On the Spherical Symmetry of Perfect-Fluid Stellar Models in General Relativity

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To all the women I have loved.

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Don't try to be a good man, just be a man.

-Zefram Cochrane

Abstract

It is well known in Newtonian theory that static self-gravitating perfect fluids in a vacuum are necessarily spherically symmetric. The necessity of spherical symmetry of perfect-fluid static spacetimes with constant density in general relativity is shown.

Contents

Chapter 1

Introduction

The Schwarzschild spacetime models the most basic perfect-fluid static stellar model in general relativity. Several assumptions were made when searching for a solution to this model, including the assumption that the spacetime was spherically symmetric, i.e. $SO(3)$ acts on the underlying spacelike manifold by isometries. It seems almost natural that rotational symmetry would follow from the other assumptions provided that it does in Newtonian theory of gravity. This desired result, however, does not extend "naturally" to general relativity.

The work of Carleman (1919) and Lichtenstein (1918) demonstrated the necessity of spherical symmetry for an isolated static perfect-fluid body in Newtonian gravity. Using potential theory, they were able to show that a spherically symmetric static finite body of perfect-fluid matter uniquely minimizes the gravitational potential energy. An analogous result for general relativity was first conjectured by Lichnerowicz (1955) and included on Yau's 1982 list of unsolved problems in classical general relativity. Early work in the area began with two articles by Avez (1963 & 64), who, using Morse Theory, managed to prove that asymptotically flat static perfect-fluid spacetimes are diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}$, and rotationally symmetric. The result depended on the assumptions that gravitational field intensity W depended only on the gravitational potential V, and also that V has no degenerate critical points and totally umbilic level sets. Later work revealed these assumptions to be equivalent to spatial conformal flatness. A few years after Avez's publications, Israel announced his Black Hole Uniqueness Theorem, which proved the necessity of rotational symmetry for static vacuum black holes in general relativity, thereby proving the uniqueness of the vacuum black hole Schwarzschild solution. An alternate and simplified proof of Israel's theorem was given by Robinson who used divergence identities and the norm squared of the Cotton tensor (which vanishes if and only if the manifold is conformally flat). Not much progress was made on the static stellar problem for nearly a decade after Robinson's proof.

The work of Künzle and Savage (1977) showed that if a family of static perfect-fluid solutions to Einstein's equation contains one spherical solution then the entire family is isometric to the spherical solution. Special cases were thoroughly examined in the mid-eighties by Lindblom (1988) and Masood-ul-Alam (1987). Lindblom, in his 1981 paper, was able to extend Robinson's divergence identity (used in his proof of Israel's theorem) to static stellar models of constant density, which would be used later to prove rotational symmetry of the spacetime. Around the same time, Masood-ul-Alam utilized the divergence identities of Robinson and Lindblom along with the Positive Mass Theorem to prove certain stellar models are conformally flat and, therefore, rotationally symmetric. Unfortunately, as pointed out by Lindblom (1988), Masood-ul-Alam's proof relied on an "unrealistic" condition on the equation of state, namely $\frac{d\rho}{dp} \leq 0$. Realizing the significance of the Positive Mass Theorem, Lindblom (1988) was able to find a conformal transformation of the spatial metric and use the Positive Mass Theorem to prove that spacetime was conformally flat, thus spherically symmetric.

The most recent publication on the matter was put out in 2007 by Masood-ul-Alam, who instead of using the divergence identities in his proof, found a conformal metric with a nonnegative spinor-norm weighted scalar curvature integral. The assumptions of his theorem are:

1. The spacetime 4-manifold is $M^4 = N^3 \times \mathbb{R}$ with line element $ds^2 = -V^2dt^2 + g_{ij}dx^idx^j$. g is a complete Riemannian metric on N amd $V : N \to [V_{min}, 1)$

2. The 3-manifold metric and gravitational potential satisfy the asymptotic conditions:

$$g_{ij} = (1 + \frac{2m}{r})\delta_{ij} + O(r^{-2})$$
$$V = 1 - \frac{m}{r} + O(r^{-2})$$

3. The gravitational potential V and the metric g_{ij} are $C^{1,Lip}$.

4. The pressure $p = p(V)$ is a nonnegative bounded measurable function.

5. The density $\rho = \rho(p)$ is a piecewise C^1 positive non-decreasing function of p for $p > 0$.

6. On the spatial hypersurface N, the boundary of the fluid region and the sets along which ρ has discontinuity are smooth 2-surfaces. V and g_{ij} are C^3 everywhere except these surfaces.

7. There are only a finite number of these surfaces.

Theorem 1.1. *(Masood-ul-Alam 2006) A static stellar model satisfying the assumption above necessarily spherically symmetric.*

In this thesis we derive spherical symmetry in the case of constant density, a result due to Lindblom (1988). A statement of the assumptions and main theorem is now given, followed by an outline of this work.

The assumptions are:

1. The spacetime 4-manifold is $M = N^3 \times \mathbb{R}$ with line element $ds^2 = -V^2dt^2 + g_{ij}dx^idx^j$

2. The spatial metric g_{ij} on N and gravitational potential $V : N \to \mathbb{R}$ satisfy the asymptotic conditions:

$$g_{ij} = (1 + \frac{2m}{r})\delta_{ij} + O(r^{-2}) \tag{1.1}$$

$$V = 1 - \frac{m}{r} + O(r^{-2}) \tag{1.2}$$

3. The gravitational potential V and the spatial metric g_{ij} are $C^{1,Lip}$ in a neighborhood of the boundary of the fluid region, and C^2 everywhere else.

4. The star is assumed to have uniform density ρ, and pressure $p = p(V)$ related to ρ by an equation of state $\rho = \rho(p)$.

5. Let V_s be the gravitational potential on the surface of the star. The field intensity, $W = |\nabla V|^2$, satisfies the jump condition:

$$\lim_{V \to V_s^-} W^{-1}\nabla^i V \nabla_i W - \lim_{V \to V_s^+} W^{-1}\nabla^i V \nabla_i W = 8\pi V_s \rho(0) \tag{1.3}$$

Theorem 1.2. *(Lindblom 1988) Let (M^4, g) be perfect fluid static stellar model in general relativity with equation of state satisfying the conditions above. Then M is rotationally symmetric.*

Outline of Proof

I. First we decompose Einstein's Equation $G = 8\pi T$ to obtain the system:

$$\nabla^i \nabla_i V = 4\pi V(\rho + 3p)$$

$$R_{ij} = V^{-1}\nabla_i \nabla_j V + 4\pi(\rho - p)g_{ij}$$

Here V is the gravitational scalar potential, ρ is fluid's density, p is the pressure, and ∇_i and R_{ij} represent the covariant derivative and Ricci tensor with respect to the spatial metric tensor g_{ij}.

Note: Using the second contracted Bianchi identity, the system above yields: $\nabla_i p = -V^{-1}(\rho + p)\nabla_i V$ which can be integrated to give: $p = \rho V^{-1}(V_s - V)$. This is explained in chapter 2 (Corollary 2.4).

II. Use the conformal transformation $\tilde{g}_{ij} = \psi^4 g_{ij}$ where

$$\psi(V) = \begin{cases} \frac{1}{2}(1+V) & V_s \le V \le 1 \\ \frac{1}{2}(1+V_s)^{3/2}(1+3V_s-2V)^{-1/2} & 0 < V \le V_s \end{cases}$$

The conformally transformed scalar curvature (of $\tilde{g}_{ij}$) is given by $\tilde{R} = \psi^{-4}(R - 8\psi^{-1}\nabla^i\nabla_i\psi)$. This is shown to equal $\tilde{R} = 8\psi^{-5}\psi''\{W_0 - W\}$ where $W = \nabla^i V \nabla_i V = |\nabla V|^2$, and

$$W_0(V) := \begin{cases} \frac{2}{3}\pi\rho(1-V^2)^4(1-V_s)^{-3}, & V_s \le V \le 1 \\ \frac{8}{3}\pi\rho V(3V_s - V) + \frac{2}{3}\pi\rho(1-9V_s{}^2), & 0 < V \le V_s \end{cases}$$

is the field intensity in the rotationally symmetric case.

III. To show that $\tilde{R}$ is nonnegative, since ψ and ψ'' are nonnegative, it is sufficient to show that $W_0 - W$ is also non-negative in the interior. This can be proved by applying the maximum principle to the two identities:

$$\nabla^i\{V^{-1}\nabla_i(W-W_0)\} = \frac{1}{4}V^3W^{-1}R_{ijk}R^{ijk} + \frac{3}{4}V^{-1}W^{-1}\nabla_i(W-W_0)\nabla^i(W-W_0)$$

And

$$\nabla_i\left\{V^{-1}\nabla^i\left[\frac{W-W_0}{(1-V^2)^3}\right]\right\} = \frac{V^{-4}|R_{ijk}|^2 + 3\,|W_{;i} + 8VW(1-V^2)^{-1}V_{;i}|^2}{4VW(1-V^2)^3}$$

Here R_{ijk} is a $(0,3)$ tensor ("Cotton Tensor") which detects conformal flatness in dimension 3 (see chapter 4). The first applies to the interior region (fluid region), and the second applies to the exterior region.

IV. Finally, recall the Riemannian *Positive Mass Theorem* (PMT) of Schoen-Yau and Witten: A complete, asymptotically flat Riemannian metric, with nonnegative scalar curvature has nonnegative ADM mass. If the mass is zero, the manifold is flat. This is used to show that $\tilde{g}$ is flat. Hence, g is conformally flat, and Avez's result applies.

This paper assumes no prior knowledge of general relativity other than Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$. Most of the material necessary for this paper can be found in a first semester Riemannian or differential geometry course. For the most part, the paper follows the outline of the proof, beginning with four chapters (chapters 2-5) of highly detailed tensor computations and other derivations of identities necessary for the proof. Following the computations is a chapter on the application of the maximum principle to identities derived in chapter 5 to show conformal flatness, and a chapter on how conformal flatness implies rotational symmetry. Finally, regularity issues arising in the proof of the main theorem are addressed, followed by a formal statement of the positive mass theorem, and a concluding chapter with a complete proof of the main theorem.

This dissertation is expository. The goal was to derive in detail all the expressions and arguments used in Lindblom's 1988 paper. These derivations are scattered throughout literature, and are presented here in a unified and systematic way. We hope this exposition will prove useful to researchers approaching this topic for the first time, as an introduction to Masood-ul-Alam's 2007 paper and to problems of current interest in this area.

Chapter 2

Decomposition of Einstein's Equation

Given the static metric $ds^2 = -V^{-2}dt^2 + g_{ij}dx^i dx^j$, in this chapter we show (Theorem 2.5) how Einstein's equation $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$ for perfect-fluid stellar models decomposes into the system:

$$\begin{cases} \Delta_g V = 4\pi V(\rho + 3p) \\ {}^{(3)}R_{ij} = V^{-1}\nabla_i\nabla_j V + 4\pi(\rho - p)g_{ij} \end{cases} \tag{2.1}$$

where ∇_i, ${}^{(3)}R_{ij}$, and Δ_g represent the three-dimensional spatial covariant derivative, Ricci tensor, and Laplace-Beltrami operator of the metric g_{ij}. Here, the Latin indices run from 1 to 3, while the Greek indices run from 0 to 3.

We begin by contracting the Einstein equation (${}^{(4)}R = -8\pi T^{\mu}_{\mu}$), and substituting $-8\pi T^{\mu}_{\mu}$ in for the scalar curvature to obtain the alternate form of Einstein's equation:

$${}^{(4)}R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}T^{\rho}_{\rho}g_{\mu\nu}) \tag{2.2}$$

The energy-momentum tensor for a perfect fluid is given by:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\ g_{\mu\nu} \tag{2.3}$$

where u is a unit timelike vector field representing the 4-velocity of the fluid, ρ is the density and p is the pressure of the fluid. Let (e_0, e_1, e_2, e_3) be an orthonormal frame with $e_0 = u$. Then,

$$T(e_0, e_0) = (\rho + p) - p = \rho \tag{2.4}$$

$$T(e_i, e_i) = p \tag{2.5}$$

$$T(e_i, e_0) = 0 \tag{2.6}$$

By contraction we have:

$$T^\nu_\mu = (\rho + p)u_\mu u^\nu + p\ \delta^\nu_\mu \tag{2.7}$$

$$\Rightarrow T^\mu_\mu = -(\rho + p) + 4p \tag{2.8}$$

$$= -\rho + 3p \tag{2.9}$$

substituting equations (2.7) and (2.9) into (2.2) we obtain:

$${}^{(4)}R_{\mu\nu} = 8\pi[(\rho + p)u_\mu u_\nu + pg_{\mu\nu} - \frac{1}{2}(-\rho + 3p)g_{\mu\nu}] \tag{2.10}$$

$$= 8\pi[(\rho + p)u_\mu u_\nu + \frac{1}{2}(\rho - p)g_{\mu\nu}] \tag{2.11}$$

Hence, in a coordinate system whose tangent vector to the timelines $\partial_t = e_0$ we have:

$${}^{(4)}R_{00} = Ric(e_0, e_0) = 8\pi T(e_0, e_0) - \frac{1}{2}T^\mu_\mu g(e_0, e_0)$$

$$= 8\pi[\rho + \frac{1}{2}(-\rho + 3p)) = 4\pi(\rho + 3p) \tag{2.12}$$

$${}^{(4)}R_{ij} = 4\pi(\rho - p)g_{ij} \tag{2.13}$$

Lemma 2.1. *Let* $(\partial_t, \partial_1, \partial_2, \partial_3)$ *be an orthonormal frame for the tangent bundle, and* ∇ *be the Levi-Civita connection. Then,*

1. $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$, *and* $\nabla_{\partial_t}\partial_j = V^{-1}V_{;j}\partial_t$

2. $\langle \nabla_{\partial_t}\nabla_{\partial_i}\partial_j, \partial_t\rangle = -\Gamma^k_{ij}VV_{;k}$

3. $\langle \nabla_{\partial_i}\nabla_{\partial_t}\partial_j, \partial_t\rangle = VV_{;ij}$

For all $i, j \neq t$. *The semicolon denotes the covariant derivative.*

Proof.

