To the Graduate Council:

I am submitting herewith a thesis written by Debra Lorraine Coaxum entitled "Geometric Programming." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Yueh-er Kuo

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GEOMETRIC PROGRAMMING

A Thesis

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Debra Lorraine Coaxum

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DEDICATION

To My Parents

John and Stella Coaxum

also, to the memory of

Rosa Coaxum

and

Ruth F. Myers

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ABSTRACT

This thesis studies the properties of posynomial geometric programming. Detailed proofs are given to support methods for solving these types of geometric programming problems. In addition, a process for a postoptimal analysis is given. Based on these results, an inventory theory model is solved.

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CHAPTER 1.

INTRODUCTION

In engineering design problems, total cost can be represented as a sum of component costs in the following manner:

$$g_k(t) = \sum_{i=1}^{n_k} c_{ki} \prod_{j=1}^{m} t_j^{a_{kij}} \quad k = 0, 1, ..., p.$$

The coefficients c_{ki} are positive constants and the exponents a_{kij} are real constants that can be either positive or negative. The design variables t_j are assumed to be strictly positive variables. From this point on t will denote the vector $t = (t_1, ..., t_m)$. Functions $g_k(t)$ that have the above properties are called **positive polynomials** or **posynomials** for short. **Geometric programming** is a technique used to optimize posynomials subject to posynomial constraints.

Geometric programming splits the optimization problem into two components. The first, termed the **primal problem**, minimizes a posynomial $g_0(t)$ subject to posynomial constraints that are less than or equal to 1. The second component is a maximization problem called the **dual problem**. The dual problem maximizes the dual function $v(\delta)$, subject to linear constraints. As will be shown in Chapter 2, the primal problem and dual problem share a special relationship. This relationship enables one to first determine the distribution of cost among each term of $g_0(t)$

and then find the actual minimal cost. Finally, the optimum values of the design parameters t_j are determined.

The method of geometric programming allows for simple sensitivity analysis of the optimization problem. In special cases the problem can be anaylzed by inspection. Chapter 3 is devoted to sensitivity anaylsis of geometric programming problems. The method described there is due to Wilde and Beightler [2].

Despite the fact that geometric programming was developed to solve engineering problems, it has been applied to several other areas. Applications of geometric programming can be seen in operations research, management science, economics, and mathematics. Chapter 4 focuses on a inventory theory model due to Cheng[3] that is solved using geometric programming techniques.

CHAPTER 2.

MINIMIZING POSYNOMIALS

This chapter contains the major theorems and definitions associated with geometric programming[7].

The general geometric programming problem has the following form:

Minimize

$$g_0(t) = \sum_{i=1}^{n_0} c_{0i} \prod_{j=1}^{m} t_j^{a_{0ij}} \tag{2-1}$$

Subject to

$$g_k(t) = \sum_{i=1}^{n_k} c_{ki} \prod_{j=1}^{m} t_j^{a_{kij}} \leq 1 \qquad for\ k = 1, 2, ..., p. \tag{2-2}$$

In the above problem, each coefficient c_{ki} and variable t_j must be strictly greater than zero. The exponents a_{kij} are real constants that can be either positive or negative. Functions that have these qualities are called **posynomials**. Geometric programming is the minimization of a posynomial subject to posynomials that are bounded by 1.

(2-1) and (2-2) is the primal problem associated with geometric programming. The coresponding dual problem is given by:

Maximize

$$v(\delta) = \prod_{k=0}^{p} \prod_{i=1}^{n_k} \left(\frac{c_{ki}}{\delta_{ki}} \right)^{\delta_{ki}} \left(\prod_{k=1}^{p} (\lambda_k)^{\lambda_k} \right) \tag{2-3}$$

Subject to

$$\sum_{i=1}^{n_0} \delta_{0i} = 1 \qquad (normality\ condition), \tag{2-4}$$

$$\sum_{k=0}^{p} \sum_{i=0}^{n_k} a_{kij} \delta_{ki} = 0 \qquad for\ j = 1, ..., m \qquad (orthogonality\ condition), \tag{2-5}$$

$$\delta_{ki} \geq 0, \qquad for\ k = 0, 1, ...p \qquad i = 1, ...n_k,$$

where the λ's are defined in terms of the δ's as

$$\lambda_k = \sum_{i=1}^{n_k} \delta_{ki}, \qquad for\ k = 1, ...p. \tag{2-6}$$

Here the δ_{ki}'s are the components of a vector with two indices where

δ_{0i}= the weights that are associated with the objective function of the primal problem and

δ_{ki}=the weights that are associated with the constraint functions of the primal problem .

The number of variables in the dual problem is equal to n, where

$$n = \sum_{k=0}^{p} n_k.$$

The normality and orthogonality conditions form $m+1$ linear equality constraints. The difference between the number of variables and the number of independent linear equations is normally known as the number of degrees of freedom. Thus these equations have $n - (m+1)$ degrees of freedom. In geometric programming, this number $[n - (m+1)]$ is called the number of degrees of difficultly, since the larger this number, the harder the problem is to solve[2].

Since all of the constraints of the dual problems are linear, the constraints form a convex set. In general, the objective function (2-3) is neither convex nor concave, but maximizing $v(\delta)$ is equivalent to maximizing $z(\delta) = ln[v(\delta)]$. This transformation can be made since all δ_{ki}'s are between zero and one, and the $ln[v(\delta)]$ is a monotonic function in the dual variables δ. The function $z(\delta)$ is concave with respect to the weights since it is the sum convex functions. Therefore, the dual problem is a convex programming problem. Hence, any local maximum is also a global maximum[2].

The following results link the primal and the dual problems together.

Theorem 2.1. *Suppose that t is a vector feasible to the primal geometric programming problem (2-1)-(2-2) and that δ is feasible to the dual problem (2-3)-(2-6). Then $g_0(\mathbf{t}) \geq v(\delta)$.*

Proof. Define

$$u_{ki} = c_{ki} \prod_{j=1}^{m} t_j^{a_{kij}} \qquad for\ i = 1, .., n_k \qquad k = 0, 1, ..., p. \tag{2-7}$$

The arithmetic-geometric mean inequality can be written as

$$\delta_{01}V_1 + \delta_{02}V_2 + ... + \delta_{pn}V_n \geq {V_1}^{\delta_{01}}{V_2}^{\delta_{02}} * ... * {V_n}^{\delta_{pn}} \tag{2-8}$$

where $V_1, V_2, ..., V_n$ are positive variables and δ_{ki} are postive weights.

