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Jan Rosinski

Dr. Jan Rosinski, Major Professor

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A STUDY OF

A STOCHASTIC MODEL

FOR

STOCK PRICES

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee

Valerie May Beaman

August, 1994

DEDICATION

For all of your love and support,

thank you Mom and Dad.

And finally, the answer to your question is yes.

To Beth, Scott, Teresa, and Angela:

O

ACKNOWLEDGEMENTS

Special thanks to Dr. Jan Rosinski,

whose patience, time, and guidance were invaluable.

Thanks as well to

Dr. Julius Smith and Dr. Phillip Daves.

ABSTRACT

The goal of this thesis is to present, explain, and analyze a stochastic model of stock behavior.

The main body of this text is inspired by A.G. Malliaris' article entitled "Itô's Calculus in Financial Decision Making," published in volume 25, number 4, of the *SIAM REVIEW* in October, 1983.

Stochastic processes, Wiener processes, Itô's stochastic differential equation, and Itô's formula are the main mathematical concepts of this paper. This thesis will clearly present these concepts and their relation to modeling the price of a stock. An intuitive understanding of this stock model is gained through graphical analysis of computer simulations.

This stochastic model of stock behavior, though simplistic, is a commonly used one in finance theory. It is capable of predicting the probabilities of certain gains or losses in the value of the stock at any time t. This model is also used in the derivation of the Black-Scholes formula, as well as other option pricing strategies.

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1. INTRODUCTION

For as long as there have been stocks to exchange, financial investors have tried to predict the future value of certain stocks so that they would know how to invest their money and, inevitably, make a profit. Mathematical models for stock prices have been and still are valuable tools used in such predictions. Whereas these models are not deterministic, they do give us insight as to the behavior of the price of a stock. The goal of this paper is to investigate a rather simplistic probability model that involves the use of stochastic calculus and is used in many other aspects of finance theory.

The main body of this text is inspired by A.G. Malliaris' article entitled "Itô's Calculus in Financial Decision Making," published in volume 25, number 4, of the *SIAM REVIEW* in October, 1983.

This paper is organized in the following manner. The mathematical terminology necessary to understand the stochastic model of stock behavior is explained first. The bond model, wherein the stock model has its roots, is discussed, and then the stock model and its solution are examined. Lastly, a discussion involving graphical analysis of the stochastic model for stock prices is presented along with a summary of this study. The computer programs by which the simulations and lognormal distributions were graphed were written by Dr. Jan Rosinski for a course in stochastic processes and are included in this paper with his permission.

2. MATHEMATICAL TOOLS

Stochastic Process

In order to fully understand the stochastic model for the behavior of the value of a stock, one must first understand what a stochastic process is. The following definitions and explanations should be helpful.

Definition 2.1: *A stochastic process $X(t,\omega)$ is a family of random variables indexed by a set of parameters and defined on a probability space $(\Omega, \mathcal{F}, P)$.*

The most common continuous parameter set is time. Oftentimes this stochastic process is represented by $X(t)$, where the ω is suppressed and $t \in [0,\infty)$ or $t \in [0,T]$ if T is fixed. In the solution to the stock model, the value of a stock $X(t)$ is a stochastic process and its parameter t represents elapsed time.

Two assumptions of the stock model are noted here. First, it is assumed that $X(t)$ is a continuous process even though the stock market does not operate continually. Second, it is assumed that $X(t)$ takes on continuous values, even when in real life the values of stocks are usually restricted to multiples of eighths of dollars. These assumptions do not seem to affect the model in most cases.

Wiener Process

A Wiener process, also known as Brownian motion, is simply a special type of stochastic process.

__**Definition 2.2**__: *A Wiener process* $Z(t)$, *for* $t \geq 0$, *is a stochastic process that satisfies the following three conditions:*

i) $Z(0) = 0$.

ii) $Z(t)$ *is normally distributed with mean* μt *and variance* $\sigma^2 t$ *for* $t > 0$.

iii) Furthermore, $Z(t_1)$, $Z(t_2)-Z(t_1)$, $\ldots$, $Z(t_n)-Z(t_{n-1})$ *are independent for every* $t_1 \leq t_2 \leq \ldots \leq t_n$.

A Wiener process is said to be ***standard*** when $\mu = 0$ and $\sigma^2 = 1$. Although $Z(t)$ is a continuous function, it is nowhere differentiable. It impossible to take the derivative of a Wiener process or evaluate the integral $\int_0^T f(t)\,dZ(t)$ as $\int_0^T f(t)\,Z'(t)\,dt$ in the traditional manner. But graphically, particularly in computer simulations, one can still interpret $Z(t)$ as the cumulative sum of the area underneath the function $\dot{Z}(t)$, much like a cumulative distribution function. $\dot{Z}(t)$ is known as white noise and is considered to be the formal derivative of the Wiener process $Z(t)$. They are formally related by the equation

$$(2.1) \qquad dZ(t) = \dot{Z}(t)\,dt.$$

Figure 1 shows the graph of a white noise process $\dot{Z}(t)$, and Figure 2 shows the graph of the associated Wiener process $Z(t)$ (see Appendix A for program).

A Wiener process is also a Gaussian process. This fact will be used later in this paper, so its proof is presented here. But first, the definition of a Gaussian process is given.

__**Definition 2.3**__: *A stochastic process $X(t)$ is considered to be a* ***Gaussian process*** *if every finite linear combination of the random variables $X(t)$ is normally distributed.*

Proof: First, consider the arbitrary linear combination of Wiener processes denoted by

$$(2.2) \qquad \sum_{i=1}^{n} a_i Z(t_i) .$$

Let $X_1 = Z(t_1)$, $X_2 = Z(t_2) - Z(t_1)$, $\ldots$, $X_n = Z(t_n) - Z(t_{n-1})$. By ii) and iii) of Definition 2.2, it is known that $X_1, X_2, \ldots, X_n$ are normally distributed and independent. Next, each $Z(t_i)$ is renamed as a function of X's by the formula

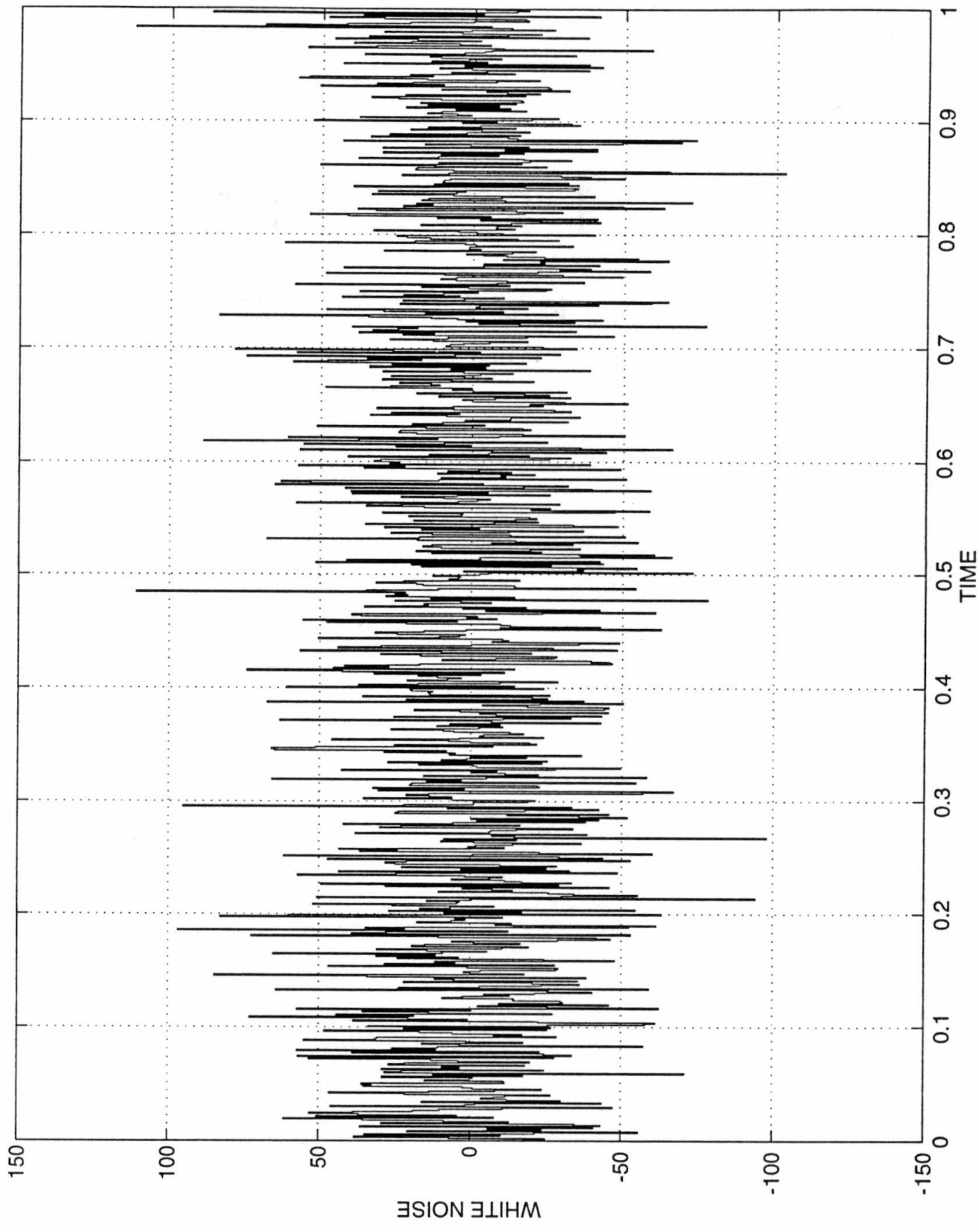

Figure 1: White Noise $\dot{Z}(t)$.

