

To the Graduate Council:

I am submitting herewith a thesis written by Shane Nicklaus Thompson entitled “Towards Continuous Variable Quantum Computation of Lattice Gauge Theories.” I have examined the final paper copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

George Siopsis, Major Professor

We have read this thesis
and recommend its acceptance:

George Siopsis

Cristian Batista

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Accepted for the Council:

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(Original signatures are on file with official student records.)

Towards Continuous Variable Quantum Computation of Lattice Gauge Theories

A Dissertation Presented for the
Doctor of Philosophy
Degree

The University of Tennessee, Knoxville

Shane Nicklaus Thompson

May 2024

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To my family

Acknowledgements

I would like to thank my thesis advisor Prof. George Siopsis for his guidance and support. I would also like to thank the members of my Ph.D. committee, Prof. Cristian Batista, Prof. Gong Gu and Prof. Thomas Papenbrock, for reviewing this Ph.D. dissertation. I am grateful for the support from and fruitful discussions with my collaborators, Dr. Felix Ringer and Dr. Raghav G. Jha from Jefferson Lab, Chinmay Mishra from the University of Tennessee, and Dr. Raphael Pooser from Oak Ridge National Laboratory (ORNL). I am also grateful for the discussions with and support from Dr. Eugene Dumitrescu, ORNL, as well as the speaking opportunities provided by Prof. Michael Geller from the University of Georgia, Athens.

Additionally, I would like to thank my previous professors from both the University of Tennessee as well as the Ohio State University. From the latter I would especially like to thank Prof. Robert Perry for his support of my undergraduate development. Finally, I thank my family, who have always supported me, believed in me, and encouraged me throughout my long endeavor to become a physicist.

Abstract

The calculation of physical quantities in a relativistic quantum field theory (rQFT) is a computationally demanding task due to the presence of an infinite number of degrees of freedom. This problem carries over to discretized versions of these theories, i.e. lattice field theories, where the Hilbert space size increases exponentially with the size of the lattice. Despite the success of some classical techniques such as Monte Carlo, their applicability is limited. For instance, Monte Carlo is associated with a sign problem that makes real-time evolution and high-fermion-density systems particularly challenging to compute. Quantum computation is a promising avenue thanks to the fact that quantum computers are quantum by nature and can efficiently simulate such systems. However, lattice gauge theories pose additional complications such as the demand for gauge invariance and the infinite Hilbert spaces corresponding to every bosonic degree of freedom. One way to address the latter is through continuous variables quantum computation, whose fundamental units of quantum information are the bosonic “qumodes.” In this dissertation, I will discuss the quantum computation of lattice field theories in both qubit and qumode settings, with a focus on methods applicable to lattice gauge theories and systems with phase structure. I will review our qubit quantum computations of an interacting fermionic model and the lattice Schwinger model, and then present CV quantum computing techniques that can more efficiently address bosonic degrees of freedom in interacting scalar field theories as well as non-linear sigma models. These algorithms make effective use of the “easy” Gaussian operations and photon number measurements, and provide setups for more complicated models such as $SU(2)$ non-Abelian gauge theory.

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Chapter 1

Introduction

Quantum field theory provides the framework for our current understanding of fundamental particle physics, and one of the most essential classes of quantum field theories required to construct the Standard Model are gauge theories. It is imperative then that efficient calculational tools are devised in order for one to be able to compute physical quantities in such theories. The introduction of lattice gauge theory (LGT) by Wilson [1] went a long way toward providing these tools and in particular the means for performing calculations in regimes inaccessible by perturbation theory.

Wilson's LGT uses the path integral formulation of quantum field theory and takes a lattice action with a Euclidean signature, allowing it to be tractable with Monte Carlo (MC) simulations in various cases. Unfortunately, these MC methods tend to exhibit the infamous sign problem in several interesting processes and regimes that we will discuss in a moment. Therefore, a Hamiltonian formulation [2] provides an alternative lattice approach for granting access to these regimes. However, the Hamiltonian formalism contains a Fock space that grows exponentially with system size, prohibiting direct calculations using conventional computational resources and techniques.

As noted by Feynman [3], it is more appropriate to use a system behaving quantum mechanically in order to simulate another quantum system. Quantum computers not only

meet the requirement of such a simulator but naturally operate according to the Hamiltonian formalism.

One important setting where a quantum computation of a gauge theory is expected to prove useful is in particle collisions. While Quantum Chromodynamics (QCD) is asymptotically free and therefore can be analyzed with perturbation theory at high energies, non-perturbative lattice calculations are needed in order to describe the hadronic phases existing in the initial and final states. Current methods used to describe these collisions rely on a number of uncontrolled approximations and are in need of first principles calculations [4, 5, 6]. In particular, quantum simulations will be most useful in describing the real-time dynamics of such processes, which as previously mentioned is generally beyond the reach of Monte Carlo methods due to the sign problem. The sign problem also occurs in cases with high fermion density, such as heavy ion collisions. On top of this, an efficient implementation of real-time dynamics is expected to prove useful in demonstrating the rapid thermalization of strongly interacting matter in heavy ion collisions [7, 8] where entanglement is proposed to play a key role in the dynamics. This then provides an excellent potential application for quantum computation. On a different note, practical simulations of strongly interacting theories (boundary system) could provide insights toward quantum gravitational theories (dual bulk system), via the AdS/CFT correspondence [9].

While several theoretical schematics have been devised for such quantum calculations over the past two decades (see, for example, [10, 11, 12, 13]), the recent emergence and continued improvement of real quantum hardware has resulted in intense efforts at realizing practical computations on existing devices [14, 15, 16, 17, 18]. Due to the limited efficacy of these devices, current hardware has been termed as noisy intermediate scale quantum (NISQ) devices, requiring current quantum computations to somehow overcome this hurdle. Gauge theories present the additional challenge of requiring that quantum computers respect gauge invariance, that is, correctly prepare and process the physical states which belong only to a small subspace of the full Fock space of the theory.

Requiring a Hamiltonian lattice gauge theory as a framework for digital quantum simulations, most approaches aimed at these simulations begin with the Kogut-Susskind lattice Hamiltonian [2]. On the Kogut-Susskind lattice, the fermionic degrees of freedom exist on the lattice sites, while the gauge degrees of freedom occupy the links connecting them, acting as parallel transporters. Implementations based on this formulation typically rely on a basis which diagonalizes the “electric” part of the Hamiltonian. Usually the temporal gauge $A_0 = 0$ is taken and the Gauss law constraint must be imposed on the physical states. Since the electric fields and charges generate gauge transformations, it becomes clear why an electric field basis can be useful. The electric Fock space is constructed by acting on the links with “link” operators an arbitrary number of times. The “magnetic” part of the Hamiltonian, as well as the interactions with fermionic matter, facilitate transitions between these eigenstates.

One of the main challenges being the infinite-dimensional local Hilbert space of the bosonic degrees of freedom (e.g. local electric basis), some approaches sought to eliminate the gauge degrees of freedom via Gauss’s law, if possible. This “pure fermionic” approach has made itself very attractive in both quantum and classical adaptations of the Kogut-Susskind Hamiltonian, due to the transformation of the system in to a pure spin Hamiltonian. It has been applied both in the case of $U(1)$ [19, 20, 14] as well as $SU(2)$ [21, 22] gauge theory. Clearly in a qubit-oriented research environment, in this case there is a natural mapping between the simulator and target system.

Unfortunately, the spin formulation has its drawbacks. Most notably the Hamiltonian contains long-range couplings between spins, which means quantum devices with limited qubit connectivity, such as superconducting qubit systems, require a costly number of operations in order to entangle distant qubits. Trapped ion devices on the other hand do allow for this level of connectivity, through the use of Mølmer Sørensen gates. Thus a trapped ion device is utilized in the quantum computation of Ref. [14]. Also, not all spin states are physical (satisfy the Gauss law constraint). On actual (noisy) quantum hardware, therefore, “leakage” into such unphysical states is inevitable even if the quantum evolution

is intended to be gauge invariant. In this case one must implement postselection in which the results of the computation are projected into the physical subspace.

When a pure fermion formulation is not an option (which is often the case, especially in higher dimensional systems), one must develop a truncation scheme for the remnant gauge fields in order to take advantage of digital hardware. A large array of proposals have been made in this arena, and a looming question is which schemes are the most efficient in the near and far term as well as what physical questions are each of them best suited for. In the electric basis the simplest answer is to limit the number of times link operators can act on the system. This was the case in the quantum simulations of the Schwinger model in Ref. [16] (see also [19] for the similar truncation of the “loop” variable in its DMRG approach) and of $SU(2)$ pure gauge theory in Ref. [17].

Another truncation method is the Quantum Link Model (QLM) formalism [23]. Here the algebra of the target theory is reproduced by the generators of a compact Lie group, at the cost of a loss in unitarity of the link operators. The simplest example is the $U(1)$ case, where all we need are the S_x, S_y, S_z for spin systems. The eigenvalues of the electric field correspond to the eigenvalues of S_z in a $(2l + 1)$ dimensional representation, where l is the spin. This formulation was the starting point for the analog quantum simulation of Ref. [18] for the Schwinger model, which used an optical lattice, as well as some additional proposed analog quantum simulations [24, 25].

As previously mentioned, another important issue to address is maintaining gauge invariance in the states. In the pure fermion formulation, gauge invariance is implemented at the operator level, and is the reason the gauge degree of freedom is “integrated out.” In small enough systems sizes one can also pick out the gauge invariant states by hand and accordingly reduce resource requirements by e.g. restricting to the physical subspace [16] (thus avoiding the “leakage” issue associated with the pure spin formulation) or performing a “gauge variant completion” [17]. In the calculation of Ref. [16], the gauge degrees of freedom are restored for the Schwinger model and the basis was written as a set of product states composed of spin states, representing the fermionic degrees of freedom, as well as the

electric field configuration. The authors employed symmetry arguments to reduce the size of the Hilbert space even further, focusing on appropriate momentum and parity sectors. In regards to performing a quantum computation on a NISQ device, all of these reductions are crucial steps so that shallow quantum circuits may be utilized, in order to minimize machine error. A very small number of qubits were required for the calculations performed.

The techniques of [16, 17] are not expected to scale well with system size, and therefore it is perhaps more worthwhile to implement Gauss’s law site-locally. Meanwhile, global constraints can be enforced during run-time through methods such as the addition of penalty terms. An example of a site-local basis is given in [26, 27] for tensor network calculations, which begins with the quantum link model formalism. While one could go ahead and encode the spin states themselves, there is still the Gauss law constraint to enforce. In order to facilitate a gauge invariant basis it has proven convenient to reformulate the link spin states as either Schwinger bosons [28] or “Rishon” modes [29], each living at a link end. In the former, the $SU(2)$ representation is given by the occupation numbers of two uncoupled harmonic oscillators. In the latter the modes obey fermionic anti-commutation relations as long as they are located at different sites or link ends. In both methods the advantage is the realization of a local constraint where the total number of particles on the site (matter fermions as well as Schwinger bosons / Rishon modes) is conserved. In a quantum computation, the Rishon formulation becomes quite attractive in the sense that the Hamiltonian can be written in terms of local operators that are bilinears of fermions/Rishons. With this one avoids the non-local Jordan-Wigner transform typically found in the representation of fermionic theories. The number of local qubits required is significantly reduced with respect to the original unconstrained basis, which is critical in a NISQ setting.

In the non-Abelian case, a site-local gauge invariant basis was developed in [30, 31] for $SU(2)$ gauge theory. The work of Ref. [31] incorporates fermionic matter and is known as the “loop-string-hadron” (LSH) formulation. One works purely with gauge invariant degrees of freedom ($SU(2)$ singlets), and the Fock space is constructed by acting with these singlets

site-locally on a vacuum state. Its generalization to higher dimensions employs a point-splitting technique, which was used in [32] for pure gauge theory. This technique is used to avoid relying on a highly-constrained loop basis (Mandelstam constraints).

The basis which diagonalizes the electric Hamiltonian is useful for strong coupling calculations. However at weak coupling the wavefunctions typically broaden within this representation to the point where any truncation of this basis demanded by resource limitations will not accurately capture the physics. The need to explore the weak coupling regime arises most prominently when it corresponds to the continuum limit, which is the case in $U(1)$ theory in one and two spatial dimensions as well as the (asymptotically free) $SU(2)$ theory in three spatial dimensions [33]. One approach to weak coupling is the use of a group element basis which diagonalizes both the magnetic Hamiltonian as well as the interaction term between the gauge and matter fields. A notable feature of this approach is its close relation to regular lattice gauge theory which employs path integrals. The Hamiltonian itself can be recovered using the transfer matrix formalism [34]. Discrete subgroups of the continuous gauge groups provide a structure for truncation, but this typically introduces an unphysical “freezing” phase transition toward weak coupling that must be either avoided or removed through additional terms in the action [35, 36, 37]. A “fuzzy” approach has also been introduced which construct models that differ from the lattice theories but agree in the continuum limit [38].

An approach which appears useful for calculations in more than one spatial dimension is investigated in Refs. [33, 39, 40, 41]. In this construction, the gauge degrees of freedom are given in the electric representation by the eigenstates of local “rotor” operators. The rotors describe the transverse electric fields and are canonically conjugate with the magnetic field. In two spatial dimensions, these conjugates are scalar valued, yielding a “dual lattice” formulation where the gauge degrees of freedom are moved from the links to the plaquette centers. The remaining longitudinal degrees of freedom are neutralized by Gauss’s law and are expressed in terms of the charge distribution, similar to the pure fermion formulation of the Schwinger model.

A Fourier transform yields a basis which diagonalizes the magnetic term, which is reminiscent of the gauge group basis discussed above. In Ref. [40], this basis is digitized by first approximating the $U(1)$ gauge group with the discrete $\mathbb{Z}(2L+1)$ subgroup and then truncating the basis based on resource availability. Proposals for quantum computation using trapped ion devices, which address both the electric and magnetic representations, can be found in Ref. [42].

The quality of calculations in the magnetic basis is determined by the delicate interplay between the resolution L and the truncation level, and one sees poor results at strong coupling in addition to the aforementioned freezing effect at weak coupling. The freezing effect is due to the unphysical gap in the magnetic Hamiltonian introduced by discretization. Ref. [41] gets around this issue by directly imposing a cutoff on the magnetic field strength before digitization, instead of truncating at a given resolution L . The optimal cutoff employed (depending on coupling strength and truncation level) is determined from analytical considerations. It is shown that the approach is effective at all values of gauge coupling. Recently [43], a “mixed basis” construction was introduced which also seeks to satisfy a large range of couplings. This is done with the help of a maximal tree approach [44].

In all of the above discussion, the inclusion of gauge degrees of freedom in the target theory of the quantum simulation required some sort of truncation scheme in order to realize a computation on a universal digital quantum computer. This situation can be traced back to just a simple scalar field $\phi(\mathbf{x})$ which is continuous-valued at each point $\mathbf{x}$ in space. This means that even on a finite lattice the field can take on an infinite range of values. Digitization requires one to choose a finite subset of this range, which typically involves defining upper and lower bounds on the field values as well as a discretization on the values in between [45]. However, recent advances in fields such as quantum optics has provided an approach to quantum computation which allows one to eliminate the need for a field digitization, thereby achieving what resembles an “analog” form of quantum computation. Known as Continuous Variables (CV) quantum computation [46, 47, 48, 49], this possibility

immediately eliminates the need for truncation in lattice gauge theories, at least in principal. Instead of qubits/qudits, this method relies on fundamental units of quantum information called “qumodes” which obey boson statistics. Each qumode represents a complete harmonic oscillator system and therefore contains infinite degrees of freedom. In quantum optics, these qumodes are modes of the electromagnetic field, with the continuous variables corresponding to the electric field associated with the light beam. It is also useful to realize a discrete set of states, such as boson/photon number, given the existence of robust photon counting devices [50, 49]. In Chapter 4, we will briefly discuss other platforms for continuous variables such as Bose-Einstein condensates [51], ion trap devices [52], and microwave cavities [53].

It turns out that one can employ qumodes for universal quantum computation [46]. In the qubit case, universality is achieved by combining what is known as the Clifford group, which contains gates such as the Hadamard gate and CNOT gate, with a non-Clifford operation such as the T gate. According to the Gottesman-Knill theorem [54], a quantum algorithm implementing only Clifford operations is simulable on a classical computer, while a quantum advantage (i.e., exponential scaling of complexity on a classical computer) is possible when non-Clifford operations are implemented. There is an equivalent for qumodes: a CV quantum algorithm requiring only Gaussian operations is classically simulable. One can see this by noting that such operations act as an affine transformation on the space of quadrature field operators (phase space) [55]. Put simply, these operations map polynomials of quadrature operators to polynomials of the same degree or lower.

To get universal quantum computation with CV, we just need one non-Gaussian gate, a “non-linear” operation that can increase the degree of the resulting polynomial. As will become important later on, utilizing such an operation in experiment is not straightforward and may require a measurement based approach [56, 57]. A collection of notable and useful CV gates is given in Appendix B. The combination of linear operations on the phase space followed by a non-linear operation reveals a natural realization of quantum neural networks. Some fundamental machine learning protocols such as Principal Component Analysis (PCA) have been analyzed in a hybrid setting [58, 59].

Additionally, the flexibility of an enlarged hilbert space offered by qumodes is known to be a candidate for quantum error correction (QEC). In particular, one can embed a logical qubit inside the space of a qumode, with the additional degrees of freedom providing the necessary redundancy for error correction, replacing the need for a very large number of qubits to realize a single logical qubit [56]. These logical qubits are known as Gottesman-Kitaev-Preskill (GKP) states.

On the other hand, it is natural to use continuous variables to match the infinite hilbert spaces associated with bosonic degrees of freedom, or to simulate them directly. Some work has been done on this already. A CV algorithm was proposed in Ref. [57] for computing scattering amplitudes in a scalar field theory. This was extended to scalar QED in [60]. A quantum imaginary time evolution (QITE) algorithm for scalar field theory was described in Ref. [61], where it was found that a Gaussian Ansatz was effective for computing energies. We find from these works that continuous variables offers a promising alternative to qubits in simulating bosonic degrees of freedom. Thus one need not rely on a carefully implemented truncation scheme which as noted above can even lead to problems in reproducing the target theory.

The subject of this dissertation is the development of variational quantum algorithms for the quantum computation of relativistic quantum field theories which share certain features with lattice gauge theories. The discussion proceeds as follows. I begin in Chapter 2 by reviewing my work on the quantum computation of an interacting fermionic model without gauge symmetry. The discussion follows Ref. [62], which was done in collaboration with C. Mishra, R. Pooser, and G. Siopsis. The mass gap was computed at various coupling strengths on the device IBM Q Boeblingen. This discussion provides an opportunity to introduce several techniques that will be used in later chapters. In Chapter 3, I introduce the lattice Schwinger model, a popular “starting point” for the quantum simulation of lattice gauge theories. I will discuss my work on the Schwinger model from Ref. [63], which was done in collaboration with G. Siopsis. In contrast with previous quantum calculations, a momentum space approach was taken. This approach has a unique set of properties, including a careful

treatment of the zero mode of the gauge field. The work also focuses on a topological “theta” parameter which, when set to $\theta = \pi$, gives the Schwinger model a phase transition. The critical point of this transition was calculated using a quantum computation. Calculations were performed on the device IBM Q Lima.

In Chapter 4, after introducing details regarding the progress and methods of CV quantum computing, I propose a CV approach to computing the phase transition in the ϕ^4 scalar field theory. This discussion follows Ref. [64], also worked on in collaboration with G. Siopsis. The calculation is based on a rather simple Gaussian Ansatz, allowing us to compute a Gaussian Effective Potential (GEP). Our approach is compared with a qubit quantum computation on IBM Q Kolkata and IBM Q Mumbai. In Chapter 5 I introduce CV algorithms for computing the $O(3)$ non-linear sigma model. These algorithms were proposed in Ref. [65], with collaborators R. G. Jha, F. Ringer, and G. Siopsis. The $O(3)$ model is depicted as the limit of a three-component scalar field theory where the unit vector constraint is realized through squeezing operations. As will be discussed, the relationship between sigma models and gauge theories encourages further application of the methods developed here toward lattice gauge theories. I conclude in Chapter 6 and introduce several interesting directions for future work, with the primary vision being the simulation of realistic gauge theories.

Chapter 2

Massive Thirring Model

Before diving into lattice theories containing the complications associated with bosonic degrees of freedom, we shall first review the computation in Ref. [62] of the energy levels of an interacting fermionic theory resembling the massive Thirring Model. This work was carried out in collaboration with Chinmay Mishra, Raphael Pooser, and George Siopsis.

C.M. initiated this project idea and worked on lattice discretization and other preliminary materials. He applied the Jordan-Wigner map (Ref. [66]) and developed Python code for the quantum circuits and variational quantum algorithm described in this paper. R.P. assisted in the interfacing with quantum hardware as well as with error mitigation techniques. G.S. guided the project, made important theoretical observations, and introduced some of the quantum circuits used in the paper. He also proposed the process described in this Chapter of expressing some trial states as sums of multiple quantum circuits to decrease quantum noise. I performed classical lattice calculations (both exact diagonalization and variational), pre-processed the Hamiltonian to extract only the terms relevant for quantum computation, formulated some of the quantum circuits used in this paper, contributed to writing code for error mitigation, submitted quantum circuit jobs to IBM Q Boeblingen (which C.M. assisted me with, having ran preliminary jobs himself), and analyzed the results of the computation. C.M, G.S. and I all contributed to the writing of the article, however I credit C.M. and G.S. for being the more primary contributors in Sections 2.1 and 2.2 below.

2.1 The Model

The Thirring model is an exactly solvable model of interacting fermions in $1 + 1$ spacetime dimensions. The Lagrangian for the massive Thirring model is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m_0)\psi + \frac{g^2}{4}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi, \quad (2.1)$$

where $\mu = 0, 1$ and $\bar{\psi} = \psi^\dagger\gamma^0$. For the Dirac matrices, we use the representation $\gamma^0 = \sigma^2$, $\gamma^1 = -i\sigma^1$. The system contains two parameters, m_0 , which is the bare mass, and the dimensionless coupling constant g^2 .

The components of the fermionic field ψ obey equal-time anti-commutation relations,

$$[\psi_\alpha^\dagger(x), \psi_\beta(x')]_+ = \delta_{\alpha\beta}\delta(x - x') \quad (2.2)$$

with all other anti-commutators vanishing, where $\alpha, \beta = 1, 2$ (ψ consists of two components).

It is convenient to expand ψ as

$$\psi(x) = \int \frac{dk}{2\pi\sqrt{2\omega(k)}} [b(k)u(k)e^{ikx} + c^\dagger(k)v(k)e^{-ikx}] \quad (2.3)$$

where $u(k)e^{ikx}$ ($v(k)e^{ikx}$) are plane-wave positive (negative) energy solutions of the Dirac equation $(i\gamma^\mu\partial_\mu - m_0)\psi = 0$. The operators $b^\dagger(k)$ ($c^\dagger(k)$) create a fermion (anti-fermion) of momentum k and energy $\pm\omega(k)$, where $\omega(k) = \sqrt{k^2 + m_0^2}$. The non-vanishing anti-commutation relations for the fermionic modes are

$$[b^\dagger(k), b(k')]_+ = [c^\dagger(k), c(k')]_+ = 2\pi\delta(k - k') \quad (2.4)$$

2.2 Lattice

To study the system on a digital computer, we discretize space into N sites, so that the spatial coordinate $x \rightarrow x_n \equiv na$, where $n \in \mathbf{Z}_N$ is the site index and a is the lattice spacing.

Having a finite lattice spacing amounts to introducing an ultraviolet cutoff $\Lambda \sim \mathcal{O}(1/a)$ in the system. The other parameter in the system that has non-vanishing dimension (the bare mass m_0) can be replaced by the dimensionless combination $m_0 a$. To simplify notation, we set the lattice spacing $a = 1$, and denote the spatial sites by $x \in \mathbf{Z}_N$. With this choice of units, the continuum limit is recovered as $N \rightarrow \infty$ and $m_0 \rightarrow 0$ (i.e., $m_0 a \rightarrow 0$).

The discretization implies a modification of the dependence of energy on momentum, which would otherwise be given by Einstein's dispersion relation. As is well-known, unlike in the case of scalar fields, this modification leads to the problem of fermion doubling in which infinite-momentum, but low energy, modes remain as $a \rightarrow 0$ [67]. This can be resolved by incorporating the Wilson term into the Hamiltonian:

$$\mathcal{H}_{\text{Wilson}} = -\frac{\xi}{2} \bar{\psi}_x \nabla^2 \psi_x \quad (2.5)$$

where the Wilson parameter ξ is arbitrary ($0 < \xi < 1$), and ∇ denotes finite difference, $\nabla \psi_x = \psi_{x+1} - \psi_x$. The Wilson term introduces a gap that prevents the creation of unphysical modes, and vanishes in the continuum limit.

Assuming periodic boundary conditions (PBC's), the momentum k is also discretized taking values on the dual lattice, $\frac{2\pi}{N}k$, where $k \in \mathbf{Z}_N$ is the momentum quantum number.

The dispersion relation on the lattice is given by

$$\omega_k = \sqrt{\tilde{m}_k^2 + \sin^2 \frac{2\pi k}{N}}, \quad \tilde{m}_k = m_0 + 2\xi \sin^2 \frac{\pi k}{N}. \quad (2.6)$$

The expansion of the fermionic field (2.3) is also modified to

$$\psi_x = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \frac{1}{\sqrt{2\omega_k}} \left[b_k u_k e^{2\pi i k x / N} + c_k^\dagger v_k e^{-2\pi i k x / N} \right] \quad (2.7)$$

where

$$u_k = \begin{pmatrix} \sqrt{\omega_k - \sin \frac{2\pi k}{N}} \\ i \sqrt{\omega_k + \sin \frac{2\pi k}{N}} \end{pmatrix}, \quad v_k = u_k^* \quad (2.8)$$

The anti-commutation relations on the lattice read

$$[b_k^\dagger, b_{k'}]_+ = [c_k^\dagger, c_{k'}]_+ = \delta_{kk'} \quad (2.9)$$

where $b_k^\dagger$ ($c_k^\dagger$) creates a fermion (anti-fermion) with momentum quantum number k (i.e., momentum $2\pi k/N$).

Expressing the Hamiltonian in terms of fermionic modes, we obtain $H = H_0 + H_{\text{int}}$ where the non-interacting part is diagonalized,

$$H_0 = \sum_{k=0}^{N-1} \omega_k \left[b_k^\dagger b_k + c_k^\dagger c_k \right] \quad (2.10)$$

An important physical quantity is the conserved total fermion charge,

$$Q_f = \sum_{k=0}^{N-1} \left(b_k^\dagger b_k - c_k^\dagger c_k \right) \quad (2.11)$$

which counts the total number of fermions minus the number of antifermions in a state. In the absence of interactions ($g = 0$), the common eigenstates can be specified by the set of occupation numbers $\{n_i\}$, $i = 0, 1, \dots, 2N-1$, where $n_i = 0, 1$. We will order the occupation numbers so that an even (odd) site refers to a fermion (anti-fermion), i.e., $n_{2k} = 1$ ($n_{2k+1} = 1$) means that the state contains a fermion (anti-fermion) of momentum quantum number k . In terms of modes,

$$|\{n_i\}\rangle = (b_0^\dagger)^{n_0} (c_0^\dagger)^{n_1} \dots (b_{N-1}^\dagger)^{n_{2N-2}} (c_{N-1}^\dagger)^{n_{2N-1}} |\Omega\rangle \quad (2.12)$$

where $|\Omega\rangle = |00 \dots 0\rangle$ is the vacuum state annihilated by all modes. The states $|\{n_i\}\rangle$ are not energy eigenstates for $g \neq 0$. However conservation of Q_f means fermions and anti-fermions are always created in pairs.

2.3 Exact Diagonalization

In order to diagonalize the Hamiltonian numerically, one needs to employ a representation of the fermionic modes b_k and anti-fermionic modes c_k . We will use the Jordan-Wigner transform [68], in which different modes are represented by Pauli matrices X, Y, Z . The Hilbert space is mapped onto a tensor product of $2N$ two-dimensional spaces, each spanned by the computational basis states, $\{|0\rangle, |1\rangle\}$, which are eigenstates of Z with corresponding eigenvalues ± 1 . We map $b_0 \mapsto X_0^-$, and

$$b_k \mapsto \prod_{i=0}^{2k-1} (-Z_i) X_{2k}^- \quad (k > 0), \quad c_k \mapsto \prod_{i=0}^{2k} (-Z_i) X_{2k+1}^- \quad (2.13)$$

where $X^\pm = \frac{1}{2}(X \pm iY)$, and indices denote the position of the two-dimensional space in the tensor product on which the Pauli matrix acts.

The basis states (4.28) are naturally mapped onto the computational basis states $|n_0 n_1 n_2 \dots n_{2N-2} n_{2N-1}\rangle$ where each entry can be 0 or 1, and n_{2k} (n_{2k+1}) denotes the number of fermions (anti-fermions) of momentum quantum number k .

The resulting Hilbert space is 2^{2N} -dimensional and the Hamiltonian is mapped onto a $2^{2N} \times 2^{2N}$ matrix. The ground and first excited state energies E_0 and E_1 as well as their difference, the mass gap m , can be obtained from exact diagonalization of the Hamiltonian. They are displayed in Figs. 2.6 and 2.7 in Section 2.6 for $N = 3$ and for the choice of parameters $m_0 = 10$, $\xi = 0.7$. It is clear from these figures that the model exhibits some phase structure. Indeed for any bare mass m_0 the gap m will vanish at some value of g^2 . For large m_0 the critical value of the coupling is given by $g_{\text{crit}}^2 \approx 2m_0$.

In our region of interest, $g^2 < g_{\text{crit}}^2$, we expect good agreement between perturbative and exact results for small values of the coupling constant g^2 and also at large bare mass parameter m_0 . This leaves only a small region where perturbation theory diverges from exact results. It turns out that, even in that region, the discrepancy is small. This makes perturbation theory a good guide towards quantum computation to which we turn our attention next.

2.4 Quantum Computation and VQE

In order to compute the ground and first excited state energies, we apply the Variational Quantum Eigensolver Algorithm (VQE) [69]. First, one introduces a parameterized trial wavefunction $|\psi_{\text{trial}}\rangle = U(\boldsymbol{\theta})|00\dots 0\rangle$, where U is a unitary that depends on a set of variational parameters $\boldsymbol{\theta}$. The energy can then be estimated using $\langle\psi_{\text{trial}}|H|\psi_{\text{trial}}\rangle = \langle 0|U^\dagger(\boldsymbol{\theta})HU(\boldsymbol{\theta})|0\rangle$. For quantum computation, one expands the Hamiltonian $H = \sum_j c_j H_j$, where each H_j is a string of Pauli matrices, which is unitary. Each H_j is diagonalized using single-qubit operations in the Pauli Z basis, $H_j = V_j^\dagger h_j V_j$, where h_j is a string of Pauli Z matrices. Final quantum circuits are formed by acting with $V_j U(\boldsymbol{\theta})$ on $|0\rangle = |00\dots 0\rangle$.

Since some Pauli strings bit-wise commute, they can be diagonalized by the same quantum circuit $V_j U|0\rangle$. Once all necessary circuits are written down, we prepare them on a quantum computer and measure all qubits in the computational basis. The expectation value of each string of Pauli Z matrices in an n -qubit circuit is given by

$$\langle Z_0^{x_0} Z_1^{x_1} \dots Z_n^{x_n} \rangle = \sum_{i_0, i_1, \dots, i_n} \left(P_{i_0, i_1, \dots, i_n} \prod_{j|x_j=1} (-1)^{i_j} \right) \quad (2.14)$$

where each of the x_i is either 0 or 1, corresponding to the absence or presence of Z_i in the string, respectively, and $P_{i_0, i_1, \dots, i_n}$ is the probability obtained from measurement counts for the computational basis state $|i_0, i_1, \dots, i_n\rangle$. Now equipped with all expectation values $\langle 0|U^\dagger V_j^\dagger h_j V_j U|0\rangle$, the expectation value of the full Hamiltonian is then given by

$$\sum_j c_j \langle 0|U^\dagger V_j^\dagger h_j V_j U|0\rangle \quad (2.15)$$

Here, we introduce slightly more general Ansätze by writing the trial function as a superposition of trial functions that need two (or more) unitaries to specify them. This has the advantage of reducing the number of CNOT gates, which are significant sources of errors in NISQ quantum computers, at the expense of increasing the classical part of the

computation. Another feature of this approach is that the size of the register required for a quantum computation is reduced through such Ansätze. We focus on the $N = 3$ case.

2.4.1 Ground State

The ground state lives in the vanishing charge sector $Q_f = 0$. This fact, along with the requirement of charge conjugation symmetry as well as insights from perturbation theory, inform the Ansatz for the ground state

$$|\psi_{\text{trial}}\rangle = a |000000\rangle + \frac{b}{\sqrt{2}} [|000110\rangle + |001001\rangle] , \quad (2.16)$$

where $a, b \in \mathbb{R}$, and $a^2 + b^2 = 1$. In the quantum computation, we work with two different but equivalent expressions for the ground state following the above Ansatz.

The trial wavefunction can be implemented through the Ansatz requiring a single CNOT gate,

$$|\psi_{\text{trial}}\rangle = \sin \theta \, U_1(\phi) |000000\rangle + \cos \theta \, U_2 |000000\rangle \quad (2.17)$$

where θ and ϕ are variational parameters, and U_1 and U_2 are implemented with the quantum circuits displayed in Fig. 2.1. In the Figure, $R_Y(\phi)$ implements a spinor a rotation around the y -axis of angle $\frac{\phi}{2}$. Note that these circuits are independent and implemented separately, and that each requires only two qubits for this particular Ansatz.

For quantum computation, this is nontraditional in the sense that the two-term Ansatz (2.17) is not given by a single unitary transformation acting on $|0\rangle$, as has been standard practice in quantum chemistry [70]. Instead, the quantum computation involves expectation values of the form

$$\langle 0 | U_m^\dagger H_j U_n | 0 \rangle \quad (2.18)$$

where the H_j are strings of Pauli matrices contributing to the Hamiltonian, and $m, n = 1, 2$.

If $m = n$, we work in the standard way to diagonalize $H_j = V_j^\dagger h_j V_j$, where h_j is diagonal in the computational basis (a string of Pauli Z matrices).

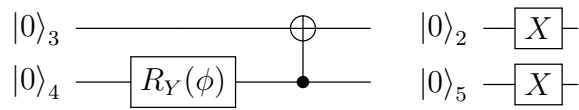


Figure 2.1: Quantum circuit Ansatz for the ground state with one CNOT.

This is achieved by using

$$X = \mathbf{H}Z\mathbf{H}, \quad Y = \mathcal{Y}^\dagger Z\mathcal{Y} \quad (2.19)$$

where $\mathbf{H}$ is the Hadamard matrix and $\mathcal{Y} = e^{i\frac{\pi}{4}X}$ [71]. Similar rotations can be used in the case $m \neq n$, except that in this case, we need to diagonalize the product $U_m^\dagger H_l U_n$, instead of just H_l .

The classical part of the algorithm can be significantly reduced by adopting an Ansatz which consists of a single term, e.g., using

$$|\psi_{\text{trial}}\rangle = U(\theta, \phi) |000000\rangle \quad (2.20)$$

where the unitary U is implemented by the circuit in Fig. 2.2.