1. By definition we have

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{0l}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}) \\
&= \frac{1}{2}g^{00}(\partial_i g_{j0} + \partial_j g_{0i} - \partial_t g_{ij}) \\
&= 0
\end{aligned}$$

The last line follows from the fact that $\partial_t V = \partial_t g_{ij} = 0$. Thus $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k + \Gamma^0_{ij}\partial_t = \Gamma^k_{ij}\partial_k$. We now compute the Christoffel symbols Γ^i_{0j} and Γ^0_{0j}:

$$\begin{aligned}
\Gamma^0_{0j} &= \frac{1}{2}g^{0l}(\partial_j g_{0l} + \partial_t g_{lj} - \partial_l g_{0j}) \\
&= \frac{1}{2}g^{00}(\partial_j g_{00} + \partial_t g_{0j} - \partial_l g_{0j}) \\
&= \frac{1}{2}g^{00}(\partial_j g_{00} + 0 - 0) \\
&= \frac{1}{2}V^{-2}\partial_j(V^2) \\
&= V^{-1}V_{;j}
\end{aligned}$$

$$\Gamma^i_{0j} = \frac{1}{2} g^{il}(\partial_t g_{il} + \partial_j g_{l0} - \partial_l g_{0j}) = 0$$
$$\Rightarrow \nabla_{\partial_t} \partial_j = \Gamma^i_{0j} \partial_i + \Gamma^0_{0j} \partial_t = V^{-1} V_{;j} \partial_t$$

2. Using the result from part i. we have

$$\begin{aligned}
\langle \nabla_{\partial_t} \nabla_{\partial_i} \partial_j, \partial_t \rangle &= \langle \nabla_{\partial_t} \Gamma^k_{ij} \partial_k, \partial_t \rangle \\
&= \Gamma^k_{ij} \langle \nabla_{\partial_t}, \partial_t \rangle = \Gamma^k_{ij} \langle \nabla_{\partial_k} \partial_t, \partial_t \rangle \\
&= \frac{1}{2} \Gamma^k_{ij} \partial_k \langle \partial_t, \partial_t \rangle = \frac{1}{2} \Gamma^k_{ij} \partial_k (-V^2) \\
&= -\Gamma^k_{ij} V V_{;k}
\end{aligned}$$

3.

$$\begin{aligned}
\langle \nabla_{\partial_i} \nabla_{\partial_t} \partial_j, \partial_t \rangle &= \langle \nabla_{\partial_i} V^{-1} V_{;j} \ \partial_t, \partial_t \rangle \\
&= \langle V^{-1} V_{;j} \nabla_{\partial_i} \partial_t + \partial_i (V^{-1} V_{;j}) \partial_t, \partial_t \rangle \\
&= \langle V^{-1} V_{;j} \nabla_{\partial_i} \partial_t, \partial_t \rangle + \langle \partial_i (V^{-1} V_{;j}) \partial_t, \partial_t \rangle \\
&= V^{-1} V_{;j} \langle \nabla_{\partial_i} \partial_t, \partial_t \rangle - V^2 (\partial_i (V^{-1} V_{;j}) \\
&= V^{-1} V_{;j} (V V_{;i}) + (-1) V^2 V^{-2} V_{;i} V_{;b} + V V_{;ij} \\
&= V_{;i} V_{;j} - V_{;i} V_{;j} + V V_{;ij} \\
&= V V_{;ij}
\end{aligned}$$

$\square$

Proposition 2.2. $\langle R(\partial_t, \partial_i)\partial_j, \partial_t\rangle = V\nabla_i\nabla_j V$.

Proof.

$$\begin{aligned}
\langle R(\partial_t, \partial_i)\partial_j, \partial_t\rangle &= \langle \nabla_{\partial_t}\nabla_{\partial_i}\partial_j - \nabla_{\partial_i}\nabla_{\partial_t}\partial_j - \nabla_{[\partial_t,\partial_i]}\partial_j, \partial_t\rangle \\
&= \langle \nabla_{\partial_t}\nabla_{\partial_i}\partial_j - \nabla_{\partial_i}\nabla_{\partial_t}\partial_j, \partial_t\rangle \\
&= \langle \nabla_{\partial_t}\nabla_{\partial_i}\partial_j, \partial_t\rangle - \langle \nabla_{\partial_i}\nabla_{\partial_t}\partial_j, \partial_t\rangle \\
&= -\Gamma^k_{ij} V V_{;k} + V V_{;ij} \\
&= V(V_{;ij} - \Gamma^k_{ij} V_{;k}) \\
&= V(\nabla_i\nabla_j\ V)
\end{aligned}$$

□

Theorem 2.3. *For perfect-fluid static stellar models, Einstein's equation* $G_{\mu\nu} = 8\pi T_{\mu\nu}$ *decomposes into the the following system:*

1. ${}^{(3)}\Delta V = \nabla^i\nabla_i V = 4\pi V(\rho + 3p)$
2. ${}^{(3)}R_{ij} = V^{-1}\nabla_i\nabla_j V + 4\pi(\rho - p)g_{ij}$

Proof.

1. Using Proposition 2.4 , we compute the R_{00} component of the Ricci tensor, and set it equal to equation (2.12).

$$\begin{aligned}
R_{00} &= g^{ij}\langle R(\partial_t, \partial_i)\partial_j, \partial_t\rangle \\
&= g^{ab} V\nabla_i\nabla_j V \\
&= V^{-1}\nabla^i\nabla_i V \\
&= 4\pi(\rho + 3p) \\
\Rightarrow \nabla^i\nabla_i V &= 4\pi V(\rho + 3p)
\end{aligned}$$

2. We will use equation (2.12) and the fact that the four dimensional Ricci tensor can be decomposed as

$$^{(4)}R_{ij} = g^{00}\langle R(\partial_t, \partial_i)\partial_j, \partial_t\rangle + g^{kl}\langle R(\partial_k, \partial_i)\partial_j, \partial_l\rangle$$

where $i, j, k, l \neq t$.

$$\begin{aligned} ^{(4)}R_{ij} &= V^{-2}V\nabla_i\nabla_j V +^{(3)} R_{ij} \\ &= -V^{-1}\nabla_i\nabla_j V +^{(3)} R_{ij} \end{aligned}$$

From equation (2.13) we have:

$$\begin{aligned} ^{(4)}R_{ij} &= 4\pi(\rho - p)g_{ij} \\ &= -V^{-1}\nabla_i\nabla_j V +^{(3)} R_{ij} \\ \Rightarrow\ ^{(3)}R_{ij} &= V^{-1}\nabla_i\nabla_j V + 4\pi(\rho - p)g_{ij} \end{aligned}$$

□

Corollary 2.4.

1. The scalar curvature $R = 16\pi\rho$*.*

2. Euler's equation for static fluids: $\nabla_i p = V^{-1}(\rho + p)\nabla_i V$

Proof.

1. Contract R_{ij} and substitute $\Delta V = 4\pi V(\rho + 3p)$ into the contracted expression to obtain $R = 16\pi\rho$.

2. We begin by computing $\nabla^i\nabla_i\nabla_j V$.

$$\begin{aligned} \nabla^i\nabla_i\nabla_j V = \nabla^i\nabla_j\nabla_i V &= g^{ik}\nabla_k\nabla_j\nabla_i V \\ &= g^{ik}(\nabla_j\nabla_k\nabla_i V + R^l_{kji}V_l = \partial_j(\Delta V) + R^l_j\nabla_l V \\ &= 4\pi V\partial_j(\rho + 3p) + 4\pi(\rho + 3p)\nabla_k V + R^l_j\nabla_l V \end{aligned} \tag{2.14}$$

Now consider the twice contracted differential Bianchi identity.

$$
\begin{aligned}
0 &= \nabla^i R_{ij} - \frac{1}{2}\nabla_j R \\
&= \nabla^i [V^{-1}\nabla_i \nabla_j V + 4\pi(\rho - p)g_{ij}] - \frac{1}{2}\partial_j [16\pi\rho] \\
&= -V^{-2}\nabla^i V \nabla_i \nabla_j V + V^{-1}\nabla^i \nabla_i \nabla_j V + 4\pi\partial_j(\rho - p) - 8\pi\partial_j \rho
\end{aligned}
$$

Here, we can substitute equation (2.14) into the above equation and use the fact that $\nabla_i \nabla_j V = V R_{ij} - 4\pi V(\rho - p)g_{ij}$ to get:

$$
\begin{aligned}
0 =& - V^{-1}\nabla^i R_{ij} + 4\pi V^{-1}(\rho - p)\nabla_j V + 4\pi\partial_j(\rho + 3p) \\
&+ 4\pi V^{-1}(\rho + 3p)\nabla_j V + V^{-1}R^l_j \nabla_l V - 4\pi\partial(\rho + p) \\
=& \; 8\pi\partial_j p + 8\pi V^{-1}(\rho + p)\nabla_j V \\
\Rightarrow& \; \nabla_j p = -V^{-1}(\rho + p)\nabla_j V
\end{aligned} \tag{2.15}
$$

□

With $p = p(V)$, (2.15) becomes a separable ordinary differential equation.

$$
\begin{aligned}
\int \frac{dp}{\rho + p} &= \int \frac{-dV}{V} \\
\log|\rho + p| &= -\log(V) + C \\
\Rightarrow \rho + p &= \frac{c}{V}
\end{aligned} \tag{2.16}
$$

The constant of integration can be found using the boundary condition $p(V_s) = 0$. Thus the equation for pressure in the interior region is completely determined by the gravitational potential V and the density ρ (assumed to be constant) as:

$$
p = \rho V^{-1}(V_s - V) \tag{2.17}
$$

Chapter 3

The Conformal Transformation

3.1 Derivation of the Conformal Factor

We derive the equation for the conformal factor ψ necessary for the proof. This is done by working backwards, that is, we assume the model g is spherical and construct a factor ψ such that $g = \psi^{-4}\tilde{g}$ where $\tilde{g}$, is a flat metric. Assuming the model is static and spherically symmetric the metric can be represented in two different forms:

$$\begin{aligned} ds^2 &= -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2 d\Omega^2 && (3.1) \\ &= -V^2 dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2 && (3.2) \end{aligned}$$

Here ν and λ are both functions of the radial coordinate r, $V = V(r)$ is the gravitational potential, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Given the orthonormal frame $e_0 = e^{-\nu/2}\partial_t,\ e_1 = e^{-\lambda/2}\partial_r,\ e_2 = r^{-1}\partial_\theta, e_3 = \frac{1}{r\sin\theta}\partial_\phi$, we have the following components for the $(0,2)$ Einstein tensor (Dalarsson & Dalarsson 2005):

$$\begin{aligned} G(e_0, e_0) &= \frac{1}{r^2} + e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) && (3.3) \\ G(e_1, e_1) &= -\frac{1}{r^2} + e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) && (3.4) \end{aligned}$$

Recall the energy-momentum tensor for a perfect fluid is:

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta}$$

thus, with $e_0 = (u^\alpha)$ and $\langle e_0, e_0 \rangle = u_\alpha u^\alpha = -1$ we have

$$T(e_0, e_0) = \rho = \rho(r) \tag{3.5}$$

$$T(e_1, e_1) = p = p(r) \tag{3.6}$$

By substituting equations (3.3), (3.4), (3.5), and (3.6) into Einstein's equation $G_{\nu\mu} = 8\pi T_{\nu\mu}$ we obtain the system of equations:

$$\frac{1}{r^2} + e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) = 8\pi\rho \tag{3.7}$$

$$-\frac{1}{r^2} + e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) = 8\pi p \tag{3.8}$$

Equation (3.7) is equivalent to

$$e^{-\lambda}(r\lambda' - 1) + 1 = 8\pi\rho r^2 \tag{3.9}$$

$$\Rightarrow \quad \frac{d}{dr}(r - re^{-\lambda}) = 8\pi r^2 \rho \tag{3.10}$$

Let $e^{-\lambda} = 1 - \frac{2m(r)}{r}$, and note that the $m(r)$ is a function of r. Solving for $2m(r)$ and substituting that into (3.10) yields:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \tag{3.11}$$

Equation (3.8) is equivalent to :

$$\frac{\nu'}{r} + \frac{1}{r^2} = e^{\lambda}\left(\frac{1}{r^2} + 8\pi p\right) \tag{3.12}$$

$$= \left(1 - \frac{2m(r)}{r}\right)^{-1}\left(\frac{1}{r^2} + 8\pi p\right) \tag{3.13}$$

From (3.1) and (3.2) we have

$$V = e^{\nu/2} \tag{3.14}$$

$$\Rightarrow \frac{2V'}{V} = \nu' \tag{3.15}$$

Solving for ν' in (3.12) and setting equal to (3.15) we have

$$\frac{2V'}{V} = \nu' = \frac{2m + 8\pi r^3 p}{r(r-2m)} \tag{3.16}$$

If $\nu' > 0$ (i.e. $r > 2m$) then $V = V(r)$ is invertible to $r = V(r)$ and the metric can be written in terms of V.

Remark 3.1. *Combining (3.16) with the fluid equation of motion* $p' = -\frac{1}{2}(p+\rho)\nu'$ *we get,*

$$p'(r) = p' = -(\rho + p)\frac{m + 4\pi r^3 p}{r(r-2m)} \tag{3.17}$$

In terms of V equation (3.16) can be written in the form

$$\frac{dr}{dV} = \frac{r(r-2m)}{V(m + 4\pi r^3 p)} \tag{3.18}$$

and using the chain rule and equation (3.18), (3.11) takes the form

$$\frac{dm}{dV} = 4\pi r^3 \rho(r)\frac{dr}{dV} = \frac{4\pi r^3 \rho(r)(r-2m)}{V(m + 4\pi r^3 p)} \tag{3.19}$$

Together (3.18) and (3.19) form a dynamical system for $r(V)$ and $m(V)$. The pressure, $p = p(V)$, can be solved for fairly easily using (3.16) and (3.17).

$$\frac{dp}{dr} = -\frac{(\rho + p)}{V}\frac{dV}{dr} \tag{3.20}$$

$$\Rightarrow \frac{d}{dr}[\log(V)] = \frac{-1}{\rho + p}\frac{dp}{dr} \tag{3.21}$$

$$\Rightarrow \log(V) = -\int_0^p \frac{d\bar{p}}{\rho[\bar{p}] + \bar{p}} := h(p) \tag{3.22}$$

Where $h(p)$ is invertible, so ρ and p can be expressed as:

$$p(V) = h^{-1}[\log V] \tag{3.23}$$

$$\rho(V) = \rho[p(V)] \tag{3.24}$$

The second equation comes from the assumed equation of state $\rho = \rho(p)$. Thus the system (3.18) and (3.19) is well defined for $r(V)$ and $m(V)$.