Now let $u_{ki} = \delta_{ki}V_i,\ for\ i = 1, 2, ..., n.$ (2-8) can be rewritten as

$$u_{01} + u_{02} + ... + u_{pn} \geq \left(\frac{u_{01}}{\delta_{01}}\right)^{\delta_{01}} \left(\frac{u_{02}}{\delta_{02}}\right)^{\delta_{02}} * ... * \left(\frac{u_{pn}}{\delta_{pn}}\right)^{\delta_{pn}} \tag{2-9}$$

Define

$$\lambda_k = \sum_{i=1}^{n_k} \gamma_{ki} \qquad for\ k = 1, ..., p \tag{2-10}$$

where the γ_{ki} are the weights that are not normalized. Hence from (2-6),

$$\lambda_k = \sum_{i=1}^{n_k} \delta_{ki} = \sum_{i=1}^{n_k} \gamma_{ki} \tag{2-11}$$

Therefore δ_{ki} is related to γ_{ki} in the following manner:

$$\delta_{ki} = \frac{\gamma_{ki}}{\lambda_k} \tag{2-12}$$

(2-11) implies that,

$$\sum_{i=1}^{n_k} \frac{\gamma_{ki}}{\lambda_k} = 1. \tag{2-13}$$

Substituting (2-12) and (2-13) into (2-9) yields the following inequality:

$$u_{01} + u_{02} + ... + u_{pn} \geq \left(\frac{u_{01}}{\gamma_{01}}\right)^{\frac{\gamma_{01}}{\lambda_0}} \left(\frac{u_{02}}{\gamma_{02}}\right)^{\frac{\gamma_{02}}{\lambda_0}} * ... * \left(\frac{u_{pn}}{\gamma_{pn}}\right)^{\frac{\gamma_{pn}}{\lambda_p}} \lambda_k. \tag{2-14}$$

Now raising both sides of (2-14) to the λ_k power,

$$[u_1 + u_2 + ... + u_n]^{\lambda_k} \geq \left(\frac{u_1}{\gamma_1}\right)^{\gamma_1} \left(\frac{u_2}{\gamma_2}\right)^{\gamma_2} * ... * \left(\frac{u_n}{\gamma_n}\right)^{\gamma_n} {\lambda_k}^{\lambda_k} \qquad for\ k = 0, 1, ..., p. \tag{2-15}$$

Now consider the primal problem (2-1) and (2-2). It can be written as

Minimize

$$[g_0(t)]^{\lambda_0} = \left(\sum_{i=1}^{n_0} u_{0i}\right)^{\lambda_0} \tag{2-16}$$

Subject to

$$[g_k(t)]^{\lambda_k} \leq 1 \qquad for\ k = 1, .., p. \tag{2-17}$$

Applying (2-15) to the objective function yields,

$$g_0^{\lambda_0} \geq \left(\frac{u_{01}}{\gamma_{01}}\right)^{\gamma_{01}} \left(\frac{u_{02}}{\gamma_{02}}\right)^{\gamma_{02}} * ... * \left(\frac{u_{0n}}{\gamma_{0n}}\right)^{\gamma_{0n}} \lambda_0^{\lambda_0} \tag{2-18}$$

where $g_0(t) = u_{01} + u_{02} + ... + u_{0n_0}$. Similarly, if

$$g_k(t) = u_{k1} + u_{k2} + ... + u_{kn_k} \qquad for\ k = 1, ..., p$$

the constraint inequality can be written as,

$$1 \geq g_k^{\lambda_k} \geq \left(\frac{u_{ki}}{\gamma_{k1}}\right)^{\gamma_{k1}} \left(\frac{u_{k2}}{\gamma_{k2}}\right)^{\gamma_{k2}} * ... * \left(\frac{u_{kn_k}}{\gamma_{kn}}\right)^{\gamma_{kn}} \lambda_k^{\lambda_k} \qquad for\ k = 1, ...p. \tag{2-19}$$

Multiplying the left-hand side of (2-18) by the left-hand side of (2-19) and the right-hand side of (2-18) by the right-hand side of (2-19) gives,

$$g_0^{\lambda_0} \geq \left(\frac{u_{01}}{\gamma_{01}}\right)^{\gamma_{01}} \left(\frac{u_{02}}{\gamma_{02}}\right)^{\gamma_{02}} * ... * \left(\frac{u_{pn}}{\gamma_{pn}}\right)^{\gamma_{pn}} \lambda_0^{\lambda_0} \lambda_1^{\lambda_1} * ... * \lambda_p^{\lambda_p}. \tag{2-20}$$

Since δ_{ki} must be feasible to the dual problem, from (2-4)

$$\lambda_0 = \sum_{i=1}^{n_0} \delta_{01} = 1$$

Using this, (2-12) can be rewritten as

$$\frac{\gamma_{ki}}{\lambda_0} = \delta_{ki}$$

(2-20) becomes

$$g_0(t) \geq \left(\frac{u_{01}}{\delta_{01}}\right)^{\delta_{01}} \left(\frac{u_{02}}{\delta_{02}}\right)^{\delta_{02}} * ... * \left(\frac{u_{pn}}{\delta_{pn}}\right)^{\delta_{pn}} \prod_{k=1}^{p} \lambda_k^{\lambda_k}. \tag{2-21}$$

Substituting in (2-7) and applying condition (2-5), (2-21) becomes

$$g_0(t) \geq \prod_{k=0}^{p} \prod_{i=1}^{n_k} \left(\frac{c_{ki}}{\delta_{ki}} \right)^{\delta_{ki}} \prod_{k=1}^{p} \left(\lambda_k^{\lambda_k} \right) = v(\delta). \tag{2-22}$$

Hence

$$g_0(t) \geq v(\delta).$$

□

The next theorem shows that the primal and dual problems have an even stronger relationship.

