To the Graduate Council:

I am submitting herewith a thesis written by M. Saeed Bejnood entitled 'Constants of Motion in Five Component Models of Two-Dimensional Turbulence.' I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Boris Kup

Boris Kupershmidt, Major Professor

We have read this thesis
and recommend its acceptance:

Accepted for the Council:

Vice Provost
and Dean of the Graduate School

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for a Master's degree at The University of Tennessee, Knoxville, I agree that the Library shall make it available to borrowers under rules of the Library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of source is made.

Permission for extensive quotation from or reproduction of this thesis may be granted by my major professor, or in his absence, by the Head of Interlibrary Services when, in the opinion of either, the proposed use of the material is for scholarly purposes. Any copying or use of the material in this thesis for financial gain shall not be allowed without my written permission.

Signature M. Saeed Behrooz

Date Dec. 87

CONSTANTS OF MOTION IN FIVE COMPONENT MODELS OF TWO DIMENSIONAL TURBULENCE

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

M. SAEED BEJNOOD

June 1988

This work is dedicated to my beloved brother:

Abdulali Bejnood

ABSTRACT

A system of equations derived by truncating a Fourier representation of the two-dimensional Euler equations using five frequencies is investigated. Five vectors are constructed which satisfy a pair of triad interactions. The general form of constants of motion in terms of the five frequencies is presented. The energy and enstrophy have been expressed as linear combinations of quadratic integrals and also have been expressed as functions of vorticity.

TABLE OF CONTENTS

CHAPTER PAGE

I. INTRODUCTION 1

II. DERIVATION OF THE EQUATIONS OF MOTION 3

III. GENERAL FORM OF THE FIVE-VECTOR MODELS 6

IV. CONCLUSION 9

BIBLIOGRAPHY 10

APPENDICES 12

APPENDIX A 13

APPENDIX B 14

APPENDIX C 17

VITA 18

LIST OF SYMBOLS

a, b, c, d, e, f, g	vector components
E	energy
H	constant of cubic integral
k, p, q,	wave vectors
ρ	Density
ν	kinematic viscosity
ν_k	fourier coefficients
ξ	vorticity
ψ	stream function
ω	fourier components of vorticity
ω^*	complex conjugate of Fourier coefficient of vorticity
Ω	enstrophy
Θ_k	quadratic integrals
γ_k	complex constants of motion

CHAPTER I

INTRODUCTION

Many theoretical and experimental investigations of turbulence use the spectral representation of the Euler equations, see, e.g., Kraichnan (1967), Lee (1975), Fox and Orszag (1973), and Basdevant and Sadourney (1975). For two-dimensional flows this representation gives rise to a dynamical system which conserves energy and enstrophy. Kraichnan (1967) has offered the conjecture that non-equilibrium flows should evolve toward absolute equilibrium. This has been interpreted to mean that the flow on the intersection of the surfaces corresponding to the constants of motion is mixing. The validity of this interpretation is hard to assess, see Orszag (1970). Hald (1976) has found several spectral models which have additional constants of motion besides energy and enstrophy.

The problem investigated in this thesis is the general form of the constants of motion of five Fourier components of the vorticity in models of two-dimensional turbulence. Hald (1976) originally found several examples, based on the classical infinite system of the equations of motion

$$\dot{\omega}_k = \frac{1}{2} \sum_{k=p+q} \left(\frac{1}{q^2} - \frac{1}{p^2} \right) |p,q| \omega_p \omega_q \tag{1.1}$$

where $k, p, q \in \mathbf{Z}^2 - \{0\}$ are nonzero vectors on the two dimensional lattice, $|p,q| = p_1 q_2 - p_2 q_1$, for vectors $p = (p_1, p_2)$ and $q = (q_1, q_2)$, $q^2 = q_1^2 + q_2^2$, ω_k are the Fourier components of vorticity, and overdot denotes the time derivative. This equation is universal for all spectral models. (The sign of time in (1.1) is taken to be opposite to the physical one, for aesthetic reasons.)