This reduces the diagonalization process to the simpler standard method (e.g. $H_j = V_j^\dagger h_j V_j$). The price one pays for the simplification, in addition to the expansion of the register to four qubits, is the increase in the number of CNOT gates from one for the two-term Ansatz (2.17) to three for the single-term Ansatz (2.20), which increases two-qubit errors in NISQ quantum computers.

2.4.2 First Excited State

The first excited state corresponding to a fermion excitation receives contributions from computational basis states with fermion number +1. This guarantees that any Ansatz will be orthogonal to the ground state. From perturbation theory, we are led to consider the Ansatz

$$|\psi_{\text{trial}}\rangle = a |100000\rangle + \frac{b}{\sqrt{2}} [|100110\rangle + |101001\rangle] + c |011010\rangle, \quad (2.21)$$

where $a, b, c \in \mathbb{R}$ and $a^2 + b^2 + c^2 = 1$. We take a two-term Ansatz given by

$$|\psi_{\text{trial}}\rangle = \sin \chi \, U'_1(\theta, \phi) |000000\rangle + \cos \chi \, U'_2 |000000\rangle \quad (2.22)$$

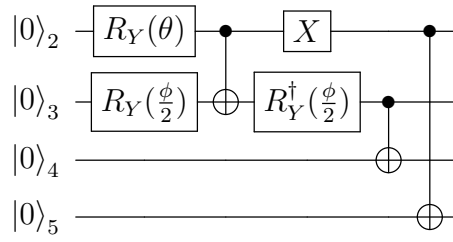


Figure 2.2: Quantum circuit Ansatz for the ground state with three CNOTs.

where the unitaries U'_1 and U'_2 are implemented by the left and right circuits of Fig. 2.3. The single-term Ansatz contains 8 CNOT gates and is implemented with the quantum circuit in Fig. 2.4 in terms of the same three variational parameters. Both of the Ansätze require six qubits, but the latter contains significantly more CNOT gates.

2.5 Error Mitigation

Raw hardware runs contain large errors even in the absence of algorithmic error, and so we employed two forms of error mitigation. Read-out (RO) error, also known as measurement error, corresponds to the error in the measurement process wherein there is a finite probability for the occurrence of a classical bit flip. One can attempt to mitigate this by running calibration experiments where a computational basis state is prepared and then measured. The probabilities of this state being read-out as other computational basis states are contained in a calibration matrix which can be inverted and then applied to measurement results to obtain approximate values for the true probabilities.

We will take this up later, but it is easy to see that this approach scales exponentially with the number of qubits being utilized. Instead, by assuming that RO errors are limited in their correlations, one can reduce the RO error mitigation overhead to be polynomial in the number of qubits. This is explored in Refs. [72, 73, 74]. Let $p(0|1)$ and $p(1|0)$ denote bit flip probabilities, where $p(0|1)$ is the probability that a true measurement outcome of 1 is read-out as a 0 instead, and similarly for $p(1|0)$. Following [73], we define for qubit i the symmetric and anti-symmetric likelihoods $p_i^\pm = p_i(0|1) \pm p_i(1|0)$. The RO error corrected expectation value of a string of Pauli-Z operators $\langle Z_0^{x_0} Z_1^{x_1} \dots Z_5^{x_5} \rangle$ (where again each of the x_i is either 0 or 1) is then given by a modified version of Eq. (2.14),

$$\sum_{i_0, i_1, \dots, i_5} \left\{ P_{i_0, i_1, \dots, i_5} \prod_{j|x_j=1} \left[\frac{(-1)^{i_j} - p_j^-}{1 - p_j^+} \right] \right\} \quad (2.23)$$

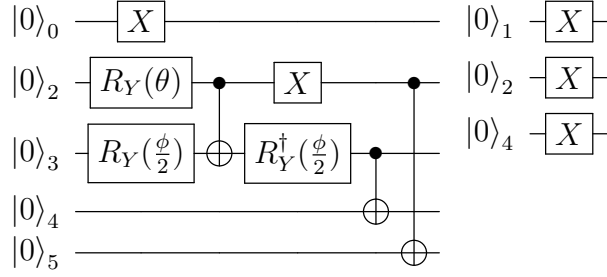


Figure 2.3: Quantum circuit Ansatz for the first excited state with three CNOTs.

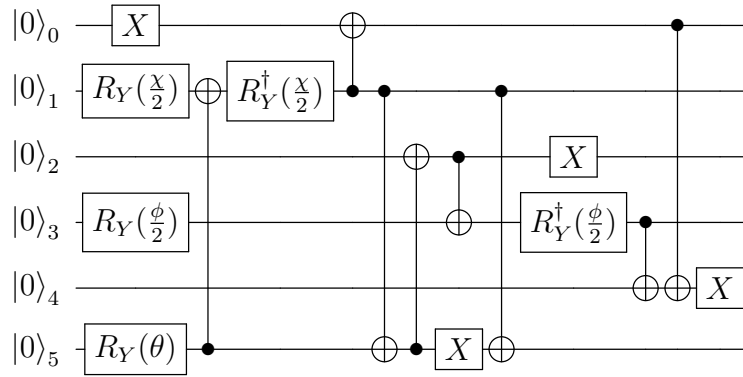


Figure 2.4: Quantum circuit Ansatz for the first excited state with eight CNOTs.

We note that this error mitigation scheme assumes local, rather than correlated, RO errors, and so requires a number of “calibration experiments” that scales only polynomially in the number of qubits.

The RO corrected data thus obtained still suffers from significant CNOT errors. In order to compensate for these errors, following [75, 76] we employ a zero noise extrapolation (ZNE) scheme wherein we include additional pairs of CNOT gates for every CNOT gate in our Ansatz. Note that in the absence of noise these are just applications of an identity operation. We take five such runs so that we go up to 9 CNOT gates per CNOT gate in the original Ansatz. Applying a linear fit to these energies, we obtain the approximate zero-noise value of the energy by extrapolating to the 0-CNOT-gate case. Extrapolation results for this model are shown for the single-term Ansätze in Fig. 2.5, where it is clear that extrapolation is more effective with fewer CNOTs.

2.6 Results

The results obtained on the IBM Boeblingen quantum device for each of the four Ansätze discussed in Section 2.4 are shown in Figs. 2.6 and 2.7. The “raw” data correspond to data obtained before CNOT extrapolation was applied but after RO error was accounted for. The error bars are statistical error bars obtained after several runs of the minimization on the device. In order to reduce computational overhead, we expressed each of the variational parameters in each circuit in terms of θ only, based on approximations derived from noiseless simulations.

Unsurprisingly, we saw the best agreement with exact results for the case where the Ansätze had two terms, i.e., the ones which contained fewer CNOT gates. For the ground state, the single-term Ansatz did not introduce significant errors since only a couple of CNOT gates were added. However, switching to a single-term Ansatz for the first excited state introduced significant errors as eight CNOT gates were required. In this case, linear extrapolation is not sufficient to match the accuracy obtained using the two-term first-excited-state Ansatz.

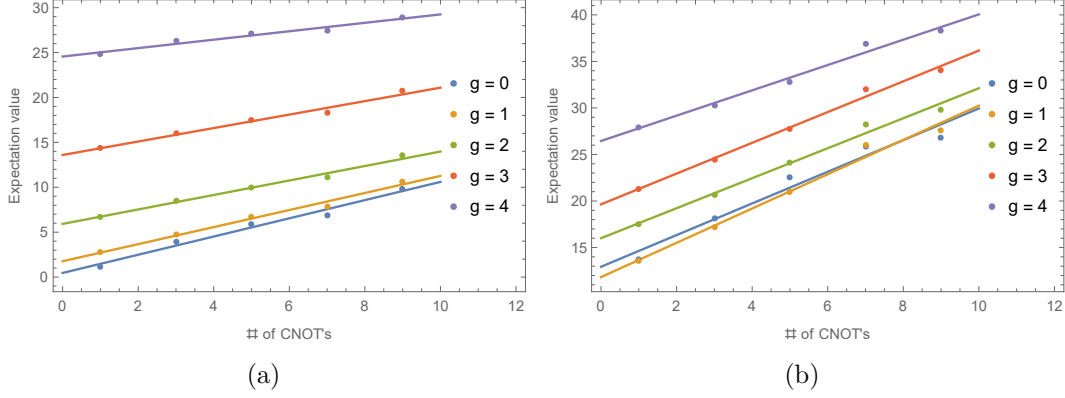


Figure 2.5: CNOT extrapolation for the minimized energy expectation for the single-term Ansätze. (a) corresponds to ground state energy E_0 and (b) to the first excited state energy E_1 . The energy expectation is plotted as a function of the number of CNOT gates per CNOT in the original Ansatz, for different values of the interaction strength g . Here $m_0 = 10$ and $\xi = 0.7$.

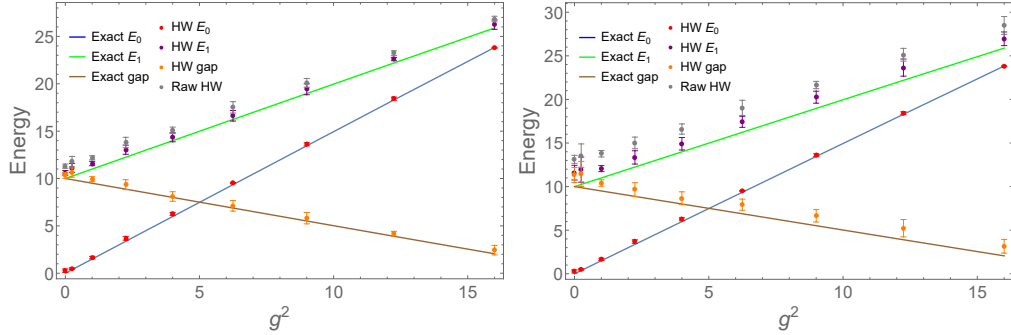


Figure 2.6: E vs g^2 for the two-term ground state Ansatz. E is the minimized energy functional for a given Ansatz. Here E_0 is the ground-state energy and E_1 is the first-excited-state energy from the two-term (left) and single-term Ansatz (right). We do not give “raw” data for the E_0 results as it overlaps with CNOT-extrapolated data. The mass gap is $m = E_1 - E_0$. “HW” denotes Boeblingen hardware. The parameters are $m_0 = 10$ and $\xi = 0.7$.

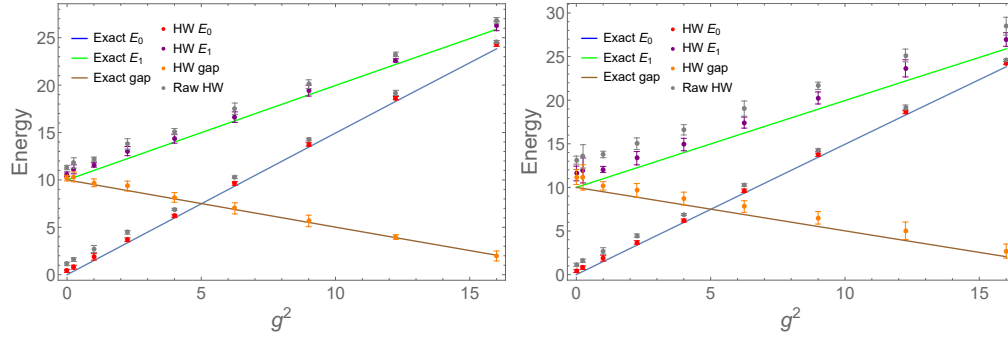


Figure 2.7: E vs g^2 for the single-term ground state Ansatz. E is the minimized energy functional for a given Ansatz. Here E_0 is the ground-state energy and E_1 is the first-excited-state energy from the two-term (left) and 1-term Ansatz (right). The mass gap is $m = E_1 - E_0$. “HW” denotes Boeblingen hardware. The parameters are $m_0 = 10$ and $\xi = 0.7$.

2.7 Discussion

In this Chapter we illustrated the quantum computation of an interacting fermionic model in 1+1 dimensions. While such a model is far from a realistic lattice gauge theory, lacking even bosonic degrees of freedom, this discussion still provided us with an opportunity to introduce several of the main approaches for taking a lattice field theory and running calculations on a quantum computer. We shall adopt many of these techniques as we move forward in the dissertation.

On one hand, we carefully wrote down a lattice Hamiltonian lacking a fermion doubling problem and expanded it in modes according to solutions of the free theory. We will repeat this procedure in later Chapters. Also, while we do not require it for the rest of this document, it was important to introduce the Jordan-Wigner transform (2.13) as it is used quite often in the literature and is expected to re-appear in future work concerned with fermionic degrees of freedom

Next we applied the Variational Quantum Eigensolver (VQE) algorithm. We will implement this algorithm in some form in all of the remaining Chapters. Finally, we performed RO and ZNE error mitigation techniques, which are the primary error mitigation strategies we adopt in the coming Chapters. Now that we have established some of the calculational tools that we will be using, it is time to move forward and take a look at a model containing a gauge symmetry and phase transition in order to understand the complications that were not yet encountered in this chapter.

Chapter 3

Schwinger Model

Let us now begin discussing the main goal presented in this dissertation, the quantum computation of lattice gauge theories. We begin with the Schwinger model [77, 78, 79], which is Quantum Electrodynamics (QED) in just one spatial dimension. In this Chapter, after introducing the model, we will cover the work of Ref. [63] which was done in collaboration with George Siopsis. This project was suggested by G.S. as a starting point for work on lattice gauge theories. I obtained the results presented in this paper, with G.S. providing key theoretical insights. We both contributed to the writing of the article.

The Schwinger model provides an excellent starting point for approaching LGT's since it serves as a toy model for Quantum Chromodynamics (QCD). It exhibits interesting phenomena such as the Higg's mechanism, charge screening, “quark” confinement, and spontaneous chiral symmetry breaking. On the other hand, it contains several simplifications with respect to QCD which make it far more adaptable to quantum computation. First, it is a $U(1)$ Abelian gauge theory, in which elements of the gauge group commute. This is not the case for QCD which is based on the gauge group $SU(3)$. Second, the one (spatial) dimensional nature of the model allows the gauge field to be nearly or completely eliminated via constraints derived from Gauss's law. Such constraints correspond to the requirement of gauge invariance.

The model features topological “ θ -vacua.” The case $\theta = \pi$ exhibits a phase transition when the mass gap vanishes somewhere between the asymptotic limits $e \gg m$ and $e \ll m$ [79, 80, 81, 82]. The latter limit corresponds to a phase where charge conjugation and parity (C/P) symmetry is spontaneously broken. In Refs. [80] and [19], the transition was shown to belong in the universality class of the Ising model.

The Lagrangian density for the model is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial_\mu - eA_\mu)\gamma^\mu\psi - m\bar{\psi}\psi, \quad (3.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ is an Abelian gauge potential, and ψ is a two-component fermionic field. $\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^2$ are Dirac matrices in one space and time dimension, and $\bar{\psi} \equiv \psi^\dagger\gamma^0$. There are two parameters in the model, the coupling strength e and the bare fermion mass m , both with mass dimension one (the only dimensionless parameter in the model is their ratio). In the case $m = 0$, the model can be solved exactly and is equivalent to a massive scalar boson of mass $\frac{e}{\sqrt{\pi}}$ [83].

Gauss’s Law is the constraint

$$\partial_1 E = ej_0 \quad (3.2)$$

where $E = F_{01}$ is the electric field and $j_0 = \psi^\dagger\psi$ is the charge density. The solution to Gauss’s Law involves a constant of integration which is the background electric field [79]

$$F = \frac{e\theta}{2\pi} \quad (3.3)$$

written in terms of a third parameter θ that determines the system.

The Hamiltonian is

$$H = \int_0^{2L} dx \left(\frac{1}{2}E^2 + i\bar{\psi}\gamma^1(\partial_1 + ieA_1)\psi + m\bar{\psi}\psi \right) \quad (3.4)$$

where $2L$ is the length of the spatial dimension. We apply periodic boundary conditions (PBC’s).

The Gauss law constraint generates gauge transformations. Thus in the quantum theory physical (gauge invariant) states $|\Psi\rangle$ must satisfy

$$\mathcal{G}(x)|\Psi\rangle = 0, \quad \mathcal{G}(x) = \partial_1 E(x) - ej_0(x) \quad (3.5)$$

After applying Gauss's law, there is one residual gauge degree of freedom, the gauge zero mode

$$q = \frac{1}{2L} \int_0^{2L} dx A_1 \quad (3.6)$$

It determines the exponent of the Wilson loop operator $e^{ie \oint A dl}$ which is a gauge-invariant quantity [1, 84].

3.1 Lattice

On the lattice, instead of using Wilson fermions, most approaches to lattice gauge theories coupled to Dirac fermions start with the Kogut-Susskind “staggering” formulation [2, 67]. Here the components of the original Dirac spinors are distributed among neighboring lattice sites, resulting in a one-component (spinless) fermionic field. This reduction in the degrees of freedom directly eliminates the unphysical modes from fermion doubling. Meanwhile, the gauge degrees of freedom reside on the links connecting the matter sites and act as parallel transporters.

The Schwinger model yields a simple Kogut-Susskind Hamiltonian given by [85]

$$H = H_f + H_{EM} \quad (3.7)$$

where

$$\begin{aligned} H_f &= \frac{i}{2} \sum_{r=0}^{2L-1} (\psi^\dagger(r) e^{ieA_1(r)} \psi(r+1) - \text{h.c.}) + m \sum_{r=0}^{2L-1} (-)^r \psi^\dagger(r) \psi(r) \\ H_{EM} &= \frac{1}{2} \sum_{r=0}^{2L-1} E^2(r) \end{aligned} \quad (3.8)$$

We work in units in which the lattice spacing (between fermionic sites) is $a = 1$, so L is the number of spatial sites and $2L$ is the number of fermionic sites. Both e and m are now dimensionless.

Accompanying this Hamiltonian is the Gauss law constraint given in (3.5), with the derivative on the electric field replaced by the finite difference $E(r) - E(r - 1)$. Gauss's law determines which states are physical.

We see that the first term in H_f is a hopping term, made gauge invariant by the inclusion of “link” operators $e^{ieA_1(r)}$. The second term is the mass term, and the $(-)^r$ shows that even and odd fermionic sites are treated differently in the staggering formalism. At strong coupling the ground state of the theory is the state $|0\rangle$ with

$$\psi(r)|0\rangle = 0, \text{ } r \text{ even}, \quad \psi^\dagger(r)|0\rangle = 0, \text{ } r \text{ odd}, \quad E|0\rangle = 0, \quad (3.9)$$

Thus the charge density $j_0(r)$ is typically normal-ordered with respect to this state. The only excitations at even sites are positrons while the only excitations at odd sites are electrons.

H_{EM} is the pure gauge field term. Each time the operator $e^{ieA_1(r)}$ acts on a link, an additional electric flux line of magnitude e is added to that link, since $[E(r), e^{ieA_1(r')}] = e \times e^{ieA_1(r')} \delta_{r,r'}$. As shown in Fig. 3.1, such an addition must be accompanied by fermionic excitations on the ends of the flux line in order for the full excitation to be gauge invariant.

3.2 Our Approach to Schwinger

In the work of Ref. [63], we demonstrated that even though the $\theta = \pi$ phase transition occurs in the continuum, the critical point can be estimated on the lattice with a small number of sites (as low as four). We studied the transition using quantum computation, and performed this computation on quantum hardware (IBM Q Lima) by minimizing the number of qubits and circuit depth to avoid quantum error.

We achieved this minimization by employing the formalism developed in [86], adapted to accommodate a spatial lattice.

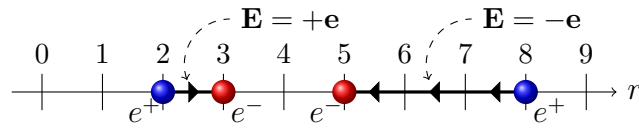


Figure 3.1: Eigenstate of the Schwinger model at strong coupling. A positron at an even site is connected to an electron at an odd site by an electric flux string, satisfying Gauss's law.

After reducing the gauge degrees of freedom to one in a fashion similar to [87], this alternative formalism allows for a careful treatment of the gauge zero mode which is present in the case of PBC's. Additionally, this formalism enforces Gauss's law automatically in the encoded basis states and facilitates the restriction of the Hilbert space to states obeying certain symmetries. Thus, we adopted the advantages of both [14] and [16] while expanding our analysis to include the dynamics of the zero mode. A brief review of this work will now be given.

3.3 Momentum space approach

In the spirit of Ref. [86], we analyze the Hamiltonian in momentum space. The fermionic part of (3.7) is simplified by defining

$$\tilde{\psi}(r) = e^{ie \sum_{r'=0}^{r-1} A_1(r')} \psi(r) \quad (3.10)$$

This transformation is similar to that used in [87], in which a purely fermionic formulation is obtained for open boundary conditions (OBC's). A free parameter exists at the boundary, which Refs. [14] and [87] set to zero. In our approach, we use PBC's which results in a single gauge degree of freedom (zero mode).

Using (3.10), the fermionic part of the Hamiltonian H_f simplifies to

$$H_f = \frac{i}{2} \sum_{r=0}^{2L-1} \left(\tilde{\psi}^\dagger(r) \tilde{\psi}(r+1) - \text{h.c.} \right) + m \sum_{r=0}^{2L-1} (-1)^r \tilde{\psi}^\dagger(r) \tilde{\psi}(r) \quad (3.11)$$

It is convenient to express the fields in momentum space. We have

$$\tilde{\psi}(r) = \frac{1}{\sqrt{2L}} \sum_{l=0}^{2L-1} b_l e^{-ik_l r}, \quad k_l = \frac{(2l+1)\pi}{2L} - eq \quad (3.12)$$

where q is the zero mode of the gauge field A_1 (Eq. (3.6)), while for the fermion expansion coefficients the usual anti-commutation relations hold:

$$\{b_l, b_{l'}^\dagger\} = \delta_{ll'} \quad (3.13)$$

Since $A(r)$ obeys PBC's, the expansion (3.12) gives anti-periodic boundary conditions for $\psi(r)$. We obtain

$$H_f = \sum_{l=0}^{2L-1} b_l^\dagger b_l \sin k_l + m \sum_{l=0}^{2L-1} b_{l+L}^\dagger b_l \quad (3.14)$$

For the gauge part, we introduce the Fourier expansion of the electric field,

$$E(r) = \frac{p + (-1)^r a_L}{2L} - \frac{i}{2L} \sum_{l=1}^{L-1} (a_l e^{\pi i l r / L} - \text{h.c.}) \quad (3.15)$$

The modes p, a_L are hermitian, and p is the momentum conjugate to the gauge zero mode q , $[q, p] = i$. Gauss's Law reads

$$E(r) - E(r-1) = e :j(r): \quad (3.16)$$

where $j(r) = \psi^\dagger(r)\psi(r)$ is the charge density and $: :$ indicates normal ordering ($:\mathcal{O}:= \mathcal{O} - \langle \mathcal{O} \rangle$, where $\langle \mathcal{O} \rangle$ is the vacuum expectation value of the operator $\mathcal{O}$).

In terms of modes, (3.16) becomes

$$a_l = \frac{ie :j_l:}{1 - e^{-\pi i l / L}}, \quad a_L = \frac{e}{2} :j_L:, \quad j_l = \sum_{s=0}^{2L-1} b_{s+l}^\dagger b_s \quad (3.17)$$

where indices are mod(2L) and $j_l = \sum_{r=0}^{2L-1} j(r) e^{-\pi i l r / L}$ are Fourier modes of the charge density $j(r)$. Notice that $j_{2L-l} = j_l^\dagger$. It is also easy to show that $[j_l, j_{l'}] = 0, \forall (l, l')$.

Moreover, the total charge vanishes,

$$Q = :j_0: = 0 \quad (3.18)$$

We adopt the ground state from Eq. (3.9), which has $\langle j(r) \rangle = r \bmod(2)$. In Ref. [85] it was shown that this is the ground state in the strong-coupling limit $e \gg 1$. Therefore,

$$\langle j_l \rangle = L(\delta_{l,0} - \delta_{l,L}) \quad (3.19)$$

The gauge part of the Hamiltonian is written entirely in terms of fermionic modes and the bosonic zero mode momentum p ,

$$H_{EM} = \frac{p^2}{4L} + \frac{e^2}{8L} \sum_{l=1}^{L-1} \csc^2 \frac{\pi l}{2L} j_l^\dagger j_l + \frac{e^2}{16L} (j_L + L)^2 \quad (3.20)$$

The remaining constraint (3.18) needs to be imposed on all physical states. From (3.19), we see that the requirement of vanishing charge restricts our Hilbert space to half-filling: $j_0 = L$.

By solving Gauss's law, we have restricted the Hilbert space to states that are invariant under “small” gauge transformations. Next consider a general gauge transformation,

$$\psi(r) \rightarrow e^{ie\omega(r)} \psi(r) , \quad A(r) \rightarrow A(r) - \omega(r+1) + \omega(r) \quad (3.21)$$

From this we have

$$q \rightarrow q - \frac{\omega(2L) - \omega(0)}{2L} , \quad \tilde{\psi}(r) \rightarrow e^{ie\omega(0)} \tilde{\psi}(r) \quad (3.22)$$

Gauge transformations are classified by their winding numbers $w \in \mathbb{Z}$,

$$\omega(2L) - \omega(0) = \frac{2\pi}{e} w \quad (3.23)$$

transforming

$$q \rightarrow q - \frac{\pi}{eL} w \quad k_l \rightarrow k_{l+w} , \quad b_l \rightarrow b_{l+w} \quad (3.24)$$

with indices taking values $\bmod(2L)$.

“Large” gauge transformations (LGTs) correspond to $w \neq 0$. Let U_1 implement the LGT with $w = 1$. Notice that $U_1^{2L} = \mathbb{I}$. Physical quantities must be invariant under U_1 . For

physical states, this implies

$$U_1 |\text{phys}\rangle = e^{i\theta} |\text{phys}\rangle \quad (3.25)$$

Owing to the periodicity $U^{2L} = \mathbb{I}$, θ can take on $2L$ values,

$$\theta = 0, \pm\frac{\pi}{L}, \dots, \pm\frac{(L-1)\pi}{L}, \pi \quad (3.26)$$

Each θ labels a distinct sector of the Hilbert space defining an independent physical system. The parameter θ is related to the background electric field. The system considered in [14, 16] corresponds to the $\theta = 0$ sector.

The Fock space is a tensor product of fermionic states and a one-dimensional system described by the residual gauge degree of freedom q . A basis for fermionic states is provided by the states

$$|\mathbf{x}\rangle \equiv |x_0 x_1 \dots x_{2L-1}\rangle = (b_0^\dagger)^{x_0} (b_1^\dagger)^{x_1} \dots (b_{2L-1}^\dagger)^{x_{2L-1}} |\Omega\rangle \quad (3.27)$$

where $|\Omega\rangle$ is the fermionic vacuum annihilated by all b_l , and $x_l = 0, 1$ ($l = 0, 1, \dots, 2L-1$) are occupation numbers.

For the gauge degree of freedom, a basis is provided by plane waves

$$\langle q | p = ne \rangle = \frac{1}{\sqrt{2\pi}} e^{ineq}, \quad n \in \mathbb{Z} \quad (3.28)$$

Imposing gauge invariance under LGTs, we obtain states in the θ sector,

$$|p = ne, \mathbf{x}; \theta\rangle \propto \sum_{l=0}^{2L-1} e^{il\theta} U_1^l |p = ne\rangle \otimes |\mathbf{x}\rangle, \quad (3.29)$$

In order to greatly reduce qubit costs, we encoded this basis directly as our computational basis for quantum computation (instead of the Fock states). Under the LGT U_1 we have

$$U_1 |n\theta\rangle = e^{-i\theta} |n\theta\rangle \quad (3.30)$$

for the values (3.26) of the parameter θ .

It is convenient to introduce lattice translation and parity operators, $\mathcal{T}, \Pi$. These commute with H and are gauge invariant for $\theta = 0, \pi$. Their definitions are given in [63]. To reduce the size of the Hilbert space for the calculation of the critical point, we invoke the symmetries of the lattice Schwinger model under $\mathcal{T}$ and Π , enabling us to focus on the sectors that contain the ground state ($\mathcal{T} = 0, \Pi = 1$) and first excited state ($\mathcal{T} = 0, \Pi = -1$).

Details of the calculation of matrix elements of the Hamiltonian can be found in the Appendices of [63]. To calculate eigenvalues numerically using these matrix elements, it is necessary to truncate the infinite dimensional Hilbert space to a cutoff, $n \leq n_{\max}$. For $e \gg 1$ and/or $m \gg 1$ we obtain, via exact diagonalization, accurate values of the mass gap (difference between the energies of the first excited state and ground state) with modest values of the cutoff.

In Fig. 3.2, the mass gap is plotted as a function of e and m , for three spatial points ($L = 3$), with the Hilbert space truncated to $n_{\max} = 20$. For $\theta = \pi$ (right panel), the gap approaches zero asymptotically for all values of e , as m varies from $m \ll e$ (where the gap is noticeably finite) to $m \gg e$ indicating the presence of a phase transition.

It turns out that a large cutoff in the Fock space is not needed when working in the $\theta = \pi$ sector, for a wide range of parameters (see Appendix C in Ref. [63] for some calculations). Accurate results can be obtained for $n_{\max} \sim 2L$. This is advantageous for quantum computation on NISQ devices.

3.3.1 Calculation of phase transition

As shown by exact diagonalization (classical calculations) in Fig. 3.3, the mass gap Δ approaches zero for $m > 0$ for $\theta = \pi$, as expected. For other values of θ , this behavior is not observed, which is consistent with the absence of a phase transition. Adopting the methods of finite-size scaling theory [88, 89], we estimate the critical point by finding values of the bare mass m and coupling constant e such that the scaled energy-gap ratio $R = 1$, where for

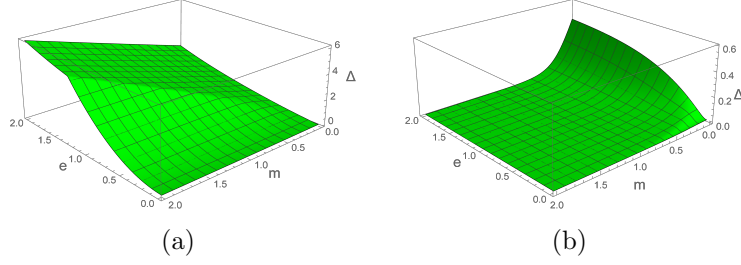


Figure 3.2: Mass gap Δ for $L = 3$. The gap is displayed as a function of coupling e and bare mass m , for (a) $\theta = 0$ and (b) $\theta = \pi$. The Hilbert space is truncated to $n_{\max} = 20$.

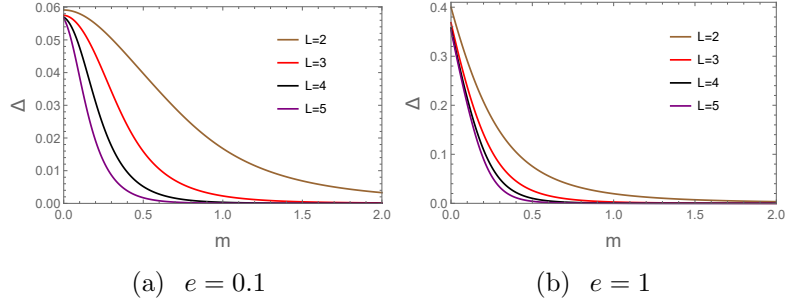


Figure 3.3: Mass gap Δ *vs.* bare mass m in the $\theta = \pi$ sector. Numerical results are displayed for $L = 2, 3, 4, 5$. The coupling is set at (a) $e = 0.1$ and (b) $e = 1$.

a given lattice size L , the ratio is defined by

$$R_L(m/e) \equiv \frac{L\Delta_L(m/e)}{(L-1)\Delta_{L-1}(m/e)} \quad (3.31)$$

and Δ_L is the mass gap. Solutions to $R_L = 1$ are known as “pseudo-critical” points, and they approximate the critical point obtained in the continuum limit $L \rightarrow \infty$. Pseudo-critical points are plotted for $L = 4$ and $e \in [0.1, 1]$ in Fig. 3.4. A weighted linear regression, using the deviation of the $L = 3$ points as an estimate of the error, yields a value of $m/e = 0.339$ for $e = 0$ which corresponds to the continuum limit (since the dimensionless quantity $ea \rightarrow 0$, where a is the lattice spacing). This result agrees with Ref. [90] where the critical value was found to be 0.325 ± 0.02 . It is also a good approximation to 0.3335 ± 0.0002 , obtained in Ref. [19]

Quantum Computation

In order to obtain the ratios R_4 via a quantum computation, we calculated energies with the VQE algorithm adopted in the previous Chapter. However from now on we will only be employing “single-term” trial states which follows the usual form of VQE and is more efficient. Due to hardware constraints, it was necessary to reduce the calculation to three qubits and encode only the states which contributed the most to the true eigenstates. The choice of states is informed from perturbation theory and verified with exact diagonalization. This reduction in available states remains effective in the range of e in Fig. 3.4 where the error bars are small, $e \in [0.5, 1.0]$ (the region unaffected by the use of a small lattice size).

In order to invoke VQE, we defined Ansatz states to represent the ground and first excited states of both $L = 3$ and $L = 4$, and then implemented them with quantum circuits. When a quantum computation is performed on a NISQ device, it is important that these circuits are shallow in depth and involve only a few variational parameters. The following quantum circuits turn out to be effective near the pseudo-critical points. Implementing these circuits required a permutation of our basis.

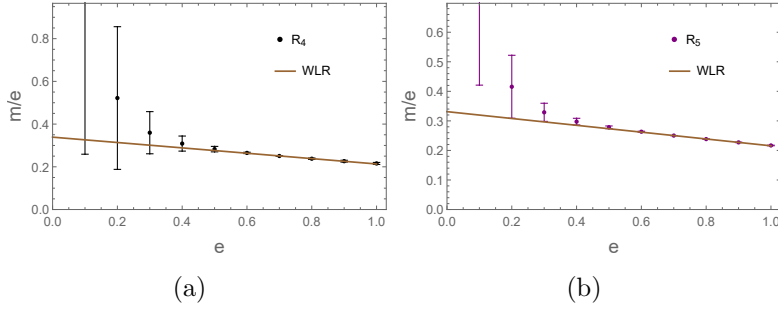


Figure 3.4: Pseudo-critical points *vs.* coupling constant e . The spatial length is $L = 4$ (a) and $L = 5$ in (b). The fit represents a weighted linear regression. The error bars are estimated as the deviation of the $L = 3$ and $L = 4$ points, respectively. While errors increase as $e \rightarrow 0$, the weighted linear regression yields the values 0.339 for $L = 4$ and 0.331 for $L = 5$ at $e = 0$, in good agreement with the continuum result 0.3335 ± 0.0002 [19].

Instead of including the permutation unitary in the Ansätze, which increases circuit depth, the Hamiltonian was transformed to the permuted basis classically.

For $L = 3$ we used the same circuit for both the ground and first excited states. The circuit is shown in Fig. 3.5. We used separate circuits for the $L = 4$ ground and first excited states, see Fig. 3.6

Due to long queue and run times on IBM’s quantum hardware, it was not practical to locate the pseudo-critical points solely on hardware. Not only were there many points in the two-dimensional parameter space (e - m plane) to sample, but with run times scaling with the number of shots, results with error bars small enough to distinguish each point in the space were inaccessible.