Theorem 2.2. *Suppose that t is a vector feasible to the primal problem and δ is feasible to the dual problem. Then $g_0(t) = v(\delta)$ if and only if*

$$\delta_{0i} = \frac{c_{0i} \prod_{j=1}^{m} (t_j)^{a_{0ij}}}{g_0(t)} \qquad i = 1, .., n_0 \tag{2-23}$$

and

$$\delta_{ki} = \lambda_k c_{ki} \prod_{j=1}^{m} (t_j)^{a_{kij}} \qquad i = 1, ..., n_k \quad k = 1, ..., p. \tag{2-24}$$

Proof. Suppose $g_0(t) = v(\delta)$.

From Theorem 2.1,

$$g_0(t) = \sum_{i=1}^{n_0} c_{0i} \prod_{j=1}^{m} t_j^{a_{0ij}} \geq v(\delta) = \prod_{k=0}^{p} \prod_{i=1}^{n_k} \left(\frac{c_{ki}}{\delta_{ki}} \right)^{\delta_{ki}} \left(\prod_{k=1}^{p} (\lambda_k)^{\lambda_k} \right) \tag{2-25}$$

Since $g_0(t) = v(\delta)$, the above inequality is actually an equality. By the properties of the arithmetic-geometric mean, this is only true if

$$\delta_{ki} = B_k c_{ki} \prod_{j=1}^{m} t_j^{a_{kij}} \tag{2-26}$$

for $B_k > 0$ and k=0,1,..,p. Summing (2-26) over i,

$$\sum_{i=1}^{n_k} \delta_{ki} = B_k \sum_{i=1}^{n_k} c_{ki} \prod_{j=1}^{m} (t_j)^{a_{kij}} \tag{2-27}$$

When k=0, from the normality condition, (2-27) becomes

$$1 = B_0 \sum_{i=1}^{n_0} c_{0i} \prod_{j=1}^{m} t_j^{a_{0ij}}. \tag{2-28}$$

Thus

$$B_0 = \frac{1}{g_0(t)}. \tag{2-29}$$

Substituting (2-29) into (2-26),

$$\delta_{0i} = \frac{1}{g_0(t)} \, c_{0i} \prod_{j=1}^{m} t_j^{a_{oij}}$$

which shows (2-23) is true.

From the feasibility condition (2-2), $g_k(t) \leq 1$ for k=1,...,p. Since it is assumed that $g_0(t) = v(\delta)$ then equality can be assumed in (2-2). Thus,

$$g_k(t) = \sum_{i=1}^{n_k} c_{ki} \prod_{j=1}^{m} t_j^{a_{kij}} = 1 \qquad k = 1, ..., p.$$

Hence, (2-27) becomes

$$\sum_{i=1}^{n_k} \delta_{ki} = B_k.$$

Now using (2-6),

$$\lambda_k = \sum_{i=1}^{n_k} \delta_{ki} = B_k$$

or $\lambda_k = B_k$. Therefore,

$$\delta_{ki} = \lambda_k c_{ki} \prod_{j=1}^{m} t_j^{a_{kij}}.$$

For sufficiency, suppose

$$\delta_{0i} = \frac{c_{0i} \prod_{j=1}^{m} (t_j)^{a_{0ij}}}{g_0(t)} \qquad i = 1, ..., n_0 \tag{2-30}$$

and

$$\delta_{ki} = \lambda_k c_{ki} \prod_{j=1}^{m} (t_j)^{a_{kij}} \qquad i = 1, .., n_k \qquad k = 1, ..., p \tag{2-31}$$

Summing (2-30) over i,

$$\sum_{i=1}^{n_0} \delta_{0i} = \frac{1}{g_0(t)} \sum_{i=1}^{n_0} c_{0i} \prod_{j=1}^{m} (t_j)^{a_{0ij}} \tag{2-32}$$

From condition (2-4),

$$\sum_{i=1}^{n_0} \delta_{0i} = 1$$

Thus (2-32) becomes,

$$\sum_{i=1}^{n_0} \delta_{0i} = \sum_{i=1}^{n_0} c_{0i} \prod_{j=1}^{m} (t_j)^{a_{0ij}} = g_0(t). \tag{2-33}$$

Now consider (2-31). Rewriting yields,

$$\frac{1}{\lambda_k \prod_{j=1}^{m} (t_j)^{a_{kij}}} = \frac{c_{ki}}{\delta_{ki}}. \tag{2-34}$$

From Theorem 2.1, $g_0(t) \geq v(\delta)$ or

$$g_0(t) = \left(\sum_{i=1}^{n_0} c_{0i} \prod_{j=1}^{m} (t_j)^{a_{0ij}} \right) \geq \prod_{k=0}^{p} \prod_{i=1}^{n_k} \left(\frac{c_{ki}}{\delta_{ki}} \right)^{\delta_{ki}} \left(\prod_{k=1}^{p} \lambda_k^{\lambda_k} \right) = v(\delta).$$

Substituting (2-33) and (2-34) into the above inequality,

$$g_0(t) = \left(\sum_{i=1}^{n_0} \delta_{0i}\right) \geq \prod_{k=0}^{p} \prod_{i=1}^{n_k} \left(\frac{1}{\lambda_k \prod_{j=1}^{m} (t_j)^{a_{kij}}}\right)^{\delta_{ki}} \prod_{k=1}^{p} \lambda_k^{\lambda_k} = v(\delta).$$

From (2-5) and (2-6),

$$\sum_{k=0}^{p} \sum_{i=1}^{n_k} a_{kij} \delta_{ki} = 0$$

and

$$\lambda_k = \sum_{i=1}^{n_k} \delta_{ki}.$$

Therefore

$$g_0(t) = \left(\sum_{i=1}^{n_0} \delta_{0i}\right) \geq \prod_{k=1}^{p} \left(\frac{1}{\lambda_k}\right)^{\lambda_k} \prod_{k=1}^{p} {\lambda_k}^{\lambda_k} = v(\delta).$$

