

Construction and Properties of Hussain Chains for Quotients of Projective Planes

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A $(0,2)$ -graph is a graph in which any two vertices have either zero or two common neighbors. A semibiplane is a symmetric 1-design such that any two points belong to either zero or two common blocks, and any two blocks intersect in either zero or two points. The incidence graph of a semibiplane is a $(0,2)$ -graph. In this paper, we study a particular type of semibiplane, namely quotients of projective planes by involutions. This construction always yields a semibiplane, and thus a $(0,2)$ -graph. We describe a construction of Hussain chains for quotients of projective planes, similar to Marshall Hall's construction for biplanes. We prove some results for the Hussain chains and define a boundary map, making the Hussain chains for quotients of projective planes into a type of complex.

1 Preliminary definitions and statements

In his paper, Marshall Hall describes a construction of Hussain chains for biplanes. The initial goal of this project was to extend Hall's construction to semibiplanes, and then to define a boundary map on the Hussain chains to obtain a type of complex. However, while the extension of Hall's construction is rather natural, such a boundary map does not exist in general for semibiplanes. The focus was narrowed to quotients of projective planes by involutions, due to the regularity and symmetry of these objects. A boundary map does exist on the Hussain chains corresponding to quotients of projective planes; at the end of this paper, we define this map and prove that it is a boundary map. To provide a bit of context, we first present some basic definitions and concepts.

Definition A *graph* Γ consists of a set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$, which is made up of two-element subsets of $V(\Gamma)$. Two vertices $x, y \in V(\Gamma)$ are *adjacent* if $\{x, y\} \subset E(\Gamma)$. A graph is *bipartite* if $V(\Gamma) = U \cup W$, $U \cap W = \emptyset$, such that any edge has one vertex in each of U and W .

Definition Given a graph Γ , a *path* between two vertices $x, y \in V(\Gamma)$ is a sequence of edges $e_1 e_2 \dots e_k$ such that the initial point of e_1 is x , the initial point of e_i is the terminal point of e_{i-1} and the terminal point of e_k is y . Also, for any $z \in V(\Gamma)$, z is contained in either 1 or 0 of the edges of the path (i.e. no vertex is used twice).

We will only be interested in *connected* graphs:

Definition A graph Γ is *connected* if for any $x, y \in V(\Gamma)$, x and y are connected by a path of edges. A graph is *disconnected* if it is not connected.

Definition Let Γ be a connected graph. The *distance* between two points $x, y \in V(\Gamma)$ is the minimum number of edges that form a path from x to y .

We are interested in a particular type of graph, namely *(0,2)-graphs*:

Definition A *(0,2)-graph* is a graph Γ such that for any $x, y \in V(\Gamma)$, x and y have either 0 or 2 common neighbors (adjacencies).

Definition A $t - (v, k, \lambda)$ *design* is a point-block incidence structure where v is the number of points, k is the number of points contained in each block and every subset of t points is incident with λ blocks. A design is called *semisymmetric* when any two distinct blocks meet in either 0 or λ points. For $\lambda = 2$ a semisymmetric design is called a *semibiplane*.

Alternatively, a semibiplane is a connected bipartite graph such that any two vertices have 0 or 2 common neighbors. This is equivalent to the definition above.

Definition The *incidence graph* of a design is a graph such that the points of the design correspond to the vertices of the incidence graph, and the blocks to edges. Incidence is determined in the natural way: A vertex v is incident with an edge e if the point corresponding to v in the design is incident with the block corresponding to e .

The incidence graph of a semibiplane is a bipartite $(0,2)$ -graph [4].
We now define the notion of a finite projective plane.

Definition A *finite projective plane of order n* , denoted $PG(2, n)$, is a set of $n^2 + n + 1$ points with the following properties:

1. Any two points determine a line, and any two lines meet at exactly one point.
2. Every point has $n + 1$ lines on it.
3. Every line contains $n + 1$ points.

A projective plane $PG(2, n)$ is a $2 - (n^2 + n + 1, n + 1, 1)$ design, if the lines are considered as blocks.

Definition A *collineation* of $PG(2, n)$ is a pointwise permutation which maps lines onto lines. A collineation of order 2 (that is, applying the collineation twice yields the original structure) is called an *involution*. An involution of $PG(2, n)$ is a *homology* when n is odd, an *elation* when n is even, and a *Baer involution* when n is a square.