In any possible spectral model, two constants: Energy,

$$E = \frac{1}{2} \sum_{k} k^{-2} |\omega_k|^2 \qquad (1.2),$$

and Enstrophy,

$$\Omega = \sum_{k} |\omega_k|^2, \qquad (1.3)$$

are conserved. It is believed that the infinite system (1.1) has no other constants of motion and, in general, exhibits chaotic behavior (see Kraichnan (1967)). For the five-vector models, other possible forms of the constants of motion is the question which is analyzed below.

The plan of the presentation is as follows. In Section II we recall the classical derivation of the equations of motion (1.1). In Section III we will present a general construction of the five vectors which will establish the general form of the system of motion equations and their coefficients. Then we will demonstrate the general form of the four real constants of motion for the five-vector models. Section IV summarizes this work.

CHAPTER II

DERIVATION OF THE EQUATIONS OF MOTION

The Navier–Stokes equations for two-dimensional, incompressible flow can be formulated as,

$$\partial U(x,t)/\partial t + (U(x,t)\cdot\nabla)U(x,t) = -\nabla p(x,t) + \nu\nabla^2 U(x,t) \qquad (2.1)$$

$$\nabla\cdot U(x,t) = 0; \qquad (2.2)$$

The constant density ρ is taken to be 1. The associated two-dimensional Euler equations correspond to the inviscid flow, $\nu = 0$, in the above equations,

$$u_t + uu_x + vu_y = -p_x, \qquad (2.3)$$

$$v_t + uv_x + vv_y = -p_y, \qquad (2.4)$$

$$u_x + v_y = 0, \qquad (2.5)$$

where u and v are the components of the velocity in the x and y direction respectively, and p is the pressure. We define the vorticity $\xi = u_y - v_x$. The pressure term can be eliminated by differentiating equations (2.3) and (2.4) with respect to y and x respectively, and subtracting one from another. This results in the vorticity equation,

$$\xi_t + u\xi_x + v\xi_y = 0. \qquad (2.6)$$

Since the flow is incompressible, there exists a function ψ, called the stream function, such that $\psi_y = u$, $\psi_x = -v$. Thus, $\xi = \psi_{yy} + \psi_{xx}$. We assume that the flow is periodic in each direction with the period 2π. That is,

$$\xi(x+2\pi, y+2\pi, t) = \xi(x,y,t) \qquad (2.7)$$

$$\psi(x+2\pi, y+2\pi, t) = \psi(x,y,t)$$

$$u(x+2\pi, y+2\pi, t) = u(x,y,t)$$
$$v(x+2\pi, y+2\pi, t) = v(x,y,t).$$

Let us decompose ξ in the Fourier series,

$$\xi(\mathbf{r},t) = \sum_{0\neq\mathbf{k}\in\mathbf{Z}^2} \omega_k(t)e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{2.8}$$

The constant term, ω_0, vanishes, since $\xi = \Delta(\psi)$.

Now,

$$\xi_x = u_{xy} - v_{xx} = -v_{yy} - v_{xx} = -\Delta(v), \tag{2.9}$$
$$\xi_y = u_{yy} - v_{xy} = u_{yy} + u_{xx} = \Delta(u),$$

so that

$$u = \Delta^{-1}(\xi_y),\ v = -\Delta^{-1}(\xi_x). \tag{2.10}$$

Also,

$$\Delta^{-1}\sum_{\mathbf{k}\neq 0} \nu_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} = -\sum_{\mathbf{k}\neq 0} \frac{\nu_k}{\|\mathbf{k}\|^2} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.11}$$