Recall by normality that $\sum_{i=1}^{n_0} \delta_{0i} = 1$. Hence

$$g_0(t) = v(\delta).$$

□

Theorem 2.3. *Let δ^* be a solution to the dual problem (2-3)-(2-6). Then any vector $t > 0$ that satisfies the systems of equations*

$$c_{0i} \prod_{j=1}^{m} (t_j)^{a_{oij}} = v(\delta^*) {\delta_{oi}}^* \qquad i = 1, ..., n_0 \tag{2-35}$$

$$c_{ki} \prod_{j=1}^{m} (t_j)^{a_{kij}} = \frac{{\delta_{ki}}^*}{{\lambda_k}^*} \qquad i = 1, ..., n_k \tag{2-36}$$

for all k where $\lambda_k{}^* > 0$ *and* $g_k(t) \leq 1$ *for all k such that* $\lambda_k{}^* = 0$ *is a minimizing vector for the primal problem.*

Proof. Summing (2-36) over i yields

$$\sum_{i=1}^{n_k} c_{ki} \prod_{j=1}^{m} (t_j)^{a_{kij}} = \frac{1}{\lambda_k{}^*} \sum_{i=1}^{n_k} \delta_{ki}{}^* \quad k = 1, ..., p. \tag{2-37}$$

Since

$$\lambda_k{}^* = \sum_{i=1}^{n_k} \delta_{ki}$$

(2-36) is really

$$\sum_{i=1}^{n_k} c_{ki} \prod_{j=1}^{m} (t_j)^{a_{kij}} = 1.$$

Hence the primal constraints are satisfied for those k where $\lambda_k{}^* > 0$ and t is a feasible point.

Now summing (2-35) over i,

$$\sum_{i=1}^{n_0} c_{0i} \prod_{j=1}^{m} (t_j)^{a_{0ij}} = v(\delta^*) \sum_{i=1}^{n_0} \delta_{0i}{}^*.$$

From the normality condition

$$\sum_{i=1}^{n_0} \delta_{0i} = 1$$

the above equation is really

$$g_0(t) = v(\delta^*).$$

By Theorem 2.1, t is the optimal point. □

Equations (2-35) and (2-36) are used to obtain an optimal vector t from the dual solution. Taking the natural logarithm of both sides of eqautions (2-35) and (2-36) results in a system of linear equation in t_j. Specifically,

$$\sum_{j=1}^{m} a_{0ij}\ ln(t_j) = ln[v(\delta^*)] + ln({\delta_{0i}}^*) - ln(c_{0i})$$

$$\sum_{j=1}^{m} a_{kij}\ ln(t_j) = ln({\delta_{ki}}^*) - ln({\lambda_k}^*) - ln(c_{ki})$$

for all k where is ${\lambda_k}^* > 0$. Linear algebra techniques can be used to solve the system of equations for the t_j's.

To illustrate the theory previously developed, consider the following examples.

Example 2.1.

Minimize

$$g_0(t) = {t_3}^{0.8}{t_4}^{1.4}$$

Subject to

$$g_1(t) = {t_1}^{-2}{t_3}^{-1} + t_2{t_3}^{-1} \leq 1$$

$$g_2(t) = t_1{t_4}^{-1} + {t_2}^{-1}{t_4}^{-1} \leq 1$$

Solution.

From Definition (2.1), D=0.

The corresponding dual problem is

Maximize

$$v(\delta) = \left(\frac{1}{\delta_{01}}\right)^{\delta_{01}} \left(\frac{1}{\delta_{11}}\right)^{\delta_{11}} \left(\frac{1}{\delta_{12}}\right)^{\delta_{12}} \left(\frac{1}{\delta_{21}}\right)^{\delta_{21}} \left(\frac{1}{\delta_{22}}\right)^{\delta_{22}} {\lambda_1}^{\lambda_1} {\lambda_2}^{\lambda_2}$$

Subject to

$$\begin{aligned}
-2\delta_{11} \qquad\qquad + \delta_{21} &= 0 \\
+\delta_{12} \qquad\qquad &= 0 \\
0.8\delta_{01} - \delta_{11} - \delta_{12} \qquad &= 0 \\
1.4\delta_{01} \qquad\qquad - \delta_{21} - \delta_{22} &= 0 \\
\delta_{01} &= 1 \\
\lambda_1 = \delta_{11} + \delta_{12} & \\
\lambda_2 = \delta_{21} + \delta_{22} &
\end{aligned}$$

Solving the above set of linear equations yields,

$$\delta_{01} = 1; \quad \delta_{11} = 0.6; \quad \delta_{12} = 0.2; \quad \delta_{21} = 1.2; \quad \delta_{22} = 0.2$$

Therefore, $v(\delta^*) = 2.78$. From Theorem 2.2, $v(\delta^*) = 2.78 = g_0(t^*)$. Now using Theorem 2.3 to recover the optimal vector t^*, (2-35) and (2-36) yield the following system of equations:

$${t_3}^{0.8} {t_4}^{1.4} = 2.78$$

$$t_2 {t_3}^{-1} = 0.25$$

$$t_2{}^{-1}t_4{}^{-1} = 0.143.$$

Hence,

$$t^* = (0.055, 108.3, 433.21, 0.065).$$

Example 2.2.

Minimize

$$g_0(t) = t_1{}^{-1}t_2{}^{-1}t_3 + 25t_1t_2{}^{-1}$$

Subject to

$$g_1(t) = 0.1t_1t_2 \leq 1$$

$$g_2(t) = 5t_1t_3{}^{-1} + t_2{}^2t_3{}^{-1} \leq 1$$

Solution.

D=1.