The spatial metric g_{ij} in terms of V has the line element

$$ds^2 = \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\Omega^2 = W^{-1} dV^2 + r^2 d\Omega^2 \tag{3.25}$$

Where $W = |\nabla_g V|_g^2$. Using the reciprocal of (3.18), W can also be written as

$$W^{-1} dV^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \tag{3.26}$$

$$\Rightarrow \; W = \left(1 - \frac{2m}{r}\right)\left(\frac{dV}{dr}\right)^2 \tag{3.27}$$

For constant density ρ, $r(V)$ can be solved for explicitly. The integral of (3.23) yields:

$$
\begin{aligned}
-\log(V) + \log(V_s) &= \log(\rho + p)\Big|_0^p && (3.28)\\
\log\left(\frac{V_s}{V}\right) &= \log(\rho + p) - \log(p) && (3.29)\\
\Rightarrow p &= \rho\left(\frac{V_s}{V} - 1\right) && (3.30)
\end{aligned}
$$

Naturally from equation (3.11) we have (for constant density ρ):

$$
m(r) = \frac{4}{3}\pi\rho r^3 \tag{3.31}
$$

The following computation will be useful. From (3.30)

$$
m + 4\pi r^3 p = \frac{4\pi\rho r^3}{3}\left(\frac{3V_s}{V} - 2\right) \tag{3.32}
$$

Substituting (3.32) into (3.18) and separating the variables we have,

$$
\begin{aligned}
\frac{dV}{V\ \frac{4\pi\rho r^3}{3}\left(3\frac{V_s}{V} - 2\right)} &= \frac{dr}{r(r-2m)} && (3.33)\\
\Rightarrow \frac{dV}{3V_s - 2V} &= \frac{4\pi\rho r^3}{3r(r-2m)} && (3.34)\\
&= \frac{4\pi\rho r\, dr}{3(1 - \frac{8\pi}{3}\rho r^2)} && (3.35)
\end{aligned}
$$

which integrates to:

$$
(3V_s - 2V)^2 = A(1 - \frac{8}{3}\pi\rho r^2) \tag{3.36}
$$

for some constant $A > 0$. Given the initial condition $V = V_s$ at $R = R_s$ we have two equations for V_s:

$$V_s^2 = 1 - \frac{2M_s}{R_s} \tag{3.37}$$

$$(V_s)^2 = A(1 - \frac{8}{3}\pi\rho R_s^2) \tag{3.38}$$

Thus,

$$\frac{(3V_s - 2V)^2}{V_s^2} = \frac{1 - \frac{8\pi\rho r^2}{3}}{1 - \frac{8\pi\rho R_s^2}{3}} \tag{3.39}$$

$$= \frac{1 - \frac{2m}{r}}{1 - \frac{2M_s}{R_s}} \tag{3.40}$$

$$\Rightarrow (3V_s - 2V)^2 = 1 - \frac{2m}{r} \tag{3.41}$$

$$= 1 - \frac{8}{3}\pi\rho r^2 \tag{3.42}$$

So $r = r(V)$ is implicitly defined by:

$$\Rightarrow 3V_s - 2V = \sqrt{1 - \frac{2m}{r}} \tag{3.43}$$

The goal is to find a conformal factor ψ such that

$$\psi^4(g_{rr}dr^2 + r^2 d\Omega^2) = ds^2 + s^2 d\Omega^2 \tag{3.44}$$

Setting the terms equal gives the system:

$$\sqrt{g_{rr}}dr = \psi^2 ds \tag{3.45}$$

$$s = r\psi^2 \tag{3.46}$$

Differentiating the second equation yields:

$$ds = \psi^2 dr + 2\psi r d\psi \tag{3.47}$$

And from (3.46),

$$\sqrt{g_{rr}}dr = dr + \frac{2r}{\psi}d\psi \tag{3.48}$$

$$\Rightarrow \psi(\sqrt{g_{rr}} - 1)dr = 2rd\psi \tag{3.49}$$

$$\Rightarrow \psi' = \frac{1}{2r}\psi(\sqrt{g_{rr}} - 1)r' \tag{3.50}$$

$$= \frac{\psi}{2r\sqrt{W}}\left(1 - \sqrt{1 - \frac{2m}{r}}\right) \tag{3.51}$$

Lemma 3.2. *If ρ is constant then,* $\frac{2r\sqrt{W}}{1-\sqrt{1-\frac{2m}{r}}} = 3V_s - 2V + 1$.

Proof.

$$\begin{aligned}
\frac{2}{1 - \sqrt{1 - \frac{2m}{r}}} &= \frac{r}{m}\left(1 + \sqrt{1 - \frac{2m}{r}}\right) \\
\Leftrightarrow \frac{2r\sqrt{W}}{1 - \sqrt{1 - \frac{2m}{r}}} &= \frac{r^2\sqrt{W}}{m}\left(1 + \sqrt{1 - \frac{2m}{r}}\right) \\
&= \frac{r^2\sqrt{W}}{m}(1 + 3V_s - 2V)
\end{aligned}$$

The last line comes from (3.43). On the other hand from (3.18), (3.27), (3.32) and (3.43) we have,

$$\begin{aligned}
\sqrt{W} &= \sqrt{1-\frac{2m}{r}} \cdot \frac{dV}{dr} \\
&= \sqrt{1-\frac{2m}{r}} \left(\frac{V}{r(r-2m)}\right) \frac{4}{3}\pi\rho r^3 (\frac{3V_s}{V} - 2) \\
&= \frac{r}{\sqrt{1-\frac{2m}{r}}} \frac{4\pi\rho}{3}(3V_s - 2V) \\
\Leftrightarrow \frac{r^2}{m}\sqrt{W} &= \frac{4\pi\rho r^3}{3m} \cdot \frac{1}{\sqrt{1-\frac{2m}{r}}}(3V_s - 2V) = 1 \cdot 1 = 1
\end{aligned}$$

□

It follows from the Lemma 3.08, and equation (3.51) that

$$\frac{\psi'}{\psi} = \frac{1}{3V_s - 2V + 1} \tag{3.52}$$

$$\Rightarrow \psi(V) = C(3V_s - 2V + 1)^{-1/2} \tag{3.53}$$

$$\tag{3.54}$$

for some constant $C > 0$. With the initial condition,

$$\psi(V_s) = \frac{1}{2}(1 + V_s) \tag{3.55}$$

the constant C can be solved for, and the conformal factor ψ is found to be :

$$\psi(V) = \frac{1}{2}(1 + V_s)^{3/2}(1 + 3V_s - 2V)^{-1/2} \tag{3.56}$$

The initial condition is obtained from the fact that the exterior metric (in the rotationally symmetric case) is the spatial Schwarzschild metric.

3.2 Derivation of the Rotationally symmetric Field Intensity

In this section we derive the expression for the gravitational field intensity W_0 (in the rotationally symmetric case) as a function of V.

The Exterior

The Schwarzschild spacetime metric with mass parameter $m > 0$, has the classical expression (on $\mathbb{R}^3 \times \mathbb{R}$; polar coordinates (s, ω) in $\mathbb{R}^3$):

$$ds^2_{schw} = -\left(1 - \frac{2m}{s}\right) dt^2 + \frac{1}{1 - \frac{2m}{s}} ds^2 + s^2 d\omega^2, \quad s > 2m \tag{3.57}$$

Thus the potential $V(s)$ is given by:

$$V(s) = \sqrt{1 - \frac{2m}{s}} \quad , \quad 1 - V^2 = \frac{2m}{s} \qquad (s > 2m) \tag{3.58}$$

The spatial Schwarzschild metric is conformal to the Euclidean metric:

$$\frac{1}{1 - \frac{2m}{r}} ds^2 + s^2 d\omega^2 = ds^2_{schw} = \psi^{-4} ds^2_{eucl} = \psi^{-4}(dr^2 + r^2 d\omega^2), \quad r > 0 \tag{3.59}$$

where $\{s > 2m\}$ and $\{r > \frac{m}{2}\}$ are related by a diffeomorphism:

$$r + m + \frac{m^2}{2r} = \left(1 + \frac{m}{2r}\right)^2 r = s, \quad \frac{1}{\sqrt{1 - \frac{2m}{s}}} ds = \left(1 + \frac{m}{2r}\right)^2 dr^2 \tag{3.60}$$

Therefore V as a function of $r > \frac{m}{2}$ is given by:

$$1 - V^2 = \frac{2m}{r + m + \frac{m^2}{4r}}, \text{ or } V = \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \tag{3.61}$$

and the conformal factor ψ is given by:

$$\psi = \frac{1}{1+\frac{m}{2r}} = \frac{1}{2}(1+V) \tag{3.62}$$

The field intensity $W_0 = |\nabla V|^2_{schw}$ is given by:

$$W_0(s) = \left(1 - \frac{2m}{s}\right) V'(s)^2 = \frac{m^2}{s^4} \tag{3.63}$$

Since

$$(1-V^2)^4 = \left(\frac{2m}{s}\right)^4 = \frac{16m^4}{s^4} \tag{3.64}$$

we have as a function of V:

$$W_0(V) = \frac{1}{16m^2}(1-V^2)^4 \tag{3.65}$$

The Schwarzschild radial coordinate, s, at the surface of the star ($s = R_s$) is related to V_s, ρ and the mass of the star m by the two equations:

$$1 - \frac{2m}{R_s} = V_s^2, \qquad \text{and} \qquad \frac{4\pi\rho}{3}R_s^3 = m \tag{3.66}$$

$$\Rightarrow m = \frac{4\pi\rho}{3}\left(\frac{2m}{1-V_s^2}\right) \tag{3.67}$$

$$\Leftrightarrow 16m^2 = \left(\frac{3(1-V_s^2)^3}{2\pi\rho}\right) \tag{3.68}$$

So W_0 can also be expressed as a function in terms of V_s. From (3.63), and (3.68) we obtain an alternate form of W_0 :

$$W_0(V) = \frac{2\pi\rho}{3}(1-V_s^2)^{-3}(1-V^2)^4 \tag{3.69}$$

The Interior

Consider the splitting of the metric along level sets of V:

$$ds^2 = h^2 dV^2 + g_{ab} dx^a dx^b \tag{3.70}$$

where g_{ab} is the induced metric on level sets of V, and $h^2 = W^{-1} = |\nabla V|_g^{-1}$. The Laplacian of V with respect to the metric splitting * of is:

$$\Delta V = h^{-1} K - h^{-3} \partial_3 h \tag{3.71}$$

where K is the mean curvature of level sets. From $\Delta V = 4\pi V(\rho + 3p)$, and (3.71) we have:

$$K - h^{-2} \partial_3 h = 4\pi V h(\rho + 3p) \tag{3.72}$$

Since $\partial_3 h = \partial_3 [W^{-1/2}] = \frac{-1}{2} W^{-3/2} \partial_3 W$, (3.72) becomes:

$$\begin{aligned} K + \frac{1}{2} W^{-1/2} \partial_3 W &= 4\pi V(\rho + 3p) \\ \Rightarrow \partial_3 W &= -2W^{-1/2} K + 8\pi V(\rho + 3p) \end{aligned} \tag{3.73}$$

Turning to the spherically symmetric case with $W_0 = W_0(V)$ we have:

$$\frac{dW_0}{dV} = -2W_0^{1/2} K_0 + 8\pi V(\rho + 3p) \tag{3.74}$$

Writing the spherically symmetric metric in the form (3.2), we find from (3.18):

$$16\pi p = \frac{4}{rV} \left(1 - \frac{2m}{r}\right) \frac{dV}{dr} - \frac{4m}{r^3} \tag{3.75}$$

*See chapter 4 for the coefficients of the connection of the metric splitting.

For the metric (3.2) we have:

$$W_0 = g^{rr}\left(\frac{dV}{dr}\right)^2 = \left(1-\frac{2m}{r}\right)\left(\frac{dV}{dr}\right)^2 \tag{3.76}$$

$$\Rightarrow W_0^{-1/2} = \left(1-\frac{2m}{r}\right)^{1/2}\frac{dV}{dr} \tag{3.77}$$

$$K_0 = \frac{2}{r}\sqrt{g^{rr}} = \left(1-\frac{2m}{r}\right)^{1/2}\frac{2}{r} \tag{3.78}$$

Substituting in (3.75) we obtain:

$$\frac{2}{V}W_0^{1/2}K_0 = \frac{4m}{r^3} + 16\pi p \tag{3.79}$$

Using this in (3.74) we have:

$$\frac{dW_0}{dV} = -16\pi Vp - \frac{4mV}{r^3} + 8\pi V(\rho+3p) = \frac{4mV}{r^3} + 8\pi V(\rho+p) \tag{3.80}$$

$$\tag{3.81}$$

In the case of constant density we can use the second equation in (3.66) to yield:

$$\begin{aligned}\frac{dW_0}{dV} &= 8\pi V(\rho+3p) - \frac{16\pi p}{3}V\\ &= \frac{8\pi}{3}V(\rho+3p)\\ &= \frac{8\pi\rho}{3}(3V_s-2V)\end{aligned} \tag{3.82}$$

The last equality comes from (3.30). By integrating we have:

$$W_0(V) = \frac{8\pi\rho V}{3}(3V_s - V) + C \tag{3.83}$$

where C is a constant that can be found from the continuity of W_0 at $V = V_s$. With respect to the exterior, $W_0(V_s) = \frac{2\pi\rho}{2}(1-V_s)$, thus $C = \frac{2\pi\rho}{3}(1-9V_s)^2$ ensures the continutiy of W_0 at the surface. Finally we obtain the expression for W_0:

$$W_0(V) = \frac{8\pi\rho}{3} V(3V_s - V) + \frac{2\pi\rho}{3}(1 - 9V_s^2) \tag{3.84}$$

Chapter 4

The Cotton Tensor

We introduce a special rank 3 tensor that is invariant under conformal transformations, and vanishes for manifolds of dimension 3 if and only if the manifold is locally conformally flat.

Definition 4.1. *The Cotton tensor is a* $(0,3)$ *tensor defined by:*

$$R_{ijk} := \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{2(n-1)}(\nabla_j R g_{ik} - \nabla_k R g_{ij})$$

Proposition 4.2. *For static perfect-fluid stellar models, the following identity holds:*

$$R_{ijk}R^{ijk} = V^{-4}\left\{8W^2\hat{\Omega}_{ij}\hat{\Omega}^{ij} + \partial_a W \partial^a W\right\}$$

Where $\hat{\Omega}_{ij} = \Omega_{ij} - \frac{K}{2}g_{ij}$, Ω_{ij} *is the second fundamental form of level sets* $\{V = const\}$, *and* K *is the mean curvature.*

Proof

Case I: The Vacuum case

Einstein's equation for the exterior region (no matter) decomposes into the following system of equations:

$$\begin{cases} \Delta V = 0 \\ R_{ij} = V^{-1}\nabla_i\nabla_j V = V^{-1}V_{;ji} \end{cases} \tag{4.1}$$

By contracting R_{ij} and using $\Delta V = 0$, it is clear that the scalar curvature vanishes.