(the zero–coefficient terms in u and v vanish since $u = \psi_y, v = -\psi_x$), where the summation is taken over $\mathbf{k} = (k_1,k_2)$ for all pairs of integers $(k_1,k_2) \neq (0,0)$. By expressing Fourier coefficients for u and v in terms of ξ, (2.8),

$$u = \frac{\partial}{\partial y}\sum_{\mathbf{k}\neq\mathbf{0}} \frac{-\omega_k}{\|\mathbf{k}\|^2} e^{i\mathbf{k}\cdot\mathbf{r}}$$
$$v = -\frac{\partial}{\partial x}\sum \frac{-\omega_k}{\|\mathbf{k}\|^2} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.12}$$

and substituting the result into the vorticity equation, (2.6), we get

$$\dot{\xi} - \left(\frac{\partial}{\partial y}\sum_{p\neq 0}\frac{\omega_p}{p^2}e^{i\mathbf{p}\cdot\mathbf{r}}\right)\left(\frac{\partial}{\partial x}\sum_{q\neq 0}\omega_q e^{i\mathbf{q}\cdot\mathbf{r}}\right) + \left(\frac{\partial}{\partial x}\sum_{p\neq 0}\frac{\omega_p}{p^2}e^{i\mathbf{p}\cdot\mathbf{r}}\right)\left(\frac{\partial}{\partial y}\sum_{q\neq 0}\omega_q e^{i\mathbf{q}\cdot\mathbf{r}}\right) = 0,$$

or

$$\sum_{k\neq 0} \dot{\omega}_k e^{i\mathbf{k}\cdot\mathbf{r}} - \sum_{p,q\neq 0} \frac{\omega_p}{p^2} ip_2 e^{i\mathbf{p}\cdot\mathbf{r}} \cdot \omega_q iq_1 e^{i\mathbf{q}\cdot\mathbf{r}} + \sum_{p,q\neq 0} \frac{\omega_p}{p^2} ip_1 e^{i\mathbf{p}\cdot\mathbf{r}} \cdot \omega_q iq_2 e^{i\mathbf{q}\cdot\mathbf{r}} = 0. \quad (2.13)$$

The above equation (after changing t into $-t$) decomposes into the system

$$\dot{\omega}_k = \frac{1}{2} \sum_{k=p+q} \left(\frac{1}{q^2} - \frac{1}{p^2}\right) |p, q| \omega_p \omega_q, \quad (2.14)$$

with $\omega_{-k} = \omega_k^*$; here μ^* denoted the complex conjugate of μ, for any $\mu \in \mathbf{C}$.

The system (2.14) is infinite. The basic dynamical variables ω_k are parametrized by the elements $\mathbf{k} \in \mathbf{Z}^2 - \{0\}$; moreover, each sum in the right-hand side of (2.14) is also infinite. The basic idea of various models is to choose, in a special way, a finite number of vectors $\mathbf{k}$ in $\mathbf{Z}^2 - \{0\}$ and to restrict the infinite system (2.14) onto only these vectors; the resulting system will be finite-dimensional; this system is not a truncation of the infinite system (2.14) in any sense. The minimal number of such vectors, three, leads to the well known triad interaction equations, see Hald (1976), Kraichnan (1967), Lee (1975). The next in complexity choice is the case of five vectors.

CHAPTER III

GENERAL FORM OF THE FIVE-VECTOR MODELS

In this section, the construction of the dynamical system based on a special choice of five vectors will be shown. These vectors satisfy six different triad interaction conditions, $\mathbf{k}+\mathbf{p}+\mathbf{q}=0,\ \mathbf{k},\mathbf{p},\mathbf{q} \in \mathbf{Z^2}-\{0\}$. The vectors $\mathbf{K_1},\mathbf{K_2},\mathbf{K_3},\mathbf{K_4}$ and $\mathbf{K_5}$, will be defined as follows:

$$
\begin{aligned}
\mathbf{K_1} &= (a,b) \\
\mathbf{K_2} &= (c,d) \\
\mathbf{a} &= (f,g) \\
\mathbf{K_3} &= \mathbf{K_1}+\mathbf{a} \\
\mathbf{K_4} &= \mathbf{K_2}-\mathbf{a} \\
\mathbf{K_5} &= -\mathbf{K_1}-\mathbf{K_2} = -\mathbf{K_3}-\mathbf{K_4}
\end{aligned}
\tag{3.1}
$$