Specifically, involutions act as follows on $PG(2, n)$:

1. An elation (n even) fixes all points on a line, and all lines through a point on the pointwise-fixed line.
2. A homology (n odd) fixes all points on a line, and all lines through a point *not* on the pointwise-fixed line. This point is called the *axis point*.
3. A Baer involution (n a square) fixes a subplane isomorphic to $PG(2, \sqrt{n})$. For example, a Baer involution of $PG(2, 9)$ fixes a subplane isomorphic to $PG(2, 3)$.

We will not consider Baer involutions in this paper.

Given a projective plane of finite order and an involution, one may construct a semiplane whose incidence graph has diameter at most 4; see Muskovin [4] for a method and proofs.

2 Constructing Hussain chains for quotients of projective planes

A *Hussain chain complex* is a collection of graphs corresponding to a design. Each individual graph is called a *Hussain chain*. Marshall Hall describes

the construction of Hussain chains for biplanes [3]; our goal is to illustrate a similar construction for the quotient of a projective plane by an involution, i.e. identifying the points (and therefore blocks) that are swapped by the involution. First note that a projective plane $PG(2, n)$ is a symmetric $2-(n^2 + n + 1, n + 1, 1)$ design; as such, it has $n^2 + n + 1$ points and blocks.

The following is a description of the construction of Hussain chains for the quotient of a projective plane by an involution. To make the steps clearer, we will construct as an example the Hussain chains for the quotient of $PG(2, 5)$ by a particular homology. Using the computer algebra program GAP with the DESIGN package[2, 5], we find the blocks of $PG(2, 5)$ and a homology written in cycle notation. In the example below, the numbers 1, 2, ..., 31 correspond to the points of $PG(2, 5)$; each bracketed set of six points (i.e. [1, 2, 3, 4, 5, 6]) corresponds to a block.

Blocks of $PG(2, 5)$:

[1, 2, 3, 4, 5, 6], [1, 7, 8, 9, 10, 11],
[1, 12, 13, 14, 15, 16], [1, 17, 18, 19, 20, 21],
[1, 22, 23, 24, 25, 26], [1, 27, 28, 29, 30, 31],
[2, 7, 12, 17, 22, 27], [2, 8, 13, 18, 23, 28],
[2, 9, 14, 19, 24, 29], [2, 10, 15, 20, 25, 30],
[2, 11, 16, 21, 26, 31], [3, 7, 13, 19, 25, 31],
[3, 8, 14, 21, 22, 30], [3, 9, 16, 20, 23, 27],
[3, 10, 12, 18, 26, 29], [3, 11, 15, 17, 24, 28],
[4, 7, 14, 20, 26, 28], [4, 8, 16, 17, 25, 29],
[4, 9, 15, 18, 22, 31], [4, 10, 13, 21, 24, 27],
[4, 11, 12, 19, 23, 30], [5, 7, 15, 21, 23, 29],
[5, 8, 12, 20, 24, 31], [5, 9, 13, 17, 26, 30],
[5, 10, 16, 19, 22, 28], [5, 11, 14, 18, 25, 27],
[6, 7, 16, 18, 24, 30], [6, 8, 15, 19, 26, 27],
[6, 9, 12, 21, 25, 28], [6, 10, 14, 17, 23, 31],
[6, 11, 13, 20, 22, 29]

Homology:

(2,3)(4,5)(8,10)(9,11)(12,13)(14,15)(17,19)
(20,21)(22,25)(23,26)(27,31)(28,29)

So the involution swaps points 2 and 3, 4 and 5, etc. Before we give the construction, we must first make another definition:

Definition A *Hussain n-chain* is recursively defined as a Hussain chain that has as its boundary (n-1)-chains. For any collection of Hussain chains, the

-1-chain corresponds to the empty set, i.e. it does not appear in a graph representation of Hussain chains. We often think of 0-chains as vertices, 1-chains as edges with 0-chains as vertices, 2-chains as two-dimensional shapes with 1-chains as edges, etc.

Now, to construct the Hussain chains corresponding to $PG(2, n)$:

1. Begin with the $n^2 + n + 1$ points and $n^2 + n + 1$ blocks of a projective plane, and an involution. Note that the involution fixes a block pointwise; call this block B_F . If the involution is a homology, we also have an axis point, and all lines through the axis point are fixed. If the involution is an elation, then all lines through a point on B_F are fixed. For simplicity's sake, we shall also call this point on B_F the axis point.

