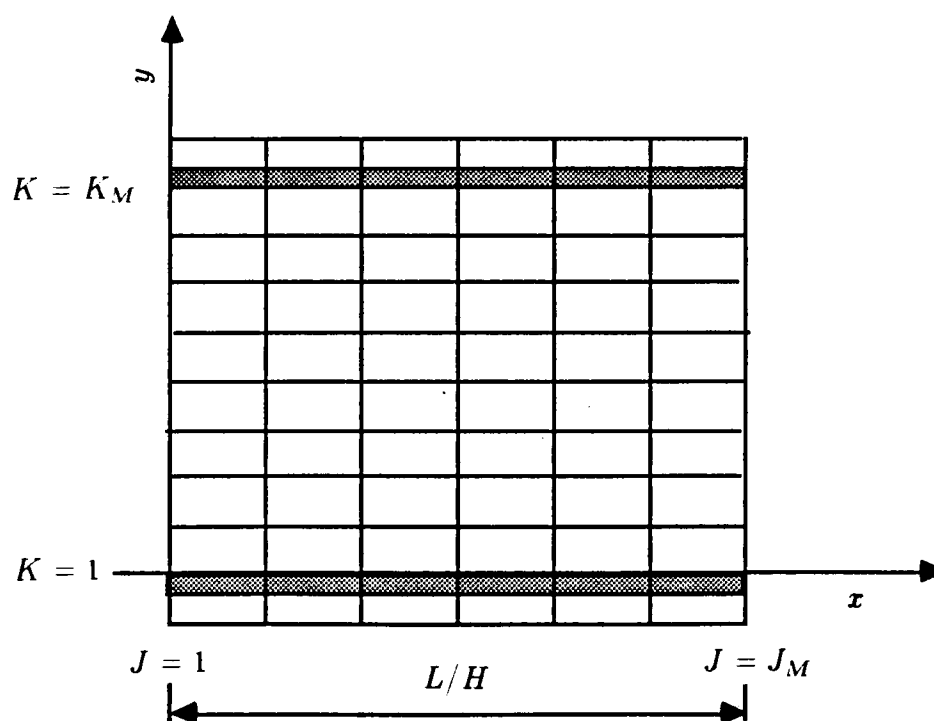


Figure 4.1 Schematic of grids for a 2-D straight duct

The terms in Equation (3.7) are computed using the following difference representation:

- 1) $\frac{\partial^2 \beta}{\partial x^2}, \frac{\partial^2 \beta}{\partial y^2}, \frac{\partial^2 \beta}{\partial t^2}$ central difference;
- 2) $\frac{\partial \beta}{\partial x}, \frac{\partial \beta}{\partial y}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$ central difference;
- 3) $\frac{\partial \phi}{\partial t}, \frac{\partial \beta}{\partial t}$ backward difference;
- 4) $\frac{\partial^2 \beta}{\partial x \partial t}$ backward in time and central in space.

These terms are evaluated to obtain an explicit time marching scheme as explained in the Appendix. To apply the hard wall boundary conditions hypothetical nodes are used as shown in Figure 4.1. A first order boundary condition was tried earlier. The boundary condition, $\phi_{J,K_M}^N = \phi_{J,K_M-1}^N$ made the solution unstable. Hence, a second order boundary condition, $\phi_{J,K_M+1}^N = \phi_{J,K_M-1}^N$ is used and nodes on $K = K_M$ and $K = 1$ are used in the field points.

A relaxation in time is used in the form

$$\phi_{J,K}^{\overline{N+1}} = \alpha \phi_{J,K}^{N+1} + (1 - \alpha) \phi_{J,K}^N \quad (4.1)$$

The parameter α is varied from 0 to a value of α_1 ($0.7 \leq \alpha_1 \leq 0.9$) over 300 – 400 time steps to start the solution slowly. Later α is kept constant at the value of α_1 .

Spatial averaging is used to stabilize the solution as

$$\phi_{J,K}^{\overline{\overline{N+1}}} = (1 - \epsilon) \phi_{J,K}^{\overline{N+1}} + \frac{\epsilon}{4} \{ \phi_{J+1,K}^{\overline{N+1}} + \phi_{J-1,K}^{\overline{N+1}} + \phi_{J,K+1}^{\overline{N+1}} + \phi_{J,K-1}^{\overline{N+1}} \} \quad (4.2)$$

The parameter ϵ is in the range 0.05 – 0.3

The steps indicated above are also used in evaluating β at the new time steps. The values of ϕ and β are evaluated one after the other during each time step.

4.1.1 Initial Conditions

Usually the duct is assumed to be quiescent. But $\phi = 0$ cannot be used as an initial value, since evaluating β at each new time step involves division by ϕ . Hence, a small value for ϕ in the range 0.001 – 0.01 is used as the initial value. The phase-like term β is set equal to zero.

4.2 Cylindrical Duct

The grid points used for a cylindrical duct for $m = 0$ and $m = 1$ are shown in the Figure 4.2.

The same difference technique as used on a 2-D duct is used in this case. The boundary conditions are set as follows:

- 1) For $m = 0$, to avoid the regular singularity, the grid points corresponding to $K = 1$ are considered to be located at a distance δ from $K = 1$ line. $\delta \sim \Delta r / 10000$. For this case, $\frac{\partial \phi_m}{\partial r} = 0$ is set using hypothetical points below the axis.
- 2) For $m = 1$, the value of $\phi_m = 0$ is set on the axis. The phase-like term β_m is defined by interpolation to the axis after each time-step.

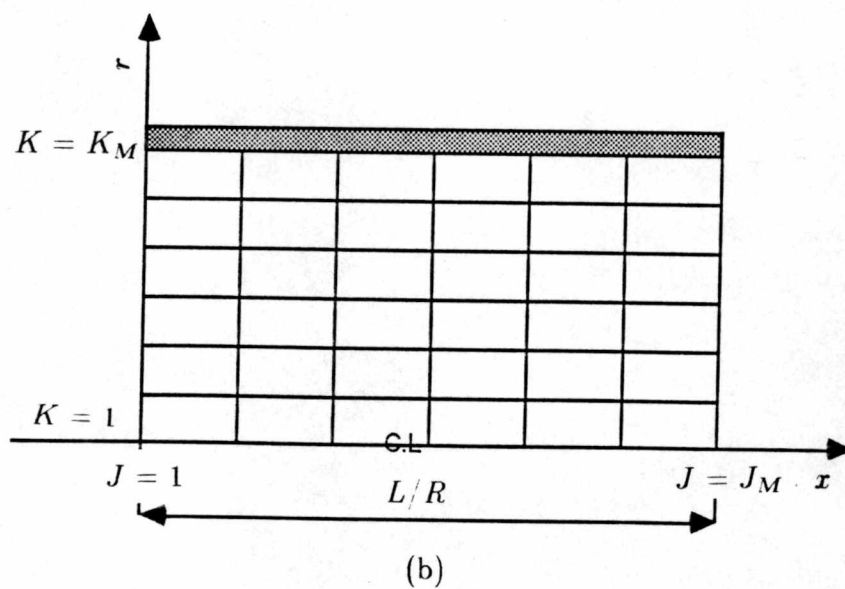
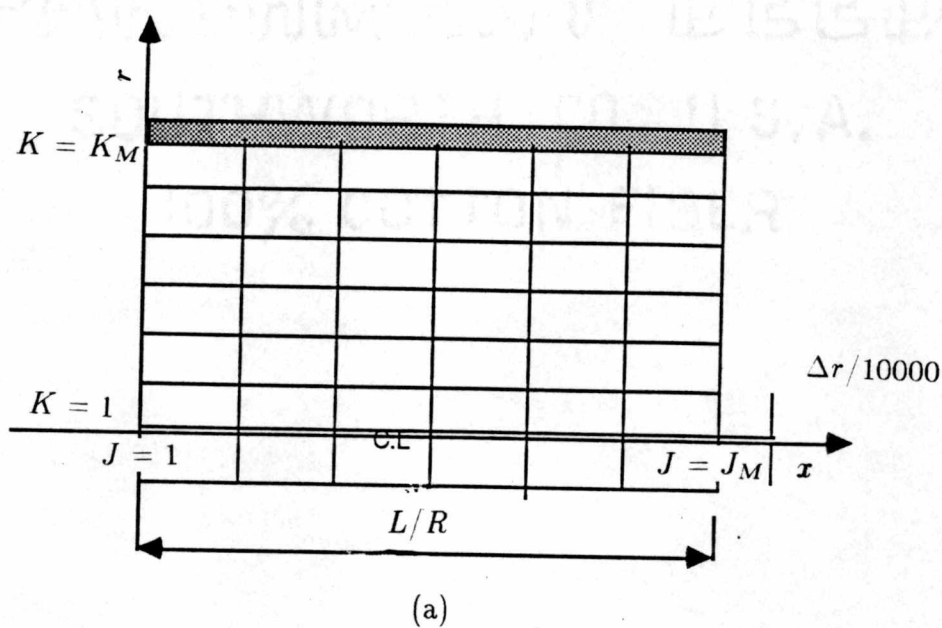


Figure 4.2 Schematics of grids for a cylindrical duct.

(a) for $m = 0$

(b) for $m \neq 0$

CHAPTER 5

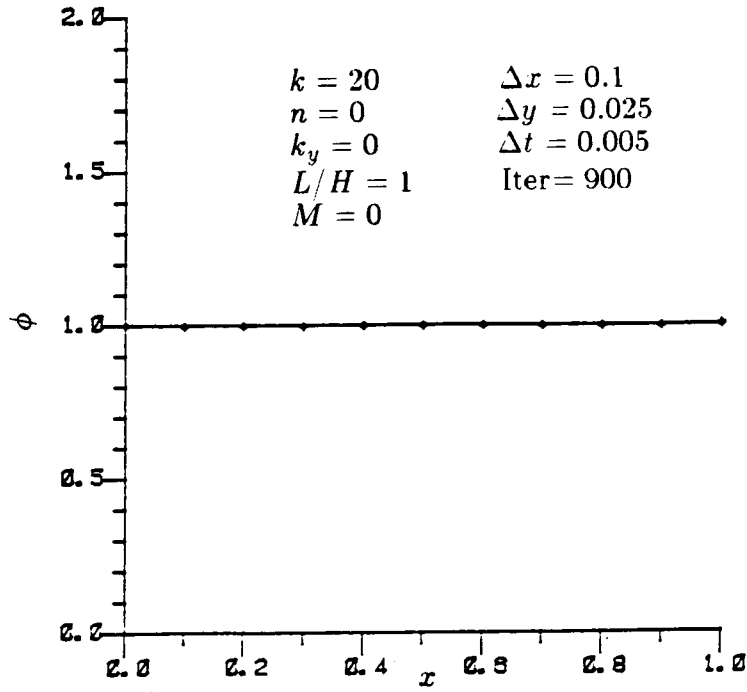
DISCUSSION OF RESULTS

The analytical and numerical values of the potential are calculated and compared for a 2-D straight duct and a cylindrical duct. The symbols in the plotted results represent the numerical values calculated at the grid points whereas the solid line represents the analytical value. The schematics of the duct grids are shown in Figures 4.1,4.2