Thus, the free parameters of the model are six components of the vectors $\mathbf{K_1},\mathbf{K_2},\mathbf{a} \in \mathbf{Z^2}-\{0\}$. By using the general form of the equations of motion, the five Fourier coefficients ω_k of the vorticity satisfy the following dynamical system:

$$
\begin{aligned}
\dot{\omega}_1 &= C_1\omega_2^*\omega_5^* : \mathbf{K_1} = (-\mathbf{K_2})+(-\mathbf{K_5}), \\
\dot{\omega}_2 &= C_2\omega_1^*\omega_5^* : \mathbf{K_2} = (-K_1)+(-K_5), \\
\dot{\omega}_3 &= C_3\omega_4^*\omega_5^* : \mathbf{K_3} = (-K_4)+(-K_5), \\
\dot{\omega}_4 &= C_4\omega_3^*\omega_5^* : \mathbf{K_4} = (-K_3)+(-K_5), \\
\dot{\omega}_5 &= D_1\omega_1^*\omega_2^* + D_2\omega_3^*\omega_4^* : \mathbf{K_5} = [(-\mathbf{K_1})+(-\mathbf{K_2})] = [(-\mathbf{K_3})+(-\mathbf{K_4})]
\end{aligned}
\tag{3.2}
$$

where the concrete form of the coefficients C_1, C_2, C_3, C_4, D_1 and D_2 is given in the Appendix A. ω_k is a complex function, $\omega_k = \alpha_k + i\beta_k$, where α_k and β_k are some real functions.

Theorem 1

$$H = Im(\omega_1^* \omega_2^* \omega_5^* + \omega_3^* \omega_4^* \omega_5^*) \tag{3.3}$$

is a constant of motion.

Proof:

We have,

$$\frac{dH}{dt} = Im(\dot{\omega}_1^* \omega_2^* \omega_5^* + \omega_1^* \dot{\omega}_2^* \omega_5^* + \omega_1^* \omega_2^* \dot{\omega}_5^* + \dot{\omega}_3^* \omega_4^* \omega_5^* + \omega_3^* \dot{\omega}_4^* \omega_5^* + \omega_3^* \omega_4^* \dot{\omega}_5^*). \tag{3.4}$$

Substituting (3.2) into (3.4), we get

$$\frac{dH}{dt} = Im[\omega_5 \omega_5^* (\omega_1 \omega_1^* + \omega_2 \omega_2^* + \omega_3 \omega_3^* + \omega_4 \omega_4^*)]. \tag{3.5}$$

Since expression in the straight brackets in the right hand side of above equation is real, we find that $\frac{dH}{dt} = 0$, so that H is a constant of motion.

Theorem 2

The following are quadratic constants of motion:

$$\Theta_1 = C_2 |\omega_1|^2 - C_1 |\omega_2|^2 \tag{3.6}$$

$$\Theta_2 = C_4 |\omega_3|^2 - C_3 |\omega_4|^2 \tag{3.7}$$

$$\Theta_3 = C_1 C_3 |\omega_5|^2 - C_3 D_1 |\omega_1|^2 - C_1 D_2 |\omega_3|^2 \tag{3.8}$$

Proof:

Take the derivative of Θ_1, Θ_2 and Θ_3 with respect to time, and use equations (3.2). For Θ_1, we obtain

$$\begin{aligned} \frac{d\Theta_1}{dt} &= C_2(\dot{\omega}_1 \omega_1^* - \omega_1 \dot{\omega}_1^*) - C_1(\dot{\omega}_2 \omega_2^* + \omega_2 \dot{\omega}_2^*) \\ &= C_2 C_1 \omega_1^* \omega_2^* \omega_5^* + C_2 C_2 \omega_1 \omega_2 \omega_5 - C_1 C_2 \omega_1^* \omega_2^* \omega_5^* - C_1 C_2 \omega_1 \omega_2 \omega_5 = 0. \end{aligned}$$