Example: We can see that the points 6, 7, 16, 18, 24, 30 are not moved by the involution, so the block $B_F = [6, 7, 16, 18, 24, 30]$. Since we are working with a homology, it fixes a point not on B_F , namely the point 1. This is our axis point, so all lines incident with 1 are fixed.

2. Choose two points (a, b) that are identified via the involution from a single block incident with the axis point; label this pair as $\emptyset$. We consider this a Hussain chain with dimension -1. We will call the block containing both a and b B_1 . The other points in the block are either fixed or will pair up via the involution. In the incidence graph, the other pairs of identified points on B_1 will be at distance 4 from our chosen pair (a, b) .

Example: We will choose the pair $(2, 3)$ to be the -1-chain, so $B_1 = [1, 2, 3, 4, 5, 6]$. In this block, the points $(4, 5)$ are identified via the involution, while the points 1 and 6 are fixed. In the incidence graph, $(4, 5)$ will be at distance 4 from our chosen pair $(2, 3)$.

3. Consider the blocks that are incident with either a or b ; excluding B_1 , there will be $2n$ such blocks. A block incident with a will be paired with a block incident with b if the two blocks contain the same point in B_F . After obtaining a pair of blocks, if $a < b$, write the block containing a above the block containing b such that a is directly above b . The points that are now vertically aligned will be the same, if that point is fixed, or they will be paired via the involution. Number these pairs from 1 to n ; these are the Hussain chains with dimension 0 (0-chains). In terms of graphs, the 0-chains are the vertices for the 1-chains, 2-chains and 3-chains.

Example: Not counting B_1 , there are 10 blocks incident with either 2 or 3. They are:

$[2, 7, 12, 17, 22, 27], [2, 8, 13, 18, 23, 28],$
 $[2, 9, 14, 19, 24, 29], [2, 10, 15, 20, 25, 30],$
 $[2, 11, 16, 21, 26, 31], [3, 7, 13, 19, 25, 31],$
 $[3, 8, 14, 21, 22, 30], [3, 9, 16, 20, 23, 27],$
 $[3, 10, 12, 18, 26, 29], [3, 11, 15, 17, 24, 28].$

We pair these blocks as described above. The assigned number appears in brackets to the left of the pair.

$\langle 1 \rangle [2, 7, 12, 17, 22, 27] \quad \langle 2 \rangle [2, 8, 13, 18, 23, 28]$
 $[3, 7, 13, 19, 25, 31], \quad [3, 10, 12, 18, 26, 29],$
 $\langle 3 \rangle [2, 9, 14, 19, 24, 29] \quad \langle 4 \rangle [2, 10, 15, 20, 25, 30]$
 $[3, 11, 15, 17, 24, 28], \quad [3, 8, 14, 21, 22, 30],$
 $\langle 5 \rangle [2, 11, 16, 21, 26, 31]$
 $[3, 9, 16, 20, 23, 27]$

Note that the two blocks in pair $\langle 1 \rangle$ have the fixed point 7 in common, the two blocks in pair $\langle 2 \rangle$ have the fixed point 18 in common, etc.

4. In the above pairs of blocks, observe that the paired blocks are actually identified, as each point in a block is identified via the involution with another point in the other block. Each pair of points occurs in exactly two blocks. These pairs will be labeled with two integers, one from each pair of blocks containing the pair of points. In the labeling, the lower number comes first (i.e. if a pair of points occurs in the 1st and 2nd pairs of blocks above, label this pair of points by "12," not "21"). These pairs of points are the 1-dimensional Hussain chains (1-chains), and shall serve as edges in a graph representation.

Example: Notice that in $\langle 1 \rangle$ above, we have the pairs $(12,13)$, $(17,19)$, $(22,25)$ and $(27,31)$. These pairs also occur in $\langle 2 \rangle$, $\langle 3 \rangle$, $\langle 4 \rangle$, and $\langle 5 \rangle$, respectively. Hence $(12,13)$ will be labeled with $\langle 12 \rangle$, $(17,19)$ with $\langle 13 \rangle$, and so on. The full list of pairs and their labels appears below; the label appears above the pair of points.