Instead, we first located the pseudo-critical points using simulations without machine noise (i.e. simulations mimicking an ideal quantum computation, which still contains shot noise). The pseudo-critical points found from these simulations yielded a continuum critical point of $m/e \simeq 0.32$, in good agreement with the classical numerical result $m/e = 0.3335$. We then proceeded to check whether the results obtained on hardware for the scaled gap ratio (3.31), evaluated at these points, contained the value $R_4 = 1$ within error bars.

Error Mitigation

Similar to Section 2.5 we addressed RO error and employed zero noise extrapolation (ZNE). In Chapter 2 we suggested that for a small number of qubits it is straightforward to run a set of calibration circuits to characterize RO error. One constructs a “calibration matrix” and applies the inverted matrix to the raw hardware results. IBM Qiskit allows the user to automatically incorporate this calibration process in the jobs they submit to the quantum computer. This is the approach we followed for our three qubit case, instead of employing Eq. 2.23. For the Richardson extrapolation procedure we applied both linear and quadratic extrapolation to achieve two levels of zero-noise extrapolation.

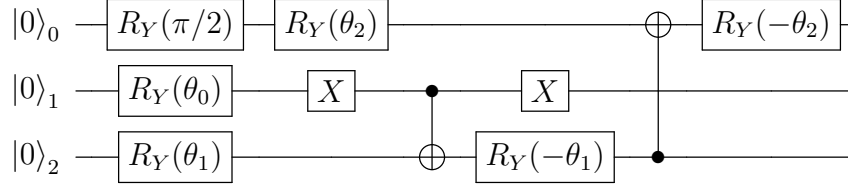


Figure 3.5: Quantum circuit for $L = 3$ spatial sites. This circuit is used for both the ground and first excited states.

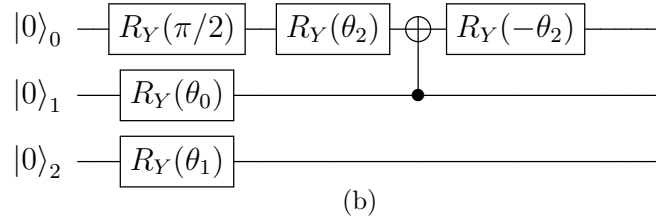
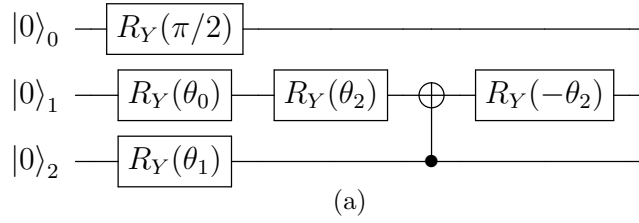


Figure 3.6: Quantum circuits for $L = 4$ spatial sites. Circuit (a) is used for the ground state and circuit (b) is used for the first excited state.

Results

For our quantum computation we used the device IBM Q Lima. In Figs. 3.7-3.9 the statistical means and error bars correspond to 15 hardware runs. In Fig. 3.7, we plot the ground state energies for $L = 3, 4$. We find that extrapolation improves the accuracy of the results for the energies. The mass gap, which is the difference between the ground and first excited state energies, is shown in Fig. 3.8. In Fig. 3.9 we plot the ratio R_4 obtained from the mass gaps in Fig. 3.8, using Eq. (3.31). From Figs. 3.8 and 3.9, we see that the mass gaps and ratio are very sensitive to errors and variations in the energies. It can also be seen that extrapolation is a somewhat unstable technique, sometimes producing significantly larger error bars for certain points when compared with no extrapolation. Also given are results from noisy simulations, which overlap with hardware results as expected.

3.4 Discussion

In this Chapter we took a model, the massive Schwinger model, with a gigantic “raw” Fock space and essentially searched for a subspace sufficiently small for a quantum computation of the phase transition. This took several steps. First, we restricted to small lattice sizes, $L = 3, 4$ and saw from classical calculations that this was sufficient to obtain a reasonable estimate of the critical point using equations from finite-size scaling theory. Second, we neutralized gauge freedom under small gauge transformations by enforcing Gauss’s law. This involved both specifying the background static charge distribution as well as expressing the non-zero gauge modes in terms of the fermions. The latter resulted in the loss of the need to specify the configuration of the non-zero gauge modes in our set of states, so long as we re-wrote the Hamiltonian to incorporate Gauss’s law as well, see Eqs. (3.14) and (3.20).

Next, we found that unlike the well-known case of open boundary conditions, we still had one gauge (bosonic) mode left over whose infinitely large hilbert space we somehow needed to encode. Once we identified a basis for the bosonic mode that we wanted to use, we eliminated gauge freedom under large gauge transformations by constructing properly symmetrized

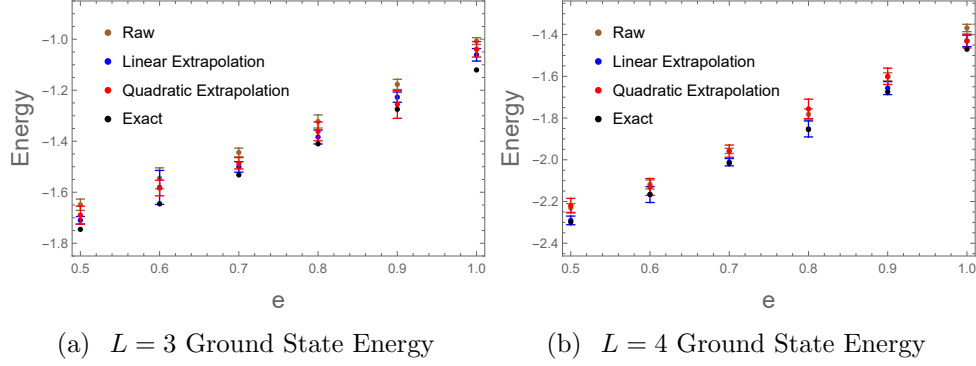


Figure 3.7: Hardware results for the $L = 3$ and $L = 4$ ground state energies vs. coupling e . For each e , m is chosen so that the points (e, m) in the parameter space are pseudo-critical points as identified from simulations without machine noise. The means and error bars describe the data from 15 runs. Results without Richardson extrapolation and with linear and quadratic extrapolation are compared with the eigenenergies found by exact diagonalization of the Hamiltonian. Results are obtained from IBM Q Lima.

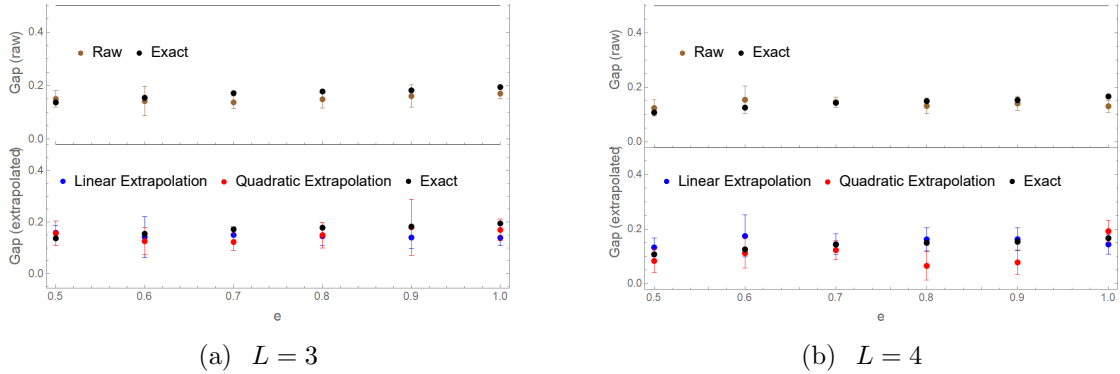


Figure 3.8: Hardware results for the $L = 3$ and $L = 4$ mass gaps Δ , vs. coupling e . These are found by subtracting the ground state energies from the first excited state energies. The energies used in this calculation are displayed in Fig. 3.7.

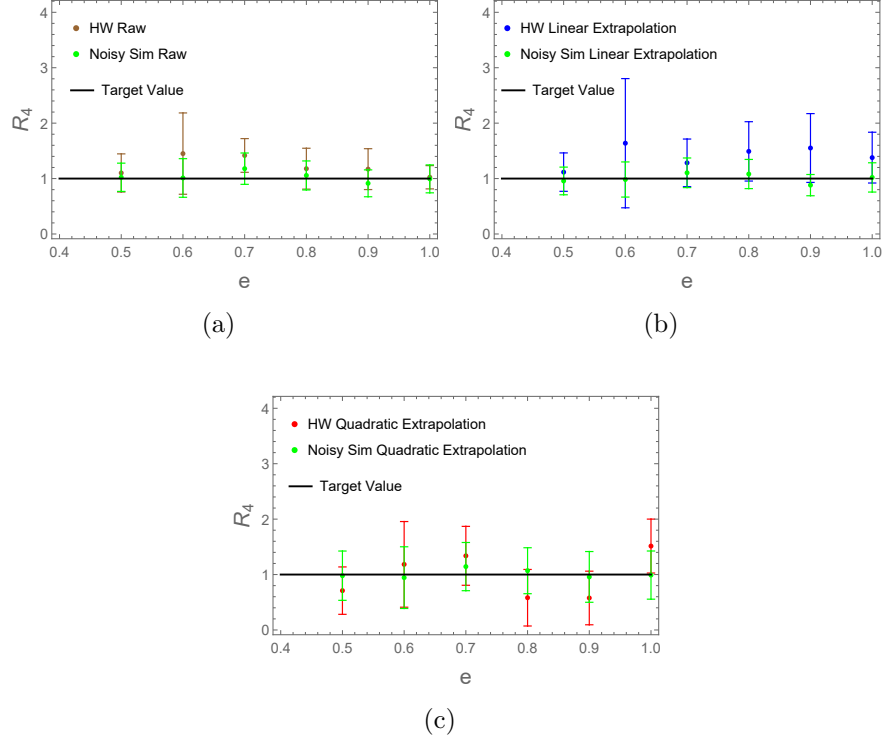


Figure 3.9: Hardware (HW) results for the scaled energy-gap ratio R_4 . (a) contains no Richardson extrapolation, while (b) and (c) correspond to linear and quadratic extrapolation, respectively. The mass gaps for $L = 3, 4$ used to compute the ratio are those from Fig. 3.8. The parameter values (e, m) we evaluate this ratio at should correspond to pseudo-critical points, and so we expect $R_4 \approx 1$. Also given are results from noisy simulations.

states. This construction required knowledge on how the fermionic states and bosonic wave-functions transformed under large gauge transformations. By choosing the bosonic wave-functions to correspond to the strong-coupling basis in the literature, we found that in most cases a large truncation in this basis would be required to approach the continuum limit $e \rightarrow 0$. However the sector of interest, $\theta = \pi$ seemed to be a sector where we could get away with a lower truncation and therefore significantly decrease the size of the hilbert space even when using the strong-coupling boson wave-functions (plane waves).

Thus in this case we were lucky that a restriction to three qubits did not invoke too much algorithmic error. However, we see that this situation was very delicate and is not expected to apply or be easily reproducible in more complicated gauge theories where gauge bosons play a more dominant role. An alternative handling of boson degrees of freedom is discussed in the next Chapter, where we introduce the advantage of continuous variables (CV) quantum computation.

Chapter 4

Interacting Scalar Field Theory

Even in the Schwinger model we are still left with one gauge degree of freedom when we employ PBC's. The role of this bosonic degree of freedom becomes increasingly relevant at small lattice spacing and has prevented us from adequately exploring this regime with a quantum computer. In order to better incorporate this single bosonic mode, we would have to somehow encode larger portions of its infinite dimensional Hilbert space, which would require an unfeasible number of qubits when utilizing a *discrete variable* (DV) quantum device.

This is just a small part of the main concern when it comes to truncation. Once we add extra dimensions to the system, the ability to eliminate (or in our case, almost eliminate) the gauge degrees of freedom is no longer present. As the necessity of truncation comes into play, one must find a truncation scheme that is efficient but gives rise to the correct physics and continuum limit. As discussed in Chapter 1, this is not a straightforward thing to do, and various truncation schemes have been developed over the years. However, there is another option that entirely eliminates the need for truncation whatsoever. We may perform field theoretical calculations using a *continuous variable* (CV) quantum device. This term refers to devices which operate on bosonic “qumodes,” each of which lives in an infinite dimensional Hilbert space instead of the two dimensional space allowed by a single qubit.

The continuous nature of the qumode is revealed by the “quadrature” field operators q, p which are simply the analogs of the position and momentum operators for a harmonic oscillator. In place of using multiple qubits just to simulate a single bosonic degree of freedom, we can instead just use one “qumode.” We will show how this may be taken advantage of by following Ref. [64], done in collaboration with George Siopsis. In this project, I introduced the possibility of obtaining an approximate critical point in a one-dimensional bosonic field theory using a CV quantum computation with Gaussian Ansätze, based on the findings of Ref. [91]. G.S. performed the analytical lattice calculation below (Section 4.3) and pointed out the basic CV circuit structure for generating Gaussian states. I worked on developing the other algorithms (e.g. energy and gradient measurements) and obtaining results for quantum calculations.

4.1 CV quantum computation

The rest of the dissertation focuses heavily on the utilisation of continuous variables (CV) quantum computing. Therefore, before proceeding, it will be helpful to provide some context and discuss some of the progress that has been made in this field. To do this, we must identify what hardware exists, or is expected to be available in the near future, which might allow one to perform quantum computations with continuous variables. As mentioned in Chapter 1, the most apparent realization of CV quantum information processing lies in quantum photonics [48, 92, 49]. In the standard setup, squeezed light is produced in a process called spontaneous parametric down conversion (SPDC) [93] which involves the generation of photon pairs from a pump source acting through a non-linear medium. The optical frequencies are determined from phase-matching conditions. As discussed, the continuous-valued quadrature field operators characterize the electric field strength in the electromagnetic field. Schematically (reducing to one dimension and ignoring polarization), the plane-wave representation of a monochromatic electric field is given by

$$E(x) = A \cos(kx - \delta) = q \cos kx + p \sin kx, \quad q = A \cos \delta, \quad p = A \sin \delta \quad (4.1)$$

The name “quadrature” therefore refers to the fact that the trigonometric functions associated with each operator are out of phase by a quarter of a cycle. After quantization, these quadratures become

$$q = \frac{1}{\sqrt{2}} (a^\dagger + a) \ , \ p = \frac{i}{\sqrt{2}} (a^\dagger - a) \ , \quad (4.2)$$

as can be seen by passing from the usual exponential Fourier representation of the quantum field to the trigonometric one. Therefore, they obey position-momentum commutation relations $[q, p] = i$. Measurements of these continuous variables is achievable through homodyne detection. A common approach to homodyne measurements is to interfere the qumode with an intense, controllable coherent beam called a local oscillator (LO) and obtain information by measuring the resulting photo-currents using photo-diodes. [94, 49]. Due to the continuous nature of these measurements it is not straightforward to obtain reliable information from them, as for example reconstructing an approximation of the Wigner function/density matrix from this data requires non-trivial methods [94].

Therefore one could also directly apply photon number measurements in order to organize measurements into a discrete basis. This will be our method of measurement as we move forward, but one should be wary of the fact that distinguishing precise photon numbers is difficult in practice. It is typically more feasible to use avalanche photo-detectors that click when a photon is present, regardless of the actual number of photons, making it difficult to determine detect $n = 2$ or $n = 3$ photons, and so forth. One way to resolve photon number would be to use beamsplitters to distribute the photons to separate detectors so that each measures only one photon [50, 49].

While photonic technology has come a long way and is continuing to develop further, other proposals for uncovering continuous variables have come to light and come with their own set of possible advantages. For example, the quadrature operators may be realized in an atomic setting using spinor Bose-Einstein condensates of e.g. Rubidium atoms trapped in an optical lattice [95, 96, 51]. The accessible modes of the atoms are the hyperfine spin states, which are populated through squeezing via spin-changing collisions. One possible

advantage of this method is the ability to apply such squeezing for longer interaction times than in SPDC for photons. Schemes for homodyne measurement as well as the detection of entangled pairs have been considered.

Additionally, “hybrid” quantum set-ups supply a very promising setting not only for CV quantum computing but also for the coupling of qumodes to qubits. One such example are ion-trap quantum devices, which as discrete variable (DV) quantum computers facilitate qubit (ion) interactions via collective vibrational modes. However one can introduce qumodes by taking these phonon modes as dynamical in the target theory. This has been investigated in [52] where tranverse local phonon modes are made to couple with the ion’s internal (qubit) degrees of freedom to simulate gauge-fermion coupling in the Schwinger model, Eq. (3.8). The on-site gauge term, e.g. $\sum_x E^2(x)$, can be implemented with a “standing-wave gate” that in terms of universal quantum computation provides a purely bosonic non-Gaussian operation (recall that the electric field E resembles a number operator in the lattice Schwinger model).

Another possibility is to use circuit quantum electrodynamics (cQED) [53]. In this technology a microwave cavity interacts with an artificial atom, a superconducting qubit. As with its predecessor, cavity QED [97], as well as ion-trap devices, interactions can be described via the Jaynes-Cummings model [98]. Like with the ion-trap case, we are introduced to several unique additional gates, all non-Gaussian in nature, involving the coupling of photons to qubits [99]. For instance the “SNAP” gate, discussed in Appendix A.2, can allow one to selectively manipulate a qubit depending on the Fock state of a qumode.

4.2 The Model

We will now discuss details of a CV quantum algorithm for estimating the phase transition in ϕ^4 scalar field theory, and this description will follow Ref. [64]. In this first section, we introduce ϕ^4 scalar field theory in $1 + 1$ dimensions as well as its phase transition. In one

spatial dimension, the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{\lambda}{4!}\phi^4 \quad (4.3)$$

where m_0 is the bare mass and λ is the coupling strength of mass dimension two. The field ϕ and its conjugate momentum π satisfy the canonical commutation relations

$$[\phi(x), \pi(x')] = i\delta(x - x') \quad (4.4)$$

The quartic self-interaction term in (4.3) induces mass renormalization. It is convenient to split the Hamiltonian density into non-interacting and interacting parts,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \quad (4.5)$$

with

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{1}{2}m^2\phi^2, \quad \mathcal{H}_{\text{int}} = \frac{1}{2}\delta_m\phi^2 + \frac{\lambda}{4!}\phi^4 \quad (4.6)$$

where m is the renormalized mass and $\delta_m = m_0^2 - m^2$ is a counterterm parameter.

It is also useful to expand $\phi(x)$ and $\pi(x)$ in modes,

$$\begin{aligned} \phi(x) &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega(k)}} (a^\dagger(k)e^{-ikx} + a(k)e^{ikx}) \\ \pi(x) &= i \int \frac{dk}{2\pi} \sqrt{\frac{\omega(k)}{2}} (a^\dagger(k)e^{-ikx} - a(k)e^{ikx}) \end{aligned} \quad (4.7)$$

where

$$\omega(k) = \sqrt{m^2 + k^2} \quad (4.8)$$

is the dispersion relation. It follows that the commutation relations for creation and annihilation operators is given by

$$[a(k), a^\dagger(k')] = 2\pi \delta(k - k') \quad (4.9)$$

Fig. 4.1 depicts the only irreducible divergent diagram for the system. According to Coleman [100], this divergence may be removed by normal-ordering. That is, we first take $m = \mu$ in Eq. 4.8 for some value μ , and this defines the a 's and $a^\dagger$'s that enter into Eqs. (4.25). Normal-ordering then means that we move all of the $a^\dagger$'s to the left. As found by Coleman, this procedure permits the removal of all divergences in a quantum field theory in $1 + 1$ dimensions containing no interactions involving derivatives of the fields. In our case, we write

$$\mathcal{H} = N_\mu \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right], \quad (4.10)$$

where the subscript μ on N indicates that we are normal-ordering with respect to mass μ . Next, we follow the discussion of Ref. [91] to describe the phase transition in the model. First, consider the potential classically, see Fig. 4.2. For $m^2 > 0$, shown in panel (a), the classical potential is minimized at $\phi = 0$. For small fluctuations, the potential corresponds to a harmonic oscillator with frequency m , showing that in the weak coupling limit $\lambda/m^2 \ll 1$, we find a non-interacting boson theory with boson mass m . If $m^2 < 0$ (panel (b)), the classical potential is minimized at $\phi = \pm\sqrt{\frac{-6m^2}{\lambda}}$. Small fluctuations around the minimum have frequency $\mu = \sqrt{-2m^2}$ indicating free bosons of mass μ in the weak coupling limit.

While Fig. 4.2 gives us a visual of the breaking of $\mathbb{Z}_2$ (reflection) symmetry, the two panels correspond to two different classical systems. However, we are interested in fixing the mass and varying the coupling λ . We also know that one must account for quantum corrections as this coupling is increased. It turns out that the freedom associated with normal-ordering allows us to relate these two systems. This relation was demonstrated in [91]. The requirement of equivalence is given by

$$\frac{1}{\lambda} (2m^2 + \mu^2) = \frac{1}{4\pi} \ln \frac{\mu^2}{m^2} \quad (4.11)$$

In the strong-coupling limit $\lambda/m^2 \gg 1$ of the system with $m^2 > 0$, this relation results in two solutions for λ/μ^2 , with one corresponding to the weak coupling limit $\lambda/\mu^2 \ll 1$, see Fig. 4.3. With this relation, we may now focus on the case $m^2 > 0$. At weak coupling, the expectation

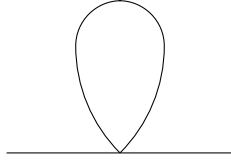


Figure 4.1: Only divergent irreducible diagram

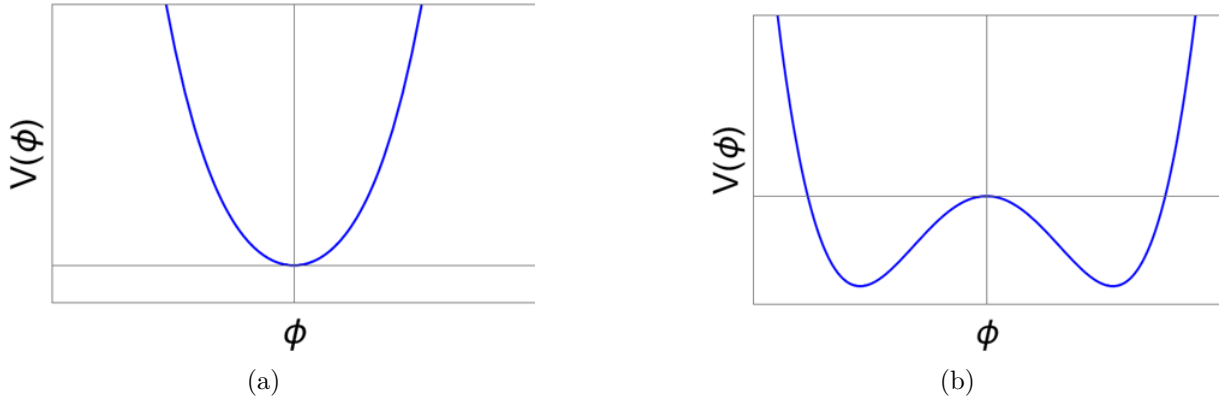


Figure 4.2: Schematic plot of the classical potential $V(\phi)$ for ϕ^4 scalar field theory in $1 + 1$ dimensions. In (a), we have $m^2 > 0$ and for (b) we have $m^2 < 0$.

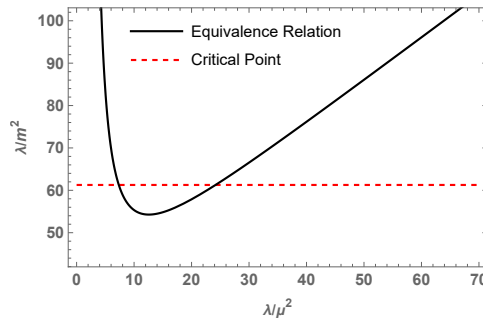


Figure 4.3: Equivalence relation between systems with $m^2 > 0$ and $m^2 < 0$. The weak coupling limit for the $m^2 < 0$ system corresponds to the strong-coupling limit of the $m^2 > 0$ system. No relation exists below the intermediate coupling $\lambda/m^2 \sim 55$. The critical point (4.15) obtained in Refs. [91, 101] is indicated with the dashed line.

value of the field $\phi(x)$ in the ground state vanishes ($\langle \text{GS} | \phi(x) | \text{GS} \rangle = 0$). If we fix the mass m^2 and increase the coupling λ , then at large coupling we find symmetry breaking indicated by the non-vanishing vacuum expectation value, $\langle \text{GS} | \phi(x) | \text{GS} \rangle = \pm \sqrt{\frac{-6m^2}{\lambda}}$. Without loss of generality, We concentrate on the case where the expectation is positive $\langle \text{GS} | \phi(x) | \text{GS} \rangle = +\sqrt{\frac{-6m^2}{\lambda}}$.

Thus, as we drive up the coupling away from $\lambda = 0$, we expect to see a phase transition from a symmetric phase to a symmetry-broken phase. We can locate the critical point, a dimensionless ratio $\frac{\lambda_c}{m^2}$ at fixed m^2 , by computing the effective potential and searching for the value of the coupling where the order parameter $\langle \text{GS} | \phi(x) | \text{GS} \rangle$ becomes non-vanishing. The effective potential is defined as [102, 101]

$$V_{\text{eff}}(\phi_C) = \min_{|\Psi\rangle} \langle \Psi | \mathcal{H} | \Psi \rangle, \quad \langle \Psi | \phi(x) | \Psi \rangle = \phi_C \quad (4.12)$$

Notice that both of these expectation values are independent of the position x . This is a consequence of the requirement of translational invariance for the ground state even in the case of spontaneous symmetry breaking, so that momentum conservation is always respected [103]. The value of ϕ_C that minimizes the effective potential is the expectation $\langle \text{GS} | \phi(x) | \text{GS} \rangle$, which we identify as the order parameter. We have seen, however, that at both weak and strong coupling, the ground state of the system is approximated by that of a non-interacting theory with some mass parameter Ω . Such a state can be obtained from the weak-coupling vacuum by applying Gaussian operations. Thus, we can approximate the ground state by a Gaussian functional of the field ϕ for all values of the coupling. This leads us to consider the Gaussian effective potential (GEP),

$$V_G(\phi_C) = \min_{\Omega} \langle \Psi_{\phi_C, \Omega} | \mathcal{H} | \Psi_{\phi_C, \Omega} \rangle \quad (4.13)$$

where $|\Psi_{\phi_C, \Omega}\rangle$ is the ground state of the non-interacting Hamiltonian

$$H_{\phi_C, \Omega} = \frac{1}{2} \int dx \left[\pi^2(x) + (\partial_x \phi(x))^2 + \Omega^2 (\phi(x) - \phi_C)^2 \right] \quad (4.14)$$

The critical point corresponding to purely GEP calculations can be obtained analytically [91, 101], and has been found to be

$$\tilde{\lambda} \equiv \frac{\lambda}{m^2} = 61.27 \quad (4.15)$$

This turns out to be very close to the values computed using Monte Carlo and DMRG [104, 105], despite the fact that the GEP calculation incorrectly yields a first-order phase transition.

4.3 Lattice

We proceed with writing down the lattice form of the Hamiltonian (4.3). For convenience and reference, we supply our own analytical minimization of the effective potential using the lattice construction, to be compared with our quantum computation which we will soon discuss.

As usual, space is discretized with coordinate $x = 0, 1, \dots, L - 1$, where L is the length of the spatial dimension in units in which the lattice spacing is $a = 1$. The momentum is described by the dual lattice $(\frac{2\pi}{L}k, k = 0, 1, \dots, L - 1)$ and identified by the integer k . The scalar field $\phi(x)$ and its conjugate momentum $\pi(x)$ are once more expanded in terms of creation and annihilation operators obeying commutation relations $[a(k), a^\dagger(k')] = \delta_{kk'}$ as

$$\phi(x) = \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} \frac{1}{\sqrt{2\omega(k)}} a^\dagger(k) e^{-2\pi i k x / L} + \text{h.c.}, \quad \pi(x) = \frac{i}{\sqrt{L}} \sum_{k=0}^{L-1} \sqrt{\frac{\omega(k)}{2}} a^\dagger(k) e^{-2\pi i k x / L} + \text{h.c.} \quad (4.16)$$

The dispersion relation (4.8) is modified to

$$\omega(k) = \sqrt{\Omega^2 + 4 \sin^2 \frac{\pi k}{L}}, \quad (4.17)$$

where Ω is an arbitrary mass parameter that will play the role of the variational parameter when minimizing the GEP. The Hamiltonian (4.14) with $\phi_C = 0$ is diagonal in the momentum

representation,

$$H_{\phi_C=0,\Omega} = \sum_k \omega(k) \left(a^\dagger(k)a(k) + \frac{1}{2} \right) \quad (4.18)$$

We work with the fields

$$q(k) = \frac{1}{\sqrt{2}} [a^\dagger(k) + a(k)] \quad , \quad p(k) = \frac{i}{\sqrt{2}} [a^\dagger(k) - a(k)] \quad (4.19)$$

which obey the commutation relations $[q(k), p(k')] = i\delta_{k,k'}$. in terms of which the Hamiltonian (4.18) is written as

$$H_{\phi_C=0,\Omega} = \sum_k \frac{\omega(k)}{2} [p^2(k) + q^2(k)] \quad . \quad (4.20)$$

Note here that the variational parameter appears in the Hamiltonian itself. This stems from our freedom in expanding $\phi(x)$ and $\pi(x)$ in terms of creation and annihilation operators that correspond to a renormalized mass. This picture will change in our quantum computation where it is usually preferable to fix the Hamiltonian and allow the quantum circuit to be parameterized. Now, the ground state of (4.20) can be written as a product of single-mode squeezed states that are Gaussian functions,

$$|\mathbf{0}\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes \cdots \otimes |0\rangle_{L-1} \quad , \quad \langle \mathbf{q} | \mathbf{0} \rangle = \frac{1}{\pi^{L/4}} e^{-\mathbf{q}^2/2} \quad (4.21)$$

where $\mathbf{q} = (q(0), \dots, q(L-1))$. We refer to these as squeeze states due to the fact that each term in (4.20) may be written as

$$\frac{\omega(k)}{2} [p^2(k) + q^2(k)] = \frac{1}{2} [p'^2(k) + \omega^2(k)q'^2(k)] \quad , \quad q'(k) = \frac{1}{\sqrt{\omega(k)}}q(k), \quad p'(k) = \sqrt{\omega(k)}p(k) \quad (4.22)$$

Identifying q', p' with the quadrature operators for a reference harmonic oscillator Hamiltonian, we find that the eigenfunctions of (4.22) are the harmonic oscillator wave-functions of the reference system but with $q(k) \rightarrow \sqrt{\omega(k)}q(k)$, giving us squeezed wave-functions.

The obtain the ground state of the Hamiltonian with $\phi_c \neq 0$ by applying a displacement,

$$|\Psi_{\phi_C, \Omega}\rangle = D\left(\sqrt{\frac{L\Omega}{2}}\phi_C\right)|\mathbf{0}\rangle = e^{-i\phi_C\sqrt{L\Omega}p(0)}|\mathbf{0}\rangle, \quad (4.23)$$

and this results in a squeezed coherent state. Additionally, under this displacement we have (See Appendix B for details)

$$a(0) \rightarrow a(0) + \sqrt{\frac{L\Omega}{2}}, \quad a^\dagger(0) \rightarrow a^\dagger(0) + \sqrt{\frac{L\Omega}{2}} \quad (4.24)$$

Since everything else is unchanged, it follows from (4.16) that $\langle\Psi_{\phi_C, \Omega}|\phi(x)|\Psi_{\phi_C, \Omega}\rangle = \phi_C$, as desired.

The fields in the position representation, $\phi(x)$ and $\pi(x)$, are expressed in terms of the fields $q(k), p(k)$ as

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} \frac{1}{\sqrt{\omega(k)}} \left[q(k) \cos \frac{2\pi kx}{L} - p(k) \sin \frac{2\pi kx}{L} \right], \\ \pi(x) &= \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} \sqrt{\omega(k)} \left[p(k) \cos \frac{2\pi kx}{L} + q(k) \sin \frac{2\pi kx}{L} \right]. \end{aligned} \quad (4.25)$$

Define the potential

$$V_G(\phi_C, \Omega) = \frac{1}{L} \langle\Psi_{\phi_C, \Omega}| H |\Psi_{\phi_C, \Omega}\rangle \quad (4.26)$$

as the expectation value of the full Hamiltonian

$$H = \sum_{x=0}^{L-1} \left[\frac{1}{2} \pi^2(x) + \frac{1}{2} [\nabla \phi(x)]^2 + \frac{m_0^2}{2} \phi^2(x) + \frac{\lambda}{4!} \phi^4(x) \right]. \quad (4.27)$$

After some algebra, we obtain

$$V_G(\phi_C, \Omega) = \frac{1}{2} m_0^2 \phi_C^2 + \frac{\lambda}{4!} (\phi_C^2 + 6I_0) \phi_C^2 + I_1(\Omega) + \frac{m_0^2 - \Omega^2}{2} I_0(\Omega) + \frac{\lambda}{8} I_0^2(\Omega) \quad (4.28)$$

where

$$I_0(\Omega) = \frac{1}{2L} \sum_k \frac{1}{\omega(k)}, \quad I_1(\Omega) = \frac{1}{2L} \sum_k \omega(k) \quad (4.29)$$

These expressions diverge in the continuum limit where k can be arbitrarily large. The GEP (4.13) corresponds to the minimum of (4.28). Setting $\frac{\partial V_G}{\partial \Omega} = 0$, and using $\frac{dI_1}{d\Omega} = \Omega I_0$, we find

$$m_0^2 - \Omega^2 + \frac{\lambda}{2} \phi_C^2 + \frac{\lambda}{2} I_0 = 0 \quad (4.30)$$

which can be solved to express Ω in terms of ϕ_C . It is more convenient to express ϕ_C in terms of Ω ,

$$\phi_C^2 = \frac{2(\Omega^2 - m_0^2)}{\lambda} - I_0(\Omega) \quad (4.31)$$

and express the GEP in terms of Ω , instead,

$$V_G(\Omega) = \frac{1}{2} m_0^2 \left[\frac{2(\Omega^2 - m_0^2)}{\lambda} - I_0(\Omega) \right] + \frac{\lambda}{4!} \left[\frac{2(\Omega^2 - m_0^2)}{\lambda} - I_0(\Omega) \right]^2 + I_1(\Omega) - \frac{\lambda}{8} I_0^2(\Omega) \quad (4.32)$$

The mass parameter Ω is in the interval $\Omega \in [\Omega_0, \infty)$, where Ω_0 corresponds to $\phi_C = 0$ through Eq. (4.31),

$$m_0^2 = \Omega_0^2 - \frac{\lambda}{2} I_0(\Omega_0) \quad (4.33)$$

Ω_0 is actually the renormalized mass ($\Omega_0 = m$). To see this, we use Eqs. (4.31) and (4.32) to obtain

$$\left. \frac{dV_G}{d\phi_C^2} \right|_{\Omega=\Omega_0} = \left. \frac{dV_G/d\Omega^2}{d\phi_C^2/d\Omega^2} \right|_{\Omega=\Omega_0} = \frac{1}{2} \Omega_0^2 \quad (4.34)$$

Thus, the GEP has a minimum at $\Omega = \Omega_0$. There is another minimum at $\Omega = \Omega_1 > \Omega_0$ at which $\frac{dV_G}{d\Omega^2} = 0$ (please refer to Appendix A.1 in [64] for details). For a fixed mass Ω_0 , as we vary the coupling constant λ , the difference between the two minima,

$$\Delta V_G = V_G(\Omega_1) - V_G(\Omega_0) \quad (4.35)$$

changes sign. The critical point Ω_c is found at $\Delta V_G = 0$. Using $\frac{dV_G}{d\Omega^2} = 0$, we deduce the critical coupling

$$\lambda_c = \frac{\Omega_c^2 + 2\Omega_0^2}{I_0(\Omega_0) - I_0(\Omega_c)} \quad (4.36)$$

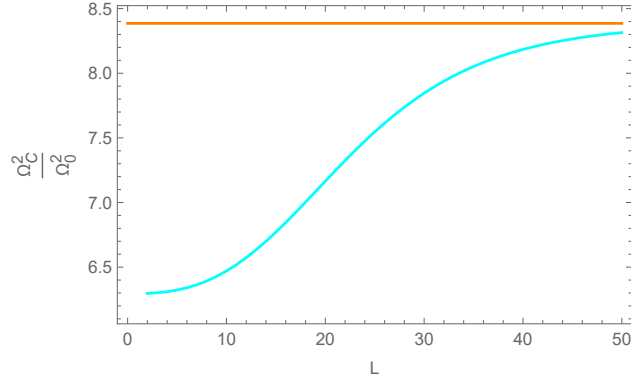
Fig. 4.4 shows how the critical values vary with lattice size for renormalized mass $\Omega_0 = m = 0.1$. We find $\frac{\Omega_c^2}{m^2} = 8.4$ and $\frac{\lambda_c}{m^2} = 60.8$ in the large L limit. The continuum limit is obtained in the limits $L \rightarrow \infty$, $m \rightarrow 0$ (in units in which the lattice spacing is $a = 1$), where we expect to recover the continuum result for the critical point, $\frac{\lambda_c}{m^2} = 61.2$.