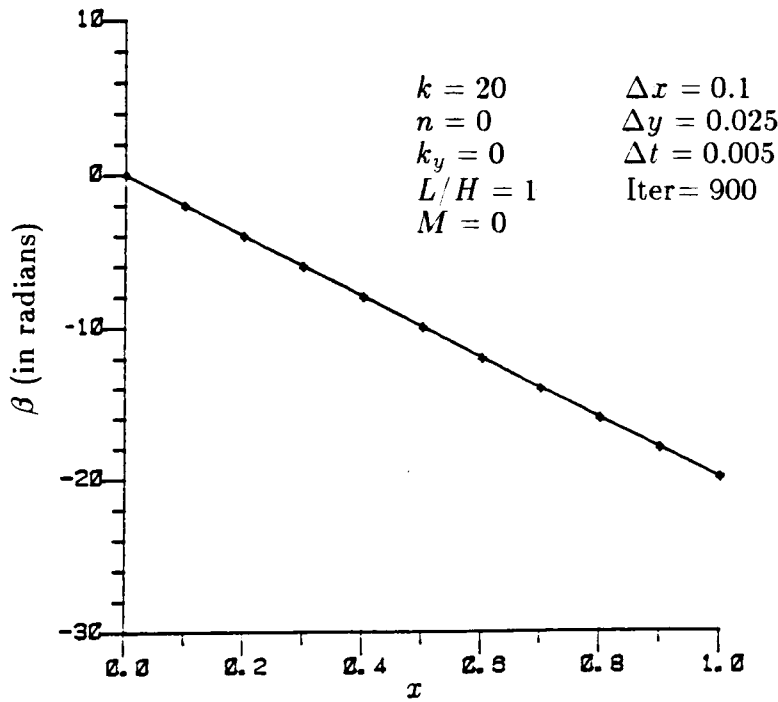
5.1 2-D Hard Walled Duct

Numerical and analytical values of the potential are computed for hard walled ducts. The calculations are made for a duct of L/H ratio equal to 1 and the cross duct wave numbers $n = 0$ and $n = 1$ with and without flow. The amplitude like term ϕ and the phase like term β are plotted *wrt* x at $y = 0.5$ and *wrt* y at $x = 0.5$ in Figure 5.1 for the case $n = 0$, $k = 20$ and $M = 0$. The phase like term β plotted in all the cases is in radians. The averaging parameters ϵ used in the above calculation is 0.15. The numerical results show excellent agreement with the analytical result.

For the case $n = 1$, the inlet wave amplitude at $J = 1$ varies as $|\cos(k_y y)|$ and the phase remains zero for $y < \frac{1}{2}$ and the phase is π for $y > \frac{1}{2}$. Instead of considering the actual amplitude and the actual phase the amplitude like term ϕ is allowed to take negative values in order to prevent 180° discontinuities in the phase like term, β . Hence a negative sign in the results imply a shift of π radians in the phase. A reduced frequency $k = 20$ and a flow in the negative x -direction with a Mach number $M = 0.5$ are considered here. The results for this above case are plotted at $x = 0.5$ ($J = 6$) *wrt* y and at $K = 26$ *wrt* x in Figure 5.2. The numerical results are in very good agreement with analytical results. This second

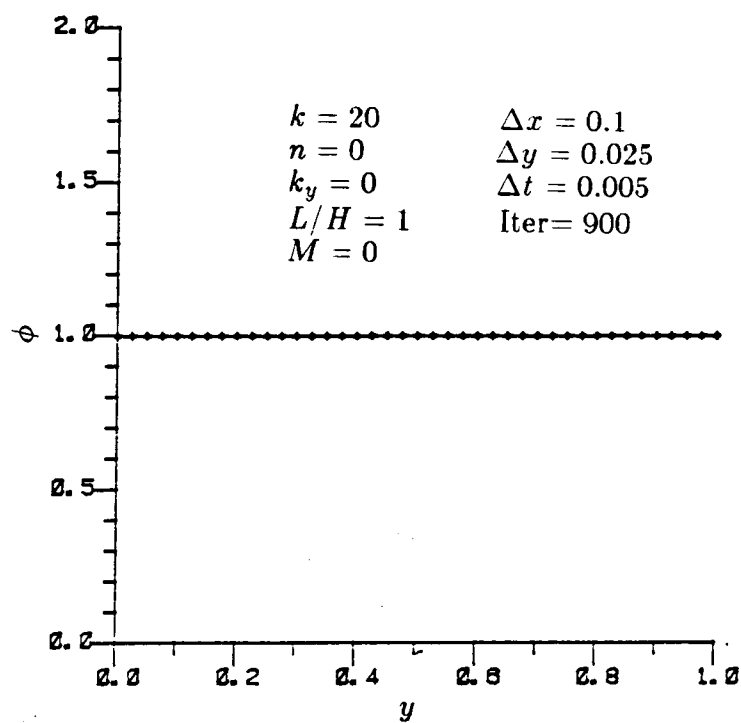


(a)

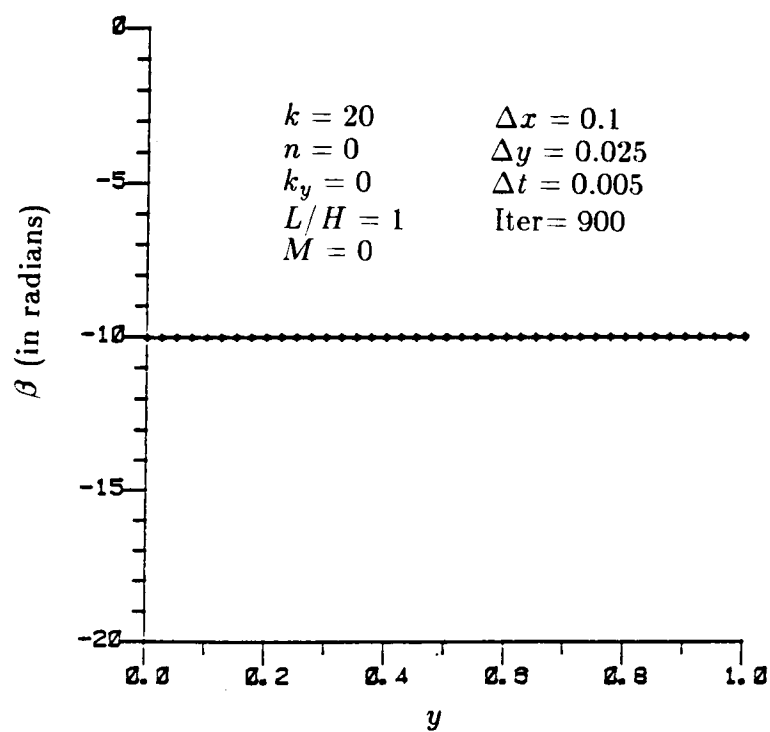


(b)

Figure 5.1 Comparison of numerical and analytical results for sound radiation in a 2-D straight duct without flow, for crossduct wavenumber $n = 0$ and reduced frequency $k = 20$



(c)



(d)

Figure 5.1 (continued)