$\langle 24 \rangle$	$\langle 35 \rangle$	$\langle 12 \rangle$	$\langle 34 \rangle$	$\langle 13 \rangle$	$\langle 45 \rangle$
$(8,10)$	$(9,11)$	$(12,13)$	$(14,15)$	$(17,19)$	$(20,21)$
$\langle 14 \rangle$	$\langle 25 \rangle$	$\langle 15 \rangle$	$\langle 23 \rangle$		
$(22,25)$	$(23,26)$	$(27,31)$	$(28,29)$		

5. To obtain the 2-dimensional Hussain chains, we again create pairs of blocks. In step 2, we note that the other points in B_1 are either fixed or are paired via the involution. For each fixed point in B_1 (except the axis point), look at the remaining blocks which contain that fixed point. One of these blocks will be the block that is fixed pointwise; the others will be identified via the involution, and will be paired. We call these pairs *non-boundary 2-chains*; as we will see, the pairs of points corresponding to 3-chains are not incident with any of the blocks comprising these 2-chains, so they will not contribute to the boundary of any 3-chain.

Example: In our case, the only fixed non-axis point on B_1 is 6. Aside from B_F , we have four blocks that are incident with 6:

$$\begin{aligned} & [6, 8, 15, 19, 26, 27], [6, 9, 12, 21, 25, 28], \\ & [6, 10, 14, 17, 23, 31], [6, 11, 13, 20, 22, 29]. \end{aligned}$$

These are paired as follows:

$$\begin{aligned} & [6, 8, 15, 19, 26, 27] \quad [6, 9, 12, 21, 25, 28], \\ & [6, 10, 14, 17, 23, 31], [6, 11, 13, 20, 22, 29]. \end{aligned}$$

Now consider the pairs of points from B_1 , excluding our initial (a, b) . For each pair of points (x_i, y_i) , pair blocks as in step 3. Note that each pair of blocks is identified pointwise via the involution. Label the pairs of blocks by a sequence of the labels from step 4, as follows: After obtaining a pair of blocks, write the pair the same way as the pairs from step 3. Again, the points that are now vertically aligned will be the same, if that point is fixed, or they will be paired via the involution. If the point labeled with the smaller integer occurs in the top block, write the label for that pair of points with the smaller integer coming first. If the point labeled with the larger integer occurs in the top block, write the label for that pair of points "backwards," with the larger integer coming first.

After writing the labels in such a way, note that the edges form a two-dimensional object. These 2-chains are called *boundary 2-chains*, since they will be boundary elements of a 3-chain.

Example: From B_1 we only have one more pair of points, namely $(4, 5)$. The blocks incident with either 4 or 5, excluding B_1 , are:

$[4, 7, 14, 20, 26, 28]$, $[4, 8, 16, 17, 25, 29]$,
 $[4, 9, 15, 18, 22, 31]$, $[4, 10, 13, 21, 24, 27]$,
 $[4, 11, 12, 19, 23, 30]$, $[5, 7, 15, 21, 23, 29]$,
 $[5, 8, 12, 20, 24, 31]$, $[5, 9, 13, 17, 26, 30]$,
 $[5, 10, 16, 19, 22, 28]$, $[5, 11, 14, 18, 25, 27]$.

Pairing them as in step 3, we get the following pairs:

$[4, 7, 14, 20, 26, 28]$ $[4, 8, 16, 17, 25, 29]$
 $[5, 7, 15, 21, 23, 29]$, $[5, 10, 16, 19, 22, 28]$,

 $[4, 9, 15, 18, 22, 31]$ $[4, 10, 13, 21, 24, 27]$
 $[5, 11, 14, 18, 25, 27]$, $[5, 8, 12, 20, 24, 31]$,

 $[4, 11, 12, 19, 23, 30]$
 $[5, 9, 13, 17, 26, 30]$.

Labeling is as follows: In our first pair, we have the points $(14,15)$, $(20,21)$, $(23,26)$, and $(28,29)$. For all pairs except $(23,26)$, the smaller number occurs in the top block, so we do not reverse the order of the labeling. Since 26 is written above 23, we do reverse the labeling for that pair. (Keep in mind that this is a well-defined action, since we require that 4 be written above 5.) So our first pair will be denoted: $\langle 34, 45, 52, 23 \rangle$. Similarly, our second pair will be denoted $\langle 24, 41, 13, 32 \rangle$. Here is a full list of our 2-chains, with their labels written above:

$\langle 34, 45, 52, 23 \rangle$ $\langle 24, 41, 13, 32 \rangle$
 $[4, 7, 14, 20, 26, 28]$ $[4, 8, 16, 17, 25, 29]$
 $[5, 7, 15, 21, 23, 29]$, $[5, 10, 16, 19, 22, 28]$,

 $\langle 35, 51, 14, 43 \rangle$ $\langle 42, 21, 15, 54 \rangle$
 $[4, 9, 15, 18, 22, 31]$ $[4, 10, 13, 21, 24, 27]$
 $[5, 11, 14, 18, 25, 27]$, $[5, 8, 12, 20, 24, 31]$,

 $\langle 53, 31, 12, 25 \rangle$
 $[4, 11, 12, 19, 23, 30]$
 $[5, 9, 13, 17, 26, 30]$.

Thinking of the 0-chains as vertices and the 1-chains as edges, we see that the boundary 2-chains in this example are quadrangles. We can

also associate a shape to the non-boundary 2-chains by labeling them in the same fashion:

$$\begin{array}{cc} \langle 24, 43, 31, 15, 52 \rangle & \langle 35, 54, 41, 12, 23 \rangle \\ [6, 8, 15, 19, 26, 27] & [6, 9, 12, 21, 25, 28], \\ [6, 10, 14, 17, 23, 31] & [6, 11, 13, 20, 22, 29]. \end{array}$$

In this case, the five 1-chains form a pentagonal shape.

6. The 3-dimensional Hussain chains correspond to the (x_i, y_i) pairs on B_1 from step 5. The 3-chain corresponding to (x_i, y_i) has as its boundary all pairs of blocks incident with (x_i, y_i) .

Example: Our lone 3-chain in this example corresponds to the pair $(4, 5)$. It has as its boundary the five boundary 2-chains given above.

Since the diameter of the incidence graph of the semiplane obtained from the quotient of a projective plane is at most 4, no projective plane of finite order will yield Hussain chains of dimension higher than 3. This completes the construction of Hussain chains for quotients of finite projective planes.

3 Results

We state some results concerning the Hussain chains for quotients of projective planes of finite order.

Proposition 3.1 *For the quotient of $PG(2, n)$ by a homology (n odd), the Hussain chain construction yields 1 -1-chain, n 0-chains, $(n^2 - n)/2$ 1-chains, $(n^2 - 2n - 1)/2$ 2-chains, and $(n - 3)/2$ 3-chains.*

Proof By construction, we choose 1 -1-chain, corresponding to a pair (a, b) from B_1 . The n 0-chains are obtained from pairings of the $2n$ blocks (leaving out B_1) incident with either a or b . By construction, B_1 is also incident with the axis point, and the remaining (non-fixed) points in B_1 are paired to form the 3-chains. So we see that none of the $n + 1$ points in B_1 contribute to the pairings that form the 1-chains, nor do the $n + 1$ points from B_F , the block fixed by the involution. Since $|B_1 \cap B_F| = 1$, we have:

$$n^2 + n + 1 - (n + 1) - n = n^2 - n$$

remaining points. These are paired to form the $(n^2 - n)/2$ 1-chains. The 2-chains come from pairs of blocks that do not correspond to the 0-chains

and are not fixed. B_1 cannot be paired, nor can B_F , nor the $2n$ blocks that make up the 0-chains, nor the remaining n blocks that are incident with the axis point and thus fixed. So we have:

$$n^2 + n + 1 - 2n - n - 1 - 1 = n^2 - 2n - 1$$

remaining blocks, which are paired to form the $(n^2 - 2n - 1)/2$ 2-chains. The 3-chains correspond to the non-fixed points of B_1 , except for the two initially chosen. Of the $n + 1$ points in B_1 , 2 are chosen, 1 is fixed since B_1 is incident with B_F at one point, and the axis point is also fixed. $n + 1 - 2 - 1 - 1 = n - 3$, and those $n - 3$ points are paired to form the $(n - 3)/2$ 3-chains. ■

Proposition 3.2 *For the quotient of $PG(2, n)$ by an elation (n even), the Hussain chain construction yields 1 -1-chain, n 0-chains, $(1/2)(n^2 - n)$ 1-chains, $(1/2)(n^2 - 2n)$ 2-chains, and $(1/2)(n - 2)$ 3-chains.*