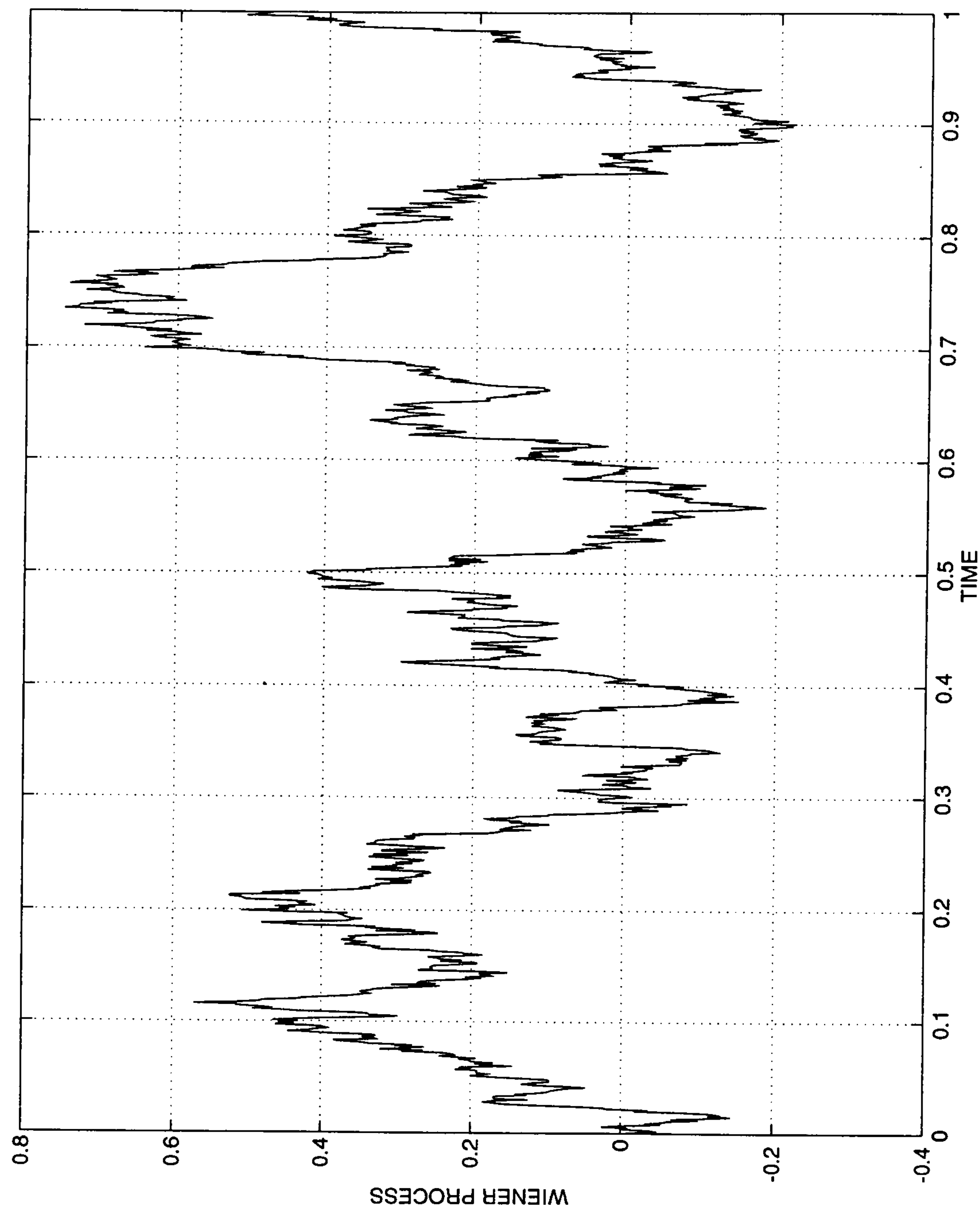

Figure 2: Wiener Process $Z(t)$.

$$(2.3) \qquad Z(t_i) = X_1 + X_2 + \ldots + X_i .$$

By substituting this result into (2.2), one obtains

$$(2.4) \qquad \sum_{i=1}^{n} a_i (X_1 + X_2 + \ldots + X_i) = \sum_{k=1}^{n} b_k X_k , \quad \text{where } b_k = \sum_{i=k}^{n} a_i .$$

Now, the original linear combination of Wiener processes has been transformed into a linear combination of independent random variables, which is known to have a normal distribution (see Larsen, pp. 318-320). ●

Itô's Stochastic Differential Equation

Recall from calculus the ordinary differential equation

$$(2.5) \qquad dx(t) = f(t, x(t))\, dt,$$

with initial condition $x(0) = x_0$ and $t \in [0, T]$. A unique solution $y(s)$ exists to (2.5) if $f(t,x)$ and $f_x(t,x)$ are continuous in a neighborhood of $(0, x_0)$.

Now consider the differential equation

$$(2.6) \qquad dX(t) = f(t, X(t))\, dt + \sigma(t, X(t))\, dZ(t)$$

with initial condition $X(0) = x_0$ where $Z(t)$ is a Wiener process. It is similar to (2.5) except that the second term on the right-hand side is added, thus giving it a stochastic quality.

A stochastic process X(s) is said to be the solution to (2.6) if

$$(2.7) \qquad X(t)-X(0) = \int_0^t f(s,X(s))\,ds + \int_0^t \sigma(s,X(s))\,dZ(s)$$

holds for all $t \in [0,T]$ with probability 1. The second integral was defined by Itô and is called Itô's integral. Notice here that the second integral on the right-hand side of (2.7) can not be interpreted as a Reimann-Stieltjes integral because $dZ(s)$ exists only as a formality. A precise definition and the properties of Itô's integral that allow (2.7) to be solved are given in Gikhman and Skorokhod, 1969 and Malliaris, 1983.

It will be shown in section 4 that (2.6) can represent a model for stock prices, where $Z(t)$ is a standard Wiener process, $f(t,X(t)) = \alpha X(t)$, $\sigma(t,X(t)) = \sigma X(t)$, and (2.6) specifically becomes

$$(2.7) \qquad \begin{aligned} dX(t) &= (\alpha X(t))\,dt + (\sigma X(t))\,dZ(t) \\ &= (\alpha\,dt + \sigma\,dZ(t))\,X(t) \\ &= (\alpha + \sigma \dot{Z}(t))\,X(t)\,dt, \end{aligned}$$

where $\dot{Z}(t)$ is white noise. A solution to (2.7) is guaranteed to exist because it meets the conditions for existence and uniqueness of a solution to Itô's stochastic differential equation (see Gikhman and Skorokhod, 1969).

Itô's Formula

Recall the chain rule from ordinary calculus that is used to calculate the differential dy of $y(t) = u(t,x(t))$

$$dy = u_t(t,x(t))\,dt + u_x(t,x(t))\,dx(t)\,. \tag{2.8}$$

Here $u(t,x)$ is a differentiable function and u_t and u_x denote its partial derivatives. Suppose it is known that $dx(t)$ follows a stochastic differential equation such as (2.6). Making this modification and formal substitution, dy is calculated by

$$\begin{aligned} dy &= u_t(t,x(t))\,dt \\ &\qquad + u_x(t,x(t))\,[f(t)\,dt + \sigma(t)\,dZ(t)] \\ &= [u_t(t,x(t)) + u_x(t,x(t))\,f(t)]\,dt \\ &\qquad + u_x(t,x(t))\,\sigma(t)\,dZ(t)\,. \end{aligned} \tag{2.9}$$

But because the function $u(t,x(t))$ is a stochastic process associated with a nondifferentiable Wiener process rather than a differentiable function, the original assumption of differentiability no longer holds and apparently (2.9) is no longer valid. Itô discovered the correct version of the chain rule for stochastic differentials, known as Itô's formula. Itô's formula for computing the stochastic differential of a new stochastic process $Y(t) = u(t,X(t))$ with respect to the standard Wiener process $Z(t)$, given the fact that $dX(t)$ follows the stochastic differential equation (2.6) is given by

$$dY(t) = [u_t(t,X(t)) + u_x(t,X(t))\, f(t) + \tfrac{1}{2} u_{xx}(t,X(t))\, \sigma^2(t)]\, dt + u_x(t,X(t))\, \sigma(t)\, dZ(t). \tag{2.10}$$

A rigorous proof of (2.10), known as Itô's formula, which is based on Taylor's expansion can be found in Gikhman and Skorokhod, pp. 387-389. Itô's formula will be used later to find a solution to the stock model differential equation.

3. BOND MODEL

The stochastic differential equation that models the behavior of the value of a stock has its roots in the separable differential equation

$$(3.1) \qquad dX(t) = \beta X(t)\,dt.$$

Equation (3.1) represents a fixed interest investment such as one would find offered by a bank. Here, β represents a fixed interest rate per unit time, and $X(t)$ is the function that represents the value of the initial investment at any elapsed time t.