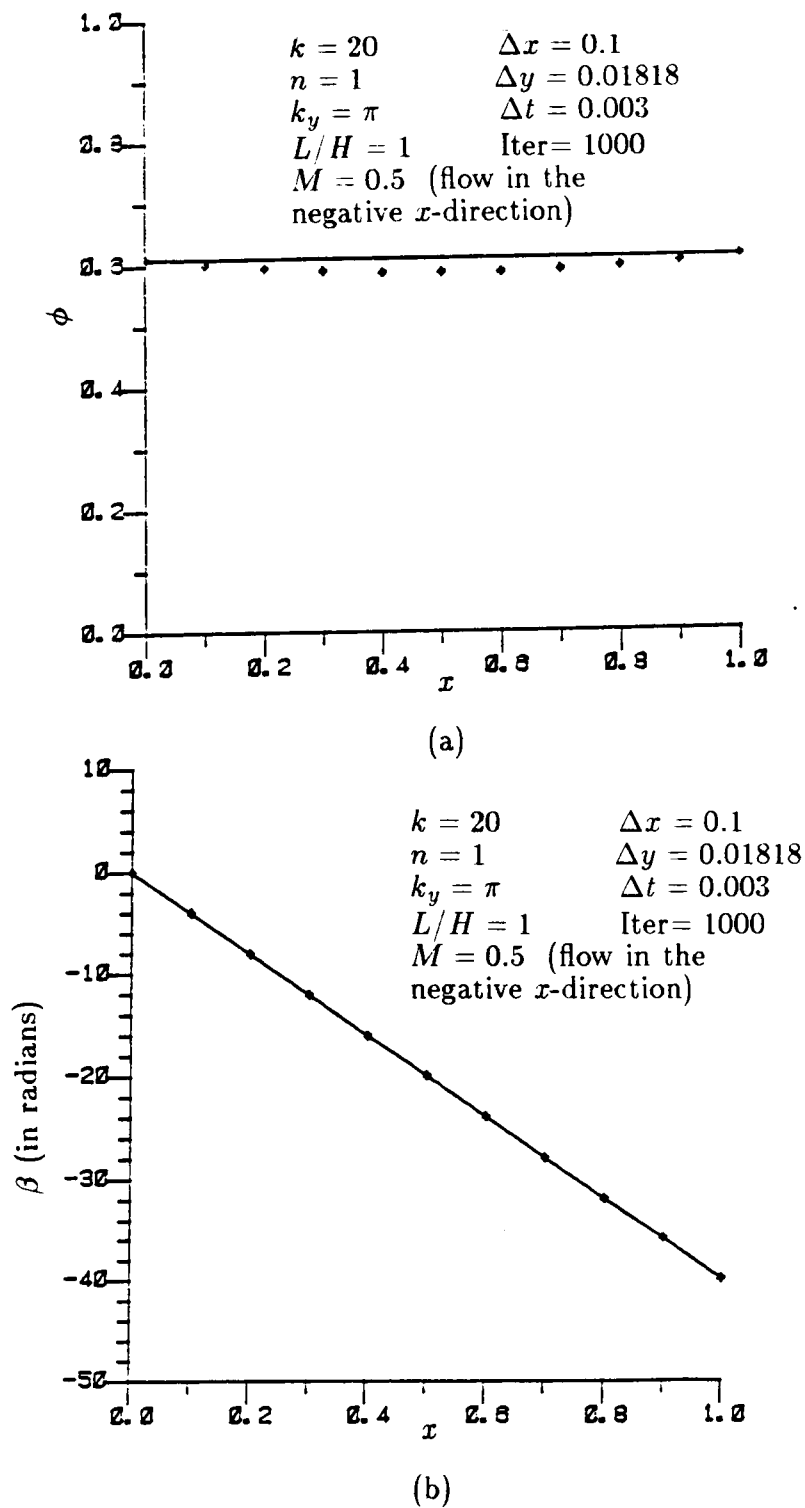
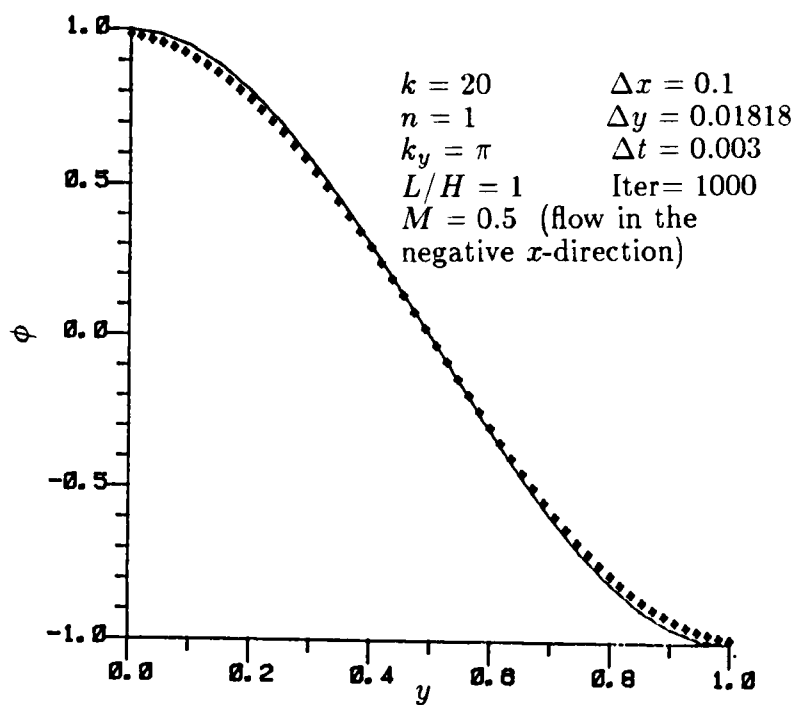
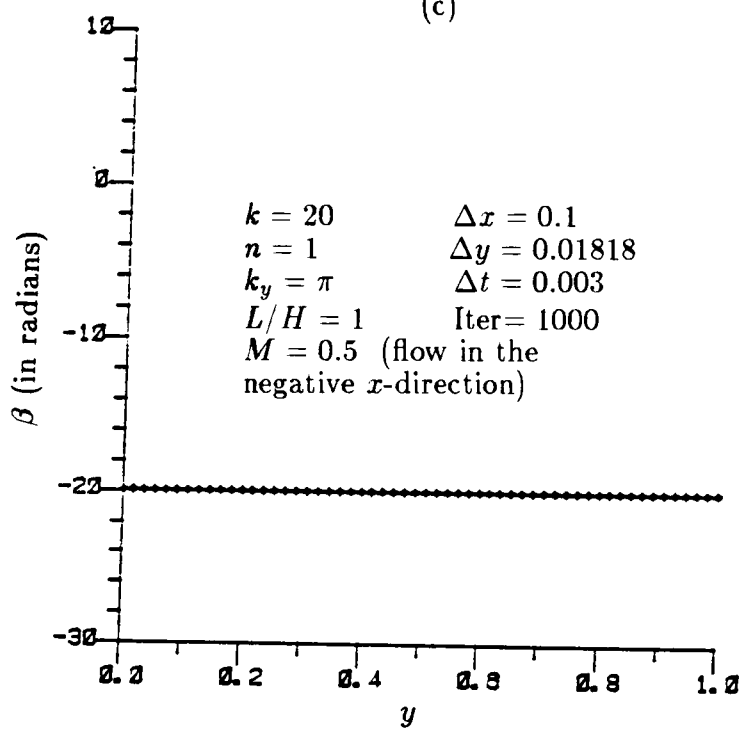


Figure 5.2 Comparison of numerical and analytical results for sound radiation in a 2-D straight duct with flow in the negative x -direction ($M = 0.5$), for crossduct wavenumber $n = 1$ and reduced frequency $k = 20$



(c)



(d)

Figure 5.2 (continued)

case was solved with 56 points in y direction and an $\epsilon = 0.2$ and $\alpha_1 = 0.7$. The value of α is varied slowly from 0 to 0.7 in starting 400 steps. The effect of varying the time step for these parameters showed that the solution is stable for $\Delta t \leq 0.005$.

5.2 Cylindrical Duct

In the cylindrical duct, calculations are made for spinning modes of $m = 0$ and $m = 1$ and transverse modes of $n = 1$ and 2 with and without flow for an $\frac{L}{R}$ ratio of 1.

For the case $m = 0$ and $n = 1$, $k_{mn} = 0$ the results, plotted in Figure 5.3, show excellent agreement with the analytical results. For the case $m = 1$ and $n = 1$, $k_{mn} = 1.8418$ (from [12]), the results are plotted in Figure 5.5. There is a very small variation (less than a percent) in the value of ϕ and the same small variation is observed for $m = 0$ and $n = 2$, $k_{mn} = 3.8317$ in Figure 5.4. The parameters $\epsilon = 0.15$ and $\alpha_1 = 0.7$ are used. The value of α is varied slowly from 0 to 0.7 in starting 400 steps. The effect of varying the time step for these parameters showed that the solution is stable for $\Delta t \leq 0.011$. For the case $m = 1$ and $n = 2$, $k_{mn} = 5.33144$ we observe a small drop in the value of ϕ towards the middle of the duct (towards $x = 0.5$) in Figure 5.6. This is attributable to the smoothing used to stabilize the numerical solution method.

5.3 Reduced Frequency $k = 100$

To check the feasibility of the method for very high reduced frequencies, two sample cases are considered. The results for a 2-D hard walled duct for a reduced frequency $k = 100$, and crossduct wavenumber $n = 1$ without flow are shown in the Figure 5.7. This case was run with a smaller time step $\Delta t = .001$, with 10 points in the axial direction and 56 points in the transverse direction. There is a small drop

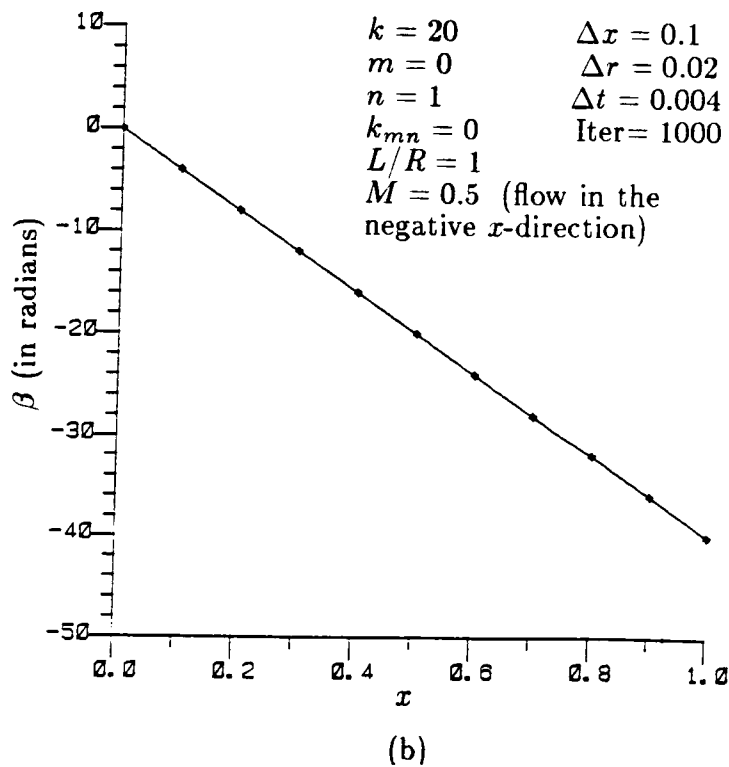
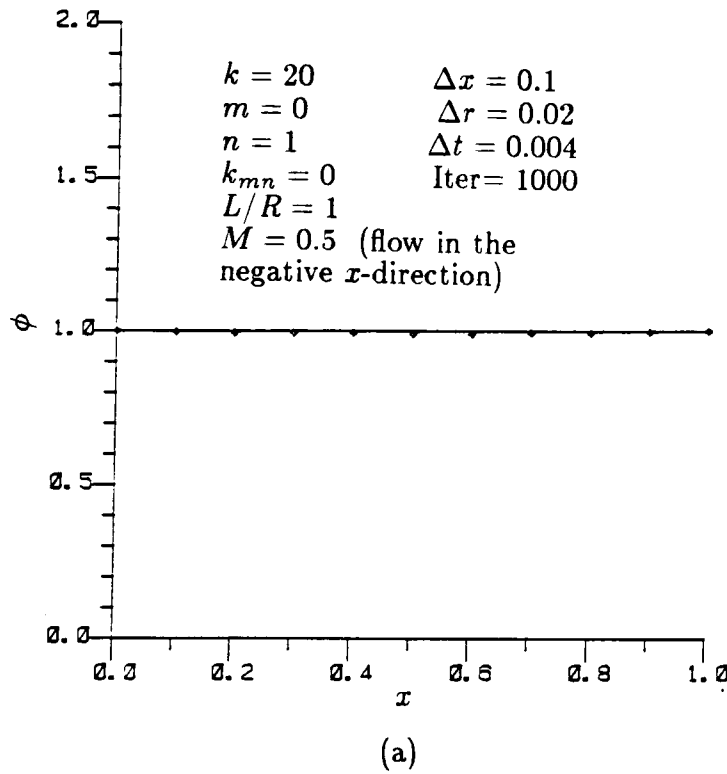
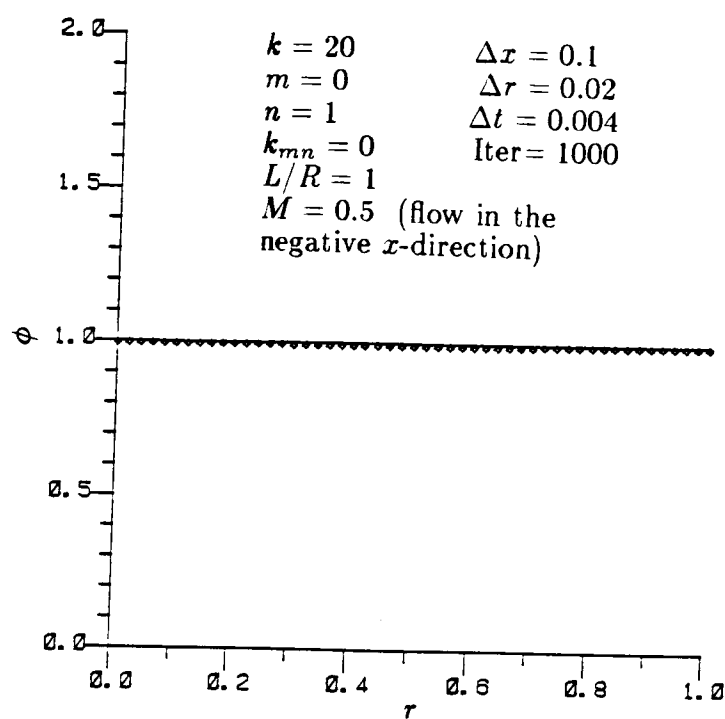
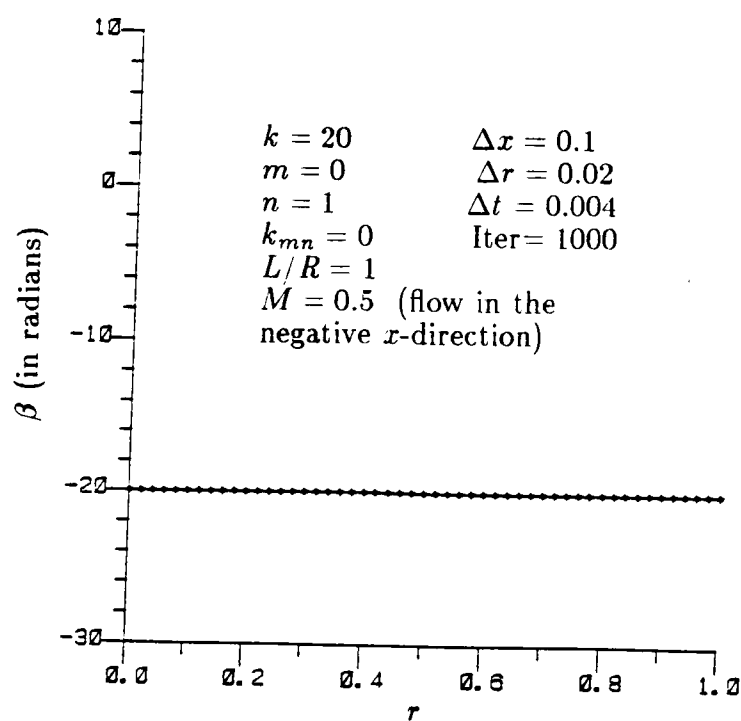