4.4 CV Quantum Algorithm

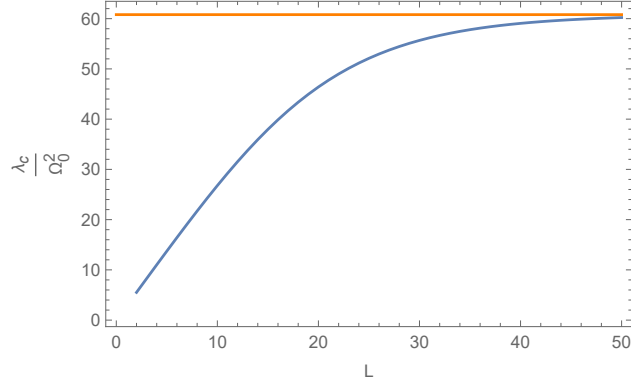
We now carry over the calculation in the previous section to a quantum computation. While we saw that the problem was quite tractable analytically, our goal here is to present the calculation as a quantum computation so that it can be generalized to more complicated Ansätze and/or Hamiltonians. Namely, the inclusion of entanglement as well as non-Gaussian operations would result in exponential computational complexity due to the Gottesman-Knill Theorem for continuous variables [106].

Our calculation of the GEP on quantum hardware relies on the Variational Quantum Eigensolver (VQE) algorithm [69]. To this end, we vary the mass parameter Ω and obtain its optimal value. Technically, a natural direction is to define the bare mass m_0 and perform the minimization to identify the renormalized mass $m = \Omega_0$, which is the required value of Ω to obtain the effective potential at $\phi_C = 0$. In the following, however, we save on resources, and reduce errors, by defining the system with the renormalized mass $m = \Omega_0$ as well as the coupling λ . The bare mass m_0 that characterizes the Hamiltonian (4.27) follows then from the analytical result Eq. (4.33).

Starting from the reference value $\Omega_0 = m$, and associated fields $q(k), p(k)$, we can prepare a system with a different value of the mass parameter Ω , with associated fields $q'(k), p'(k)$,



(a) variational mass parameter, $\frac{\Omega_c^2}{m^2}$



(b) coupling constant, $\frac{\lambda_c}{m^2}$

Figure 4.4: Critical values of the variational mass parameter and coupling constant. These are given for different lattice sizes L and for renormalized mass $\Omega_0 = m = 0.1$.

by realizing that a Bogoliubov transformation relates the two,

$$q'(k) = \cosh r(k)q(k) - \sinh r(k)q(L-k), \quad p'(k) = \cosh r(k)p(k) + \sinh r(k)p(L-k), \quad (4.37)$$

where

$$r(k) = \frac{1}{2} \log \frac{\omega(k)}{\omega'(k)}. \quad (4.38)$$

$\omega(k), \omega'(k)$ are provided by the dispersion relation (4.17) for mass m, Ω , respectively. For $k \neq 0, \frac{L}{2}$, this transformation is realizable via a two-mode squeezer $S_2(k, L-k; r(k))$, where

$$S_2(k, k'; r) = e^{r[a^\dagger(k)a^\dagger(k') - a(k)a(k')]} \quad (4.39)$$

Equivalently, it can also be prepared with beam splitters and single-mode squeezers as

$$S_2(k, k'; r) = BS \cdot S(k; r)S(k'; -r) \cdot BS, \quad (4.40)$$

where $S(k; r) = e^{r(a^{\dagger 2}(k) - a^2(k))}$ is a single-mode squeezer, and BS is a 50/50 beam splitter. For $k = 0, \frac{L}{2}$, the transformation (4.37) reduces to single-mode squeezing, and we have

$$\Omega = e^{-r(0)}m \quad (4.41)$$

See Appendix B for information on squeezer gates.

Thus, the state $|\Psi_{\phi_C, \Omega}\rangle$ (Eq. (4.23)) is prepared using L qumodes using the quantum circuit shown in Fig. 4.5. In the Figure, the argument of the displacement is identical to Eq. (4.23), which follows from Eqs. (4.37) and (4.38) with $k = 0$. However this new form enables us to express everything in terms of quadratures for the system with $\Omega = m$, which is how we choose to expand the Hamiltonian in a fixed form for our quantum computation. Equipped with the Ansatz state $|\Psi_{\phi_C, \Omega}\rangle$, we now need to determine how to compute its energy $\langle H \rangle$. To calculate expectation values on a CV substrate, we proceed by performing photon-number measurements. (A.3),

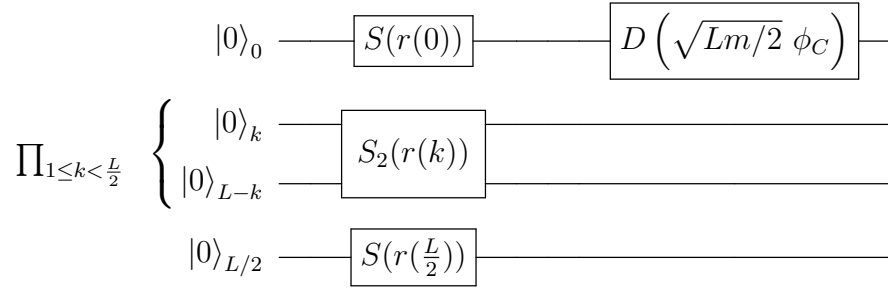


Figure 4.5: CV quantum circuit for the Gaussian Ansatz state (4.23). It contains a qumode for each lattice site, two ancillary qumodes, and features single- and two-mode squeezers, and a displacement.

We thus seek to transform the computation into one that relies on the calculation of expectation values of number operators (or their powers) for the various modes labeled by k ,

$$N(k) = a^\dagger(k)a(k) = \frac{1}{2} [q^2(k) + p^2(k) - 1] \quad (4.42)$$

Expectation values of terms in the Hamiltonian that are proportional to $N(k)$, such as the k^{th} term in the non-interacting Hamiltonian are straightforward to compute with photon-number measurements of the mode k . A little more work is required for computing expectation values of the quadratures, such as $\langle q^{2n}(k) \rangle$. To understand the course of action we take to deal with such terms, consider how expectation values can be calculated on a qubit quantum device. In Chapters 2 and 3, we similarly noted that Pauli strings containing only Pauli Z operators could be calculated by directly implementing and measuring the variational circuit we developed in those Chapters, while we had to append extra single-qubit gates for strings containing Pauli X or Y operators.

For instance, Hadamard gates H were applied when X operators were present, due to the identity $X = HZH$.

We take a similar approach for qumodes, that is, append additional gates to the circuit to address Hamiltonian terms not immediately expressible in terms of number operators $N(k)$. Suppose we apply the single-qumode quadratic phase gate $P(\Gamma) \equiv e^{i\Gamma q_k^2/2}$ (see Appendix B) to each qumode q_k independently. Note that the quadratic phase gate can be decomposed in terms of a single rotation and single-mode squeezer gate. It transforms the number operator, N_k , as

$$\begin{aligned} N_k = \frac{1}{2} (q^2(k) + p^2(k) - 1) &\rightarrow e^{-i\Gamma q_k^2/2} N_k e^{i\Gamma q_k^2/2} \\ &= \frac{1}{2} [q^2(k) + (p(k) + \Gamma q(k))^2 - 1] \\ &= N_k + \frac{1}{2} \Gamma (q(k)p(k) + p(k)q(k)) + \frac{\Gamma^2}{2} q^2(k), \end{aligned} \quad (4.43)$$

where we used the transformations (B.18) induced by the P gate in the second line. The generation of the third term without purely squeezing away the p quadrature is the key to

how we can measure expectation values of (even powers of) the quadratures with realistic technology. In fact, we can isolate this third term by taking derivatives with respect to Γ . However, it is experimentally easier to implement an evolution Hamiltonian that does not contain quadratic powers of creation and annihilation operators per qumode. Therefore, we will instead construct our quantum circuits out of “CX” gates. They can be decomposed in terms as a two-mode squeezer sandwiched between two beam splitters, and each of these operations takes an evolution Hamiltonian which is bi-linear in the qumode quadratures. The result has a similar effect to (4.43), and relies on ancillary qumodes.

To calculate $\langle q^2(k) \rangle$, the ancilla is prepared in some initial state $|\phi\rangle_{\text{anc}}$. It is entangled with our input state $|\Psi_{\phi_C, \Omega}\rangle$ using the CX gate $e^{-i\Gamma p_{\text{anc}} \otimes q(k)}$. We obtain the state

$$|\Phi(\Gamma)\rangle = e^{-i\Gamma p_{\text{anc}} \otimes q(k)} |0\rangle_{\text{anc}} \otimes |\Psi_{\phi_C, \Omega}\rangle \quad (4.44)$$

Measuring the photon number of the ancillary qumode, $N_{\text{anc}} = a_{\text{anc}}^\dagger a_{\text{anc}}$, we obtain the expectation value $\langle \Phi(\Gamma) | N_{\text{anc}} | \Phi(\Gamma) \rangle$. Similar to Eq. (4.43), the Heisenberg picture yields

$$\begin{aligned} N_{\text{anc}} &= \frac{1}{2} (q_{\text{anc}}^2 + p_{\text{anc}}^2 - 1) \rightarrow e^{i\Gamma p_{\text{anc}} \otimes q(k)} N_{\text{anc}} e^{-i\Gamma p_{\text{anc}} \otimes q(k)} \\ &= \frac{1}{2} [(q_{\text{anc}} + \Gamma q(k))^2 + p_{\text{anc}}^2 - 1] \\ &= N_{\text{anc}} + \Gamma q_{\text{anc}} q(k) + \frac{\Gamma^2}{2} q^2(k) \end{aligned} \quad (4.45)$$

where we used the transformations (B.20) induced by the CX gate in the second line. Thus,

$$\begin{aligned} \langle \Phi(\Gamma) | N_{\text{anc}} | \Phi(\Gamma) \rangle &= {}_{\text{anc}} \langle \phi | N_{\text{anc}} | \phi \rangle_{\text{anc}} + \Gamma {}_{\text{anc}} \langle \phi | q_{\text{anc}} | \phi \rangle_{\text{anc}} \langle \Psi_{\phi_C, \Omega} | q(k) | \Psi_{\phi_C, \Omega} \rangle \\ &\quad + \frac{\Gamma^2}{2} \langle \Psi_{\phi_C, \Omega} | q^2(k) | \Psi_{\phi_C, \Omega} \rangle. \end{aligned} \quad (4.46)$$

This procedure becomes ideal if the initial state of the ancilla $|\phi\rangle_{\text{anc}}$ is the vacuum state $|0\rangle_{\text{anc}}$ and the first two terms drop out. After differentiating with respect to Γ twice and

setting $\Gamma = 0$, the ancillary qumode decouples and we obtain the desired expectation value,

$$\langle q^2(k) \rangle = \frac{d^2}{d\Gamma^2} \langle \Phi(\Gamma) | N_{\text{anc}} | \Phi(\Gamma) \rangle \Big|_{\Gamma=0} \quad (4.47)$$

Expectation values of higher powers of the quadratures can be obtained in a similar fashion by generalizing the above procedure and working with higher-order derivatives (see Appendix A.1).

In our case, this procedure is followed for quadratures corresponding to the $k = 0, \frac{L}{2}$ qumodes, assigning one ancillary qumode to each physical mode. For $k \neq 0, \frac{L}{2}$, we can take advantage of a pairing of the $k, L - k$ modes, which is present in the Hamiltonian, and let them behave as one another's ancillary qumode. This helps us to reduce resources for the computation. The terms in the Hamiltonian containing the $(k, L - k)$ qumode pairs can be measured by computing the expectation values $\langle N(k) \pm N(L - k) \rangle$ and $\langle [N(k) \pm N(L - k)]^2 \rangle$. The required CX gates is $e^{-i\Gamma p(k) \otimes q(L-k)}$.

The complete circuit, including the Gaussian Ansatz state, appended CX gates, and photon-number measurements, is shown on Fig 4.6. The beam splitter (*BS*) gates are justified in Appendix A.1.

Let us take a moment to review the practicality running this circuit in a laboratory setting. Without the CX gates, which are responsible for taking a prepared quantum state and processing it for an expectation value calculation, the circuit has a very experimentally realizable form. It begins with a set of two-mode squeezed vacua each of which can be prepared using non-degenerate parametric down-conversion (Appendix B). Afterwards, only beam splitters and a displacement are required, and these can be realized with linear optical elements. Unfortunately, the CX gates appended to the circuit require “in-line” squeezing in their implementation. A direct in-line squeeze is difficult if not unfeasible due to the need to inject an arbitrary quantum state into a non-linear medium supplied with a strong pump mode, which introduces significant noise [107]. An alternative is to use a resource squeezed state prepared offline. Using homodyne measurements and feed-forward, one can realize an effective in-line squeezer [108].

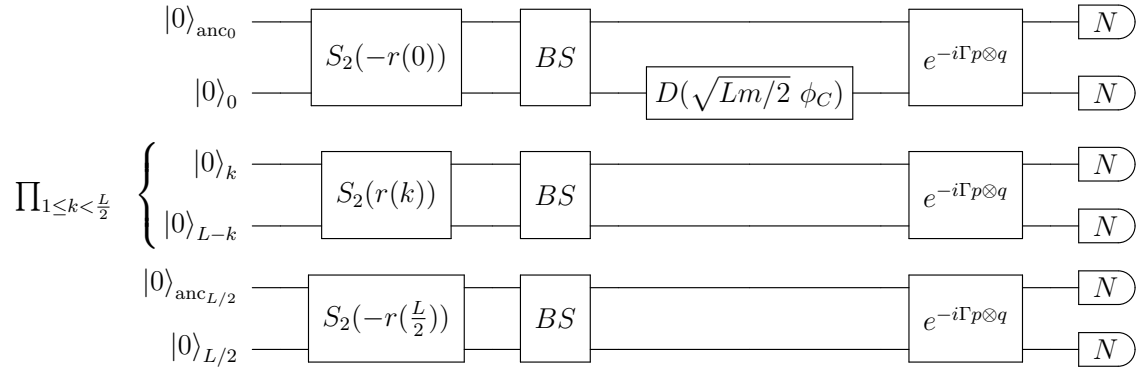


Figure 4.6: CV quantum circuit required for computing expectation values of the various terms in the Hamiltonian. It contains a qumode for each lattice site, two ancillary qumodes, and features two-mode squeezers S_2 , a displacement, 50/50 beam splitters, CX gates and photon-number measurements.

The same method has even been shown to produce the CX gate itself using two offline squeezers [109]. In any case, it is important to note that the CX gate ($e^{-ip \otimes q}$) depicted in Figure 4.6 is not necessarily something applied directly, but may require resource states for its execution.

One way to compute derivatives with respect to the parameter Γ (Eq. (4.47)) is to approximate them using finite differences for small Γ . This results in significant error when applied in a noisy setting. Instead, we construct appropriate linear combinations of expectation values,

$$\sum_i c_i \langle \Phi(\Gamma_i) | f(N) | \Phi(\Gamma_i) \rangle, \quad (4.48)$$

where $f(N)$ is a polynomial function of number operators. This form allows us to evaluate expectation values of quadratures exactly. This form bears a resemblance to the CV version of “parameter shift rules” [110], and they may be used to compute some derivatives of expectation values exactly. Conveniently, the set of CX gates form the final layer in the circuit before measurement. It follows that parameter shifts of Γ will be useful to us regardless of the form of the trial state. This can be seen with an example. The expectation value (4.47) can be computed using the exact expression

$$\langle q^2(k) \rangle = \frac{1}{2s^2} [\langle \Phi(s) | N_{\text{anc}} | \Phi(s) \rangle + \langle \Phi(-s) | N_{\text{anc}} | \Phi(-s) \rangle - 2 \langle \Phi(0) | N_{\text{anc}} | \Phi(0) \rangle] \quad (4.49)$$

Similar linear combinations can be found for all other expectation values of quadratures required for computing the energy of the trial state $\langle H \rangle$. Details can be found in Appendix A.1. For our calculations we simply took $s = 1$. This value is large enough to reduce effects of sampling error (the same errors that would prevent an effective use of finite differences), and small enough to keep truncation error under control in simulations. By expressing the CX gate in terms of beam splitters and squeezers [111], we see that the value $s = 1$ corresponds to about 4 dB squeezing which is well within the ~ 10 -15 dB limit of today’s technology [112, 113].

For the variational algorithm, we also strive to compute exact derivatives with respect to the mass parameter Ω . This helps reduce sampling error in our VQE results, as we will see below. All gates in our Ansatz are Gaussian, and this allows us to apply a parameter shift rule for derivatives with respect to the $r(k)$, each of which in turn contains the single parameter Ω .

Consider the derivative

$$\frac{d}{dr(k)} \langle \Phi(\Gamma) | N_{\text{anc}} | \Phi(\Gamma) \rangle \quad (4.50)$$

of the expectation value that contributes to $\langle q^2(k) \rangle$ (Eq. (4.49)). It can be computed with the following parameter shift rule:

$$\frac{d}{dr(k)} \langle N_{\text{anc}} \rangle_{\phi_C, r(k)} = \frac{1}{\sinh t} \left[\langle N_{\text{anc}} \rangle_{\phi_C \cosh \frac{t}{2}, r(k) + \frac{t}{2}} - \langle N_{\text{anc}} \rangle_{\phi_C \cosh \frac{t}{2}, r(k) - \frac{t}{2}} \right] \quad (4.51)$$

Therefore, our derivative calculation requires us to take $\phi_C \rightarrow \phi_C \cosh \frac{t}{2}$ in our circuits. To see why this is, let X and Y denote a pair of quadratures (q or p) each which can operate on either of the qumodes that are being coupled in Fig. 4.6. According to the Figure, the two qumodes are squeezed using equal and opposite squeezing parameters with magnitude $|r(k)| = r$. Therefore,

$$XY \rightarrow e^{2\sigma r} XY, \quad \sigma = 0, \pm 1 \quad (4.52)$$

under the two squeezes. If we then subtract the result of $r \rightarrow r - t/2$ from $r \rightarrow r + t/2$, we get

$$2\sigma \sinh t e^{2\sigma r} XY = \sinh t \frac{d}{dr} (e^{2\sigma r} XY) \quad (4.53)$$

On the other hand, if we only have one quadrature X and a constant c , and do the same, we find

$$2\sigma \sinh \frac{t}{2} e^{\sigma r} cX = 2 \sinh \frac{t}{2} \frac{d}{dr} (e^{\sigma r} cX) \quad (4.54)$$

Therefore, rescaling c by $\cosh \frac{t}{2}$ will allow us to recover the factor of $\sinh t$ in (4.53). Now, the operand of the derivative in Eq. (4.50) may be written as

$$\langle 0 | S_a^\dagger(r) S_b^\dagger(-r) \mathcal{O} S_a(r) S_b(-r) | 0 \rangle, \quad (4.55)$$

where $\mathcal{O}$ is a polynomial of degree 2 in the quadratures and a, b are the coupled qumodes. Taking X and Y to be the appropriate quadratures in each term of this polynomial, and $c \rightarrow \phi_C$, we recover (4.51).

For the terms quadratic in the number operators, we similarly employ the rule

$$\frac{d}{dr(k)} \langle N_{\text{anc}}^2 \rangle_{\phi_C, r(k)} = \frac{2}{\sinh t} \left[\langle N_{\text{anc}}^2 \rangle_{\phi_C \sqrt{\cosh \frac{t}{2}}, r(k) + \frac{t}{4}} - \langle N_{\text{anc}}^2 \rangle_{\phi_C \sqrt{\cosh \frac{t}{2}}, r(k) - \frac{t}{4}} \right]. \quad (4.56)$$

In this case the parameter shift does not give the correct factors for the odd powers of $q(k)$ in the derivative of $\langle \Phi(\Gamma) | N_{\text{anc}}^2 | \Phi(\Gamma) \rangle$. However, these terms do not contribute since odd powers of q map even number states to odd number states, the latter of which do not contribute to the squeezed vacuum of a single qumode. These expressions allow the calculation of derivatives with respect to squeezing parameters, and consequently the mass parameter Ω , using qumode photon number expectation values.

In the general case, one would encounter non-Gaussian gates in the Ansatz circuit, in which case effective parameter shift rules would be more difficult to formulate. Ref. [114], for instance, proposes an Ansatz where a non-Gaussian operator follows the initial squeeze of the vacuum. In Appendix A.2 we write down quantum circuits for computing derivatives with respect to Ω in the presence of non-Gaussian gates. This circuit was used in our simulation results.

We can play the same game with derivatives with respect to the field shift ϕ_C . For example, the derivative of a single power of N_{anc} can be computed using the displacement parameter shift rule:

$$\frac{d}{d\phi_C} \langle N_{\text{anc}} \rangle_{\phi_C, r(k)} = \frac{1}{2t} \left[\langle N_{\text{anc}} \rangle_{\phi_C + t, r(k)} - \langle N_{\text{anc}} \rangle_{\phi_C - t, r(k)} \right] \quad (4.57)$$

However, the fact that the displacement is applied last in the circuit (at least before diagonalizing operations from the CX gates) gives us an opportunity to simplify the calculation, and compute the derivative in a way that does not require preparing and measuring circuits with two separate parameters. It is realized by directly transforming the quadrature operators, i.e.,

$$q(k) \rightarrow q(k) + \sqrt{Lm/2} \phi_C, \quad p(k) \rightarrow p(k). \quad (4.58)$$

This results in a polynomial in ϕ_C which is straightforward to differentiate. This approach applies to any variational calculation of the effective potential where ϕ_C is directly controlled by a final displacement. We note that in momentum space this also only needs to be applied to the zero mode $q(0)$. Details relevant for our system are given in Appendix A.2, and they were used in derivative calculations for our CV simulations.

Let $\Omega = \Omega_1$ be the value of the variational mass parameter that minimizes the energy $\langle H \rangle$ at some displacement ϕ_C . As discussed in the previous Section (Eq. (4.35)), we obtain the critical coupling by identifying the point at which the difference

$$\Delta \langle H \rangle = \langle H \rangle_{\Omega=\Omega_1, \phi_C} - \langle H \rangle_{\Omega=m, \phi_C=0}, \quad (4.59)$$

in the case the ϕ_C is the minimum of the effective potential, changes sign as we vary the dimensionless coupling $\tilde{\lambda} = \frac{\lambda}{m^2}$. Recall that we chose the initial value of the variational mass parameter $\Omega = m$ which is related to the bare mass m_0 through (4.33). This keeps us from having to apply a variational algorithm to compute the minimum $\langle H \rangle$ for $\phi_C = 0$. This in turn helps reduce the effects of noise, both from sampling error, as well as from machine noise on qubit hardware which we employ in the next Section.

For small lattice sizes, it is possible to reduce the amount of computational resources even further by setting the squeezing parameters $r(k) = 0$ for values of k close to $\frac{L}{2}$ (a momentum cutoff), with little shift in the location of the critical point. The reason for this can be seen

from Eq. (4.38). Recall that to track the continuum limit we set our renormalized mass $m \ll 1$. We find that for $k \sim \frac{L}{2}$ the value of $r(k) \sim m^2$.

This results in the replacement of the expectation values of these modes with their vacuum expectation values, e.g., $\langle \Psi | q^2(k) | \Psi \rangle \rightarrow \langle 0 | q^2(k) | 0 \rangle = \frac{1}{2}$. This reduction is also necessary when implementing the algorithm on a qubit architecture in order to reduce machine noise to a manageable level. For lattice size $L \lesssim 10$, it turns out that there is no harm in doing this for all $k \neq 0$. For larger L , the error increases and it becomes necessary to compute expectation values away from the vacuum for some of these nonzero modes.

Displayed in Figure 4.7 are results from noiseless simulations of the CV algorithm, for $L = 10$ sites. They were obtained with the assistance of CV gates included in *bosonic qiskit* [99]. Since our interest is in locating the minimum of the effective potential, the VQE algorithm was employed to minimize the energy functional $\langle H \rangle_{\Omega=\Omega_1, \phi_C}$ with respect to both the mass parameter Ω as well as the field shift ϕ_C . In panel (a), we compare VQE results from a non-gradient based optimizer (“COBYLA”) with those from gradient descent (GD). We see that significantly less shots (2048) are needed to identify the optimal parameters in the gradient descent method. However, a large number of shots (100,000) are required when evaluating $\Delta \langle H \rangle$ at these parameters. We obtain the critical point by extrapolating the GD VQE energies to the x -intercept. The dashed blue curve is classical and corresponds to the case $r(k) = 0$ for all $k \neq 0$. We find a value of $\tilde{\lambda} = 28.0 \pm 0.7$, which contains the value obtained from the dashed curve, $\tilde{\lambda} = 27.5$, within its error bars. In panel (b), VQE energies are computed for larger values of $\tilde{\lambda}$, and a quadratic extrapolation is performed. We obtain a value of 28.9 ± 1.0 . The low end of this range has about a 1.5% error with the value obtained from the dashed curve. In all of these calculations, a Hilbert space cutoff of $n = 32$ is used. In *bosonic qiskit*, this is achieved by assigning 5 qubits for each qumode. In Figure 4.8 we display results for a larger lattice size, $L = 30$. Here, we use only gradient descent for minimization. It is necessary to increase the number of modes with $r(k) \neq 0$ up to $k = 3$ in order to obtain a good approximation of the case where all modes are squeezed.

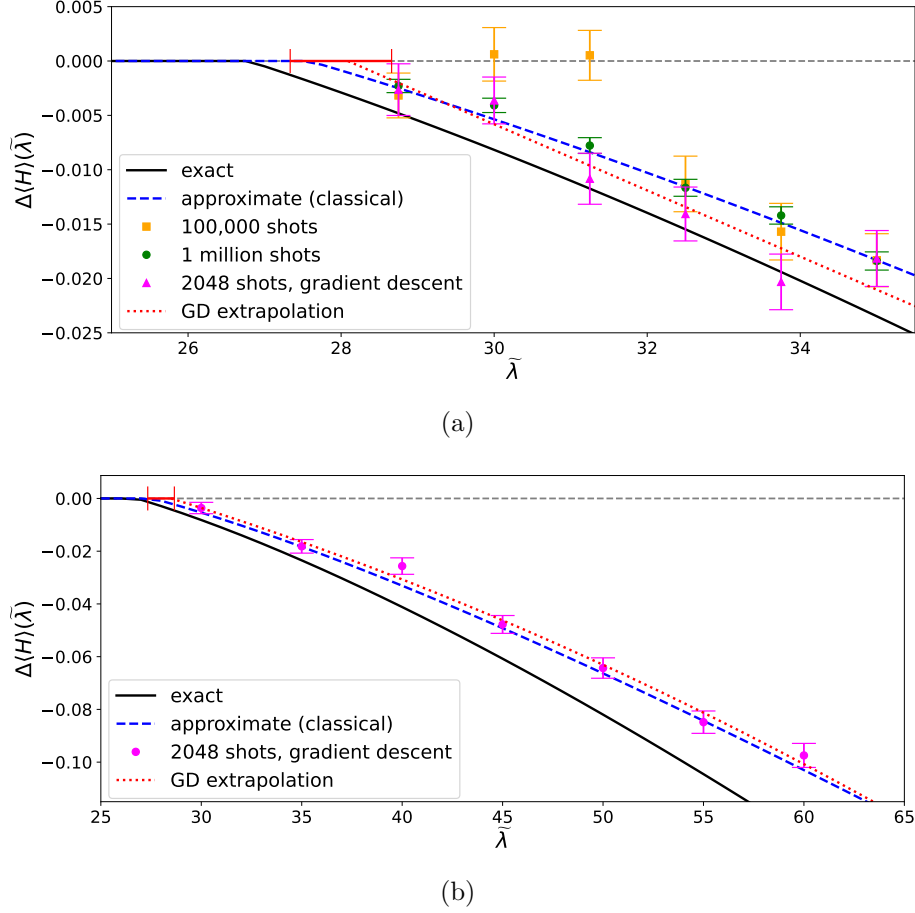


Figure 4.7: CV simulation results for lattice size $L = 10$ and renormalized mass $m = \Omega_0 = 0.1$. The solid black curve is the exact difference $\langle H \rangle_{\text{VQE}} - \langle 0 | H | 0 \rangle$ at a Hilbert space truncation of $n = 32$, while the dashed blue curve sets the squeezing parameter $r(k) = 0$ for all modes with $k \neq 0$. Displayed in panel (a) are points obtained using the COBYLA optimizer for two different numbers of shots, as well as points obtained using gradient descent (GD). In the latter, we take 2048 shots to find the optimal parameters and 100,000 shots to evaluate the expectation value at the optimal parameters. Points are plotted near the transition and the GD results are linearly extrapolated (dotted red line) to the critical point. In panel (b), points obtained using gradient descent are plotted on a larger scale, and quadratic extrapolation (dotted red line) is used to find the critical point. The value of $\langle H \rangle$ at $\phi_C = 0$ is obtained classically to reduce sampling error. The CX gate parameter shift is $s = 1$. Error bars were obtained using bootstrap re-sampling [115].

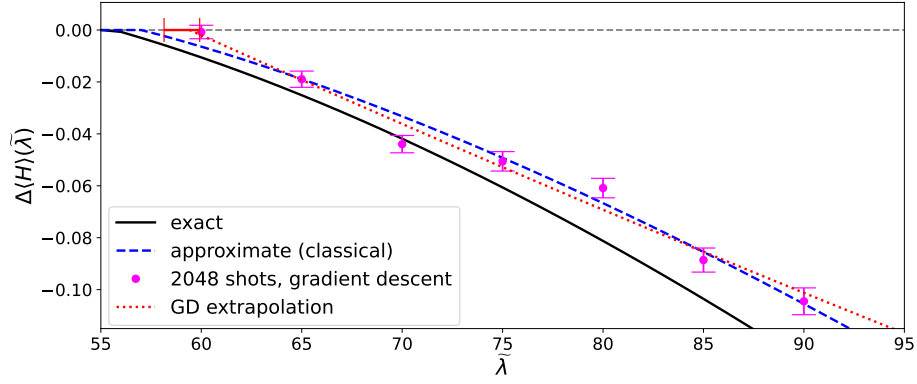


Figure 4.8: CV simulation results for lattice size $L = 30$ and renormalized mass $m = \Omega_0 = 0.1$. The solid black curve is the exact difference $\langle H \rangle_{\text{VQE}} - \langle 0 | H | 0 \rangle$ at a Hilbert space truncation of $n = 32$, while the dashed blue curve sets the squeezing parameter $r(k) = 0$ for modes with $k > 3$. 2048 shots were used to obtain the optimal parameters. Quadratic extrapolation (dotted red line) was used to find the critical point. The value of $\langle H \rangle$ at $\phi_C = 0$ was obtained classically to reduce sampling error. The CX gate parameter shift is $s = 1$. Error bars were obtained using bootstrap re-sampling [115].

We locate the critical point at $\tilde{\lambda} = 59 \pm 0.9$. The low end of the range has about a 1.9% error with the value obtained from the dashed curve, $\tilde{\lambda} = 57$.

To get a closer look at the continuum limit, we increased the lattice size to $L = 76$, with the necessary squeezes applied to all qumodes, in contrast to Figures 4.7 and 4.8 where only a few modes were squeezed. Simulation time was reduced by employing a *statevector*-based simulator which corresponds to the infinite shot limit. This made it possible to reduce truncation error by taking $s = 0.1$ instead of $s = 1$ due to the lack of sampling error, which in turn allowed us to use only 4 qubits per qumode instead of 5. This also allowed us to take advantage of the COBYLA optimizer, which in our simulations was more efficient but behaved poorly in the presence of noise, see Figs. 4.7 and 4.8. The results are displayed in Figure 4.9. We found a level crossing at $\tilde{\lambda} = 61.1$, which is very close to the limiting value obtained for $\Omega_0 = 0.1$, $\tilde{\lambda} = 60.8$ (see panel (b) in Figure 4.4), and has a 0.33% error with the continuum critical value $\tilde{\lambda} = 61.3$.

4.5 DV Quantum Algorithm

It is worth investigating how this problem may be approached a discrete-variable (DV) approach in order to better understand the advantages and disadvantages of each method. In this Section, we calculate the phase transition on two IBM devices, IBM Q Kolkata and IBM Q Mumbai. We will find that very significant (and yet successful) simplifications will allow us to approach the calculation in this manner.

To begin, it turns out that much can be achieved by assigning just two qubits to each pair of modes $(k, L - k)$. This is possible for two reasons. First, we are able to remove the final displacement operation from the quantum circuit and apply it backwards as a similarity transformation on the Hamiltonian,

$$H \rightarrow \tilde{H} = D^\dagger \left(\sqrt{Lm}\phi_C \right) \cdot H \cdot D \left(\sqrt{Lm}\phi_C \right) , \quad (4.60)$$

where D acts only on the zero mode.