The solution of (3.1) is now given. First, dividing both sides of (3.1) by $X(t)$ and assuming $X(0) > 0$ yields

$$(3.2) \qquad \frac{dX(t)}{X(t)} = \beta\,dt.$$

Transforming (3.2) into an integral equation and then introducing a change of variable produces

$$(3.3) \qquad \int_0^t \frac{dX(s)}{X(s)} = \int_0^t \beta\,ds.$$

Now computing the definite integral and simplifying gives the explicit formula for $X(t)$

$$(3.4) \qquad X(t) = X(0)\,e^{\beta t}.$$

For the sake of notation simplicity, the constant $X(0)$ will be written as x_0, thus (3.4) will become

$$X(t) = x_0 e^{\beta t}. \tag{3.5}$$

Equation (3.5) may look familiar, as it is commonly called the continuous compound interest formula and is typically used by financial institutions to approximate earnings on certain types of bonds, savings and interest-bearing checking accounts, certificates of deposits, loans, and other services. The value of the investment $X(t)$ in (3.5) strictly increases as time goes by. The slope of the tangent line also increases as time goes by. This can be shown mathematically by computing the first and second derivatives of (3.5) and noticing that they are both positive, implying that $X(t)$ is strictly increasing and concave up. In monetary terms, this implies that the initial investment will grow faster and faster as time goes on.

4. STOCK MODEL

Fluctuating Rate of Return

In Figure 3, the daily closing price of Boston Celtics stock is documented for 1993. From Figure 3, one can see that the value of the stock fluctuates and therefore does not follow the same strictly increasing formula as given by the solution to the bond model in (3.5). The question that naturally arises is thus: Can one modify the bond model given by (3.1) in order to find a satisfactory solution that will model the behavior of the value of a stock?

Intuitively, one realizes that the major difference between a stock investment and a fixed interest rate bond investment is that the stock does not have a fixed pattern of growth, whereas it has already been shown that the bond model increases exponentially. As a matter of fact, one can not even be assured of any growth in a stock investment. Furthermore, from day to day, hour to hour, or any given unit of time, the rate of return a stock investment may change dramatically. This implies that if one were to attempt to use the bond model to determine the future value of a stock, the fixed interest rate variable β that was noted in the solution to the bond model in (3.5) may not be strictly positive, will not be a constant, and most importantly, can not even be determined explicitly. A modification to (3.1) is necessary

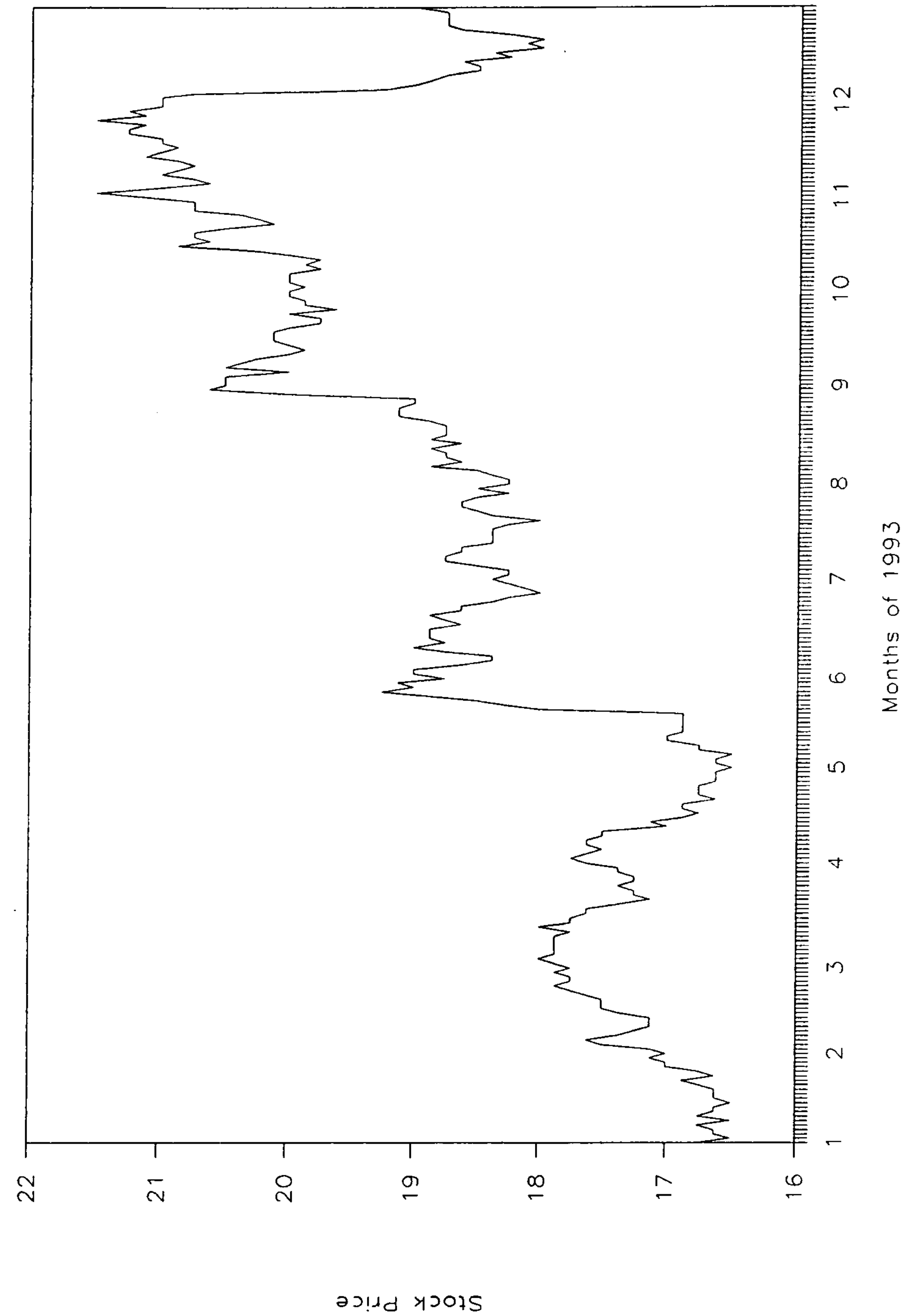

Figure 3: Daily Prices of Boston Celtics Stock in 1993.

to accommodate this variable rate of return.

Suppose one was to estimate the rate of return of the stock in question by averaging its daily rate of increase and/or decrease over a period of time. Then, by adding a random component to this average, a fluctuating interest rate is obtained. Algebraically, the once constant β in (3.1) is thus transformed into a random variable

$$\beta = \alpha + \sigma \dot{Z}(t) . \tag{4.1}$$

Here, α represents the aforementioned expected rate of return over a period of time, and $\dot{Z}(t)$ represents white noise, which accounts for the fluctuation of the rate of return. And lastly, σ is a scalar representing the magnitude of volatility of the stock. A further interpretation of σ is given in section 5.

Now that the interest rate has been adapted to fit the behavior of the rate of return of a stock, (3.1) becomes

$$dX(t) = (\alpha + \sigma \dot{Z}(t)) X(t) dt. \tag{4.2}$$

Distributing $X(t) dt$ and making a formal use of (2.1), (4.2) becomes

$$dX(t) = \alpha X(t) dt + \sigma X(t) dZ(t) , \tag{4.3}$$

which is a special form of Itô's stochastic differential equation (2.6). Now that the origin of the stochastic differential equation (4.3) has been established, its solution will now be attempted.

Solution

To gain some insight about the form of a solution of (4.3), the formal method of solving the bond equation is applied to (4.2). Treating (4.2) as a separable differential equation and using a change of variable gives

$$
\begin{aligned}
&\frac{dX(s)}{X(s)} = [\alpha + \sigma \dot{Z}(s)]\, ds \\
(4.4) \qquad \Rightarrow &\int_0^t \frac{dX(s)}{X(s)} = \int_0^t [\alpha + \dot{Z}(s)]\, ds \\
\Rightarrow &\quad X(t) = x_0 e^{\alpha t + \sigma Z(t)}.
\end{aligned}
$$

This solution will now be verified to see if (4.4) satisfies equation (4.3). Let

$$(4.5) \qquad Y(t) = x_0 e^{at + \sigma Z(t)} = u(t, Z(t)).$$

It is now necessary to check if $dY(t) = \alpha Y(t)\, dt + \sigma Y(t)\, dZ(t)$, as in (4.3) with a change of variable. To find $dY(t)$, we apply Itô's formula with

$$(4.6) \qquad u(t,x) = x_0 e^{\alpha t + \sigma x} \text{ and } X(t) = Z(t).$$

The partial and second partial derivatives of $u(t,x)$ are computed as

$$
(4.7)\qquad \begin{aligned} u_t(t,x) &= \alpha x_0 e^{\alpha t+\sigma x} \\ u_x(t,x) &= \sigma x_0 e^{\alpha t+\sigma x} \\ u_{xx}(t,x) &= \sigma^2 x_0 e^{\alpha t+\sigma x}. \end{aligned}
$$