Figure 5.3 Comparison of numerical and analytical results for sound radiation in a Cylindrical duct with flow in the negative x -direction ($M = 0.5$), for spinning mode $m = 0$, crossduct wavenumber $n = 1$ and reduced frequency $k = 20$

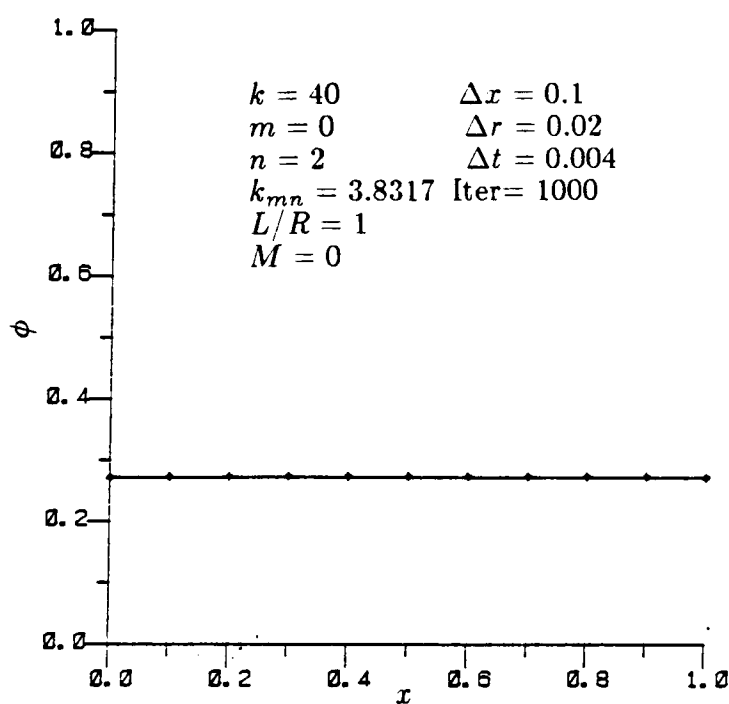


(c)

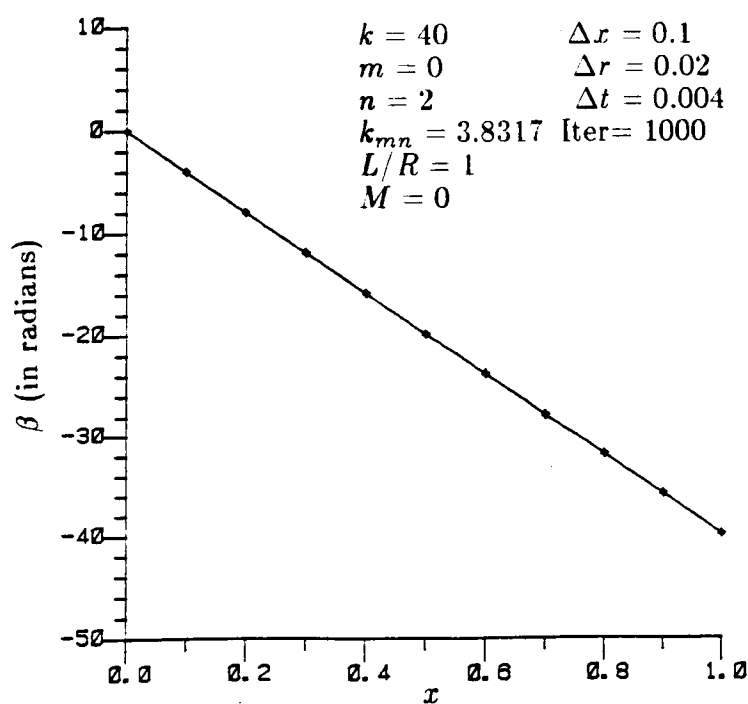


(d)

Figure 5.3 (continued)

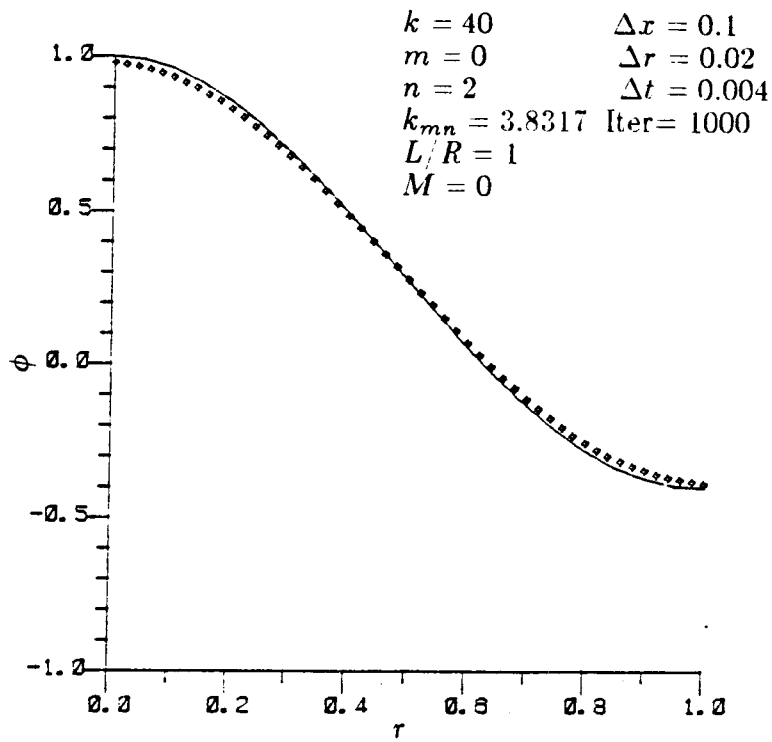


(a)

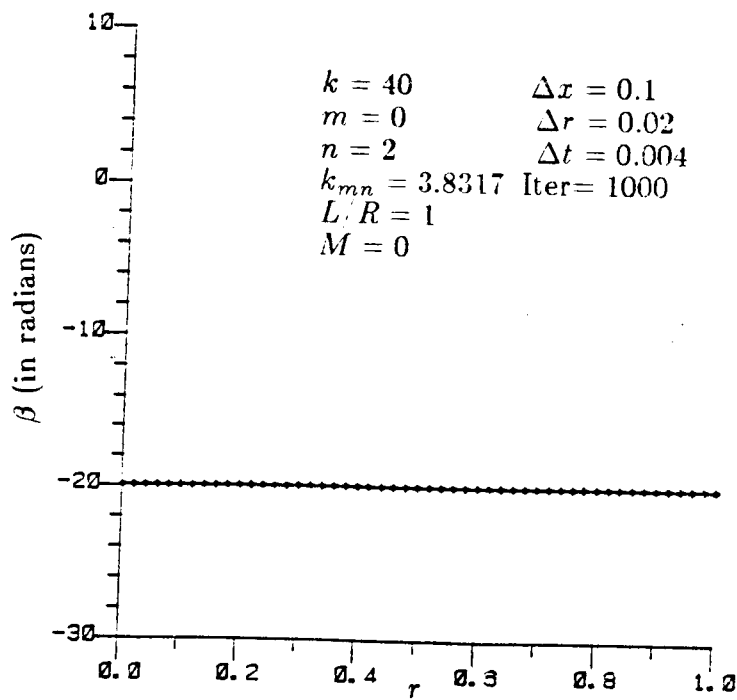


(b)

Figure 5.4 Comparison of numerical and analytical results for sound radiation in a Cylindrical duct without flow, for spinning mode $m = 0$, crossduct wavenumber $n = 2$ and reduced frequency $k = 40$



(c)



(d)

Figure 5.4 (continued)