Proof As above, we choose 1 -1-chain, corresponding to a pair (a, b) from B_1 . The n 0-chains are obtained from pairings of the $2n$ blocks (leaving out B_1) incident with either a or b . Again, none of the $n + 1$ points from B_1 contribute to the pairings that form the 1-chains, nor do the $n + 1$ points from B_F , the block fixed by the involution. Since $|B_1 \cap B_F| = 1$, we have:

$$n^2 + n + 1 - (n + 1) - n = n^2 - n$$

remaining points. These are paired to form the $(n^2 - n)/2$ 1-chains. As with the homology case, the 2-chains come from pairs of blocks that do not correspond to the 0-chains and are not fixed. B_1 cannot be paired, nor can B_F , nor the $2n$ blocks that make up the 0-chains, nor the remaining $n - 1$ blocks that are incident with the axis point ($n + 1$ blocks are incident with each point; since the axis point lies on B_F , we have both B_1 and B_F incident, leaving $n - 1$ other blocks incident with the axis point). So we have:

$$n^2 + n + 1 - 2n - (n - 1) - 1 - 1 = n^2 - 2n$$

remaining blocks, which are paired to form the $(n^2 - 2n)/2$ 2-chains. The 3-chains correspond to the non-fixed points of B_1 , except for the two initially chosen. Of the $n + 1$ points in B_1 , 2 are chosen, and one is the axis point, which is fixed. $n + 1 - 2 - 1 = n - 2$ remaining points in B_1 , which are paired to form the $(n - 2)/2$ 3-chains. ■

Proposition 3.3 *For the quotient of $PG(2, n)$ by either a homology or an elation, each 3-chain has as its boundary n 2-chains, with $n - 1$ edges per*

2-chain. If an edge appears in a boundary 2-chain, it also appears in another boundary 2-chain, and these two occurrences are "opposite," i.e. if $\langle i_1 i_2 \rangle$ appears in a boundary 2-chain, then $\langle i_2 i_1 \rangle$ appears in another boundary 2-chain.

Proof Each 3-chain is represented by a pair of points (x, y) , and the boundary chains are represented by pairs of blocks, with one block from the pair incident with x and the other with y . Each of x and y are incident with $n + 1$ blocks; since they are both incident with a single block, the total number of such blocks is $2n + 1$. However, the block containing both x and y does not contribute to the pairing, so $2n$ blocks form the n pairs corresponding to boundary chains.

Each block in a boundary chain pairing has $n + 1$ points. One of these points is x , and another is a point fixed by the involution, leaving $n - 1$ points per block. These are paired in the construction to form the $n - 1$ edges of each boundary chain.

Now suppose we have $\langle i_1 i_2 \rangle$ as an edge to a boundary 2-chain. Then the points i_1 and i_2 are in different blocks that have been paired according to the construction. In order to pair the blocks, we look at a pair of points (x, y) in B_1 different from our initial indexing pair. Without loss of generality, say x and i_1 occur in the same block B_x , and y and i_2 occur in a different block B_y . If we show that there must exist a block containing both x and i_2 , the proof will be finished, since this block will be matched with a block containing y and i_1 by the involution, hence we will have an edge $\langle i_2 i_1 \rangle$. There are $n + 1$ blocks containing the point i_2 , and they all intersect B_x ; no two of these $n + 1$ blocks can intersect B_x at the same point, since those two blocks will then intersect at two points, which is impossible. Thus there exists a block containing i_2 intersecting B_x at x . ■

Proposition 3.4 *For the quotient of $PG(2, n)$ by a homology (n odd), the Hussain chain construction yields $(n - 1)/2$ non-boundary 2-chains, and each edge appears in exactly one non-boundary 2-chain. For the quotient of $PG(2, n)$ by an elation, the Hussain chain construction yields no non-boundary 2-chains.*

Proof We consider the case of homology first. The non-boundary 2-chains come from pairings of blocks incident with $B_1 \cap B_F = \{z\}$. There are $n + 1$ such blocks; since B_1 and B_F are not paired, $n - 1$ blocks are paired, yielding $(n - 1)/2$ total non-boundary 2-chains. Each edge must appear only one time; if an edge appeared more than once, then we would have two blocks whose intersection is two points (z and a point corresponding to the common edge), which violates the conditions for finite projective planes.