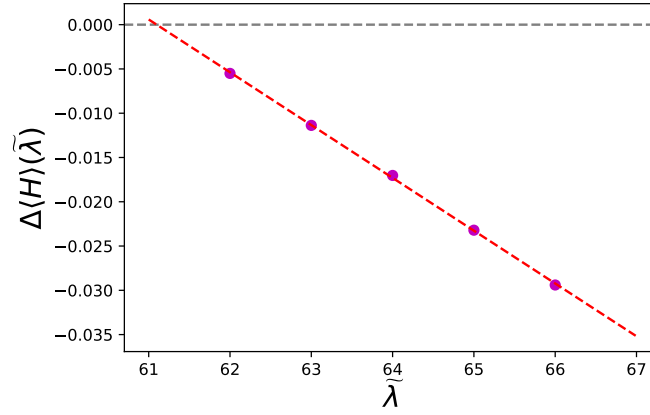
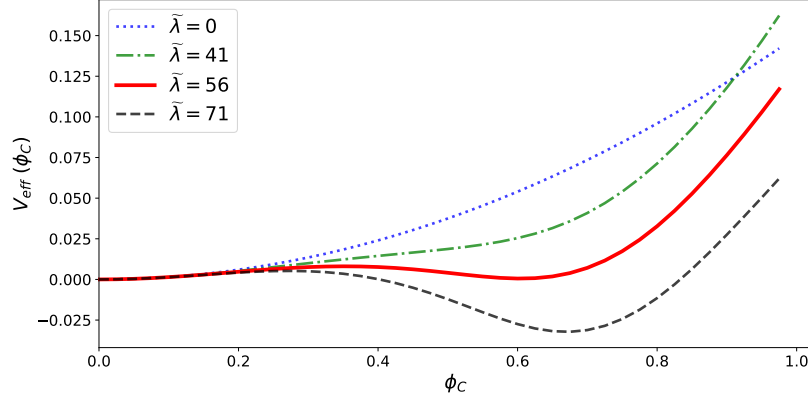


Figure 4.9: Noiseless CV simulation results for lattice size $L = 76$ and renormalized mass $m = \Omega_0 = 0.1$. The minimized energy differences are extrapolated to the x -axis, where a level-crossing at $\tilde{\lambda} = 61.1$ is identified. We used a CX gate parameter shift of $s = 0.1$ and a qumode Hilbert space dimension cutoff of $n = 16$. These results correspond to the infinite shot limit.

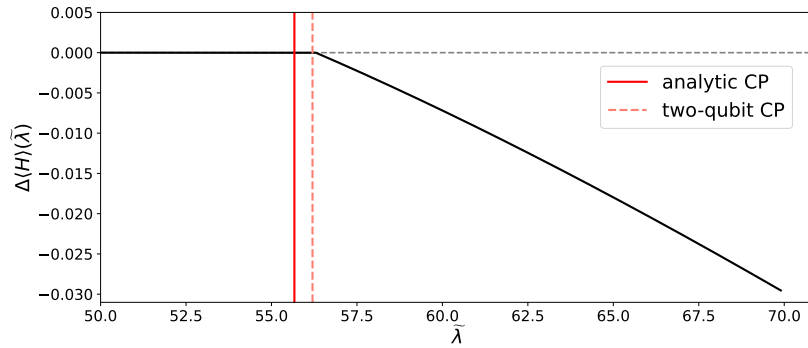
In the previous section, we used this property to simplify the ϕ_C derivative calculation. Second, the remaining operations in the quantum circuit only produce photons in pairs. On one hand, for $k = 0, \frac{L}{2}$, a single squeeze operator creates only even-photon-number states from the vacuum. On the other, the two-mode squeezes acting on the $(k, L - k)$ pairs of modes with $k \neq 0, \frac{L}{2}$ produce the collective number states $|00\rangle, |11\rangle, |22\rangle, \dots$, and so a reduced number of qubits encode such states at a given truncation level. We will treat the $(k, L - k)$ pair of modes as a single entity (“mode”) labeled by k ($1 \leq k < \frac{L}{2}$), and use two qubits to represent it.

The effectiveness of this truncation may be seen from Figure 4.10 where we see that, when disregarding noise, two qubits bring us to a level crossing that is within 1% of the true value for lattice size $L = 30$ and mass $m = 0.1$. By level crossing, we are referring to the value of the coupling $\tilde{\lambda} = \frac{\lambda}{m^2}$ where the absolute minimum of the effective potential first appears at a non-zero value of ϕ_C , when turning up the coupling. Note also that this crossing is already within 10% of the continuum critical point $\tilde{\lambda} = 61.2$. Increasing the lattice size to $L \gtrsim 40$, the truncation error turns out to have a more significant effect, and it becomes necessary to include more of the infinite Hilbert space of each qumode to remain within 1% of the true level crossings for the lattice theory.

For the sake of computing the 10-site case on quantum hardware, two qubits are enough. Unlike the CV case, it is not necessary to entangle different modes (that is, after characterizing the $(k, L - k)$ pair as a single “mode”, for $1 \leq k < \frac{L}{2}$). Therefore, each mode requires a circuit that contains only one CNOT, and this is the maximum number required to prepare a two-qubit state from the qubit $|00\rangle$ state. One may see this by considering a Schmidt decomposition of the two-qubit system [116], the corresponding circuit shown in Figure 4.11. As an aside, Schmidt circuit structures for arbitrary three- and four-qubit circuits have been shown to require less CNOT’s than the circuits output by IBM qiskit’s *isometry* function, which are based on the circuit decomposition methods of Ref. [117]. In any case, it is desirable to write down the most shallow circuit representing the squeezed vacuum state so that we may compute the ϕ^4 phase transition with current quantum hardware. This



(a) GEP



(b) $\Delta\langle H\rangle$

Figure 4.10: Gaussian Effective Potential (GEP) and $\Delta\langle H\rangle$ vs $\tilde{\lambda} \equiv \frac{\lambda}{m^2}$. The numerical results are given for $L = 30$ sites, with renormalized mass $m = \Omega_0 = 0.1$, determined using two qubits per qumode. The potential in (a) is rescaled by L and plotted for several values of the coupling $\tilde{\lambda}$. In (b), the first-order level crossing occurs near $\tilde{\lambda} = 56$, to be compared with the continuum phase transition at $\tilde{\lambda} = 61.2$. The solid black curve shows energy differences for the two-qubit truncation, with the level crossing identified with the dashed vertical line. The analytical critical point (CP) is indicated with the solid vertical line (see panel (b) in Figure 4.4).

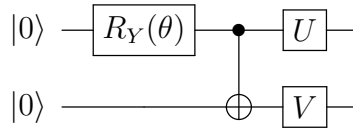


Figure 4.11: Preparation of an arbitrary two-qubit state. The circuit constructs the Schmidt decomposition of the bipartite system, as described in Ref. [116]. $R_Y(\theta)$ is a y -rotation and U, V are single-qubit unitary gates. Only one CNOT gate is needed.

approach thus differs from what is done in Bosonic Qiskit [99], where the necessary DV gates are applied so that unitary operations simulating the action of CV gates are precisely realized, up to a given numerical truncation (number of qubits per qumode).

The calculation of powers of quadrature operators $q(k), p(k)$ proceeds in a fashion similar to the approaches used in Chapters 2 and 3. We expand the quadratures in the Hamiltonian as linear combinations of two-qubit Pauli strings, e.g.,

$$q^2(k) = \sum_{i,j=0}^3 c_{i,j}^k \sigma_i \otimes \sigma_j, \quad (4.61)$$

where σ_i is a Pauli matrix and $\sigma_0 = \mathbf{I}_{2 \times 2}$. Since we have moved the final displacement into the Hamiltonian as a similarity transformation, the coefficients $c_{i,j}^0$ of the zero-mode quadrature expansions are functions of the displacement parameter ϕ_C . All operators that we are concerned with are real, and so there are only ten Pauli operators that we need to worry about. As before, we evaluate the expectation values of these operators by diagonalizing them in the computational (Pauli Z) basis using single qubit operations. An example relevant to the current case is to evaluate $\langle \Psi | X \otimes Z | \Psi \rangle$ using

$$\langle \Psi | X_0 Z_1 | \Psi \rangle = \langle \Psi | H_0 Z_0 Z_1 H_0 | \Psi \rangle, \quad (4.62)$$

where X and Z are Pauli matrices, H is the Hadamard gate and $H_0 | \Psi \rangle$ is the circuit to be measured on the quantum computer. Some operators commute in a bit-wise fashion, so their expectations may be calculated using measurement results from the same circuit. For our case, only five circuits need to be run on the quantum computer, for each mode and iteration.

Like the previous Section, we desire a gradient based optimization for VQE. For the squeezing parameter $r(k)$ of mode k , we used

$$\frac{d}{dr(k)} \langle S^\dagger(r(k)) A S(r(k)) \rangle = \frac{i}{2} \langle S^\dagger(r(k)) [A, q(k)p(k) + p(k)q(k)] S(r(k)) \rangle, \quad (4.63)$$

where $S(r) = e^{\frac{i}{2}r(qp+pq)}$ is a single-mode squeezer (the two-mode squeezers are handled similarly). Since the Hilbert space has been substantially truncated, we cannot simplify the commutator using the fundamental commutation relations $[q(k), p(k')] = i\delta_{k,k'}$. Instead, similar to the original quadrature operators, we expanded the matrix for the commutator in terms of the ten two-qubit Pauli operators. We then used the results of the same five circuits as above to calculate derivatives of expectation values. The form (4.63) is key in implementing ADAPT-VQE [118], an iterative algorithm where new candidate gates are applied to the circuit at each iteration, and the derivatives with respect to their parameters are computed for selecting the best candidate to append to the circuit for the next iteration. In our case, the circuit remains fixed, but Eq. (4.63) is still relevant as the single- and two-mode squeezes are in the last (and only) layer of the circuit.

Derivatives with respect to ϕ_C are evaluated by obtaining the commutators of zero mode quadratures with the displacement generator, and then making the Pauli expansion coefficients of the resulting matrix be functions of the displacement parameter as before.

Error Mitigation

Despite the rather simple structure of our qubit calculation (two-qubit circuits with only one CNOT gate each), it is still necessary to apply our usual error mitigation techniques. This is because we are searching for a value of the dimensionless coupling $\tilde{\lambda} = \frac{\lambda}{m^2}$, where $\Delta\langle H \rangle$ (Eq. (4.59)) becomes non-zero. In the presence of noise, it is challenging to resolve this point precisely. It is our goal, therefore, to reduce this noise. This includes sampling error, which can be decreased by taking a large number of shots. $\Delta\langle H \rangle$ was evaluated at the optimal parameters using 100,000 shots. However, as with the CV case, 2048 shots was sufficient for obtaining the optimal parameters when using gradient descent (GD) optimization.

For machine noise, we once more apply readout (RO) and zero noise (ZNE) error mitigation. In the case of RO mitigation, only four calibration experiments had to be run given our two-qubit setting, so we made use of IBM Qiskit's RO mitigation routine once more.

As for zero noise extrapolation, Figures 4.12 and 4.13 show that the total number of pairs to add, as well as the order of the extrapolation, depends on the quantum device the calculation is performed on and the severity of the errors encountered at the time the device is accessed. In Figure 4.12 and panel (a) of Figure 4.13, we see that even for 9 CNOTs per CNOT, the errors do not grow too large, and therefore a linear extrapolation incorporating all five points helps to get a good ZNE. On the other hand, panel (b) in Figure 4.13 depicts significantly more growth in the error, even when using the same qubits as in panel (a). Therefore, linear extrapolation does not give an accurate ZNE when going up to 9 CNOTs per CNOT and a better estimate is obtained with fewer points.

Limiting the maximum number of additional CNOT gates such that the error growth remains under control is the focus of Refs. [119, 120]. The alternatives presented in these References reduce the number of CNOT gates needed for extrapolation at the expense of increased sampling error. However, we proceeded with the usual form of ZNE since we already required a large number of shots to deal with the sampling error. However, a higher truncation level or more complicated Ansätze will yield deeper circuits, and these alternatives might prove to be advantageous in those cases.

Quantum Hardware Results

Results from IBM Q hardware are displayed in Figure 4.14. As with the CV quantum algorithm for 10 sites, we set $r(k) = 0$, for all $k \neq 0$, and replace expectation values of nonzero modes with their vacuum expectation values. VQE minimization is performed using GD, and derivatives are computed using only 2048 shots. Furthermore, ZNE was not necessary for locating reasonable optimal parameters. Several devices were used for minimization: IBM Perth, Nairobi and Lagos. To judge how well the minimization step was performed, we evaluated classically the value of $\Delta\langle H \rangle$ at the obtained optimal parameters. These values are given in Figure 4.14, labeled “hybrid”, and are seen to agree with fully classical results.

We also computed the value of $\Delta\langle H \rangle$ at the optimal parameters using quantum hardware. To get acceptable results, we needed 100,000 shots as well as ZNE (see Figures 4.12 and

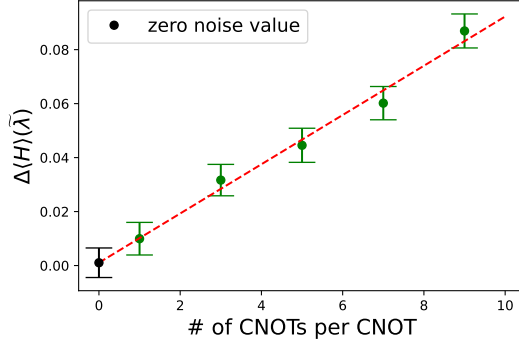


Figure 4.12: CNOT extrapolation on IBM Q Mumbai. The extrapolation is performed for the evaluation of $\Delta\langle H \rangle$ at coupling $\tilde{\lambda} = 30$ evaluated at the optimal parameters for lattice size $L = 10$ and renormalized mass $m = 0.1$. Five points (up to 9 CNOTs per CNOT) were used for the extrapolation.

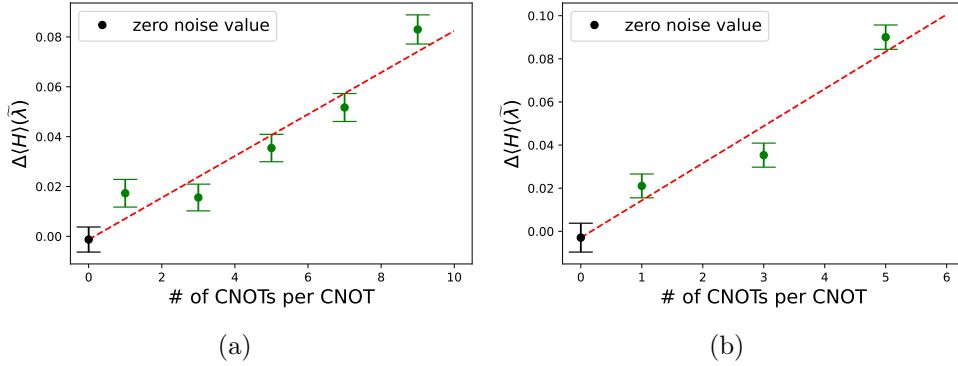
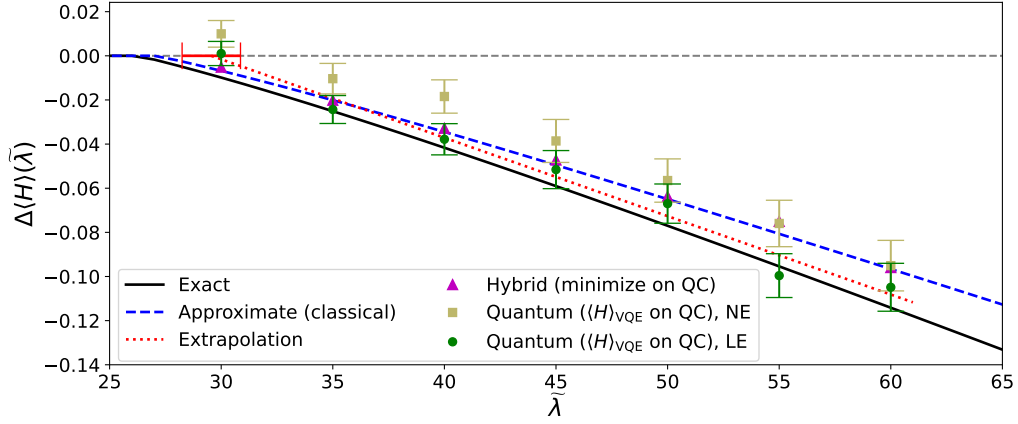
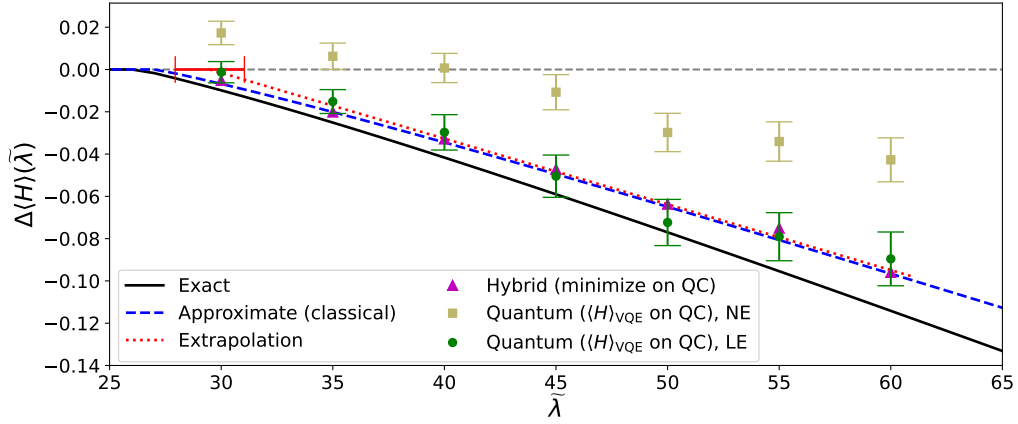


Figure 4.13: CNOT extrapolation on IBM Q Kolkata. The extrapolation is performed for the evaluation of $\Delta\langle H \rangle$ at coupling $\tilde{\lambda} = 30$ evaluated at the optimal parameters for lattice size $L = 10$ and renormalized mass $m = 0.1$. In (a), five points (up to 9 CNOTs per CNOT) were used for extrapolation. In (b), only three points (up to 5 CNOTs per CNOT) were used. While the device and qubits used were the same, the circuit jobs were run on different days.



(a)



(b)

Figure 4.14: Quantum hardware results for lattice size $L = 10$ and renormalized mass $m = 0.1$. The solid black curve is the exact difference $\langle H\rangle_{\text{VQE}} - \langle 0|H|0\rangle$ while the dashed blue curve sets the squeezing parameter $r(k) = 0$ for all modes with $k \neq 0$. Minimization is done on the IBM Q devices: Perth, Nairobi, and Lagos. The value of $\langle H\rangle$ at the optimal parameters was obtained both classically (purple triangles), and on two separate quantum computers: (a) IBM Q Mumbai and (b) IBM Q Kolkata. The error bars were obtained by bootstrap re-sampling. Displayed are results with no CNOT extapolation (NE) and linear extrapolation (LE). A linear extrapolation was performed on the LE points to locate the critical point (dotted red line).

4.13). Extrapolating the results and locating the x -intercept, we identified a critical point at $\tilde{\lambda} = 29.6 \pm 1.3$ in panel (a). The lower end of this range has a 4.4% error with the value obtained from the dashed curve, $\tilde{\lambda} = 27.1$. Again, the dashed curve is the fully classical result for the case where $r(k) = 0$ for all $k \neq 0$. In panel (b), we obtained a value of $\tilde{\lambda} = 29.5 \pm 1.5$, the lower end of which had about a 3.3% error with the value from the dashed curve. The values of $\Delta\langle H \rangle$ were obtained using IBM Q Mumbai for panel (a) and IBM Q Kolkata for panel (b) of Figure 4.14.

Like for CV, we desire to get closer to the continuum limit by increasing the lattice size to $L = 30$. We performed a noisy simulation for this case. The results are displayed in Figure 4.15. As discussed earlier, this is about the largest lattice size we can compute using only two qubits per qumode. Also, as with the CV case, we needed to increase the number of modes with $r(k) \neq 0$ up to $k = 3$. Extrapolating the data to the x -intercept, we found a level crossing at $\tilde{\lambda} = 62.3 \pm 1.3$, the lower end of which had about a 6.3% error with the value obtained from the dashed curve, $\tilde{\lambda} = 57.4$. We note that the latter value itself has about a 6% error with the continuum critical point, demonstrating that a lot can be done with just two qubits per qumode and limited applications of squeezing operations.

4.6 Discussion

The takeaway from this Chapter is the approaches we took to compute the ϕ^4 scalar field theory phase transition using Continuous Variables quantum computing. While we found that we needed relatively few resources for the computation, regardless of whether we used qubits or qumodes, we consider this calculation as a possible starting point for the CV quantum computation of non-perturbative properties in more complicated systems.

As mentioned in the previous section, the boson number truncation to two qubits per mode only appeared to work for small lattice sizes. The problem of truncating the bosonic Hilbert space goes all the way to current work in lattice gauge theories as described in Chapter 1.

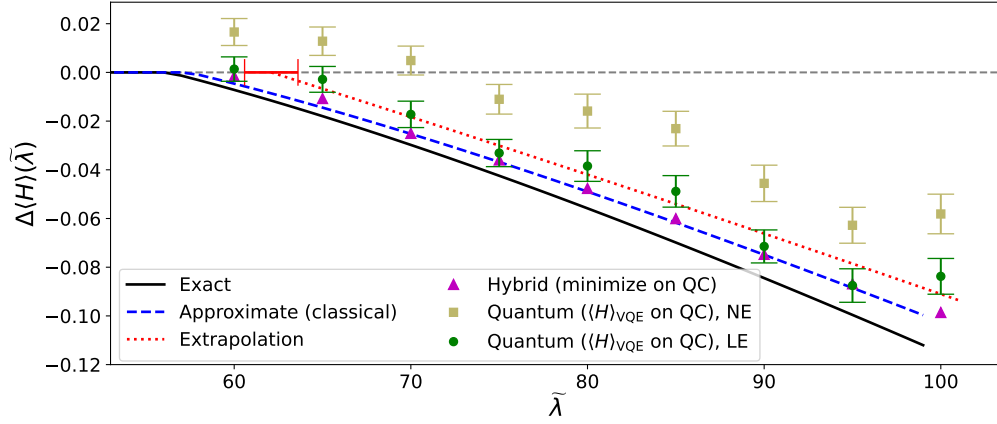


Figure 4.15: Noisy DV simulation results for lattice size $L = 30$ and renormalized mass $m = 0.1$. The solid black curve is the exact difference $\langle H \rangle_{\text{VQE}} - \langle 0 | H | 0 \rangle$ while the dashed blue curve sets the squeezing parameters $r(k) = 0$, for all modes with $k > 3$. The noise model derives from calibrations of the device IBM Q Mumbai. The error bars were obtained by bootstrap re-sampling. Displayed are results with no CNOT extapolation (NE) and linear extrapolation (LE). A linear extrapolation was performed on the LE points to locate the critical point (dotted red line). The value of $\langle H \rangle$ at the optimal parameters was also computed classically (purple triangles).

Furthermore, once one requires a non-separable Ansatz in a more complicated theory, we would no longer be able to keep using the same qubits on hardware for each qumode like we did in our case. The growing number of total required qubits then is expected to make it increasingly challenging to detect phase transitions on current noisy qubit hardware.

Therefore, it was worth discussing how a CV quantum computation would proceed for this rather simple system. We thus keep in mind two important points in regard to this Chapter. First, we prepared a simplistic, experimentally realizable Gaussian state under the assumption that the exact ground state of the system had high fidelity with the ground state of a massive scalar field theory with some mass Ω .

A particularly encouraging aspect of this was that our CV quantum circuit was initiated with two-mode squeezed vacuum states which are possible to prepare with non-degenerate parametric down conversion. Also, while our Ansatz was separable in this case, the application of linear optical operations such as beam splitters and phase rotators for a more general and entangled Gaussian Ansatz is straightforward, see Appendix B.

It has been demonstrated [114] that non-Gaussian operations might be essential in probing the second-order nature of the ϕ^4 phase transition, and so we expect such elements to play an important role as we add complexity in future work. Additionally, as suggested by the CV equivalent of the Gottesman-Knill Theorem [54, 106] mentioned in Chapter 1 and Section 4.4, the presence of non-Gaussian gates is essential in order for a CV quantum computer to demonstrate a quantum advantage in processing entangled quantum states. Demonstrating such an advantage will be key in verifying the utility of CV quantum computation applied to relativistic quantum field theories. We will see multiple examples of non-Gaussian operations in the work described in the next Chapter.

The second point to keep in mind is the methods we devised to measure expectation values. In Section 4.4 we laid out a protocol for using photon number measurements to determine expectation values of powers of the quadrature operators q, p . This involved appending CX gates between pairs of qumodes and measuring photon numbers on circuits with different CX gate parameters. We noted that the technique is similar to the use of CV

parameter shift rules [110] and that it is also a CV analog to the addition of single-qubit rotations for the diagonalization of Pauli matrices in the Z basis, for the DV case. These methods will carry over to our discussion of the $O(3)$ non-linear sigma model, which is the subject of the next Chapter.

Chapter 5

$O(3)$ Model with Continuous Variables

Let us now return to the idea of being able to perform a quantum computation of a lattice gauge theory. Instead of directly trying to apply the techniques of CV quantum computation towards some non-Abelian gauge theory, such as $SU(2)$, we can take a similar route as the Schwinger model and focus our attention on some simple system which shares several notable properties with QCD.

A model satisfying these conditions is the $O(3)$ nonlinear sigma model in 1+1 dimensions. While the model does not contain a gauge symmetry, it is still associated with a non-Abelian symmetry group and like the Schwinger model exhibits phenomena that is also encountered in four-dimensional QCD. Notably, it is asymptotically free and demonstrates dynamical mass generation. Furthermore, one may add a topological theta term, acknowledging the existence of instanton solutions [\[121, 122\]](#).

In this Chapter, we review the protocols for computing spectra and implementing time evolution introduced in Ref. [\[65\]](#), the work of which was done in collaboration with Raghav G. Jha, Felix Ringer, and George Siopsis. R. G. J. supplied excellent reference material and was the one to propose trying a coupled-cluster approach for state preparation. He and F.R. made significant writing contributions regarding the background of the $O(3)$ model and prior work done on it. They also guided the directions taken in the paper. G.S. presented the structural connection between sigma models and $SU(2)$ lattice gauge theory (see next

Section). He was also vital in determining how we can construct, using continuous variables, the non-interacting ground state of the system (e.g. Eqs. (5.36) and (5.37) below). I filled in additional details of this construction and worked on the other algorithms in the article including the engineering of the coupled-cluster state (and its measurement) as well as time evolution. I obtained all numerical results that appeared in this paper. While we all contributed to the writing of this article, I again credit my three co-authors for the writing of the sections concerned with the background and relevance of sigma models. We expect that the methods presented will generalize to other non-linear sigma models as well as gauge theories, as we will motivate in the following Section.

5.1 $O(3)$ non-linear sigma model

Non-linear sigma models have been examined at great lengths due to their resemblance to gauge theories, however they are simpler systems due to the lack of a gauge symmetry. Their form is inspired by the low-energy dynamics of pions described by an effective chiral Lagrangian density given schematically by $\mathcal{L} = \frac{1}{4}\text{Tr}[\partial_\mu U \partial^\mu U^\dagger]$, where U is an isospin $SU(2)$ matrix. It has a global $SU(2)_L \times SU(2)_R$ symmetry which coincides with the $O(4)$ symmetry of the sigma model. In the latter, the classical configurations are points on the 4-sphere. One can see this by parametrizing the isospin matrix by $U = n_0 \mathbb{I} + i \mathbf{n} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices, and $n_a n^a \equiv n_0^2 + \mathbf{n}^2 = 1$. Switching to the Hamiltonian formalism and discretizing on a spatial lattice, we obtain the Hamiltonian

$$H = \frac{1}{2g^2} \sum_{a,b} J_{ab}^2 - g^2 \sum_{\langle x, x' \rangle} n_a(x) n^a(x') \quad (5.1)$$

where

$$J_{ab} = -i \left(n_a \frac{\partial}{\partial n_b} - n_b \frac{\partial}{\partial n_a} \right), \quad a, b = 0, 1, 2, 3, \quad (5.2)$$

is identified as the angular momentum and g is a coupling constant. The potential is bilinear in the vectors n_a that act as coordinates of the system. Here we've defined g^2 to be

the inverse of what usually appears in the literature (c.f. Refs. [89, 123]). This choice will become convenient when we discuss our approach.

Similarly, the $SU(2)$ non-Abelian gauge theory, containing a local $SU(2)$ symmetry, can also be formulated in terms of parametrized $SU(2)$ matrices. As in the Schwinger model, the matrices U reside on the links in the Kogut-Susskind formulation, and the angular momenta J_{ab} are similarly defined on these links. The Hamiltonian is

$$H = \frac{1}{2g^2} \sum_{\text{links}} J^2 - \frac{g^2}{2} \sum_{\text{plaquettes}} \text{Tr}[U(1)U(2)U(3)U(4)] \quad (5.3)$$

where the second term is the sum of Wilson loops over each plaquette. Parameterizing in terms of the four-dimensional vectors $n_a(i)$ with $i = 1, 2, 3, 4$, one finds an expression for the plaquette term that is quadri-linear in the coordinates

$$\begin{aligned} \frac{1}{2} \text{Tr}[U(1)U(2)U(3)U(4)] &= n^a(1)n_a(2) n^b(3)n_b(4) - n^a(1)n_a(3) n^b(2)n_b(4) \\ &\quad + n^a(1)n_a(4) n^b(2)n_b(3) + \epsilon^{abcd} n_a(1)n_b(2)n_c(3)n_d(4). \end{aligned} \quad (5.4)$$

The states obey the constraint $\epsilon^{abcd} J_{ab} J_{cd} |\Psi\rangle = 0$. Additionally, the system obeys Gauss's Law which further constrains the Hilbert space to the gauge singlet sector [2].

With the structure of the Hamiltonian in mind, we now simplify the system to the $O(3)$ model, the nearest-neighbor version of which has a Hamiltonian given by [124]

$$H = \frac{1}{2g^2} \sum_i \mathbf{L}_i^2 - g^2 \sum_{\langle i,j \rangle} \mathbf{n}_i \cdot \mathbf{n}_j. \quad (5.5)$$

Here g^2 is the coupling constant, i and j index nearest neighbor sites on a one-dimensional lattice. $\mathbf{n}_i$ is a unit vector on the 2-sphere and $\mathbf{L}_i$ is the angular momentum operator at each site, $L^a = \frac{1}{2} \epsilon^{abc} J_{bc}$. The main difference between this Hamiltonian and that of the $SU(2)$ non-Abelian gauge theory is that the only constraint is $\mathbf{n}^2 = 1$ must be satisfied on each site. It is important to note that the continuum limit corresponds to $g^2 \rightarrow \infty$.

The interaction term can be written in terms of $n^\pm = (n^x \pm in^y)/\sqrt{2}$ as [124]

$$\mathbf{n}_i \cdot \mathbf{n}_j = n_i^+ n_j^- + n_i^- n_j^+ + n_i^z n_j^z. \quad (5.6)$$

Therefore, the lattice Hamiltonian may be expressed as

$$H = \frac{1}{2g^2} \sum_{x=0}^{L-1} \mathbf{L}^2(x) - g^2 \sum_{x=0}^{L-1} \left(n^+(x) n^-(x+1) + n^-(x) n^+(x+1) + n^z(x) n^z(x+1) \right), \quad (5.7)$$

The eigenvalues of the kinetic term are proportional to $l(l+1)$ where $l = 0, 1, \dots, \infty$ denote the energy levels based on the irreducible representations of the $O(3)$ symmetry. For practical calculations, it is conventional to impose a cutoff, which we refer to as $l_{\max}$. Define

$$n^\pm = \mp X_{\pm 1}, \quad n^z = X_0 \quad (5.8)$$

It is known [125] that the matrix elements of $\mathbf{n}$ can be expressed in terms of Wigner-3j symbols,

$$\langle l_1, m_1 | X_M | l_2, m_2 \rangle = (-1)^{m_1} \sqrt{(2l_1+1)(2l_2+1)} \begin{pmatrix} l_1 & 1 & l_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & 1 & l_2 \\ -m_1 & M & m_2 \end{pmatrix}. \quad (5.9)$$

To compute energies we adopt the coupled-cluster (CC) method which takes a state of the form [126]

$$|\psi\rangle \propto e^{\alpha \hat{T}} |\psi_0\rangle \quad (5.10)$$

where $\hat{T}$ is the cluster operator built from the single interaction terms appearing in the Hamiltonian (Eq. (5.6)) and α is a tunable parameter. We can include higher-interaction terms, known as double-excitation terms in the literature. However we will see that the simple bilinear expression in $\mathbf{n}$ suffices. The CC Ansatz has already seen use in the investigation of $O(N)$ non-linear sigma models [127, 128, 129]. In other contexts, unitary CC Ansätze (where $\hat{T}$ is anti-Hermitian) have been prepared as effective variational Ansätze

in quantum computations [130, 76]. In our case, the construction of the CC state begins with the tensor product of the states $|l = 0, m = 0\rangle$ defined at each site

$$|\Omega_0\rangle \equiv \bigotimes_{x=0}^{L-1} |l(x) = 0, m(x) = 0\rangle, \quad (5.11)$$

which is the weak coupling ground state with $\hat{T} = 0$. Expressing $\hat{T}$ in terms of the site-coupling term of the Hamiltonian, we obtain the following coupled-cluster Ansatz [131]:

$$|\text{CC}\rangle \propto \prod_x e^{\frac{\alpha g^2}{L} \mathbf{n}(x) \cdot \mathbf{n}(x+1)} |\Omega_0\rangle, \quad (5.12)$$

We take α to be a parameter for a variational energy calculation.

For the first excited state, we apply the CC Ansatz:

$$|\text{CC}_1\rangle \propto e^{\frac{\alpha g^2}{L} \sum_x \mathbf{n}(x) \cdot \mathbf{n}(x+1)} \sum_x n_3(x) |\Omega_0\rangle \quad (5.13)$$

Since $n_3 = \cos \theta$, the inclusion of $n_3(x)$ replaces $Y_{00}(\theta, \phi)$ with $Y_{10}(\theta, \phi)$ at lattice site x , thus introducing an angular momentum excitation of $l = 1$ there. The resulting state is orthogonal to the CC Ansatz for the ground state $\langle \text{CC} | \text{CC}_1 \rangle = 0$, since the cluster operator is invariant under global rotations while $n_3(x)$ increases the angular momentum representation from $l = 0$ to $l = 1$ on site x .

The ground-state energy obtained by minimizing the energy functional corresponding to the CC state and the Hamiltonian (5.5) is shown in Figure 5.1(a). We plot $E_0(\alpha)/L$ as a function of g^2 for $L = 2$ and $L = 3$ compared to the results from exact diagonalization (ED). Using a cutoff $l_{\text{max.}} = 3$, for small g^2 , we see excellent agreement between the CC and ED results up to $g^2 \approx 4$. In both methods, the uncertainty in our results increases with g^2 . In exact diagonalization, this is because the angular momentum truncation error increases with g^2 . Therefore, the difference between the results corresponding to $l_{\text{max.}} - 1$ and $l_{\text{max.}}$, which we take as our error estimate, increases as well. For the coupled-cluster method, we applied quasi-Monte Carlo integration [132] to obtain the $L = 3$ energies.