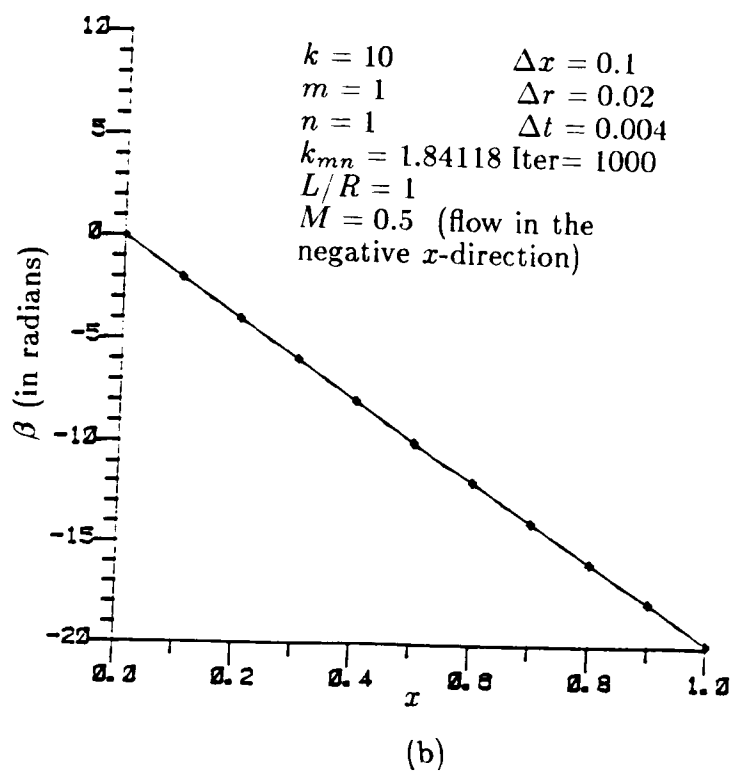
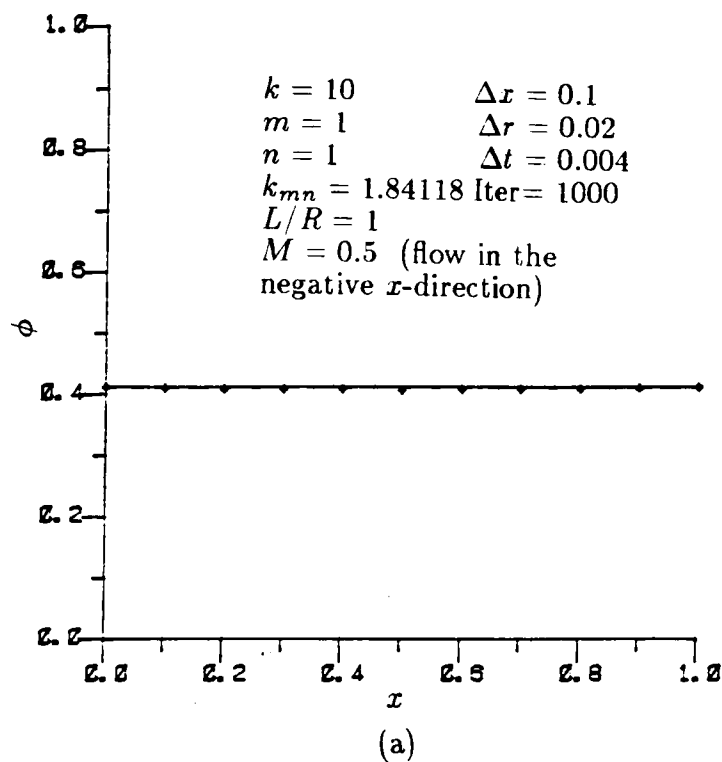
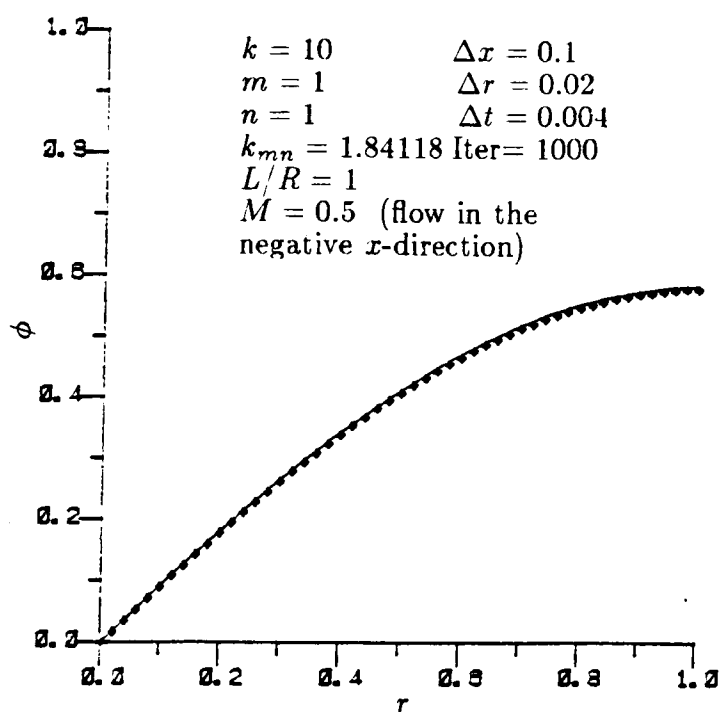
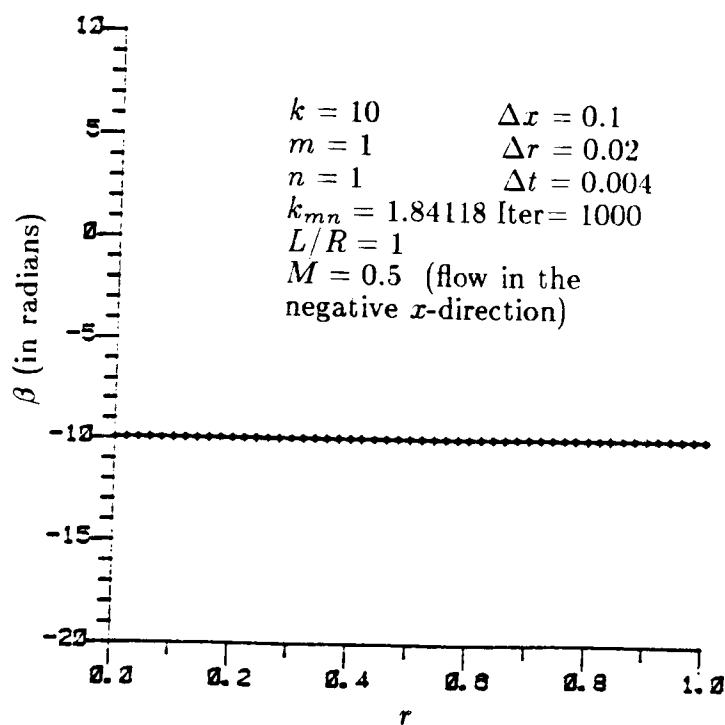


Figure 5.5 Comparison of numerical and analytical results for sound radiation in a Cylindrical duct with flow in the negative x -direction ($M = 0.5$), for spinning mode $m = 1$, crossduct wavenumber $n = 1$ and reduced frequency $k = 10$

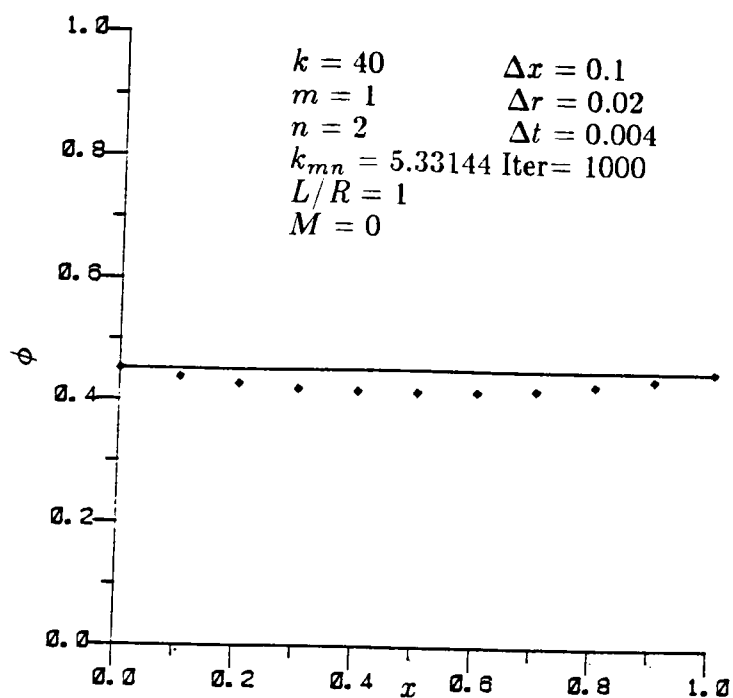


(c)

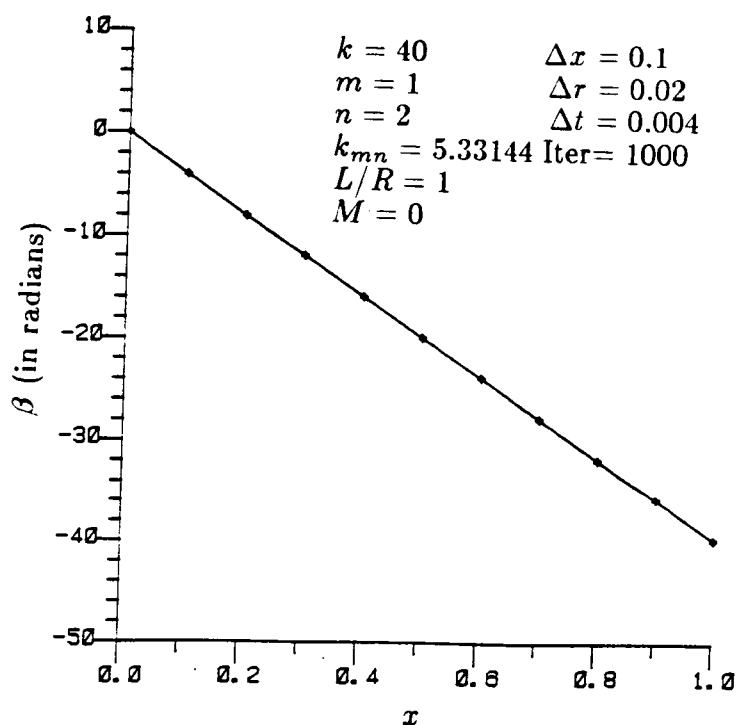


(d)

Figure 5.5 (continued)



(a)



(b)

Figure 5.6 Comparison of numerical and analytical results for sound radiation in a Cylindrical duct without flow, for spinning mode $m = 1$, crossduct wavenumber $n = 2$ and reduced frequency $k = 40$