The dual problem is to

Maximize

$$v(\delta) = \left(\frac{1}{\delta_{01}}\right)^{\delta_{01}} \left(\frac{25}{\delta_{02}}\right)^{\delta_{02}} \left(\frac{0.1}{\delta_{11}}\right)^{\delta_{11}} \left(\frac{5}{\delta_{21}}\right)^{\delta_{21}} \left(\frac{1}{\delta_{22}}\right)^{\delta_{22}} \lambda_1{}^{\lambda_1}\lambda_2{}^{\lambda_2}$$

Subject to

$$-\delta_{01} + \delta_{02} + \delta_{11} + \delta_{21} \qquad = 0$$

$$-\delta_{01} - \delta_{02} + \delta_{11} \qquad + 2\delta_{22} = 0$$

$$\delta_{01} \qquad - \delta_{21} - \delta_{22} = 0$$

$$\delta_{01} + \delta_{20} = 1$$

$$\lambda_1 = \delta_{11}$$

$$\lambda_2 = \delta_{21} + \delta_{22}.$$

Solving the constraints in terms of δ_{01},

$\delta_{02} = -\delta_{01} + 1$, $\delta_{11} = \frac{2}{3}\delta_{01} - \frac{1}{3}$, $\delta_{21} = \frac{4}{3}\delta_{01} - \frac{2}{3}$, and $\delta_{22} = -\frac{1}{3}\delta_{01} + \frac{2}{3}$.

Substituting these values into the transformed dual,

$$z(\delta_{01}) = \delta_{01} ln\left(\frac{1}{\delta_{01}}\right) + (-\delta_{01} + 1)\, ln\left(\frac{25}{-\delta_{01} + 1}\right) +$$
$$\left(\frac{2}{3}\delta_{01} + \frac{1}{3}\right) ln\left(\frac{0.1}{\frac{2}{3}\delta_{01} - \frac{1}{3}}\right) + \left(\frac{4}{3}\delta_{01} - \frac{2}{3}\right) ln\left(\frac{5}{\frac{4}{3}\delta_{01} - \frac{2}{3}}\right) +$$
$$\left(-\frac{1}{3}\delta_{01} + \frac{2}{3}\right) ln\left(\frac{1}{-\frac{1}{3}\delta_{01} + \frac{2}{3}}\right) + \left(\frac{2}{3}\delta_{01} - \frac{1}{3}\right) ln\left(\frac{2}{3}\delta_{01} - \frac{1}{3}\right) + \delta_{01} ln(\delta_{01}).$$

Now using MapleV to take the first derivative,

$$z'(\delta_{01}) = ln\left(\frac{1}{\delta_{01}}\right) - ln\left(\frac{25}{-\delta_{01} + 1}\right) + \frac{2}{3} ln\left(\frac{0.1}{\frac{2}{3}\delta_{01} - \frac{1}{3}}\right) + \frac{4}{3} ln\left(\frac{5}{\frac{4}{3}\delta_{01} - \frac{2}{3}}\right)$$
$$-\frac{1}{3} ln\left(\frac{1}{-\frac{1}{3}\delta_{01} - \frac{2}{3}}\right) + \frac{2}{3} ln\left(\frac{2}{3}\delta_{01} - \frac{1}{3}\right) + ln(\delta_0 1).$$

Setting $z'(\delta_{01}) = 0$ and solving, implies $\delta_{01} = 0.586$. Thus,

$$\delta_{01} = 0.586; \quad \delta_{02} = 0.414; \quad \delta_{11} = 0.057; \quad \delta_{21} = 0.114; \quad \delta_{22} = 0.471$$

$$v(\delta^*) = 10.5 = g_0(t^*)$$

and

$$t^* = (1.32, 7.58, 61.56).$$

CHAPTER 3.

SENSITIVITY ANALYSIS

Often there are constant terms in optimization problems that are independent of the problem variables. Yet, these constants are closely related to the actual performance of the model, and in most cases they cannot be predicted before the model is put into practice. For example in engineering design problems, the cost of some components fluctuate. Prior to preceeding with a project, it is important for a decision maker to estimate the impact of all of these uncertainties upon the calculated optimum value.

Definition 3.1. *Determining the rate of change of the optimum value with respect to perturbations in the constraint function is called a* **sensitivity analysis** *or* **postoptimality.**

The approach used here was first derived in full detail by Beightler, Phillips, and Wilde [1]. Basically, the independent variables are first divided into two groups: **decision variables (d)** and **state** or **solution variables (s)**. As their names suggest, decision variables can be manipulated freely; whereas, state variables adjust to keep new points feasible. After the variables have been divided up, the linear terms of the Taylor expansions for the differentials of the constraint and objective functions are then solved for the state differentials as functions of the

decision differentials.

Recall from (2-2) that

$$\sum_{i=1}^{n_k} c_{ki} \prod_{j=1}^{m} t_j^{a_{kij}} \leq 1 \qquad k = 1, .., p.$$

Rearranging (2-2) yields,

$$g_k(t) = 1 - \sum_{i=1}^{n_k} c_{ki} \prod_{j=1}^{m} t_j^{a_{kij}} \geq 0.$$

Hence, the primal problem can be written:

Minimize

$$y = g_0(t) = \sum_{i=1}^{n_0} c_{0i} \prod_{j=1}^{m} t_j^{a_{0ij}} \tag{3-1}$$

Subject to

$$g_k(t) \geq 0 \qquad k = 1, .., p \tag{3-2}$$

Introducing slack variables V_k into (3-2), implies

$$g_k(t) - V_k = 0 \qquad k = 1, ..., p; \tag{3-3}$$

where

$$V_k \geq 0$$

For any feasible t, let M of the p slack variables V_k be zero. These variables will be indexed by $w = 1, ..., M$. So now,

$$V_w = 0; \qquad w = 1, ..., M \tag{3-4}$$

and

$$V_l > 0 \qquad l = M+1, ..., p. \tag{3-5}$$

The first M constraints are called **tight** for they are binding at optimality or

$$g_w(t) = 0 \qquad w = 1, ..., M.$$

The other constraints are **loose** for they are nonbinding at optimality or

$$g_l(t) > 0 \qquad l = M+1, ..., p.$$

Now let

$$s_w \equiv t_j \qquad w, j = 1, ..., M$$

be solution variables, and let

$$d_r \equiv t_j \qquad r, j = M+1, ..., m$$

be the decision variables. Now consider the effects of small perturbations $\partial \mathbf{s}$ and $\partial \mathbf{d}$ on the objective function $y(\mathbf{s}, \mathbf{d})$ and the tight constraint functions $g_w(\mathbf{s}, \mathbf{d})$. The feasible region consists of all points t that satisfy $g_w(t) = 0$. Thus, every point is a boundary point. In every neighborhood of a feasible point t there are both infeasible points where some of the differentials $\partial g_w \neq 0$, and feasible points where all $\partial g_w = 0$. Consider only the points in the feasible neighborhood that satisfies $\partial g = 0$.