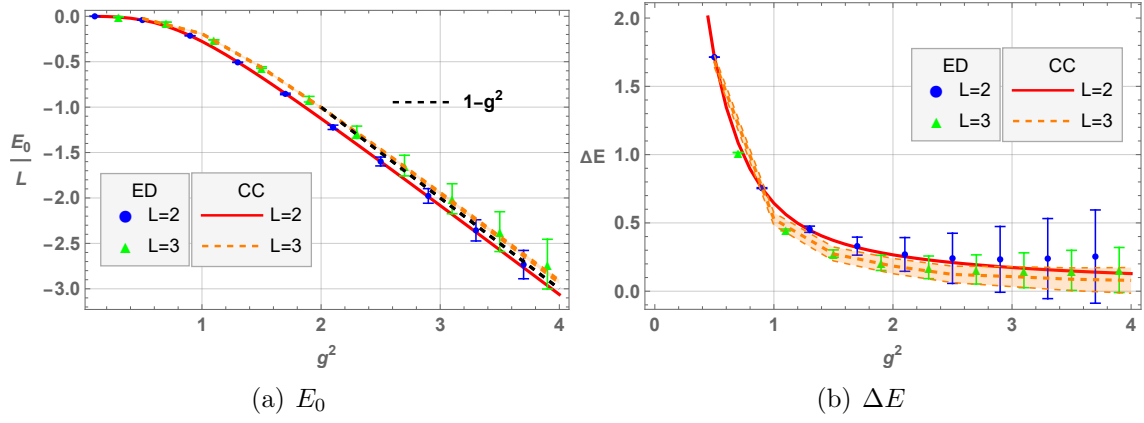


Figure 5.1: The ground state energy E_0 and mass gap $\Delta E = E_1 - E_0$ for lattice sizes $L = 2, 3$. Both ED and CC numerical results are displayed. Angular momentum cutoff set at $l_{\text{max.}} = 3$ for ED, with error bars given by difference from $l_{\text{max.}} = 2$ result. 500,000 sample points for MC integration for $L = 3$. $L = 2$ CC is analytic.

The uncertainty therefore derives from the fact that quasi-Monte Carlo, or at least the form employed, does not contain importance sampling, whereas a uniform distribution is more effective when $g^2 \rightarrow 0$. The mass gaps $\Delta E = E_1 - E_0$ obtained from the CC and ED methods are shown in Figure 5.1(b).

5.1.1 Realizing a Continuous Variable calculation

It is possible to write the $O(3)$ Hamiltonian given by (5.7) in terms of bosonic creation and annihilation operators at each site using the Schwinger boson formalism [133]. This is due to the well-known Jordan-Schwinger map [28] which allows one to express the angular momentum eigenstates as well as the angular momentum operators in terms of a pair of harmonic oscillators, say a and b . The angular momentum representation is given by half the total number of bosons,

$$j = \frac{1}{2} (a^\dagger a + b^\dagger b) \quad (5.14)$$

and the components of the the angular momentum operator are

$$J_i = \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \sigma_i \begin{pmatrix} a \\ b \end{pmatrix} \quad (5.15)$$

The angular momentum eigenbasis can be constructed by acting with the creation operators corresponding to the two oscillators on the vacuum

$$|j, m\rangle = \frac{a^\dagger{}^{j+m} b^\dagger{}^{j-m}}{\sqrt{(j+m)!(j-m)!}} |N_a = 0, N_b = 0\rangle \quad (5.16)$$

Therefore, it is possible to realize a qumode encoding of the exact hilbert space of the $O(3)$ model for a continuous variable calculation, and the simulation requires only two qumodes for each lattice site. The downside of this approach, however, is that the interaction term (the second term in (5.5)) is difficult to represent naturally in such a formalism, as the coordinates $\mathbf{n}(x)$ appearing in this term are non-polynomial functions of the number operator.

Another way of thinking is to express the $O(3)$ model as a limit of a three-component scalar field theory in 1+1 dimensions, requiring three qumodes per site. The $O(3)$ limit is achieved by approximating the constraint $\mathbf{n}^2 = 1$ via the introduction of appropriate Ansätze. This is the approach taken in this Chapter, and we will find that the calculation of energies proceeds along similar lines as the previous Chapter. We will also introduce a time evolution protocol for this model.

5.2 Scalar field theory formulation on the 2-sphere

We begin with a model comprised of real scalar fields $\phi^a(x)$ ($a = 1, 2, 3$) where x denotes the spatial position in one dimension. As usual, we take PBC's with lattice positions $x = 0, 1, \dots, L - 1$ and use units where the lattice spacing is $a = 1$. Let π^a be the conjugate momentum to ϕ^a obeying the canonical commutation relations

$$[\phi^a(x), \pi^b(x')] = i\delta^{ab}\delta_{xx'}. \quad (5.17)$$

We introduce the Hamiltonian

$$H = \frac{1}{2g^2} \sum_x \mathbf{L}^2(x) + \sum_x \left[\frac{1}{2} (\phi(x) - \phi(x+1))^2 - g^2 \right]. \quad (5.18)$$

Here the angular momentum operator $\mathbf{L}$ denotes the cross product of the vector field and its conjugate momentum,

$$\mathbf{L}(x) = \boldsymbol{\phi}(x) \times \boldsymbol{\pi}(x), \quad (5.19)$$

where we defined a triplet notation for the fields, $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, and similarly for $\boldsymbol{\pi}$.

It is useful to define local sets of basis vectors which resemble those relevant to the $O(3)$ model. This can be done by letting the field triplets $\boldsymbol{\phi}$ define a vector in three dimensional space, $|\boldsymbol{\phi}\rangle$, or writing down a set of wave-functions on these states which take the form

spherical harmonics,

$$|l, m; \Lambda\rangle = \frac{1}{\sqrt{\mathcal{N}}} \int dr r^2 d^2 \mathbf{n} e^{-\frac{\Lambda^2}{8g^2}(r^2-g^2)^2} Y_{lm}(\mathbf{n}) |r, \mathbf{n}\rangle . \quad (5.20)$$

Here $\mathbf{n}$ is a unit 3-vector, r is the radial direction, and Λ is a (radial) momentum cutoff scale restricting the wave function on a sphere of radius $r = g$. We will see that in a hardware implementation, the parameter Λ characterizes an energy scale and expresses the strength of the squeezes/non-linearities required in the state preparation process.

The state described in (5.20) can be separated into radial and angular parts denoted by Y_{lm} . The radial integrand is not unique and any function that converges to a δ -function centered at $r = g$ as the cutoff $\Lambda \rightarrow \infty$ would work in practice.

The states in (5.20) are orthogonal and satisfy

$$\mathcal{N} = \frac{g^2 \sqrt{\pi}}{\Lambda} + \mathcal{O}(\Lambda^{-3}) . \quad (5.21)$$

This can be computed by expanding the radial part of the integrand in (5.20) around $r = g$ and calculating Gaussian integrals on each term of the expansion. The matrix elements of the kinetic energy are exact and independent of the cutoff Λ since the radial part of the wave functions decouples,

$$\frac{1}{2g^2} \langle l, m; \Lambda | \mathbf{L}^2 | l', m'; \Lambda \rangle = \frac{l(l+1)}{2g^2} \delta_{l,l'} \delta_{m,m'} . \quad (5.22)$$

According to the Hamiltonian (5.18), the interaction term contains a local contribution with matrix elements given by

$$\langle l, m; \Lambda | (\phi^2 - g^2) | l', m'; \Lambda \rangle = \mathcal{O}(\Lambda^{-2}) \delta_{l,l'} \delta_{m,m'} . \quad (5.23)$$

The elements clearly vanish in the limit of large values for the cutoff parameter $\Lambda \rightarrow \infty$. Now consider two adjacent sites labeled by i and j . The remaining contribution to the interaction

term in (5.18) has matrix elements

$$\begin{aligned} & \langle l_i, m_i; \Lambda | \langle l_j, m_j; \Lambda | \boldsymbol{\phi}(x_i) \cdot \boldsymbol{\phi}(x_j) | l'_i, m'_i; \Lambda \rangle | l'_j, m'_j; \Lambda \rangle \\ &= g^2 (1 + \mathcal{O}(\Lambda^{-2})) \langle l_i, m_i | \langle l_j, m_j | \mathbf{n}_i \cdot \mathbf{n}_j | l'_i, m'_i \rangle | l'_j, m'_j \rangle . \end{aligned} \quad (5.24)$$

It has the same structure as the potential term of the $O(3)$ model discussed in the previous section, the latter of which is recovered in the limit $\Lambda \rightarrow \infty$. We therefore find that the matrix elements of the Hamiltonian in the basis (5.20) deviate from the matrix elements of the $O(3)$ Hamiltonian only by an amount $\mathcal{O}(\Lambda^{-2})$, which also vanishes in the limit $g \rightarrow 0$.

Fixing the cutoff Λ (fixing the radial wavefunction), these states form a restricted basis for representing the Hamiltonian (5.18) which can be used to compute energies which approximate the $O(3)$ energies. The approximation is improved by increasing Λ for the restricted basis. The ideal restricted basis is one where the radial wavefunction is a delta function centered at $|\boldsymbol{\phi}| = g$.

Alternatively, we may construct a variational Ansatz by extending the CC method used for the $O(3)$ model in Section 5.1. Analogous to (5.11), we start with a reference state $|\Omega(\Lambda)\rangle$ defined as a tensor product over the lattice sites x of the local states $|l(x) = 0, m(x) = 0; \Lambda\rangle$ (Eq. (5.20)),

$$|\Omega(\Lambda)\rangle \equiv \bigotimes_{x=0}^{L-1} |l(x) = 0, m(x) = 0; \Lambda\rangle . \quad (5.25)$$

This is the weak coupling ($g \rightarrow 0$) vacuum state which corresponds to the vanishing cluster operator $\hat{T} = 0$. A visualization of the wave-function for $|l = 0, m = 0; \Lambda\rangle$ is shown in panel (a) of Fig. 5.2. It becomes sharply peaked at $|\boldsymbol{\phi}| = g$ as Λ becomes large. We define the exponent of the cluster operator, $\hat{T}$, in terms of the potential term in the Hamiltonian and adopt the CC Ansatz:

$$|\text{CC}(\Lambda)\rangle \propto e^{-\frac{\alpha}{2L} \sum_x (\boldsymbol{\phi}(x) - \boldsymbol{\phi}(x+1))^2} |\Omega(\Lambda)\rangle , \quad (5.26)$$

which reduces to the CC Ansatz for the $O(3)$ model (5.12) in the limit $\Lambda \rightarrow \infty$.

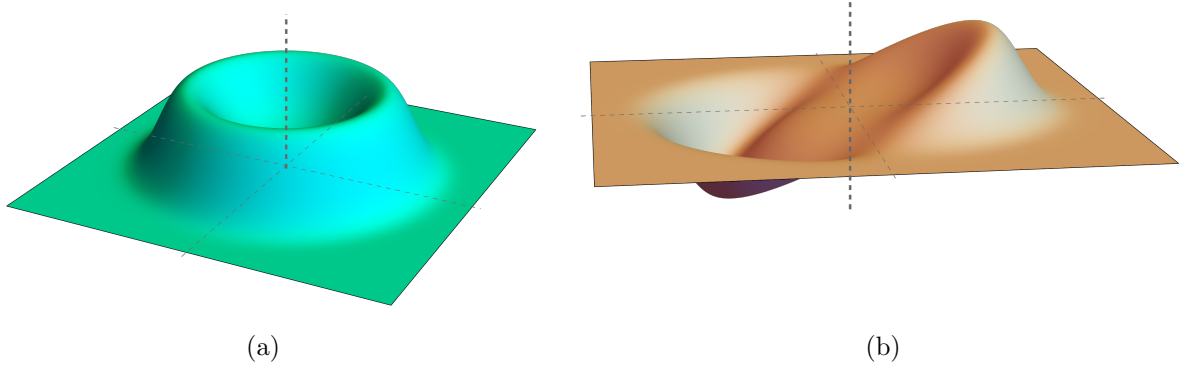


Figure 5.2: Wave-functions corresponding to the state (a) $|l = 0, m = 0; \Lambda\rangle$ and (b) $|l = 1, m = 0; \Lambda\rangle$. Displayed is the wave-function restricted to the x - z plane (horizontal axes).

We take a similar approach to Eq. (5.13) to compute excited state energies. First, from Eq. (5.20), we obtain

$$|l = 1, m; \Lambda\rangle \propto Y_{1m}(\mathbf{n}) |l = 0, m = 0; \Lambda\rangle. \quad (5.27)$$

A visualization of the wave-function is shown in panel (b) of Fig. 5.2. For arbitrary coupling the first excited state is degenerate with respect to L_z . Therefore, we concentrate on $m = 0$ (the other cases can be treated similarly) and define

$$|\Omega_1(x, \Lambda)\rangle \propto \phi_3(x) |\Omega(\Lambda)\rangle. \quad (5.28)$$

The first excited state is obtained by applying the CC Ansatz

$$|\text{CC}_1(\Lambda)\rangle \propto e^{-\frac{\alpha}{2L} \sum_x (\phi(x) - \phi(x+1))^2} \sum_x |\Omega_1(x, \Lambda)\rangle. \quad (5.29)$$

This Ansatz reduces to the one for the $O(3)$ model, see Eq. (5.13), in the limit $\Lambda \rightarrow \infty$. The state $|\Omega_1(x, \Lambda)\rangle$, given by (5.28), is the weak-coupling eigenstate with site x in the state $|l(x) = 1, m(x) = 0; \Lambda\rangle$, and all others in the $|l(x') = 0, m(x') = 0; \Lambda\rangle$ state ($x' \neq x$).

Let us now compare ED with the coupled-cluster method as we did with the $O(3)$ case. The results are shown in Fig. 5.3, which displays the ground state energy density for $L = 2, 3$, and $l_{\text{max.}} = 3$ for ED. Here we consider two values of Λ , 1 and 10, and display the energies up to $g^2 = 4$. Fig. 5.4 displays mass gaps for $L = 2$ and $L = 3$.

For ED, the Hamiltonian (5.18) is taken in the local basis of states (5.20), for given Λ , and we compute the ground state energy by diagonalizing the resulting matrix. To obtain the CC energies, we compute expectation values with numerical integration. For $L = 2$, the angular portion of the integral is done analytically. Monte Carlo integration is used for $L = 3$ sites. More specifically, we first perform quasi-Monte Carlo integration [132] to estimate the optimal value of α . In quasi-Monte Carlo integration, the sample points are given by a “quasi-random” sequence designed for faster convergence than uniform sampling.

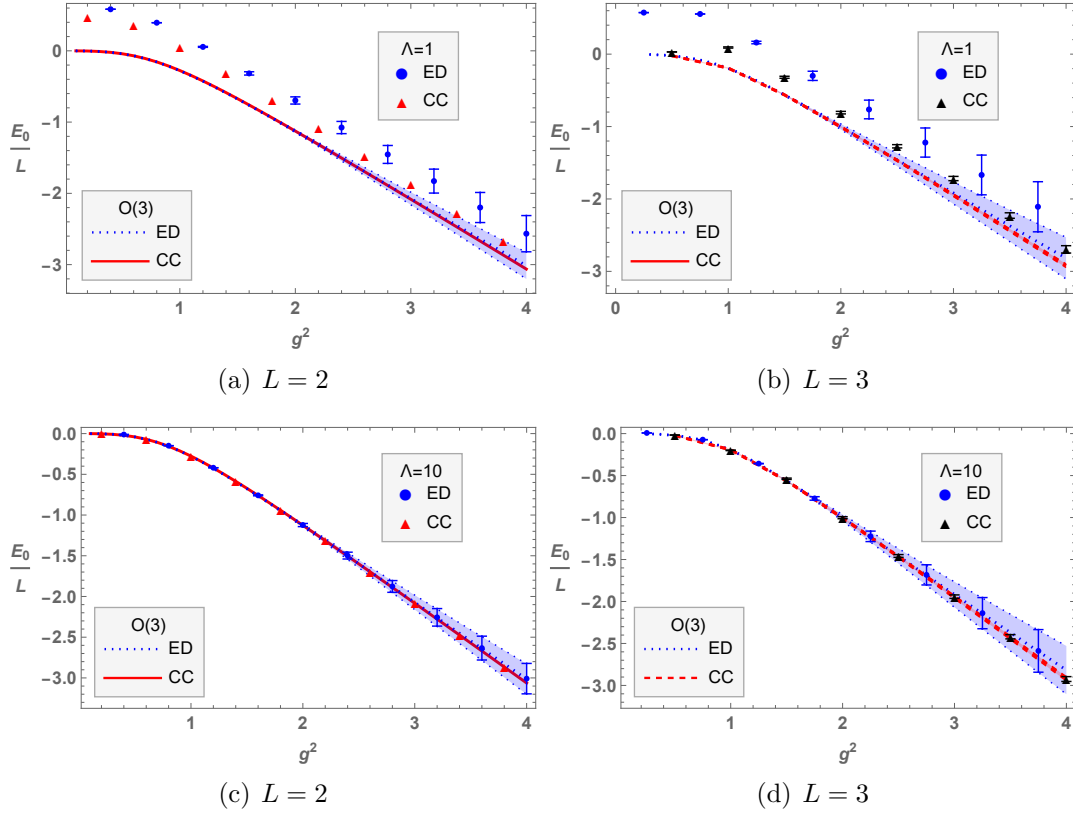


Figure 5.3: Ground state energy E_0/L for $L = 2, 3$ vs g^2 . It is computed with ED and the CC method, for $\Lambda = 1, 10$. ED is taken at $l_{\max.} = 3$, with error bars given by difference from $l_{\max.} = 2$ result. CC error bars for $L = 3$ is statistical uncertainty from Monte Carlo integration.

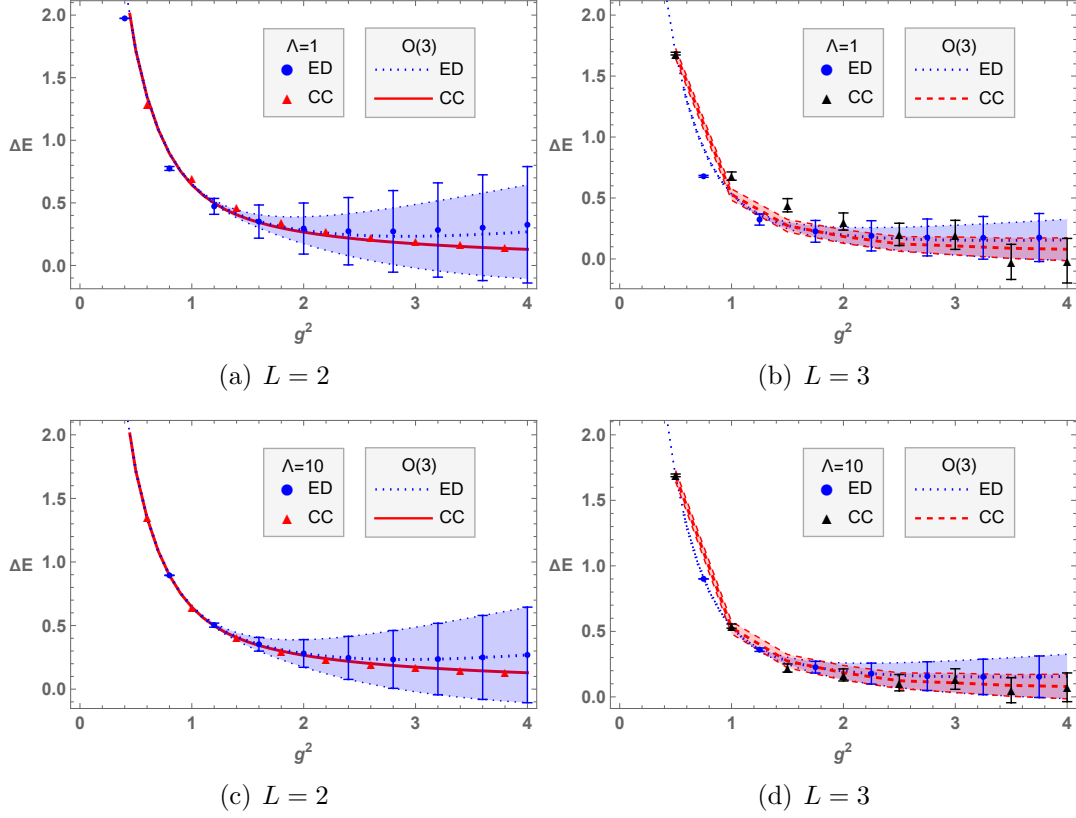


Figure 5.4: Mass gap ΔE for $L = 2, 3$ vs g^2 . It is computed with ED and the CC method, for $\Lambda = 1, 10$. ED is taken at $l_{\max.} = 3$, with error bars given by difference from $l_{\max.} = 2$ result. CC error bars for $L = 3$ is statistical uncertainty from Monte Carlo integration.

The fixed set of sampling points eliminates statistical noise during energy minimization. Next the energy expectation is then re-evaluated at the obtained value of α by sampling from a distribution whose radial component approximates the true radial wave function. For large values of Λ , a Gaussian distribution with mean g and width $\sim \frac{1}{\Lambda}$ is sufficient, and is what is used in Figs. 5.3 and 5.4(d). For smaller values of Λ , the radial wave function can not be accurately approximated by a Gaussian. In this case, we use the full radial wave function at $\alpha = 0$, which is the radial part of the integrand in Eq. (5.20). This is what is used in Figs. 5.3 and 5.4(b).

While we focus on intermediate values of g^2 given the current literature (see Ref. [123]), it is still important to test the effectiveness of the CC Ansatz for larger values. It becomes necessary in this case to use a significantly higher value of $l_{\text{max.}}$ to provide a value from an ED calculation to compare the CC result against. In Fig. 5.5, we set $l_{\text{max.}} = 10$ and compute the ground state energies and mass gaps for two values of g^2 and three values of Λ . We not only find agreement between the two methods as Λ becomes large, but we also find that the results of the CC calculation outperform the ED results for the $O(3)$ model with $l_{\text{max.}} = 3$, as indicated by the light-blue lines in the two panels on the left-hand side.

Fig. 5.6 displays additional numerical results, namely the ground and first excited state energies and mass gap as a function of the lattice size L . In the Figure, Monte Carlo numerical integration (using the same methods as with $L = 3$ in Fig. 5.3) is employed for all L , as ED quickly becomes challenging beyond $L = 4$, $l_{\text{max.}} = 2$. We observe the accumulation of numerical errors as the number of lattice sites is increased, particularly in the case of the lower two panels where $g^2 = 4$.

5.3 Quantum simulation using continuous variables

In this section, we discuss the simulation of the $O(3)$ model using CV gates. After constructing the CC Ansatz, we develop methods for computing energy levels. Next, we describe the relevant circuits for the time evolution within the CV approach and present

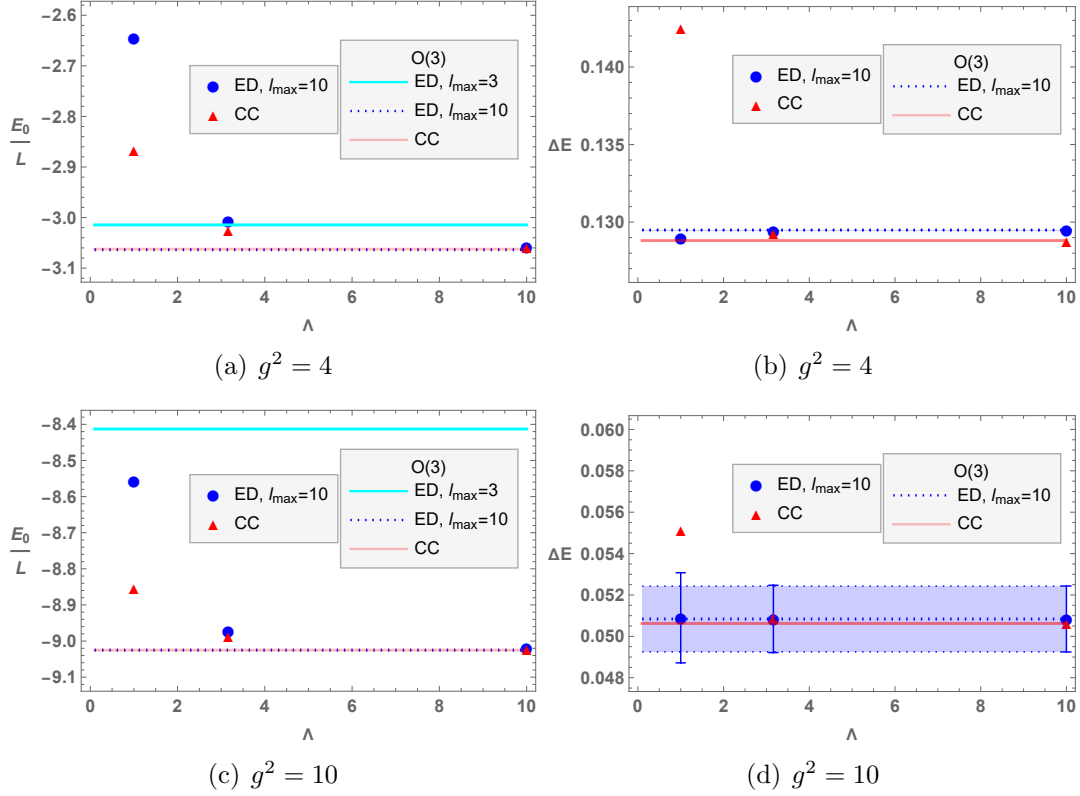


Figure 5.5: Ground state energy $\frac{E_0}{L}$ and mass gap ΔE for $L = 2$ sites. Numerical results are shown for $g^2 = 4$ (upper panels) and $g^2 = 10$ (lower panels) with $l_{\max.} = 10$. The displayed results include the $O(3)$ limit as well as three values for Λ : 1, 3.2 and 10. Error bars for ED results are computed by finding the deviation from the $l_{\max.} = 9$ results.

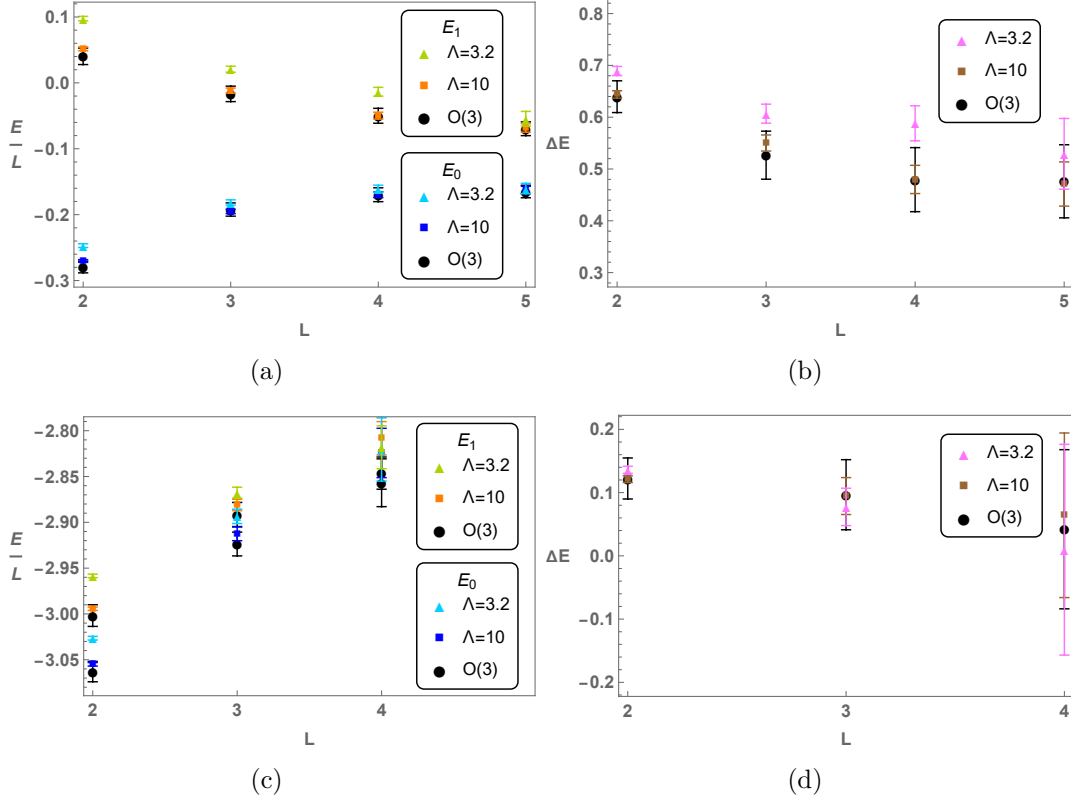


Figure 5.6: Ground and first excited state energies, as well mass gaps, as a function of L . We take the CC Ansatz and results are displayed for both the $O(3)$ model and qumode formulation. These are shown for two values of Λ , and we take the lattice size up to $L = 5$. (a) and (b) correspond to $g^2 = 1$. We used Monte Carlo integration with 500,000 sample points. (c) and (d) correspond to $g^2 = 4$ and 5,000,000 sample points are used for the Monte Carlo integration.

numerical results, which use the STRAWBERRY FIELDS quantum simulator. More detailed discussions about CV gates can be found in Appendix B.

5.3.1 Coupled-Cluster Ansatz

To engineer the CC Ansatz in (5.26), we must first prepare the single-site weak-coupling ground state, $|l=0, m=0; \Lambda\rangle$, which is required in Eq. (5.25). Taking three qumodes of quadratures (q_a, p_a) ($a = 1, 2, 3$), collectively denoted as $(\mathbf{q}, \mathbf{p})$, we initialize the system in the vacuum state

$$|0\rangle \equiv \bigotimes_{a=1}^3 |0\rangle_a = \frac{1}{\pi^{3/4}} \int d^3 \mathbf{q} e^{-\frac{1}{2} \mathbf{q}^2} |\mathbf{q}\rangle \quad (5.30)$$

We attach an ancilla qumode of quadratures (q_b, p_b) , also initialized in the vacuum state $|0\rangle$. We apply the product of two-mode entangling non-Gaussian unitaries

$$U_{ab} = e^{-i \frac{\Lambda}{\sqrt{2}g} q_a^2 p_b}, \quad (5.31)$$

followed by the application of a translation operator on the ancilla qumode

$$U_b = e^{ig \frac{\Lambda}{\sqrt{2}} (1 + \frac{2}{\Lambda^2}) p_b}. \quad (5.32)$$

These operators transform the state of the ancilla from the vacuum to a coherent state

$$|\alpha\rangle_C = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (5.33)$$

where the value of α is conditioned on q_a . Namely, we obtain

$$U_b \prod_{a=1}^3 U_{ab} |0\rangle |0\rangle_b = \int d^3 \mathbf{q} e^{-\mathbf{q}^2} |\mathbf{q}\rangle \left| \frac{\Lambda}{2g} \mathbf{q}^2 - g \frac{\Lambda}{2} \left(1 + \frac{2}{\Lambda^2} \right) \right\rangle_{b;C}, \quad (5.34)$$

where the subscript C on the ancilla state $|\alpha\rangle_{b;C}$ indicates that it is the coherent state with parameter α , while $|\mathbf{q}\rangle$ indicates an eigenstate of the three-component quadrature operator

$\mathbf{q}$. Next, we measure the ancilla qumode projecting it onto the state $|0\rangle_b$. Given

$$\langle 0|\alpha\rangle_C = e^{-|\alpha|^2/2}, \quad (5.35)$$

the state then collapses to

$${}_b\langle 0| U_b \prod_{a=1}^3 U_{ab} |\mathbf{0}\rangle |0\rangle_b \propto \int d^3\mathbf{q} e^{-\frac{\Lambda^2}{8g^2}(\mathbf{q}^2 - g^2)^2} |\mathbf{q}\rangle. \quad (5.36)$$

Thus, we were able to construct the state $|\omega(\Lambda)\rangle = |l=0, m=0, \Lambda\rangle$ (Eq. (5.20)) which contributes to the state $|\Omega(\Lambda)\rangle$ (Eq. (5.25)) used in the coupled-cluster Ansatz. Notice that the particular form of U_{ab} in (5.31) allows us to remove the original vacuum exponential of Eq. (5.30). A schematic circuit is shown in Fig. 5.7, but needs to be expressed in terms of fundamental CV gates.

The non-Gaussian unitaries U_{ab} in (5.31) is expressible in terms of cubic phase gates. Using the relation

$$6q_a^2 p_b = (q_a + p_b)^3 - (q_a - p_b)^3 - 2p_b^3, \quad (5.37)$$

we obtain the following gate decomposition of U_{ab} :

$$U_{ab} = F_b^\dagger \cdot BS_{ba} \cdot V_a \left(-\frac{\Lambda}{g} \right) \cdot V_b \left(\frac{\Lambda}{g} \right) \cdot BS_{ab} \cdot V_b \left(\frac{\Lambda}{\sqrt{2}g} \right) \cdot F_b. \quad (5.38)$$

where F is the Fourier transform operator (see (B.8)), BS_{ab} implements a 50:50 beam splitter ($\theta = \frac{\pi}{4}$ in the definition (B.10)), and we introduced the cubic phase gate

$$V(s) = e^{i\frac{s}{3}q^3}. \quad (5.39)$$

The purpose of the combination of BS with F is to rotate the argument of $V(q)$ to incorporate the p_b appearing in Eq.(5.37). See Appendix B to see how such rotations are performed. The quantum circuit for this unitary is shown in Fig. 5.8.

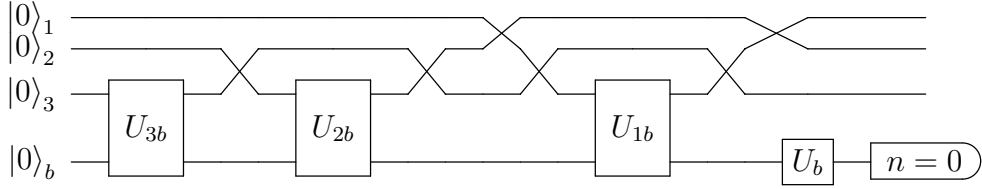


Figure 5.7: CV quantum circuit for the generation of the state $|\omega(\Lambda)\rangle = |l = 0, m = 0, \Lambda\rangle$ (5.36). It corresponds to Eq. (5.36) and is relevant for the construction of the coupled-cluster Ansatz.

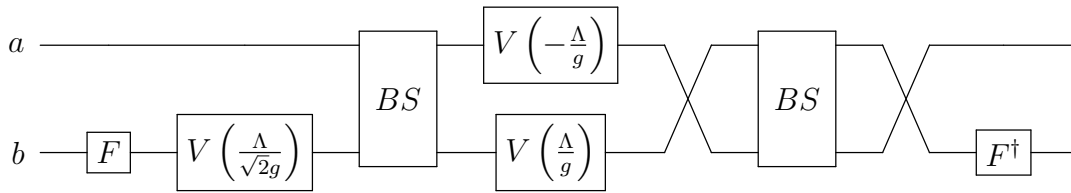


Figure 5.8: CV quantum circuit (5.38) implementing the non-Gaussian unitary operator U_{ab} .