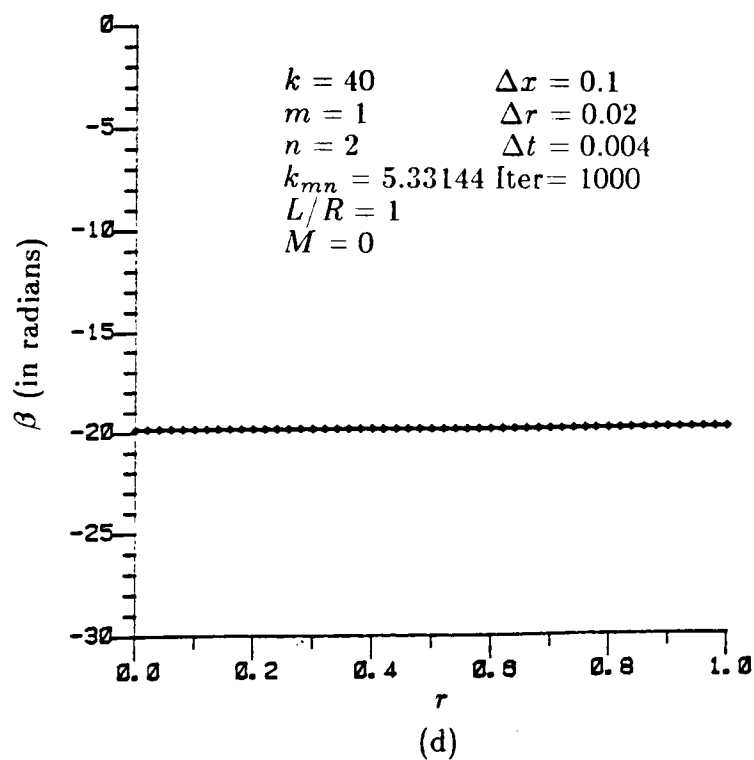
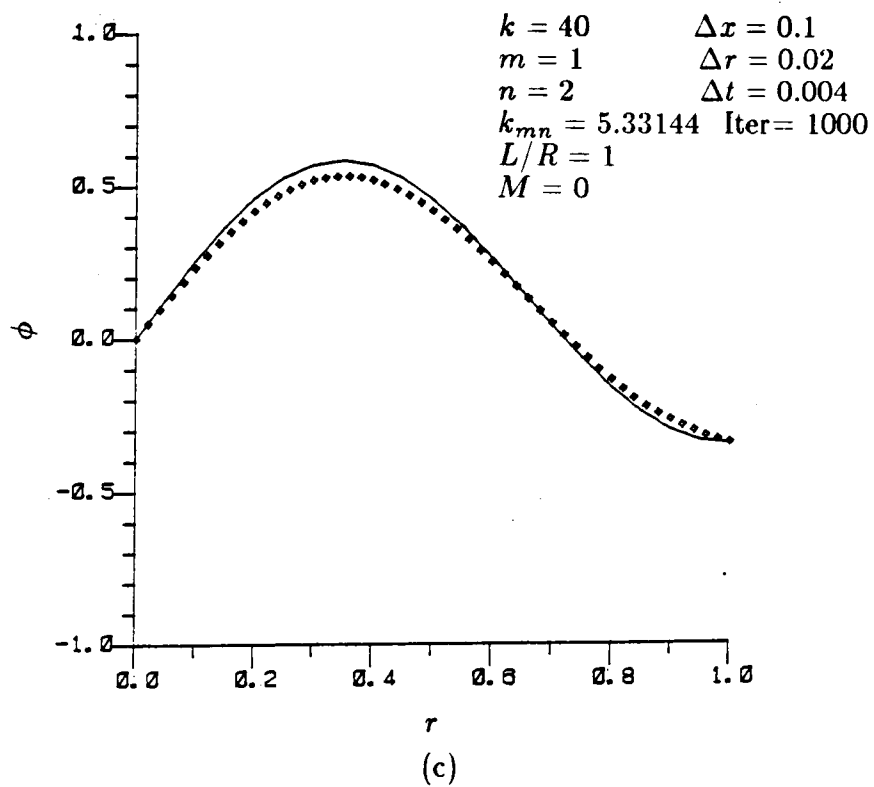
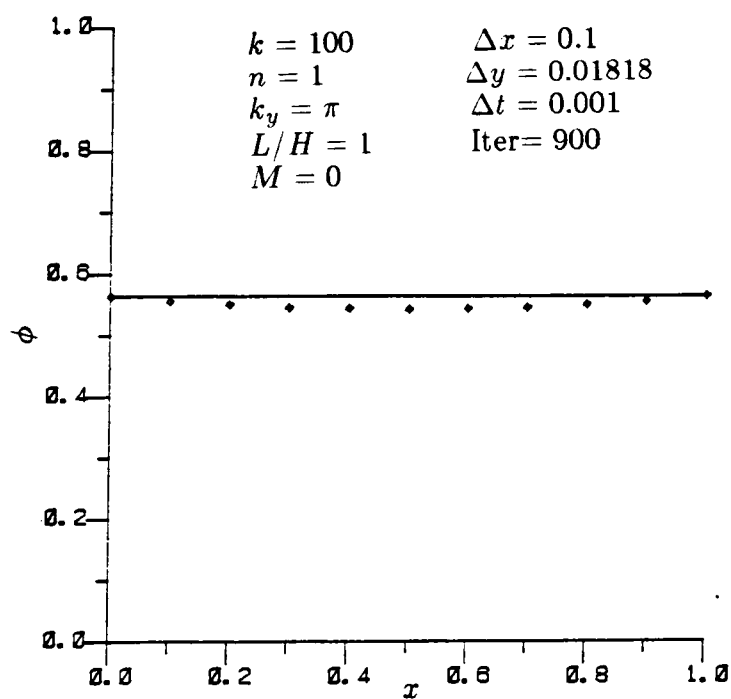
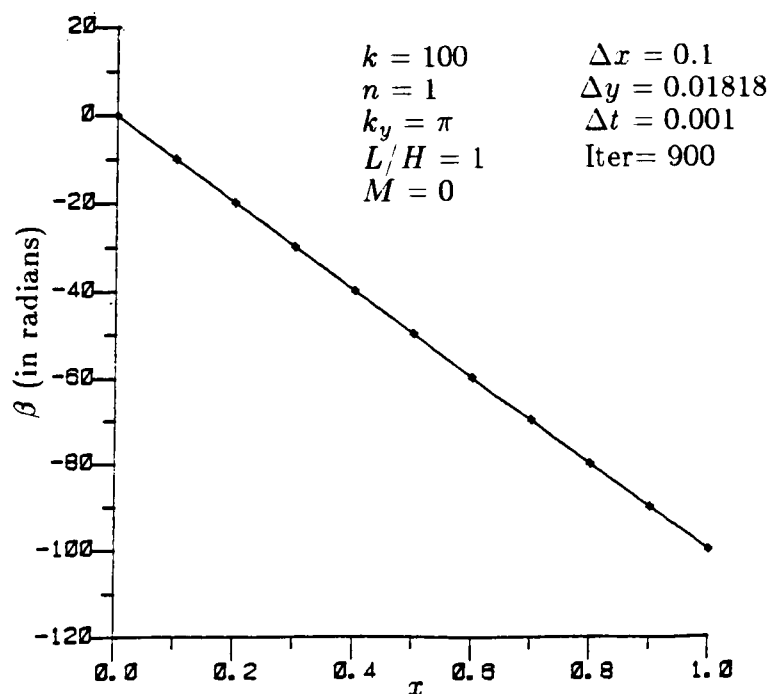


Figure 5.6 (continued)

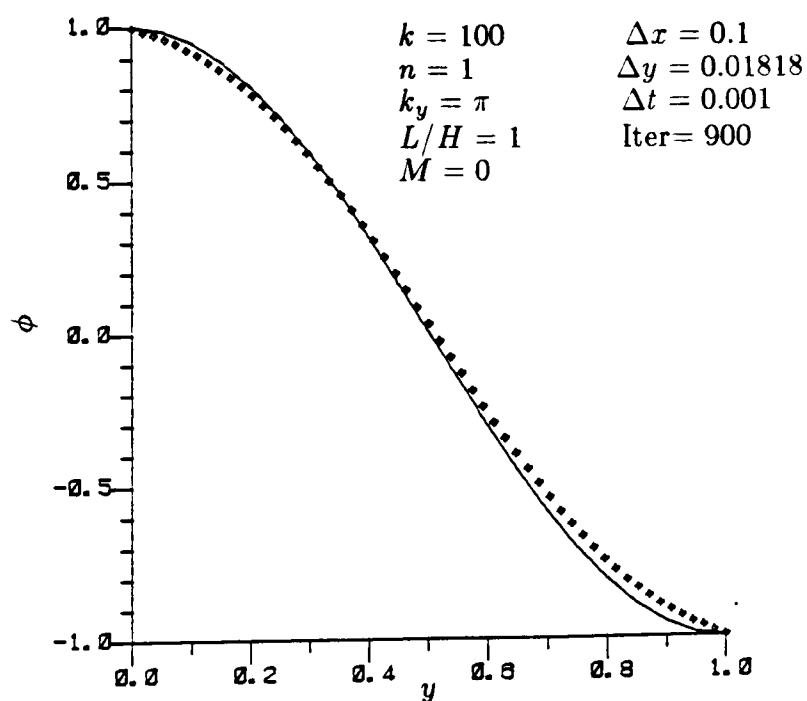


(a)

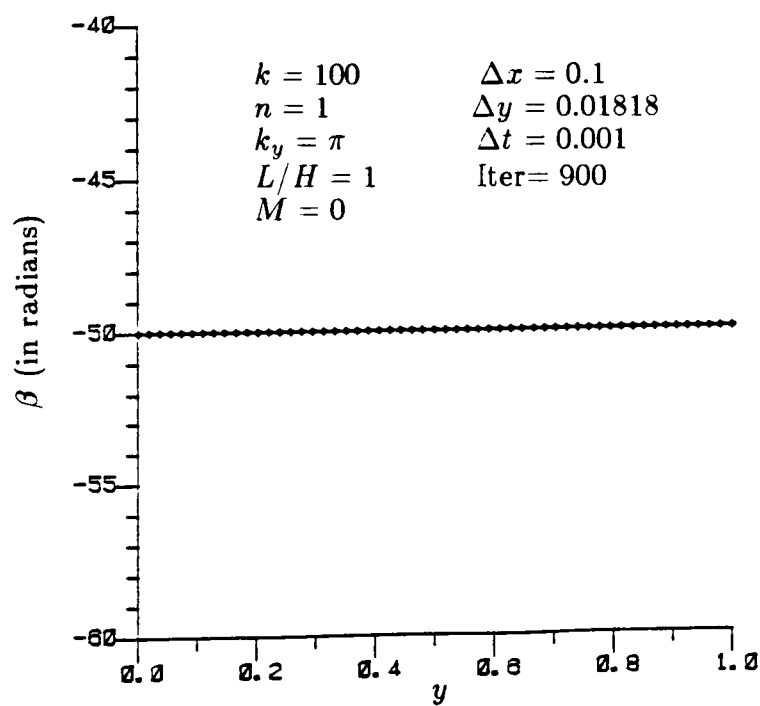


(b)

Figure 5.7 Comparison of numerical and analytical results for sound radiation in a 2-D straight duct for a reduced frequency $k = 100$



(c)



(d)

Figure 5.7 (continued)

in ϕ towards the middle of the duct (along the x-direction). The numerical results are in good agreement with the analytical results.