For Θ_2, we get

$$\begin{aligned} \frac{d\Theta_2}{dt} &= C_4 \dot{\omega}_3 \omega_3^* + C_4 \omega_3 \dot{\omega}_3^* - C_3 \dot{\omega}_4 \omega_4^* - C_3 \omega_4 \dot{\omega}_4^* \\ &= C_4 C_3 \omega_3^* \omega_4^* \omega_5^* + C_4 C_3 \omega_3 \omega_4 \omega_5 - C_3 C_4 \omega_3^* \omega_4^* \omega_5^* - C_3 C_4 \omega_3 \omega_4 \omega_5 = 0. \end{aligned}$$

Finally, for Θ_3 we find

$$\frac{d\Theta_3}{dt} = C_1C_3(\dot{\omega}_5\omega_5^* + \omega_5\dot{\omega}_5^*) - C_3D_1(\dot{\omega}_1\omega_1^* + \omega_1\dot{\omega}_1^*) - C_1D_2(\dot{\omega}_3\omega_3^* + \omega_3\dot{\omega}_3^*)$$

$$= C_1C_3D_1\omega_1^*\omega_2^*\omega_5^* + C_1C_3D_2\omega_3^*\omega_4^*\omega_5^* + C_1C_3D_1\omega_1\omega_2\omega_5 + C_1C_3D_2\omega_3\omega_4\omega_5\omega_5$$

$$- C_1C_3D_1\omega_1^*\omega_2^*\omega_5^* - C_1C_3D_1\omega_1\omega_2\omega_5 - C_1C_3D_2\omega_3^*\omega_4^*\omega_5^* - C_1C_3D_2\omega_3\omega_4\omega_5 = 0.$$

Thus, we have shown that Θ_1, Θ_2 and Θ_3 are quadratic constants of motion for arbitrary parameters $\mathbf{K_1}, \mathbf{K_2}, \mathbf{a}$.

Four other constants of motion could be obtained, when the condition $C_1C_2 = C_3C_4$ holds for the general form of the coefficients related to the five differential equations (3.2).

Theorem 3

When $C_1C_2 = C_3C_4$, the quantities

$$\gamma_1 = C_3\omega_1\omega_4^* - C_1\omega_3\omega_2^*$$

$$\gamma_2 = C_4\omega_1\omega_3^* - C_1\omega_2^*\omega_4$$

are (complex) constants of motion.

Proof:

Take the derivative of γ_1 and γ_2 with respect to time, and substitute for $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \dot{\omega}_4$, and $\dot{\omega}_5$ their expressions from the equation (3.2). For γ_1, we get

$$\frac{d\gamma_1}{dt} = C_3\dot{\omega}_1\omega_4^* + C_3\omega_1\dot{\omega}_4^* - C_1\dot{\omega}_3\omega_2^* - C_1\omega_3\dot{\omega}_2^*$$

$$= C_3C_1\omega_2^*\omega_4^*\omega_5^* + C_3C_4\omega_1\omega_3\omega_5 - C_1C_3\omega_2^*\omega_4^*\omega_5^* - C_1C_2\omega_2\omega_3\omega_5 = 0.$$

Similarly, for γ_2 we obtain

$$\frac{d\gamma_2}{dt} = C_4\dot{\omega}_1\omega_3^* + C_4\omega_1\dot{\omega}_3^* - C_1\dot{\omega}_2^*\omega_4 - C_1\omega_2^*\dot{\omega}_4^*$$

$$= C_4C_1\omega_2^*\omega_3^*\omega_5^* + C_4C_3\omega_1\omega_4\omega_5 - C_1C_2\omega_1\omega_4\omega_5 - C_1C_2\omega_2^*\omega_3^*\omega_5^* = 0.$$

Energy and enstrophy can be expressed as linear combinations of $\Theta_1, \Theta_2, \Theta_3$ (see Appendix B).

CHAPTER IV

CONCLUSION

The goal of this thesis was to find the general form for the constants of motion for the five Fourier components of the vorticity in models of two-dimensional turbulence.

