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I am submitting herewith a dissertation written by Chang-Hyuk An entitled "The Study of MHD Instabilities in Elongated Tokamaks." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

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THE STUDY OF MHD INSTABILITIES IN ELONGATED TOKAMAKS

A Dissertation
Presented for the
Doctor of Philosophy
Degree
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Chang-Hyuk An

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ABSTRACT

Fixed boundary ideal MHD instabilities are studied analytically and computationally for highly elongated axisymmetric toroidal equilibria.

The computational studies of $n = 1$ large-scale instabilities show that even a slightly diamagnetic plasma with broad current profile and highly elongated cross section is subject to ballooning modes for q values well above unity at the magnetic axis. The maximum stable average betas are achieved with a broad current profile and either a paramagnetic plasma ($\beta_p < 1$) with a highly elongated cross section ($b/a \approx 3$) or a diamagnetic plasma ($\beta_p > 1$) with only a mildly elongated cross section ($b/a < 2$).

In the analytic study of high n ballooning modes, the marginal pressure gradient is derived as a function of equilibrium parameters such as safety factor, shift, elongation, and inverse aspect ratio. It is found that high elongation reduces the stability by decreasing the stabilizing effects of field line bending and local shear. The stability of a highly elongated plasma depends sensitively on the profile of the safety factor, $q(\psi)$, due to the weakness of the field line bending effect.

The analytic results for high n ballooning modes are compared with computational results obtained by using the ORNL BALOON code. The computational results reveal that high elongation, low aspect ratio, and broad pressure profile enhance the marginal beta for β_p less than unity but severely reduce the marginal beta for β_p larger than unity for the equilibrium profiles used in this study.

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CHAPTER I

INTRODUCTION

Elongation of the plasma cross section in tokamaks has been proposed^{1,2} as a way of increasing the average beta of a plasma without violating the Kruskal-Shafranov stability criterion near the magnetic axis, $q\text{-axis} \geq 1$. The analysis of Lortz and Nührenberg³ indicates that by properly shaping the plasma cross section it is also possible to make the plasma arbitrarily elongated without violating the Mercier-necessary⁴ or Lortz-sufficient⁵ stability criteria in the neighborhood of the magnetic axis. However, Freidberg, Haas, Marder, Goedbloed, and Grossmann⁶⁻⁹ have predicted that the pressure in a toroidal plasma should be limited by the onset of ballooning instabilities driven by the plasma pressure gradient in regions of convex curvature. Using a surface current model for toroidal plasmas with a variety of specified cross sectional shapes, they find that sufficiently elongated equilibria are unstable at all q -values, and that the optimum shape for maximizing beta (the plasma pressure relative to magnetic pressure) is a mildly elongated cross section with rounded ends. This optimum elongation and maximum beta depends on the details of the plasma cross sectional shape and can be significantly altered by driving force-free currents outside the high pressure plasma.⁹ A number of other computational and analytic predictions¹⁰⁻¹² for highly elongated cross sections with diffuse current profile indicate that there may be windows of stability and enhanced beta

depending on the current profile, plasma shape, and proximity of the conducting wall.

Along with the elongation, higher poloidal beta $\beta_p^{5,13,14}$ (the ratio of plasma pressure and poloidal B field energy) and lower aspect ratio^{9,15,17} R_c/a have been regarded as ways of increasing β from the relation that the equilibrium average beta is proportional to β_p and $(a/R_c)^2$. The high equilibrium β_p was thought to be restricted by the appearances of the reverse plasma current^{13,18} and the second magnetic axis¹⁹ in the plasma, but the restriction can be avoided by the process of flux conserving equilibrium.²⁰ Mikhailovskij²¹ shows that for β_p larger than $(R_0/a)^{2/3}$ the tokamak with circular cross section is more stable than for small values of β_p . Lortz and Nührenberg³ show that high elongation and diamagnetism is favorable for stability using Mercier-necessary and Lortz-sufficient criterion near the magnetic axis. However Freidberg⁹ et al. predict that if β is on the order of a/R_c ballooning instability appears. And the ballooning instability is computationally found by Bateman et al.²² and by Todd et al.²³ when the stability of high beta flux conserving equilibria is studied. They find that high beta is restricted not by forming the equilibrium but by the ballooning instability. Analytic studies of ideal ballooning mode have been done by Dobrott et al.²⁴ and Connor et al.²⁵ In the limit of high toroidal mode number n , the problem reduces to a second order ordinary differential equation on each flux surface for the marginal stability criterion. The analytic studies are complementary to large two-dimensional codes such as PEST,²⁶ ERATO,²⁷ and the initial value codes.^{12,28}

For the optimization to get high stable beta, several problems have to be solved. Is there an optimum elongation for all equilibria? If not, how does the optimum elongation depend on equilibrium? Are high elongation, high β_p , and low aspect ratio as a whole beneficial for high beta or is it needed to have a balance between these equilibrium parameters? How does each equilibrium parameter affect the stability and what equilibrium parameter plays a crucial role to determine the stability of elongated plasma? Many analytic and computational studies provide information about the stability of high β plasmas in elongated tokamaks. However, because of the complicated interdependence of equilibrium parameters, it is difficult to understand their individual effects on stability for various equilibrium profiles. In order to study the high beta plasma more systematically and to understand and predict computational and experimental results correctly, it is important to know how each equilibrium parameter affects the stability.

In this work a reasonably broad range of equilibrium parameters are considered for the study of optimization and stability in elongated tokamaks. The primary concern of this work is to find the trend of stability in the equilibrium parameter space. We mainly study the effects of plasma current profile and diamagnetism on the ideal internal MHD stability of elongated plasmas ($2 \leq b/a \leq 4$) for various equilibrium profiles and different aspect ratio. The ORNL linear initial value code^{12,28} is used to study macroscopic stability in a broad range of equilibrium parameter space and detailed stabilizing and driving mechanism of instabilities are clarified by the analytic calculation.

The analytic calculation is compared with the computational results using ORNL BALOON^{24,25} code.

It is found that the optimum elongation depends on the diamagnetism and plasma current profile, and the highly elongated plasma strongly depends on the plasma current profile and enhances the ballooning character with β_p slightly larger than 1.

This work consists of three parts. Chapter II deals with the equilibrium of elongated plasma with rectangular and elliptic cross section. The effects of diamagnetism, current profile, and global shear on the properties of equilibrium are presented. In Chapter III the stabilities of elongated plasma are studied computationally with ORNL linear initial value code. The effects of current profile and diamagnetism on the stability and beta value are presented and their results are compared with other works. In Chapter IV an analytic study of ballooning mode in highly elongated plasma is given to elucidate the effects of high elongation and other equilibrium parameters on the stability. The ordinary differential equation for marginal stability against high n ballooning mode by Connor et al.²⁵ is used for the study and the results are compared with the computational results using ORNL BALOON code. In the Appendix a detailed calculation for the marginal pressure gradient in Chapter IV is given.

CHAPTER II

EQUILIBRIUM

The possibility to get stable high beta β (the ratio of plasma pressure and magnetic energy) plasma in tokamaks by controlling the equilibrium profiles^{29,30} motivates the extensive study of equilibrium in a wide range. Moreover, detailed study of equilibrium properties is essential to verify the roles of equilibrium parameters on the stability. In order to study equilibrium in axisymmetric geometry we solve the Grad-Shafranov equilibrium equation,

$$RJ_t = -\Delta^*\psi = R^2 P'(\psi) + II'(\psi) \quad (II.1)$$

$$\Delta^*\psi \equiv R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial y^2} \quad (II.2)$$

$$\vec{B} = \nabla \times (\psi \nabla \phi) + B_t \hat{e}_\phi \quad (II.3)$$

for stream function ψ which is proportional to the poloidal flux ψ_p , $\psi = \psi_p/2\pi$. Here J_t is the toroidal current density, $P(\psi)$ is the plasma pressure, $I(\psi) \equiv RB_t$ is proportional to the poloidal current I_p , $I(\psi) = I_p/2\pi$, and ϕ is toroidal angle.

Equation (II.1) can be solved generally by prescribing the source functions $P'(\psi)$ and $I(\psi)$ and giving boundary conditions on ψ . Since

Eq. (II.1) is a second order and generally nonlinear partial differential equation it is very difficult to solve this equation analytically. Even though several analytic methods have been developed, we have to heavily depend on computational methods for the solution of Eq. (II.1). In this chapter we briefly mention the analytic calculations and present computational results. For the computational study of equilibrium we mainly concentrate on the effects of plasma current profile and diamagnetism on the equilibrium configurations in highly elongated (elongation between 2 and 4) plasmas. Both the plasma current profile and diamagnetism are crucial factors needed to determine the equilibrium configuration and the stability of elongated plasma. The effects of plasma current profile are studied by appropriately varying the source functions $P'(\psi)$ and $II'(\psi)$ in Eq. (II.1) or specifying $q(\psi)$ with fixed $P'(\psi)$ as a function of ψ .

1. REVIEW OF ANALYTIC STUDIES

In this section several analytic calculations of equilibrium in various papers are briefly mentioned and their results are discussed.

A closed form of analytic solution³¹⁻³³ can be obtained by determining source terms $P'(\psi)$, $II'(\psi)$ as linear functions of ψ . This analytic result with closed form is interesting for the study of stability, transport, etc.

For the general form of source functions $P'(\psi)$, $II'(\psi)$, asymptotic method³⁴ is applied by assuming that aspect ratio is very large. Even though a detailed equilibrium configuration cannot be obtained with this

method, several important relations between equilibrium parameters can be formulated.

Multipole moments^{35,36} method from momentum and magnetic field equation is also a powerful method for the study of relations between equilibrium parameters. Especially this method is good to determine the cross sectional form of a tokamak plasma from the measurement of the magnetic field outside the plasma.

This method is applied to the study of flux conserving equilibria of circular²⁰ and elliptically elongated³⁷ toroidal plasma. Several important equilibrium relations between shift and elongation, between pressure parameter and plasma current, between poloidal beta β_p and pressure parameter, etc. are obtained.

For a highly elongated plasma (i.e., Belt Pinch configuration) a cylindrical slab model³⁸ is used for the calculation of equilibria with flat current profile. Comparison between results using cylindrical slab model and computational results with same current profile shows good agreement. But this agreement decreases as plasma current profile becomes peaked.

These analytic studies are mainly devoted to formulate equilibrium parameters, shift, distortion of flux surface, vertical B field to keep plasma in equilibrium, etc., with measurable or known equilibrium quantities. Critical β , β_p are calculated with constraints of no reverse current or no second magnetic axis. Brief summaries about analytic results are as follows.

With increasing pressure the magnetic axis shifts outward and reverse current appears with β_p about half of aspect ratio ASP. Plasma

shift is proportional to inverse aspect ratio. A second magnetic axis is never found in the equilibria with source terms $P'(\psi)$, $II'(\psi)$ given as a linear function of ψ . The β value does not increase linearly with β as indicated by the relation $\beta = \beta_p / (qA)^2$ where q is the safety factor and A is the aspect ratio. Rather, there is an optimum β_p for high β and the relation $\beta = \beta(\beta_p)$ sensitively depends on current profile. FCT equilibria do not have β_p limit by the appearance of second magnetic axis in the plasma.

The analytic calculations give good indications to understand the properties of equilibrium, but very restricted classes of equilibria are available for the analytic calculation. The studies of equilibria in a broad range can be made by solving Eq. (II.1) computationally. In the following sections the equilibrium computer codes used for the study of equilibrium and stability are mentioned and their results are compared.

2. EQUILIBRIUM COMPUTER CODES

Two different equilibrium computer codes have been used to study equilibrium and stability. In this section brief description of the codes is given.

1. Equilibrium Code by Bateman at ORNL

This code is used mainly to study the MHD instabilities in elongated plasma with rectangular cross section. A Buneman fast Poisson inverter³⁹ is used to solve Eq. (II.1). The stream function ψ is taken to be zero on the rectangular boundary and maximum at the magnetic axis. The stream function is normalized in the source functions $P'(\psi)$, $II'(\psi)$

in order to improve the numerical convergence as the Grad-Shafranov equation is solved iteratively. This code is convenient to study effects of diamagnetism but is hard to study $q(\psi)$ profile effects. This is no simple way to figure out the profile of $q(\psi)$ with given source functions $P'(\psi)$ and $II'(\psi)$.

2. Island Code by Stevens⁴⁰ at NYU, Modified by Nelson⁴¹ at ORNL

The main purpose of this code is to study adiabatic compression with arbitrary cross section. But by choosing $\gamma = 0$ in a source function $\mu = P(\frac{dV}{d\psi})^\gamma$, usual ideal MHD equilibrium can be studied. Equation (II.1) is solved using Buneman solver just the same as in the equilibrium code by Bateman. However, there are several differences between the Island code and the Bateman code. First, the source functions are different. Instead of defining $P'(\psi)$ and $II'(\psi)$ as functions of ψ , safety factor $q(\psi)$ and $P(\psi)(\frac{dV}{d\psi})^\gamma$ are specified (here with $\gamma = 0$). Second, any plasma shape (circles, ellipse, Dees, triangles, rectangles) can be obtained by determining parameters in the boundary equation of flux,

$$\psi = b^2 x^2 + a^2 y^2 + adxy^2 - qx^2 y^2 \quad (II.4)$$

where b/a is elongation, d is triangularity, and q is squareness. Since safety factor $q(\psi)$ is specified as a source function and plasma shape is easily controlled using Eq. (II.4), this code is effective for studying the effects of shear and shaping the plasma cross section on stability. This code is mainly used in this report to study high n ballooning mode in elliptically elongated plasma.

Several results from the Bateman code and the Island code are presented and these equilibrium results will be used for the study of linear instabilities in elongated plasma.

3. Computational Study of Equilibrium in Elongated Plasma

The effects of toroidal current profile and diamagnetism on the equilibrium flux configuration are studied using the Bateman code by specifying the source functions $P'(\psi)$ and $II'(\psi)$ as functions of ψ as

$$P'(\psi) = A(\psi/\psi_a)^N \quad (II.5)$$

$$I(\psi) = 1 + c(\psi/\psi_a)^M \quad (II.6)$$

where ψ_a is ψ at the magnetic axis, A is the amplitude of pressure gradient, c determines the paramagnetism ($c > 0$) or diamagnetism ($c < 0$) of the plasma. The powers N and M control the width of the pressure profile and poloidal current profile. As the plasma is made more diamagnetic, by making the coefficient c more negative, the magnetic axis shifts outwards along the major radius and the toroidal current becomes hollow at the midplane on the inner side of the toroidal plasma.^{13,42} This hollowness of the toroidal current profile becomes more pronounced as the pressure profile is made broader relative to the width of the toroidal magnetic well by increasing M over N in Eqs. (II.5) and (II.6). A small negative toroidal current forms near the inner wall when, for example, $c < -0.2$ for $N = 1$ and $M = 2$. Under these conditions,

the negative diamagnetic term $II'(\psi)$ balances much of the pressure term $R^2 P'(\psi)$ on the average, but because of the radial dependence of the pressure term, the diamagnetic term overcompensates at the inner edge of the toroid. A second magnetic axis appears from the inner wall if the depth of the magnetic well is increased further ($c < -0.55$ for $N = 1$ and $M = 2$). The appearance of a second magnetic axis is contrary to several analytic results³¹⁻³³ with $P'(\psi)$ and $II'(\psi)$ linear functions of ψ .

For further systematic study, equilibria are divided into para- and diamagnetism with peaked and broad current profiles. The properties of these equilibria chosen for the study of MHD instabilities with toroidal mode number $n = 1$ are presented in Table I, and their profiles are shown in Fig. II.1. Pressure profiles are almost fixed for these four cases.

The poloidal beta β_p is defined as follows:

$$\beta_p \equiv 1 + \mu_p, \quad (II.7)$$

where

$$\mu_p \equiv \frac{1}{2} \left\langle \left(\frac{RB_t}{R_0} \right)_w^2 - \left(\frac{RB_t}{R_0} \right)^2 \right\rangle / \overline{B_p^2}, \quad (II.8)$$

and

$$\overline{B_p^2} \equiv \oint d\ell \, B_p^2 / \oint d\ell. \quad (II.9)$$

Table I. Cases of equilibria

$$\text{ASP} = 4$$

$$(b/a)_w = 3$$

CASE	N	M	A	C	β_p	$(b/a)_a / (b/a)_w$	Δ/a	CONFIGURATION
CASE 1	1	3	1	0.3	0.83	0.66	0.099	PEAKED PARA
CASE 2	1	2	1	0.3	0.81	0.92	0.093	BROAD PARA
CASE 3	1.5	3	1	-0.2	1.42	0.80	0.172	PEAKED DIA
CASE 4	1	3	1	-0.3	1.32	1.57	0.185	BROAD DIA

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EQUILIBRIUM PROFILES

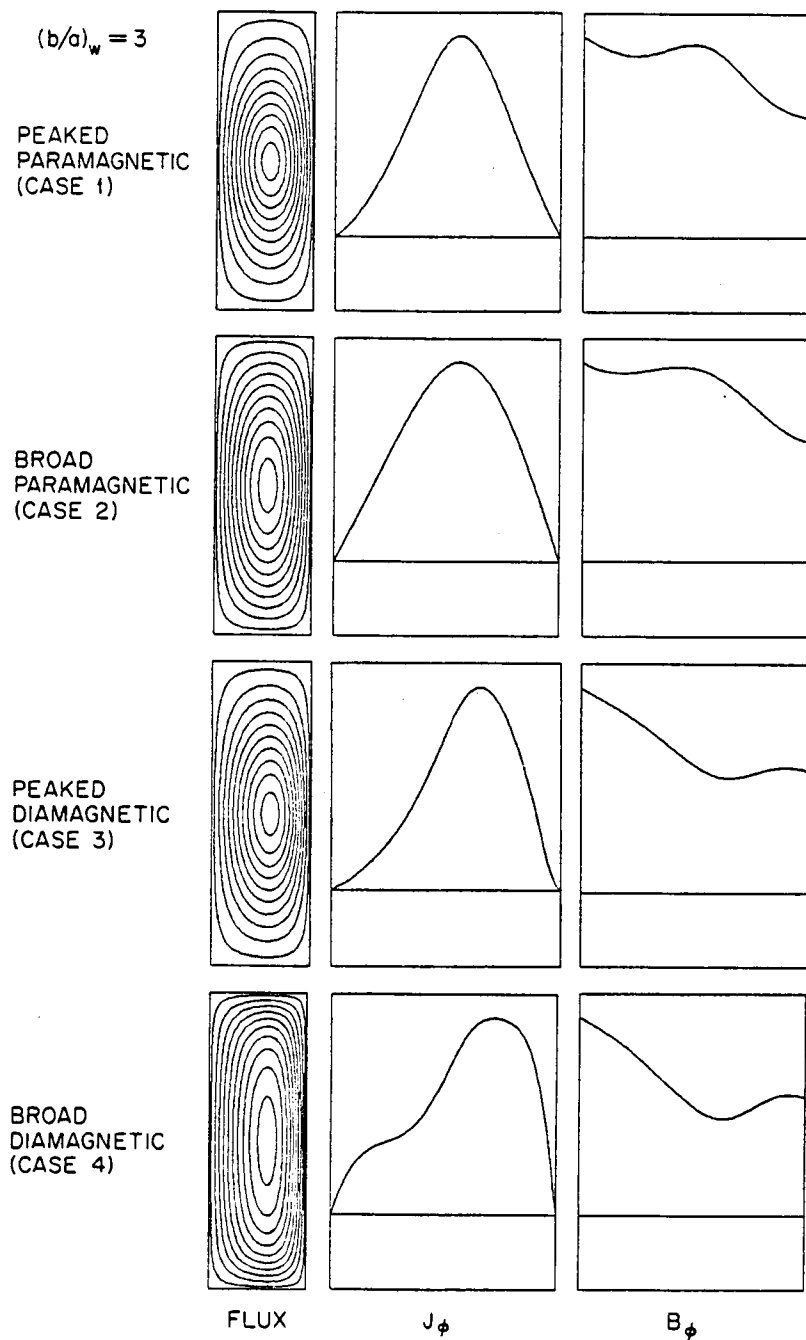


Figure II.1. Equilibrium profiles for the four cases studied in this paper, with elongation $(b/a)_w = 3$. Flux contour over a cross section of the plasma is shown on the left. The toroidal current density and magnetic field at the midplane are shown on the right.

$\oint d\ell$ is a line integral along a poloidal flux line near the plasma edge and $\langle \rangle$ is an average over the cross section of the plasma. Δ/a represents the shift of the magnetic axis from the geometric center of the domain relative to the minor radius of the wall. The subscript w represents a value near the plasma edge (or conducting wall) and the subscript a stands for a value at the magnetic axis. As the toroidal current profile becomes more peaked at the magnetic axis in equilibria with highly elongated walls, the flux surfaces near the magnetic axis become more circular⁴³⁻⁴⁶ as illustrated in Fig. II.1. The elongation of the plasma is physically produced by image currents in the walls, and these image currents become less effective as the toroidal current becomes more concentrated. Increasing the diamagnetism also raises the elongation of inner flux surfaces.¹³

Both diamagnetism and broad current profile have the effect of elongating the inner flux surfaces and the combined effects of these makes the inner flux surfaces in case 4 (see Table I) more elongated than the outer flux surfaces and the wall. The elongation of the flux surfaces in the neighborhood of the magnetic axis $(b/a)_a$ relative to the elongation of the wall $(b/a)_w$ is represented here by $(b/a)_a/(b/a)_w$, where b and a are the half height and half width of a flux surface. The ratio $(b/a)_a/(b/a)_w$ decreases as the wall is elongated if $(b/a)_a/(b/a)_w$ is less than unity, but the ratio increases as the wall is elongated if $(b/a)_a/(b/a)_w$ is greater than unity. The relative shrinking of $(b/a)_a/(b/a)_w$ with $(b/a)_w$ becomes more pronounced with more peaked current profile, as illustrated in Fig. II.2. The effects of current

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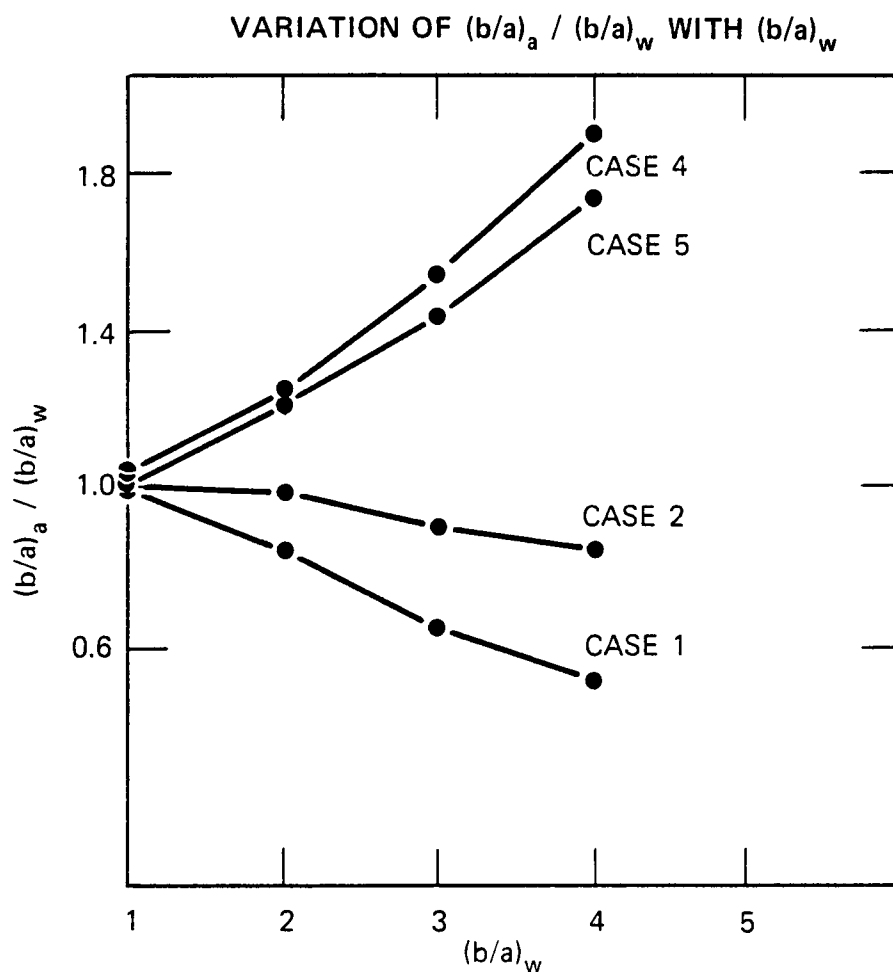


Figure II.2. Variation of the relative elongation at the magnetic axis as a function of the elongation at the wall for the four cases studied in this paper.

profile and diamagnetism on the equilibrium are studied using Bateman code. However, with this code it is difficult to figure out the safety factor $q(\psi)$ and to study the equilibrium with prescribed $q(\psi)$. As a supplement to the Bateman code, Island code is used to examine the equilibria of the elliptically elongated plasma with specified $q(\psi)$ profiles. The effects of current profile on the equilibrium can also be postulated from the effects of $q(\psi)$ profile, since $q(\psi)$ profile strongly depends on the toroidal current profile. By determining $q(\psi)$ profiles as $q_1(\psi) = 3-2\psi$, $q_2(\psi) = 3-2\psi^2$, and $q_3(\psi) = 4-3\psi$, the dependence of equilibrium on $q(\psi)$ profile is studied. The $q_1(\psi)$ and $q_2(\psi)$ have the same q_a , q_w but different functions of ψ , and $q_1(\psi)$ and $q_3(\psi)$ are all linear functions of ψ with the same q_a but q_3 has higher shear and larger q_w than $q_1(\psi)$ has.