Since X(t) = Z(t), it is known that

$$
(4.8)\qquad dX(t) = dZ(t) = 0\cdot dt+1\cdot dZ(t)\,,
$$

which gives f(t) = 0 and σ(t) = 1 in the use of Itô's formula. Making these substitutions into (2.10) to obtain dY(t) yields

$$
(4.9)\qquad \begin{aligned} dY(t) &= [u_t(t,Z(t))+u_x(t,Z(t))\,f(t)+\tfrac{1}{2}u_{xx}(t,Z(t))\,\sigma^2(t)]\,dt \\ &\qquad +u_x(t,Z(t))\,\sigma(t)\,dZ(t) \\ &= [\alpha x_0 e^{\alpha t+\sigma Z(t)}+0+\tfrac{1}{2}\sigma^2 x_0 e^{\alpha t+\sigma Z(t)}]\,dt \\ &\qquad +\sigma x_0 e^{\alpha t+\sigma Z(t)}\,dZ(t) \\ &= [(\alpha+\tfrac{1}{2}\sigma^2)\,x_0 e^{\alpha t+\sigma Z(t)}]\,dt+\sigma x_0 e^{\alpha t+\sigma Z(t)}\,dZ(t) \\ &= (\alpha+\tfrac{1}{2}\sigma^2)\,Y(t)\,dt+\sigma Y(t)\,dZ(t)\,. \end{aligned}
$$

The final result of (4.9) is not equivalent to equation (4.3). Suppose X(t)=Z(t) as before but

$$
(4.10)\qquad Y(t) = x_0 e^{\alpha_1 t+\sigma Z(t)} = u(t,Z(t))
$$

and

$$
(4.11)\qquad u(t,x) = x_0 e^{\alpha_1 t+\sigma x}.
$$

Then using the same type of computations as in (4.7) - (4.9), Itô's formula yields

$$(4.12) \qquad dY(t) = (\alpha_1 + \tfrac{1}{2}\sigma^2)\, Y(t)\, dt + \sigma\, Y(t)\, dZ(t)\,.$$

Choosing $\alpha_1 = \alpha - \frac{1}{2}\sigma^2$, then the conclusion obtained in (4.12) is the equivalent to (4.3) and we have shown that

$$(4.13) \qquad Y(t) = X(t) = x_0\, e^{(\alpha - \frac{1}{2}\sigma^2)\, t + \sigma Z(t)}$$

is the correct solution to (4.3).

5. ANALYSIS OF SOLUTION

Expectation

Now that a correct solution to (4.3) has been derived in the form of the stochastic process $X(t)$ given in (4.13), our first goal is to compute the mean of this stock market investment. Its mean, or expectation at any given time t is expressed by

$$(5.1) \qquad \mathbf{E}(X(t)) = \mathbf{E}(x_0 e^{(\alpha - \frac{\sigma^2}{2})t + \sigma Z(t)}).$$

Since x_0 is constant and t is given, one can use the laws of expectation to rewrite (5.1) as

$$(5.2) \qquad \begin{aligned} \mathbf{E}(X(t)) &= \mathbf{E}(x_0 e^{(\alpha - \frac{\sigma^2}{2})t} e^{\sigma Z(t)}) \\ &= x_0 e^{(\alpha - \frac{\sigma^2}{2})t} \mathbf{E}(e^{\sigma Z(t)}). \end{aligned}$$

Now consider the moment generating function for the normally distributed random variable Y that has mean μ_Y and variance σ_Y^2 given by

$$(5.3) \qquad \mathbf{M}_Y(s) = \mathbf{E}(e^{sY}) = e^{(\mu_Y s + \frac{\sigma_Y^2 s^2}{2})}.$$

If one lets $Y = Z(t)$, then Y will be a normally distributed random variable with mean $\mu_Y = 0$ and variance $\sigma_Y^2 = t$ since $Z(t)$

is a standard Wiener process. Allowing $s=\sigma$, (5.3) becomes

$$(5.4)\qquad \begin{aligned} \mathbf{M}_Y(\sigma) &= \mathbf{E}(e^{\sigma Z(t)}) \\ &= e^{0\cdot\sigma+\frac{t\sigma^2}{2}} \\ &= e^{\frac{t}{2}\sigma^2}. \end{aligned}$$

Substituting the result of (5.4) into the right-hand side of (5.2) gives

$$(5.5)\qquad \begin{aligned} \mathbf{E}(X(t)) &= x_0 e^{(\alpha-\frac{\sigma^2}{2})t} e^{\frac{\sigma^2 t}{2}} \\ &= x_0 e^{\alpha t-\frac{\sigma^2}{2}t+\frac{t}{2}\sigma^2} \\ &= x_0 e^{\alpha t}. \end{aligned}$$

It is important to notice here that the result in (5.5) is precisely the solution to the bond model given in (1.8) where $\beta = \alpha$. The interpretation is that one would expect the average of the stochastic process $X(t)$ in the stock case to result in a bond case with a fixed interest rate of α.

Variance

For any fixed time t, the variance of the solution $X(t)$ to the stock model can be calculated as

$$(5.6)\qquad VAR(X(t)) = \mathbf{E}(X(t)^2)-\mathbf{E}(X(t))^2.$$

Substituting the results of (4.13) and (5.5) into (5.6) yields

$$(5.7)\qquad VAR(X(t)) = \mathbf{E}(x_0^2 e^{2(\alpha-\frac{\sigma^2}{2})t+2\sigma Z(t)}) - x_0^2 e^{2\alpha t}.$$

Using laws of expectation and exponents and then factoring (5.7) gives

$$(5.8)\qquad \begin{aligned} VAR(X(t)) &= x_0^2 e^{2(\alpha-\frac{\sigma^2}{2})t}\,\mathbf{E}(e^{2\sigma Z(t)}) - x_0^2 e^{2\alpha t} \\ &= x_0^2 e^{2\alpha t}[e^{-\sigma^2 t}\,\mathbf{E}(e^{2\sigma Z(t)}) - 1]. \end{aligned}$$

Consider (5.3) where $Y = Z(t)$ and $s = 2\sigma$. Recall $Z(t)$ is a standard Wiener process, therefore $\mu_{Z(t)} = 0\ \mu_Y$ and $\sigma^2_{Z(t)} = t = \sigma^2_Y$. Now

$$(5.9)\qquad \begin{aligned} \mathbf{E}(e^{2\sigma Z(t)}) &= \mathbf{M}_Y(2\sigma) \\ &= e^{0\cdot(2\sigma)+\frac{t(2\sigma)^2}{2}} \\ &= e^{2t\sigma^2}. \end{aligned}$$

Substituting the result of (5.9) into the right-hand side of (5.8) and simplifying exponents yields

$$(5.10)\qquad \begin{aligned} VAR(X(t)) &= x_0^2 e^{2\alpha t}[e^{-\sigma^2 t} e^{2\sigma^2 t} - 1] \\ &= (x_0 e^{\alpha t})^2 [e^{\sigma^2 t} - 1]. \end{aligned}$$

In order to establish an interpretation of the parameter σ, the $\lim_{t\to 0} \frac{VAR(X(t))}{t}$ is calculated here with the help of l'Hôpital's Rule:

$$(5.11) \qquad \lim_{t\to 0} \frac{VAR(X(t))}{t} = \lim_{t\to o} \frac{(x_0 e^{\alpha t})^2 [e^{\sigma^2 t}-1]}{t}$$
$$= X_0^2 \lim_{t\to o} \frac{e^{2\alpha t}[e^{\sigma^2 t}-1]}{t}$$
$$= X_0^2 \lim_{t\to 0} \frac{e^{(2\alpha+\sigma^2)t}-e^{2\alpha t}}{t}$$
$$= X_0^2 \lim_{t\to 0} \frac{(2\alpha+\sigma^2)e^{(2\alpha+\sigma^2)t}-2\alpha e^{2\alpha t}}{1}$$
$$= x_0^2(2\alpha+\sigma^2-2\alpha)$$
$$= x_0^2\sigma^2.$$

Solving (5.11) for σ^2 gives

$$(5.12) \qquad \sigma^2 = \frac{1}{X_0^2} \lim_{t\to 0} \frac{VAR(X(t))}{t}$$

This result shows that a large (or small) σ in the stock model solution (4.13) will correspond to large (or small) fluctuations in the price of the stock $X(t)$ over a small period of time. Appropriately, σ is sometimes referred to as the instantaneous volatility parameter.

Lognormal

The distribution of $X(t)$ is also of interest. A new variable $Y(t) = \ln x_0 + (\alpha - \frac{1}{2}\sigma^2)t + \sigma Z(t)$ is introduced to help with the analysis. Rewriting the solution $X(t)$ in terms of $Y(t)$ gives

$$X(t) = e^{\ln x_0} e^{(\alpha - \frac{1}{2}\sigma^2) t + \sigma Z(t)}$$
$$(5.13) \qquad = e^{\ln x_0 + (\alpha - \frac{1}{2}\sigma^2) t + \sigma Z(t)}$$
$$= e^{Y(t)}.$$

Due to Y's Gaussian property, it is known that Y is normally distributed. Furthermore, for a fixed t, Y has mean

$$\mu_Y = \mathbf{E}[\ln x_0 + (\alpha - \tfrac{1}{2}\sigma^2) t + \sigma Z(t)]$$
$$(5.14) \qquad = \ln x_0 + (\alpha - \tfrac{1}{2}\sigma^2) t + \sigma \mathbf{E}(Z(t))$$
$$= \ln x_0 + (\alpha - \tfrac{1}{2}\sigma^2) t$$

and variance

$$\sigma_Y^2 = VAR[\ln x_0 + (\alpha - \tfrac{1}{2}\sigma^2) t + \sigma Z(t)]$$
$$(5.15) \qquad = 0 + 0 + \sigma^2 VAR(Z(t))$$
$$= \sigma^2 t.$$

<u>Definition 5.1</u>: A positive random variable V is said to be ***lognormal*** if $\ln V$ is normal.

In other words, if V is lognormal and $W = \ln V$, then $V = e^{\ln V} = e^W$, and the exponent W is a normal random variable. Therefore since Y is normal, the solution to the stock model $X(t)$ is lognormal.