The result of the task is summarized as follows:

$$\Theta_1 = C_2|\omega_1|^2 - C_1|\omega_2|^2,$$

$$\Theta_2 = C_4|\omega_3|^2 - C_3|\omega_4|^2,$$

$$\Theta_3 = C_1C_3|\omega_5|^2 - C_3D_1|\omega_1|^2 - C_1D_2|\omega_3|^2,$$

are quadratic integrals;

$$H = Im\,(\omega_1^*\omega_2^*\omega_5^* + \omega_3^*\omega_4^*\omega_5^*)$$

is a cubic integral; when $C_1C_2 = C_3C_4$, we obtain two more constants of motion:

$$\gamma_1 = C_3\omega_1\omega_4^* - C_1\omega_3\omega_2^*,$$

$$\gamma_2 = C_4\omega_1\omega_3^* - C_1\omega_2^*\omega_4.$$

The energy and enstrophy integrals can be expressed through the Θ's as follows

$$E = P\Theta_1 + Q\Theta_2 + R\Theta_3,$$

$$\Omega = S\Theta_1 + T\Theta_2 + W\Theta_3,$$

where the constants P, Q, R, S, T, W are given in Appendix B.

BIBLIOGRAPHY

BIBLIOGRAPHY

Basdevant, C. and Sadourny R. (1975): "Ergodic Properties of Inviscid Truncated Models of Two-Dimensional Incompressible Flows," Journal of Fluid Mechanics, vol. 69, part 4, pp. 673–688.

Fox, D. G. and Orszag, S. A. (1973): "Inviscid Dynamics of Two-Dimensional Turbulence," The Physics of Fluids, vol. 16, No. 2, pp. 169–171.

Hald, Ole. H. (1976): "Constants of Motion in Models of Two-Dimensional Turbulence," The Physics of Fluids, vol. 19, No. 6, pp. 914–915.

Kraichnan, H. (1963): "Direct-Interaction Approximation for a System of Several Interacting Simple Shear Waves," The Physics of Fluids, vol. 6, No. 11, pp. 1603–1609.

Lee, J. (1975): "The Triad-Intraction Representation of Homogeneous Turbulence," Journal of Mathematical Physics, vol. 16, No. 7, pp. 1359–1373.

APPENDICES

APPENDIX A

The General Form of the Coefficient $C_1, C_2, C_3, C_4, D_1, D_2$

From the formula (1), the coefficients $C_1, C_2, C_3, C_4, D_1, D_2$ in the equation (3.2) are

$$C_1 = -\frac{(ad - bc)}{2\left[(a+c)^2 + (b+c)^2\right]} \left(\frac{a^2 + b^2 + 2ac + 2bd}{c^2 + d^2} \right) \tag{A.1}$$

$$C_2 = \frac{(ad - bc)}{2\left[(a+c)^2 + (b+c)^2\right]} \left(\frac{c^2 + d^2 + 2ac + 2bd}{a^2 + b^2} \right) \tag{A.2}$$

$$C_3 = \left\{ \frac{\left[(b+d)(f-c) - (g-d)(a+c)\right]}{2\left[(a+c)^2 + (b+d)^2\right]} \right\} \cdot \left\{ \frac{(f-c)^2 + (g-d)^2 - (a+c)^2 - (b+d)^2}{(f-c)^2 + (g-d)^2} \right\} \tag{A.3}$$

$$C_4 = \left\{ \frac{-\left[(a+f)(b+d) - (b+g)(a+c)\right]}{2\left[(a+c)^2 + (b+d)^2\right]} \right\} \cdot \left\{ \frac{(a+f)^2 + (b+g)^2 - (a+c)^2 - (b+d)^2}{\left[(a+f)^2 + (b+g)^2\right]} \right\} \tag{A.4}$$

$$D_1 = \frac{(ad - bc)}{2} \left[\frac{a^2 + b^2 - c^2 - d^2}{(c^2 + d^2)(a^2 + b^2)} \right]$$

$$D_2 = \left\{ \frac{\left[(a+f)(d-g) - (b+g)(c-f)\right]}{2} \right\} \cdot \left\{ \frac{(f+a)^2 + (g+b)^2 - (f-c)^2 - (g-d)^2}{\left[(f-c)^2 + (g-d)^2\right]\left[(f+a)^2 + (g+b)^2\right]} \right\} \tag{A.5}$$