The first-order terms of the Taylor expansions give

$$\partial y = \left(\frac{\partial y}{\partial \mathbf{s}}\right) \partial \mathbf{s} + \left(\frac{\partial y}{\partial \mathbf{d}}\right) \partial \mathbf{d} \tag{3-5}$$

$$\partial \mathbf{g} = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right) \partial \mathbf{s} + \left(\frac{\partial \mathbf{g}}{\partial \mathbf{d}}\right) \partial \mathbf{d} \tag{3-6}$$

Here $\mathbf{g}$ denotes a vector of the constraint functions g_k. In addition, the partial derivatives are presumed to be known constants; therefore, Equations (3-5) and (3-6) constitute $M+1$ linear equations with $m+1$ unknown differentials. Assume the equations are linearly independent, if they are not, then take the largest number that form an independent set and discard the rest as redundant. This automatically rules out the case where $M > m$. If $M = m$, then the uninteresting case of $\partial \mathbf{s_w} = 0$ arises. This case implies that there is no feasible region at all. Hence from now on assume only $M < m$. The difference $m - M$ actually coresponds to the degree of difficulty of the geometric programming problem.

Rearranging Equations (3-5) and (3-6)

$$-\partial y + \left(\frac{\partial y}{\partial \mathbf{s}}\right) \partial \mathbf{s} = -\left(\frac{\partial y}{\partial \mathbf{d}}\right) \partial \mathbf{d} \tag{3-7}$$

$$\left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right) \partial \mathbf{s} = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{d}}\right) \partial \mathbf{d}. \tag{3-8}$$

Since the equations are linearly independent, the Jacobian ($\mathbf{J}$) is not zero. Thus

$$\left|\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right| \equiv |\mathbf{J}| \equiv \begin{vmatrix} \frac{\partial g_1}{\partial s_1} & \cdots & \frac{\partial g_1}{\partial s_M} \\ \vdots & & \vdots \\ \frac{\partial g_M}{\partial s_1} & \cdots & \frac{\partial g_M}{\partial s_M} \end{vmatrix} \equiv \frac{\partial(g_1, ..., g_M)}{\partial(s_1, ..., s_M)}$$

Multiplying (3-8) by $\left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right)^{-1}$,

$$\partial \mathbf{s} = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right)^{-1}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{d}}\right)\partial \mathbf{d} \equiv \frac{\partial \mathbf{s}}{\partial \mathbf{d}}\partial \mathbf{d}$$

Now substituting this into (3-7),

$$\partial y = \left[-\frac{\partial y}{\partial \mathbf{d}} + \frac{\partial y}{\partial \mathbf{s}}\left(\frac{\partial \mathbf{s}}{\partial \mathbf{d}}\right)\right]\partial \mathbf{d} \equiv \frac{\partial z}{\partial \mathbf{d}}\partial \mathbf{d} \tag{3-9}$$

Here $z \equiv z(\mathbf{d})$ is the reduced objective function that only depends on decision variables. Now multiply (3-6) by $\left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right)^{-1}$ and rearrange to get

$$\partial \mathbf{s} = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right)^{-1}\partial \mathbf{g} - \left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right)^{-1}\frac{\partial \mathbf{g}}{\partial \mathbf{d}}\partial \mathbf{d}.$$

Substituting this into equation (3-7) yields

$$\partial z = \frac{\partial y}{\partial \mathbf{s}}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right)^{-1}\partial \mathbf{g} + \frac{\partial z}{\partial \mathbf{d}}\partial \mathbf{d}$$

Since t^* is a local optimum $\frac{\partial z}{\partial \mathbf{d}}$ must be zero there, so

$$\partial y^* = \frac{\partial y}{\partial \mathbf{s}}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right)^{-1}\partial \mathbf{g} \equiv \left(\frac{\partial z}{\partial \mathbf{g}}\right)^*\partial \mathbf{g}.$$

Thus,

$$\left(\frac{\partial z}{\partial \mathbf{g}}\right)^* = \left(\frac{\partial y}{\partial \mathbf{s}}\right)^*\left(\frac{\partial \mathbf{g}}{\partial \mathbf{s}}\right)_*^{-1} \tag{3-10}$$

Equation (3-10) is a matrix of **sensitivity coefficients** since its elements give the change in the optimal value of y with respect to perturbations in the constraint functions $g_k(t)$. Since the constrained derivative $\left(\frac{\partial z}{\partial d}\right)$ vanishes at the optimum,

$\left(\frac{\partial z}{\partial \mathbf{g}}\right)$ does not depend on the set of variables chosen to be the states. Hence one could assume that there could be a different value of the sensitivity coefficients for every possible set of decisions.

To illustrate this method of performing a sensitivity analysis, consider **Example 2.2** from **CHAPTER 2.**

Example 3.1.

Minimize

$$y = {t_1}^{-1}{t_2}^{-1}t_3 + 25t_1{t_2}^{-1}$$

Subject to

$$0.1t_1t_2 \leq 1$$

$$5t_1{t_3}^{-1} + t_2^2{t_3}^{-1} \leq 1$$

Recall that for this problem D=1 and that the solution is $v(\delta^*) = 10.5$ and that $t^* = (1.32, 7.58, 61.56)$.

As a sensitivity analysis, consider the effects of changes in the first constraint on $g_0(t)$. Let t_3 be the decision variable and let the other variables be state variables.