In dimension 3, the $(0,4)$ Riemannian tensor can fully be expressed in terms of the $(0,2)$ Ricci tensor (using the Kulkarni-Nomizo product $\odot$) as:

$$\begin{aligned} R_{ijkl} &= -[\frac{R}{4}(g\odot g)_{ijkl} + (Ric\odot g)_{ijkl}] \\ &= R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il} + \frac{R}{2}(g_{il}g_{jk} - g_{ik}g_{jl}) \end{aligned} \tag{4.2}$$

Or,

$$R_{jkil} = R_{ij}g_{lk} + R_{lk}g_{ij} - R_{jl}g_{ik} - R_{ik}g_{jl} + \frac{R}{2}(g_{jl}g_{ik} - g_{ij}g_{kl}) \tag{4.3}$$

In the exterior region we can use the Ricci tensor from (4.1), and equation (4.2) to recover the $(0,4)$ Riemannian tensor.

$$R_{jkil} = V^{-1}[V_{;ij}g_{lk} + V_{;lk}g_{ij} - V_{;jl}g_{ik} - V_{;ik}g_{lj}] \tag{4.4}$$

From the Ricci identity and (4.4) we have:

$$\begin{aligned} \nabla_k\nabla_j V_{;i} - \nabla_j\nabla_k V_{;i} &= -R^l_{jki}V_l = -R_{jkil}V^l \\ &= -V^{-1}V^{;l}[V_{;ij}g_{lk} + V_{;lk}g_{ij} - V_{;jl}g_{ik} - V_{;ik}g_{lj}] \\ &= V^{-1}[V_{;ik}V_{;j} - V_{;k}V_{;ij} + V^{;l}(V_{;jl}g_{ik} - V_{;lk}g_{ij})] \end{aligned} \tag{4.5}$$

To compute the Cotton tensor, we must first take the covariant derivative of the Ricci tensor.

$$\begin{aligned}\nabla_k R_{ij} &= V^{-1}\nabla_k\nabla_i\nabla_j V - V^{-2}\nabla_k V\nabla_i\nabla_j V \\ &= V^{-1}\nabla_k\nabla_j V_{;i} - V^{-2}V_{;k}V_{;ij} \qquad (4.6)\end{aligned}$$

It will be helpful to define a new variable W as the norm squared of $V_{;i}$, that is $W := V_{;i}V^{;i}$. Taking the first and second covariant derivative of W, we have:

$$W_{;k} = \partial_k W = \partial_k(V_{;j}V^{;j}) = 2V^{;j}\nabla_j\nabla_k V = 2V^{;j}V_{;kj} \qquad (4.7)$$

$$\Leftrightarrow W^{;k} = 2V_{;j}\nabla^k\nabla^j V = 2V_{;j}V^{;kj} \qquad (4.8)$$

$$\begin{aligned}\Rightarrow W_{;kl} &= \nabla_l(W_k) = 2(\nabla_l\nabla_k\partial_j V)(V^{;j}) + 2V_{;kj}(\nabla_l\partial^j V) \\ &= 2(\nabla_j\nabla_l V_{;k} + R^m_{ljk}V_{;m})V^{;j} + 2V_{;kj}V^{;j}_{;l} \\ \Rightarrow \Delta W &= g^{kl}W_{;kl} = g^{kl}[2(\nabla_j\nabla_l V_{;k} + R^m_{ljk}V_{;m})V^{;j} + 2V_{;kj}V^{;j}_{;l}] \\ &= (2\nabla_j\Delta V + 2R_{jm}V^{;m})V^{;j} + 2V_{;kj}V^{;kj} \\ &= 0 + 2V^{-1}V_{;jm}V^{;m}V^{;j} + 2V_{;kj}V^{;kj} \\ &= V^{-1}W_{;j}V^{;j} + 2V_{;kj}V^{;kj} \\ &\Rightarrow V_{;kj}V^{kj} = \frac{1}{2}\Delta W - \frac{1}{2}V^{-1}W_{;j}V^{;j} \qquad (4.9)\end{aligned}$$

Using the definition of the Cotton tensor and equations (4.5), (4.6), and (4.7) we have:

$$
\begin{aligned}
R_{ijk} =& \nabla_k R_{ij} - \nabla_j R_{ik} \\
=& V^{-1}\nabla_k\nabla_j V_i - V^{-2}V_{;k}V_{;ij} - V^{-1}\nabla_j\nabla_k V_i + V^{-2}V_{;j}V_{ik} \\
=& V^{-1}(\nabla_k\nabla_j V_{;i} - \nabla_j\nabla_k V_{;i}) + V^{-2}V_{;j}V_{ik} - V^{-2}V_{;k}V_{ij} \\
=& V^{-1}(-V^{;l}R_{jkil}) + V^{-2}V_{;j}V_{;ik} - V^{-2}V_{;k}V_{;ij} \\
=& V^{-2}[V_{;ik}V_{;j} - V_{;k}V_{;ij} + V^{;l}(V_{;jl}g_{ik} - V_{;lk}g_{ij})] \\
& + V^{-2}V_{;j}V_{;ik} - V^{-2}V_{;k}V_{;ij} \\
=& V^{-2}[2V_{;j}V_{;ik} - 2V_{;k}V_{;ij} + V^{;l}(V_{;jl}g_{ik} - V_{;kl}g_{ij})] \\
=& V^{-2}[2V_{;j}V_{;ik} - 2V_{;k}V_{;ij} + \frac{1}{2}(W_{;j}g_{ik} - W_{;k}g_{ij})]
\end{aligned}
\tag{4.10}
$$

We can now compute $R_{ijk}R^{ijk}$.

$$
\begin{aligned}
V^4 R_{ijk}R^{ijk} =& [2V_{;j}V_{;ik} - 2V_{;k}V_{;ij} + \frac{1}{2}(W_{;j}g_{ik} - W_{;k}g_{ij})] \\
& \cdot [2V^{;j}V^{;ik} - 2V^{;k}V^{;ij} + \frac{1}{2}(W^{;j}g^{ik} - W_{;k}g^{ij})] \\
=& 4WV_{;ik}V^{;ik} - W_{;i}W^{;i} + (\Delta V)V_{;j}W^{;j} + \frac{1}{2}W_{;k}W^{;k} \\
& - W_{;i}W^{;i} + 4WV_{;ij}V^{;ij} - \frac{1}{2}W_{;j}W^{;j} \\
& + (\Delta V)V_{;k}W^{;k} + (\Delta V)W_{;j}V^{;j} - \frac{1}{2}W_{;j}W^{;j} \\
& + \frac{3}{4}W_{;j}W^{;j} - \frac{1}{4}\delta^j_k W_{;j}W^{;k} - \frac{1}{2}W_{;k}W^{;k} \\
& + (\Delta V)V^{;k}W_{;k} - \frac{1}{4}\delta^k_j W_{;k}W^{;j} + \frac{3}{4}W_{;k}W^{;k}
\end{aligned}
\tag{4.11}
$$

Combining like terms $R_{ijk}R^{ijk}$ reduces to:

$$
R_{ijk}R^{ijk} = V^{-4}[8WV_{ij}V^{ij} - 3W_{;k}W^{;k} + 4(\Delta V)V_{;k}W^{;k}] \tag{4.12}
$$

Finally from (4.9), and (4.12) we have:

$$R_{ijk}R^{ijk} = V^{-4}[4W\Delta W - 4WV^{-1}W_{;j}V^{;j} - 3W_{;j}W^{;j}] \tag{4.13}$$

Now consider the splitting of the metric along level sets of V,

$$\begin{aligned} ds^2 =& h^2(V, x^n)dV^2 + d\hat{s}^2, \qquad (h^2 = W^{-1}) \\ =& h^2(V, x^n)dV^2 + g_{ab}dx^a dx^b \end{aligned} \tag{4.14}$$

The connection coefficients are:

$$\begin{aligned} \Gamma^n_{nn} = h^{-1}\partial_n h, \quad \Gamma^n_{na} = h^{-1}\partial_a h, \quad \Gamma^a_{nn} = h\partial^a h \\ \Gamma^b_{na} = h\Omega^b_a, \quad \Gamma^n_{ab} = -h^{-1}\Omega_{ab}, \quad \Gamma^c_{ab} = \hat{\Gamma}^c_{ab} \end{aligned} \tag{4.15}$$

Where Ω_{ab} is the second fundamental form for level surfaces of V. The Hessian of V has components:

$$\begin{aligned} \nabla_a\nabla_b V &= h^{-1}\Omega_{ab} \\ \nabla_a\nabla_3 V &= -h^{-1}\partial_a h \\ \nabla_3\nabla_3 V &= h^{-1}\partial_3 h \\ \Rightarrow \Delta V &= g^{33}(V_{;33}) + g^{ab}(\nabla_a\nabla_b V) \\ &= \frac{k}{h} - \frac{\partial_3 h}{h^3} \end{aligned} \tag{4.16}$$

So,

$$\begin{aligned} V_{;ij}V^{;ij} &= (g^{33})^2(\nabla_3\nabla_3 V)^2 + 2g^{33}g^{ab(}\nabla_3\nabla_a V)(\nabla_3\nabla_b V) + (\nabla^a\nabla^b V)(\nabla_a\nabla_b V) \\ &= h^{-4}(h^{-1}\partial_3 h)^2 + 2h^{-2}g^{ab}(h^{-1}\partial_a h)(h^{-1}\partial_b h) + h^{-2}\Omega_{ab}\Omega^{ab} \end{aligned} \tag{4.17}$$

Using the fact that $\partial_3 h = h^2 k - h^3(\Delta V)$ (where $\Omega^a_a = k$) and $W = h^{-2}$, (4.17) can be written as:

$$V_{;ij}V^{;ij} = Wk^2 - 2WhK(\Delta V) + (\Delta V)^2 + \frac{1}{2}W^{-1}\partial_a W \partial^a W + W\Omega_{ab}\Omega^{ab} \tag{4.18}$$

Also,

$$\begin{aligned}
\partial_j W \partial^j W &= g^{33}(\partial_3 W)^2 + \partial_a W \partial^a W \\
&= W\left(2\Delta V - \frac{2K}{h}\right)^2 + \partial_a W \partial^a W \\
&= 4W^2k^2 - \frac{8KW(\Delta V)}{h} + 4W(\Delta V)^2 + \partial_a W \partial^a W \qquad (4.19)\\
V_{;k}W^{;k} &= g^{33}(\partial_3 W) = 2W(\Delta V) - \frac{2Wk}{h} \qquad (4.20)
\end{aligned}$$

By substituting (4.18), (4.19), and (4.20) into (4.12) we obtain:

$$\begin{aligned}
V^4 R_{ijk}R^{ijk} =& 8WV_{ij}V^{ij} - 4W_{;k}W^{;k} + 4(\Delta V)V_{;k}W^{;k} \\
=& 8W^2k^2 - 16Wh^{-1}k\Delta V + 8W(\Delta V)^2 + 4\partial_a W \partial^a W \\
&+ 8W^2\Omega_{ab}\Omega^{ab} + 8W(\Delta V)^2 - 8Wh^{-1}k(\Delta V) \\
&- 12W^2k^2 + 24Wk^{-1}h(\Delta V) - 12W(\Delta V)^2 - 3\partial_a W \partial^a W \\
=& 8W^2\Omega_{ab}\Omega^{ab} - 4W^2k^2 + \partial_a W \partial^a W + 4W(\Delta V)^2 \qquad (4.21)
\end{aligned}$$

Let $\hat{\Omega}_{ab} := \Omega_{ab} - \frac{k}{2}g_{ab}$. Then,

$$\begin{aligned}
|\hat{\Omega}|^2 = \hat{\Omega}_{ab}\hat{\Omega}^{ab} &= (\Omega_{ab} - \frac{k}{2}g_{ab})(\Omega^{ab} - \frac{k}{2}g^{ab}) \\
&= \Omega_{ab}\Omega^{ab} - \frac{k^2}{2} \qquad (4.22)
\end{aligned}$$

Thus (4.21) in terms of $\hat{\Omega}_{ab}$ is:

$$\begin{aligned} V^4 R_{ijk}R^{ijk} &= 8W^2\hat{\Omega}_{ab}\hat{\Omega}^{ab} + \partial_a W\partial^a W + 4(\Delta V)^2 \\ &= 8W^2\hat{\Omega}_{ab}\hat{\Omega}^{ab} + \partial_a W\partial^a W \end{aligned} \tag{4.23}$$

Case II: The Interior Region

The Einstein equation for the interior region decomposes as

$$\begin{cases} \tilde{R}_{ij} = V^{-1}\nabla_i\nabla_j V + 4\pi(\rho - p)g_{ij} = V^{-1}V_{ij} + 4\pi(\rho - p)g_{ij} \\ \Delta V = 4\pi V(\rho + 3p) \end{cases} \tag{4.24}$$

The following calculations will be useful.

$$\tilde{R}^i_i = \tilde{R} = 16\pi\rho \tag{4.25}$$

$$p_{;k}p^{;k} = V^{-2}(\rho + p)^2 W \tag{4.26}$$

$$V_{;k}p^{;k} = -V^{-1}(\rho + p)W \tag{4.27}$$

$$\begin{aligned} \nabla_k\tilde{R}_{ij} &= \nabla_k[V^{-1}\nabla_i\nabla_j V + 4\pi(\rho - p)g_{ij}] \\ &= V^{-1}\nabla_k\nabla_i\nabla_j V - V^{-2}V_{;k}V_{;ij} + 4\pi\rho_{;k}g_{ij} - 4\pi p_{;k}g_{ij} \end{aligned} \tag{4.28}$$

The $(0,4)$ Riemannian tensor can be recovered from the Ricci tensor using equation (4.2):

$$\begin{aligned}
\tilde{R}_{jkil} &= \tilde{R}_{ij}g_{lk} + \tilde{R}_{lk}g_{ij} - \tilde{R}_{jl}g_{ik} - \tilde{R}_{ik}g_{jl} + \frac{\tilde{R}}{2}(g_{jl}g_{ik} - g_{ij}g_{kl}) \\
&= (V^{-1}V_{;ij} + 4\pi(\rho - p)g_{ij})g_{lk} + (V^{-1}V_{;lk} + 4\pi(\rho - p)g_{lk})g_{ij} \\
&- (V^{-1}V_{;jl} + 4\pi(\rho - p)g_{jl})g_{ik} - (V^{-1}V_{;ik} + 4\pi(\rho - p)g_{ik})g_{jl} \\
&+ 8\pi\rho(g_{jl}g_{ik} - g_{ij}g_{lk}) \\
&= R_{jkil} + 8\pi p(g_{jl}g_{ik} - g_{ij}g_{lk})
\end{aligned} \tag{4.29}$$

We now define a tensor C_{ijk} in the same way we defined the Cotton tensor for the vacuum case.