APPENDIX B

The quadratic integrals of energy, (1.2), and enstrophy, (1.3), can be expressed as linear combinations of the quadratic integrals Θ_1, Θ_2, and Θ_3, as follows:

$$E = \frac{1}{2}\sum_k k^{-2}|\omega_k|^2 = P\Theta_1 + Q\Theta_2 + R\Theta_3, \tag{B.1}$$

where,

$$\begin{aligned} P &= -\frac{k_2^{-2}}{2C_1}, \\ Q &= -\frac{k_4^{-2}}{2C_3}, \\ R &= \frac{k_5^{-2}}{2C_1C_3}, \end{aligned} \tag{B.2}$$

and

$$\Omega = \sum_k |\omega_k|^2 = S\Theta_1 + T\Theta_2 + W\Theta_3,$$

where,

$$\begin{aligned} S &= -\frac{1}{C_1}, \\ T &= -\frac{1}{C_3}, \\ W &= \frac{1}{C_1C_3}. \end{aligned} \tag{B.3}$$

<u>Proof</u>:

We have,

$$\begin{aligned} P\Theta_1 + Q\Theta_2 + R\Theta_3 = &-\frac{k^{-2}}{2C_1}\left[C_2|\omega_1|^2 - C_2|\omega_2|^2\right] - \frac{k_4^{-2}}{2C_3}\left[C_4|\omega_3|^2 - C_3|\omega_4|^2\right] \\ &+ \frac{k_5^{-2}}{2C_1C_3}\left[C_1C_3|\omega_5|^2 - C_3D_1|\omega_1|^2 - C_1D_2|\omega_3|^2\right] = \\ &\left(-\frac{C_2k_2^{-2}}{2C_1} - \frac{D_1k_5^{-2}}{2C_1}\right)|\omega_1|^2 + \frac{k_2^{-2}C_2}{2C_1}|\omega_2|^2 - \left(-\frac{k_4^{-2}C_4}{2C_3} - \frac{k_5^{-2}D_2}{2C_3}\right)|\omega_3|^2 - \\ &\frac{k_4^{-2}}{2}|\omega_4|^2 + \frac{k_5^{-2}}{2}|\omega_5|^2|\omega_5|^2. \end{aligned}$$

Substitute for the constants $C_i (i = 1,2,3,4)$, and $D_j (j = 1,2)$ from Appendix A and simplify for each coefficient of $|\omega_k|^2$. We get

$$P\Theta_1 + Q\Theta_2 + R\Theta_3 = \frac{1}{2(a^2+b^2)}|\omega_1|^2 + \frac{1}{2(c^2+d^2)}|\omega_2|^2 + \frac{1}{2\left[(f+a)^2+(g+b)^2\right]}|\omega_3|^2 +$$

$$+ \frac{1}{2\left[(f-c)^2+(g-d)^2\right]}|\omega_4|^2 + \frac{1}{2\left[(a+c)^2+(b+d)^2\right]}|\omega_5|^2$$

$$= \frac{1}{2}\sum_{i=1}^{5} k_i^{-2}|\omega_i|^2,$$

Since

$$\frac{1}{2(a^2+b^2)} = \frac{1}{2|k_1|^2},$$

$$\frac{1}{2(c^2+d^2)} = \frac{1}{2|k_2|^2},$$

$$\frac{1}{2\left[(a+f)^2+(b+g)^2\right]} = \frac{1}{2|k_3|^2},$$

$$\frac{1}{2\left[(c-f)^2+(d-g)^2\right]} = \frac{1}{2|k_4|^2},$$

$$\frac{1}{2\left[(a+c)^2+(b+d)^2\right]} = \frac{1}{2|k_5|^2},$$

This proves (B.1).