The relations between β and β_J are studied for different shear at aspect ratio 4. Figure II.3 shows the dependence of β on β_J for each $q(\psi)$. Here β_J is defined as

$$\beta_J \equiv \frac{\langle p \rangle}{\frac{1}{2} I_p^2}.$$

I_p is the total plasma current and $\langle p \rangle$ is a volume average of plasma pressure. The total plasma current I_p for the highly elongated elliptic plasma is calculated by assuming that elongation e is on the order of $(R_c/a)^{1/2}$ and $\epsilon \equiv a/R_c \ll 1$. Using Eq. (IV.9) and neglecting the terms of order ϵ^2 , I_p is turned out to be

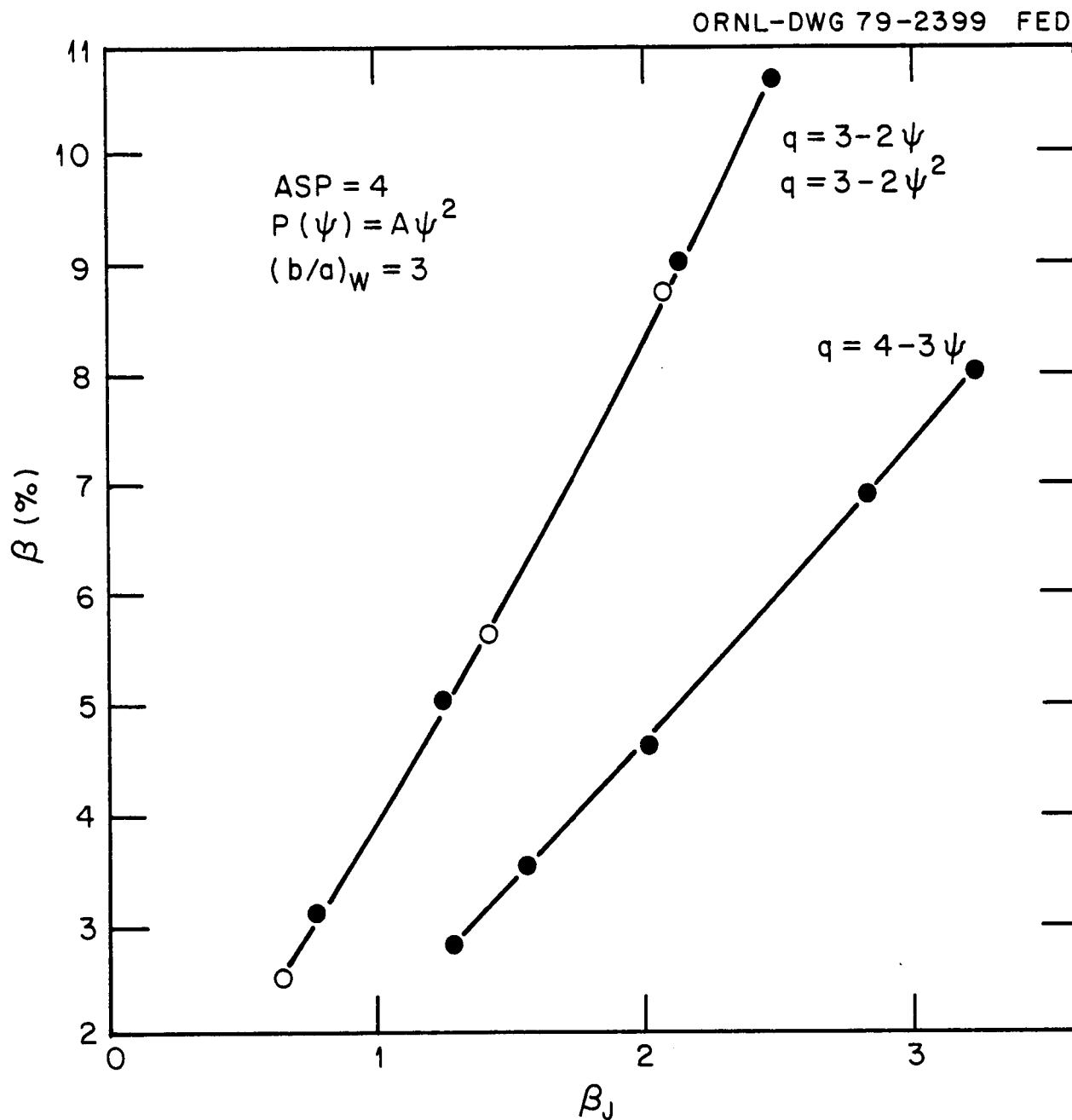


Figure II.3. The dependence of β on β_J for different shear, $q(\psi) = 3 - 2\psi$, $3 - 2\psi^2$, and $4 - 3\psi$.

$$I_p = \left(\frac{a^2 e^2 I(\psi) \pi}{R_c^2 q(\psi)} \right)_w \left[1 + \frac{3e'}{2e} \dot{s} a^2 + \frac{1}{e^2} \right]_w. \quad (\text{II.10})$$

The definition of β_J is different from that of β_p in Eq. (II.7) but the difference is not large.

There are several things to consider from the graph. First, β increases linearly with β_J for all the $q(\psi)$ profiles as $\beta \propto \frac{1}{2} \beta_J$. This linear dependence is somewhat contrary to the computational result by Callen and Dory,¹³ and the analytic explanation by B. J. Greene et al.⁴⁷ These studies show that β does not increase linearly with β_J , but β decreases for further increase of β_J over a critical value of β_J . The discrepancy comes from the way to increase β_J . According to B. J. Greene et al.,⁴⁷ if β_J increases by fixing $I(\psi)$ and q_a (instead of fixing $q(\psi)$), the current density on the magnetic axis decreases followed by the decrease of total plasma current I_p in order to maintain q_a . From the definition of average beta $\beta \left(\equiv \frac{\langle p \rangle}{\frac{1}{2} B_0^2} \right)$ and the definition of β_J , the relation between β and β_J is $\beta = \frac{I_p^2}{2 B_0^2} \beta_J$. Here B_0 is a vacuum B field. However, if $q(\psi)$ profile is fixed, the total plasma current I_p of Eq. (II.10) is unchanged with β_J to some extent where plasma does not extremely distort. Therefore β increases almost linearly with β_J . Next, we can notice that the dependences of β on β_J for $q_1(\psi)$ and $q_2(\psi)$ are the same in a broad range of β_J , while the dependences are much different between $q_3(\psi)$ and $q_1(\psi)$ or $q_2(\psi)$. It seems that the properties of equilibrium sensitively depend on the value of q_a and q_w , but not on the profile of $q(\psi)$ as a function of ψ . Further studies about the

properties of equilibrium with different $q(\psi)$ are done by examining the relations between shift Δ and β_J and between the elongation near the magnetic axis e and β_J . Figures II.4, II.5, and II.6 show that the relations between Δ and β_J and between e and β_J are not significantly different from each other for $q_1(\psi)$ and $q_2(\psi)$. The comparisons between the curves for $q_3(\psi)$ and $q_1(\psi)$ (or $q_2(\psi)$) in Figs. II-4-II-6 show that the plasma shift does not sensitively depend on $q(\psi)$ profile, and the elongation near the magnetic axis strongly depends on the values of q_a and q_w . As a result, the equilibrium is insensitive to the profile of $q(\psi)$ as a function of ψ but strongly depends on the values of q_a and q_w , since the ratio of q_a and q_w determines the elongation of inner flux surface and q_w determines the total plasma current through Eq. (II.10).

The effects of changing the aspect ratio on the equilibrium are studied by fixing the profile of $q(\psi)$ and $p(\psi)$ as shown in Figs. II-7-II-10. The effects of changing the aspect ratio are big in the relations between β and β_J , between Δ and β_J , and between e and β_J . The big effects of aspect ratio can be understood by noticing the following; β_J is proportional to the average pressure $\langle p \rangle$ as

$$\beta_J \propto (R_c/a)^4 \langle p \rangle \quad (\text{II.11})$$

from the definition of β_J and Eq. (II.10), the plasma shift increases with aspect ratio at the same $\langle p \rangle$ because increasing the aspect ratio relaxes the tension of field lines in major radial direction, and the

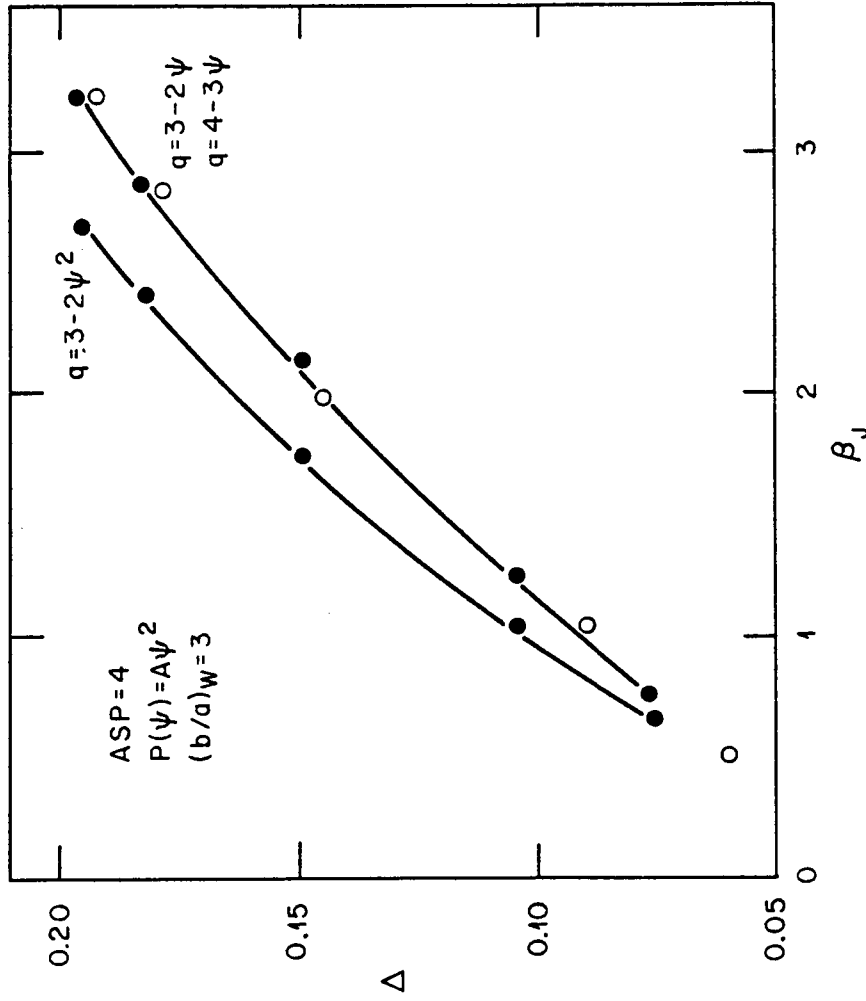


Figure II.4. The dependence of shift Δ on β_J for different shear. The vacant circles along the line of $q = 3-2\psi$ are for $q = 4-3\psi$.

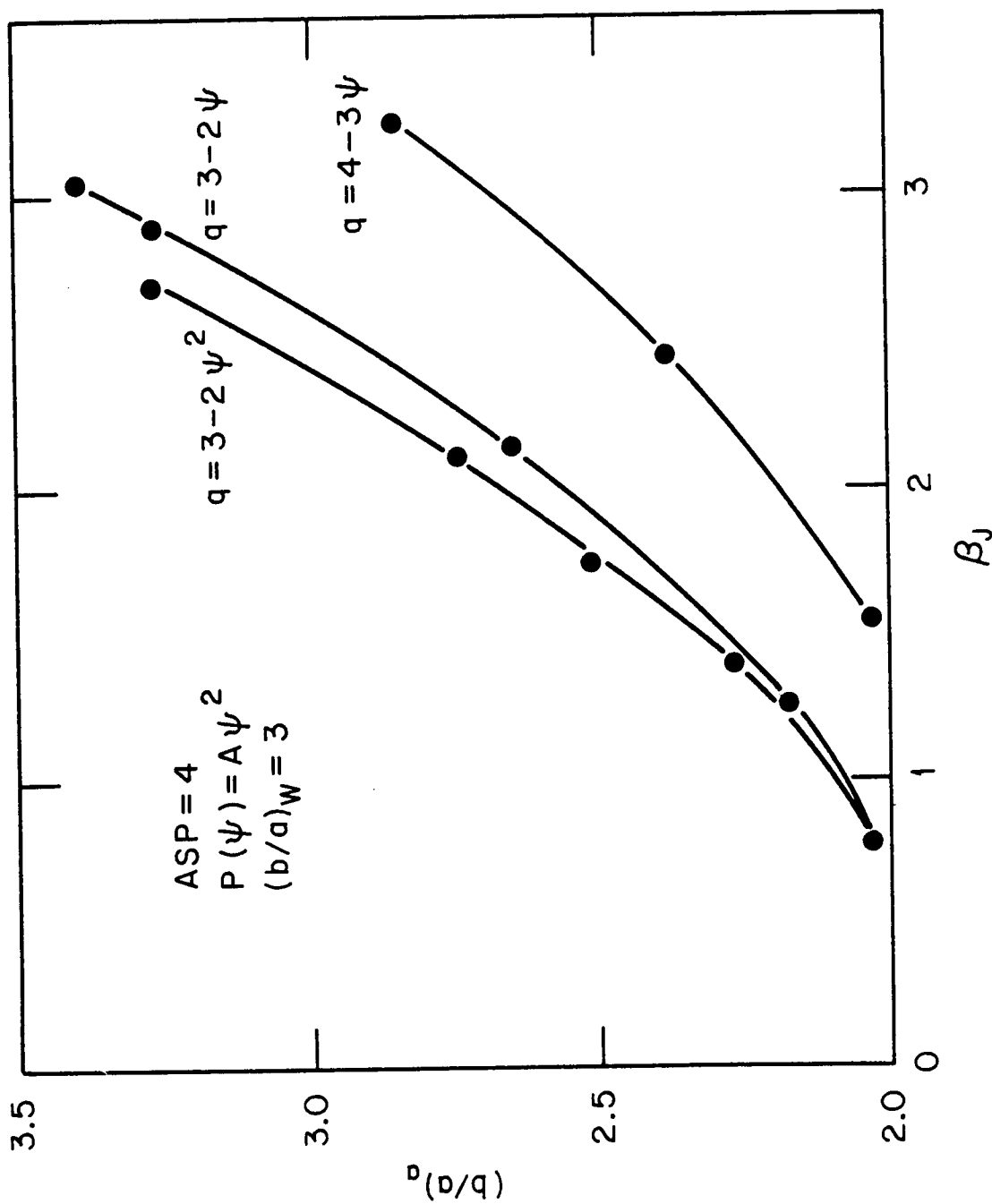


Figure II.5. The relations between elongation near the magnetic axis e and β_J for different shear.

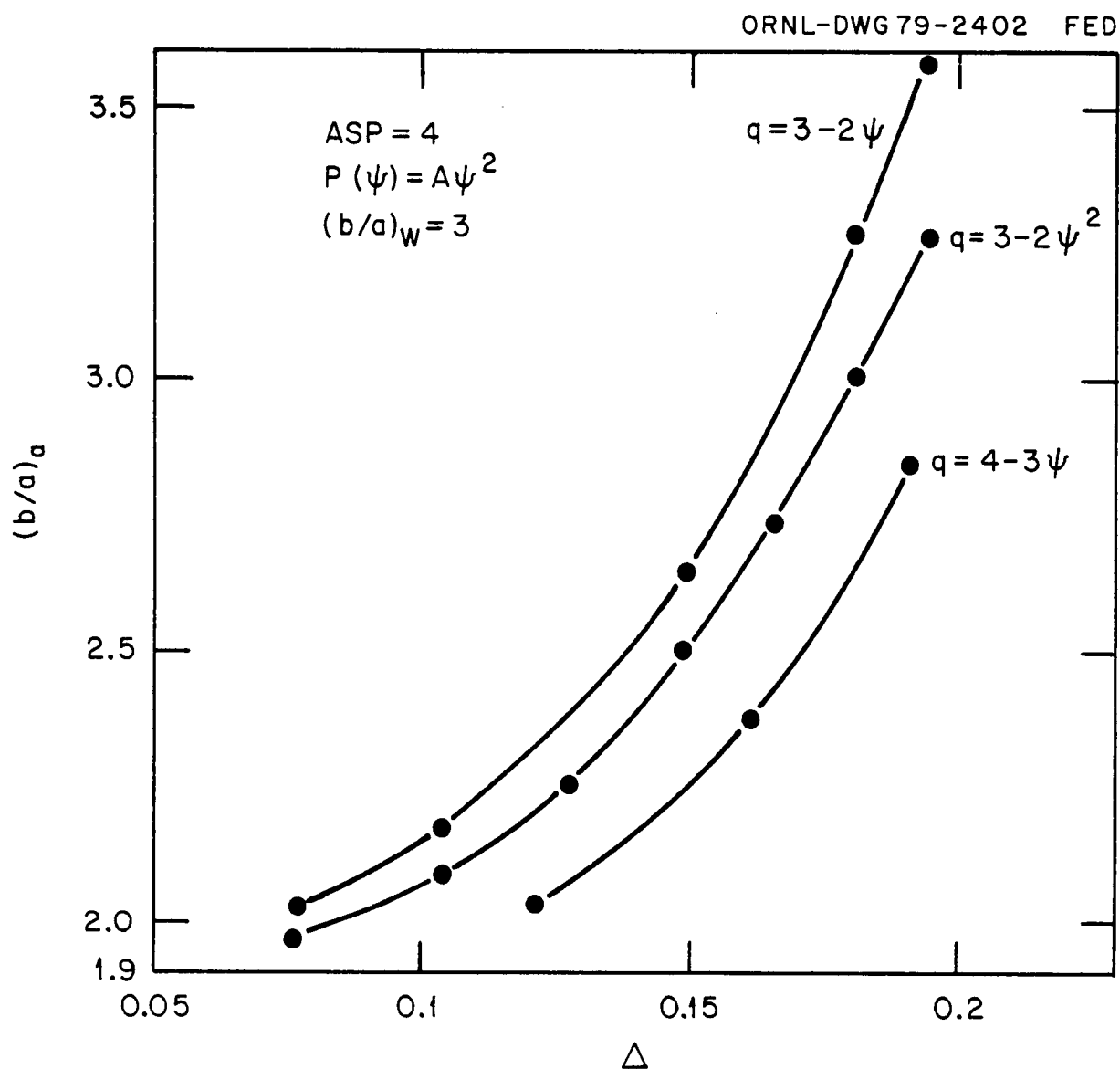


Figure II.6. The relations between e and Δ for different shear. Pressure profile $p(\psi) = A\psi^2$, aspect ratio $ASP = 4$, and $(b/a)_w = 3$ are fixed through Figs. II.3-II.6.

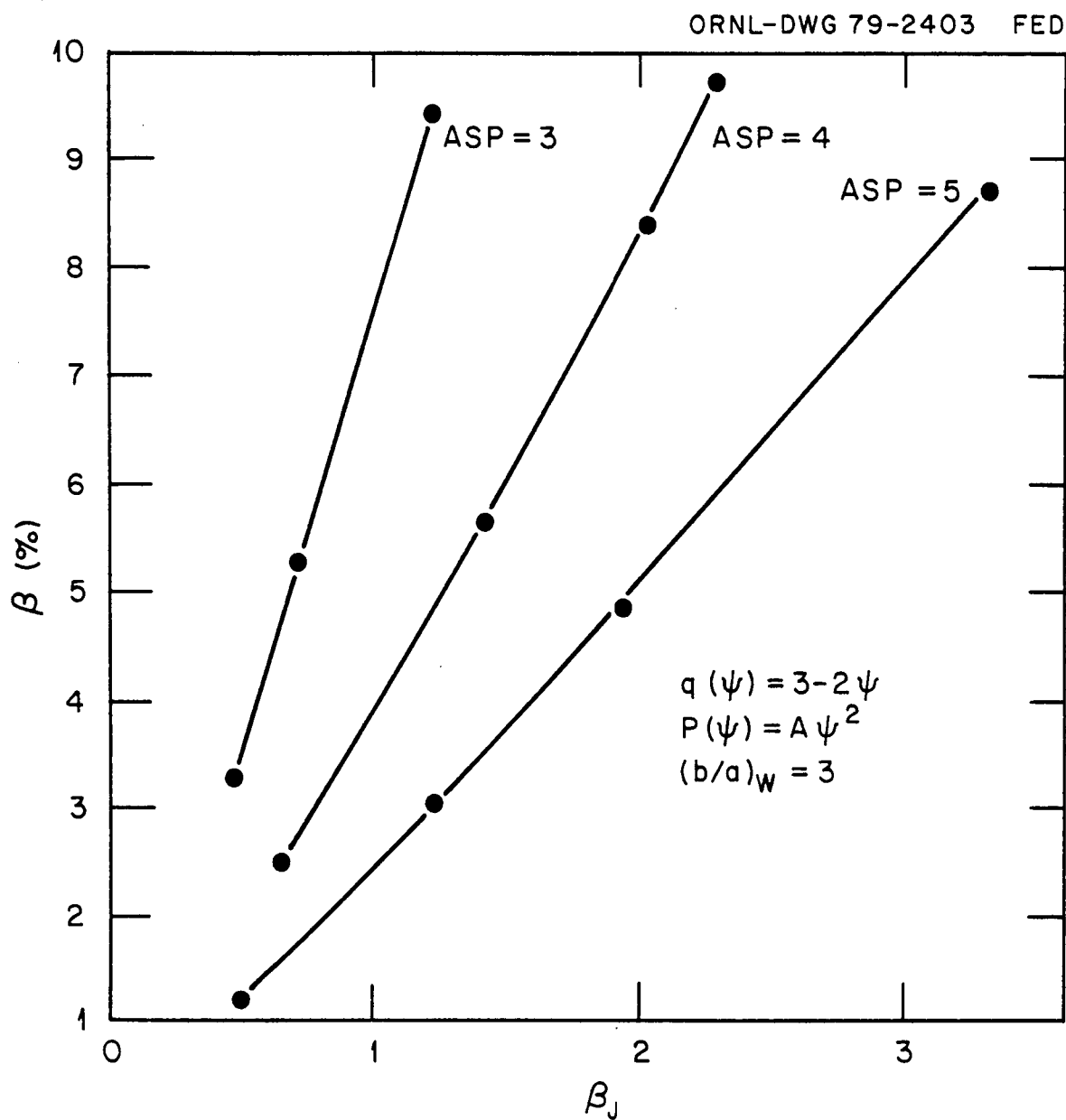


Figure II.7. The effects of changing the aspect ratio on the relation between β and β_J .

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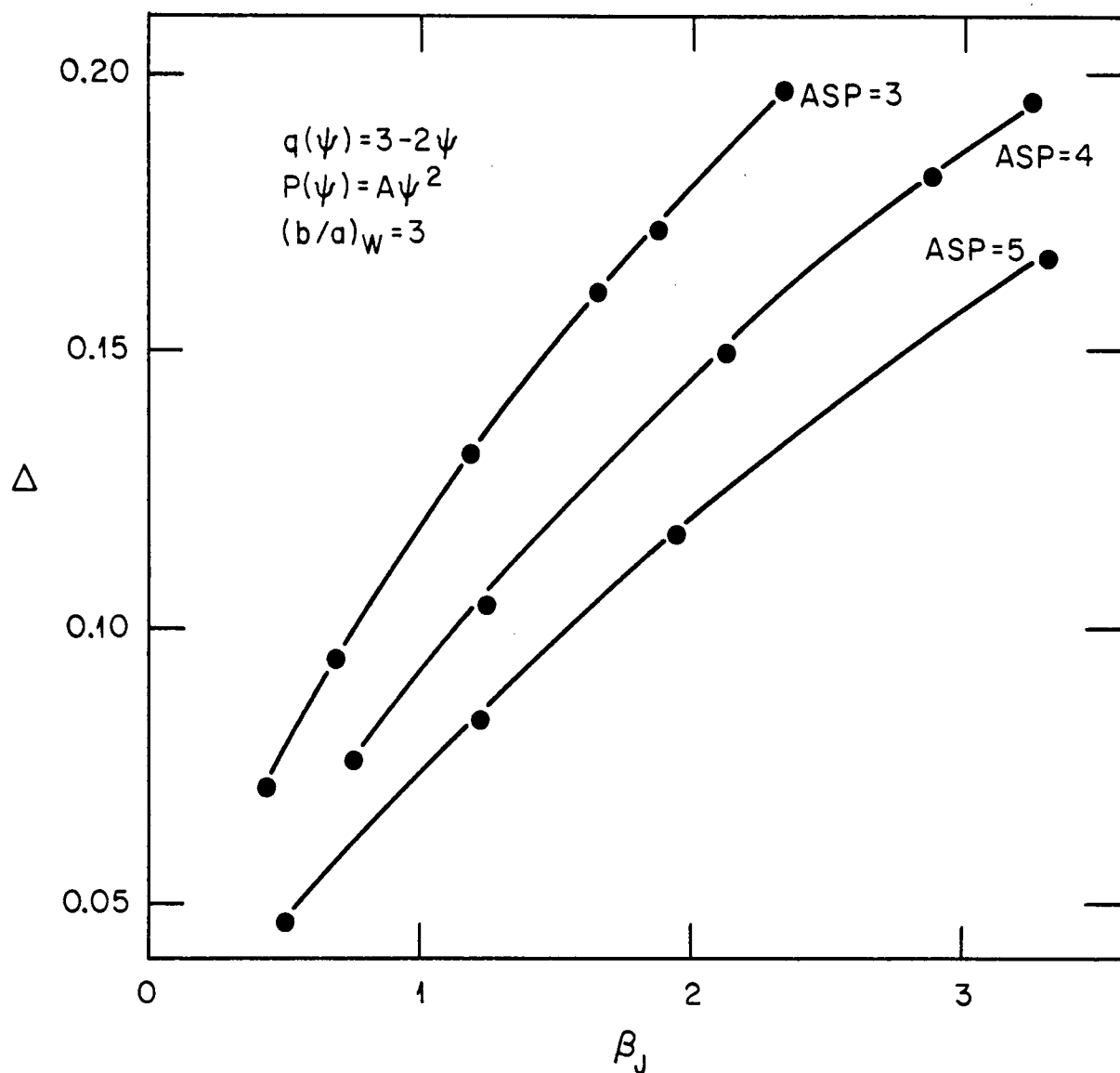


Figure II.8. The effects of changing the aspect ratio on the relations between Δ and β_J .