$$\begin{aligned}
C_{ijk} :&= V^{-1}\nabla_k\nabla_j V_{;i} - V^{-2}V_{;k}V_{;ij} - V^{-1}\nabla_j\nabla_k V_{;i} + V^{-2}V_{;j}V_{;ik} \\
&= V^{-1}(\nabla_k\nabla_j V_{;i} - \nabla_j\nabla_k V_{;i}) + V^{-2}V_{;k}V_{;ik} - V^{-2}V_{;k}V_{;ij} \\
&= V^{-1}(-V^{;l}\tilde{R}_{jkil}) + V^{-2}V_{;j}V_{;ik} - V^{-2}V_{;k}V_{;ij}
\end{aligned} \tag{4.30}$$

Since the $(0,4)$ Riemannian tensor inside the star is different from the exterior case, C_{ijk} will not equal R_{ijk}. Substituting the $(0,4)$ Riemannian tensor from (4.29) into (4.30) we have:

$$\begin{aligned}
C_{ijk} &= -V^{-1}V^{;l}[R_{jkil} + 8\pi p(g_{jl}g_{ik} - g_{ij}g_{lk})] + V^{-2}(V_{;j}V_{;ik} - V_{;k}V_{;ij}) \\
&= R_{ijk} + 8\pi pV^{-1}(V_{;k}g_{ij} - V_{;j}g_{ik})
\end{aligned} \tag{4.31}$$

The norm squared of C_{ijk} will be used later.

$$\begin{aligned}
C_{ijk}C^{ijk} &= [R_{ijk} + 8\pi pV^{-1}(V_{;k}g_{ij} - V_{;j}g_{ik})][R^{ijk} + 8\pi pV^{-1}(V^{;k}g_{ij} - V^{;j}g^{ik})] \\
&= R_{ijk}R^{ijk} + 16\pi pV^{-1}R_{ijk}(V^{;k}g^{ij} - V^{;j}g^{ik}) \\
&+ 64\pi^2p^2V^{-2}(V_{;k}g_{ij} - V_{;j}g_{ik})(V^{;k}g^{ij} - V^{;j}g^{ik})
\end{aligned} \tag{4.32}$$

The middle term reduces to:

$$
\begin{aligned}
16\pi p V^{-1} R_{ijk}(V^{;k}g^{ij} - V^{;j}g^{ik}) \\
&=16\pi p V^{-3}[V^{;k}g^{ij} - V^{j}g^{ik}] \\
&\quad \cdot [2V_{;j}V_{;ik} - 2V_{;k}V_{;ij} + \frac{1}{2}(W_{;j}g_{ik} - W_{;k}g_{ij})] \\
&=16\pi p V^{-3}[W_{;k}V^{;k} - 2W\Delta V + \frac{1}{2}V^{;k}W_{;j}\delta^{j}_{k} - \frac{3}{2}W_{;k}V^{;k} \\
&\quad - 2W\Delta V + W_{;j}V^{;j} - \frac{3}{2}W_{;j}V^{;j} + \frac{1}{2}V^{;j}W_{;k}\delta^{k}_{j}] \\
&=16\pi p V^{-3}(-4W\Delta V) \\
&=-256\pi^2 p V^{-2} W(\rho + 3p) \qquad (4.33)
\end{aligned}
$$

Expanding the last term in (4.32), we see,

$$
\begin{aligned}
&64\pi^2 p^2 V^{-2}(V_{;k}g_{ij} - V_{;j}g_{ik})(V^{;k}g^{ij} - V^{;j}g^{ik}) \\
&= 64\pi^2 p^2 V^{-2}[3W - \delta^{k}_{j}V_{;k}V^{;j} - \delta^{j}_{k}V_{;j}V^{;k} + 3W] \\
&= 256\pi^2 p^2 V^{-2} W \qquad (4.34)
\end{aligned}
$$

The norm squared of the Cotton tensor in the exterior has already been computed, we can therefore substitute (4.23), (4.33) and (4.34) into (4.32) to obtain:

$$
\begin{aligned}
C_{ijk}C^{ijk} =& V^{-4}[8W^2\hat{\Omega}_{ab}\hat{\Omega}^{ab} + \partial_a W \partial^a W + 4W(\Delta V)^2] - 256\pi^2 p V^{-2} W(\rho + 3p) \\
&+ 256\pi^2 p^2 V^{-2} W \qquad (4.35)
\end{aligned}
$$

We can now compute the Cotton tensor for the interior case.

$$
\begin{aligned}
\tilde{R}_{ijk} =& \nabla_k \tilde{R}_{ij} - \nabla_j \tilde{R}_{ik} + \frac{1}{4}(\nabla_j \tilde{R} g_{ik} - \nabla_k \tilde{R}_{ij}) \\
=& V^{-1}\nabla_k \nabla_j V_{;i} - V^{-2} V_{;k} V_{;ij} + 4\pi \rho_{;k} g_{ij} - 4\pi p_{;k} g_{ij} \\
& - V^{-1}\nabla_j \nabla_k V_{;i} + V^{-2} V_{;j} V_{;ik} - 4\pi \rho_{;j} g_{ik} + 4\pi p_{;j} g_{ik} \\
& + 4\pi(\rho_{;j} g_{ik} - \rho_{;k} g_{ij}) \\
=& C_{ijk} + 4\pi(g_{ik} p_{;j} - g_{ij} p_{;k})
\end{aligned} \tag{4.36}
$$

The norm squared of the Cotton tensor is then,

$$
\begin{aligned}
\tilde{R}_{ijk}\tilde{R}^{ijk} =& C_{ijk}C^{ijk} + 8\pi C_{ijk}(p^{;j} g^{ik} - p^{;k} g^{ij}) \\
& + 16\pi^2 (p_{;j} g_{ik} - p_{;k} g_{ij})(p^{;j} g^{ik} - p^{;k} g^{ij})
\end{aligned} \tag{4.37}
$$

Expanding the middle term we have:

$$
\begin{aligned}
& 8\pi C_{ijk}(p^{;j} g^{ik} - p^{;k} g^{ij}) \\
& \quad = 8\pi V^{-2}[2V_{;j}V_{;ik} - 2V_{;k}V_{;ij} + \frac{1}{2}(W_{;j} g_{ik} - W_{;k} g_{ij})] \\
& \quad \cdot [p^{;j} g^{ik} - p^{;k} g^{ij}] + 64\pi^2 p V^{-1}[V_{;k} g_{ij} - V_{;j} g_{ik}] \cdot [p^{;j} g^{ik} - p^{;k} g^{ij}] \\
& \quad = 8\pi V^{-2}[2V_{;j} p^{;j} \Delta V - W_{;k} p^{;k} - W_{;j} p^{;j} + 2p^{;k} V_{;k} \Delta V \\
& \quad \frac{1}{2}(3W_{;j} p^{;j} - W_{;j} p^{;k} \delta^j_k - W_{;k} p^{;j} \delta^k_j + 3W_{;k} p^{;k})] \\
& \quad + 64\pi^2 p V^{-1}[V_{;k} p^{;j} \delta^k_j - 3V_{;k} p^{;k} - 3V_{;j} p^{;j} + p^{;k} V_j \delta^j_k] \\
& \quad = -128\pi^2 V^{-2} W(\rho + p)(\rho + 3p) + 256\pi^2 V^{-2} W p(\rho + p)
\end{aligned} \tag{4.38}
$$

The last term simplifies to:

$$
\begin{aligned}
16\pi^2 (p_{;j} g_{ik} - p_{;k} g_{ij})(p^{;j} g^{ik} - p^{;k} g^{ij}) &= 16\pi^2 (4 p_{;j} p^{;j}) \\
&= 64\pi^2 V^{-2} W(\rho + p)^2
\end{aligned} \tag{4.39}
$$

Finally the norm squared of $\tilde{R}_{ijk}$ is:

$$\begin{aligned}
\tilde{R}_{ijk}\tilde{R}^{ijk} =& V^{-4}[8W^2\hat{\Omega}_{ab}\hat{\Omega}^{ab} + \partial_a W \partial^a W + 4W(\Delta V)^2] - 256\pi^2 V^{-2} W p(\rho + 3p) \\
& + 256\pi^2 p^2 V^{-2} W - 128\pi^2 V^{-2} W(\rho + p)(\rho + 3p) \\
& + 256\pi^2 V^{-2} W p(\rho + p) + 64\pi^2 V^{-2} W(\rho + p)^2 \\
=& V^{-4}[8W^2\hat{\Omega}_{ab}\hat{\Omega}^{ab} + \partial_a W \partial^a W + 4W(\Delta V)^2] - 64\pi^2 V^{-2} W(\rho + 3p)^2 \\
=& V^{-4}[8W^2\hat{\Omega}_{ab}\hat{\Omega}^{ab} + \partial_a W \partial^a W + 4W(\Delta V)^2] - 4V^{-2}(\Delta V)^2 \\
=& V^{-4}[8W^2\hat{\Omega}_{ab}\hat{\Omega}^{ab} + \partial_a W \partial^a W]
\end{aligned} \tag{4.40}$$

Corollary 4.3. *For perfect-fluid stellar models, the metric on spacelike sections is conformally flat if and only if the level sets of the potential function V are totally umbilic and the* field intensity *W is constant on level sets of V.*

Proof.

For manifolds of dimension 3, the Cotton tensor R_{ijk} vanishes if and only if the manifold is conformally flat (Eisenhart 1949). Using proposition 1, and the fact that W is constant on level sets of V, the norm squared of the Cotton tensor becomes $|R_{ijk}|^2 = 8V^{-4}W^2|\hat{\Omega}_{ij}|^2$. Thus by the definition of totally umbilic and the conformal properties of the Cotton tensor, the level sets of V are totally umbilic if and only if the manifold is conformally flat.

Chapter 5

The Israel-Robinson Identities

We establish two identities concerning elliptic operators acting on $W - W_0$, where

$$W_0(V) = \begin{cases} \frac{2}{3}\pi\rho(1-V^2)^4(1-V_s^2)^{-3}, & V_s \leq V \leq 1 \\ \frac{8}{3}\rho V(3V_s - V) + \frac{2}{3}\pi\rho(1-9V_s^2), & 0 \leq V \leq V_s \end{cases} \tag{5.1}$$

The Exterior

Lemma 5.1. *(Lindblom 1980) For the exterior region, $V_s \leq V \leq 1$, the following identity holds:*

$$\nabla_i \left\{ V^{-1} \nabla^i \left[\frac{W - W_0}{(1-V^2)^3} \right] \right\} = \frac{V^{-4}|R_{ijk}|^2 + 3\left|W_{;i} + 8VW(1-V^2)^{-1}V_{;i}\right|^2}{4VW(1-V^2)^3} \tag{5.2}$$

Proof.

First rewrite (5.2) as:

$$\nabla_i \left\{ \frac{1}{V(1-V^2)^3} \nabla^i (W - W_0) + \frac{6(W-W_0)}{(1-V^2)^4} \nabla^i V \right\} = \frac{1}{V(1-V^2)^3} \cdot \frac{V^4}{4} W^{-1} |R_{ijk}|^2$$
$$+ \frac{3}{4} \cdot \frac{1}{V(1-V^2)^3} W^{-1} \left|W_{;i} + 8VW(1-V^2)^{-1}V_{;i}\right|^2 \tag{5.3}$$

Now consider identities of the form:

$$\nabla_i \left\{ k_2(V)\nabla^i W + k_1(V,W)\nabla^i V \right\}$$
$$= Q_1(V)\frac{V^4}{4}W^{-1}|R_{ijk}|^2 + Q_2(V)W^{-1}|\nabla_i W - F(V,W)\nabla_i V|^2 \tag{5.4}$$

The right-hand side of (5.3) is of this form with $Q_1(V) = \frac{1}{V(1-V^2)^3}$ and $Q_2(V) = \frac{3}{4}\frac{1}{V(1-V^2)^3}$. Starting with the left side of (4) we have:

$$k_2\nabla_i\nabla^i W + \frac{\partial k_2}{\partial V}\nabla_i V\nabla^i W + \frac{\partial k_1}{\partial V}\nabla_i V\nabla^i V + \frac{\partial k_1}{\partial W}\nabla_i W\nabla^i V + k_1\nabla_i\nabla^i V \tag{5.5}$$

Note that the $\Delta V = 0$ in the exterior, so (5.5) reduces to:

$$k_2\nabla_i\nabla^i W + \frac{\partial k_2}{\partial V}\nabla_i V\nabla^i W + \frac{\partial k_1}{\partial V}W + \frac{\partial k_1}{\partial W}\nabla_i W\nabla^i V \tag{5.6}$$

On the right side of (4.5) we have:

$$Q_1[\nabla_i\nabla^i W - V^{-1}\nabla^i V\nabla_i W - \frac{3}{4}W^{-1}\nabla^i W\nabla_i W]$$
$$+ Q_2W^{-1}\nabla_i W\nabla^i W - 2Q_2FW^{-1}\nabla_i V\nabla^i W + Q_2F^2W^{-1}\nabla_i V\nabla^i V \tag{5.7}$$

Comparing the coefficients of $\nabla^i\nabla_i W$, $\nabla_i V\nabla^i W$, and $W^{-1}\nabla_i W\nabla^i W$ from (5.6) and (5.7) with (5.3) we get the following conditions:

$$\begin{cases} k_2 = Q_1 \\ 0 = Q_2 - \frac{3}{4}Q_1 \\ \frac{\partial k_1}{\partial V}W = Q_2F^2 \\ \frac{\partial k_1}{\partial W} + \frac{\partial k_2}{\partial V} = -V^{-1}Q_1 - 2W^{-1}Q_2F \end{cases} \tag{5.8}$$

A solution to this system will prove existence of identities of the form in equation (5.4). To find a suitable choice for F, we first consider the fact that in the rotationally symmetric case $\nabla_i W = \frac{dW_0}{dV}\nabla_i V$, because $W_0 \equiv \textit{constant}$ on level sets of V. Thus F is chosen so that in the rotationally symmetric case $|\nabla_i W - F\nabla_i V|^2 = 0$, and it coincides with $\frac{dW_0}{dV}$. Here are two possibilities then for F:

$$F(V,W) = \frac{dW_0}{dV} \tag{5.9}$$

or

$$F(V,W) = \left(\frac{W_0}{W}\right)^n \frac{dW_0}{dV} + (W - W_0)^m \tag{5.10}$$

For the exterior region, we know that if the spacetime is spherically symmetric then it must also satisfy the vacuum Schwarzschild solution, and $W_0 = \frac{1}{16m^2}(1-V^2)^4$ (see the Remark in Chapter 3). Taking the derivative of W_0 with respect to V we get:

$$\frac{dW_0}{dV} = \frac{-V(1-V^2)^3}{2m^2} = -\frac{8VW_0}{1-V^2} \tag{5.11}$$

Using (5.3), (5.10), and (5.11) we now define F as:

$$F(V,W) := \frac{W}{W_0}\frac{dW_0}{dV} = \frac{-8VW}{1-V^2} \tag{5.12}$$

with this F and $Q_1(V) = \frac{1}{V(1-V^2)^3}$ the solution to (5.8) is:

$$k_1(V,W) = \frac{6W}{(1-V^2)^4} \tag{5.13}$$

$$k_2(V) = Q_1 = \frac{4}{3}Q_2 = \frac{1}{V(1-V^2)^3} \tag{5.14}$$

Finally, note that since

$$\nabla_i\left(\frac{W_0}{(1-V^2)^4} + \nabla^i V\right) = \frac{1}{16m^2}\Delta V = 0 \tag{5.15}$$

and,

$$\frac{W_0'}{V(1-V^2)^3} = -\frac{1}{2m^2} \tag{5.16}$$

we may introduce W_0 on the left-hand side of (5.4), bringing it into the form of (5.3).