Similarly for Ω,

$$S\Theta_1 + T\Theta_2 + W\Theta_3 = -\frac{1}{C_1}\left[C_2|\omega_1|^2 - C_1|\omega_2|^2\right] - \frac{1}{C_3}\left[C_4|\omega_3|^2 - C_3|\omega_4|^2\right] +$$

$$+ \frac{1}{C_1C_3}\left[C_1C_3|\omega_5|^2 - C_3D_1|\omega_1|^2 - C_1D_2|\omega_3|^2\right] =$$

$$= \left(-\frac{C_2}{C_2} - \frac{D_1}{C_1}\right)|\omega_1|^2 + |\omega_2|^2 + \left(-\frac{C_4}{C_3} - \frac{D_2}{C_3}\right)|\omega_3|^2 +$$

$$+ |\omega_4|^2 + |\omega_5|^2.$$

Substitute for the constants C_1, C_2, C_3, C_4 and D_1, D_2, from Appendix A and simplify. We get

$$-\frac{C_2}{C_2}-\frac{D_1}{C_1}=1,$$

$$-\frac{C_4}{C_3}-\frac{D_2}{C_3}=1.$$

Thus, we will have

$$\begin{aligned} S\Theta_1+T\Theta_2+W\Theta_3 &= |\omega_1|^2+|\omega_2|^2+|\omega_3|^2+|\omega_4|^2+|\omega_5|^2 = \\ &= \sum_{i=1}^{5}|\omega_i|^2, \end{aligned}$$

which proves (B.2).

APPENDIX C

Energy and enstrophy could be expressed in terms of ξ, as follows:

$$E = \frac{-1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\xi\Delta^{-1}(\xi)dxdy, \tag{C.1}$$

and

$$\Omega = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\xi^2 dxdy, \tag{C.2}$$

where Δ^{-1} is defined as before:

$$\Delta^{-1}(\sum_{k\neq 0}\nu_k e^{ik\cdot\mathbf{r}}) = \sum_{k\neq 0}\frac{-\nu_k}{k^2}e^{ik\cdot\mathbf{r}} \tag{C.3}$$

<u>Proof:</u>

Substituting for ξ and $\Delta^{-1}(\xi)$ in (C.1), we get

$$\begin{aligned} E &= \frac{-1}{(2\pi)^2}\sum_{k_1,k_2}\int_0^{2\pi}\int_0^{2\pi}\omega_{k_1}e^{ik\cdot\mathbf{r}}\cdot\frac{-\omega_{k_2}}{k_2^2}e^{ik_2\cdot\mathbf{r}}dxdy \\ &= \frac{1}{(2\pi^2_)}\sum_{k_1,k_2}\frac{1}{k_2^2}\int_0^{2\pi}\int_0^{2\pi}\omega_{k_1}\omega_{k_2}e^{i(k_1+k_2)}dxdy. \end{aligned} \tag{C.4}$$

Since

$$\int_0^{2\pi}\int_0^{2\pi}e^{ik\cdot\mathbf{r}}dxdy = \begin{cases}(2\pi)^2, & k=0,\\ 0, & k\neq 0,\end{cases} \tag{C.5}$$

We get

$$E = \sum k^{-2}|\omega_k|^2, \tag{C.6}$$

This proves (C.1).

Similarly, for Ω,

$$\Omega = \frac{1}{(2\pi)^2}\sum_{k_1,k_2}\int_0^{2\pi}\int_0^{2\pi}\omega_{k_1}\omega_{k_2}e^{i(k_1+k_2)\cdot\mathbf{r}}dxdy = \sum|\omega_k|^2,$$

This proves (C.2).

VITA

M. Saeed Bejnood was born in Tehran, Iran. He attend elementary school and high school in Tehran and received his high school diploma from Hadaff-No.1. He further pursued his education at Pahlavi University(Shiraz), Iran. He later attend the University of Oklahoma, Norman, Oklahoma where he received a Bachelor of Science degree in Mechanical Engineering. He enterd graduate school at the University of Tennessee, Knoxville, Tennessee where he continued studies in Mechanical Engineering. After an intteruption in his education and a brief stay in Alberta, Canada he returned to the United State and worked as adjunct faculty at Walter State Community College, Morristown, Tennessee. In September 1986 he returned to the University of Tennessee Space Institute at Tullahoma, Tennessee. He received his Master of Science in Mathematics(Applied), March 1988.