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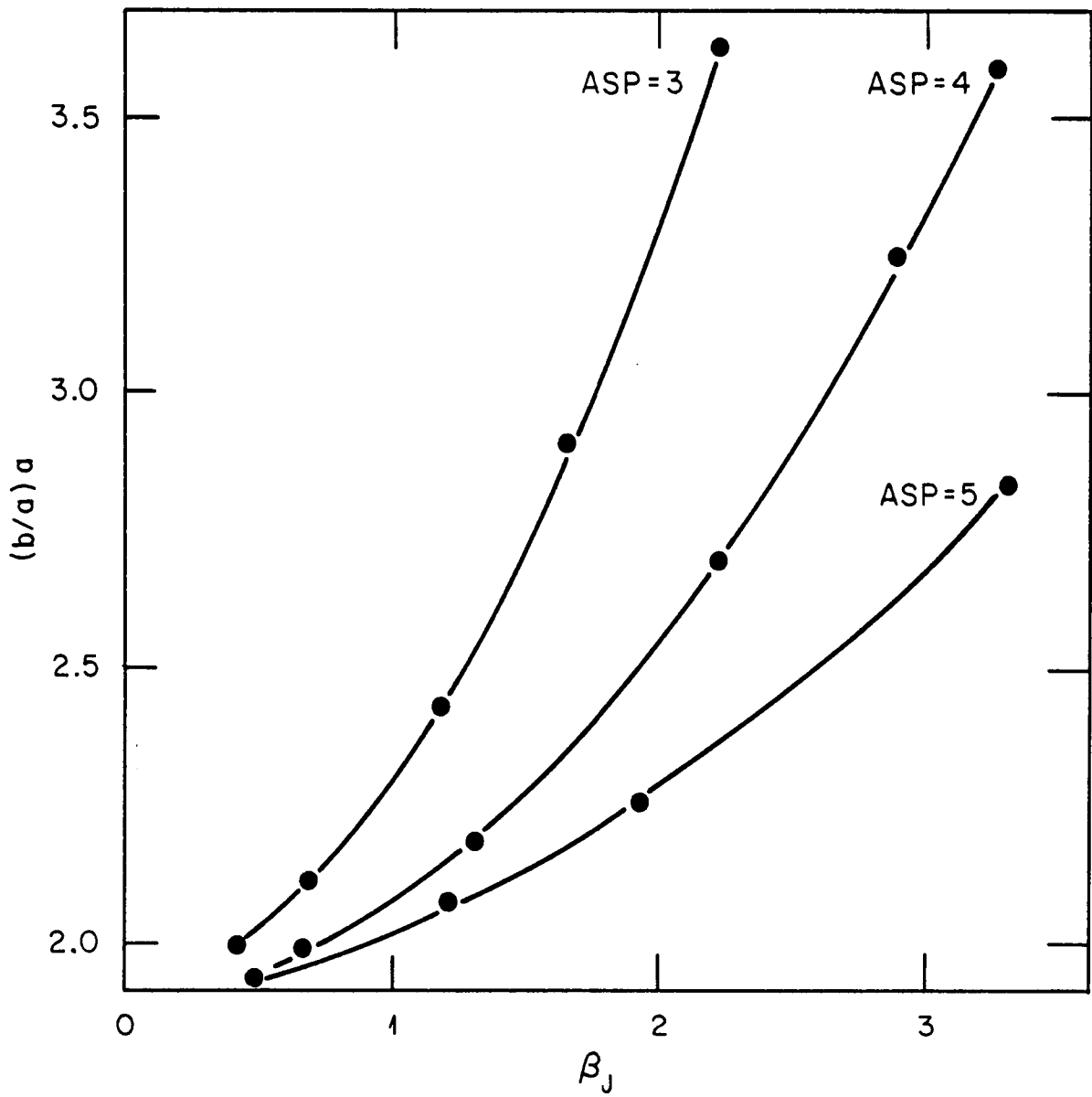


Figure II.9. The effects of changing the aspect ratio on the relations between e and β_J .

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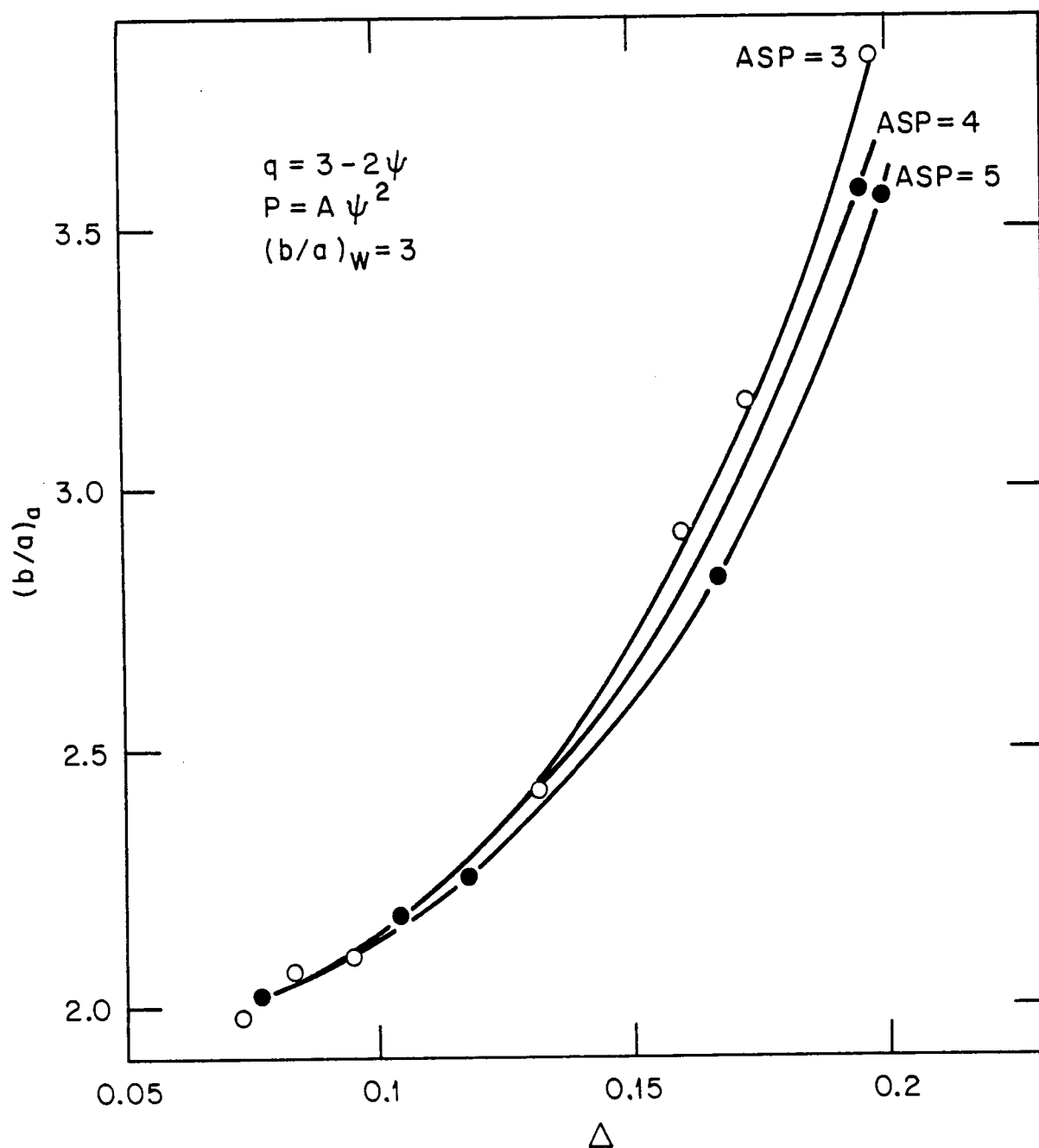


Figure II.10. The effects of changing the aspect ratio on the relations between e and Δ . $P(\psi) = A\psi^2$, $q(\psi) = 3 - 2\psi$, and $(b/a)_w$ are fixed through Figs. IV.7-IV.10.

ratio of increasing β_p with $\langle p \rangle$ is much larger than the ratio of increasing the shift Δ with $\langle p \rangle$ for large aspect ratio because of the factor $(Rc/a)^4$ in Eq. (II.11). The dependence of elongation on the plasma shift is insensitive to the aspect ratio for given equilibrium profiles, the increase of elongation comes mainly from the plasma shift.

In summary, a detailed study about the properties of equilibrium has been done using computational methods. The effects of current profile and diamagnetism are considered through the studies about the effects of $q(\psi)$ profile for different β_j . The effects of changing the aspect ratio are also studied by fixing $q(\psi)$ and $p(\psi)$. The study of the equilibrium properties is important to understand and analyze the computational results of stability and the result will be used for the study of stability in the next chapters.

CHAPTER III

COMPUTATIONAL STUDY OF LINEAR INSTABILITIES IN ELONGATED TOKAMAKS

In the previous chapter the equilibrium of highly elongated plasma has been studied and the effects of current profile, safety factor, diamagnetism on the equilibrium properties were determined. The stability of these equilibria will be tested with computational and analytic methods and the results will be analyzed to clarify the mechanisms which drive and stabilize instabilities in highly elongated plasma.

1. METHODS FOR THE STUDY OF STABILITY

Before going into the problems of ideal MHD instabilities, let us briefly review techniques available for the study of stability.

One of the most useful ways to study the stability of complicated equilibria is to use the energy principle⁴⁸ technique. Its advantage lies in the fact that if one seeks only to determine stability, and not growth rate, it is only necessary to determine whether there is any perturbation which decreases the potential energy from its equilibrium value. The energy principle is closely related to Rayleigh's principle.⁴⁹ The advantage of Rayleigh's principle lies in the fact that if the ratio of potential energy $\delta w(\vec{\xi}, \vec{\xi})$ and kinetic energy $k(\vec{\xi}, \vec{\xi})$

$$\omega^2 = \frac{\delta w(\vec{\xi}, \vec{\xi})}{k(\vec{\xi}, \vec{\xi})}$$

possesses a minimum, the minimum value of ω^2 can be used as a growth rate of instability. The minimization of ω^2 is just the same as the minimization of $\delta w(\vec{\xi}, \vec{\xi})$ with the normalized condition $k(\vec{\xi}, \vec{\xi}) = 1$. On the other hand, the energy principle is not restricted to the normalized condition $k(\vec{\xi}, \vec{\xi}) = 1$, but chooses any convenient condition for analytic simplification, which does not alter the sign of $\delta w(\vec{\xi}, \vec{\xi})$. Much work has been done to derive necessary as well as sufficient stability conditions from the energy principle for circular and toroidal geometry and for local and global criterion. Big and sophisticated two-dimensional MHD stability computer codes such as PEST²⁶ and ERATO²⁷ also use the energy principle (or more correctly Rayleigh's principle) with the normalization condition $k(\vec{\xi}, \vec{\xi}) = 1$.

Another powerful way to study the stability is to integrate the MHD equations forward in time with an arbitrary initial perturbation, i.e., to solve the initial boundary value problem^{12,28} using a computational method. The MHD equations are linearized and one Fourier component in the ignorable direction of the equilibrium is used to reduce the problem to a two-dimensional problem. The most unstable mode ultimately dominates over all other modes and is picked up and its spatial form and growth rate are calculated by the computational calculation. This method is directly extended to the problem of nonlinear instabilities which are impossible to study using the energy

principle. Furthermore, nonideal effects can be implemented directly without much changing the computer code.

These two methods have mainly been used to study the stability of complicated equilibria in tokamaks. Other methods for MHD stability can be found in the reference.⁵⁰

In this chapter we study the stability of elongated tokamaks using the ORNL linear initial value code^{12,28} and in the next chapter we use the differential equation for marginal stability against high n ballooning mode derived from the energy principle by Connor et al.²⁵

2. COMPUTATIONAL STUDY

Fixed-boundary magnetohydrodynamic instabilities with one wavelength around the toroid ($n = 1$) are investigated here using the Oak Ridge linear initial-value magnetohydrodynamic instability code.^{12,28} The effects of elongation, breadth of the toroidal current profile, and diamagnetism are studied as they relate to the growth rate of the fastest growing $n = 1$ magnetohydrodynamic instability, the spatial structure of the eigenfunctions, and the volume-averaged beta at marginal stability. Results are presented primarily for the four cases described in the last chapter (see Fig. II.1 and Table I, pages 12 and 13). These results are found to be representative of a broad class of similar equilibria.

1. Growth Rates

Growth rate curves are presented in Figs. III.1-III.4 for the four cases described in the last chapter, with three elongations of the wall,

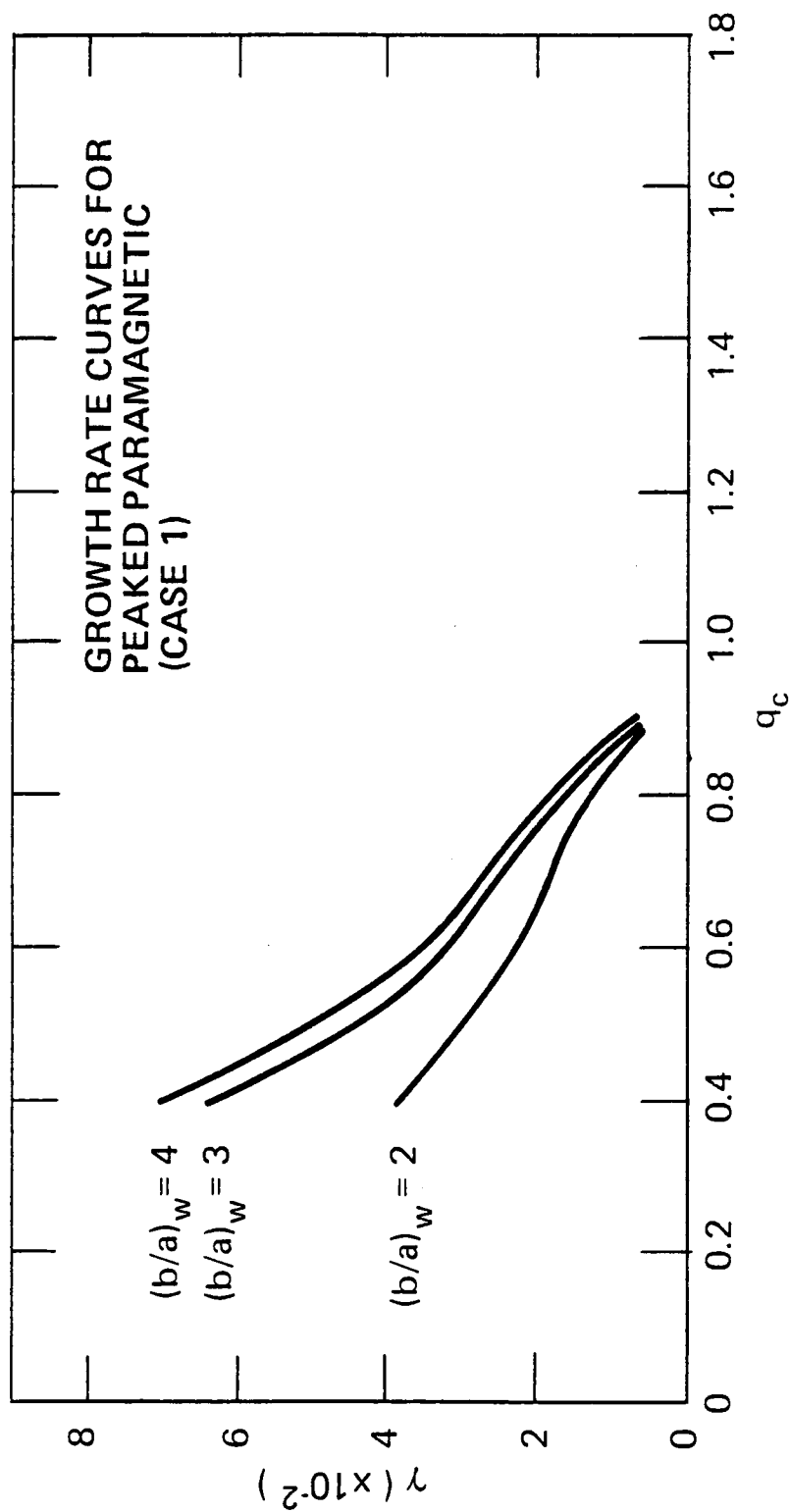


Figure III.1. Growth rate as a function of q value at the magnetic axis for the peaked-current paramagnetic equilibrium of case 1.

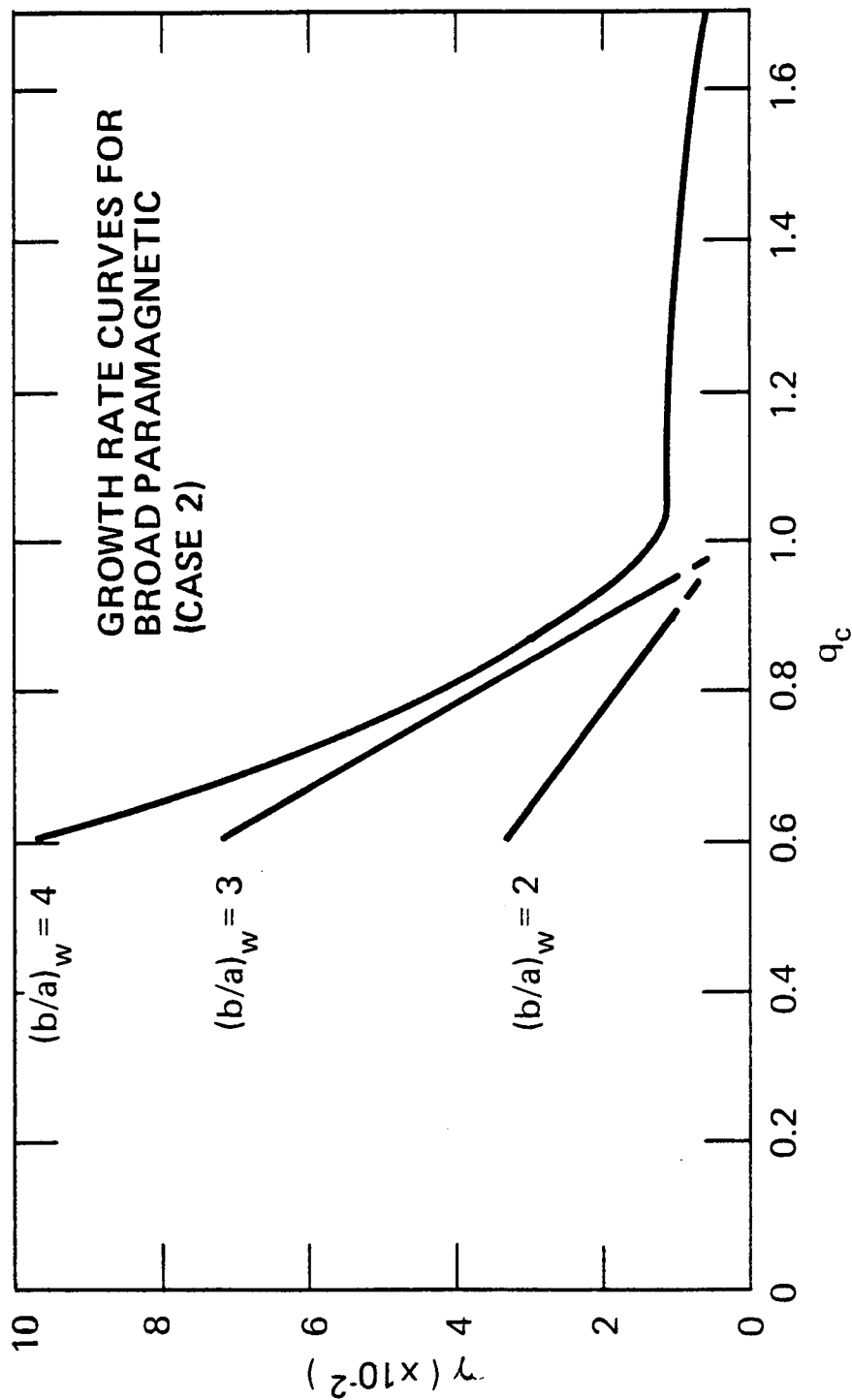


Figure III.2. Growth rate as a function of q value at the magnetic axis for the broad-current paramagnetic equilibrium of case 2.

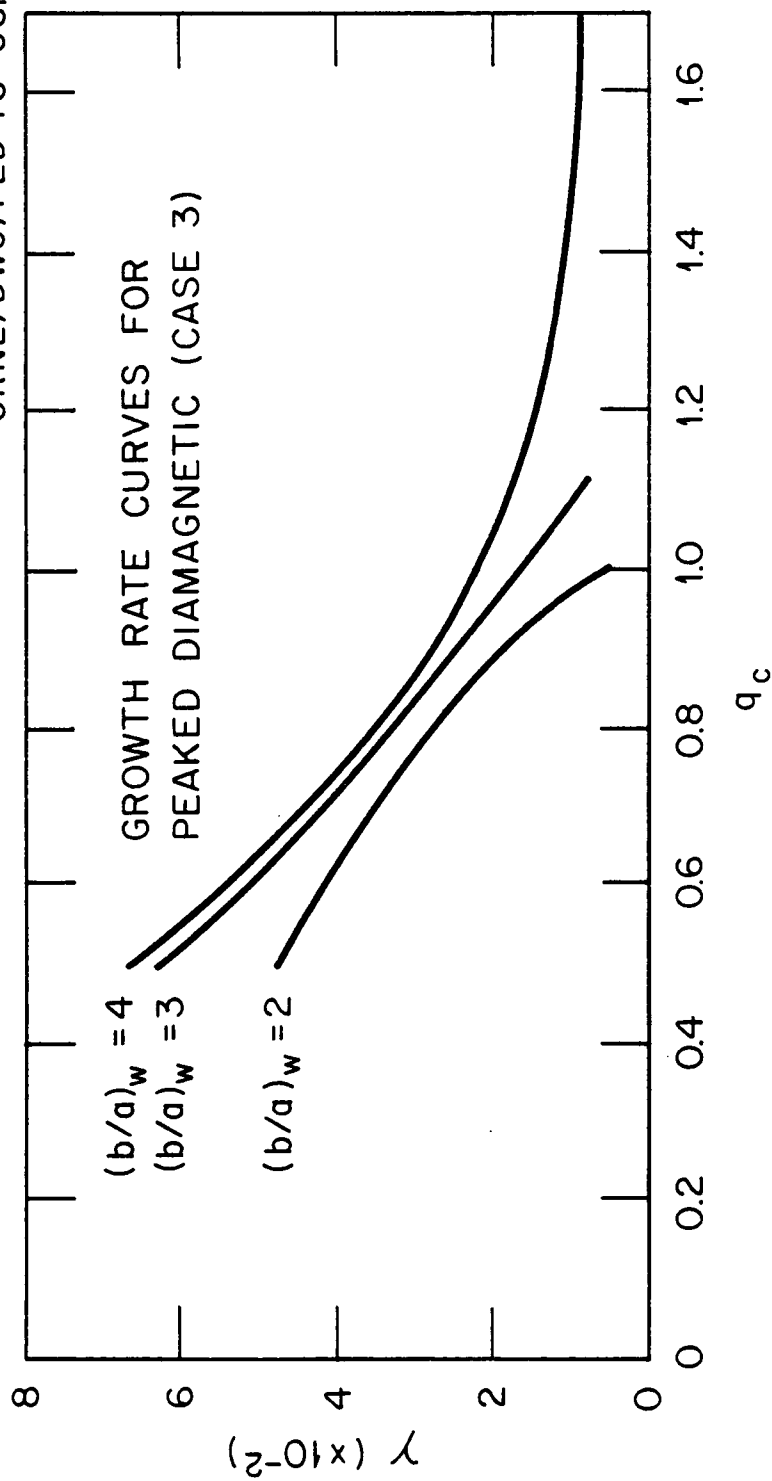


Figure III.3. Growth rate as a function of q value at the magnetic axis for the peaked-current diamagnetic equilibrium of case 3.

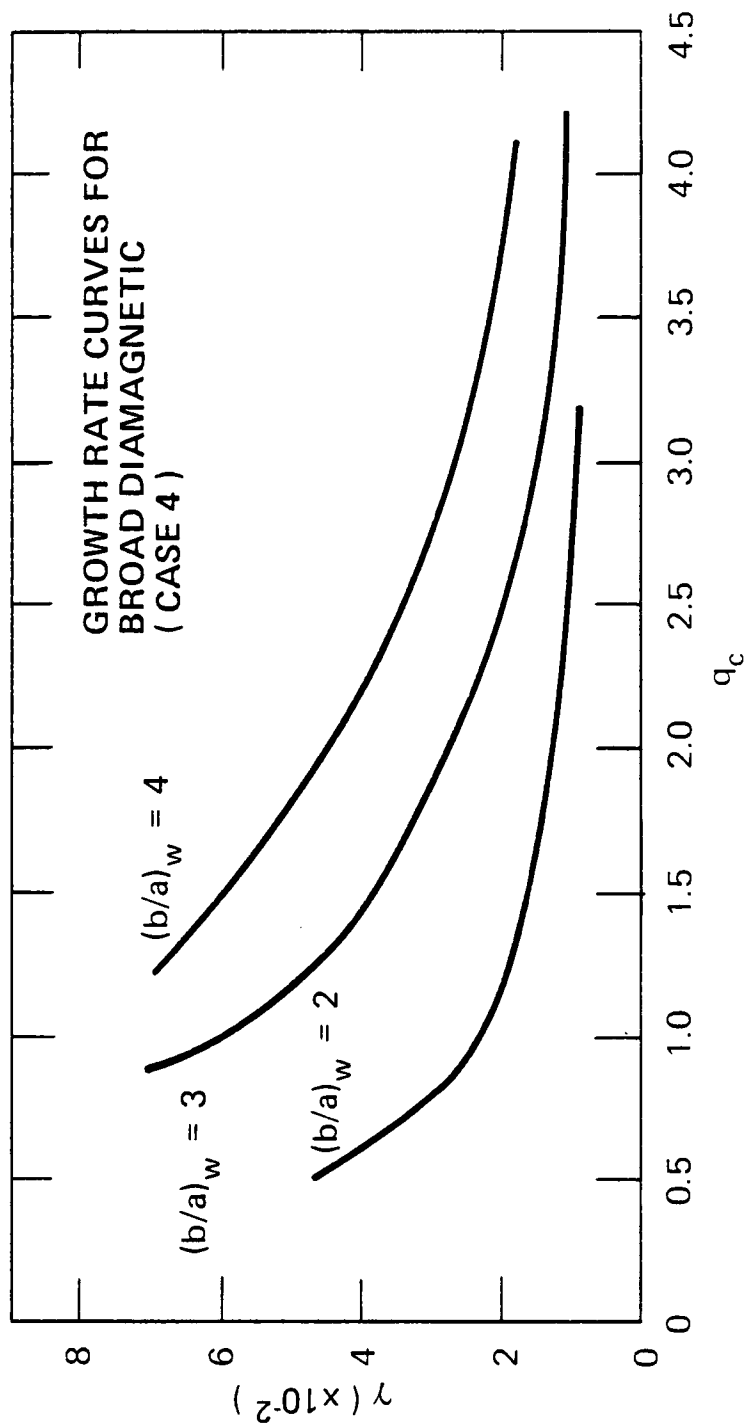


Figure III.4. Growth rate as a function of q value at the magnetic axis for the broad-current diamagnetic equilibrium of case 4. Note that the range of q values extends much higher here than in Figs. III.1-III.3.