The challenge now is to determine a way to implement the cubic phase gate V , which in fact is parameterized by the cutoff parameter Λ . As discussed in Appendix B, this gate is not straightforward to realize directly and possibly requires further ancilla measurements. Instead, Ref. [134] demonstrates how the cubic phase gate may be constructed from the non-linearity in the Kerr effect, in addition to Gaussian operations such as squeezing and displacement. From the cubic term in the non-linearity, it is shown that the cubic gate parameter is

$$s = \frac{3}{\sqrt{2}} \chi \tau \alpha \lambda^3, \quad (5.40)$$

where α is a displacement parameter and λ corresponds to a squeeze parameter $r = \ln \lambda$. χ characterizes the Kerr non-linearity and τ the Kerr gate time. Achieving high values of χ and τ without significant noise is impractical, and so the value of s is essentially controlled by the squeezing. As shown in the Reference, the parameter α can be adjusted to reduce undesired terms in the evolution Hamiltonian and in the noiseless case should be set to be $\alpha \sim \lambda^3$.

Sufficiently high squeezing is still required to reduce the contribution of the non-cubic terms to implement the cubic gate to arbitrary precision. To use their scheme for our purposes and construct the state $|\omega(\Lambda)\rangle$, we take a small value of the product $\chi\tau$ to construct the non-linear unitary $U_{\text{NL}}(s)$, and apply squeezing to re-scale the gate parameter s by the appropriate amount to obtain our desired value of Λ . In Fig. 5.9, we estimate achievable values of the cutoff Λ as a function of squeezing, in the absence of losses. The squeezing is calculated by first considering the amount of squeezing required to implement the V gate with gate parameter $s = 0.3$ at $\approx 99\%$ and $\approx 99.9\%$, the approximate values of which are displayed in the Reference. We then add the additional number of decibels to increase the gate parameter to $s = \frac{\Lambda}{g}$.

Unfortunately, while realistic squeezing levels appear sufficient for reasonable values of the cutoff Λ , this procedure has resulted in the need for “in-line” squeezers, even for the state preparation procedure.

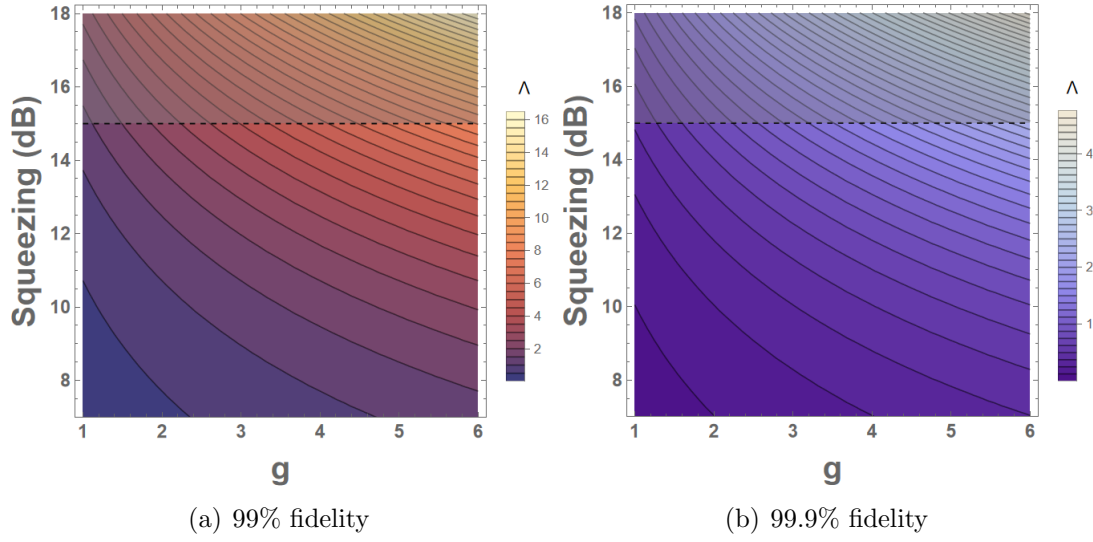


Figure 5.9: Achievable values of Λ corresponding for the state $|\omega(\Lambda)\rangle$ as a function of coupling g and squeezing in dB. Panels (a) & (b) correspond to 99% and 99.9% cubic gate fidelity. Estimates are based on computed fidelities for gate parameter $s = 0.3$ from Ref. [134], with the Kerr strength and gate time held to the values used for $s = 0.3$. Displayed is the ideal (noiseless) case.

As discussed in the previous chapter, this introduces additional challenges when executing on hardware, and may require extra components for the setup such as squeezed resource states.

Now that we have constructed the single-site non-interacting ground state (5.25), we now need to apply the cluster operator. As this operator only couples nearest neighbors, it suffices to consider just two lattice sites, each occupied by qumode triplets of quadratures (q_a, p_a) and $(q_{a'}, p_{a'})$, respectively with $a, a' = 1, 2, 3$. We collectively denote them by $(\mathbf{q}, \mathbf{p})$ and $(\mathbf{q}', \mathbf{p}')$. We start by engineering the multi-site non-interacting state

$$|\omega(\Lambda)\rangle_a \otimes |\omega(\Lambda)\rangle_{a'} = |l=0, m=0; \Lambda\rangle_a \otimes |l'=0, m'=0; \Lambda\rangle_{a'} , \quad (5.41)$$

which is given in terms of a direct tensor product of the states constructed for each lattice site, see (5.36). Now introduce a triplet of ancilla qumodes of quadratures $(\mathbf{q}_c, \mathbf{p}_c)$, initialized in the vacuum state, and apply the Gaussian unitary (a product of three CX gates (B.21))

$$W_{ac} = e^{-i\sqrt{\frac{2\alpha}{L}}\mathbf{q}\cdot\mathbf{p}_c} \quad (5.42)$$

followed by the analogous unitary $W_{a'c}^\dagger$ which couples the other lattice site with the ancilla. These two operators generate a coherent state on the ancilla qumode whose parameter is proportional to the difference of the quadratures of the physical qumodes. That is, we prepare the state

$$\begin{aligned} W_{a'c}^\dagger W_{ac} |\omega(\Lambda)\rangle_a \otimes |\omega(\Lambda)\rangle_{a'} &= \int d^3\mathbf{q} d^3\mathbf{q}' e^{-\frac{\Lambda^2}{8g^2}(\mathbf{q}^2 - g^2)^2} e^{-\frac{\Lambda^2}{8g^2}(\mathbf{q}'^2 - g^2)^2} \\ &\quad \times |\mathbf{q}\rangle |\mathbf{q}'\rangle \left| \sqrt{\frac{\alpha}{L}}(\mathbf{q} - \mathbf{q}') \right\rangle_{c,C} . \end{aligned} \quad (5.43)$$

After measuring the ancilla qumodes and projecting them onto the vacuum state $|\mathbf{0}\rangle_c$, we obtain the state

$$e^{-\frac{\alpha}{2L}(\mathbf{q}-\mathbf{q}')^2} |\omega(\Lambda)\rangle_a \otimes |\omega(\Lambda)\rangle_{a'} . \quad (5.44)$$

After identifying $\phi(x) = \mathbf{q}$, $\phi(x+1) = \mathbf{q}'$, we have thus engineered the desired state $|\text{CC}(\Lambda)\rangle$ for two lattice sites, see (5.26). The schematic circuit to apply the cluster operator is shown in Fig. 5.10. For an arbitrary 1-dimensional lattice, we repeat this procedure for each pair of adjacent sites $(x, x+1)$ (with an ancilla measurement required for each pair).

As our scheme is not fully unitary and requires ancilla measurements, it is important to analyze the success rate of our algorithm. Clearly, the assignment of separate ancillae to each spatial site suggests that the success rate decreases with lattice size. Notice that the circuits of Figs. 5.7 and 5.10 both require ancilla measurements on each site. Fig. 5.7 corresponds to the generation of resource states for the calculation. Since they are site-local, this generation can be done in parallel and offline. For instance, for $g = 1$ and $\Lambda = 3.2$ the success rate to generate $|\omega(\Lambda)\rangle$ is ~ 0.4 , so that it would only take ~ 2 -3 runs to obtain the state. Of course the number of runs would still increase with g and Λ . In any case, we move forward and compute the success rate of Fig. 5.10 assuming that we have access to the resource states $|\omega(\Lambda)\rangle$. These rates are shown in Fig. 5.11. Fitting to an exponential in the lattice size L , the rate of decay appears to be $g^2 - 1$. Given the exponential decrease in the success probabilities with lattice size, it may be worth identifying a unitary coupled-cluster (UCC) Ansatz through which we would not need to perform so many photon number measurements for large L . This is a subject for future work.

Next, we briefly consider the construction of the first excited state, see (5.29) above, which is a simple extension of the ground state case. After obtaining $|\Omega(\Lambda)\rangle$ at each lattice site, we add a single ancilla of quadratures (q_c, p_c) prepared in the vacuum state $|0\rangle_c$.

$$\prod_x e^{-i\gamma\phi_3(x)\otimes p_c}, \quad \gamma \ll 1, \quad (5.45)$$

We then apply the following string of Gaussian unitaries (CX gates) and measure the added ancilla in the photon number basis. This time, we look for a projection onto the single-photon state. As any measurement result will contain an undesired exponential of $\gamma^2\phi_3^2(x)$ (Eq. (5.33)), it is important that γ is chosen to be small.

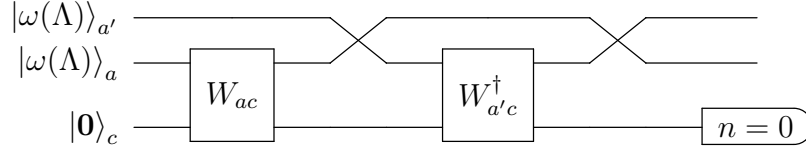


Figure 5.10: CV quantum circuit for constructing the CC state for two lattice sites in (5.44). Each line represents three qumodes.

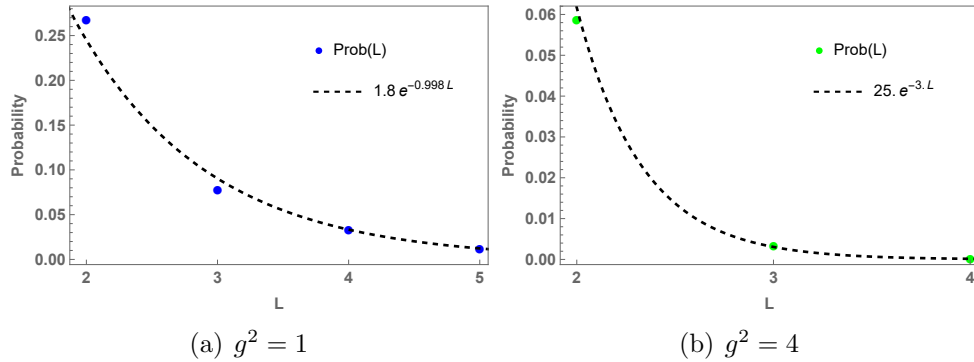


Figure 5.11: Success probability of obtaining the CC state (at the optimal value of α) as a function of lattice size L , given an ensemble of resource states $|\omega(\Lambda)\rangle$ (prepared using Fig 5.7). The points are fitted to an exponential function of L .

In this case, we obtain the following uniform superposition of states

$${}_c\langle 1 | \prod_x e^{-i\gamma\phi_3(x)\otimes p_c} |\Omega(\Lambda)\rangle |0\rangle_c \propto \sum_x \phi_3(x) |\Omega(\Lambda)\rangle , \quad (5.46)$$

on which the CC Ansatz for the first-excited state (5.29) is based. The quantum circuit for $L = 2$ lattice sites is shown in Fig. 5.12. After preparing the state in (5.46), we simply apply the procedure depicted in Fig. 5.10 to apply the cluster operator and thus obtain the CC Ansatz for the first excited state.

5.3.2 Quantum computation of energy levels

Having constructed the CC Ansatz, we now turn to our procedure for obtaining the ground state energy of the Hamiltonian given in (5.18) using quantum resources. This procedure follows the same principles we used to compute expectation values for CV in Chapter 4.

We first consider the interaction term in (5.18), which is given by $\frac{1}{2} \sum_x (\phi(x) - \phi(x+1))^2$ (plus a constant). As in the previous chapter, we may append single- or two- quode gates to write the expectation of this operator in terms of expectations of number operators.

For instance, after engineering the CC Ansatz (5.26), we focus on a single value of x and create the state

$$|\Psi(x, \Gamma)\rangle \equiv \prod_a P_{x,a}(\Gamma) \cdot BS_{x,x+1}^a |\text{CC}\rangle \quad (5.47)$$

by acting on the CC Ansatz first with a series of 50:50 beam splitters with neighboring lattice sites as input ports followed by quadratic phase gates $P_{x,a}(\Gamma) = e^{i\Gamma\phi_a^2(x)/2}$, where Γ is a real parameter (see Eq. (B.17)). As before, we can then compute

$$\begin{aligned} \frac{1}{\Gamma^2} \sum_x \sum_a \Big[& \langle \Psi(x, \Gamma) | N_{x,a} | \Psi(x, \Gamma) \rangle + \langle \Psi(x, -\Gamma) | N_{x,a} | \Psi(x, -\Gamma) \rangle \\ & - 2 \langle \Psi(x, 0) | N_{x,a} | \Psi(x, 0) \rangle \Big] , \end{aligned} \quad (5.48)$$

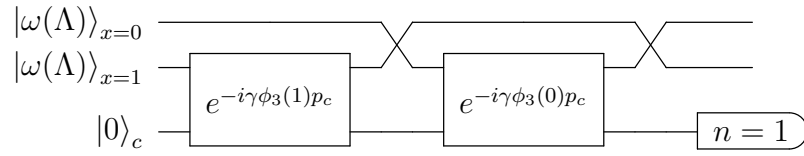


Figure 5.12: CV quantum circuit relevant for constructing the CC ansatz for the first excited state. It is shown here for $L = 2$ lattice sites.

where $N_{x,a}$ is the number operator for the qumode labeled by x, a , which can be written as $N_{x,a} = \frac{1}{2}(\pi_a^2(x) + \phi_a^2(x))$ with $a = 1, 2, 3$. We obtain the parameter shift rule

$$\phi_a^2(x) = \frac{1}{\Gamma^2} [P^\dagger(\Gamma)NP(\Gamma) + P^\dagger(-\Gamma)NP(-\Gamma) - 2N] . \quad (5.49)$$

Applying the 50:50 beam splitters results in

$$\phi_a(x) \rightarrow \frac{1}{\sqrt{2}} (\phi_a(x) - \phi_a(x+1)) \quad (5.50)$$

Therefore, we obtain the desired expression for each one of the interaction terms (each value of x and a) in the Hamiltonian (5.18). A quantum circuit for the above calculation is shown in Fig. 5.13 for qumodes $\phi_a(x)$ and $\phi_a(x+1)$.

However, similar to the previous chapter, we follow a calculation based on CX gates to reduce numerical error. That is, we add an ancilla qumode of quadratures (q_c, p_c) prepared in the vacuum state and make the replacement $e^{i\Gamma\phi_a^2(x)/2} \rightarrow e^{-i\Gamma p_c \otimes \phi_a(x)}$. As can be seen from Eq. 4.46, initiating the ancilla in the vacuum state allows us to choose only one value of Γ and compute

$$\frac{2}{\Gamma^2} \langle \Psi(\Gamma) | N | \Psi(\Gamma) \rangle . \quad (5.51)$$

thereby removing the need for a parameter shift rule. While this allows us to avoid errors from unphysical terms, with only one ancilla this alternate method requires us to compute $\langle (\phi_a(x) - \phi_a(x+1))^2 \rangle$ for each value of x and a . The circuit for this computation is given in Fig. 5.14.

Next, we are going to consider the kinetic energy term in the Hamiltonian (5.18), which is given by

$$\frac{1}{2g^2} \sum_x \mathbf{L}^2(x) = \frac{1}{2g^2} \sum_x (\boldsymbol{\phi}(x) \times \boldsymbol{\pi}(x))^2 . \quad (5.52)$$

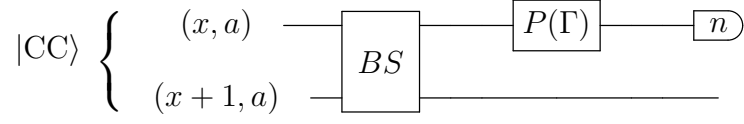


Figure 5.13: Quantum circuit for the calculation of the expectation value of the interaction term in the Hamiltonian (5.18) involving $\phi_a(x)$ and $\phi_a(x + 1)$.

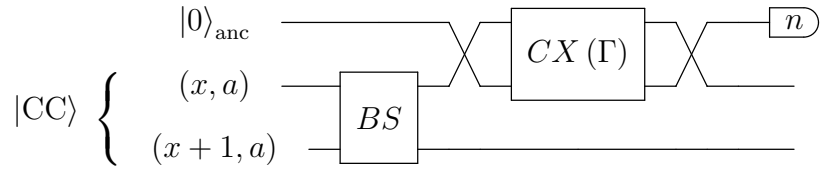


Figure 5.14: Alternate quantum circuit for the calculation of the expectation value of the interaction term in the Hamiltonian (5.18) involving $\phi_a(x)$ and $\phi_a(x + 1)$. Here we couple the system to an ancilla qumode and measure it in the photon number basis.

We first point out that in the general case, the angular momentum $\mathbf{L} = \mathbf{q} \times \mathbf{p}$ can be diagonalized in the photon number basis without the need for parameter shifts,

$$L_i = \epsilon_{ijk} F_j^\dagger BS_{jk}^\dagger (N_k - N_j) BS_{jk} F_j, \quad (5.53)$$

where F is again the Fourier transform operator $F = e^{i\frac{\pi}{4}(p^2+q^2)}$ and BS is a 50:50 beam splitter. This follows easily from the CV gate transformation properties reviewed in Appendix B. It is straightforward then to find the correct calculation to compute the kinetic term. As an example, fix x and $a = 3$, and consider $\langle L_3^2(x) \rangle$. The contributions of the other components can be treated similarly. We find

$$\langle \text{CC} | L_3^2(x) | \text{CC} \rangle = \langle \Psi | (N_2(x) - N_1(x))^2 | \Psi \rangle \quad (5.54)$$

where

$$|\Psi\rangle \equiv BS_x^{12} \cdot F_{x,1} | \text{CC} \rangle. \quad (5.55)$$

Thus, the angular momentum for the $a = 3$ direction is computed by measuring the square of the photon number difference for the $a = 1, 2$ directions. The expectation values of $L_{1,2}^2(x)$ can be obtained by cyclic permutation of the indices. The required circuit and measurements is displayed in Fig. 5.15.

Next, we present numerical results using the quantum simulator STRAWBERRY FIELDS [111] for two values of the coupling, $g^2 = 1$ and $g^2 = 4$. For these simulations, we need to impose a Fock space cutoff, which we denote by $n_{\max}$. The results for the ground state using the CC ansatz are shown in Fig. 5.16. These were found by computing the energy expectation near the optimal value of α obtained from numerical integration.

For the coupling term, we employed an ancilla and CX gates instead of P gates, see Eq. (5.51) and Fig. 5.14. The displayed error bars were obtained by computing the energy along each Cartesian direction. Since the CC ground state is invariant under global rotations, each result (multiplied by 3) is valid in the limit $n_{\max} \rightarrow \infty$, and any variation reflects truncation errors.

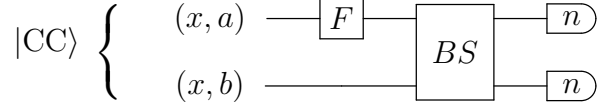


Figure 5.15: Quantum circuit for the calculation of the expectation value of the kinetic term in the Hamiltonian (5.18) involving $L_3^2(x)$.

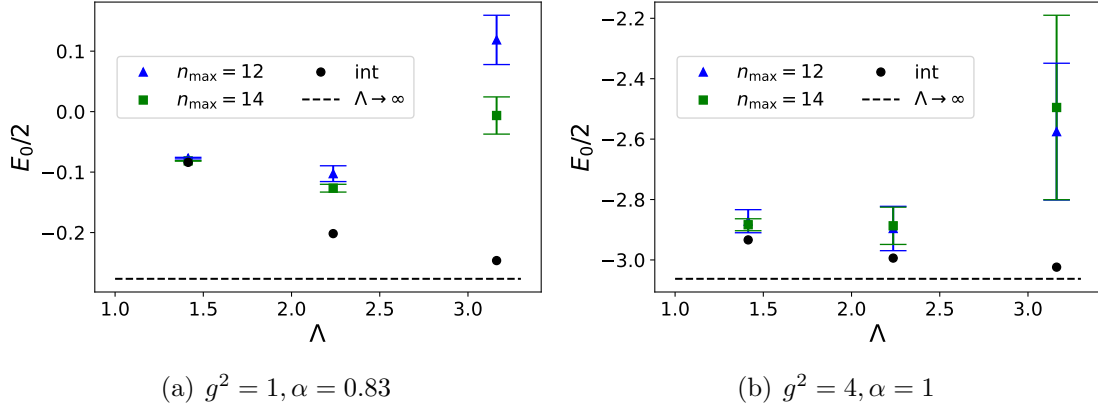


Figure 5.16: STRAWBERRY FIELDS simulation results for the calculation of the ground state energy using the CC Ansatz. Calculations were performed near the minimizing value of α for (a) $g^2 = 1$ and (b) $g^2 = 4$ and $L = 2$. The black “int” points were obtained with numerical integration over the radial coordinates, and the dashed line obtained using an analytical calculation.

Even at $n_{\max} = 14$, which requires a Hilbert space dimension of 14^7 ($3L$ physical qumodes plus an ancilla, summing to 7), we see that there is very significant truncation error for $\Lambda = 3.2$.

Results seem to be improved somewhat if we compute $\langle \mathbf{L}^2 \rangle$ using Eq. (5.54) but by breaking up the expectation value as

$$\langle \text{CC} | L_3^2(x) | \text{CC} \rangle = 2 \left(\langle \Psi | N_1^2(x) | \Psi \rangle + \langle \Psi | N_2^2(x) | \Psi \rangle \right) - \langle \text{CC} | (N_1 + N_2)^2 | \text{CC} \rangle, \quad (5.56)$$

see Fig. 5.17. Here the last term is an expectation in the state $|\text{CC}\rangle$ rather than $|\Psi\rangle$ (Eq. (5.55)), to be compared with Fig. 5.16 which uses $|\Psi\rangle$ for all terms:

$$\langle \text{CC} | L_3^2(x) | \text{CC} \rangle = \langle \Psi | N_1^2(x) | \Psi \rangle + \langle \Psi | N_2^2(x) | \Psi \rangle - 2 \langle \Psi | N_1 N_2 | \Psi \rangle \quad (5.57)$$

In any case, we note that this significant truncation error is just in the photon number basis. The previously obtained ED results were truncated in the spherical harmonic basis. The discrepancy follows from the fact that the states relevant to our calculation are highly non-trivial in the photon number basis, requiring non-linearities as well as significant amounts of squeezing and displacing (see e.g. Figs. 5.2 and 5.9). Such states are expected to require a spread of contributions from large n number states.

5.3.3 Time evolution circuits and photonic quantum simulator results

Given our qumode Hamiltonian (5.18), it is important to understand how times evolution can be studied in this framework. In this section, we perform the time evolution for the Hamiltonian given in (5.18) in terms of small time steps Δt .

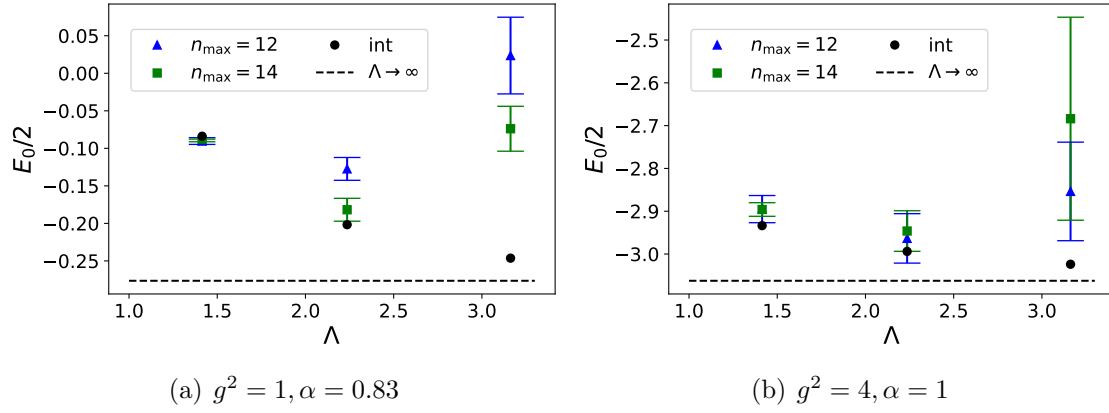


Figure 5.17: STRAWBERRY FIELDS simulation results for the calculation of ground state energy using Eq. (5.56) for finding the kinetic energy. Calculations were performed near the minimizing value of α for (a) $g^2 = 1$ and (b) $g^2 = 4$ and $L = 2$. The black “int” points were obtained with numerical integration over the radial coordinates, and the dashed line obtained using an analytical calculation.

In order to promote the interaction term of the Hamiltonian (5.18) to a time-evolution operator, we apply the string of unitaries

$$U_I \equiv \prod_x \prod_a U_a(x), \quad (5.58)$$

where $U_a(x)$ is obtained by applying 50:50 beam splitters and a quadratic phase gate

$$U_a(x) \equiv e^{-i\frac{\Delta t}{2}(\phi_a(x) - \phi_a(x+1))^2} = BS_{x+1,x}^a \cdot P_{x,a}(-2\Delta t) \cdot BS_{x,x+1}^a. \quad (5.59)$$

A quantum circuit implementing this Trotter step is shown in Fig. 5.18 for qumodes $\phi_a(x)$ and $\phi_a(x+1)$.

The time evolution of the kinetic term can be implemented with the aid of non-Gaussian Kerr gates, $K(s) = e^{isN^2}$, and Cross-Kerr gates, $CK(s) = e^{isN_1N_2}$. For a given lattice site x , each Trotter step will contain the following string of unitaries

$$U_{31}(x) \cdot U_{23}(x) \cdot U_{12}(x). \quad (5.60)$$

Using Eq. (5.53) applied to the $\mathbf{L}^2$ operator, we can write the unitaries as

$$U_{ab}(x) = F_{x,a}^\dagger \cdot BS_x^{ba} \cdot CK_x^{ab}\left(\frac{\Delta t}{g^2}\right) \cdot K_{x,a}\left(-\frac{\Delta t}{2g^2}\right) \cdot K_{x,b}\left(-\frac{\Delta t}{2g^2}\right) \cdot BS_x^{ab} \cdot F_{x,a}. \quad (5.61)$$

A quantum circuit implementing the Trotter step for the kinetic energy (5.60) is shown in Fig. 5.19 for the three qumodes at a given lattice site x .

Consider the state $|l, m; \Lambda\rangle$ given by Eq. (5.20). Its wave function factorizes into radial and angular parts

$$\langle r, \mathbf{n} | l, m; \Lambda \rangle = \psi_\Lambda(r) Y_{lm}(\mathbf{n}), \quad \psi_\Lambda(r) = \frac{1}{\sqrt{\mathcal{N}}} e^{-\Lambda^2(r^2 - g^2)^2 / 8g^2}, \quad (5.62)$$

where the normalization constant is given in (5.21) for large values of the cutoff Λ . The radial wave function is centered at $r = g$ with a spread $\Delta r \sim \mathcal{O}(1/\Lambda)$.

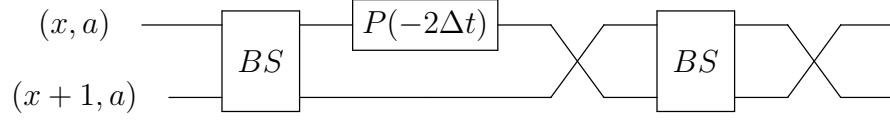


Figure 5.18: Quantum circuit for a Trotter step of the interaction term (5.59) in the Hamiltonian (5.18) for qumodes $\phi_a(x)$ and $\phi_a(x+1)$.

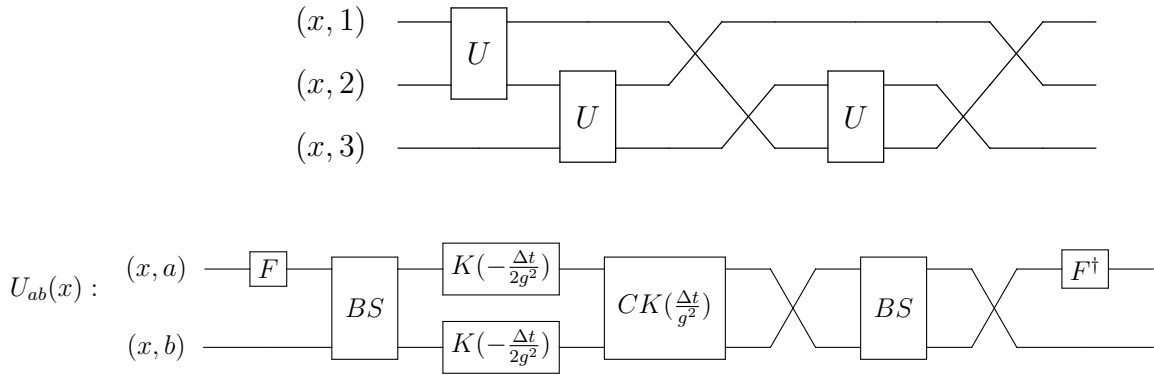


Figure 5.19: Quantum circuit for a Trotter step of the kinetic energy term (5.60) of the Hamiltonian (5.18) for the three qumodes $\phi_a(x)$.

The matrix elements of the evolution operator in the basis of (5.20) can be written as

$$\langle \mathbf{l}, \mathbf{m}; \Lambda | e^{-itH} | \mathbf{l}', \mathbf{m}'; \Lambda \rangle = \int \prod_x dr(x) r^2(x) |\psi_\Lambda(r(x))|^2 \langle \mathbf{l} \mathbf{m} | e^{-itH} | \mathbf{l}' \mathbf{m}' \rangle . \quad (5.63)$$

Here $|\mathbf{l}, \mathbf{m}; \Lambda\rangle = \bigotimes_x |l(x), m(x); \Lambda\rangle$, and similarly for the other states. The Hamiltonian H is a function of the radial coordinates through its interaction term which is quadratic in the fields. Therefore, the radial spread in the exponent of the evolution operator is $t \cdot \mathcal{O}(r\Delta r) \sim \mathcal{O}(gt/\Lambda)$, where we used $r \sim g$ and $\Delta r \sim 1/\Lambda$. It follows that, given t , we need to choose Λ so that $t \lesssim \Lambda/g$. To show this numerically, we computed the probability corresponding to the transition amplitudes in (5.63) for $g^2 = 1$ and several values of Λ . The results are shown in Fig. 5.20.

In this Figure, we computed this probability for $l = l' = m = m' = 0$ and $L = 2$. As in (5.63), this calculation was performed by expressing the time evolution operator as a matrix in the basis of spherical harmonics, and as a function of the continuous parameter r . Monte Carlo integration was then performed over the r coordinate to complete the computation of the amplitude. We find that greater values of Λ are required for large times t as well as for values of t for which the probability of returning to the original state is large.

On a CV quantum computer, we may compute the probability by applying a factor of $\sqrt{2}$ to the arguments of U_b and V in Figs. 5.7 and 5.8, respectively, but not yet measuring the ancillae in Fig. 5.7. After applying the time evolution, we measure all qumodes including the ancilla. The probability of all qumodes being measured in the $|0\rangle$ state is

$$P(|\mathbf{0}\rangle) = \mathcal{N}^{2L} |\langle \mathbf{0}, \mathbf{0}; \Lambda | e^{-itH} | \mathbf{0}, \mathbf{0}; \Lambda \rangle|^2, \quad (5.64)$$

where $\mathcal{N}$ is the normalization constant in Eq. (5.62).

In Fig. 5.21 we show results for a similar time evolution calculation, using the STRAWBERRY FIELDS quantum simulator.

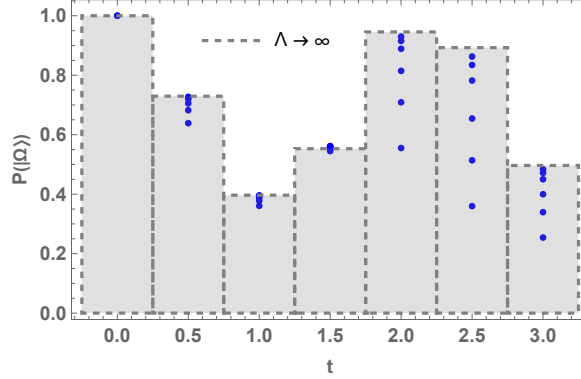


Figure 5.20: The probability of obtaining the original state $|\Omega(\Lambda)\rangle$ (return probability) as a function of time t . Here we take $L = 2, l_{\max.} = 3, g^2 = 1$. The blue points correspond to $\Lambda = 3.2, 4.5, 6.3, 10, 14, 20$ that converge to the $O(3)$ result shown by the horizontal dashed line in the large Λ limit.

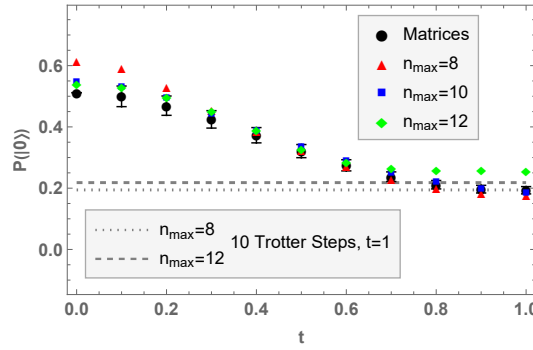


Figure 5.21: Probability of obtaining the vacuum state $|0\rangle$, after evolving the state $|\Omega(\Lambda)\rangle$ for time t . Here we take $L = 2, g^2 = 1, \Lambda = 3.2$. The black points are obtained in the same way as in Fig. 5.20 (matrices in spherical harmonic basis with truncation $l_{\max.} = 3$, Monte Carlo integration over the radial direction). The points without error bars indicate results obtained with the quantum simulator STRAWBERRY FIELDS with two Trotter steps for each time measurement, for different Fock space cutoffs $n_{\max.}$. We also show the result for $n_{\max} = 8$ and $n_{\max} = 12$ with 10 Trotter steps, for $t = 1$.

Here we constructed and evolved the state $|\mathbf{0}, \mathbf{0}; \Lambda\rangle$ and computed the overlap with the photon number vacuum state

$$\langle \mathbf{r} | \mathbf{0} \rangle \propto e^{-\mathbf{r}^2/2} |\Omega_0\rangle \quad (5.65)$$

where $|\Omega_0\rangle$ is defined in Eq. (5.11) and $\mathbf{r}$ is the vector of radial coordinates over all sites. Even with just two Trotter steps for each time measurement (colored markers) and a small truncation in the photon number basis, the results in Fig. 5.21 display good agreement with results obtained using matrices and Monte Carlo integration, the latter of which contains no Trotter error.

5.4 Discussion

The previous Chapter laid out a CV quantum computing algorithm for approximating the phase transition of ϕ^4 scalar field theory. However, our ultimate goal is to obtain an efficient simulation of a lattice gauge theory. Therefore, the theme of this chapter was to apply some of the techniques we used for ϕ^4 theory toward an interesting model sharing features with four-dimensional QCD. As mentioned at the beginning of this Chapter, the $O(3)$ model shares similarities with four-dimensional QCD including the presence of instanton solutions, asymptotic freedom, and the presence of a dynamically generated mass gap.