The Interior

Lemma 5.2. *If the density ρ is constant, then the following identity holds in the interior region $0 \leq V \leq V_s$:*

$$\nabla^i\{V^{-1}\nabla_i(W-W_0)\} = \frac{1}{4}V^3W^{-1}\,|R_{ijk}|^2 + \frac{3}{4}V^{-1}W^{-1}|\nabla_i(W-W_0)|^2 \tag{5.17}$$

Before the proof of Lemma 5.2 is given, we must first compute the Laplacian of W_0. The spatial metric is:

$$ds_g^2 = W^{-2}dV^2 + \tilde{g}_{ab}dx^a dx^b \tag{5.18}$$

where $\tilde{g}_{ab}$ is the induced metric on level sets of $V \equiv const$ and $g_{33} = W^{-1}$. The metric determinant of g is:

$$\sqrt{|g|} = W^{-1/2}\sqrt{|\tilde{g}|} \tag{5.19}$$

The second fundamental form on level sets of V is:

$$\Omega_{ab} = \langle \nabla_{\partial_a} n, \partial_b \rangle = W^{1/2}\partial_n g_{ab} \tag{5.20}$$

where $n = W^{1/2}\partial_3$ is the unit outward normal. Taking the trace of Ω_{ab} with respect to the metric $\tilde{g}_{ab}$, we can calculate the mean curvature:

$$K := \frac{1}{2}g^{ab}\Omega_{ab} = \frac{\partial_n\sqrt{|\tilde{g}|}}{\sqrt{|\tilde{g}|}} = W^{1/2}\frac{\partial_3\sqrt{|\tilde{g}|}}{\sqrt{|\tilde{g}|}} \tag{5.21}$$

The next identity will be used to compute Laplacian of W_0

$$
\begin{aligned}
W\frac{\partial_3\sqrt{|g|}}{\sqrt{|g|}} &= \frac{W}{\sqrt{|g|}}\partial_3(W^{1/2}\sqrt{|\tilde{g}|}) \\
&= \frac{W}{\sqrt{|g|}}[-\frac{1}{2}W^{-3/2}(\partial_3 W)\sqrt{|\tilde{g}|}+W^{-1/2}\partial_3\sqrt{|\tilde{g}|}] \\
&= -\frac{1}{2}\partial_3 W + W^{1/2}K
\end{aligned}
\tag{5.22}
$$

Using (5.19) and (5.23) we have:

$$
\begin{aligned}
\Delta W_0 &= \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j W_0) \\
&= \partial_i(g^{ij}\partial_j W_0) + \left(\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\right)g^{ij}\partial_j W_0 \\
&= \partial_3(W(W_0)') + \frac{\partial_3\sqrt{|g|}}{\sqrt{|g|}}WW_0' \\
&= WW_0'' + (\partial_3 W)W_0' + (-\frac{1}{2}\partial_3 W + W^{1/2}K)W_0'
\end{aligned}
\tag{5.23}
$$

Proof of Lemma 5.2.

We begin be expanding the left hand side of equation (5.17).

$$
\begin{aligned}
&\nabla^i(V^{-1}\nabla_i W) - \nabla^i(V^{-1}\nabla_i W_0) \\
=&V^{-1}\Delta W - V^{-2}\nabla^i V\nabla_i W - V^{-1}\Delta W_0 + V^{-2}\nabla^i V\nabla_i V(W_0') \\
=&V^{-1}\Delta W - V^{-2}W\partial_3 W - V^{-1}\Delta W_0 + V^{-2}WW_0'
\end{aligned}
\tag{5.24}
$$

Recall the identity derived earlier:

$$
\begin{aligned}
V^{-1}\Delta W =&\frac{1}{4}V^3 R_{ijk}R^{ijk} + V^{-2}\nabla^i V\nabla_i W + \frac{3}{4}V^{-1}W^{-1}\nabla_i W\nabla^i W \\
&- 8\pi V^{-1}W(\rho+p) - 4\pi W^{-1}(\rho+3p)\nabla^i V\nabla_i W \\
&+ 16\pi V(\rho+3p)^2 + 8\pi\nabla^i V\nabla_i\rho
\end{aligned}
\tag{5.25}
$$

Note that,

$$\nabla^i V \nabla_i W = g^{ij}(\nabla_i V \nabla_j W) = g^{33}(\frac{dV}{dV}\nabla_3 W) = W\partial_3 W$$

Also $\rho = \rho(V)$, so (5.25) can be simplified with respect to the metric splitting to:

$$\begin{aligned} V^{-1}\Delta W =& \frac{1}{4}V^3W^{-1}|R_{ijk}|^2 + V^{-2}W\partial_3 W + \frac{3}{4}V^{-1}W^{-1}|\nabla_i W|^2 - 8\pi V^{-1}W(\rho+p) \\ & - 4\pi(\rho+3p)\partial_3 W + 16\pi^2 V(\rho+3p)^2 + 8\pi W\rho'(V) \end{aligned} \tag{5.26}$$

By substituting (5.26) into (5.25) we get the following expression for the left hand side of (5.17).

$$\begin{aligned} & \nabla^i(V^{-1}\nabla_i W) - \nabla^i(V^{-1}\nabla_i W_0) \\ & = \frac{1}{4}V^3W^{-1}|R_{ijl}|^2 - V^{-1}\Delta W_0 + V^{-2}W(W_0') + \frac{3}{4}V^{-1}W^{-1}|\nabla_i W|^2 \\ & - 8\pi V^{-1}W(\rho+p) - 4\pi(\rho+3p)\partial_3 W + 16\pi^2 V(\rho+3p)^2 + 8\pi W\rho'(V) \end{aligned} \tag{5.27}$$

Expanding the right hand side of (5.17), we have:

$$\begin{aligned} & \frac{1}{4}V^3W^{-1}R_{ijk}R^{ijk} + \frac{3}{4}V^{-1}W^{-1}|\nabla_i(W-W_0)|^2 \\ & = \frac{1}{4}V^3W^{-1}R_{ijk}R^{ijk} + \frac{3}{4}V^{-1}W^{-1}(\nabla_i W\nabla^i W - 2\nabla_i W\nabla^i W_0 + \nabla_i W_0\nabla^i W_0) \\ & = \frac{1}{4}V^3W^{-1}R_{ijk}R^{ijk} + \frac{3}{4}V^{-1}W^{-1}\nabla_i W\nabla^i W - \frac{3}{2}V^{-1}(\partial_3 W)W_0' + \frac{3}{4}V'(W_0)^2 \end{aligned} \tag{5.28}$$

Subtracting (5.27) from (5.28) we get:

$$\begin{aligned} & V^{-1}\Delta W_0 - V^{-2}W(W_0)' + 8\pi V^{-1}W(\rho+p) + 4\pi W^{-1}(\rho+3p)\nabla^i V\nabla_i W \\ & - 16\pi V(\rho+3p)^2 - 8\pi\nabla^i V\nabla_i\rho - \frac{3}{2}V^{-1}(\partial_3 W)W_0' + \frac{3}{4}V'(W_0')^2 \end{aligned} \tag{5.29}$$

To show this equals zero we must use the integrated form of Euler's equation for static fluids and the derivative of W_0.

$$p = \rho V^{-1}(V_s - V) \;\Leftrightarrow\; \rho + 3p = \frac{\rho}{V}(3V_s - 2V) \tag{5.30}$$

$$W_0' = \frac{8\pi\rho}{3}(3V_s - 2V) \Rightarrow W_0'' = \frac{-16\pi\rho}{3} \tag{5.31}$$

Thus,

$$4\pi(\rho + 3p)\partial_3 W = 4\pi\rho V^{-1}(3V_s - 2V)\partial_3 W = \frac{3}{2}V^{-1}(\partial_3 W)W_0 \tag{5.32}$$

$$\begin{aligned} 8\pi V^{-1}W(\rho + p) &= V^{-2}W\frac{8\pi\rho}{3}(3V_s - 2V) + \frac{16\pi\rho}{3}V^{-1}W \\ &= V^{-2}W(W_0)' - V^{-1}WW_0'' \end{aligned} \tag{5.33}$$

$$16\pi^2 V(\rho + 3p)^2 = 16\pi^2\rho^2 V^{-1}(3V_s - V)^2 = \frac{9}{4}V^{-1}(W_0)^2 \tag{5.34}$$

Recall that from the Einstein equation and (4.16) we have:

$$\Delta V = 4\pi V(\rho + 3p) = W^{1/2}K - W^{3/2}\partial_3 W^{-1/2} \tag{5.35}$$

From (5.30) we know, $\rho + 3p = \frac{\rho}{V}(3V_s - 2V)$, thus:

$$W^{1/2}K - W^{3/2}\partial_3 W^{-1/2} = 4\pi\rho(3V_s - 2V) \tag{5.36}$$

Substituting (5.23), (5.32), (5.33), (5.34), and (5.36) into (5.29) we have:

$$\begin{aligned} &V^{-1}\Delta W_0 - V^{-1}WW_0' - \left(\frac{2}{3}\right)16\pi^2 V^{-1}(3V_s - 2V)^2 + 8\pi W\rho'(V) \\ &= V^{-1}\frac{1}{2}(\partial_3 W + W^{1/2}K)W_0' - \left(\frac{2}{3}\right)16\pi^2 V^{-1}(3V_s - 2V)^2 + 8\pi W\rho'(V) \\ &= 4\pi\rho V^{-1}(3V_s - 2V)^2\frac{8\pi\rho}{3} - \left(\frac{2}{3}\right)16\pi^2\rho^2 V^{-1}(3V_s - 2V)^2 + 8\pi\rho'(V) \\ &= 8\pi\rho'(V) = 0 \end{aligned} \tag{5.37}$$

The last equality comes from the assumption that $\rho \equiv const$. QED

Chapter 6

The Maximum Principle

We have already derived the conformal factor ψ, and the Israel-Robinson identities for the interior and exterior cases which will now be used to show that the conformally transformed scalar curvature is nonnegative. This is needed so we can apply the Positive Mass Theorem to conclude the space is spherically symmetric.

Proposition 6.1. *Let (M, g) be a Riemannian manifold with scalar curvature R and $\tilde{g} = \psi^4 g$ be a conformal metric. The scalar curvature $\tilde{g}$ is given by:*

$$\tilde{R} = \psi^{-4}\left[R - 8\psi^{-4}(\Delta\psi)\right] \tag{6.1}$$

Proof.

Given the conformal metric $\tilde{g}_{ij} = \psi^4 g_{ij}$, the coefficients of the connection for the metric $\tilde{g}_{ij}$, i.e. the Christoffel symbols, are:

$$\begin{aligned}\tilde{\Gamma}^k_{ij} &= \frac{1}{2}\psi^{-4}g^{ij}[\partial_i(\psi^4 g_{jk} + \partial_j(\psi^4 g_{mj}) - \partial_m(\psi^4 g_{ij})] \\ &= \Gamma^k_{ij} + 2\psi^{-1}[\psi_{;i}\delta^k_j + \psi_{;j}\delta^k_i - \psi_{,m}g^{mk}g_{ij}]\end{aligned} \tag{6.2}$$

By substituting the transformed Christoffel symbols in the definition of the Riemann tensor, contracting twice and simplifying we find that the transformed scalar curvature is indeed $\tilde{R} = \psi^{-4} \left[R - 8\psi^{-4}(\Delta\psi) \right]$ *. □

Proposition 6.2. *The conformally transformed scalar curvature for uniform density perfect fluid static stellar models is:*

$$\tilde{R} = 8\psi^{-5}\psi''\{W_0(V) - W\} \tag{6.3}$$

Where ψ is the conformal factor derived earlier, and W_0 is defined as:

$$W_0(V) := \begin{cases} \frac{8}{3}\pi\rho V(3V_s - V) + \frac{2}{3}\pi\rho(1 - 9V_s^2), & 0 \le V \le V_s \\ \frac{2}{3}\pi\rho(1 - V^2)^4(1 - V_s^2)^{-3}, & V_s \le V \le 1 \end{cases} \tag{6.4}$$

Note that W_0 is zero at $V = V_s$.

Remark. $W_0(V)$ corresponds to W in the spherically symmetric (in particular to the Schwarzschild metric in the exterior region

Proof.

Case I: The Exterior Region

For the exterior region, $\psi'' = 0$, thus $\tilde{R} = 0$.

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*See Eisenhart 1949 for details

Case II: The Interior Region

For the interior, $V_s \leq V \leq 1$, we have the following:

$$\psi = \frac{1}{2}(1+V_s)^{3/2}(1+3V_s-2V)^{-1/2} \tag{6.5}$$
$$\psi' = \frac{1}{2}(1+V_s)^{3/2}(1+3V_s-2V)^{-3/2} \tag{6.6}$$
$$\psi'' = \frac{1}{2}(1+V_s)^{3/2}(1+3V_s-2V)^{-5/2} \tag{6.7}$$
$$= 3\psi(1-3V_s-2V)^{-2} = 3\psi'(1+3V_s-2V)^{-1} \tag{6.8}$$
$$\Delta\psi = \psi''\nabla_i V\nabla^i V + \psi'\nabla_i\nabla^i V = \psi'' W + \psi\Delta V \tag{6.9}$$
$$R = 16\pi\rho \tag{6.10}$$
$$\Delta V = 4\pi V(\rho + 3p) = 4\pi\rho(V_s - 2V) \tag{6.11}$$

The last equality comes from the integrated Euler's equation, $p = \rho V^{-1}(V_s - V)$. By substituting the equations above into (5.1), we have:

$$\begin{aligned}\tilde{R} &= \psi^{-4}[R - 8\psi^{-1}\Delta\psi] \\ &= \psi^{-4}[16\pi\rho - 8\psi^{-1}(\psi'' W + \psi'\Delta V)] \\ &= 8\psi^{-5}[2\pi\rho\psi - 4\pi\rho\psi'(3V_s - 2V) - \psi'' W]\end{aligned} \tag{6.12}$$

The first two terms in the brackets can be simplified to $\psi'' W_0$ by using equation (6.8).

$$\begin{aligned}2\pi\rho\psi - 4\pi\rho\psi'(3V_s - 2V) &= \frac{\psi''}{3}\pi\rho[2(1+3V_s-2V)^2 - 4(3V_s-2V)(1+3V_s-2V)] \\ &= \frac{\psi''}{3}\pi\rho[2 - 18V_s^2 + 24VV_s - 8V^2] \\ &= \psi'' W_0\end{aligned} \tag{6.13}$$

Finally by substituting (6.13) into (6.12) we obtain the desired result. □

Proposition 6.3. *The conformally transformed scalar curvature satisfies $\tilde{R} \geq 0$.*

Proof.