From (3-3), the constraints must be rewritten as

$$g_1 = 10 - t_1t_2 = 0$$

$$g_2 = 1 - 5t_1{t_3}^{-1} - t_2^2{t_3}^{-1} = 0$$

In this case Equation (3-10) becomes,

$$\frac{\partial z^*}{\partial g_1} = \frac{\begin{vmatrix} \frac{\partial y}{\partial s_1} & \frac{\partial y}{\partial s_2} \\ \frac{\partial g_2}{\partial s_1} & \frac{\partial g_2}{\partial s_2} \end{vmatrix}}{\begin{vmatrix} \frac{\partial g_1}{\partial s_1} & \frac{\partial g_1}{\partial s_2} \\ \frac{\partial g_2}{\partial s_2} & \frac{\partial g_2}{\partial s_2} \end{vmatrix}}$$

Let $s_1 = t_1$ and $s_2 = t_2$.

$$\frac{\partial y}{\partial s_1} = -{t_1}^{-2}{t_2}^{-1}t_3 + 25{t_2}^{-1} = -1.362$$

$$\frac{\partial y}{\partial s_2} = -{t_1}^{-1}t_2^{-2}t_3 - 25t_1t_2^{-2} = -0.820$$

$$\frac{\partial g_2}{\partial s_1} = -5t_3^{-1} = -0.081$$

$$\frac{\partial g_2}{\partial s_2} = -2t_2t_3^{-1} = -0.246$$

$$\frac{\partial g_1}{\partial s_1} = -t_2 = -7.58$$

$$\frac{\partial g_1}{\partial s_2} = -t_1 = -1.32$$

Therefore,

$$\frac{\partial z^*}{\partial g_1} = \frac{\begin{vmatrix} -1.362 & -0.820 \\ -0.081 & -0.246 \end{vmatrix}}{\begin{vmatrix} -7.58 & -1.32 \\ -0.081 & -0.246 \end{vmatrix}} = \frac{0.2686}{1.75776} = 0.1528$$

As the coefficient of the product t_1t_2 is increased, the value of g_1 decreases. Hence, the optimal solution decreases at a rate of 0.1528.

Now consider an example from a machine rate model solved by Petropoulos[8].

Example 3.2.

Minimize

$$y = 452t_1^{-1}t_2^{-1} + 10^{-5}t_1^{2.33}t_2^{0.4}$$

Subject to

$$1.1 * 10^4 t_1^{-1.52} t_2 \leq 1$$

$$1.93 * 10^{-2} t_1 t_2^{0.83} \leq 1$$

For this problem $D = 1$, and the solution is $g_0(t^*) = v(\delta^*) = 11.64$ and $t^* = (178, 0.24)$.

To perform a sensitivity analysis, consider the effects on y^* of a small change in $g_1(t)$. Let t_2 be the state variable.

Rewriting the constraints according to (3-3),

$$g_1 = 1 - 1.10 * 10^4 t_1^{-1.52} t_2 = 0$$

$$g_2 = 1 - 1.93 * 10^{-2} t_1 t_2^{0.83} = 0.$$

From (3-10),

$$\frac{\partial z^*}{\partial g_1} = \frac{\frac{\partial y}{\partial t_2}}{\frac{\partial g_1}{\partial t_2}}.$$

$$\frac{\partial y}{\partial t_2} = -452t^{-1}t_2^{-2} + 4 * 10^{-6} t_1^{2.33} t_2^{0.6} = -42.436$$

$$\frac{\partial g_1}{\partial t_2} = -1.10 * 10^4 t_1^{-1.52} = -4.176.$$

Therefore,

$$\frac{\partial z^*}{\partial g_1} = 10.162$$

This result indicates that an increase in the coefficient of $t_1 t_2$ would result in a decrease in the value of g_1. This results in a decrease in the value of the objective optimum at a rate of 10.162.

CHAPTER 4.

CONTROLLING INVENTORY WITH GEOMETRIC PROGRAMMING

This chapter illustrates how geometric programming can be applied to solve a real-world problem. This model was derived by T. C. E. Cheng[3].

Economic order quantity (EOQ) answers the inventory control question of how much product to order while at the same time minimizing total cost. The classical EOQ model, known as Wilson's model, explicitly assumes that demand is known, continuous and constant through time. It also assumes that the cost of each unit is constant through time and is independent of the demand for the unit. However, there are production situations where demand rate and the unit production cost are related. For example, from a manufacturing viewpoint, if the demand for a unit is high, then a company could use a more efficient production process which would lower the unit production cost. Moreover, in a distribution environment, if the demand for a unit is sufficient enough, a firm could take advantage of a long term supply contract that offers a unit price discount. Hence, it is reasonable to assume that the demand rate and unit production cost are inversely related to each other[3].

Wilson's Model has the implied assumption that items produced are of a perfect quality. Of course realistically, product quality is sometimes imperfect. The qual-

ity of the items produced depends upon the reliability of the production process. Therefore, investing in improving the reliability of the production process would increase the quality of the product. However, at the same time, the cost of the production of each unit would also increase. Thus, it could be assumed that unit production costs are an increasing function of process reliability[3].

In a situation where the demand rate is dependent upon the unit production cost and the unit production cost is dependent upon process reliability, the classical EOQ model must be modified in order to minimize long term average costs.

The following variables are needed in order to construct a model of this type of situation.

S=setup cost per batch

H=inventory carrying cost per item per unit time

D=demand rate (a decision variable)

q=production quantity per batch (a decision variable)

r=production process reliability (a decision variable)

V(D,r)=unit cost of production

C(D,q,r)=total cost of setup, production and inventory holding per unit time

The following are the basic assumptions of this EOQ model:

(1) Production is instantaneous

(2) No back-order is allowed

(3) Demand for the product is greater than supply

(4) The unit cost of production is directly related to the process reliability and inversely related to the demand rate.

The first two assumptions are common assumptions used in the classical EOQ model. The third assumption allows the manufacturer to choose a production level, since any quantity of the item placed on the market will be immediately bought. This implies that production volume equals demand rate. The fourth assumption mostly results from the third assumption. As the demand rate and thus the production volume rise, the fixed cost per unit is lowered. This leads to a lower unit production cost. However, in order to reduce the amount spent when quality products are not produced, more is invested in improving the reliability of the production cost[6].