The density function corresponding to a lognormal random variable is also of interest here. In order for the density function to be derived, the distribution function is first established by

$$
(5.16) \qquad
\begin{aligned}
F_V(v) &= P[e^W \le v] \\
&= P[W \le \ln v] \\
&= P[\frac{W-\mu_W}{\sigma_W} \le \frac{\ln v-\mu_W}{\sigma_W}] \\
&= \Phi(\frac{\ln v-\mu_W}{\sigma_W}) .
\end{aligned}
$$

By taking the derivative of $F_V(v)$, the density function of V can be obtained by

$$
(5.17) \qquad
\begin{aligned}
f_v(v) &= \frac{d}{dv} F_V(v) \\
&= \varphi(\frac{\ln v-\mu_W}{\sigma_W}) \ \frac{1}{V} \ \frac{1}{\sigma_W} \\
&= \frac{1}{\sqrt{2\pi}} \exp(-\frac{(\frac{\ln v-\mu_W}{\sigma_W})^2}{2}) \ \frac{1}{v\sigma_W} \\
&= \frac{1}{\sqrt{2\pi}\sigma_W v} \exp(-\frac{(\ln v-\mu_W)^2}{2\sigma_W^2}) .
\end{aligned}
$$

The graph of the lognormal density with $\mu_W = 0$ and $\sigma_W^2 = 1$, is shown in Figure 4.

Recalling (5.14) - (5.17), the distribution function of the stock model is

$$
(5.18) \qquad F_{X(t)}(x) = \Phi(\frac{\ln x-(\ln x_0+(\alpha-\frac{1}{2}\sigma^2)t)}{\sigma\sqrt{t}}) ,
$$

and its density function is represented by

$$
(5.19) \qquad f_{X(t)}(x) = \frac{1}{\sqrt{2\pi t}\sigma x} \exp(-\frac{[\ln x-(\ln x_0+(\alpha-\frac{1}{2}\sigma^2)t]^2}{2\sigma^2 t}) .
$$

The density function of the stock model represents the probable dispersion of the value of a stock after a specific amount x_0 is invested for a certain amount of time t.

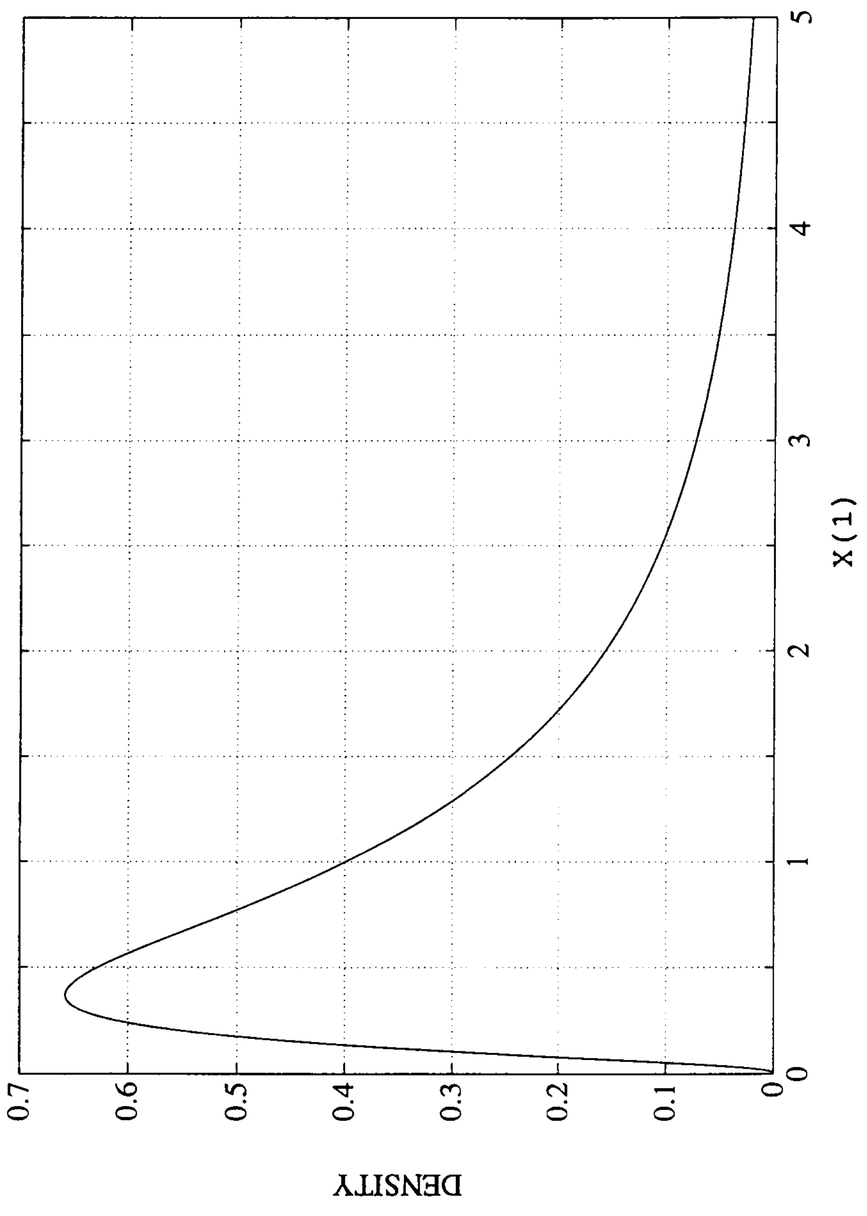

Figure 4: Lognormal Density Function with $\mu_W = 0$ and $\sigma_W^2 = 1$.

6. DISCUSSION AND SUMMARY

Several important results about the solution $X(t) = x_0 e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma Z(t)}$ of the stock model differential equation $dX(t) = (\alpha + \sigma \dot{Z}(t))X(t)\,dt$ have been obtained in the previous section. First, the mean of the solution has been found to be $\mathbf{E}(X(t)) = x_0 e^{\alpha t}$. Second, the variance of the solution has been calculated to be $VAR\,X(t) = (x_0 e^{\alpha t})^2 [e^{\sigma^2 t} - 1]$. Then it was determined that $X(t)$ is lognormal and has density function given by (5.17). These results will be used to analyze the practical implications of this mathematical model.

First, a comparison of the solutions to the bond and the stock model is considered. This is done by means of a computer simulation (see Appendix B for program). Because the stock model is a stochastic process and it varies every time it is modeled, it is also of interest to graph the expected value of the solution to the stock model. Figure 5 shows the bond model when the fixed interest rate β is 8%, and the stock model when the average rate of return α is 15%, and the volatility parameter σ is 30%. The time t is 1 year and the initial investment x_0 is \$1000 in each case. The selection of these values is somewhat arbitrary, but intuitively represents a relatively realistic scenario. It is not atypical for a