Since, ψ and ψ'' are nonnegative in both regions, and ψ is continuous at $V = V_s$, it is sufficient to show that the sign of $W_0 - W$ is also nonnegative.

The Interior

Consider the Israel-Robinson identity for the interior region:

$$\nabla^i[V^{-1}\nabla_i(W - W_0)] = \frac{1}{4}V^3W^{-1}|R_{ijk}|^2 + \frac{3|\nabla_i(W - W_0)|^2}{4VW} \tag{6.14}$$

The left hand side is an elliptic operator acting on $W - W_0$. By the maximum principle we know that the maximum of $W - W_0$ occurs somewhere on the surface $\{V = V_s\}$, and also that the gradient of $W - W_0$ is outward pointing with respect to the star or the zero vector if $W - W_0 \equiv const$.

The Exterior Region

Consider the Israel-Robinson identity for the exterior region:

$$\nabla_i\left\{V^{-1}\nabla^i\left[\frac{W - W_0}{(1 - V^2)^3}\right]\right\} = \frac{V^{-4}|R_{ijk}|^2 + 3|W_{;i} + 8VW(1 - V^2)^{-1}V_{;i}|^2}{4VW(1 - V^2)^3} \tag{6.15}$$

Since $V < 1$ the right hand side of (5.15) is nonnegative, while the left hand side is an elliptic operator acting on $\frac{W-W_0}{(1-V^2)^3}$. By the maximum principle for elliptic operators we know the maximum of $\frac{W-W_0}{(1-V^2)^3}$ must be attained on the boundary, that is on the level surface $\{V = V_s\}$ or at infinity. Suppose the maximum occurs at infinity, $V = 1$. By the asymptotic fall off condition for V, we know $(1 - V^2) = O(r^{-2}),\ W = O(r^{-4}),\ W_0 = O(r^{-4})$ all converge to 0 as $r \to \infty$, thus,

$$\max_{V_s \le V \le 1}\left\{\frac{W - W_0}{(1 - V^2)^3}\right\} = 0 \tag{6.16}$$

Therefore $W_0 - W \ge 0$ in the exterior, and by continuity also on the surface $\{V = V_s\}$. From the argument for the interior region, $\{V_{min} \le V \le V_s\}$, we know the maximum of $W - W_0$ occurs on the surface $\{V = V_s\}$. Therefore $W_0 - W \ge 0$ in the interior region as well, and hence everywhere.

Now suppose the maximum of $\frac{W-W_0}{(1-V^2)^3}$ occurs on the surface. Let $x_0 \in V_s$ be the point that maximizes $\frac{W-W_0}{(1-V^2)^3}$. In this case we know that $\left.\frac{W-W_0}{(1-V^2)^3}\right|_{V=V_s}$ must be greater than or equal to $\frac{W-W_0}{(1-V^2)^3}$ at infinity, that is :

$$\max_{V_s \leq V \leq 1} \left\{ \frac{W - W_0}{(1 - V^2)^3} \right\} \geq 0 \tag{6.17}$$

Note that since the maximum of $(1 - V^2)^3$ occurs on the surface, and $\frac{W-W_0}{(1-V^2)^3} \geq 0$ at this maximum point, the maximum of $W - W_0$ must also occur on the surface. In particular x_0 maximizes $W - W_0$ with respect to both the exterior and interior, therefore the gradient of $(W - W_0)$ is zero at x_0. The gradient of $\left.\frac{W-W_0}{(1-V^2)^3}\right|_{x_0}$ is easily calculated to be:

$$\begin{aligned} \nabla_i \left[\frac{W - W_0}{(1 - V^2)^3} \right]\Bigg|_{x_0} =& (1 - V^2)^{-3} \; \nabla_i[W - W_0]|_{x_0} \\ &+ (W - W_0)(-3)(1 - V^2)^{-4}(-2V) \; \nabla_i V|_{x_0} \\ =& \left. \frac{6V(W - W_0)}{(1 - V^2)^4} \nabla_i V \right|_{x_0} \end{aligned} \tag{6.18}$$

Since V is greater outside the star than in the interior, the gradient of V is outward pointing with respect to the interior of the star, and therefore the same holds for the gradient of $(W - W_0)(1 - V^2)^{-3}$ at x_0. The Hopf-boundary-point Lemma, however, states that the gradient of this function is either outward pointing with respect to the exterior of the star or $\vec{0}$ if $\frac{W-W_0}{(1-V^2)^3} = const$ in the exterior region. We show in chapter 8 that the gradient of $W - W_0$ is continuous at the surface. Thus by the previous two statements this gradient must be the zero vector, and $\frac{W-W_0}{(1-V^2)^3} \equiv const$ in the exterior. Since $\frac{W-W_0}{(1-V^2)^3} = 0$ at infinity, it is zero everywhere in the exterior and on the level set $\{V = V_s\}$, which implies $W \equiv W_0$ in the interior as well, hence everywhere. Thus

$\tilde{R} \equiv 0$ in this case.

In both cases the sign of $W_0 - W$ is nonnegative, therefore the conformally transformed scalar curvature $\tilde{R}$ is also nonnegative.

Chapter 7

Conformal Flatness implies Rotational Symmetry

One of the key ingredients in proving rotational symmetry for static stellar models is the fact that conformal flatness implies rotational symmetry. This was first proven by Avez (1963) who used Morse theory to prove that any such manifold (static, asymptotically flat, perfect-fluid model) was diffeomorphic to $\mathbb{R}^3$. Künzle (1979) was able to remove one of the assumptions made by Avez, namely, that V has non-degenerate critical points. Since the improvement given by Künzle is relatively minor, we present here the theorem and proof due to Avez.

Theorem 7.1. *Let (N, g) be a Riemannian manifold that satisfies the following conditions:*

1. *(N, g) is a static stellar model with constant density.*
2. *(N, g) is conformally flat.*
3. *(N, g) is complete and asymptotically Euclidean.*
4. *The gravitational potential $V : N^3 \to [V_{min}, 1) \subset \mathbb{R}$ is a proper Morse function (each critical point is isolated with a nondegenerate Hessian and $V \to 1$ at infinity.*

Then (N, g) is diffeomorphic to $(\mathbb{R}^3, \delta_{ij})$, and the pull-back metric $\psi^* g$ under this diffeomorphism is rotationally symmetric on $\mathbb{R}^3$

The proof requires a lemma involving the number of critical points of V which is now given.

Lemma 7.2. *For $r = 0, 1, 2, 3$, let $M_r := \#\{p \in N | dV(p) = 0$ and $Hess(V)|_p$ has index $r\}$, then*

1. *$M_1 = M_2 = 0$ (i.e. no saddle points, each critical point is a local maximum or local minimum).*
2. *$M_3 = 0$ (no local maxima).*
3. *$M_0 = 1$ (only one global minimum).*

Proof.

1. Consider a neighborhood of a critical point $p \in N \cup M_1 \cup M_2$. Since $W = W(V)$ we have:

$$\partial_i W = f(V)\partial_i V, \tag{7.1}$$

for some smooth function f in a neighborhood of p. Using (4.8) and the fact that $dV(p) = 0$, the covariant derivative of the left hand side of (7.1) is:

$$\begin{aligned} \nabla_k \partial_i [\partial^j V \partial_j V]\big|_p &= 2(\nabla_i \partial_j V)|_p \, (\nabla_k \partial^j V)\big|_p + 2\, (\nabla_k \nabla_i \partial_j)|_p \, (\partial^j V)\big|_p \\ &= 2(\nabla_i \partial_j V)|_p \, (\nabla_k \partial^j V)\big|_p \\ &= V_{;ij}\big|_p \, V_{;k}{}^{j}\big|_p \end{aligned} \tag{7.2}$$

where $V_{;ij}$ is the Hessian of V. Taking the covariant derivative of the right hand side of (7.1) we have:

$$\begin{aligned}\nabla_k[f(V)\partial_i V]\big|_p &= f(V)\nabla_k\partial_i V\big|_p + (\nabla_k f(V))(\partial_i V)\big|_p \\ &= f(V)(V_{;ik})\Big|_p \end{aligned} \tag{7.3}$$

Therefore,

$$2V_{;ij}V_{;k}{}^{j}\Big|_p = f(V)V_{;ik}\Big|_p \tag{7.4}$$

Since $V_{;ij}$ is nondegenerate, we know it has an inverse, thus by multiplying both sides of (7.4) by $V^{;kl}$ we have:

$$2V_{;ij}V_{;k}{}^{j}V^{;kl}\Big|_p = f(V)V_{;ik}V^{;kl}\Big|_p = f(V)\delta_i^l\Big|_p$$

Now by multiplying both sides by the metric tensor g_{lm} we have:

$$2V_{;im}\Big|_p = f(V)g_{im}\Big|_p \tag{7.5}$$

Thus $V_{;ij}$ is either positive of negative definite, hence $M_1 = M_2 = 0$ since $V_{;ij}$ is also nondegenerate.

2. Let $p \in N$ be a local maximum. The Hessian $V_{;ij}\Big|_p$ is negative definite, thus the Laplacian $\Delta V\Big|_p < 0$. This contradicts the equation for the Laplacian of V in Theorem 2.5, $\Delta V = 4\pi V(\rho + 3p) \geq 0$. Thus $M_3 = 0$.

3. By the asymptotic fall off conditions, we know $V \to 1$ as $r \to \infty$. So, there exists a V_0 close to 1 such that $D = \{p \in N^3 | V(x) \leq V_0\}$ contains all the critical points of V in its interior, and $\partial D = \{V = V_0\}$ is diffeomorphic to S^2. From Morse

theory * we have the equality:

$$M_0 - M_1 + M_2 - M_3 = \chi(D) \tag{7.6}$$

where $\chi(D)$ is the Euler characteristic. Using the fact that $\chi(D) = 1$ for 3-manifolds D with boundary diffeomorphic to S^2, $\chi(D) = 1$, and Lemma 7.1 and 7.2, we see that $M_3 = 1$.

□

We conclude that V has a unique critical point p_0 such that $V(p_0) = V_{min}$, $V_{min} < V < 1$ on $N \backslash \{p_0\}$. The proof of Theorem 1 will follow from the next lemma.

Lemma 7.3. *Let $V : N^3 \to [V_{min}, 1)$ such that $V \to 1$ at infinity. If V has no critical points other than a global minimum p_0, then the gradient flow of V defines a diffeomorphism from N^3 to $\mathbb{R}^3$, and the level sets $\{V = const | const > V_{min}\}$ are diffeomorphic to two-spheres.*

A proof of this lemma can be found in Hirsch (1991).

Proof of Theorem 7.1

By Lemma 2, let $\Phi : \{V = C_0\} \to S^2_{R_0}$ be a diffeomorphism and $\{\phi_t\}_{t\in\mathbb{R}}$ be the flow of $\frac{\nabla_g V}{|\nabla_g V|_g}$. Choose the parameter t so that

$$V(\phi_t p) \to 1 \ \text{ as } t \ \to +\infty$$
$$V(\phi_t p) \to V_{min} \ \text{ as } t \ \to -\infty$$

Let $V(\bar{p}) = c_0$ and $\Psi : N^3 \to \mathbb{R}^3$ defined by:

$$p = \phi_t(\bar{p}) \mapsto (R_0 e^t, \Phi(\bar{p})) = Y \in \mathbb{R}^3, \quad \text{(Polar coordinates)} \tag{7.7}$$

*See Hirsch 1991 pp. 133-166

where $e^t R_0 = |Y|$ and $\Phi(\bar{p}) \in S^2 \subset \mathbb{R}^3$. Now recall the splitting of the metric from (4.14):

$$\begin{aligned} ds^2 &= h^2 dV^2 + d\hat{s}^2 \\ &= h^2 dV^2 + g_{ab} dx^a dx^b \end{aligned}$$

The second fundamental form on level sets of V is:

$$\Omega_{ab} = \frac{1}{2h} \partial_n g_{ab} \tag{7.8}$$

where n is a unit vector normal to level surfaces of $V = const$. Since $\hat{\Omega}_{ab} \equiv 0$ (Corollary 4.11) we have:

$$\begin{aligned} \partial_n g_{ab} &= 2W\Omega_{ab} \\ &= WK g_{ab} \end{aligned} \tag{7.9}$$

where $W = h^{-2}$ is function of V. It follows from (2.17), (4.16), and the fact that $\rho \equiv const$ that the mean curvature K is also a function of V. The scalar curvature $\hat{R}$ (twice the Gauss curvature) for the metric g_{ab} is:

$$\hat{R} = \frac{2K}{hV} + K^2 |\Omega_{ab}|^2 + 4\pi p \tag{7.10}$$

Hence, the level sets of $V = const$ are 2-spheres with constant Gauss curvature and thus isometric to round 2-spheres in $\mathbb{R}^3$. So Φ can be an isometry:

$$\Phi : \{V = c_0, g_{ab}\} \to \{r = R_0, R_0^2 d\omega_s^2\} \tag{7.11}$$

where $d\omega^2$ is the standard metric on $S^2 \subset \mathbb{R}^3$, and R_0 is fixed by:

$$\frac{2}{R_0^2} = \hat{R}\big|_{V=c_0} \tag{7.12}$$

The linear ODE

$$\frac{dg_{ab}}{dV} = 2W(V)k(V)g_{ab} \tag{7.13}$$

can be integrated from $V = c_0$ (with a given initial condition) to the whole space $\{V > V_{min}\}$ to yield a metric $g_{ab} = g_{ab}(V)$ throughout N. Thus $(\Psi^{-1})^*g$ is constant on spheres $\{r = const\}$ hence rotationally symmetric on $\mathbb{R}^3$.