The cylindrical duct with $k = 100, m = 1, n = 1$ case shows very good agreement with the analytical results. (Figure 5.8)

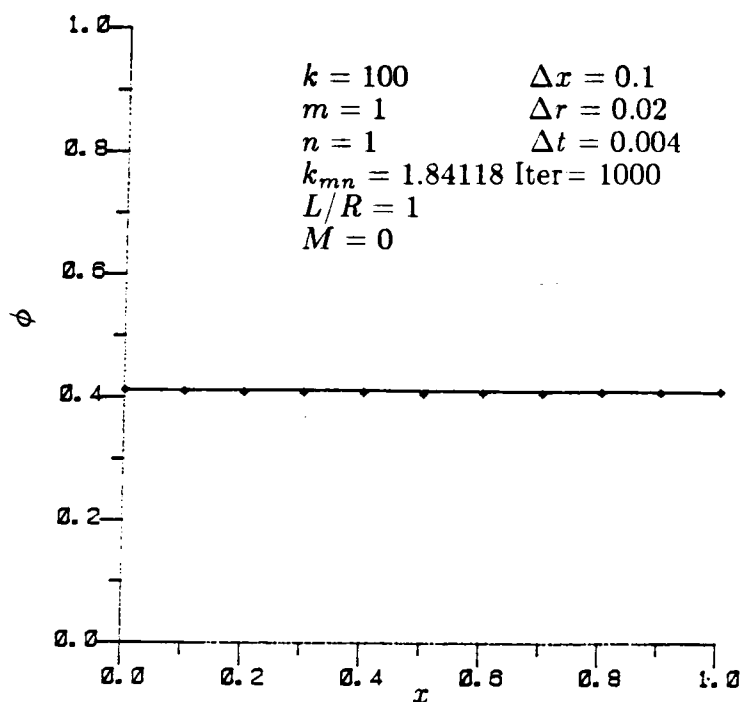
Baumeister, K. J. and Bittner, E. C. [3] have suggested an empirical relation for the minimum number of axial grid points to be used to resolve the oscillatory nature of the solution in a finite difference method (steady state solution) as

$$I > 12\eta(L/H)$$

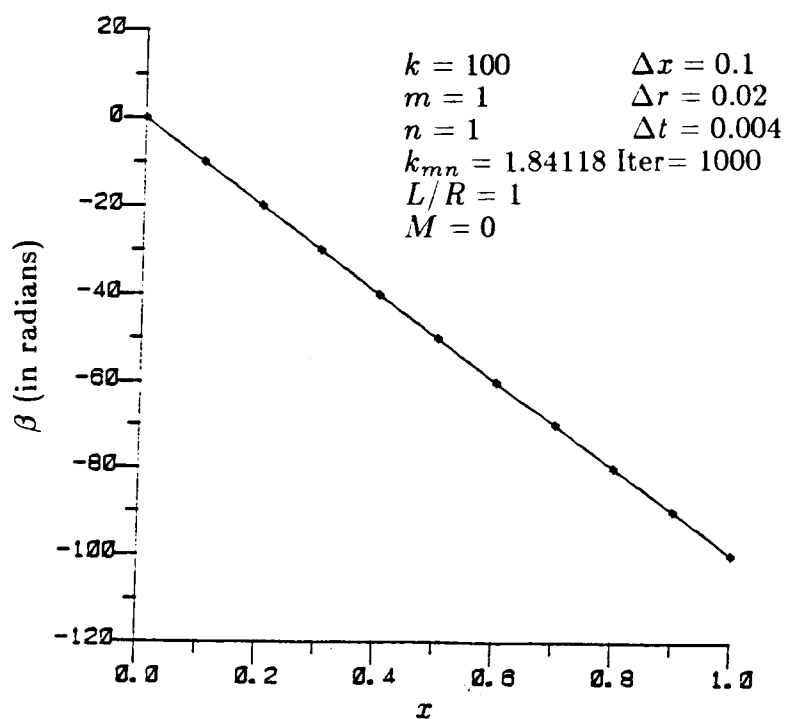
where $\eta = \frac{\omega L}{2\pi H}$.

For $\frac{L}{H} = 1, k = 100, \eta = \frac{100}{2\pi}$ the number of axial grid points required according to the above relation is approximately 192. However, the number of axial grid points used in the above calculation being only 10, shows a dramatic reduction in the number of axial grid points.

Although the present checkout cases computed are simple enough to have allowed factoring out the known oscillating behavior at the outset, the method presented here does not require such simple behavior in order to be applicable. In fact, no assumptions regarding the spatial variations of the acoustic field need be made at all!



(a)



(b)

Figure 5.8 Comparison of numerical and analytical results for sound radiation in a Cylindrical duct for a reduced frequency $k = 100$

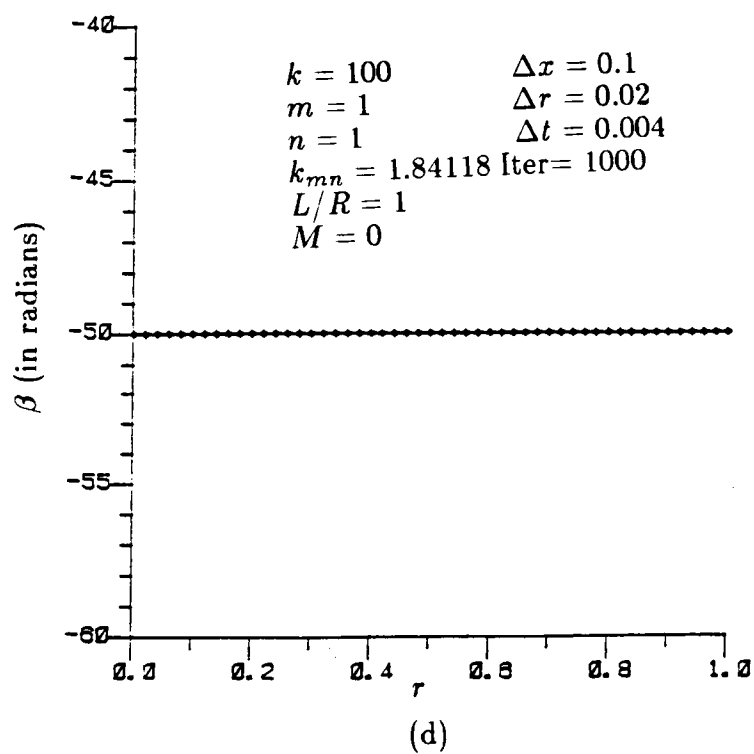
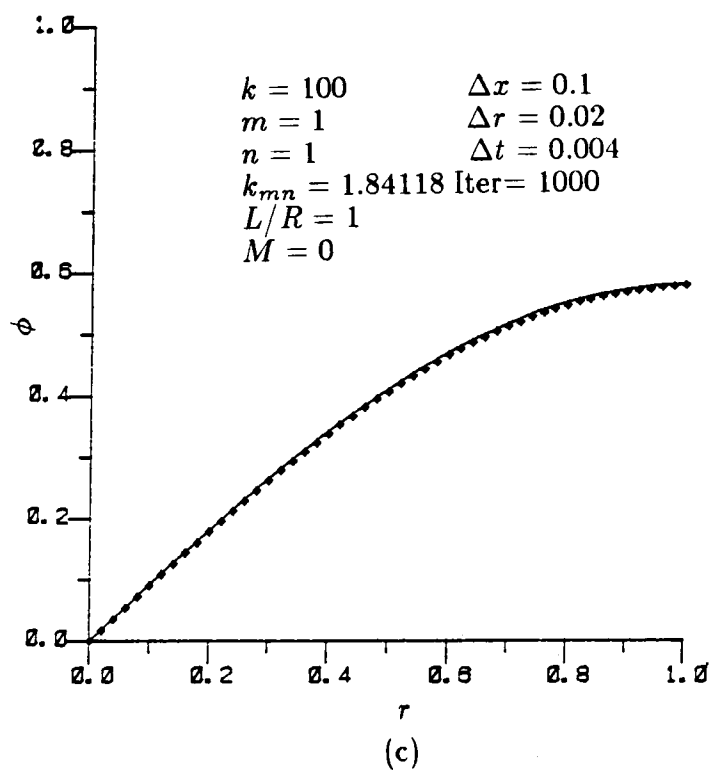


Figure 5.8 (continued)

CHAPTER 6

CONCLUSION

A major problem in any numerical scheme to analyze high frequency sound propagation is the required grid resolution. The oscillatory nature of the solution demands more gridpoints for good resolution depending on the frequency. Earlier techniques, which solved for the real and imaginary parts of the complex potential, either used more and more grid points depending on the frequency or assumed the behavior of the solution in the propagating direction to subtract out that part of the oscillation in the propagating direction. The present technique which does not need to assume any behavior of the solution in the propagating direction is shown to resolve very high frequency problem with comparatively very few grids in that direction. The case of reduced frequency $k = 100$ was solved with just 10 gridpoints in the axial direction.(the direction of propagation in the case considered.) Reduced frequency $k = 100$ being equal to 15.91 cycles of oscillation, it indicates that just 1 gridpoint per cycle is more than sufficient to resolve the above problem.

Results for the sound radiation in 2-D straight ducts and cylindrical ducts are in excellent agreement with the analytical solution. Various cases with flow and without flow for different reduced frequencies have been considered. The basic problem of high frequency sound propagation being solved,the following problems will be considered for further application :

- 1) Sound propagation in open and flanged ducts;
- 2) Sound propagation in soft walled ducts;
- 3) Sound radiation from a compressor inlet.