$(b/a)_w = 2, 3, 4$, on each graph. For each curve, the growth rate is plotted as a function of q_c , the q value at the magnetic axis, by scaling the equilibria as described in Ref. 22. Under this scaling procedure, the geometric shape of the flux surfaces and the form of the profiles remain unchanged. The growth rate is normalized to the Alfvén transit time across the minor radius of the plasma [$\gamma_A \equiv B_t/(\mu_0 \rho)^{1/2} a$]. Only instabilities with one wavelength around the toroid and, of those, only the instabilities with the lowest poloidal harmonic structure are considered here. It is possible, for example, that there are additional disconnected regions of instability at larger q values which do not appear on these illustrations. However, the instabilities with a higher toroidal mode number and a higher q value have a more complicated spatial structure which cannot be accurately computed with the grids and computer time available. This is also a problem for eigenvalue codes, such as PEST²⁶ and ERATO,²⁷ as well as initial value codes such as our own. A sample of convergence studies with finer grids is discussed later in this paper (Fig. III.5).

In every case, the growth rates of the instabilities studied increase monotonically with elongation. The increased growth rates are most pronounced in the cases with diamagnetism and a broad current profile.

There is a dramatic change in the range of the instability as a function of q value as the plasma is made diamagnetic with broad current profile. Compare, for example, Figs. III.1 and III.4. Note that the ordinate in Fig. III.4 spans a much wider range of q values than the

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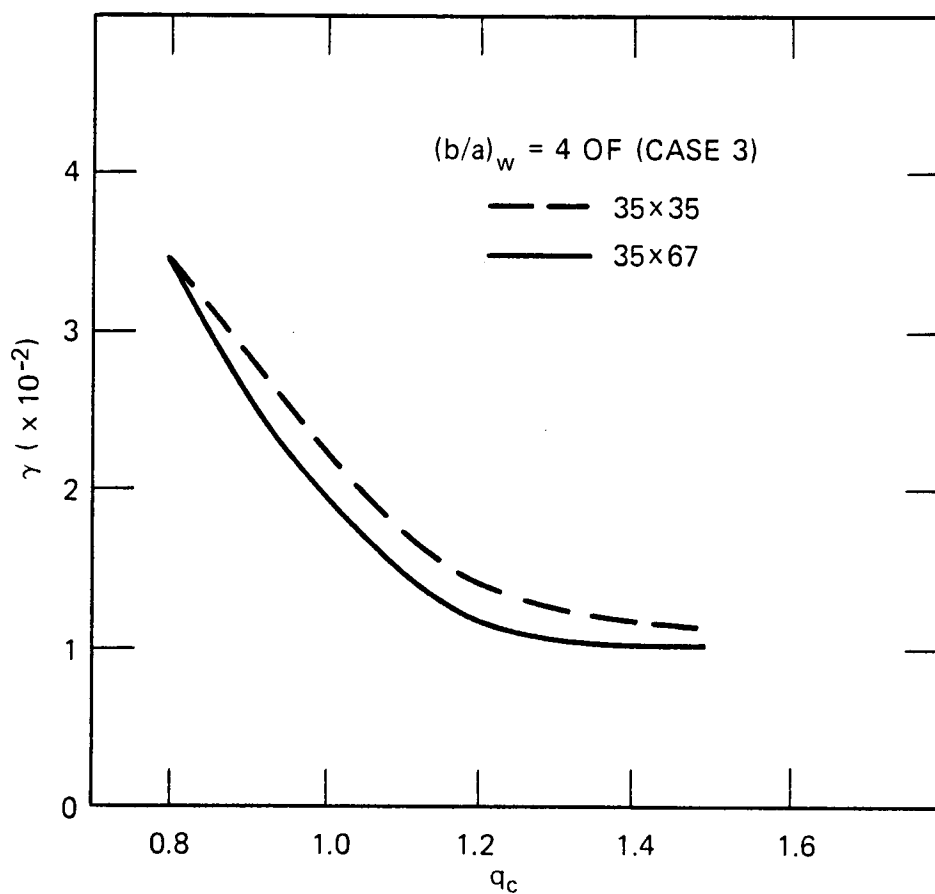
COMPARISON OF GROWTH RATES
WITH DIFFERENT GRID SIZE

Figure III.5. Growth rate curves as a function of q value at the magnetic axis for different grid resolutions. The 35 x 35 is the standard used in the rest of the paper.

corresponding ordinates in Figs. III.1-III.3. The marginal stability points are of order unity for all the elongations in the paramagnetic peaked-current profile case presented in Figure III.1. (If there is a tail on the $(b/a)_w = 4$ case in Fig. III.1, as there is in Figs. III.2 and III.3, it is below the reliable resolution $\gamma < 5 \times 10^{-3}$ of our computer code.) However, no marginal point at all could be found for the broad-current diamagnetic case of Fig. III.4, and there are strong instabilities present at q values well above unity. The cases studied in Figs. III.2 and III.3 show evidence of a transition as the toroidal current profile is made broad [by changing the powers N and M in the source functions of the Grad-Shafranov equation Eqs. (II.5)-(II.6)] or as the plasma is made diamagnetic [by changing the coefficient C in Eq. (II.6)] with a peaked-current profile, especially in the highly elongated case $(b/a)_w = 4$. This transition emerges first as a tail on a growth rate curve that would otherwise intersect the ordinate at q axis near unity, and then as a shift of the entire growth rate curve to higher q values.

The range of unstable q values and the magnitude of the growth rates depend more strongly on the elongation of the inner flux surfaces than on the diamagnetism, for the cases we have studied. This sensitivity to elongation is illustrated in Fig. III.6 where growth rate curves are presented for a variety of paramagnetic, diamagnetic, peaked current, and broad current cases, all with $(b/a)_w = 3$. Except for the two curves labeled $(b/a)_a/(b/a)_w = 0.80$ and 0.88 , the magnitudes of the growth rates and the range of the unstable q values both increase monotonically

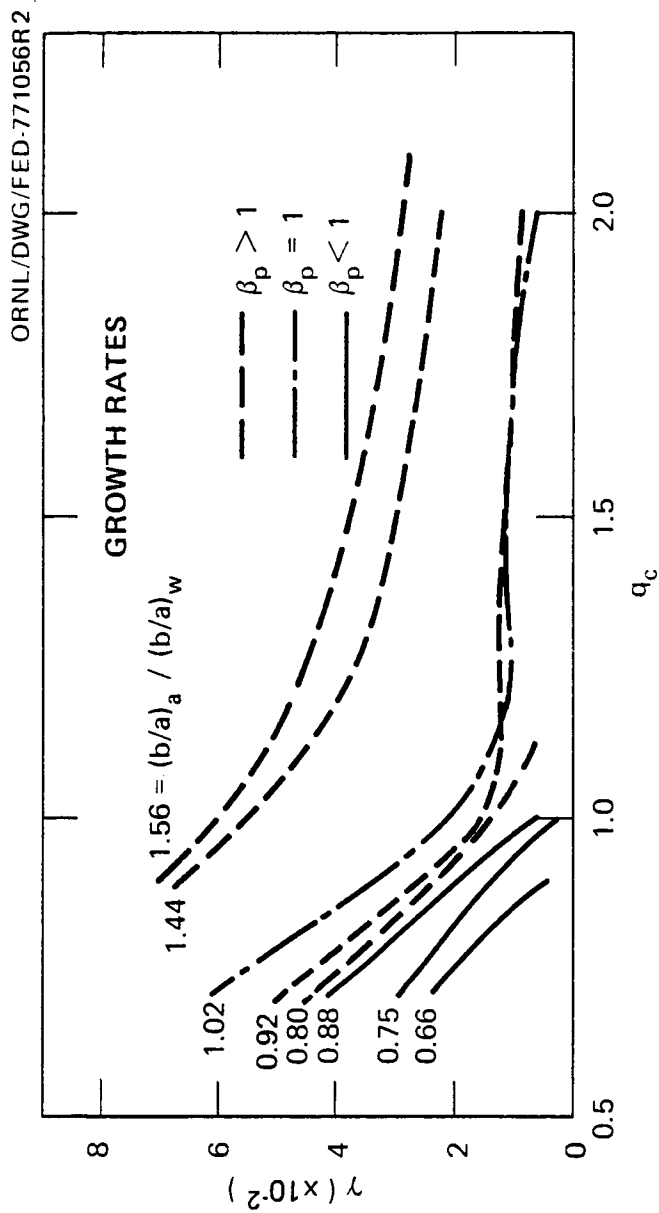


Figure III.6. Growth rate curves as a function of q value at the magnetic axis for a variety of equilibria with different relative elongations of the flux surfaces near the magnetic axis.

with elongation of the inner flux surfaces. The one exception illustrates the milder effect of diamagnetism.

When a finer 35×67 grid is used for the computation of instabilities instead of the standard 35×35 grid, individual growth rates are decreased by as much as 10%, but the forms of the growth rate curves remain unchanged, as illustrated in Fig. III.5. In particular, the tails on the growth rate curves in Figs. III.2 and III.3 remain, and there is the same general transition from kink to ballooning behavior.

2. Eigenfunctions

Velocity arrow plots and perturbed pressure contours are shown in Fig. III.7 at cross sections of the plasma one quarter of a wavelength apart for cases 2, 3, and 4 with q values close to the respective marginal points. The marginal eigenfunction for case 1 (not shown) is similar to that of case 2. This eigenfunction corresponds to an $m = 1$ kink instability whose spatial extent shrinks down to the magnetic axis as the q value is raised to the marginal point. In case 3, where the marginal q value is just above unity, the marginal eigenfunction has the character of an $m = 2$ instability (with four vortex cells and two positive and two negative perturbed pressure peaks) which have a broader spatial extent. In the broad-current diamagnetic case 4, where the marginal q value is well above unity, the instability has a strongly ballooning character^{22,23,29,51,52} with a high poloidal mode number concentrated away from the magnetic axis. The ballooning character of the instability is made evident from the fact that the perturbed pressure is strongest at the outer edge of the toroidal plasma (each

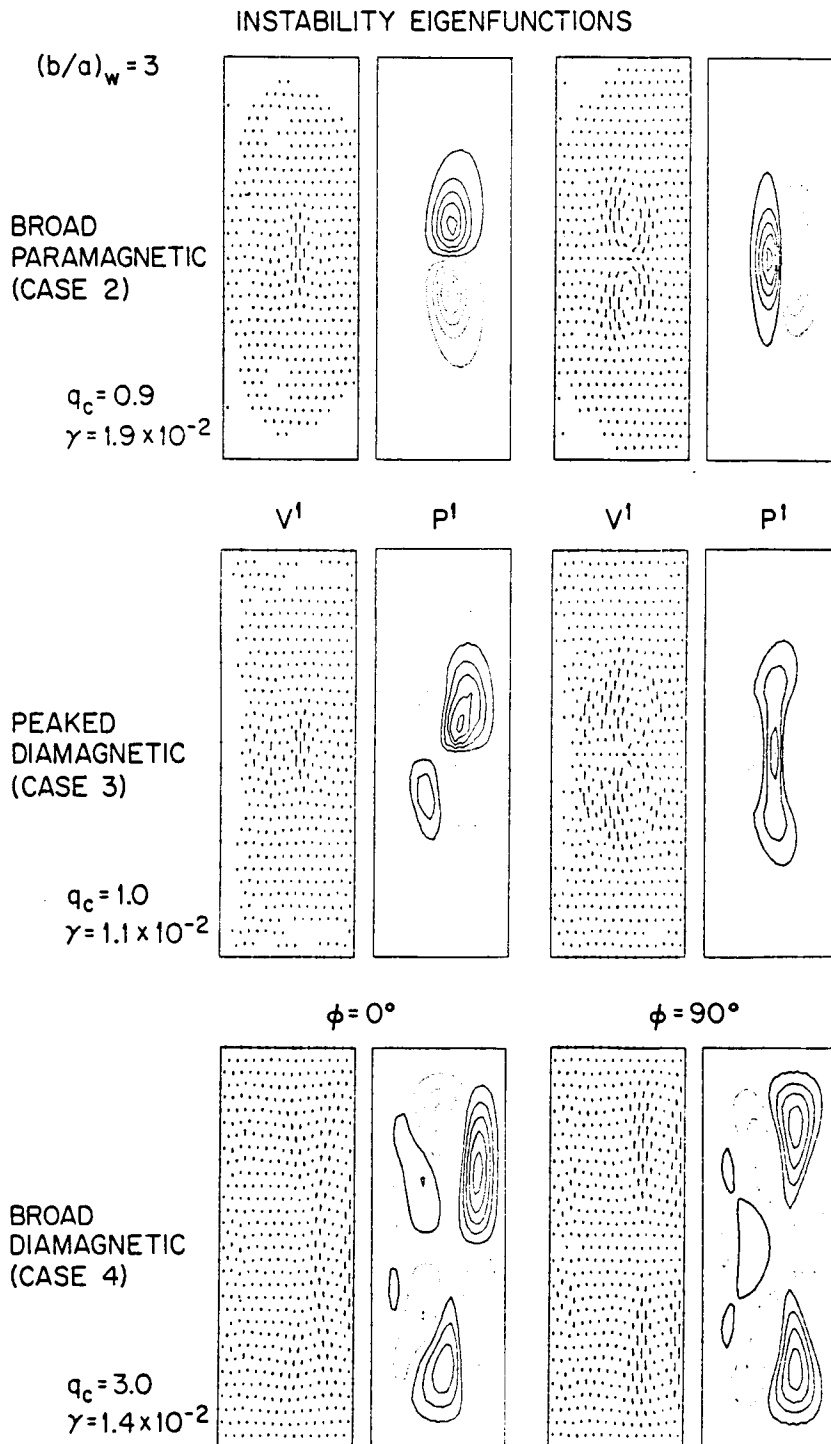


Figure III.7. Spatial structure of the velocity field and the perturbed pressure. Each is shown at two cross sections a quarter of a wavelength apart, for instabilities near the marginal points of three cases studied in this paper. Contours for positive (solid lines) and negative (dotted lines) perturbed pressures are shown. From top to bottom, the instabilities exhibit an increasingly ballooning character.

case has an aspect ratio of four, with the center line to the left). It follows that the abrupt increase in the marginal q value with broad-current diamagnetic equilibrium is accompanied by the onset of a strong ballooning character to the instability near the marginal point.

Elongation of the plasma cross section has the effect of greatly enhancing the ballooning character of the marginal instability, as shown in Fig. III.8. Perturbed pressure contours are shown here at two cross sections of the plasma for a broad-current diamagnetic equilibrium, corresponding to case 4, with $(b/a)_w = 1.5, 2, \text{ and } 3$.

In summary, then, a strongly ballooning character to the marginal instability and a shift of the marginal q value well above unity sets in for highly elongated equilibria $(b/a)_w$ with a broad toroidal current profile as soon as the plasma becomes diamagnetic ($1 < \beta_p < 2$).

3. Maximum Stable Beta

It is clear that a compromise must be struck if the volume-averaged beta ($\langle \beta \rangle \equiv 2\mu_0 \langle p \rangle / B_t^2$) is to be increased by elongating the plasma cross section, increasing diamagnetism, or making the toroidal current profile broader. For example, if the edge of a paramagnetic plasma with peaked profile is elongated, the volume-averaged beta will decrease because the central flux surfaces, where the pressure is the highest, will remain essentially circular while the additional space near the ends of the elongated wall will largely be filled with low-pressure plasma. If the diamagnetism is increased or the current profile made broader in a highly elongated plasma, the inner flux surfaces become more elongated and the volume of the high pressure region increases to

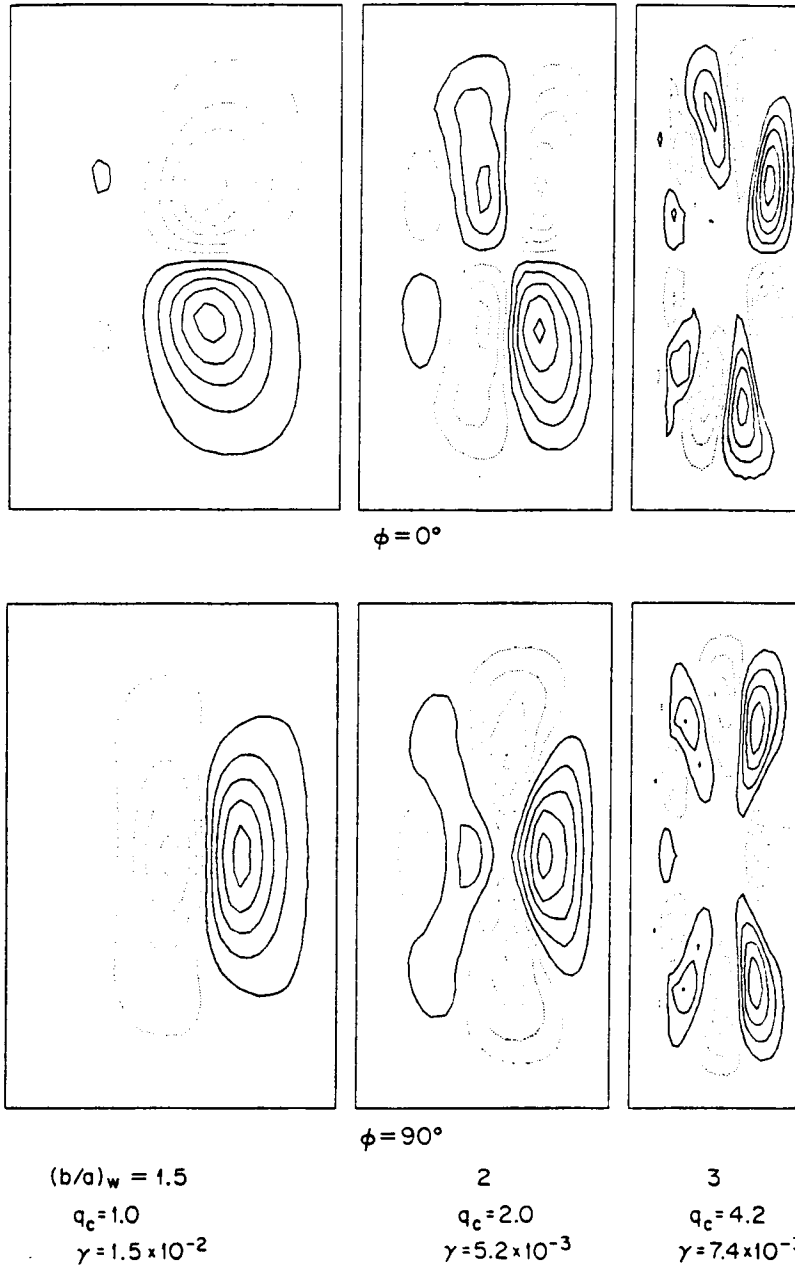
INSTABILITIES WITH DIFFERENT $(b/a)_w$ 

Figure III.8. Perturbed pressure contours for the broad-current diamagnetic equilibrium corresponding to case 5. Instabilities near the marginal point are shown at cross section a quarter of a wavelength apart. From left to right, the elongation is increased while the aspect ratio is held fixed at $R/a = 4$ (the center line is to the left).

make more efficient use of the available space, but the marginally stable q value increases dramatically so that the toroidal current and, therefore, the volume-averaged beta must be decreased substantially in order to insure stability. Hence, the optimum volume-averaged beta is achieved with a broad current profile equilibrium that is either diamagnetic with moderate elongation or paramagnetic with high elongation.

For the equilibria used in this paper, with an aspect ratio of four and variable elongation, the values of poloidal beta (defined in Eqs. (II.7)-(II.9), q value at the magnetic axis, ratio of elongations $(b/a)_a/(b/a)_w$, and volume-averaged beta near the marginal point are presented in Table II. Since it is not always possible to make an accurate extrapolation of each growth rate curve to the marginal stability point, all the values listed in Table II refer to an extrapolation of the growth rate curves to the growth rate value $\gamma = 0.007$. This is nominally the lower bound on the accuracy of our computer code with the grid size and time available. The grid size is relevant because the marginally stable eigenfunctions tend to have finer spatial structure than the eigenfunctions of more rapidly growing instabilities.

From Table II, it can be seen that the volume-averaged beta is only slightly increased by elongation in the peaked-current paramagnetic case. This is because the inner flux surfaces become only moderately elongated [$(b/a)_a = 2.08$ when $(b/a)_w = 4$] at a nearly fixed q value on axis so that the pressure on axis rises while the fraction of space

Table II. Equilibrium conditions near marginal stability

ASP = 4

GROWTH RATE = 0.007

CASE	$(b/a)_w$	$(b/a)_a$	MARGINAL q_c	β_p	$\beta_{av}(\%)$
CASE 1 PEAKED PARA	2	1.70	0.85	0.82	1.0
	3	1.98	0.88	0.83	1.2
	4	2.08	0.90	0.83	1.2
CASE 2 BROAD PARA	2	1.98	0.95	0.78	1.8
	3	2.76	1.00	0.81	3.0
	4	3.40	1.70	0.82	1.4
CASE 3 PEAKED DIA	2	1.90	0.98	1.38	2.1
	3	2.40	1.12	1.42	2.1
	4	2.64	2.00	1.50	0.6
CASE 5 BROAD DIA	1.5	1.73	1.20	1.20	2.8
	2	2.44	1.90	1.18	1.9
	3	4.32	4.50	1.13	1.2

wasted on low-pressure plasma increases at nearly the same rate. Better results are obtained in the broad-current paramagnetic example of case 2. Here, the central flux surfaces elongate together with the elongation of the wall. However, the maximum volume-averaged beta (3.0%) is achieved with $(b/a)_w = 3$ because the marginal q value at the magnetic axis rapidly increases with further elongation.

In the peaked-current diamagnetic example of case 3, the rise in the elongation of the magnetic axis competes with the marginal q value at the magnetic axis so that the volume-averaged beta remains essentially fixed up to elongations of $(b/a)_w = 3$. With further elongation of the wall, the central flux surfaces do not elongate proportionally and the q value on axis rises rapidly so that the volume-averaged beta decreases substantially.

The last entries in Table II pertain to an example of a broad-current diamagnetic plasma with less diamagnetism [smaller value of coefficient C in Eq. (II.6)] than in case 4 so that marginal stability points could be found for elongations of the wall up to three. In this case, the elongation of the central flux surfaces increases faster than the elongation of the wall, so that very effective use is being made of the volume. However, the marginal q value at the magnetic axis increases rapidly with elongation so that the volume-averaged beta decreases monotonically with elongation of the wall.

From this study it follows that the maximum stable volume-averaged beta is achieved with highly elongated paramagnetic plasmas with broad toroidal current profile or with diamagnetic plasmas with moderate or little elongation.

Up to this point we have mainly studied the effects of current profile on plasma stability with the fixed pressure profile as a function of ψ [e.g., as in Eq. (II.5) with $N = 1$]. The overall effect of current profile on stability and critical beta can be summarized as the following; a toroidal current profile determines the poloidal flux function in such a way that a broad current profile increases the elongation of the inner flux surfaces. Because of the dependence of the pressure profile on the poloidal flux, the spatial pressure profile is broadened by increasing the breadth of the current profile. A broader current profile induces more paramagnetic poloidal current to keep the plasma in equilibrium and then shifts the plasma inward of the torus for any fixed $p(\psi)$.⁵³ Therefore, broad current profile enhances the critical beta not only by reducing the driving force of ballooning mode but also by increasing the effective volume of high pressure. But an extremely broad current profile worsens the stability by raising the pressure gradient near the plasma edge and decreasing the global shear. This current profile effect is not confined to any special value of N in Eq. (II.5). Higher beta can be made by reducing N in Eq. (II.5),¹⁵ but the effect of changing the breadth of the current profile will remain qualitatively the same as presented in this paper. Similar effects have also been observed in studies now underway using the BALOON code.^{24,25} More detailed shaping and optimization of the profiles may further improve the volume-averaged beta but is not likely to reverse the trends indicated here.

3. INTERPRETATION AND COMPARISON WITH OTHER WORK

The research reported here is a computational study of fixed-boundary ideal magnetohydrodynamic instabilities in a toroidal plasma with vertically elongated, perfectly-conducting rectangular walls. We are particularly interested in the transition between internal kink instabilities, which are generally localized to the region near the magnetic axis of the plasma, to ballooning modes, which are generally strongest around the outer edge of the toroidal plasma away from the magnetic axis. There are many other research papers reporting analytic, computational, or experimental studies of similar large-scale instabilities in elongated plasmas. In this section, we shall show how our results overlap with or can be interpreted in terms of a selection of these other papers. Only those references concerned with highly elongated cross sections ($b/a > 2$), distributed current, and the ideal magnetohydrodynamic model will be considered here.