Chapter 8

Regularity Issues and the Positive Mass Theorem

Certain regularity issues arise from Euler's equation, $\nabla_i p = -V^{-1}(\rho + p)\nabla_i V$, concerning the smoothness of V and g_{ij} at the surface of the star. These issues must be addressed so that the Positive Mass Theorem can be properly applied later. It has already been shown that $p = \rho V^{-1}(V_s - V)$ in the interior region $0 \leq V \leq V_s$. The density function ρ has a jump discontinuity at $V = V_s$ (from its constant interior value to zero), hence the gravitational potential V, and metric g_{ij} can not be C^2 in a neighborhood of the surface (recall that the scalar curvature $R = 16\pi\rho$). We require only that the spacetime metric ${}^{(4)}g_{\alpha\beta}$ be $C^{1,Lip}$ in Σ-Gaussian coordinates where Σ is the evolution of the surface of the star. This is equivalent to condition that the second fundamental form Ω_{ij} be continuous across the surface of the star, as we now show.

Figure 1. depicts depicts the 3-manifold evolution of the star which we call Σ, bounded by $\{V = V_s\}$. The vector $n = \partial_{x_1}$ is a unit space-like vector normal to the surface of the star, and u is a time-like future pointing unit vector. The second

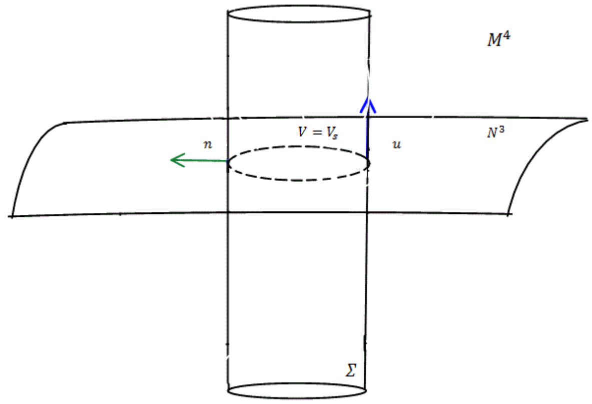

Figure 8.1: Evolution of the star

fundamental form for Σ on the level surface $V = V_s$ is:

$$
\begin{aligned}
-\Omega_{ij}|_{V=V_s} &= A(\partial_{x_i}, \partial_{x_j}) \\
&= -\langle \nabla_{\partial_{x_i}} n, \nabla_{\partial_{x_j}} \rangle \\
&= \langle \nabla_{\partial_{x_i}} \partial_{x_j}, n \rangle = \\
&= g_{ik} \Gamma^k_{ij} \\
&= \frac{1}{2} \frac{\partial g_{ij}}{\partial x^1} \qquad (8.1)
\end{aligned}
$$

This proves that ${}^{(4)}g_{\alpha\beta}$ is C^1 if and only if Ω_{ij} is continuous across Σ, and $C^{1,Lip}$ if and only if Ω_{ij} is Lipschitz continuous. From the equation $\Delta V = 4\pi V(\rho + 3p)$ (elliptic with L^∞ source term) and elliptic regularity, we see that V is C^2 everywhere on N^3.

Lemma 8.1. *The pressure p vanishes continuously on the surface on the surface of the star.*

Proof.

The energy-momentum tensor vanishes outside the star $T_{\mu\nu} = 0$ on the exterior. From the definition of the energy-momentum tensor for perfect fluids we have:

$$\begin{aligned} T(n,n)|_{V=V_s} &= -(p+\rho)\langle u,n\rangle^2 - p\langle n,n\rangle \\ &= -p|_{V=V_s} \end{aligned} \tag{8.2}$$

Therefore the pressure is zero on the exterior of the star. Now consider the ODE:

$$\frac{dp}{dV} = -\frac{\rho + p}{V}$$

Integrating from $V_1 < V_s$ to V_s:

$$\begin{aligned} p(V_s) - p(V_1) &= -\int_{V_1}^{V_s} [\rho + p(\tau)]d\tau \\ &= -\rho(V_s - V_1) - \int_{V_1}^{V_s} p(\tau)d\tau \end{aligned}$$

Letting $V_1 \nearrow V_s$, we see that $p(V)$ vanishes continuously at $V = V_s$.

$\square$

The jump condition (1.3) for the field intensity W can now be formulated using the identity derived earlier (4.20):

$$W^{-1}\nabla_i V \nabla^i W = -2W^{1/2}K + 8\pi V(\rho + 3p) \tag{8.3}$$

This identity implies that $\nabla_i W$ has a discontinuity on the surface of the star since ρ has a jump discontinuity at $V = V_s$. Since ∇V is C^1, and W, p, and K are continuous, the magnitude of the discontinuity of the gradient of W is given by:

$$\lim_{V \to V_s^+} W^{-1}\nabla_i V \nabla^i W - \lim_{V \to V_s} W^{-1}\nabla_i V \nabla^i W = -8\pi V_s \rho \tag{8.4}$$

Similarly in the rotationally symmetric W_0 must satisfy the condition:

$$\lim_{V \to V_s^+} W^{-1} \nabla_i V \nabla^i W_0 - \lim_{V \to V_s} W^{-1} \nabla_i V \nabla^i W_0 = -8\pi V_s \rho \tag{8.5}$$

Together the two conditions imply that the gradient of $W - W_0$ is continuous at the surface (This fact is used in the maximum principle argument in chapter 6).

The Positive Mass Theorem

The Riemannian Positive Mass Theorem (PMT) of Schoen-Yau (1981) and Witten (1981) is a key component of this paper and has not yet been properly stated. It should be noted that it was Masood-ul-Alam (1987) who originally applied the PMT to certain stellar models to show conformal flatness. This approach was adopted by Lindblom (1988) in his proof of Theorem 1.0.2. Before the positive mass theorem is stated, we must first define the ADM mass.

Definition 8.2. *The ADM mass at an end E of a manifold N is:*

$$C(n) m_E = \lim_{r \to \infty} \int_{S(R)} (g_{ij,j} - g_{jj;i}) dS_i \tag{8.6}$$

where $C(n)$ is a positive constant, and $S(R)$ is the coordinate sphere of radius R.

We now state the PMT in the version for non-smooth metrics needed for our main result.

Theorem 8.3. *(Theorem 3.1 in Shi and Tam 2002.) Let (N^3, g) be a an orientable,complete, and noncompact Riemannian manifold such that there is a bounded domain $\Omega \subset N^3$ with a smooth boundary $\partial\Omega$. Assume the metric satisfies the following conditions*

(i) g is smooth on $N^3 \backslash \Omega$ and $\bar{\Omega}$, and is Lipschitz near $\partial\Omega$.

(ii) The mean curvatures at $\partial\Omega$ with respect to the outward normal and with respect to the metrics $g|_{N\backslash\Omega}$ and $g|_{\bar{\Omega}}$ are the same.

(iii) *N has finitely many ends, each of which is asymptotically Euclidean in the following sense: There is a compact set K containing Ω such that $N\backslash K = \cup_{i=1}^{l} E_i$. Each E_i is diffeomorphic to $\mathbb{R}^n \backslash B_{R_i}(0)$ and in the standard coordinates in $\mathbb{R}^n$, the metric g satisfies $g_{ij} = \delta_{ij} + b_{ij}$, with*

$$|b_{ij}| + r|\partial b_{ij}| + r^2|\partial\partial b_{ij}| = O(r^{2-n})$$

where r and ∂ denotes the Euclidean distance and the standard gradient operator in $\mathbb{R}^n$ respectively.

(iv) *The scalar curvature of $N\backslash\partial\Omega$ is nonnegative and in $L^1(N)$.*

Then, $M_E \geq 0$ for any end E of N^3. Moreover, if the ADM mass of one of the ends of N^3 is zero, then N has only one end and N is flat.

We need to verify the hypotheses of Theorem 8.3 for $\tilde{g} = \psi^4 g$ in our construction. Condition (i) follows from hypothesis (3) of Theorem 1.0.2. Condition (ii) is established in equation (8.1). Condition (iii) corresponds to the hypothesis (2) of Theorem 1.0.2 (in our case, there is only one end). Positivity of the scalar curvature was established in chapter 6 (Proposition 6.3), while integrability follows from the fact that $\tilde{R} = 0$ in the exterior region.

Chapter 9

Final Proof and Conclusion

The complete proof of Theorem 1.0.2 is now given.

Proof.

Suppose $(M^4, g_{\mu\nu})$, $M^4 = N^3 \times \mathbb{R}$, is a static stellar model with spatial metric g_{ij} satisfying the hypotheses of Theorem 1.2. We define a new spatial metric $\tilde{g}_{ij}$ conformal to g_{ij}:

$$\tilde{g_{ij}} = \psi^4 g_{ij} \tag{9.1}$$

where ψ is given in (3.56). As shown in proposition 6.3, the scalar curvature $\tilde{R}$ is nonnegative. It follows from the asymptotic conditions for g_{ij} and V of assumption (2), and the fact that $\psi|_{V_s \leq V \leq 1} \sim (Const - 2V)^{-1/2}$ (equation (3.56)) that ψ must satisfy the condition:

$$\psi = 1 - \frac{m}{2r} + \phi \tag{9.2}$$

where $\phi = O(r^2)$. Thus,

$$\begin{aligned}
\tilde{g}_{ij} &= \left(1 - \frac{2m}{r} + \phi\right)^4 \left[\left(1 - \frac{2m}{r}\right)\delta_{ij} + h_{ij}\right] \\
&= \left(1 - \frac{2m}{r} + O(r^{-2})\right) \left[\left(1 - \frac{2m}{r}\right)\delta_{ij} + h_{ij}\right] \\
&= \left(1 - \frac{2m}{r} + \frac{2m}{r} + O(r^{-2})\right)\delta_{ij} + \psi^4 h_{ij} \qquad (9.3)
\end{aligned}$$

This cancellation shows that the asymptotic behavior of the conformal metric is independent of the mass of the star, i.e. $\tilde{g}_{ij} = \delta_{ij} + \tilde{h}_{ij}$, where $\tilde{h}_{ij} = O(r^{-2})$. Furthermore the ADM mass is:

$$m_{ADM} = c_n \lim_{r\to\infty} \int_{S_R} (g_{ij,j} - g_{jj,i}) ds_i \qquad (9.4)$$

where $ds^i = n^i ds \sim O(R^2)$. Since there are no linear term in the parentheses, $g_{ij,j} - g_{jj,i}$ is of order (r^{-3}) on S_r, we have $m_{ADM} = 0$. Applying the PMT to $\tilde{g}_{ij}$, we conclude that $\tilde{g}_{ij}$ is flat, $\tilde{g}_{ij} = \delta_{ij}$, and $g_{ij} = \psi^{-4}\delta_{ij}$ is conformally flat. By Corollary 4.0.11, conformal flatness implies that $W = W(V)$, and level sets of V are totally umbilic. By Theorem 7.1, g is rotationally symmetric (after pull back by diffeomorphism, or in a global coordinate system adapted to level sets of V).
QED

Remark 9.1. *In particular, in the exterior region, g_{ij} is isometric to the Riemannian Schwarzschild metric with mass m.*

Although the result seems almost natural, it took mathematicians and physicists over thirty years to establish a complete proof of Lichnerowicz's conjecture. Since the proof relies heavily on Israel's divergence identity and the PMT, it is safe to assume that without the two, a proof of the conjecture might not have been found for several more years after Lindblom's 1988 article. Both the theorem and the proof are truly remarkable and, comparatively, as beautiful as Isreal's Black Hole Uniqueness Theorem, if not more, since it utilizes newer mathematics.

Bibliography

Avez, A, (1963). Comptes Rendus de l'Académie des Sciences. **257** (3316).

Avez, A. (1964). Le ds^2 de Schwarschild parmi les ds^2 stationares. *Annales de L'institut Henri Poincare.* **1**, pp. 291-300.

Carleman, T. (1919). Über eine isoperimetrische Aufgabe und ihre physikalischen Anwendungen. *Math Zeitschrift.* **3**, pp 1-7.

Künzle, H.P., Savage, J.R. (1979). A global analysis approach to the general relativistic fluid ball problem. *General Relativity and Gravity.* **12** (12), pp. 155-174.

Dalarsson M. & Dalarsson N. (2005). *Tensors, Relativity, and Cosmology.* Burlington, MA: Elsevier Academic Press.

Eisenhart, L. (1949). *Riemannian Geometry.* Princeton, NJ: Princeton University Press.

Hirsch, M. (1991). *Differential Topology.* New York, NY: Springer-Verlag.

Lichnerowicz, A. (1955). Théories relativistes de la gravitation et de lélèctromagnetisme. Masson, Paris.

Lichtenstein, L. (1918). Über eine isoperimetriche Aufgabe der mathematischen Physik. *Math Zeitschrift.* **3**, pp. 8-10.

Lindblom, L. (1980). Some properties of static general relativistic stellar models. *Journal of Mathematical Physics.* **21** (6), pp. 1455-1459.

Lindblom, L. (1988). Static uniform-density stars must be spherical in general relativity. *Journal of Mathematical Physics.* **29** (2), pp. 436-439.

Masood-ul-Alam, A.K.M.(1987). On The spherical symmetry of static perfect fluid spacetimes and the positive-mass theorem. *Classical and Quantum Gravity.* **4**, pp. 625-633.

Masood-ul-Alam, A.K.M. (2007). Proof that static stellar models are spherical. *General Relativity and Gravitation.* **39** (1), pp. 55-85.

Robinson, D.C. (1977). A simple proof of the generalization of Israel's Theorem. *General Relativity and Gravitation.* **8** (9), pp. 695-698.

Schoen R. & Yau S.T. (1981). Proof of the positive mass theorem II. *Communications* *in Mathematical Physics*. **79** (2), pp 231-260.

Shi. L, and Tam L. (2002). Positive Mass Theorem and the Boundary Behavior of Compact Manifolds with Nonnegative Scalar Curvature. *Journal of Differential Geometry*. **62**, pp. 79-125.

Witten, E. (1981). A new proof of the positive energy theorem. *Communications in Mathematical Physics* **80** (3), pp. 381-402.

Vita

Joshua Miguel Galler Brewer was born March 14, 1988, in Cuidad de México, to parents Susan and James Brewer. Raised in a tradition of faith, Josh completed his high school career at the Christian Academy of Knoxville, graduating Magna Cum Laude in 2006. He then went on to complete a degree at King College, where he received both academic and athletic scholarships for cross country and track. In 2010, Josh received a Bachelor of Science degree in mathematics from King, with a double minor in Spanish and physics. In the summer of 2010, he began his graduate work in mathematics at the University of Tennessee, focusing on differential geometry under the guidance of Professor Alex Freire. After two and a half years, he finally received his Master's of Science in December 2012.