LIST OF REFERENCES

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- 1) Ni, R. H., and Sisto, F., "Numerical Calculation of Nonstationary Aerodynamics of Flat Plate Cascades in Compressible Flow", Journal of Engineering for Power, April 1976.
- 2) Baumeister, K. J. and Rice, E. J., "A Difference Theory for Noise Propagation in an Acoustically Lined Duct with Mean Flow," Progress in Astronautics and Aeronautics - Aeroacoustics: Jet and Combustion Noise; Duct Acoustics, Vol. 37, AIAA, New York, 1975, pp. 435-453.
- 3) Baumeister, K. J. and Bittner, E. C., "Numerical Simulation of Noise Propagation in Jet Engine Ducts," NASA TN D-7339, 1973.
- 4) Baumeister, K. J., "Analysis of Sound Propagation in Ducts Using the Wave Envelope Concept," NASA TN D-7719, 1974.
- 5) Baumeister, K. J., "Time-Dependent Wave Envelope Finite Difference Analysis of Sound Propagation," AIAA Journal Vol, 24, No. 1, January 1986.
- 6) Quinn, D. W., "Finite Difference Method for Computing Sound Propagation in Nonuniform Ducts," AIAA Paper 75-130, Jan. 1975.
- 7) Sigman, R. K., Majjigi, R. K., and Zinn, B. T., "Determination of Turbofan Inlet Acoustics Using Finite Elements," AIAA Journal, Vol. 16, No. 11, Nov. 1978.
- 8) Horowitz, S. J., Sigman, R. K., and Zinn, B.T., "An Iterative Finite Element-Integral Technique for Predicting Sound Radiation from Turbofan Inlets in Steady Flight", AIAA-82-0124, Jan. 1982.
- 9) Parrett, A. V., and Eversman, W., "Applications of Finite Element and Wave Envelope Element Approximations to Turbofan Engine Noise Radiation Including Flight Effects", AIAA-84-2333, Oct. 1984.
- 10) Baumeister, K. J., "Numerical Techniques in Linear Duct Acoustics - A Status Report", Transactions of the ASME, Vol. 103, August, 1981.
- 11) Goldstein, M.E., Aeroacoustics, McGraw-Hill Book Co., New York, 1976.
- 12) Abramowitz, M., and Stegun, I.A., Handbook of Mathematical Functions, Dover , New York, 1969.

APPENDIX

THE NUMERICAL SCHEME

The equations developed in Chapter 3, for the 2-D straight duct are

$$\begin{aligned}
 (1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - (1 - M^2) \phi \left(\frac{\partial \beta}{\partial x} \right)^2 - \phi \left(\frac{\partial \beta}{\partial y} \right)^2 + k^2 \phi + 2k \phi \frac{\partial \beta}{\partial t} \\
 = \frac{\partial^2 \phi}{\partial t^2} - \phi \left(\frac{\partial \beta}{\partial t} \right)^2 + 2M \left(\frac{\partial^2 \phi}{\partial x \partial t} - \phi \frac{\partial \beta}{\partial x} \frac{\partial \beta}{\partial t} - \phi k \frac{\partial \beta}{\partial x} \right)
 \end{aligned} \quad (3.6)$$

$$\begin{aligned}
 (1 - M^2) \phi \frac{\partial^2 \beta}{\partial x^2} + \phi \frac{\partial^2 \beta}{\partial y^2} + 2(1 - M^2) \frac{\partial \phi}{\partial x} \frac{\partial \beta}{\partial x} + 2 \frac{\partial \phi}{\partial y} \frac{\partial \beta}{\partial y} - 2k \frac{\partial \phi}{\partial t} \\
 = \phi \frac{\partial^2 \beta}{\partial t^2} + 2 \frac{\partial \phi}{\partial t} \frac{\partial \beta}{\partial t} + 2M \left(\phi \frac{\partial^2 \beta}{\partial x \partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \beta}{\partial t} + k \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial t} \frac{\partial \beta}{\partial x} \right)
 \end{aligned} \quad (3.7)$$

The terms in Eqn(3.6) are computed as follows (as mentioned in Chapter 4)

$$\begin{aligned}
 \frac{\partial^2 \phi}{\partial x^2} &= \frac{\phi_{J+1,K}^N - 2\phi_{J,K}^N + \phi_{J-1,K}^N}{\Delta x^2} \\
 \frac{\partial^2 \phi}{\partial y^2} &= \frac{\phi_{J,K+1}^N - 2\phi_{J,K}^N + \phi_{J,K-1}^N}{\Delta y^2} \\
 \frac{\partial^2 \phi}{\partial t^2} &= \frac{\phi_{J,K}^{N+1} - 2\phi_{J,K}^N + \phi_{J,K}^{N-1}}{\Delta t^2} \\
 \frac{\partial \beta}{\partial x} &= \frac{\beta_{J+1,K}^N - \beta_{J-1,K}^N}{2\Delta x} \\
 \frac{\partial \beta}{\partial y} &= \frac{\beta_{J,K+1}^N - \beta_{J,K-1}^N}{2\Delta y} \\
 \frac{\partial \beta}{\partial t} &= \frac{\beta_{J,K}^N - \beta_{J,K}^{N-1}}{\Delta t} \\
 \frac{\partial^2 \phi}{\partial x \partial t} &= \frac{\phi_{J+1,K}^N - \phi_{J-1,K}^N}{2\Delta x \Delta t} - \frac{\phi_{J+1,K}^{N-1} - \phi_{J-1,K}^{N-1}}{2\Delta x \Delta t}
 \end{aligned}$$

where the superscript N indicates time level and the subscripts J and K indicate the gridpoints along the spatial directions x and y respectively.

The above differences are used in Eqn(3.6) to give

$$\begin{aligned}
 \phi_{J,K}^{N+1} = & \{(1-M^2)a^2 - aM\}\phi_{J+1,K}^N + aM\phi_{J+1,K}^{N-1} + \{-2(1-M^2)a^2 \\
 & - 2b^2 - (\beta_{J,K+1}^N - \beta_{J,K-1}^N)^2\left(\frac{b^2}{4}\right) - (1-M^2)(\beta_{J+1,K}^N - \beta_{J-1,K}^N)^2\left(\frac{a^2}{4}\right) \\
 & + k^2\Delta t^2 + 2k\Delta t(\beta_{J,K}^N - \beta_{J,K}^{N-1}) + 2 + (\beta_{J,K}^N - \beta_{J,K}^{N-1})^2 \\
 & + Ma(\beta_{J+1,K}^N - \beta_{J-1,K}^N)(\beta_{J,K}^N - \beta_{J,K}^{N-1}) + kMe(\beta_{J+1,K}^N - \beta_{J-1,K}^N)\}\phi_{J,K}^N \\
 & + \{(1-M^2)a^2 + Ma\}\phi_{J-1,K}^N - Ma\phi_{J-1,K}^{N-1} + b^2\phi_{J,K+1}^N \\
 & + b^2\phi_{J,K-1}^N - \phi_{J,K}^{N-1}
 \end{aligned} \tag{A.1}$$

where

$$a = \Delta t / \Delta x$$

$$b = \Delta t / \Delta y$$

$$e = \Delta t^2 / \Delta x$$

The terms of Eqn(3.7) are computed as follows

$$\begin{aligned}
 \frac{\partial^2 \beta}{\partial x^2} &= \frac{\beta_{J+1,K}^N - 2\beta_{J,K}^N + \beta_{J-1,K}^N}{\Delta x^2} \\
 \frac{\partial^2 \beta}{\partial y^2} &= \frac{\beta_{J,K+1}^N - 2\beta_{J,K}^N + \beta_{J,K-1}^N}{\Delta y^2} \\
 \frac{\partial^2 \beta}{\partial t^2} &= \frac{\beta_{J,K}^{N+1} - 2\beta_{J,K}^N + \beta_{J,K}^{N-1}}{\Delta t^2} \\
 \frac{\partial \beta}{\partial x} &= \frac{\beta_{J+1,K}^N - \beta_{J-1,K}^N}{2\Delta x} \\
 \frac{\partial \beta}{\partial y} &= \frac{\beta_{J,K+1}^N - \beta_{J,K-1}^N}{2\Delta y} \\
 \frac{\partial \phi}{\partial x} &= \frac{\phi_{J+1,K}^N - \phi_{J-1,K}^N}{2\Delta x} \\
 \frac{\partial \phi}{\partial y} &= \frac{\phi_{J,K+1}^N - \phi_{J,K-1}^N}{2\Delta y} \\
 \frac{\partial \beta}{\partial t} &= \frac{\beta_{J,K}^N - \beta_{J,K}^{N-1}}{\Delta t}
 \end{aligned}$$

$$\frac{\partial \phi}{\partial t} = \frac{\phi_{J,K}^N - \phi_{J,K}^{N-1}}{\Delta t}$$

$$\frac{\partial^2 \beta}{\partial x \partial t} = \frac{\beta_{J+1,K}^N - \beta_{J-1,K}^N}{2\Delta x \Delta t} - \frac{\beta_{J+1,K}^{N-1} - \beta_{J-1,K}^{N-1}}{2\Delta x \Delta t}$$

The above differences are used in Eqn(3.7) to give

$$\begin{aligned} \phi_{J,K}^N \beta_{J,K}^{N+1} = & \{(1-M^2)a^2\phi_{J,K}^N - aM\phi_{J,K}^N - aM(\phi_{J,K}^N - \phi_{J,K}^{N-1}) + (1-M^2)(\frac{a^2}{2}) \\ & (\phi_{J+1,K}^N - \phi_{J-1,K}^N)\} \beta_{J+1,K}^N + aM\phi_{J,K}^N \beta_{J+1,K}^{N-1} + \{-2(1-M^2)a^2\phi_{J,K}^N \\ & - 2b^2\phi_{J,K}^N - aM(\phi_{J+1,K}^N - \phi_{J-1,K}^N) - 2(\phi_{J,K}^N - \phi_{J,K}^{N-1}) + 2\phi_{J,K}^N\} \beta_{J,K}^N \\ & + \{-\phi_{J,K}^N + 2(\phi_{J,K}^N - \phi_{J,K}^{N-1}) + Ma(\phi_{J-1,K}^N - \phi_{J-1,K}^{N-1})\} \beta_{J,K}^{N-1} \\ & + \{(1-M^2)a^2\phi_{J,K}^N - (1-M^2)(\frac{a^2}{2})(\phi_{J+1,K}^N - \phi_{J-1,K}^N) \\ & + aM(\phi_{J,K}^N - \phi_{J,K}^{N-1}) + aM\phi_{J,K}^N\} \beta_{J-1,K}^N - Ma\phi_{J,K}^N \beta_{J-1,K}^{N-1} \\ & + \{b^2\phi_{J,K}^N + (\frac{b^2}{2})(\phi_{J,K+1}^N - \phi_{J,K-1}^N)\} \beta_{J,K+1}^N \\ & + \{b^2\phi_{J,K}^N - \frac{b^2}{2}(\phi_{J,K+1}^N - \phi_{J,K-1}^N)\} \beta_{J,K-1}^N \\ & - 2k\Delta t(\phi_{J,K}^N - \phi_{J,K}^{N-1}) - Mka\Delta t(\phi_{J+1,K}^N - \phi_{J-1,K}^N) \end{aligned} \quad (A.2)$$

The Eqn(A.1) is used to evaluate ϕ and the Eqn(A.2) is used to evaluate β .

VITA

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