We can estimate the conditions needed for the onset of pressure-driven instabilities near the outer edge of a highly elongated toroidal plasma from the following inequality:

$$\frac{2}{R} \left| \frac{dp}{dR} \right| \leq \max \left[k_{\parallel}^2 B^2; \frac{1}{4} B_p^2 \left(\frac{1}{q_L} \frac{dq_L}{dR} \right)^2 \right], \quad (\text{III.1})$$

where R is the major radius, $k_{\parallel}$ is the wave number parallel to the magnetic field, and

$$q_L \equiv 2bB_t/2\pi RB_p \quad (\text{III.2})$$

is a measure of the helical pitch (q value) of the field lines along just the outer edge of the toroid. The expression on the left is the driving term due to a pressure gradient in the presence of toroidal curvature. The first expression on the right approximates the restoring force due to field line bending when the perturbation is a ballooning mode with one-half wavelength equal to the length along the field lines around the outer edge of the toroid

$$k_{\parallel} \approx \frac{1}{(2q_L R)} \quad (\text{III.3})$$

The second term on the right is the restoring force due to shear stabilization of a perturbation localized near a mode rational surface, as in the derivation of the Suydam criterion.⁵² A more exact combination of these stability criteria, involving both field line bending and shear stabilization, can be derived from the $n \rightarrow \infty$ limit of the energy principle,^{24,25} as will be shown in the next chapter. In Eqs. (III.1)-(III.3), the elongated toroidal plasma is approximated by a cylindrical slab with the end effects neglected.

In the spirit of dimensional analysis, Eq. (III.1) yields the approximate stability criterion

$$(\beta_p)_{\max} \lesssim \max \left[\frac{1}{4} \frac{R}{a} \left(\frac{\pi}{b/a} \right)^2 \frac{a_p}{a}; \frac{1}{4} \frac{R}{a_B} \frac{a_p}{a_B} \right], \quad (\text{III.4})$$

where the scale lengths a_p and a_B are defined by

$$a_p \equiv p_{\max} / \left| \frac{dp}{dR} \right|_{\max}, \quad (\text{III.5})$$

$$a_B \equiv (B_p)_{\max} / \left| \frac{dB_{\text{pol}}}{dR} \right|_{\max}, \quad (\text{III.6})$$

and $(\beta_p)_{\max}$ is

$$(\beta_p)_{\max} \equiv 2\mu_0 p_{\max} / (B_p^2)_{\max}. \quad (\text{III.7})$$

When the gradient scale lengths are comparable to the minor radius of the plasma ($a_p \sim a_B \sim a$), and the plasma is highly elongated, $b/a \gtrsim 3$, it can be seen that the poloidal beta must be small, $(\beta_p)_{\max} \lesssim 1$, for stability of the $R/a = 4$ plasma studied in this paper; the volume-averaged poloidal beta is even smaller. Elongation makes ballooning modes more unstable because there is less restoring force due to field line bending as the connection length becomes longer. Under these conditions, there is little to be gained by further elongation. A similar conclusion was reached by Freidberg et al.⁹ using a surface current model of the plasma.

Now let us consider other stability criteria that have appeared in the literature. Lortz and Nührenberg³ have evaluated a rigorous pair of necessary and sufficient stability criteria in the neighborhood of the magnetic axis for a toroidal plasma with any cross sectional shape and

poloidal beta. They found favorable stability for highly elongated (at the magnetic axis) diamagnetic plasmas with D shape (triangularity pointing away from the toroidal center line) for moderate q values at the magnetic axis. We find that highly elongated, diamagnetic plasmas are violently unstable even at large q values. The fact that the shape of our boundary conditions unavoidably produces flux surfaces with unfavorable triangularity (with the flat side pointing outward) probably contributes to this instability. However, a detailed study of the driving term for the most unstable mode has revealed that the region driving the instability shifts outward away from the magnetic axis, and the instability becomes predominantly pressure driven rather than parallel current driven as the plasma is made more elongated and diamagnetic. Therefore, as Lortz and Nührenberg pointed out, neither a necessary nor a sufficient stability criterion evaluated at the magnetic axis can be used to find instabilities such as ballooning modes, which are driven in regions away from the magnetic axis.

In our study of the instability driving terms, we evaluated the last two terms (separately) of Eq. (2) in Todd et al.²³ using the instability eigenfunctions determined by our initial value instability code. For the peaked profile paramagnetic equilibrium (case 1) with $(b/a)_w = 3$, the kink term is much larger than the ballooning term and both are localized near the magnetic axis. The broad-current paramagnetic equilibrium (case 2) shows a mix of parallel current and pressure-driven terms. The current-driven term is much larger than the pressure-driven term on the plane where the instability velocity field is mainly in the vertical direction

($\phi = 0$ in Fig. III.7, page 40). However, the current- and pressure-driven terms are more comparable a quarter of a wavelength around the toroid where the velocity field has a larger component along the major radius. In a toroidal environment, the mix of driving terms can vary around each flux surface.

The kink term dominates the ballooning term in the peaked-current diamagnetic equilibrium (case 3). Both kink and ballooning terms are localized near the magnetic axis, but the spatial distribution of the ballooning term is weighted more toward the outer edge of the toroid.

In the broad-current diamagnetic equilibrium (case 4), in which the instability had a strongly ballooning character, both the kink and the ballooning driving terms are distributed off the magnetic axis near the outer wall of the toroid. The pressure-driven term is several times stronger than the kink term; again, there is a strong dependence on the toroidal angle. From these observations, it can be seen how important the pressure-driven term can be in the region away from the magnetic axis for the stability of diamagnetic toroidal plasmas with broad current profiles and elongated cross sections.

The results of Laval et al.¹⁰ and Grossmann et al.¹¹ are primarily concerned with free-boundary instabilities, which are not considered in this paper. Assuming $\beta_p \sim 0$ (1), large q value at the edge of the plasma and large aspect ratio, Laval et al., find that stability is improved with a more peaked-current profile, in general agreement with our results. There is no evidence, however, that they were observing a transition between kink and pressure-driven ballooning behavior. In the

work of Grossmann et al.¹¹ the assumption that the elongation b/a scales like $\epsilon^{-1/2}$, where $\epsilon \equiv a/R$ is the reciprocal of the aspect ratio, restricts their attention to relatively broad-current profiles where the inner flux surfaces are highly elongated. Under these conditions, there is general agreement that the equilibria are unstable.

There has been a series of experiments⁵⁴⁻⁵⁸ in which highly elongated diamagnetic toroidal plasmas have been formed by shock heating and maintained for many Alfvén transit times. In the belt pinch experiments,⁵⁴⁻⁵⁸ large-scale stability is observed even in highly elongated discharges $[(b/a)_w \sim 5 \text{ to } 10]$, when the q value at the wall is greater than three. An $m = 1$ instability is observed at lower q values, although this instability is observed to saturate and then go away when the q value decreases below and then increases above the instability limit during the discharge. In high density stable discharges, filamentary structures are observed around the outer side of the plasma coincident with the helical pitch of the magnetic field lines.⁵⁴ The filaments decay as the plasma elongation contracts and as the temperature and diamagnetic currents decay. These observations are consistent with the picture that ballooning modes begin to grow and cause a filamentation of the outer side of the plasma while the plasma is diamagnetic and elongated with a broad toroidal current profile; but these ballooning modes saturate and decay as the plasma pressure and elongation decay. This interpretation is certainly reasonable to Belt Pinch I investigated by Krause, in which the plasma temperature decayed to <5 eV within 10 sec. In the larger and more optimized device, Belt

Pinch IIa, the filamentary structure is observed only in the high compression case.⁵⁶ The filaments are not observed optically in the low compression discharges in Belt Pinch IIa, but the experimental observations do not rule out the possibility of ballooning modes.⁵⁶ In particular, fluctuating magnetic fields and enhanced energy losses are observed under these conditions.

Another experiment which has produced plasmas with elongated cross section is the T-9 "Finger ring tokamak."⁵⁹⁻⁶¹ A copper shell whose cross section has the form of a segment of a circle is fitted inside the vacuum vessel to form the shape of the plasma column. During the formation of plasma, the plasma column becomes elongated in the vertical direction with elongation $(b/a)_w \geq 2$.⁶⁰ After the plasma has formed, plasma enters the various instability regions depending on the plasma density and plasma current.⁶¹ It is found that the stability region has three boundaries, as in an ordinary tokamaks; the left boundary which corresponds to low plasma densities, the upper boundary which corresponds to the maximum current in the plasma, and the right boundary which corresponds to high plasma densities. The processes operating at the left and at the upper boundary of the stability region in T-9 are generally similar to those in ordinary tokamaks. However, there are several characteristics on the right boundary of the stability region which are not observed in circular tokamaks, e.g., the different dependence of the plasma density on the plasma current, the increase in the oscillation of the plasma density, the confinement of the oscillation to the outer edge of plasma, the increase in the elongation

and safety factor q . Bortnikov et al.⁶¹ make use of Lehnert's⁶² study to explain the characteristics on the right boundary and believe that these differences are due to the onset of a ballooning instability. Further detailed study is necessary to reach their conclusion because the Lehnert's work⁶² did not take into account the shear or toroidal nature of the magnetic field. But their conclusion can be supported by our results that elongation enhances the ballooning character with relatively small β_p (≥ 1). There are other experiments^{63,64} of elongated plasma and their results consistently show the improvements of the energy confinement time and the MHD properties. But the study of MHD behavior in high beta plasma is just in an early stage.

4. CONCLUSION

Under the influence of internal $n = 1$ kink and ballooning modes in elongated toroidal equilibria, the maximum stable volume-averaged beta is achieved with a broad-current profile and either a paramagnetic plasma ($\beta_p < 1$) with a highly elongated cross section ($b/a > 2$) or a diamagnetic plasma ($\beta_p > 1$) with only a mildly elongated cross section ($b/a < 2$). These observations are summarized in Table II, page 44. Internal ballooning modes are most easily observed in highly elongated diamagnetic plasmas with $q_{axis} > 1$ and a broad-current profile. While parallel current has some destabilizing effects on these modes, the dominant driving mechanism is the pressure gradient in the toroidal curvature of the magnetic field lines.

Theoretical evidence suggests that these pressure-driven instabilities near the edge of the plasma can be very sensitive to the boundary conditions. If they are present under fixed-boundary conditions, they will certainly be present and will probably be much stronger under free-boundary conditions. Also, it is likely that a resistive instability theory will indicate the onset of resistive ballooning instabilities resulting in a deterioration of plasma confinement at lower pressures than the onset of the ideal instability. Therefore, the fixed-boundary ideal magnetohydrodynamic theory presented here probably predicts an upper bound on the pressure we can expect to confine in elongated toroidal plasmas.

CHAPTER IV

ANALYTIC AND COMPUTATIONAL STUDY OF STABILITY OF HIGH n BALLOONING MODE IN HIGHLY ELONGATED ELLIPTIC PLASMA

In this chapter we modify the ordinary differential equation for marginal stability against the high n ballooning mode^{24,25} to express the marginal pressure gradient as a function of equilibrium parameters, shift, toroidicity, elongation, and safety factor, etc. For equilibrium we follow the work done by Clarke and Sigmar²⁰ and by Mizoguchi et al.³⁷ without further solving the Grad-Shafranov equilibrium equation. Simple test functions are used to calculate the marginal pressure gradient. The effects of high elongation on the stability are primarily studied and the roles of other parameters are also considered. The analytic results are compared with the computational results using ORNL BALOON code. The effects of elongation, aspect ratio, shear, and pressure profile are shown in (β, β_p) parameter space.

1. VARIATIONAL FORM OF EQUATION FOR MARGINAL STABILITY OF HIGH n BALLOONING MODES

An ordinary differential equation for marginal stability of high n ballooning modes by Connor et al.²⁵ is modified to a variational form to study the elongation effect on stability using a simple test function. The orthogonal flux coordinate system used is defined by the metric

$$(ds)^2 = \left(\frac{d\psi}{RB_p} \right)^2 + (JB_p d\chi)^2 + (Rd\zeta)^2 . \quad (\text{IV.1})$$

Here ψ is a poloidal flux function, χ is a generalized poloidal angle, ζ is a toroidal angle, and J is Jacobian. The ordinary differential equation for marginal stability²⁵

$$\begin{aligned} \frac{1}{J} \frac{d}{dy} \left\{ \frac{1}{JR^2 B_p^2} \left[1 + \left(\frac{R^2 B_p^2}{B} \int \frac{\partial v}{\partial \psi} dy \right)^2 \right] \frac{dF}{dy} \right\} \\ + \frac{2}{RB_p} \frac{dP}{d\psi} \left\{ K_n - \frac{I(\psi) R B_p^2}{B^2} K_s \int^y \frac{\partial v}{\partial \psi} dy \right\} F = 0 \end{aligned} \quad (\text{IV.2})$$

is used to construct the variational form,

$$\frac{\int_{-\infty}^{\infty} \frac{1}{JR^2 B_p^2} \left\{ 1 + \left[\frac{R^2 B_p^2}{B} \int^y \left(\frac{\partial v}{\partial \psi} \right)_y dy \right]^2 \right\} \left| \frac{dF}{dy} \right|^2 dy}{\int_{-\infty}^{\infty} \frac{2J}{RB_p} \left\{ K_n - \frac{I(\psi) R B_p^2}{B^2} K_s \int^y \left(\frac{\partial v}{\partial \psi} \right)_y dy \right\} |F|^2 dy} \geq \left| \frac{dP}{d\psi} \right| . \quad (\text{IV.3})$$

Here K_n , K_s are normal and geodesic curvature defined as

$$\begin{aligned} K_n &= \frac{\nabla \psi}{|\nabla \psi|} \cdot (\hat{B} \cdot \nabla) \hat{B} , \\ K_s &= \frac{\nabla \psi \times \vec{B}}{|\nabla \psi| |\vec{B}|} \cdot (\hat{B} \cdot \nabla) \hat{B} . \end{aligned} \quad (\text{IV.4})$$

B_p is poloidal B field, $I(\psi) = RB_T$ (B_T is toroidal B field), and F is a test function corresponding to $RB_\chi \xi_\psi$ where ξ_ψ is the component of the displacement normal to the flux surface. The generalized poloidal angle χ is replaced with variable y whose range is from $-\infty$ to $+\infty$. v is defined as

$$v = \frac{JB_T}{R} . \quad (IV.5)$$

For equilibrium, we follow the work done by Clarke and Sigmar,²⁰ and Mizoguchi et al.³⁷ Several assumptions are made to simplify the problem and isolate the effects of each equilibrium parameters from other effects. First, flux surfaces are taken to be vertically elongated ellipses, each with different elongation and shift; second, the aspect ratio is taken to be very large so that $a/R_c \equiv \epsilon \ll 1$; third, the elongation $e(\equiv b/a)$ is taken to be large as order of $e \sim O(\epsilon^{-1/2})$; fourth, flux function is assumed to be more or less parabolic.

The assumption of elliptic cross section of each flux surface makes ψ function of $\bar{\rho}^2$, i.e.,

$$\psi = S(\bar{\rho}^2) \quad (IV.6)$$

$$\bar{\rho}^2 = x^2 + \frac{1}{e^2} z^2$$

with

$$x = R - R_c - \delta(\psi) \quad (IV.7)$$

$$e = b/a$$

where e is the elongation, $\delta(\psi)$ is the shift of a flux surface ψ and R_c is the geometric center of the plasma. (See Fig. IV.1.) ψ is assumed to increase with the increase of minor radius $a(\psi)$. B_p , B_t , and $q(\psi)$ are calculated from the relations $RB_p = |\nabla\psi|$, $RB_t = I(\psi)$, and $q(\psi) = \frac{1}{2\pi} \oint \frac{JB_t}{R} d\chi$, respectively. If we use elliptic angle θ on a flux surface ψ such as

$$x = a(\psi)\cos \theta , \quad (IV.8)$$

$$z = b(\psi)\sin \theta ,$$

B_p , B_t are expressed with (ψ, θ, ζ) by neglecting terms with order higher than $O(\epsilon^2)$,

$$B_p \approx \frac{2\dot{S}a}{R_c} \left[\cos^2\theta + \frac{1}{e^2} \sin^2\theta \right]^{1/2} \left\{ 1 - \left(\frac{a}{R_c} + 2\delta'\dot{S}a \right) \cos\theta - \frac{2e'}{e} \dot{S}a^2 \sin^2\theta \right\}, \quad (IV.9)$$

$$B_t \approx \frac{2\dot{S}q(\psi)}{e} \left(1 - \frac{2e'}{e} \dot{S}a^2 - \frac{a \cos \theta}{R_c} \right),$$

where

$$\dot{S} \equiv \frac{dS}{d\rho^2}, \quad e' \equiv \frac{de}{d\psi}, \quad \text{and} \quad \delta' \equiv \frac{d\delta}{d\psi}.$$

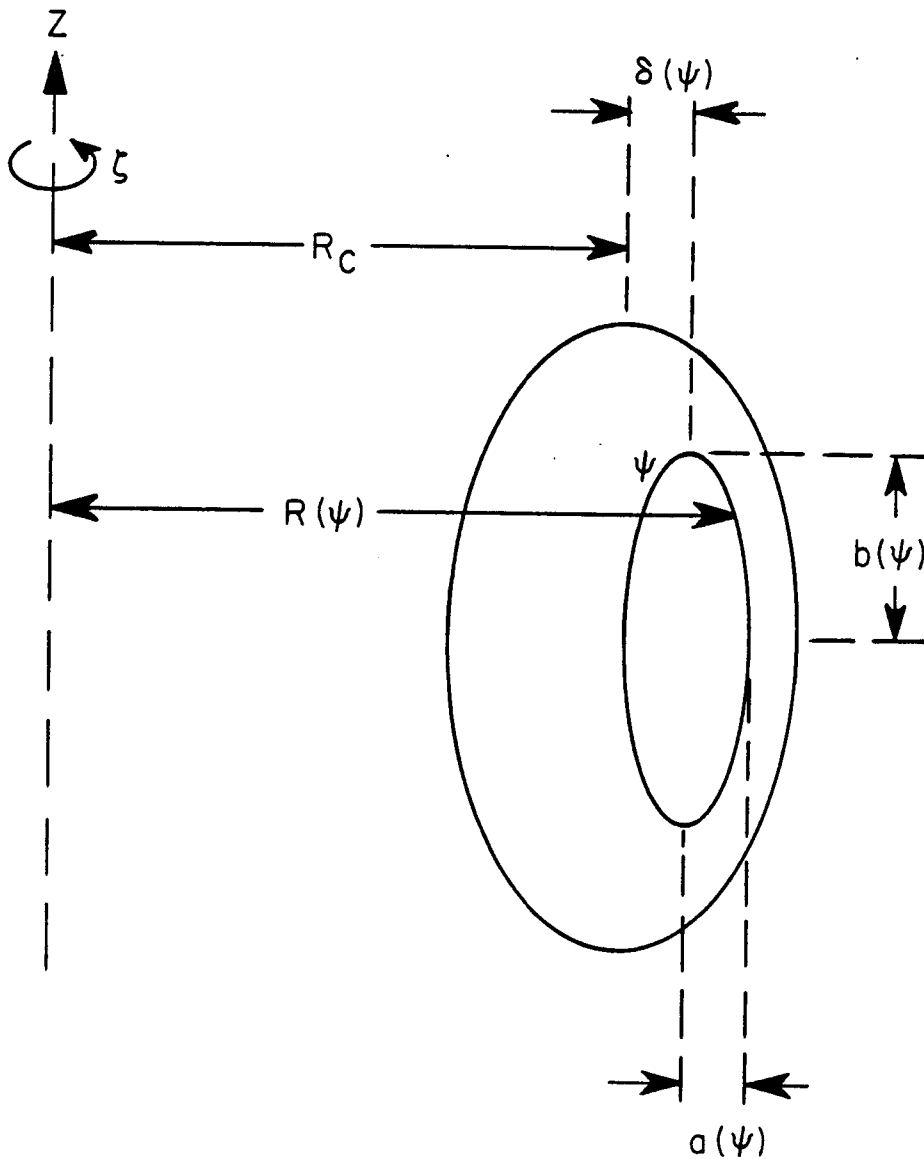


Figure IV.1. An elliptically elongated poloidal flux surface ψ shifted as $\delta(\psi)$ from the geometric center of the plasma cross section. $a(\psi)$ and $b(\psi)$ are minor and major radius of the elliptic surface ψ . R_c is a distance from the axis of the torus to the geometric center of the plasma cross section.

Poloidal flux function ψ is assumed minimum at the magnetic axis and maximum at the plasma edge. The transformation of the coordinate system from any orthogonal flux coordinate system (ψ, χ, ζ) to the nonorthogonal flux coordinate system (ψ, θ, ζ) in Eq. (IV.3) can be simply done by noticing that once χ is chosen on a given flux surface it is then determined on all the flux surfaces for a given equilibrium by the equation

$$\frac{R^2}{J} \frac{\partial}{\partial \psi} \left[J B_p^2 \right] = R^2 P' + II'(\psi) \quad (\text{IV.10})$$

and the relations

$$J = \frac{ae}{B_p} \left\{ \cos^2 \theta + \frac{1}{e^2} \sin^2 \theta \right\}^{1/2} \left(\frac{\partial \theta}{\partial \chi} \right)_{\psi}, \quad (\text{IV.11})$$

$$d\chi = \left(\frac{\partial \chi}{\partial \theta} \right)_{\psi} d\theta.$$

However, the transformation of coordinate system in $\int \left(\frac{\partial v}{\partial \psi} \right)_{\chi} d\chi$ is not simple because $\left(\frac{\partial v}{\partial \psi} \right)_{\chi} \neq \left(\frac{\partial v}{\partial \psi} \right)_{\theta}$. Direct calculation should be done by expressing v with (ψ, θ) and using Eq. (IV.11). The detailed calculation is given in the Appendix. The result, as a function of (ψ, θ, ζ) , is

$$\int^y \left(\frac{\partial v}{\partial \psi} \right)_y dy = S_1 \theta + S_2 \frac{\tan \theta}{1 + (1/e^2) \tan^2 \theta} + \sigma. \quad (\text{IV.12})$$

S_1, S_2 are zero order coefficients and stand for global shear and a geometric effect (or residual shear⁶⁶), respectively. σ is a first-order

correction and is

$$\sigma \equiv S_3 \sin \theta + S_4 \sin 2\theta + S_5 e \tan^{-1} \left(\frac{1}{e} \tan \theta \right) . \quad (\text{IV.12'})$$

The coefficients $S_1, S_2, \dots, S_5$ are shown in the Appendix. The term $\int \left(\frac{\partial v}{\partial \psi} \right)_y dy$ plays an important role in determining the stability for highly elongated plasmas. Explanation about the meaning of this term will be given later.

2. MARGINAL PRESSURE GRADIENT

In the previous section we set up an equilibrium for the calculation of marginal pressure gradient. In this section we calculate a marginal pressure gradient by determining a test function $F(\theta)$. The test function is chosen for simplicity of the calculation while giving a severe restriction for stability. To meet the requirement we choose between two simple test functions defined as follows:

$$F_1(\theta) = \begin{cases} 1 + \cos \theta & -\pi \leq \theta \leq \pi \\ 0 & \text{elsewhere} \end{cases} \quad (\text{IV.13})$$

$$F_2(\theta) = \begin{cases} 1 + \cos 2\theta & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases} .$$

These test functions would give optimistic stability criteria because of the finite range of θ .

For the test function $F_1(\theta)$, the marginal pressure gradient $\left| \frac{dP}{d\rho^2} \right|$,

$$\left| \frac{dP}{d\rho^2} \right|_1 \leq \left(\frac{\dot{S}^2}{R_c a} \right) \frac{\left\{ \frac{1}{e} + \frac{(\dot{S}a^2)^2}{2\pi} \left[10 \left(\frac{q'}{q} \right)^2 + \frac{2.4}{(\dot{S}a^2)^2} + 2.4 \left(\frac{q'}{q} \right) \frac{1}{\dot{S}a^2} + \Delta_1 \right] \right\}}{1 + \frac{5}{3} \dot{S}a^2 \left(\frac{q'}{q} \right) + \Delta_2}.$$

(IV.14)

It is compared with $\left| \frac{dP}{d\rho^2} \right|_2$ for the test function $F_2(\theta)$,

$$\left| \frac{dP}{d\rho^2} \right|_2 \leq \frac{8\pi\dot{S}^2}{5R_c a} \frac{\left\{ \frac{1}{e^2} + \frac{(\dot{S}a^2)^2}{2\pi} \left[1.6 \left(\frac{q'}{q} \right)^2 + \frac{0.8}{(\dot{S}a^2)^2} - 2.1 \left(\frac{q'}{q} \right) \frac{1}{\dot{S}a^2} + \Delta_3 \right] \right\}}{1 + 0.4 \dot{S}a^2 \left(\frac{q'}{q} \right) + \Delta_4}.$$

(IV.15)

Here Δ_1 , Δ_2 , Δ_3 and Δ_4 are correction terms with toroidicity, shift, and variation of inner elongation. Their expressions will be shown and their effects will be discussed later.

An appropriate test function is chosen by comparing $\left| \frac{dP}{d\rho^2} \right|_1$ and $\left| \frac{dP}{d\rho^2} \right|_2$ for equilibria with different shear. For the equilibria with reasonably high shear, i.e., $\left(\frac{q'}{q} \right) \dot{S}a^2 \geq 1$ and $1/e^2 \approx 0.1$ it is found that $\left| \frac{dP}{d\rho^2} \right|_2 < \left| \frac{dP}{d\rho^2} \right|_1$. In this case, the test function $F_2(\theta)$ is an appropriate test function since it more nearly minimizes the variational form. But for the equilibria with very low shear, i.e., $\left(\frac{q'}{q} \right) \frac{1}{\dot{S}a^2} \ll 1$, $F_1(\theta)$ is an appropriate one because $\left| \frac{dP}{d\rho^2} \right|_1 < \left| \frac{dP}{d\rho^2} \right|_2$. Since usual tokamaks have reasonably high shear, we concentrate on the marginal pressure gradient $\left| \frac{dP}{d\rho^2} \right|_2$, Eq. (IV.15), for further study.