For an elation, we have $(n^2-2n)/2$ total 2-chains and $(n-2)/2$ 3-chains, as proved in Proposition 1.2. By Proposition 1.3, each 3-chain has n 2-chains as its boudary. Note that, by construction, a single 2-chain does not contribute to the boundary of two different 3-chains. $(n(n-2)/2) = (n^2-2n)/2$, so non-boundary 2-chains cannot exist. ■

4 A boundary map for Hussain chains of quotients of projective planes

The initial goal of this project was to define a boundary map on the Hussain chain complexes for all semibiplanes; however, this is impossible. A counterexample can be found on Dr. Andries Brouwer's list of small (0,2)-graphs [1]. The Hussain chains corresponding to the (0,2)-graph $\Delta_{6,5}$ on his list are such that defining a boundary map is not possible; specifically, it isn't possible to define an orientation on the 2-chains such that the 1-chains will negate one another when the boundary map is applied a second time.

One can define a boundary map on the Hussain chains corresponding to quotients of projective planes. Let X be the quotient of a projective plane by an involution. In order to define a boundary map on the Hussain chain construction for X , we must first define an orientation on the 1-chains.

Definition If $\langle ij \rangle$ is a 1-chain in the Hussain chain construction for X ,

$$\langle ij \rangle = -\langle ji \rangle.$$

Definition Let C_i be the free abelian group generated by the i-chains in the Hussain chain construction for X . Note that for $i \notin \{0, 1, 2, 3\}$, $C_i = \mathbf{0}$.

We now define the boundary map.

Definition Define a boundary map $\delta_i: C_i \rightarrow C_{i-1}$ as follows:

1. If $Y \in C_3$,

$$\delta_3(Y) = D_1 + \dots + D_k,$$

where $D_i \in C_2$ is a part of the boundary of Y .

2. If $Y \in C_2$,

$$\delta_2(Y) = \langle i_1 i_2 \rangle + \langle i_2 i_3 \rangle + \dots + \langle i_{n-1} i_n \rangle + \langle i_n i_1 \rangle,$$

where $\langle i_{k-1} i_k \rangle \in C_1$ is a part of the boundary of Y .

3. If $\langle i_{k-1}i_k \rangle \in C_1$,

$$\delta_1(\langle i_{k-1}i_k \rangle) = \langle i_k \rangle - \langle i_{k-1} \rangle,$$

where $\langle i_k \rangle \in C_0$.

4. For all other values of i , $\delta_i = 0$.

Proposition 4.1 $\delta_{i-1} \circ \delta_i = 0$ for all $i \in \mathbf{Z}$.

Proof This result is nontrivial only for $i = 2$ and $i = 3$. For $i = 2$, let $Y \in C_2$. Then

$$\delta_2(Y) = \langle i_1i_2 \rangle + \langle i_2i_3 \rangle + \dots + \langle i_{n-1}i_n \rangle + \langle i_ni_1 \rangle,$$

where $\langle i_{k-1}i_k \rangle \in C_1$ is a part of the boundary of Y . Now we apply δ_1 :

$$\begin{aligned} \delta_1(\langle i_1i_2 \rangle + \langle i_2i_3 \rangle + \dots + \langle i_{n-1}i_n \rangle + \langle i_ni_1 \rangle) &= \\ \delta_1(\langle i_1i_2 \rangle) + \delta_1(\langle i_2i_3 \rangle) + \dots + \delta_1(\langle i_{n-1}i_n \rangle) + \delta_1(\langle i_ni_1 \rangle) &= \\ (\langle i_2 \rangle - \langle i_1 \rangle) + (\langle i_3 \rangle - \langle i_2 \rangle) + \dots + (\langle i_n \rangle - \langle i_{n-1} \rangle) + (\langle i_1 \rangle - \langle i_n \rangle) &= 0. \end{aligned}$$

For $i = 3$, let $Y \in C_3$. Then

$$\delta_3(Y) = D_1 + \dots + D_k,$$

where $D_i \in C_2$ is a part of the boundary of Y . Now we apply δ_2 . For each D_i ,

$$\delta_2(D_i) = \langle i_1i_2 \rangle + \langle i_2i_3 \rangle + \dots + \langle i_{n-1}i_n \rangle + \langle i_ni_1 \rangle,$$

where $\langle i_{k-1}i_k \rangle \in C_1$ is a part of the boundary of Y . By Proposition 1.3, each of these edges will appear exactly twice, and they will always be opposites and thus cancel. So we obtain the desired result:

$$\delta_3 \circ \delta_2 = 0. \quad \blacksquare$$

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