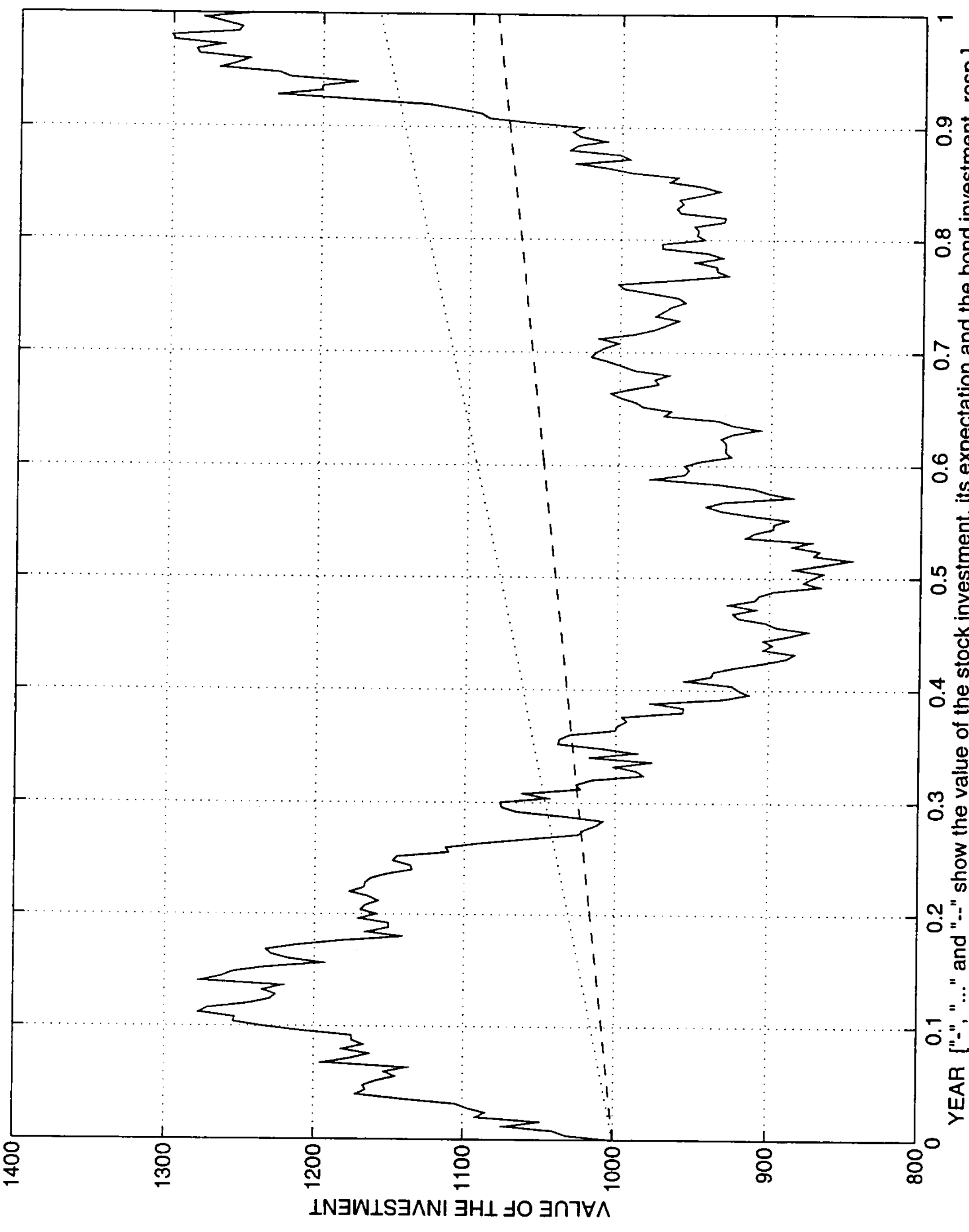

Figure 5: Graph of Solutions to Bond and Stock Models and Expectation of Stock Solution. Parameters for the bond model are t = 1 year, β = 8% and x_0 = 1,000. Parameters for the stock model are t = 1 year, α = 15%, σ = 30%, and x_0 = 1,000.

bond to yield 8% over a period of one year, and one expects a higher potential rate of return from a more risky investment such as a stock than from a secure investment such as a bond. Recalling the result of (5.12), the volatility parameter σ corresponds to an estimated 30% fluctuation of the stock price per year. Figure 5 shows that one would expect a higher profit (approximately $162) from the stock model than from the bond model (approximately $83). Indeed this is the case as the stock solution yields a profit of roughly $250 and performs even better than expected.

Keeping the same parameters, the comparison is made again by means of another computer simulation and the results are shown in Figure 6. As anticipated, the expected value of the stock solution and the bond solution remain unchanged. But in this case, the solution to the stock model is drastically different than the first simulation and what was expected. The result of the simulation this time was a loss of approximately $20 of the initial investment.

The questions that naturally arise are which situation is more typical, and more specifically, with what probability would one expect each type of result?

The lognormal density function of the solution to the stock model is useful in answering these questions. Here, only one very special case of the stock model with certain parameters, namely $\alpha = 15\%$, $\sigma = 30\%$, $t = 1$, and $x_0 = 1000$, is considered. This makes the parameters of the lognormal density

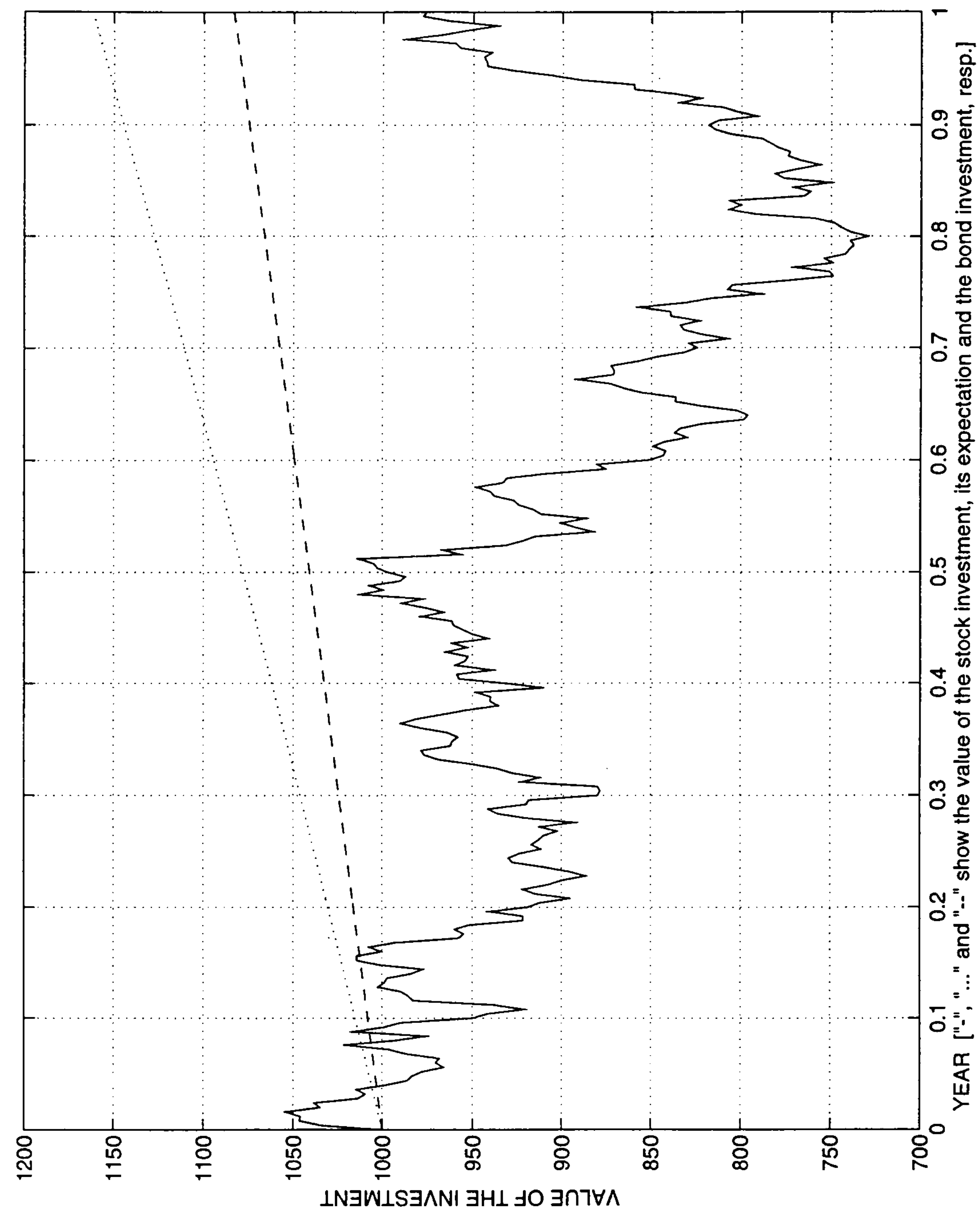

Figure 6: Graph #2 of Solutions to Bond and Stock Models and Expectation of Stock Solution. Parameters for the bond model are t = 1 year, β = 8%, and x_0 = 1,000. Parameters for the stock model are t = 1 year, α = 15%, σ = 30%, and x_0 = 1,000.

$\mu_Y = \ln 1000 + 0.105 \approx 7.01$ and $\sigma_Y^2 = 0.09$. Figure 7 (see Appendix C for program) shows the graph of the lognormal density function with these parameters as well the mean of X(1) and the median of the lognormal distribution function itself. Note the graph is skew left, and that the initial investment was $1000.

One way to determine which result is more typical is to find the median of the lognormal density function with the aforementioned parameters. The following calculations are required:

$$
\begin{aligned}
& F_X(x) = \tfrac{1}{2} \\
\Rightarrow\ & \Phi\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right) = \tfrac{1}{2} \\
\Rightarrow\ & \frac{\ln x - \mu_Y}{\sigma_Y} = 0 \\
\Rightarrow\ & \ln x = \mu_Y \\
\Rightarrow\ & x = e^{\mu_Y} \\
\Rightarrow\ & x \approx 1111.
\end{aligned}
\tag{6.1}
$$

The solid vertical line X(1) = 1111 in Figure 7 represents the median of the lognormal density function. This line represents the division of the curve where half of the probability lies to the left of the line underneath the curve, and half the probability lies to the right of the line underneath the curve. Noticing that this line is greater than 1000 means that with these parameters, one would expect to gain at least $111 half of the time. One can interpret this result in terms of Figures 5 and 6 as the fact that the simulation that gained value in Figure 5 was more of a typical