3. INTERPRETATION FOR STABILITY

The stability of highly elongated elliptic plasmas is expressed with Eq. (IV.15). The first parenthesis on the right side of Eq. (IV.15) shows the effects of toroidicity ($\sim 1/R_c$) and poloidal flux profile ($\sim \dot{S}^2$) on the marginal pressure gradient. The numerator of Eq. (IV.15) corresponds to a stabilizing mechanism while the denominator corresponds to a driving mechanism. The driving mechanism has two effects; the first comes from normal curvature and the other from geodesic curvature. There are two effects in the stabilizing mechanism, the field line bending effect which decreases as $1/e^2$ with elongation e and the local shear effect.⁶⁶ The local shear effect has three terms, global shear $(q'/q)^2$, a geometric effect $\frac{1}{(\dot{S}a^2)^2}$, and the combined effect of global shear and geometry $\left(\frac{q'}{q}\right) \frac{1}{\dot{S}a^2}$. The last two terms, $\frac{1}{(\dot{S}a^2)^2}$ and $\left(\frac{q'}{q}\right) \frac{1}{\dot{S}a^2}$, do not have correspondence to the stabilizing mechanism in the circular tokamaks with concentric flux configurations.⁶⁵ The term $\frac{1}{(\dot{S}a^2)^2}$ contributes to a stabilization and is independent of global shear. The term $\left(\frac{q'}{q}\right) \frac{1}{\dot{S}a^2}$ contributes to a destabilization for usual tokamak condition ($q' > 0$) and the destabilization is large compared with other stabilization terms in local shear effect. The reduction of field line bending effect as $1/e^2$ and additional terms, $\frac{1}{(\dot{S}a^2)^2}$ and $\left(\frac{q'}{q}\right) \frac{1}{\dot{S}a^2}$, in local shear effect are characteristics of stability in highly elongated plasma.

The appearance of these additional effects can be explained by considering the term $\left.\frac{\partial v}{\partial \psi}\right|_y$ in Eq. (IV.2) for an elliptically elongated cylinder. It is found that the term $\left.\frac{\partial v}{\partial \psi}\right|_y$ is proportional to local shear⁶⁶

and a local intersection angle between two field lines on adjacent flux surfaces during an interchange of plasma. Letting Z stand for any surface quantity ($Z = Z(\psi)$), the local shear

$$\frac{\vec{B} \times \nabla Z \cdot \nabla X(\vec{B} \times \nabla Z)}{|\nabla Z|^4},$$

would vanish if there were a set of surfaces that contained both $\vec{B}$ and ∇Z . Greene and Johnson⁶⁶ expressed this local shear in Hamada coordinates⁴ as

$$\begin{aligned} \frac{\vec{B} \times \nabla Z \cdot \nabla X(\vec{B} \times \nabla Z)}{|\nabla Z|^4} &= \frac{1}{V'(Z)} [\phi'(Z)\psi''(Z) - \psi'(Z)\phi''(Z)] \\ &\quad - \vec{B} \cdot \nabla \frac{\nabla \phi \cdot \nabla \theta \quad \nabla \psi \cdot \nabla \eta}{|\nabla Z|^2} \end{aligned} \quad (\text{IV.16})$$

where $V(Z)$ is a volume inside a surface Z , ψ is poloidal flux function, and ϕ is toroidal flux function. θ increases by unity after one trip the short way around the torus, η increases by unity after one trip the long way. The first term of the left side of Eq. (IV.16) is global shear and the last term is called residual shear. Therefore, the local shear consists of two parts, global and residual shear. We can also express the local shear using the orthogonal flux coordinate by noticing that

$$\vec{B} = \nabla \psi \chi (\nu \nabla \chi - \nabla \zeta), \quad (\text{IV.17})$$

here ν is defined in Eq. (IV.5). Using Eq. (IV.17) we get

$$\begin{aligned}\vec{B} \times \nabla\psi &= |\nabla\psi|^2 (\nu\nabla\chi - \nabla\zeta) , \\ \frac{\nabla\chi(\vec{B}\times\nabla\psi)}{|\nabla\psi|^2} &= \nu'\nabla\psi \times \nabla\chi + \frac{\nabla|\nabla\psi|^2 \times (\nu\nabla\chi - \nabla\zeta)}{|\nabla\psi|^2} .\end{aligned}\tag{IV.18}$$

From Eqs. (IV.16)-(IV.18) the local shear is

$$\begin{aligned}\frac{\vec{B} \times \nabla\psi \cdot \nabla \times (\vec{B} \times \nabla\psi)}{|\nabla\psi|^4} &= -\nu'\nabla\zeta \cdot (\nabla\psi \times \nabla\chi) \\ &= -\frac{\nu'}{J} ,\end{aligned}\tag{IV.19}$$

here J is Jacobian and ν' appears in Eq. (IV.3) and Eq. (IV.11) and ψ has been used in place of Z . We find in this calculation that $-\nu'/J$ is the same as the local shear.

It will now be shown that the term ν'/J is also proportional to a local intersection angle of two field lines on adjacent flux surfaces during an interchange of plasma.

The local intersection angle is calculated by assuming that plasma motion has only a perpendicular component to a flux surface. In an elliptic cylinder the angle α_1 between a field line on ψ and the z axis of the cylinder is (see Fig. IV.2)

$$\tan \alpha_1 = \frac{B_p(\psi, \chi)}{B_t(\psi, \chi)} .\tag{IV.20}$$

The angle α_2 between a field line moving from $\psi - \delta\psi$ to ψ and the z axis is

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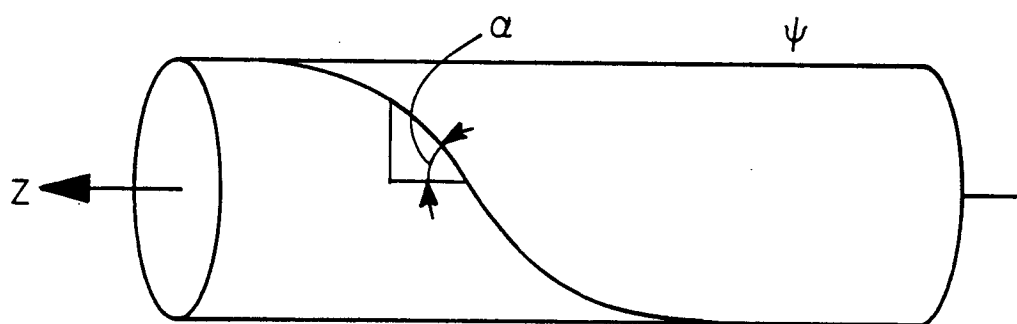


Figure IV.2. An angle α between a field line on an elliptically elongated cylindrical flux surface ψ and the z -axis.

$$\tan \alpha_2 = \frac{B_p(\psi - \delta\psi, \chi) + \frac{\vec{B}_p \cdot (\vec{\xi} \cdot \nabla) \vec{B}}{|\vec{B}_p|} + B_{1p}}{B_t(\psi - \delta\psi, \chi) + \frac{\vec{B}_t \cdot (\vec{\xi} \cdot \nabla) \vec{B}}{|\vec{B}_t|} + B_{1t}}, \quad (\text{IV.21})$$

by adopting a Lagrangian description of the plasma motion.⁴⁸ Here B_{1p} and B_{1t} are poloidal and toroidal components of $\vec{B}_1 = \nabla X(\vec{\xi} \times \vec{B})$. By neglecting displacement parallel to the flux surface and using Eqs. (IV.20), (IV.21), we get an intersection angle $\delta\alpha$,

$$\delta\alpha = \tan(\alpha_1 - \alpha_2) = \left[|\nabla\psi| \frac{\vec{\xi} \cdot \nabla\psi}{B_t^2} \right] \frac{v'}{J}. \quad (\text{IV.22})$$

Equation (IV.21) shows $\frac{v'}{J}$ is proportional to the local intersection angle. Consequently local shear is proportional to the local intersection angle.

Next, we consider v' evaluated for a special geometry, i.e., elliptically elongated toroid with very large aspect ratio to illustrate the effects of geometry and combined effect of geometry and global shear. Using Eqs. (IV.5), (IV.9), and (IV.10), and neglecting the plasma shifts and variation of inner elongation, we get

$$v = q \left(\frac{\partial\theta}{\partial\chi} \right)_\psi. \quad (\text{IV.23})$$

The derivative of v with ψ is

$$\left(\frac{\partial v}{\partial \psi}\right)_\chi = v' = q \left(\frac{\partial \theta}{\partial \chi}\right)_\psi \left\{ \left(\frac{q'}{q}\right) + \left(\frac{\partial \chi}{\partial \theta}\right)_\psi \frac{\partial}{\partial \psi} \left[\left(\frac{\partial \theta}{\partial \chi}\right)_\psi \right]_\chi \right\} \quad (\text{IV.24})$$

From Eq. (IV.16) and (IV.19), the first term (q'/q) on the right side of Eq. (IV.24) is global shear and the last term is residual shear, which has a strong dependence on geometry judging from the form of the term.

Consider the physical meaning of $\frac{\partial}{\partial \psi} \left[\left(\frac{\partial \theta}{\partial \chi}\right)_\psi \right]_\chi$ in an effort to explain the geometric effect on the stability. Figure IV-3 shows two adjacent elliptic flux surfaces $\psi - \delta\psi$ and ψ . A generalized poloidal angle χ is determined exactly the same as an elliptic angle θ on one particular flux surface $\psi - \delta\psi$. The deviation of these two angles is shown in the flux surface ψ . Figure IV.4 shows the relations between χ and θ on different flux surfaces. Because of the geometric effect, θ changes slowly with χ near the midpoint of a flux surface χ ($\theta \approx 0$) but changes rapidly near the top and bottom ($\theta \approx \pm \frac{\pi}{2}$). Figure IV.4 shows how $\left(\frac{\partial \theta}{\partial \chi}\right)_\chi$ changes for different adjacent flux surfaces, i.e., it measures $\frac{\partial}{\partial \psi} \left[\left(\frac{\partial \theta}{\partial \chi}\right)_\psi \right]_\chi$.

The stabilizing mechanism of geometry can be explained with Figs. IV.3 and IV.4. First, we consider that there is no global shear as $q' = 0$ and only consider the perpendicular displacement of plasma which mainly affects the stability. Shear stabilization comes from intersection of field lines on flux surfaces $\psi - \delta\psi$ and ψ when interchange appears by a perturbation. After interchange, plasma in a flux surface $\psi - \delta\psi$ takes positions on an adjacent flux surface ψ as shown in Fig. IV.3. Since the relative position of each element of plasma changes as shown in Fig. IV.4, there is an intersection angle between two field lines when interchange appears even when $q' = 0$. This is a stabilization due to

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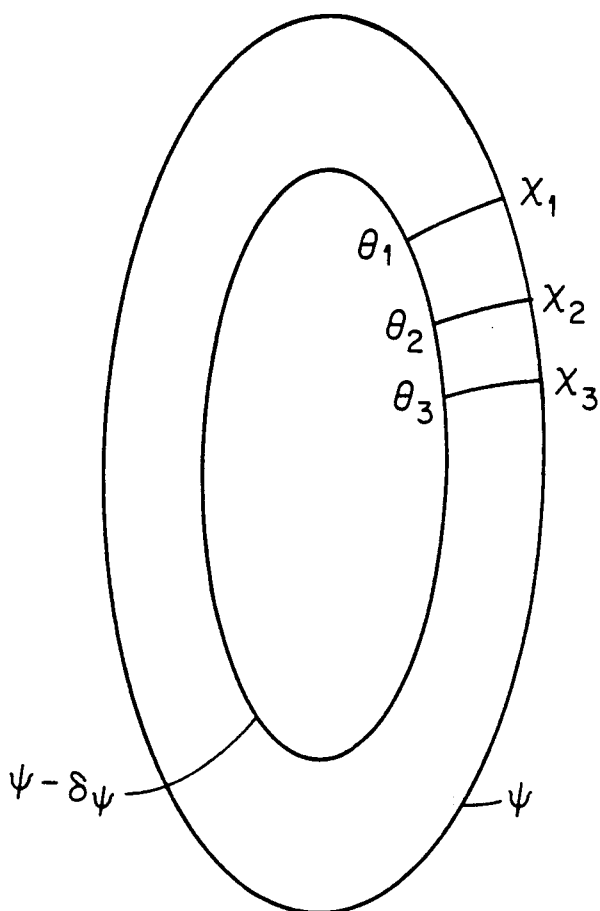


Figure IV.3. Two adjacent elliptic flux surfaces $\psi - \delta\psi$ and ψ . A generalized poloidal angle χ is determined exactly the same as an elliptic angle θ on the flux surface $\psi - \delta\psi$. The deviation of these two angles is shown in the flux surface ψ .

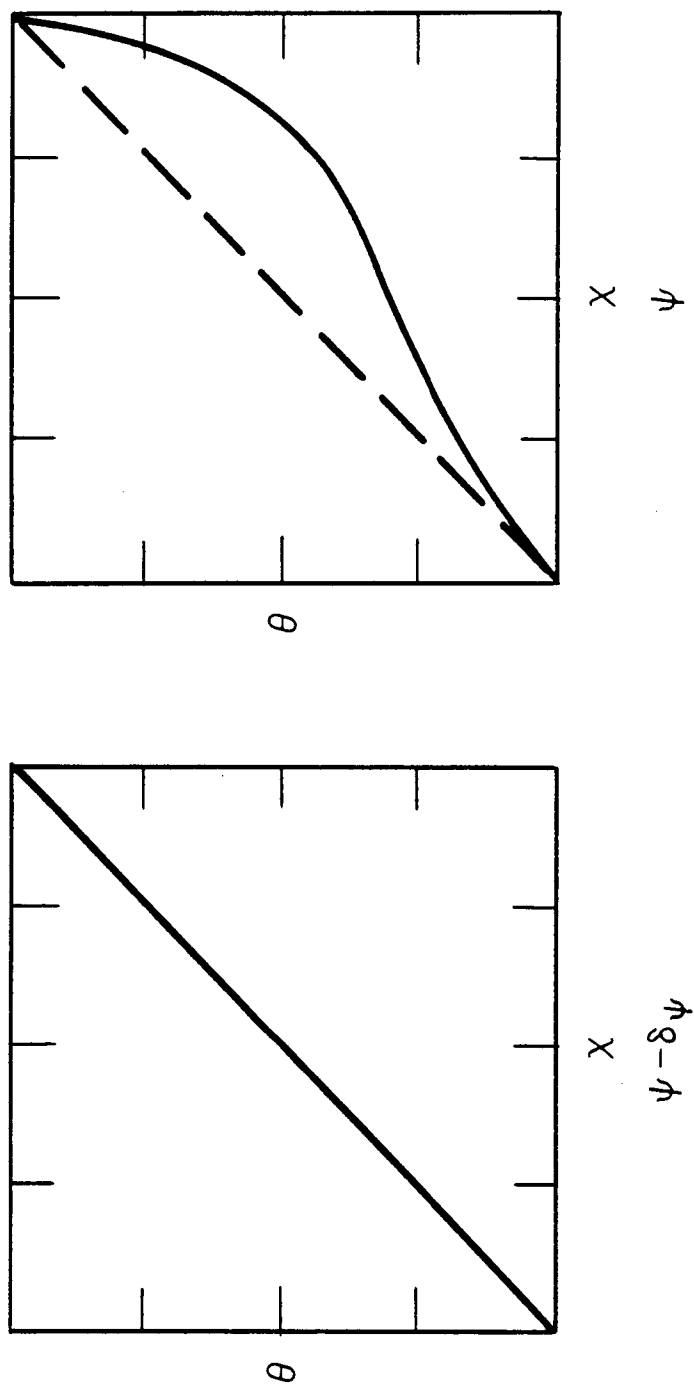


Figure IV.4. The relations between X and θ on two adjacent flux surfaces $\psi - \delta\psi$ and ψ .

residual shear (or geometric) effect, and the third term $\left(\sim \frac{1}{(\dot{S}a^2)^2} \right)$ of right side of Eq. (IV.15) shows this effect.

If $q' \neq 0$, the combined effect of geometry (or residual shear) and global shear acts to reduce stability under usual tokamak condition, $q' > 0$. Figure IV.4 shows that a residual shear $\left[\frac{\partial}{\partial \psi} \left(\frac{\partial \theta}{\partial \chi} \right) \right]_{\psi, \chi}$ is negative near $\theta = 0$ but positive near $\theta = \pm\pi/2$. Therefore, for $q' > 0$ the local intersection angle of two field lines reduces near $\theta = 0$ but increases near $\theta = \pm\pi/2$. However, we can neglect the intersection of field lines near $\theta = \pm\pi/2$ because instability is strongly localized near $\theta = 0$. Therefore, only the local intersection angle near $\theta = 0$ affects the stability in such a way that the combined effect of global and residual shear reduces stability for $q' > 0$. The fourth term of Eq. (IV.15) shows this effect.

The effects of global and residual shear have been related to the local angle of intersection. It is found that geometric effect comes from the residual shear and this effect is large for highly elongated elliptic plasma.

Up to this point we have considered the effect of high elongation on the stabilizing and driving mechanism in detail. Equation (IV.15) shows that the relative importance of shear stabilization increases while the field line bending effect decreases with elongation. Therefore, the overall marginal pressure gradient strongly depends on plasma current profile. The effects of plasma current profile are profound in the highly elongated plasma, because the current profile determines the flux configuration, elongation of inner flux surfaces, and global

shear. The sensitive dependence of stability on shear is also shown in the correction terms Δ_3, Δ_4 .

The correction terms Δ_3, Δ_4 in Eq. (IV.15) are given as follows:

$$\begin{aligned} \Delta_3 \equiv & \left(\frac{q'}{q} \right)^2 \left\{ \frac{2.8}{e^2} - 3.4 \frac{e'}{e} \dot{S}a^2 - 1.9 \frac{a}{R_c} + 0.6 \dot{S}\delta'a \right\} \\ & + \left(\frac{q'}{q} \right) \frac{1}{\dot{S}a^2} \left\{ \frac{6.9}{e^2} + 5.1 \frac{e'}{e} \dot{S}a^2 + 0.6 \frac{a}{R_c} + 4.1 \dot{S}\delta'a \right\} \\ & + \frac{1}{(\dot{S}a^2)^2} \left\{ \frac{-3.4}{e^2} - 1.6 \frac{e'}{e} \dot{S}a^2 + 0.5 \frac{a}{R_c} - 2.7 \dot{S}\delta'a \right\}, \quad (\text{IV.25}) \end{aligned}$$

$$\begin{aligned} \Delta_4 \equiv & 0.8 \frac{a}{R_c} + 3.3 \dot{S}\delta'a - 0.2 \frac{e'}{e} \dot{S}a^2 \\ & + \dot{S}a^2 \left(\frac{q'}{q} \right) \left\{ 0.6 \frac{e'}{e} \dot{S}a^2 + 1.2 \dot{S}\delta'a + 0.3 \frac{a}{R_c} \right\}. \quad (\text{IV.26}) \end{aligned}$$

Δ_3 consists largely of three parts; one is global shear $\left(\frac{q'}{q} \right)^2$, another is coupling of global and residual shear $\left(\frac{q'}{q} \right) \frac{1}{\dot{S}a^2}$, and the other is residual shear $\frac{1}{(\dot{S}a^2)^2}$. These three different effects are coupled with the effects of elongation ($\sim 1/e^2$), variation of inner elongation ($\sim \frac{e'}{e} \dot{S}a^2$), toroidicity ($\sim a/R_c$), and shift ($\sim \dot{S}\delta'a$). Because of these couplings between $\left(\frac{q'}{q} \right)$ and the equilibrium parameters, the effects of each equilibrium parameter sensitively depend on current profile.

First, we consider the shift effect. Since the flux function ψ is assumed to be maximum at the plasma edge and minimum at the magnetic axis in section 1, $a\dot{S}\delta'$ is negative for the outward shift of plasma. Therefore the shifts coupling with $\left(\frac{q'}{q} \right)^2$ and $\left(\frac{q'}{q} \right) \frac{1}{\dot{S}a^2}$ are destabilizing

but coupling with $\frac{1}{(\dot{S}a^2)^2}$ are stabilizing. For the equilibria with reasonably high shear, shift is destabilizing because the shift effect coupling with global shear dominates the shift effect with residual shear. For this reason, stability deteriorates more severely as shift increases for the equilibria with higher shear.

The toroidicity effect a/R_c appearing in the first parentheses of Eq. (IV.15) is stabilizing. On the other hand, Eq. (IV.25) shows that toroidicity a/R_c coupled with $\left(\frac{q'}{q}\right)^2$ is destabilizing and coupled with $\left(\frac{q'}{q}\right) \frac{1}{\dot{S}a^2}$ and $\frac{1}{(\dot{S}a^2)^2}$ is stabilizing. But the destabilizing term is much larger than the stabilizing term in the correction term Δ_3 . The toroidicity effects in Δ_4 are destabilizing because they increase the driving force. Therefore, for equilibria with reasonably high shear, the toroidicity effect in correction terms Δ_3, Δ_4 is destabilizing, and the destabilizing effect of toroidicity in Δ_3, Δ_4 can overcome the stabilizing effect that appears in the first parentheses of Eq. (IV.15).

The effect of the variation of inner elongation can be considered by noticing that under high shear conditions as, for example, with peaked current profile, flux surfaces near the magnetic axis are more circular than flux surfaces away from the axis. The term $\frac{e'}{e} \dot{S}a^2$ is positive for the contracting inner elongation. Therefore, the effect of the term $\frac{e'}{e} \dot{S}a^2$ coupled with $\left(\frac{q'}{q}\right)^2$ and $\frac{1}{(\dot{S}a^2)^2}$ are destabilizing but coupled with $\left(\frac{q'}{q}\right) \frac{1}{\dot{S}a^2}$ are stabilizing. As a result, the effect of decreasing the inner elongation contributes more to destabilization for the higher shear. The terms with $1/e^2$ show that the increase of elongation reduces the stability in general.

The consideration of the effects of each parameter on stability is not complete because each parameter is interdependent of each other. Mizoguchi et al. shows analytically that as plasma pressure builds up along flux conserving process, plasma shift appears without noticeable change of plasma elongation until shift reaches a critical point. This phenomenon was found in the computational study of equilibria in circular tokamaks by Dory and Callen.¹³ Therefore, the separation of the effect of $\frac{e'}{e} \dot{S} a^2$ from toroidicity and shift is reasonable to some extent. Decreasing the aspect ratio generally produces more plasma shift, and toroidicity and shift are destabilizing for reasonably high shear. Therefore, the separation of the toroidicity effect from the shift effect does not alter the net destabilizing effect of toroidicity. The toroidicity effect associated with the plasma shift only increases the destabilizing effect. From the study of the correction terms Δ_3 and Δ_4 , we find that shear effect is big for the stability of highly elongated plasma because each equilibrium parameter couples with shear. We also notice that stability of equilibrium with higher shear is deteriorated more severely with the increase of shift and toroidicity.

The equation (IV.15) for marginal pressure gradient can also be used to estimate the marginal poloidal beta β_p approximately. For an equilibrium with $\left(\frac{q'}{q}\right) \dot{S} a^2 \approx 1$ and $1/e^2 \approx 0.1$ marginal β_p is calculated using Eq. (IV.15) is, for example,

$$\beta_p \approx 0.5 \frac{a_p}{a} \left(\frac{R_c}{a} \right). \quad (\text{IV.27})$$

Here a_p is an average width of pressure profile, a is a minor radius of plasma cross section, R_c/a is aspect ratio. The marginal β_p is compared with computational results and we find that they agree well with each other for equilibria with shear $q(\psi) = A - 2\psi$ and elongation $e = 3$. However Eq. (IV.27) is not a general expression for marginal β_p ; the general expression will be a sensitive function of shear.

4. COMPARISON OF STABILITIES BETWEEN HIGHLY ELONGATED ELLIPTIC CROSS SECTION AND CYLINDRICAL SLAB

Because of the complicated geometry of the elongated toroidal plasma, a cylindrical slab model is frequently used for the study of equilibrium and stability of the plasma in highly elongated tokamaks. It is assumed that the limit of a highly elongated plasma is a cylindrical slab (see Fig. IV.5) and that field configurations are essentially the same for both models. An analytic study of equilibria for highly elongated plasma with broad current profile was done using a cylindrical slab model for the study of equilibria in a belt pinch configuration by Becker.³⁸ He found good agreement between analytic calculation of equilibrium using a cylindrical slab model and computational results for a highly elongated belt pinch configuration. The stability of kink modes in a highly elongated plasma was studied by Grossman et al.¹¹ By assuming that elongation is an order of square root of aspect ratio for large aspect ratio they got an equilibrium equation essentially the same as for a cylindrical slab. On the other hand, in the previous sections we found that the geometric effect on stability is big and stability sensitively depends on equilibrium flux configuration.