A noteworthy aspect of our approach was how we considered the $O(3)$ model as the limit of a three-component scalar field theory, the latter of which is straightforward to simulate with continuous variables. Understanding how to realize this limit, i.e. obtaining the constraint $\phi^2(x) = 1$, is vital for carrying our construction over to theories such as $SU(2)$ lattice gauge theories where the $SU(2)$ matrices can be mapped to a unit vector in $\mathbf{R}^4$. In our case, we found that a non-linearity (cubic phase gate) combined with large squeezing was required to realize the constraint, specifying the technological challenges of preparing the state in a laboratory.

Chapter 6

Conclusion and Future Work

In this dissertation, we demonstrated several methods which could be used to approach quantum computations of lattice gauge theories (LGTs). Each Chapter generally investigated separate unique features each of which adds to the complexity of a realistic model such as Quantum Chromodynamics (QCD). In the latter portion of the dissertation, we began to focus on taking advantage of the capabilities of continuous variables (CV) quantum computation.

We started with a review of standard methods for qubit quantum computation and how they could be applied to simpler relativistic quantum field theories. In Ref. [62], discussed in Chapter 2, we attacked an interacting theory of fermions in 1+1 dimensions using the well-known Jordan-Wigner transformation and applied error mitigation methods. These methods included readout error mitigation as well as zero noise extrapolation, which we relied on in later quantum calculations. Most importantly, this work contained an analysis of what happens when we replace certain parts of the quantum algorithm with classical elements by splitting Ansatz states into two separate quantum circuits. The various algorithms were run on IMBQ Boeblingen to obtain model energies, and we found that increasing the classical portion of the calculation resulted in a reduction in errors.

In Ref. [63] (see Chapter 3) we analyzed the massive Schwinger model. Like with the fermionic model, this analysis was done in a momentum space formulation, and we observed

two important complications which needed to be addressed. The first was the gauge freedom of the theory, against which we took several measures to obtain a set of computational basis states that were gauge invariant. By holding to periodic boundary conditions, which is not often discussed in the relevant literature, we encountered the second complication brought forth by the one remaining bosonic degree of freedom which exhibited an infinite hilbert space. These complications appeared to spoil a natural encoding on a quantum computer, and major truncation efforts were required to compute the phase transition on a quantum device, IBMQ Lima. It was noted that the problem of truncation extends to other theories with bosonic degrees of freedom, especially those existing in higher dimensional lattice gauge theories. Additional difficulties were introduced by our attempt to locate the critical point of the phase transition on a quantum computer, and we found that the quantities of interest, the mass gaps, were difficult to properly resolve near the transition, given the noise from the quantum computer.

Given the challenge of encoding one or more bosons, it became important to consider an alternative method for quantum computation that could accommodate these degrees of freedom. This came in the form of CV quantum computing where the fundamental unit of quantum computation was the qumode. A qumode encodes a complete harmonic oscillator system, which maps to the infinite dimensional hilbert space associated with a boson mode. After discussing the context of CV, namely possible hardware realisations, we presented a CV quantum algorithm for computing the phase transition in an interacting scalar field theory, which was discussed in Ref. [64]. This algorithm established the preparation of a Gaussian Ansatz for computing the Gaussian Effective Potential GEP and how energies can be computed in a variational setting. The Ansatz state itself was rather simple, involving a collection of squeezed vacua with the zero mode shifted by a displacement. The scheme employed photon number measurements to compute expectation values, as well as formulae resembling parameter shift rules for continuous variables. Thus we also needed to add entangling CX gates as well as photon number-resolving detectors. We then compared our

simulation results, which used Bosonic Qiskit [99], to a qubit calculation performed on IBM Q Kolkata and Mumbai.

Finally, we reviewed the CV quantum algorithms deployed in Ref. [65] for computing the $O(3)$ non-linear sigma model. Instead of applying the Schwinger boson formulation, this work considered the $O(3)$ model as the limit of a three-component scalar field theory. This limit was to be achieved as an energy cutoff $\Lambda \rightarrow \infty$. This parameter was shown to represent the squeezing required to constrain the system to a sphere. Additional non-trivial circuit elements were included for the preparation of states such as the coupled-cluster (CC) Ansatz. These included cubic phase gates and photon number measurements (with postselection applied to the latter). After discussing the accuracy of coupled-cluster spectra calculations we introduced a simple Trotterized time evolution protocol for CV quantum computation.

A direct extension of the work in Chapter 5 would be to add a θ term and investigate the effectiveness of CV in these different sectors. DV quantum circuits were introduced for $\theta = \pi$ in Ref. [135]. Included in their analysis was a look at the critical behavior as well as an identification of the vanishing of the mass gap at strong-coupling. However, as expected, the $g^2 \rightarrow \infty$ limit (in our coupling convention) is not as easy to analyze using qubit approaches due to the problem of truncation. It would then be beneficial to check whether the methods introduced in Chapter 5 are expected to demonstrate improved results in this regime.

Moving forward, some additional steps can be taken to expand our CV quantum computing vocabulary before proceeding to attack lattice gauge theories. An interesting model to consider is the Bose-Hubbard model [136, 137] which is a bosonic system exhibiting a superfluid to Mott insulator phase transition. Its bosonic nature suggests that it is amenable to CV quantum computation. While recent work has demonstrated that Gaussian states are effective at describing the ground state as well as excitations in the superfluid phase [138], we have found that unlike the ϕ^4 case, a Gaussian Ansatz is ineffective at describing the insulating phase and the phase transition.

As phase transitions were a significant focus of this dissertation, a major goal of future work would be to attempt to effectively describe phase transitions using continuous variables. While it turns out that the Gutzwiller Ansatz and its extensions [139, 140] are known to be quite effective for describing the Bose-Hubbard transition, its form is that of a non-unitary exponential which resembles the CC Ansatz of Chapter 5. Therefore we expect that a direct mapping of this Ansatz to continuous variables would result in the need for photon number measurements once more. While the Bose-Hubbard model may not have many obvious connections to LGT's, a successful CV approach to computing the phase transition in the former would go a long way toward identifying additional techniques for the latter.

It is also worth continuing to investigate the utility of hybrid systems. In Chapter 5 we made an implicit assumption that our simulator contained no capability of coupling to qubits, and perhaps assumed furthermore that we were relying on a photonic architecture. However, according to Appendix A.2 a circuit QED platform not only gives us the desired coupling, but also turned out to be quite useful in the calculations of Chapter 4. The primary advantage came from the so-called “SNAP” gate (see Appendix A.2) which allowed us to isolate particular number states from a Gaussian qumode state, which is clearly a non-Gaussian operation. This gate might also help to improve the success rates plotted in Figure 5.11 by allowing us to give preference to the $n = 0$ state.

The tools of circuit QED described in [99] also include Gaussian operations controlled by a qubit, as well as an exponential SWAP gate (for one application of the latter, see [59]). All of these are additional non-Gaussian operations. As mentioned in Section 4.1, trapped-ion systems provide another set of non-Gaussian operations, one of which is purely bosonic. As the trapped-ion system was already shown to be useful in simulating a lattice gauge theory (cf. Ref. [52]), we must take note that future quantum algorithms for LGT's, whether they primarily rely on qubits or qumodes, should still remain open to the advantages provided by a hybrid coupling. Of course, these analyses should also take into account the hardware limitations of the technology envisioned if the device is not yet available for testing.

The primary direction of all of the work described in this dissertation was of course the quantum computation of lattice gauge theories, with the “dream” being an efficient calculation of lattice QCD. In Section 5.1, we introduced a convenient form of the Kogut-Susskind Hamiltonian for lattice $SU(2)$ gauge theory (c.f. Eq. (5.3)). We noted that this form contains several similarities with the $O(3)$ and $O(4)$ models. Unfortunately, the model is equipped with a Gauss law constraint like the Schwinger model (Chapter 3). Satisfying this constraint and generating gauge-invariant states is thus an additional challenge that must be addressed before the methods of Chapter 5 may be applied. This is the subject of ongoing work. It would be particularly interesting if it can be demonstrated that continuous variables provide a superior framework to realize such constraints. However, this remains to be seen.

We would eventually like to re-couple fermions to the gauge fields to get closer to reality. Representing fermions with continuous variables does not seem natural, therefore a hybrid system would almost become necessary at this point. Finally, it will also become imperative to increase the system dimensionality at some point if we desire to simulate 3+1 dimensional QCD. In fact, it would be useful to revisit some of the systems described in this dissertation and upgrade their spatial dimension. As Coleman’s normal-ordering protocol [100] only applies for the 1+1 dimensional case, it seems necessary to test our Gaussian Ansatz prescription of Chapter 4 against the 2+1 dimensional case where there is also a phase transition [45].

Perhaps the most interesting major item in this dissertation to expand to a higher dimension is QED. Unlike the Schwinger model (QED in 1+1 dimensions) we obtain multiple bosonic degrees of freedom that cannot be removed via “fermionization” through Gauss’s Law, inspiring us to attempt a CV quantum computation. Upon arriving at the physical theory, QED in three spatial dimensions, we would be in a position to observe some interesting non-perturbative phenomena. Several of these phenomena are investigated in Ref. [27], which utilized tree tensor networks (TTN). Of particular interest are confinement properties. The massive Schwinger model exhibits confinement (linear potentials between

charges), while QED in three spatial dimensions contains a phase transition between a confining phase at strong coupling (with a non-vanishing photon mass) and a non-confining phase at weak coupling [1]. Confinement properties of pure electrodynamics in both two and three spatial dimensions have only been identified in the compact version of the theory, which is a theory of interacting rather than free photons [1, 141, 142, 27]. Another interesting object in the 2+1 dimensional case is the expectation value of the plaquette operator, as it turns out to be directly related to the running of the coupling constant [42].

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Appendices

Appendix A

Additional Details for Interacting Scalar Field Theory

In this Appendix, we provide details pertaining to the discussion of Chapter 4.

A.1 Computing Expectation Values

To calculate the expectation value of the Hamiltonian, we express it in terms of quadrature operators q and p . We first simplify this calculation by identifying which Hamiltonian terms can have non-zero expectation values with respect to our rather simple Ansatz state described in Section 4.4. Notice that the Gaussian Ansatz in Figure 4.5 is even under the reflection operators for non-zero modes, $e^{\pi i[N(k)+N(L-k)]}$ ($1 \leq k < \frac{L}{2}$) and $e^{\pi i N(\frac{L}{2})}$. Therefore, the only Hamiltonian terms contributing are those with an even number of quadrature operators acting on the $k = \frac{L}{2}$ mode or an even total number of quadrature operators acting on a $(k, L - k)$ mode pair, since each quadrature operator is odd under reflection. Since the Hamiltonian only contains an even number of quadrature operators per term, this implies that only an even number can act on the zero mode as well. We can also eliminate terms with an odd number of quadratures $p(k)$ acting on a single mode. This follows from the

fact that such an operator is purely imaginary while the Ansatz unitaries are real. These observations help us reduce the number of terms to be computed.

It is convenient to express the fields (Eq. (4.25)) as

$$\begin{aligned}
\phi(x) &= \frac{1}{\sqrt{L}} \left[\frac{q(0)}{\sqrt{\omega(0)}} + \frac{q(\frac{L}{2})}{\sqrt{\omega(\frac{L}{2})}} (-)^x \right. \\
&\quad \left. + \sum_{1 \leq k < \frac{L}{2}} \sqrt{\frac{2}{\omega(k)}} \left(q_+(k) \cos \frac{2\pi kx}{L} - p_-(k) \sin \frac{2\pi kx}{L} \right) \right] \\
\pi(x) &= \frac{1}{\sqrt{L}} \left[\sqrt{\omega(0)} p(0) + \sqrt{\omega\left(\frac{L}{2}\right)} p\left(\frac{L}{2}\right) (-)^x \right. \\
&\quad \left. + \sum_{1 \leq k < \frac{L}{2}} \sqrt{2\omega(k)} \left(p_+(k) \cos \frac{2\pi kx}{L} + q_-(k) \sin \frac{2\pi kx}{L} \right) \right] \quad (\text{A.1})
\end{aligned}$$

in terms of the linear combinations of quadratures,

$$q_{\pm}(k) = \frac{1}{\sqrt{2}} (q(k) \pm q(L-k)) \quad , \quad p_{\pm}(k) = \frac{1}{\sqrt{2}} (p(k) \pm p(L-k)) \quad , \quad 1 \leq k < \frac{L}{2} \quad (\text{A.2})$$

These are simply the transformed quadrature operators under a 50/50 beamsplitter operation, Eq. (B.11).

The Hamiltonian can be written as

$$\begin{aligned}
H = & \frac{\omega(0)}{2} [q^2(0) + p^2(0)] + \frac{\omega(\frac{L}{2})}{2} \left[q^2(\frac{L}{2}) + p^2(\frac{L}{2}) \right] \\
& + \sum_{1 \leq k < \frac{L}{2}} \frac{\omega(k)}{2} [q_+^2(k) + q_-^2(k) + p_+^2(k) + p_-^2(k)] \\
& + \frac{\Omega^2 - m^2}{2} \left[\frac{q^2(0)}{\omega(0)} + \frac{q^2(L/2)}{\omega(\frac{L}{2})} + \sum_{1 \leq k < \frac{L}{2}} \frac{1}{\omega(k)} (q_+^2(k) + p_-^2(k)) \right] \\
& + \frac{\lambda}{24L} \left[\frac{q^4(0)}{\omega^2(0)} + \frac{q^4(\frac{L}{2})}{\omega^2(\frac{L}{2})} + 6 \frac{q^2(0)q^2(\frac{L}{2})}{\omega(0)\omega(\frac{L}{2})} + 6 \left(\frac{q^2(0)}{\omega(0)} + \frac{q^2(\frac{L}{2})}{\omega(\frac{L}{2})} \right) \sum_{1 \leq k < \frac{L}{2}} \frac{1}{\omega(k)} (q_+^2(k) + p_-^2(k)) \right. \\
& + \frac{1}{2} \sum_{1 \leq k < \frac{L}{2}} \frac{1}{\omega^2(k)} \left[3 (q_+^2(k) + p_-^2(k))^2 \left(1 - \delta_{k, \frac{L}{4}} \right) + 4 (q_+^4(k) + p_-^4(k)) \delta_{k, \frac{L}{4}} \right] \\
& + 3 \sum_{1 \leq k < k' < \frac{L}{2}} \frac{1}{\omega(k)\omega(k')} \left[2 (q_+^2(k) + p_-^2(k)) (q_+^2(k') + p_-^2(k')) \right. \\
& \left. \left. + (q_+^2(k) - p_-^2(k)) (q_+^2(k') - p_-^2(k')) \delta_{k+k', \frac{L}{2}} \right] \right] \tag{A.3}
\end{aligned}$$

In the VQE circuit shown in Figure 4.6, we used a two-mode squeezer to generate all of the squeezed states, including the $k = 0, \frac{L}{2}$ modes. We also applied 50/50 beamsplitters to all pairs of qumodes in order to implement the transformation (A.2). For simulations, the circuit in Figure 4.6 results in significant sampling error from computing derivatives with respect to Γ , even when using the parameter shift rule. This sampling error can be mitigated on actual CV hardware by increasing the magnitude of the shifts on the Γ parameter (up to the limit the platform can handle). However, in a finite Hilbert space necessitated by limitations of simulations, this greatly increases truncation error. Our results therefore are based on the circuit shown in Figure A.1, instead. In this case, the ancilla is not squeezed and the state $|\phi\rangle_{\text{anc}}$ in Eq. (4.46) becomes $|0\rangle_{\text{anc}}$ and the ancilla decouples without differentiating.

We calculated expectation values of quadratures that contribute to the expectation value of the Hamiltonian using photon-number measurements. The expectation value $\langle q^2(k) \rangle$ was computed in (4.47).

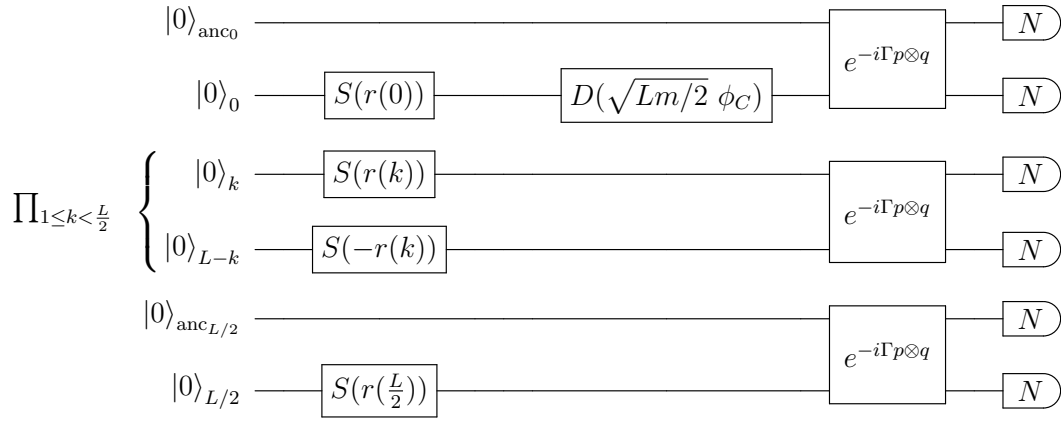


Figure A.1: Alternative quantum circuit to the one in Figure 4.6.

More generally, we have

$$\langle q^{2n}(k) \rangle = \frac{2^n}{(2n)!} \frac{d^{2n}}{d\Gamma^{2n}} \langle \Phi(\Gamma) | N_{\text{anc}}^n | \Phi(\Gamma) \rangle \quad (\text{A.4})$$

where $|\Phi(\Gamma)\rangle$ is given in Eq. (4.44). This follows from the fact that the CX gate $e^{i\Gamma p_{\text{anc}} \otimes q(k)}$ transforms $q_{\text{anc}} \rightarrow q_{\text{anc}} + \Gamma q(k)$, $p_{\text{anc}} \rightarrow p_{\text{anc}}$.

We re-expressed these derivatives as linear combinations of expectation values using CX gates with different parameters Γ ; see, Eq. (4.49) for $\langle q^2(k) \rangle$. For the quartic term, $\langle q^4(k) \rangle$, we used a five-term parameter shift:

$$\begin{aligned} \langle q^4(k) \rangle = \frac{1}{6s^4} \Big[& \langle \Phi(2s) | N_{\text{anc}} | \Phi(2s) \rangle + \langle \Phi(-2s) | N_{\text{anc}} | \Phi(-2s) \rangle - 4 \langle \Phi(s) | N_{\text{anc}} | \Phi(s) \rangle \\ & - 4 \langle \Phi(-s) | N_{\text{anc}} | \Phi(-s) \rangle + 6 \langle \Phi(0) | N_{\text{anc}} | \Phi(0) \rangle \Big] \quad (\text{A.5}) \end{aligned}$$

For modes with $1 \leq k < \frac{L}{2}$, we considered pairs $(k, L-k)$ as one another's ancilla by making the replacements $N_{\text{anc}} \rightarrow N(k) + N(L-k)$, and $e^{-i\Gamma p_{\text{anc}} \otimes q(k)} \rightarrow e^{-i\Gamma p(k) \otimes q(L-k)}$.

A.2 Derivatives for physical parameters

Here we discuss further the implementation of derivatives of expectation values with respect to physical parameters which can be computed exactly in quantum circuits. As we discussed, the introduction of parameter shift rules allowed us to compute derivatives with respect to the squeezing parameters $r(k)$ (Eq. (4.38)). This was enabled by the Gaussian form of our Ansatz. It is of great interest to extend results of exact computation of derivatives to cases where parameter shift rules are challenging to implement, such as in the presence of non-Gaussian gates.

To this end, one can leverage the tools of *bosonic qiskit* [99] which allow coupling between qubits and qumodes. Such couplings exist in hybrid quantum systems such as circuit QED [53]. In particular, one can take advantage of the Selective Number-dependent Arbitrary

Phase (SNAP) gate,

$$\text{SNAP}_n(\theta) = e^{-i\theta Z \otimes |n\rangle\langle n|} \quad (\text{A.6})$$

which involves a phase that is conditioned on the photon number state $|n\rangle$ of the physical qumode.

To compute derivatives with respect to a squeezing parameter $r(k)$, notice that for the expectation value of a Hermitian operator A in a state constructed using a squeezer $S(r) = e^{\frac{i}{2}(a^{\dagger 2} - a^2)}$ and an unspecified, possibly multi-modal, unitary U , taking a derivative with respect to r , we obtain

$$\frac{d}{dr} \langle 0 | S^\dagger(r) U^\dagger A U S(r) | 0 \rangle = \sqrt{2} \text{Re} \{ \langle 2 | S^\dagger(r) U^\dagger A U S(r) | 0 \rangle \} \quad (\text{A.7})$$

involving number states $|0\rangle$ and $|2\rangle$. For a single qumode, this is implemented with the quantum circuit in Figure A.2. This circuit allows us to compute derivatives using an ancilla. We first apply a squeezer of arbitrary squeezing t that generates all the even number states, and then we use SNAP gates to change the states of the qubits. The qubit states allow us to label the number states for the physical qumode. By projecting the ancilla qubit labeled 1 onto the state $|1\rangle$, we pick out the states of the qumode that we are interested in: $|0\rangle, |2\rangle$. Finally, we measure $Z_0 \otimes A$. To obtain (A.7), we need divide by the appropriate factor which contains the matrix elements $\langle 0 | S(t) | 0 \rangle$ and $\langle 2 | S(t) | 0 \rangle$. This method is an extension of the Hadamard test [143] to hybrid systems. We used the circuit of Figure A.2 for the $k = 0, \frac{L}{2}$ modes.

For modes with $1 \leq k < \frac{L}{2}$, we treat them as pairs $(k, L - k)$. The quantum circuit for derivatives with respect to squeezing parameters in this case is slightly modified. We need include additional circuit elements that incorporate some of *bosonic qiskit*'s controlled Gaussian operations, namely a controlled beam splitter and rotation gate. The resulting circuit is shown in Figure A.3.

Turning to derivatives with respect to ϕ_C , they can be computed by removing the displacement D from the circuit, taking $H \rightarrow D^\dagger H D$, and expressing it as a polynomial

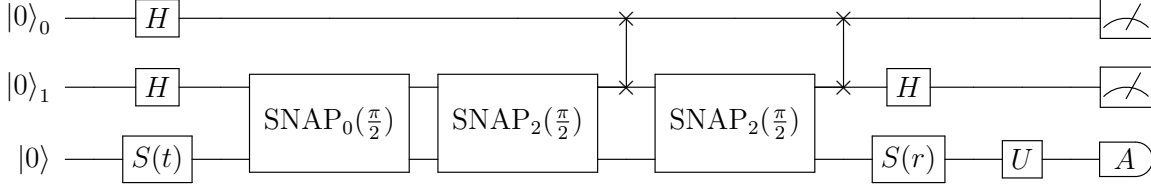


Figure A.2: Quantum circuit employing two ancilla qubits for the calculation of derivatives with respect to squeezing parameters for the modes with $k = 0, \frac{L}{2}$. It features single-mode squeezers, an unspecified qumode unitary U , Hadamard gates on the qubits, and SNAP gates entangling qumodes and qubits. At the end, qubits are measured in the computational basis whereas the (unspecified) Hermitian operator A is measured on the qumode.

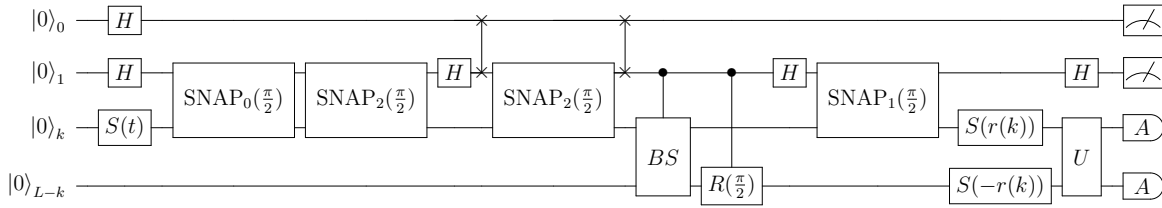


Figure A.3: Quantum circuit employing two ancilla qubits for the calculation of derivatives with respect to squeezing parameters for pairs of modes $(k, L-k)$ with $1 \leq k < \frac{L}{2}$. It features single-mode squeezers, an unspecified qumode unitary U , Hadamard gates on the qubits, and controlled beam splitter, rotation, and SNAP gates entangling qumodes and qubits. At the end, qubits are measured in the computational basis whereas the (unspecified) Hermitian operator A is measured on the qumode.

in ϕ_C . In general, derivatives with respect to ϕ_C are straightforward. The only terms affected by the displacement are those that contain the zero-mode quadrature. Thus, we need only compute $\langle D^\dagger q^2(0)D \rangle$ and $\langle D^\dagger q^4(0)D \rangle$ in a state that lies in the even parity subspace (squeezed vacuum). Dropping terms that are odd in the quadrature $q(0)$, we easily obtain

$$\frac{d}{dc} \langle D^\dagger(c) q^2(0) D(c) \rangle = 2c . \quad (\text{A.8})$$

For the quartic term, we obtain

$$\frac{d}{dc} \langle D^\dagger(c) q^4(0) D(c) \rangle = 12 \langle D^\dagger(c) q^2(0) D(c) \rangle - 8c^3 , \quad (\text{A.9})$$

expressing the derivative in terms of the expectation value of $q^2(0)$ which has already been computed (Eq. (4.49)).

Appendix B

Gate set for Continuous Variables

In this appendix, we will discuss some standard gates that are used in the gate model of continuous variables quantum computation. Additional details and gates can be found in [48, 111]. These can be subdivided into various categories, depending on the ease of implementation in a photonic device [48]. Unitary gates with an exponent that is quadratic in the quadrature operators is considered a Gaussian operation, which as noted in the Introduction will not increase the degree (in the Heisenberg representation) of any polynomial of the quadrature operators q, p . However, as we shall shortly discuss, a Gaussian unitary is not necessarily straightforward to implement in practice. In addition to the elementary gate set, we will introduce “compound” gates that proved useful in this dissertation.

B.1 Linear Gaussian gates

These gates are typically implemented with “passive” linear optical elements. A displacement is given by

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \tag{B.1}$$

which generates a coherent state from the vacuum,

$$D(\alpha) |0\rangle = |\alpha\rangle_C, \tag{B.2}$$

and translates other coherent states,

$$D(\beta) |\alpha\rangle_C = |\alpha + \beta\rangle_C \quad (\text{B.3})$$

Defining the usual quadrature operators,

$$q = \frac{1}{\sqrt{2}} (a^\dagger + a) \ , \ p = \frac{i}{\sqrt{2}} (a^\dagger - a) \ , \quad (\text{B.4})$$

we see that a displacement by a real (imaginary) α yields translations in position (momentum) space,

$$D\left(\frac{\Delta x}{\sqrt{2}}\right) = e^{-ip\Delta x} \ , \ D\left(\frac{\Delta p}{\sqrt{2}}i\right) = e^{ix\Delta p} \quad (\text{B.5})$$

The rotation gate is defined by

$$R(\theta) = e^{iN\theta} \ , \ \theta \in \mathbb{R} \ N = a^\dagger a \quad (\text{B.6})$$

It rotates the position and momentum as

$$\begin{pmatrix} q \\ p \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} . \quad (\text{B.7})$$

and thus can be implemented with (passive) optical phase-shifters. From this we define the Fourier transform F ,

$$F = R\left(\frac{\pi}{2}\right) = e^{\frac{i\pi}{4}(p^2+q^2)} . \quad (\text{B.8})$$

This gate is the continuous variable version of the Hadamard gate. Notice that R rotates coherent states to other coherent states

$$R(\theta) |\alpha\rangle_C = |e^{i\theta}\alpha\rangle_C \quad (\text{B.9})$$

A fundamental two qumode gate is the beamsplitter gate given by

$$\text{BS}_{12}(\theta) = e^{\theta(a_1 a_2^\dagger - a_1^\dagger a_2)} = \text{BS}_{21}^\dagger(\theta), \quad \theta \in \mathbb{R}. \quad (\text{B.10})$$

where the subscripts refer to the qumodes q_1 and q_2 that this gate acts on. Its action is given by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad (\text{B.11})$$

It's easy to show that q_1 and q_2 mix in the same way, and similarly for p_1 and p_2 . The beamsplitter is implemented via the passive optical element of the same name, the latter of which is characterized by its transmissivity $T = \cos \theta$ and reflectivity $R = \sin \theta$. It is convenient to consider the 50/50 beamsplitter, $\text{BS} \equiv \text{BS}(\frac{\pi}{4})$, which commonly appears in the main text.

B.2 Squeezing

Everything discussed so far is easily implemented with linear optical elements. For example, a displacement on a qumode is possible by interfering it with a strong coherent beam on a poorly reflective beamsplitter (or poorly transmissive, depending on the setup) [144]. However, to perform a general Gaussian operation we must include Bogoliubov transformations which mix creation and annihilation operators. This can be implemented using a single-mode squeezer

$$S(r) = e^{-\frac{r}{2}(a^{\dagger 2} - a^2)} \quad (\text{B.12})$$

which rescales the quadratures via

$$S^\dagger(r)qS(r) = e^{-r}q, \quad S^\dagger(r)pS(r) = e^r p. \quad (\text{B.13})$$

Unfortunately, a squeeze gate cannot be achieved using passive linear elements. Squeezed states are typically prepared using non-linear optics processes such as degenerate spontaneous

parametric down-conversion (SPDC), four-wave mixing, and the Kerr effect in optical fibers [145, 146, 47]. In the appropriate limits, such as a strong pump beam in SPDC, the resulting evolution approximates the squeezing unitary with the non-linear interaction strength and evolution time contained in the squeezing parameter. The need for non-linear media to implement squeezing has led some authors [48] to refer to it as a non-linear operation despite the fact that it results in a linear transformation of the creation and annihilation operators

$$a \rightarrow \cosh ra - \sinh ra^\dagger, \quad a^\dagger \rightarrow \cosh ra^\dagger - \sinh ra \quad (\text{B.14})$$

A similar gate is the two-mode squeeze, implementable with a non-degenerate SPDC, given by

$$S_{12}(r) = e^{\frac{r}{2}(a_1^\dagger a_2^\dagger - a_1 a_2)} \quad (\text{B.15})$$

The operator can be written in terms of 50/50 beamsplitters and two single mode squeezes,

$$S_{12}(r) = \text{BS}_{12}^\dagger S_1(r) S_2(-r) \text{BS}_{12} \quad (\text{B.16})$$

Squeezing finds use in additional compound Gaussian gates such as the quadratic phase gate

$$P(s) = e^{isq^2/2}, \quad (\text{B.17})$$

which may be written in terms of a single-mode rotation and squeeze [111], and has the transformation properties

$$P^\dagger(s)qP(s) = q, \quad P^\dagger(s)pP(s) = p + sq \quad (\text{B.18})$$

Another is the CX gate, also known as the addition or quantum nondemolition (QND) gate, given by

$$\text{CX}_{12}(s) = e^{-isq_1 p_2}. \quad (\text{B.19})$$

It can be decomposed in terms of beam splitters and single-mode squeezers. Its action is

$$\begin{aligned} q_1 &\rightarrow q_1 , & p_1 &\rightarrow p_1 - sp_2 , \\ p_2 &\rightarrow p_2 , & q_2 &\rightarrow q_2 + sq_1 . \end{aligned} \tag{B.20}$$

In “position” space, we see an addition of the value q_1 to q_2 , suggesting that this is the CV version of the CNOT gate for qubits. The analog for the controlled phase gate is the CZ operation,

$$\text{CX}_{12}(s) = e^{-isq_1q_2} . \tag{B.21}$$

which can be obtained from CX by Fourier transforming the second qumode,

$$\text{CZ}_{12} = F_2^\dagger \cdot \text{CX}_{12} \cdot F_2 \tag{B.22}$$

In this case, we see that the second qumode is multiplied by a phase in position space, with the phase dependent on the value of q_1 .

B.3 Non-gaussian gates

The discussion above contains a complete set of gates for all Gaussian operations, as well as some useful combinations of such gates. However a universal set of gates can only be obtained once we add a non-Gaussian gate to the mix. As previously mentioned, the inclusion of such gates allows us to construct quadrature polynomials of arbitrary order from another given polynomial of fixed order. The simplest such operation is provided by the single-qumode cubic phase gate

$$V(s) = e^{i\frac{s}{3}q^3} \tag{B.23}$$

whose action is

$$q \rightarrow q , \quad p \rightarrow p + sq^2 \tag{B.24}$$

Clearly, such an operation can increase the degree of a given polynomial by one (we can always apply a Fourier transform if we want to increase the degree of the q quadrature). We can construct polynomials of higher order through repeated application of this gate. Unfortunately, such a simple gate turns out to be quite difficult to realize in a laboratory setting. In Chapter 5, we discuss how one may use a Kerr gate (see below) along with some Gaussian operations to approximate the cubic phase gate. Earlier proposals exist, however, and they typically rely on photon number measurements or photon addition/subtraction to introduce the requisite non-linearity.

In the “GKP scheme” of Ref. [56], V can be implemented in a fully deterministic manner by performing photon counting on a two-mode-squeezed resource state. The resulting “cubic phase state” is then coupled to the physical qumode and generates the cubic phase gate after a homodyne measurement and feed-forward. Enough squeezing must be applied so that a sufficiently strong cubic gate phase parameter is attained, and the required squeezing has been considered to be unfeasible in a realistic setting [147]. In a different approach, Ref. [57] introduces a “repeat-until-success” method where V is factorized in a way such that each factor can be implemented after a successful photon subtraction. Failures do not destroy the calculation, hence the label for the method.

Another well-known non-Gaussian is the single-qumode Kerr gate

$$K(s) = e^{isN^2} \tag{B.25}$$

where N is the number operator. Its action on the creation and annihilation operators is

$$a \rightarrow e^{is(2N+1)}a, \quad a^\dagger \rightarrow a^\dagger e^{is(2N+1)} \tag{B.26}$$

For this reason, the Kerr gate is known as implementing “self-phase-modulation.” When combined with displacement operations it produces a “banana-shaped” quasi-probability distribution where one can witness “photon number squeezing” surpassing the capabilities

of quadrature squeezing [148]. As noted before, the Kerr effect can also be employed for implementing a single-mode squeeze [47].

Finally, we can use a two-qumode cross-Kerr gate to apply phases dependent on another qumode

$$CK(s)_{1,2} = e^{isN_1N_2} , \quad a_1 \rightarrow e^{isN_2}a_1 \quad (\text{B.27})$$

Again, we refer the reader to Ref. [111] for additional details on all of these CV gates, as well as for additional gates not discussed. Gates associated with hybrid systems can be found in Ref. [99] for the case of circuit QED and Ref. [52] for the trapped-ion case.

Vita

Shane Thompson grew up in Cincinnati, Ohio. He studied physics and mathematics at The Ohio State University and received B.S. degrees in both fields. He then continued his studies in the physics graduate program at the University of Tennessee, Knoxville, with a focus on theoretical physics. In 2019 he joined Prof. Siopsis' research group and began research on using quantum computers to simulate relativistic quantum field theories.