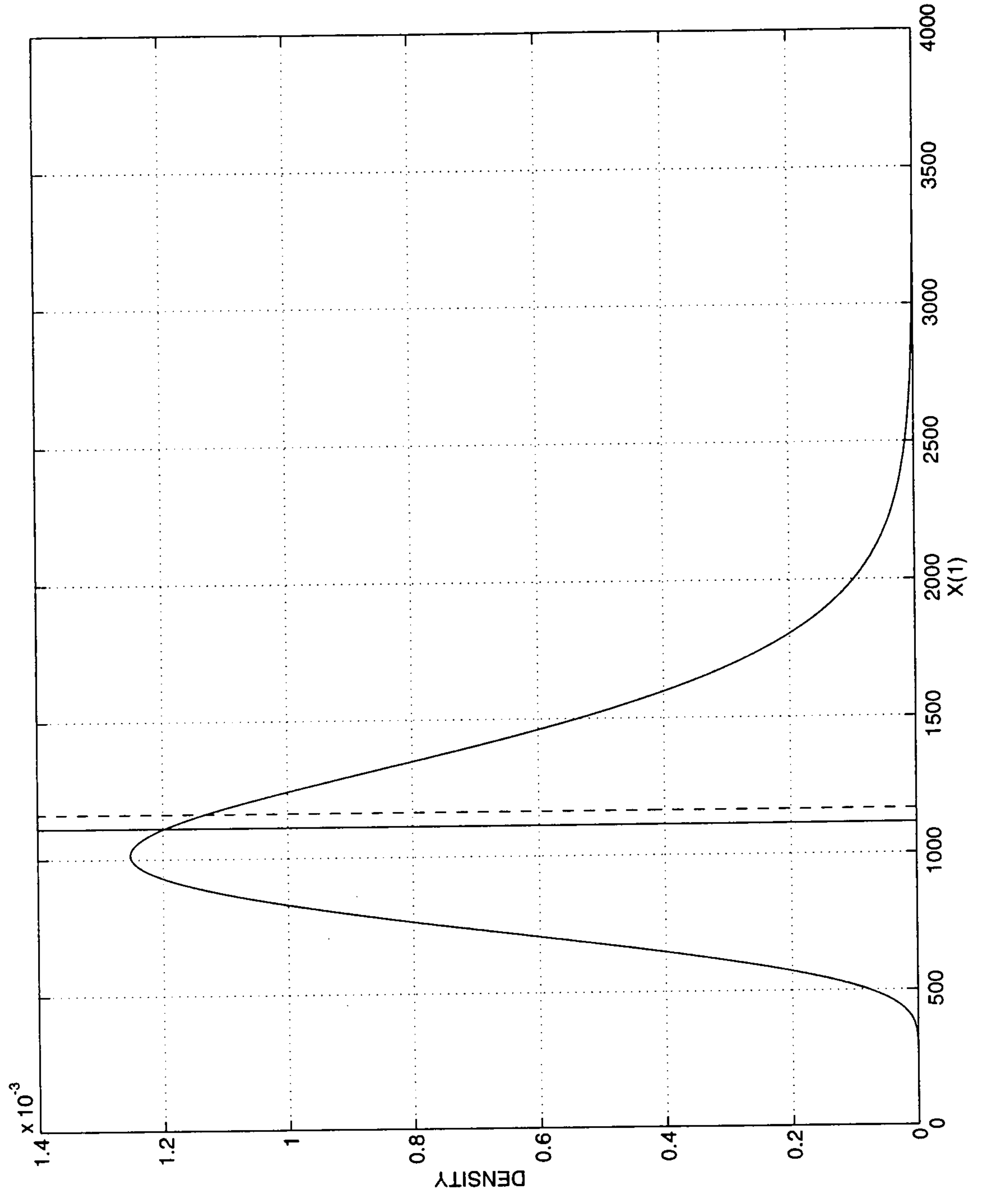

Figure 7: Lognormal Density Function for Figures 5 and 6.

result than the loss in Figure 6.

The dashed vertical line X(1) = 1162 in Figure 7 represents the mean of the random variable X(1). This weighted average shows that over many simultaneous investments of the type with the specific aforementioned parameters, one expects to average a gain of $162, whereas the median of the lognormal density function shows that in any particular investment one is likely to gain at least $111.

Another way to compare the results of the simulations is to calculate the probabilities of the severeness of the outcomes observed in Figures 5 and 6. These probabilities can be computed by

$$P(e^W > x) = P(W > \ln x) = P\left(\frac{W-\mu_W}{\sigma_W} > \frac{\ln x - \mu_W}{\sigma_W}\right) = 1 - \Phi\left(\frac{\ln x - \mu_W}{\sigma_W}\right) \tag{6.2}$$

for calculating the probability of gaining more than x, and

$$P(e^W < x) = P(W < \ln x) = P\left(\frac{W-\mu_W}{\sigma_W} < \frac{\ln x - \mu_W}{\sigma_W}\right) = \Phi\left(\frac{\ln x - \mu_W}{\sigma_W}\right) \tag{6.3}$$

for calculating the probability of losing more than x, where $e^W = X(1)$ is the solution to the stock model for $t = 1$. Notice that for $t = 1$, W equals $Y(t)$ as described in section 5, and μ_W and σ_W^2 are equivalent to μ_Y and σ_Y^2 in (5.14) and (5.15), respectively. The $\Phi(\frac{\ln x-\mu_W}{\sigma_W})$ in (6.2) and (6.3) can be found in any cumulative normal distribution table. Using (6.2) with $x = 1250$ and the appropriate parameters used in the first simulation, one finds that the probability of gaining more than \$250, as in Figure 5, is approximately 0.3483. Using (6.3) with $x = 980$ and the appropriate parameters used in the second simulation, one finds that the probability of losing more than \$20, as in Figure 6, is approximately 0.3409. A comparison of these probabilities indicates that a gain of \$250 or more is more likely than a loss of \$20 or more, but not by much. Consider the case where Figure 5 yields a profit of \$260 rather than \$250. The probability of a gain of more than \$260 is calculated by (6.2) with $x = 1260$ to be approximately 0.3372, which is less than the probability of losing more than \$20. This leads to the conclusion that even though a gain is more likely than a loss, a "big" gain of \$260 or more is less likely than a loss of \$20 to \$1000.

The preceding analysis of Figures 5, 6, and 7 leads to the conclusion that the answer to the question "Which result is more typical?" sometimes depends on one's interpretation of "typical."

The lognormal density shown in Figure 7 is also useful in the comparison of outcomes of the bond and stock models graphed in Figures 5 and 6. The bond model is considered a "safe" investment, as it will always gain approximately \$83. Through the analysis of the median, it is already known that the stock model with the aforementioned parameters will gain more than \$111 approximately 50% of the time. The exact probability of the "risky" stock model earning more than the "safe" bond model is calculated to be 0.5319 by using (6.2), with $x = 1083$ and the appropriate parameters. This type of information is useful when making decisions about whether one should invest in a bond or a stock with these parameters.

This first illustration sheds some light onto the practical interpretation of the stock model, but is only a very specific example. Another computer simulation is now investigated, this time with a larger volatility parameter that corresponds to a more risky investment. Figure 8 shows a computer simulation of the bond model, stock model and expectation of the stock model with the same parameters as in Figures 5 and 6, except this time $\sigma = 3$. The result this time is that virtually all of the original investment is lost.

Figure 9 shows the lognormal density with the same parameters $\mu_Y = \ln 1000 - 4.35 \approx 2.56$ and $\sigma_Y^2 = 9$. Note that the density shown in Figure 9 is even more skew to the left than in Figure 7. As σ increases, the "typical" gain exemplified in Figure 5 becomes smaller and smaller, and eventually a loss is more typical than a gain. In Figure 9, the median of the density function is very close to $X(1) = 13$. A practical

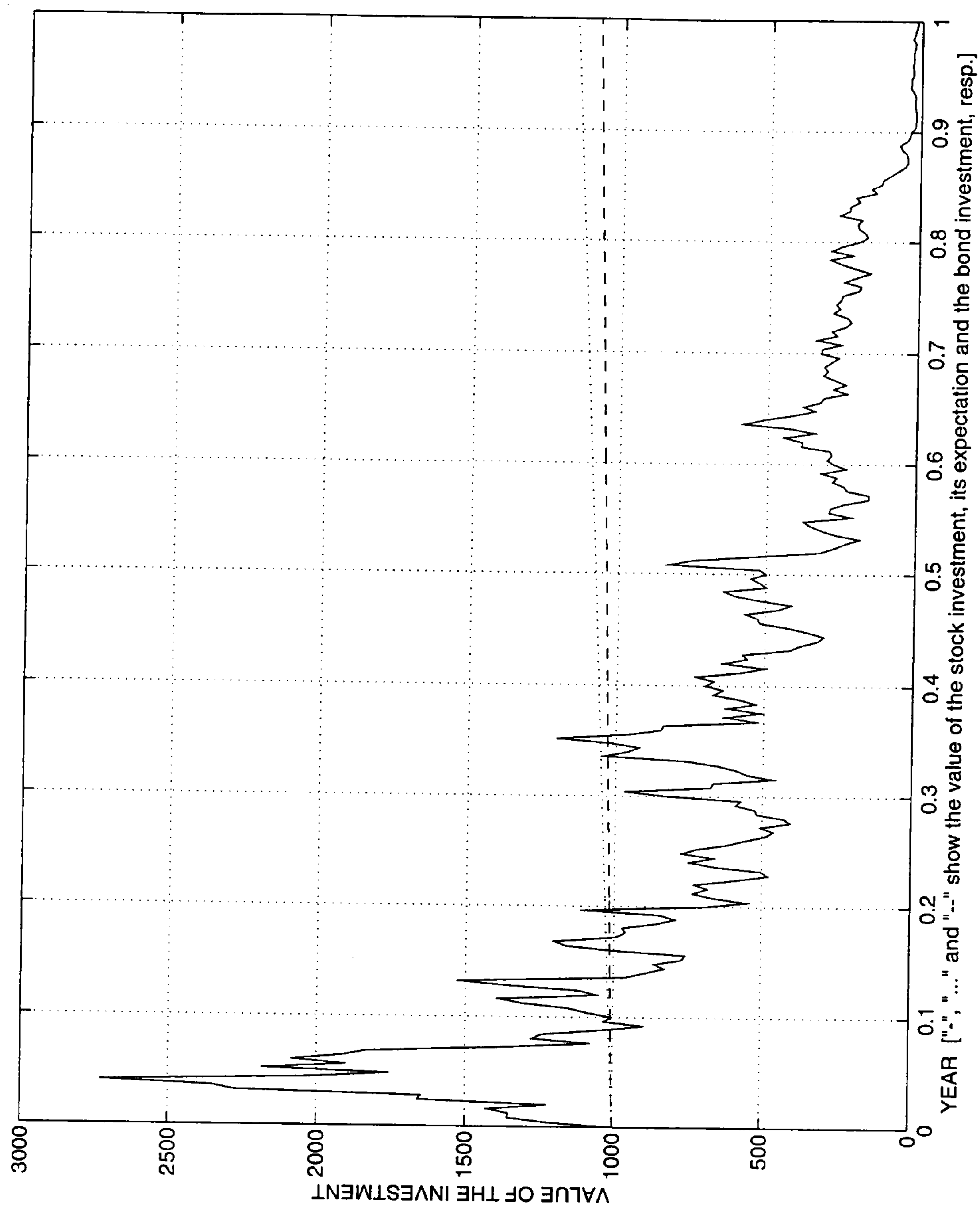

Figure 8: Graph of Solutions to Bond and Stock Models and Expectation of Stock Solution for Large σ. Parameters for the bond model are t = 1 year, β = 8%, and x_0 = 1,000. Parameters for the stock model are t = 1 year, α = 15%, σ = 30%, and x_0 = 1,000.