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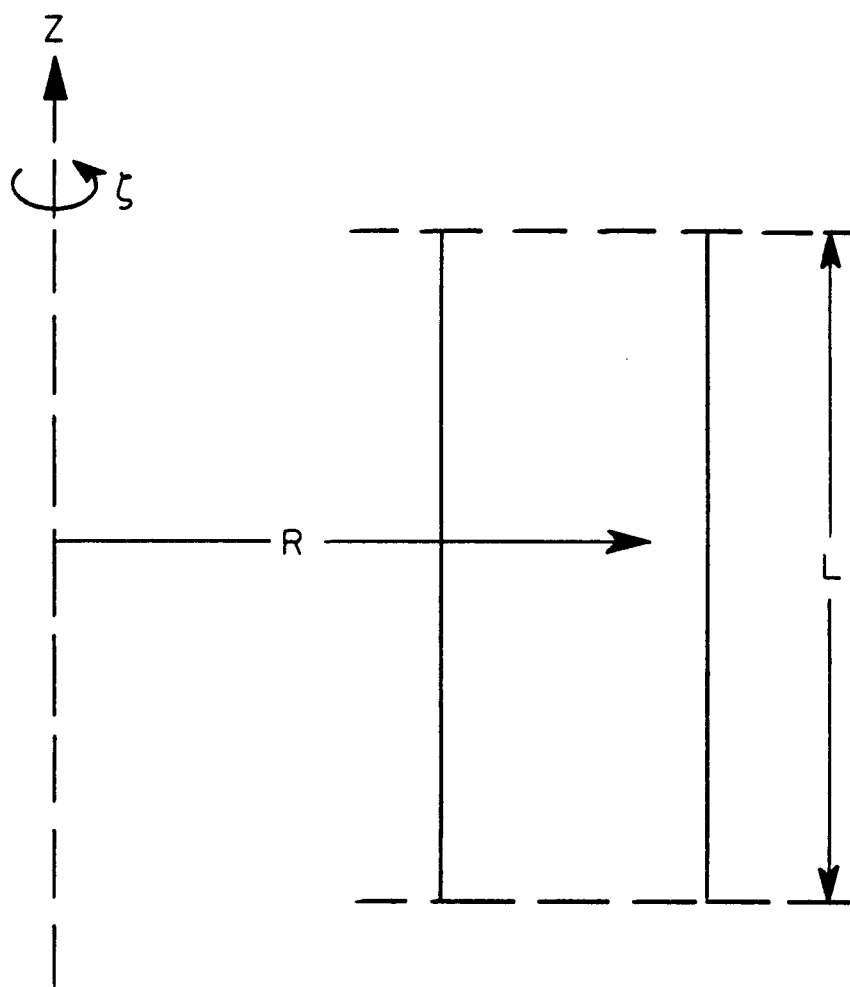


Figure IV.5. The geometry of a cylindrical slab model.

Therefore, it is interesting to study how much the stability conditions differ from each other.

The marginal pressure gradient for a cylindrical slab can be easily calculated since B_p , B_t and P are only functions of R , radial distance from the major axis of the torus to a point in plasma (see Fig. 5). The Jacobian J is calculated by equating poloidal angle χ to Z , the coordinate parallel to the axis of the cylindrical slab. Then the coordinate system (R, Z, ζ) forms an orthogonal coordinate system.

From $JB_p d\chi = dZ$ and by letting $\chi = Z$ we get

$$J = \frac{1}{B_p} . \quad (\text{IV.28})$$

From Eq. (IV.5) for v

$$v = \frac{B_t}{RB_p} . \quad (\text{IV.29})$$

The geodesic curvature K_s is equal to zero and the normal curvature K_n is

$$K_n = - \left(\frac{B_t}{B} \right)^2 \frac{1}{R} . \quad (\text{IV.30})$$

For stability it is assumed that field lines are frozen at the top and bottom of the cross section of the cylindrical slab. In this case, a reasonable test function is formed as follows:

$$F(Z) = \begin{cases} 1 + \cos kZ & -\frac{L}{2} \leq Z \leq \frac{L}{2} \quad (K \equiv \frac{\pi}{L}) \\ 0 & \text{elsewhere} \end{cases} \quad (\text{IV.31})$$

The marginal stability pressure gradient, calculated using Eqs. (IV.28)-(IV.31), is

$$\left| \frac{dP}{dR} \right| \leq \frac{2\pi^2 \dot{S}^2}{3R} \left[\frac{1}{\bar{e}^2} + \frac{a^2}{\pi^2} \left(\frac{dq}{dR} \frac{1}{q} \right)^2 \left(\frac{\pi^2}{3} - \frac{1}{2} \right) \right], \quad (\text{IV.32})$$

where $\bar{e}^2 \equiv \left(\frac{L}{2a} \right)^2$, L is the height and $2a$ is the width of the cylindrical slab. For $\frac{dq}{dR} \frac{a}{q} \approx 2$ (therefore $\frac{dq}{d\psi} \frac{1}{q} \dot{S} a^2 \approx 1$ for elliptically elongated plasma) and $1/\bar{e}^2 \approx 0.1$ cylindrical slab overestimates the marginal pressure gradient about seven times larger than the elongated elliptic cross section does. The overestimation of stability for the cylindrical slab model comes from the facts that there is no driving force due to the geodesic curvature and no big destabilizing effect due to the combined effect of global and residual shear. The shear stabilization for elliptically elongated plasma almost vanishes with $(q'/q) \dot{S} a^2 \approx 1$ due to the combined effect of global and residual shear, while the shear stabilization for cylindrical slab is order of 1.

The simple comparison shows that stability depends sensitively on flux configuration. Therefore, careful consideration is required to use cylindrical slab model for a study of stability.

5. COMPARISON WITH COMPUTATIONAL RESULTS

Computational study of high n ballooning mode is done using the ORNL BALOON code written by G. Bateman and D. B. Nelson. These results are compared with the analytic results to verify the analytic results and to clearly understand the computational results. For equilibrium we use the Island code by D. C. Stevens⁴⁰ at NYU as modified by D. B. Nelson.⁴¹ Equilibrium is calculated by prescribing the safety factor $q(\psi)$ and pressure $p(\psi)$ as functions of the stream function ψ . The boundary shape of plasma can be changed from circular and ellipse to D shape with any elongation. In this computational study we mainly concentrate on elliptically shaped flux surfaces.

The eigenfunction solution of the ordinary differential equation, Eq. (IV.2), should drop to zero as y goes to infinity.²⁵ Since the computer code must consider only finite intervals, it was set to iteratively change the eigenvalue until the solution has its first zero crossing at y equals to 3.5 times around the cross section. These results were then compared with solution integrated y equals to 28.5 times around the cross section. Larger integration angle gives more strict stability condition. Most of the computational work is done with poloidal angle three and one-half times around the cross section to save computer time without losing accuracy required in this work.

First, we study how eigenfunctions change with equilibrium. According to the analytic study shown in section 2, the form of the most appropriate test function depends on shear. We study how eigenfunctions change with shear. Figure IV.6 shows two eigenfunctions for

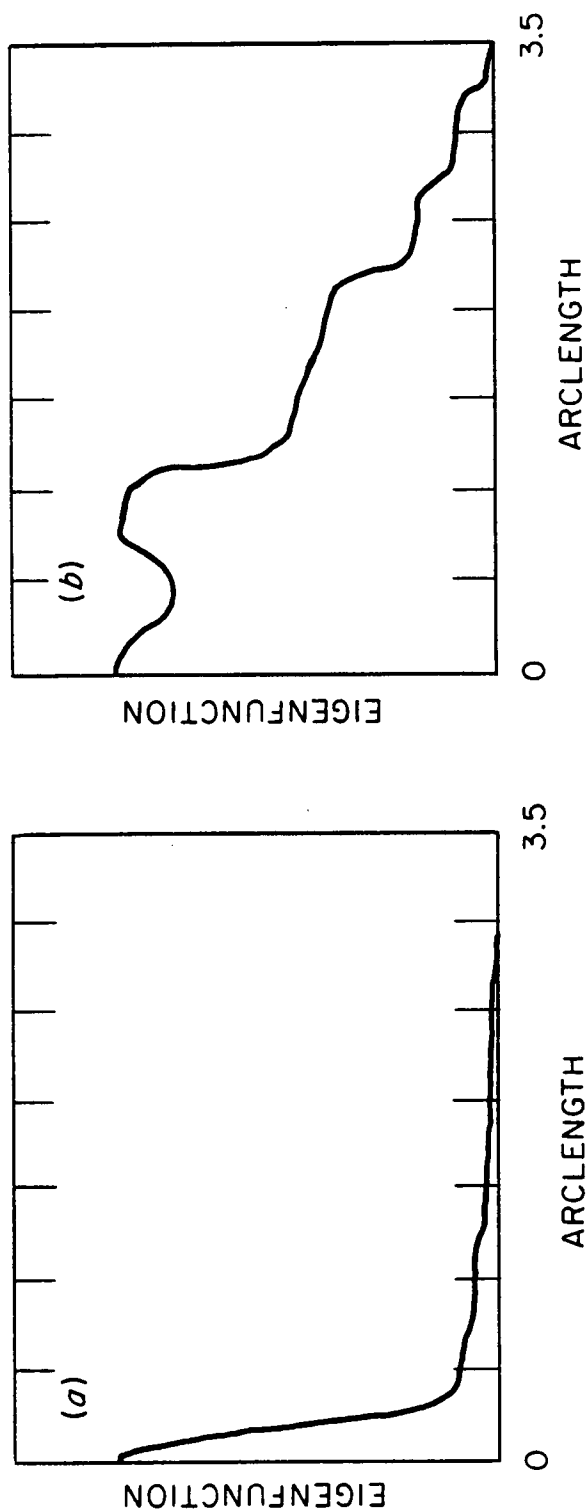


Figure IV.6. Eigenfunctions with different shears for the integration angle three and one-half times around the cross section. (A) With high shear; (B) with low shear, plotted with respect to arc length of a poloidal flux surface ψ . The elongation of the plasma edge is $(b/a)_w = 3$.

high and low shear with integration arc length three and one-half times around a poloidal flux cross section. Elongation $e = 3$, aspect ratio $ASP = 4$, and pressure profile $P(\psi) = A\psi^2$ are used for both cases. The eigenfunction for the high shear case (Fig. IV.6A) decreases to zero very rapidly while the eigenfunction for the low shear case slowly decreases to zero (Fig. 6B). The sharp change of eigenfunction for low shear comes from high curvature effect on the top and the bottom of an elliptic cross section. The eigenfunctions as a function of integration arc length with zero crossing at $y = 0.5$ are presented in Fig. IV.7 to compare with the test functions $F_1(\theta)$ and $F_2(\theta)$ defined in Eq. (IV.13). The eigenfunction for high shear goes almost to zero at the integration arc length a quarter of the cross sectional arc length. On the other hand, the eigenfunction for low shear gradually goes to zero at the integration arc length half of the cross sectional arc length. The eigenfunctions for high shear (Fig. IV.7A) and for low shear (Fig. IV.7B) look close to $F_2(\theta)$ and $F_1(\theta)$, respectively. These results agree with the analytic prediction discussed in section 2.

In Fig. IV.8 three different marginal curves for $q(\psi) = B - 2\psi^2$, $C - 2\psi$, and $D - 3\psi$ with $P(\psi) = A\psi^2$ are plotted for β as a function of q_a . The marginal β 's for different $q(\)$ are much different at a same q_a near 1 but these curves converge to one curve for q_a larger than 1. The convergence comes from the dependence of stability on shear in the form of (q'/q) as shown in Eq. (IV.15) in section 2. As q_a increases, (q'/q) and whole shear effect decrease, and the relative importance of the field line bending effect increases. Therefore stability becomes insensitive to shear for $q_a \gg 1$.

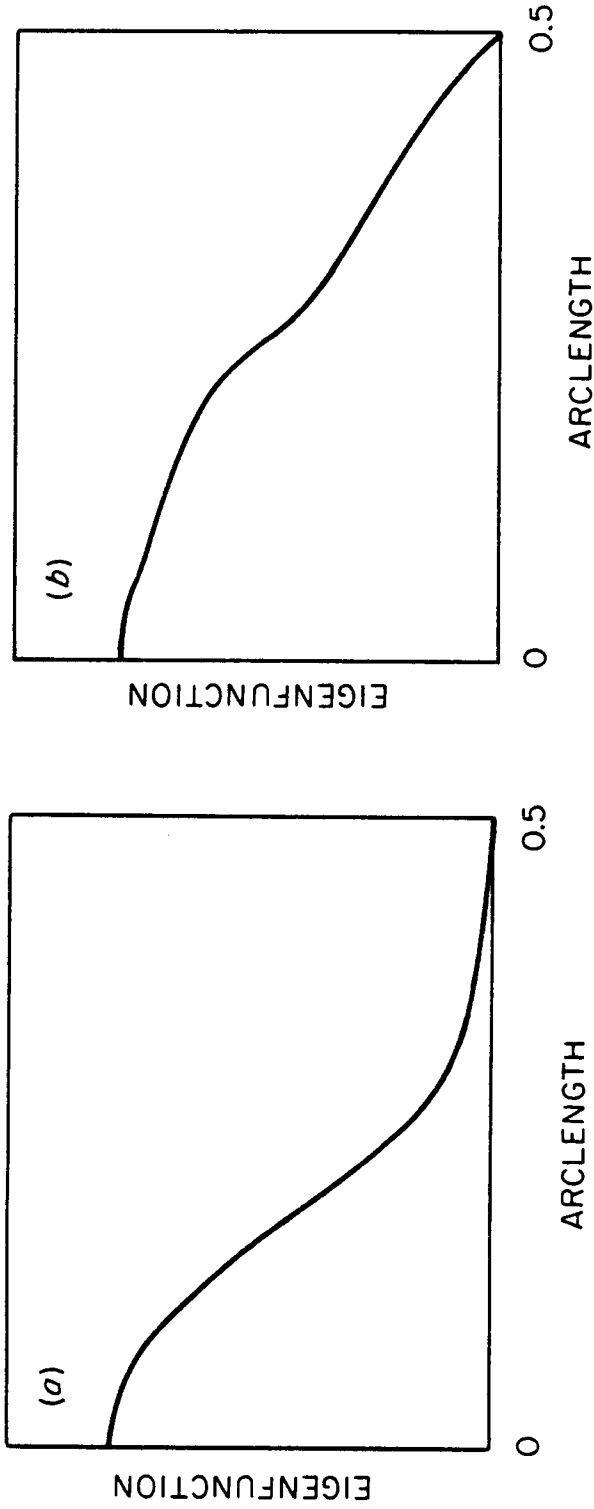


Figure IV.7. Eigenfunctions with different shears for the integration angle one-half time around the cross section. (A) With high shear; (B) with low shear, plotted with respect to arc length of a poloidal flux surface ψ .

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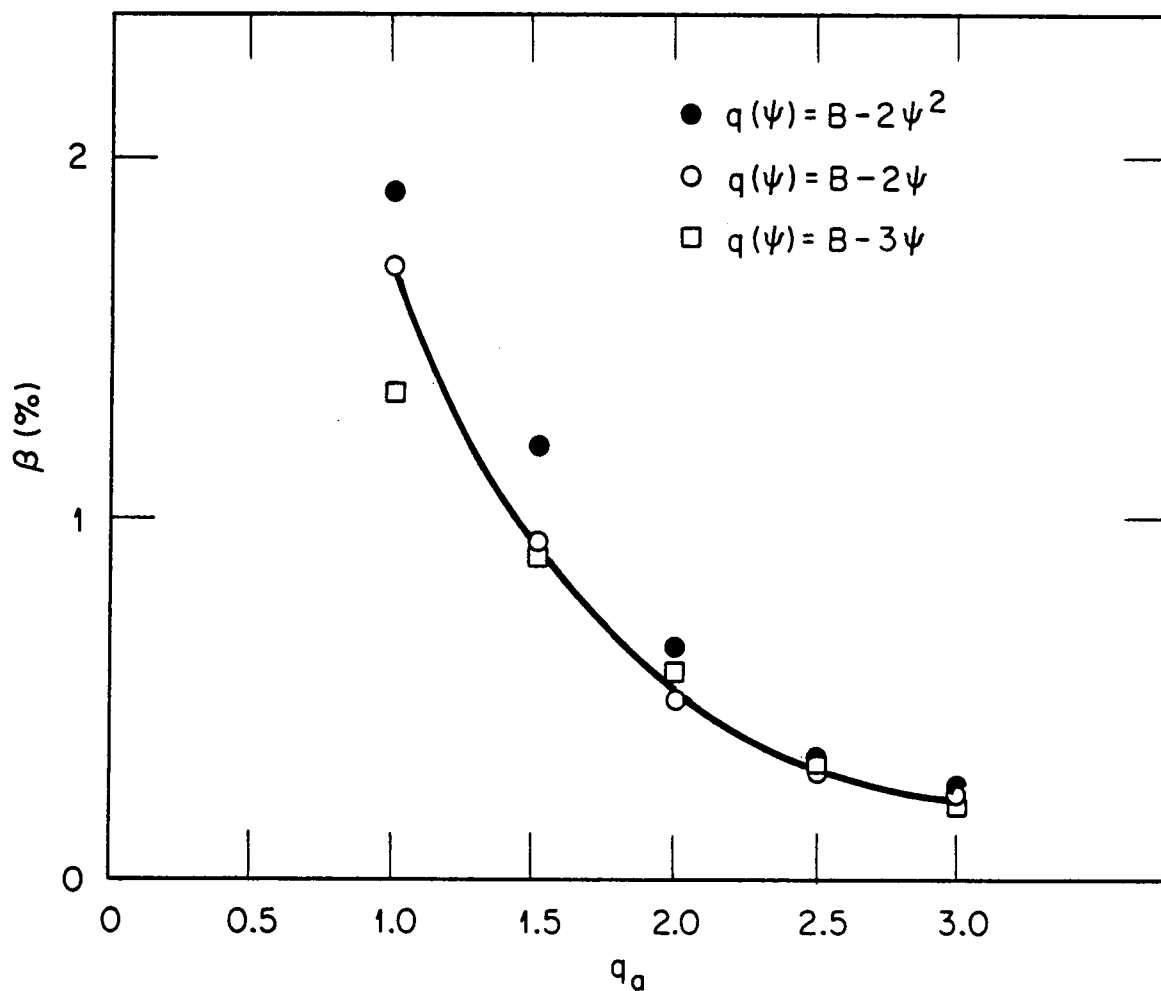


Figure IV.8. Marginal curves with same pressure profile $P(\psi) = A\psi^2$ and elongation $(b/a)_w = 3$ but different $q(\psi)$ profiles in $\beta - q_a$ plane. Three different curves converge to one curve at high q_a .

Figure IV.9 shows also the sensitive dependence of stability on shear. The curves 1, 2, and 3 have maximum values of $|\frac{q'}{q}|$ equal to 4, 3, and 2, respectively, and the values of q' for curve 1 decreases to zero as flux surfaces become close to the plasma edge while q' for the other two curves remains constant. The rapid drop of marginal β for curve 1 with β_p is due to the fact that as β_p increases the most unstable flux surface moves to the plasma edge where shear decreases to zero for the case of curve 1. Here β_p is defined as

$$\beta_p \equiv \frac{2\langle p \rangle}{\oint_c B_p^2 d\ell / \oint_c d\ell}$$

and the contour c is near the plasma edge. The slight lower marginal β 's for the curve 2 than the marginal β 's for the curve 3 with $\beta_p > 1$ are due to the effects of shift and elongation in the correction terms Δ_3 and Δ_4 of Eq. (IV.25) and Eq. (IV.26). In these correction terms, the effects of shift and shrinking the inner elongation reduce the stability by coupling with shear.

Elongation effects on the stability against the high n ballooning mode are studied in a broad range of parameter space. Figure IV.10 shows how marginal β 's change with β_p for each elongation. The graph shows that $(b/a)_w = 3$ is the most optimum elongation for the equilibrium profile $q(\psi) = A - 2\psi$, $p(\psi) = B\psi^2$ at ASP = 4. Higher elongation than $(b/a)_w = 3$ does not have any advantage for high stable beta. For an equilibrium with shear higher than that of equilibrium shown in Fig. IV.10, stability for $(b/a)_w = 2$ is better than that for $(b/a)_w = 3$

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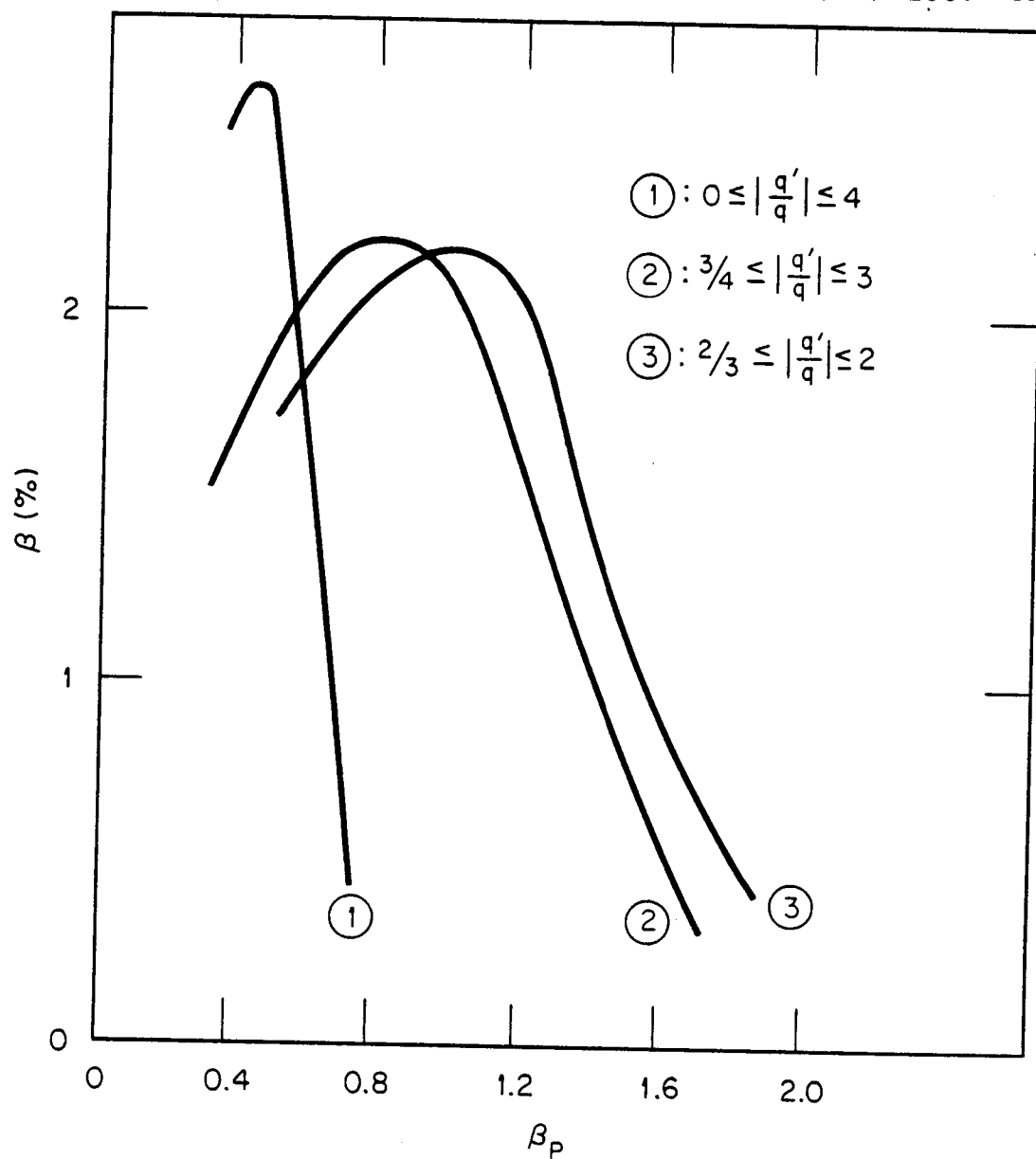


Figure IV.9. Marginal beta as functions of β_p for different shear with same pressure profile $P(\psi) = A\psi^2$, elongation $(b/a)_w = 3$.

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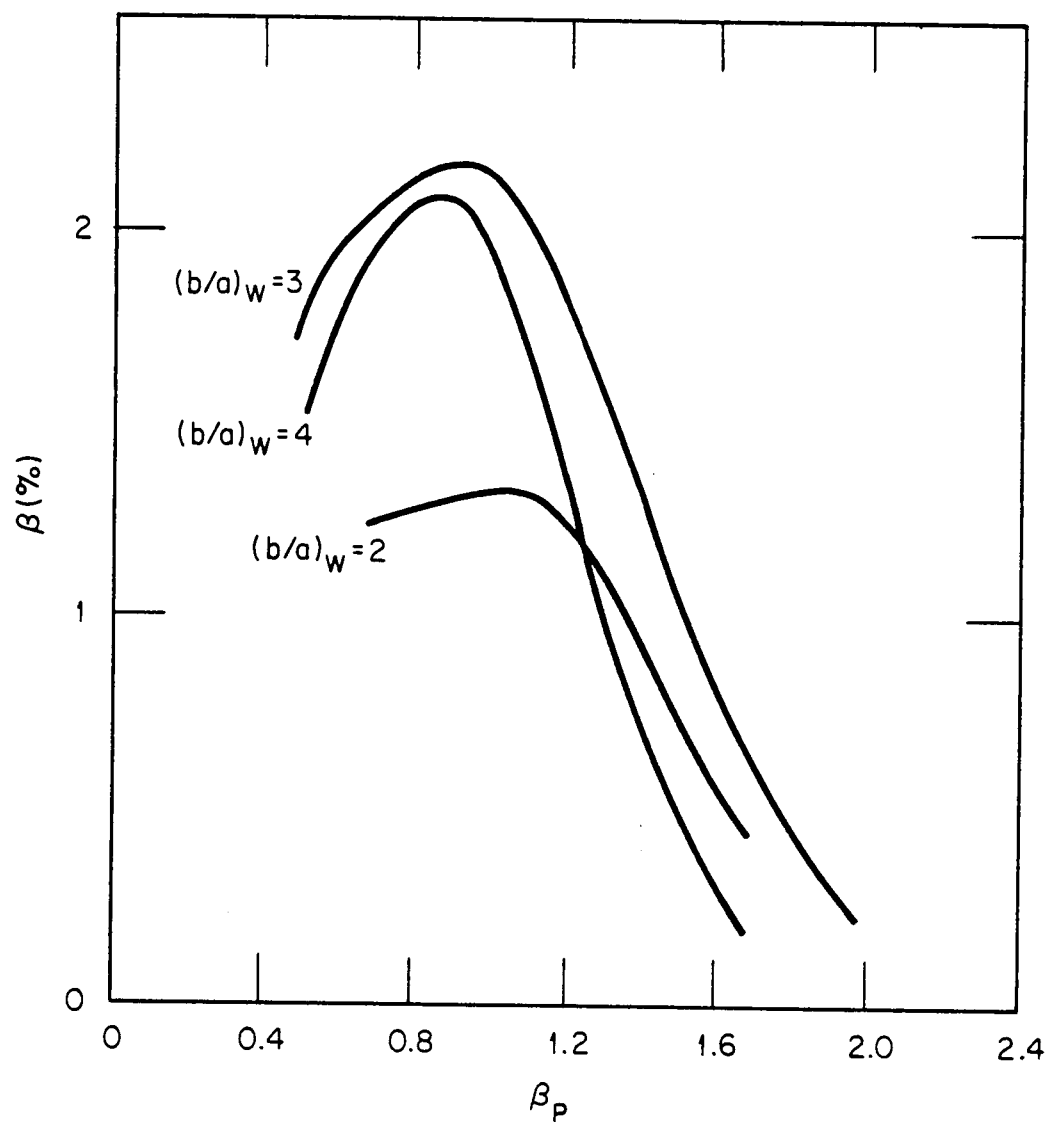


Figure IV.10. Marginal betas as functions of β_p for different elongations $(b/a)_w = 2, 3, 4$ with $q(\psi) = A - 2\psi$, $P(\psi)^p = B\psi^2$.

and 4 with $\beta_p \geq 1.2$. Higher elongation is usually unfavorable for high stable β with higher β_p ($\beta_p > 1$) but favorable with lower β_p . We can also notice that stability for higher elongation drops more rapidly with β_p .