The objective is to minimize the long term average cost. This is mathematically equivalent to minimizing the total cost per unit time. Therefore, the problem can be expressed in the following manner:

Minimize

$$C(D,q,r) = \frac{Total\ cost\ per\ cycle}{Length\ of\ a\ cycle}.$$

The total cost per cycle is equal to the sum of the setup, production, and inventory carrying costs. The setup per batch is a deterministic variable given by S. The unit production cost is $qV(D,r)$. From assumption four, $V(D,r)$ can be written as the following power function:

$$V(D,r) = aD^{-b}r^{c}$$

where a,b, and c are non-negative real numbers that are chosen to provide the best fit of the estimated cost function. The derivatives

$$\frac{\partial V(D,r)}{\partial D} = -abD^{-(b+1)}r^c < 0$$

and

$$\frac{\partial V(D,r)}{\partial r} = acD^{-b}r^{c-1} > 0$$

varify that $V(D,r)$ is in accordance to the earlier assumption that unit production cost is an increasing function to production reliability and a decreasing function to the demand rate. The last component of $C(D,q,r)$ is that inventory carrying cost. According to Hax and Candea [6], inventory carrying cost are proportional to the size of the investment in inventory. Hence, the inventory carrying cost is the carrying charge times the unit cost of the item. The number of usable products per batch is qr. The inventory carrying cost is given by $\frac{Hq^2r^2}{2D}$. Hence,

$$Total\ cost\ per\ cycle = S + V(D,r)q + \frac{Hq^2r^2}{2D}$$

In the classical EOQ model, the production cycle is implicitly assumed to repeat infinitly. Thus, it is sufficient to analyze one typical cycle. The length of a production cycle is $\frac{qr}{D}$. A process reliability level of r means that of all the items produced in a production run, only r percent are of the acceptable quality to be used to meet demand. Therefore, in attempting to minimize the total cost per unit time, the process reliability cannot exceed one hundred percent.

Combining all of the preceeding ideas, yields the following EOQ model:

Minimize

$$C(D,q,r) = \frac{\left(S + aD^{-b}r^{c}q + \frac{Hq^2r^2}{2D}\right)}{\left(\frac{qr}{D}\right)} \tag{4-1}$$

Subject to

$$r \leq 1. \tag{4-2}$$

Now to solve the model using geometric programming, let $t_1 = D$, $t_2 = q$, and $t_3 = r$. Thus the primal problem is

Minimze

$$C(\mathbf{t}) = St_1t_2^{-1}t_3^{-1} + at_3^{c-1}t_1^{-b+1} + 0.5Ht_2t_3 \tag{4-3}$$

Subject to

$$t_3 \leq 1$$

The cooresponding dual problem is

Maximize

$$v(\delta) = \left(\frac{S}{\delta_{01}}\right)^{\delta_{01}} \left(\frac{a}{\delta_{02}}\right)^{\delta_{02}} \left(\frac{0.5H}{\delta_{03}}\right)^{\delta_{03}} \left(\frac{1}{\delta_{11}}\right)^{\delta_{11}} {\lambda_1}^{\lambda_1} \tag{4-4}$$

Subject to

$$\delta_{01} + (-b+1)\delta_{02} = 0$$

$$-\delta_{01} \qquad\qquad + \delta_{03} = 0$$

$$-\delta_{01} + (c-1)\delta_{02} + \delta_{03} + \delta_{11} = 0$$

$$\delta_{01} + \delta_{02} + \delta_{03} = 1$$

$$\lambda_1 = \delta_{11}$$

Therefore

$$\delta^* = \left(\frac{b-1}{2b-1}, \frac{1}{2b-1}, \frac{b-1}{2b-1}, \frac{-c+1}{2b-1}\right) \tag{4-5}$$

and

$$v(\delta^*) = C(t^*) = [S(b-1)^{-1}]^{\left(\frac{b-1}{2b-1}\right)}[0.5H(2b-1)(b-1)^{-1}]^{\left(\frac{2b-1}{b-1}\right)}[a(2b-1)^b]^{\left(\frac{1}{2b-1}\right)}$$

Now using Theorem 2.3 to solve for $\mathbf{t}^*$,

$$St_1t_2^{-1}t_3^{-1} = v(\delta^*)\left(\frac{b-1}{2b-1}\right) \tag{4-6}$$

$$at_3^{c-1}t_1^{(-b+1)} = v(\delta^*)\left(\frac{1}{2b-1}\right) \tag{4-7}$$

$$0.5Ht_2t_3 = v(\delta^*)\left(\frac{b-1}{2b-1}\right). \tag{4-8}$$

Therefore,

$$D = t_1 = 2S^{-1}H^{-1}\left[v(\delta^*)\left(\frac{b-1}{2b-1}\right)\right]^2$$

$$q = t_2 = 2H^{-1}v(\delta^*)\left(\frac{b-1}{2b-1}\right)\left[a^{-1}v(\delta^*)\left(\frac{1}{2b-1}\right)D^{b-1}\right]^{\frac{-1}{c-1}}$$

$$r = t_3 = \left[a^{-1}v(\delta^*)\left(\frac{1}{2b-1}\right)D^{b-1}\right]^{\frac{1}{c-1}}.$$

For concreteness of the above solution, consider a particular EOQ problem where S=\$ 150/batch and H=\$ 2/item/day. In addition, it is estimated that a=5000, b=2, and c=0.5.

Now (4-5) becomes,

$$\delta^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right)$$

and

$$v(\delta^*) = C(t^*) = 272.57/day.$$

From Equations (4-6)-(4-8),

$$150 t_1 t_2^{-2} t_3^{-1} = 90.86$$

$$5000 t_3^{-0.5} t_1^{-1} = 90.86$$

$$t_2 t_3 = 90.86.$$

Thus

$$D = 55.03/day$$

$$q = 91.77/batch$$

$$r = 99\%.$$

Now since this problem has a zero degree of difficulty, the dual variables δ_{ki} are completely independent of the constant coefficients H, S, and a. Thus, any perturbation of these constants will not alter the distribution of the optimum cost among the setup, production, and inventory carrying costs.

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