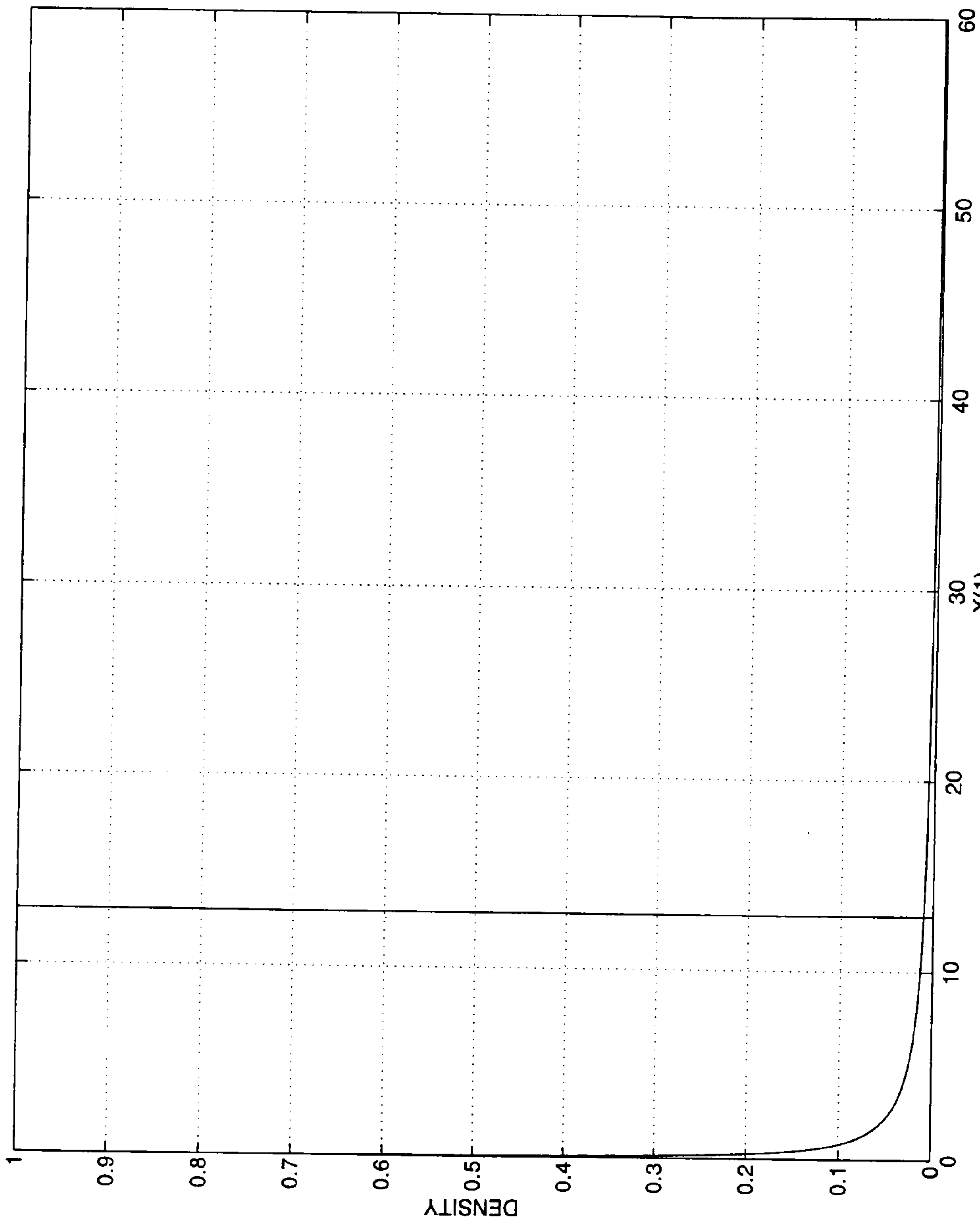

Figure 9: Lognormal Density Function for Figure 8.

interpretation of this graph is that after a year of investing \$1000 in this stock, the probability of one being left with less (or more) than about \$13 is approximately 0.5! The probability of this type of investment even being profitable is approximately 0.0735, calculated by (6.2) with x=1000. Furthermore, the probability of the stock model gaining more than the bond model is approximately 0.0694, calculated by (6.2) with x = 1083. Note in Figure 8 that the expectation of X(1) is still 1162, despite the fact that one is likely to lose money over 92% of the time. In fact, the probability of gaining \$162 or more is approximately 0.0668, and is calculated by using formula (6.2) and x = 1162. Over many simulations, this can be explained by a few extremely large gains in the price of the stock offsetting many losses and averaging out to a gain of \$162.

In conclusion, some very specific cases of the stock model have been investigated and interpreted graphically. The lognormal density function of the model also allows us to calculate specific probabilities of gaining (or losing) at least (or at most) a certain amount of money for any set of known parameters. Though there are certain characteristics of stocks that we have failed to consider in this model, it useful in its own right and serves as a building block for more complex models in the field of finance, such as the Black-Scholes formula for option pricing.

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APPENDICES

APPENDIX A

```
echo on
% Gaussian white noise and a Wiener process.

% The time interval [0,T] is partitioned into N subintervals
% of equal length h. N independent normal random variables
% WN(1), ... ,WN(N) with mean 0 and variance h^{-1/2} are
% simulated. A discrete version of the white noise is a step
% function taking on  WN(i) on the i-th subinterval.
% The Wiener process W(t) is the integral on [0,t] of the
% white noise.
% This program graphs sample paths of the white
% noise (more precisely, its discretization) and the
% corresponding Wiener process.

echo off

T=input('duration of time:  ')
N=input('number of partitions of [0,T]:  ')

h=T/N;
s=sqrt(h);
t=0:h:T;
Z=randn(1,N);
WN= Z/s;
W=[0 ((cumsum(Z'))*s)'];
clf
figure
stairs(t,[WN WN(N)])
xlabel('TIME')
ylabel('WHITE NOISE')
grid
figure
plot(t, W)
xlabel('TIME')
ylabel('WIENER PROCESS')
grid
```

APPENDIX B

```
echo on
% A simple stock market model

% A solution X(t) to the stochastic differential equation
% dX(t) = a*X(t)*dt + sigma*X(t)*dW(t) is simulated with
% 0 < t < T(years), where W is the standard Wiener process.
% This solution, showing the growth of a risky investment in
% a particular stock with volatility sigma and expected rate
% of return a, is compared with the growth of the same
% investment in a bond with a fixed rate of return b.
% The expected value of X(t) is also displayed.

echo off


X0=input('initial investment:  ')
T=input('time (in years):    ')
a=input('expected rate of return in a stock p.a.: ')
sigma=input('volatility of a stock p.a.:  ')
b=input('rate of return on a bond p.a.:  ')

n=250*T;
h=1/250;
t=0:h:T;

A=(X0)*exp(a*t);
W=[0 (sqrt(h)*cumsum(randn(n,1)))'];
X=(X0)*exp((a- ((sigma)^2)/2)*t + (sigma)*W);
B=(X0)*exp(b*t);

figure
plot(t,X, '-', t, A,':', t, B, '--')
grid
ylabel('VALUE OF THE INVESTMENT')
xlabel('YEAR  ["-", "..." and "--" show the value of the stock
investment, its expectation and the bond investment, resp.]')
```

APPENDIX C

```
% Graphs lognormal

sigma =.3;
mu = log(1000) + (.15 - sigma^2/2);

x= .01:1:4000;
f=exp(-(log(x) - mu).^2/(2*sigma^2)) ./ (sqrt(2*pi)*x*sigma);
A=1000*exp(.15)*ones(1,141);
B=0:.00001:.0014;
C=exp(-sigma^2/2)*A;
plot(x,f, A,B, '--', C,B,'-')
xlabel('X(1)')
ylabel('DENSITY')
grid
```

VITA

Valerie May Beaman was born to George and Beverly Beaman on June 5, 1968. She grew up in upstate New York and graduated from Webster High School in June, 1986.

She entered Kent State University in Kent, Ohio in August, 1986. She graduated in May of 1990, earning a Bachelor of Science in Education degree, with a major in mathematics.

After spending two years substitute teaching, coaching, and tutoring in the Rochester, New York area, she accepted a graduate teaching assistantship at the University of Tennessee, Knoxville in the fall of 1992. In August, 1994, she received a Master of Science degree, with a major in mathematics.