The effect of toroidicity is studied in $\beta - \beta_p$ parameter space with aspect ratio 3, 4, and 5, and elongation $(b/a)_w = 3$. Figure IV.11 shows that maximum stable β increases with inverse aspect ratio monotonically. However, marginal β for lower aspect ratio drops more rapidly with β_p and is much smaller than that for larger aspect ratio for β_p larger than 1.2. In other words, lower aspect ratio is generally favorable for high stable beta with lower β_p but unfavorable with higher β_p . The tendency of toroidicity effect is very similar to that of elongation effect in such a way that the stability of lower aspect ratio (higher elongation) is more vulnerable to the increase of β_p . The toroidicity effect was expected from the analysis presented in sections 2 and 3. In section 3, we find that toroidicity is destabilizing in the correction terms Δ_3, Δ_4 . Since shift is also destabilizing for equilibria with reasonably high shear, the destabilizing effect of toroidicity associated with the effect of shift can overcome the stabilizing effect of toroidicity appearing in the first parentheses of Eq. (IV.15).

Up to this point we have mainly fixed the pressure profile as a function of poloidal flux ϕ . The effect of pressure profile is also studied by fixing $q(\psi)$ as a function of ψ and determining pressure profiles as $p(\psi) = A\psi^\alpha$ with different value of α , 1.2, 1.6, and 2. Figure IV.12 shows that maximum marginal β increases by decreasing α

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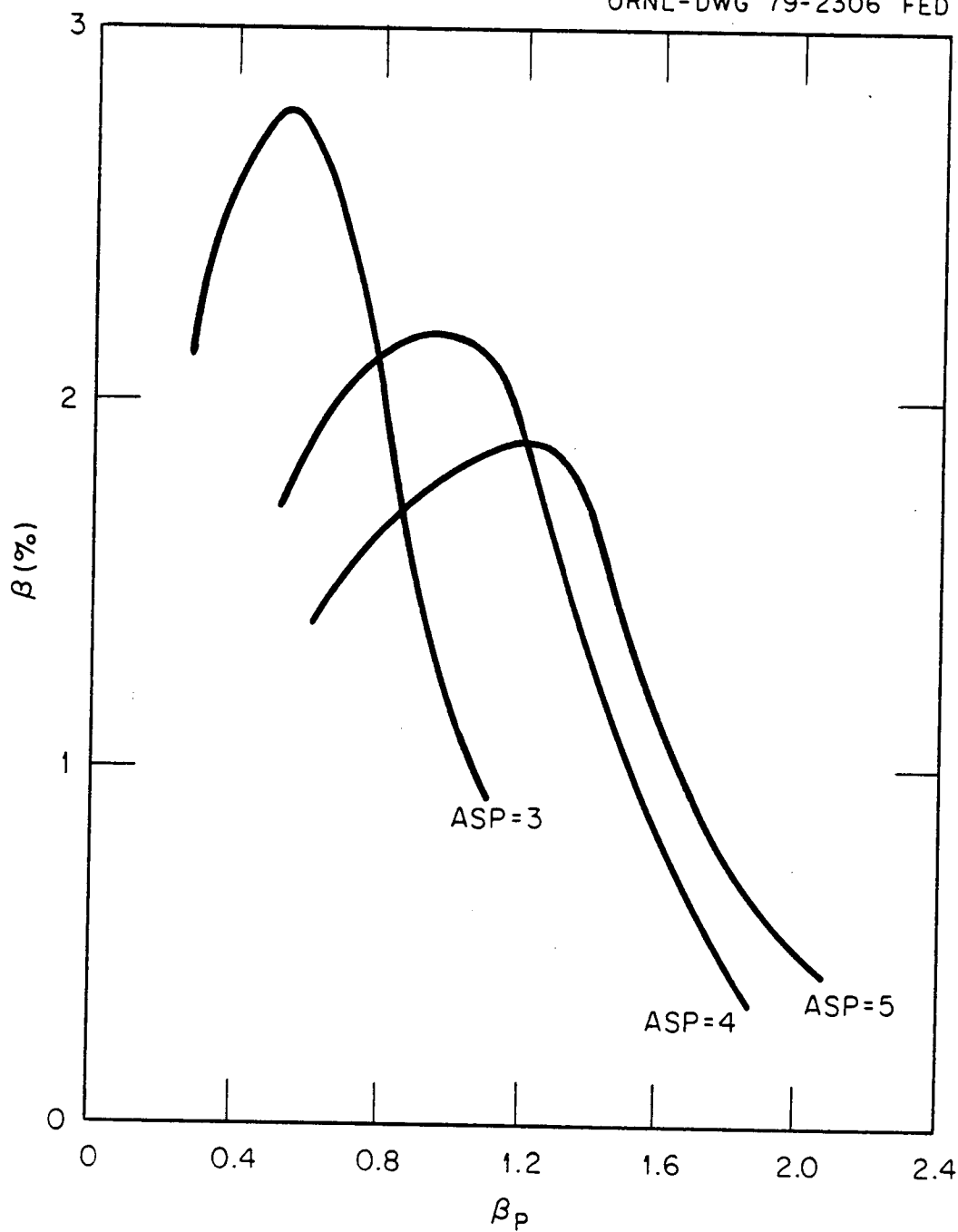


Figure IV.11. Marginal beta as functions of β_p for different aspect ratio $ASP = 3, 4, 5$ with $q(\psi) = A - 2\psi$, $P(\psi) = P_B \psi^2$, $(b/a)_w = 3$.

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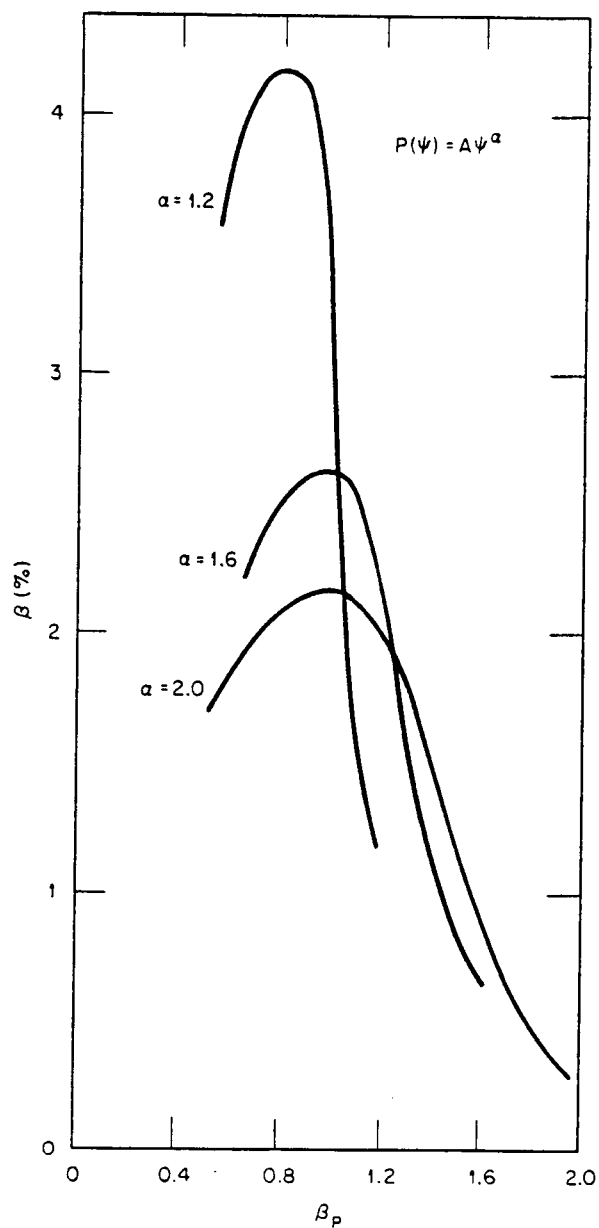


Figure IV.12. Marginal beta as functions of β_p for different pressure profiles $P(\psi) = A\psi^\alpha$ ($\alpha = 1.2, 1.6, 2$) with $q(\psi) = B-2\psi$, $(b/a)_w = 3$.

(or by broadening pressure profile). But the stability of broader pressure profile drops more rapidly with the increase of β_p . Broader pressure profile has a favorable effect on high β for lower β_p but has an unfavorable effect for higher β_p , and this tendency is similar to that of toroidicity and elongation effects with the variation of β_p .

The computational results are sensitive to the integration angle, especially near the magnetic axis. The integration angle three and one-half times around a cross section gives an optimistic stability condition. A more stringent stability condition requires a larger integration angle, but an indefinitely large integration angle is not available because of the restricted computer time. In cases where the eigenfunction spans a broad integration angle, the computational results can be supplemented by applying the Mercier stability criterion to every flux surface since Mercier criterion is a necessary condition²⁵ for the stability of a high n ballooning mode.^{25,67,68} We use a very large integration angle (twenty-eight and one-half times around the cross section) to test the accuracy of the results mentioned in this section and to see whether the Mercier stability criterion can be produced computationally with the large integration angle. These results are compared by estimating the Mercier criterion near the magnetic axis using a formula by Lörtz and Nührenberg³ and Solov'ev.⁷⁰ The solid lines of Fig. IV.13 show how marginal curves in $\beta - \beta_p$ parameter space change with different integration angles. The curves 1 and 2 are marginal curves with integration angle three and one-half times around the cross section and twenty-eight and one-half times around the cross section, respectively. The change of marginal curves with β_p larger

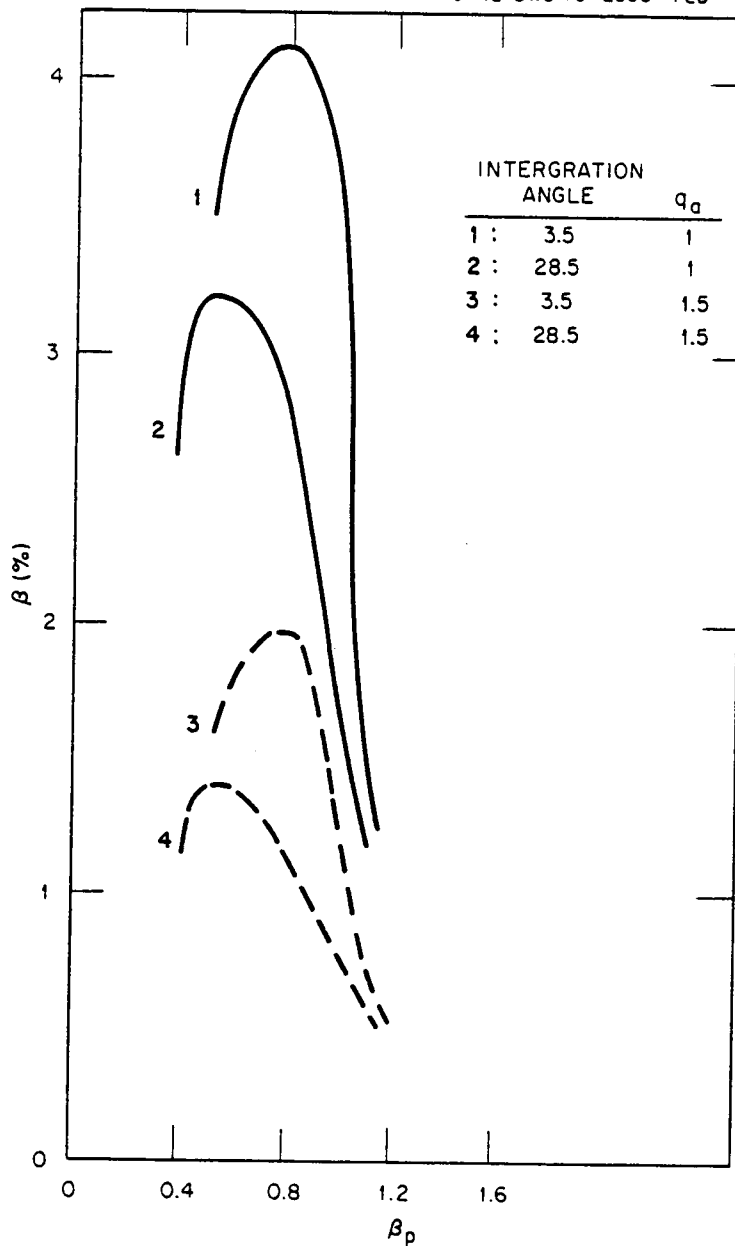


Figure IV.13. Comparison of marginal curves with different integration angles. The solid lines 1, 2 are for $q_a = 1$ and integration angles three and one-half times and twenty-eight and one-half times around the cross section, respectively. The dotted lines 3, 4 are for $q_a = 1.5$ and integration angles three and one-half times and twenty-eight and one-half times around the cross section, respectively. The marginal beta for 3 and 4 are scaled down as $1/(1.5)^2$ from the marginal beta for 1 and 2, respectively.

than 1 is smaller but the change with β_p equal to or less than 1 is significant. The significant change of stability as a function of integration angle is closely related to the distribution of the most unstable region. The most unstable regions are near the plasma edge for $\beta_p > 0.8$ but are near the magnetic axis for $\beta_p \leq 0.8$. Because there are few grid points along the arc length near the magnetic axis, the computer code does not properly pick up the instability in this region with finite integration angle. The Mercier stability criterion near the magnetic axis is tested to these marginal stability curves. It is found that these equilibria with $q_a = 1$ are all Mercier unstable near the magnetic axis but the equilibria with $q_a = 1.5$ are Mercier stable for $\beta_p > 1$. The dotted lines 3 and 4 in Fig. IV.13 are marginal curves with $q_a = 1.5$ and the integration angle three and one-half times and twenty-eight and one-half times around the cross section, respectively. The overall marginal beta is scaled down such as $\beta = \beta_1/q_a^2$ for each integration angle, where β is a marginal beta with $q_a > 1$ and β_1 is with $q_a = 1$.

From this study we conclude the following: The results depend sensitively on the integration angle if the most unstable region is near the magnetic axis. The computer code (BALOON code) cannot generate the Mercier criterion even though very large integration angle is used. The Mercier criterion near the magnetic axis can be satisfied by raising q_a sufficiently above 1 and raising q_a only scales down the β values as $\beta = \beta_1/q_a^2$ without altering the results already described.

6. CONCLUSION

An analytic study of high n ballooning mode in highly elongated elliptic plasma is done and the results are compared with computational results using BALOON code. The analytic calculation for marginal pressure gradient shows that stability sensitively depends on shear through global and residual shear stabilization and through the coupling between global shear and parameters of poloidal flux configurations, i.e., elongation, shift, toroidicity, etc. Since the global shear and poloidal flux configuration strongly depend on toroidal current profile, the effect of toroidal current profile is profound on the stability of highly elongated plasma.

The computational results are insensitive to the integration angle if the most unstable region is near the plasma edge but sensitive if the most unstable region is near the magnetic axis. The BALOON code does not produce the Mercier stability criterion even though very large integration angle is used. For the complete study of high n ballooning mode, the Mercier criterion is also considered near the magnetic axis. The Mercier criterion is satisfied for q_a sufficiently larger than unity and then the marginal beta value scales down as $1/q_a^2$ without changing the effects of equilibrium parameters mentioned in section 5.

The analytic results predict and the computational studies verify that high elongation, low aspect ratio, and broad pressure profile enhance the marginal beta value for β_p less than unity but severely reduce marginal β for β_p larger than unity. The computational results reveal that the optimization always associates with disadvantages such

as narrowing the stable regime in a parameter β_p and severe vulnerability of stability for small increase of β_p . If we consider a large range of error on setting up equilibrium parameters in real experiments and a possible nonlinear evolution of plasma to an unfavorable direction, the optimizations are always followed by some risk in experiments. Therefore optimization should also be made with this consideration as well as by considering realistic model of plasma condition for theoretical study.

Further study of optimization is required in a broad range of equilibrium parameter space along with the study of nonlinear evolution in optimized high β plasma.

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APPENDIX

APPENDIX

1. The calculation of $\int_{\chi}^{\chi} \left(\frac{\partial v}{\partial \psi} \right)_{\chi} d\chi$

Since the coordinate system we have used is not an orthogonal coordinate system and $\left(\frac{\partial v}{\partial \psi} \right)_{\chi}$ is different from $\left(\frac{\partial v}{\partial \psi} \right)_{\theta}$ we have to calculate the integral $\int \left(\frac{\partial v}{\partial \psi} \right)_{\chi} d\chi$ in a way which is independent of the coordinate system. Using $v = \frac{JB_t}{R}$ and $d\ell_p = JB_p d\chi$, $\int_{\chi}^{\chi} \left(\frac{\partial v}{\partial \psi} \right)_{\chi} d\chi$ can be changed to an expression independent of the coordinate system as

$$\int \left(\frac{\partial v}{\partial \psi} \right)_{\chi} d\chi = \int d\ell_p \left[\frac{I(\psi) J_t}{(RB_p)^3} + \frac{B_p I'(\psi)}{(RB_p)^2} - \frac{2B_p I(\psi)}{(RB_p)^3} \frac{\partial}{\partial \psi} (RB_p) \right]. \quad (A.1)$$

Here J_t is the toroidal current density and $d\ell_p$ is an infinitesimal arc length of a poloidal flux contour. We will calculate the three terms separately.

$$(a) \int d\ell_p \frac{I(\psi) J_t}{(RB_p)^3};$$

$I(\psi)$, $(RB_p)^3$ can be expressed with (ψ, θ) using Eq. (IV.9). J_t should be calculated from the equation

$$J_t = \left(\frac{\partial B_{px}}{\partial z} \right)_R - \left(\frac{\partial B_{pz}}{\partial R} \right)_z. \quad (A.2)$$

$\left(\frac{\partial B_{px}}{\partial z} \right)_R$ and $\left(\frac{\partial B_{pz}}{\partial R} \right)_z$ are transformed to (ψ, θ) system using the equations:

$$\left(\frac{\partial B_{px}}{\partial z}\right)_R = \frac{a' \cos\theta \left(\frac{\partial B_{px}}{\partial \theta}\right)_\psi + a \sin\theta \left(\frac{\partial B_{px}}{\partial \psi}\right)_\theta}{a' b \left(1 + \frac{2e'}{e} \dot{S} a^2 \sin^2\theta + \frac{\delta'}{a} \cos\theta\right)}, \quad (A.3)$$

$$\left(\frac{\partial B_{pz}}{\partial R}\right)_z = \frac{b \cos\theta \left(\frac{\partial B_{pz}}{\partial \psi}\right)_\theta - b' \sin\theta \left(\frac{\partial B_{pz}}{\partial \theta}\right)_\psi}{a' b \left(1 + \frac{2e'}{e} \dot{S} a^2 \sin^2\theta + \frac{\delta'}{a} \cos\theta\right)}. \quad (A.4)$$

B_{px} , B_{pz} can be calculated as functions of (ψ, θ) from the equation $\vec{B}_p = \Delta\zeta \times \nabla\psi$. J_t can be calculated using Eqs. (A.2)-(A.4) straightforwardly. By neglecting terms higher than $O(\epsilon)$ we get

$$\int d\ell_p \frac{I(\psi) J_t}{(RB_p)^3} = \frac{I(\psi) a e}{(2\dot{S} a)^3} \left[C_1 \theta + C_2 \sin 2\theta + C_3 e \tan^{-1}\left(\frac{1}{e} \tan\theta\right) + C_5 \log \left(\frac{1 + \frac{1}{2e^2} + \sin\theta}{1 + \frac{1}{2e^2} - \sin\theta} \right) \right] \quad (A.5)$$

Here

$$\begin{aligned} C_1 &= \frac{-8\dot{S}^2 a^2 e'}{R_c e} + \frac{4\dot{S}\dot{S}' a^2}{R_c} \\ C_2 &= \frac{4\dot{S}^2 a^2 e'}{R_c e} \\ C_3 &= \frac{2\dot{S}}{R_c} \left(1 + \frac{1}{e^2} + \frac{4e' \dot{S} a^2}{e} \right) \\ C_4 &= \frac{-2\dot{S} a}{R_c^2} \end{aligned} \quad (A.6)$$

$$(b) \int d\ell_p \frac{I'(\psi) B_p}{(RB_p)^2};$$

It is straightforward to calculate and the result is

$$\begin{aligned} \int d\ell_p \frac{B_p I'(\psi)}{(RB_p)^2} &= \frac{I'e}{2\dot{S}R_c} [\theta(1 + \frac{e'}{e} \dot{S}a^2) + (2\delta'\dot{S}a - \frac{a}{R_c})\sin\theta \\ &\quad - \frac{e'\dot{S}a^2}{2e} \sin 2\theta] \end{aligned} \quad (A.7)$$

$$(c) \int d\ell_p \frac{B_p I(\psi)}{(RB_p)^3} \frac{\partial}{\partial \psi} (RB_p)_\chi;$$

The calculation of $\frac{\partial}{\partial \psi} (RB_p)_\chi$ can be done in (ψ, θ) coordinate system using the relations

$$\frac{\partial}{\partial \psi} (RB_p)_\chi = \frac{\partial}{\partial \psi} (RB_p)|_\theta + \frac{\partial}{\partial \theta} (RB_p)|_\psi \frac{\partial \theta}{\partial \psi}|_\chi \quad (A.8)$$

and

$$\frac{\partial \theta}{\partial \psi}|_\chi = \frac{\nabla \psi \cdot \nabla \theta}{|\nabla \psi|^2}. \quad (A.9)$$

$\nabla \theta$ can be calculated using (IV.7) in section 1.

$$\nabla \theta = \frac{(\delta' + a'\cos\theta)\nabla z - b'\sin\theta \nabla R}{a'b \left[1 + \frac{2e'}{e} \dot{S}a^2 \sin^2\theta + \frac{\delta'}{a} \cos\theta \right]}. \quad (A.10)$$

The calculation of $\frac{\partial \theta}{\partial \psi}|_\chi$ and $\frac{\partial}{\partial \psi} (RB_p)_\chi$ are straightforward using Eqs. (A.8)-(A.10), and neglecting terms $O(\epsilon^2)$ the result is

$$\begin{aligned}
\int d\ell_p \frac{B_p I(\psi)}{(RB_p)^3} \frac{\partial}{\partial \psi} (RB_p) \chi = D_1 \theta + D_2 \sin 2\theta + D_3 \frac{\tan \theta}{1 + \frac{1}{2} \tan^2 \theta} \\
+ D_4 e \tan^{-1} \left(\frac{1}{e} \tan \theta \right) + D_5 \log \left(\frac{1 + \frac{1}{2e^2} + \sin \theta}{1 + \frac{1}{2e^2} - \sin \theta} \right). \quad (A.11)
\end{aligned}$$

Here

$$\begin{aligned}
D_1 &= q \left(\frac{\dot{S}'}{\dot{S}} - \frac{2e'}{e} \right) \\
D_2 &= \frac{-q}{2} \frac{e'}{e} \\
D_3 &= \frac{q}{4\dot{S}a^2} \left(1 - \frac{1}{e^2} + \frac{3e'}{e} \dot{S}a^2 \right) \\
D_4 &= \frac{q}{2\dot{S}a^2} \left(\frac{1}{2} - \frac{1}{e^2} + \frac{3}{2} \frac{e'}{e} \dot{S}a^2 \right) \\
D_5 &= \frac{-q}{4\dot{S}a} \frac{1}{R_c}.
\end{aligned} \quad (A.12)$$

From Eqs. (A.5), (A.7), and (A.11) we calculate

$$\int \left(\frac{\partial v}{\partial \psi} \right) \chi d\chi.$$

The result is

$$\begin{aligned}
\int \left(\frac{\partial v}{\partial \psi} \right) \chi d\chi = S_1 \theta + S_2 \frac{\tan \theta}{1 + (1/e^2) \tan^2 \theta} + S_3 \sin \theta + S_4 \sin 2\theta \\
+ S_5 e \tan^{-1} \left(\frac{1}{2} \tan \theta \right). \quad (A.13)
\end{aligned}$$

Here

$$S_1 = q'$$

$$S_2 = - \frac{1}{2\dot{S}a^2} \left(1 - \frac{1}{e^2} + \frac{3e'}{e} \dot{S}a^2 \right)$$

$$S_3 = q' (2\delta' \dot{S}a - a/R_c) \quad (\text{A.14})$$

$$S_4 = \frac{2e'}{e} - \frac{e'}{2e} \dot{S}a^2 \left(\frac{q'}{q} \right)$$

$$S_5 = \frac{q}{2\dot{S}a^2} \left(\frac{3}{e^2} \right) .$$

In this calculation we neglect terms with $O(\epsilon^2)$.

VITA

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