To the Graduate Council:

I am submitting herewith a thesis written by Josephine Cridlin Rowlett entitled "A Study of Monotonic Functions." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts, with a major in Mathematics.

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2

A STUDY OF MONOTONIC FUNCTIONS

A Thesis

Presented for the

Master of Arts

Degree

The University of Tennessee

Josephine Cridlin Rowlett

August 1974

ACKNOWLEDGMENTS

The author wishes to express her sincere appreciation to Dr. G. Samuel Jordan for many hours of his time and for his helpful guidance during the preparation of this thesis.

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ABSTRACT

The purpose of this investigation is to study properties of monotonic functions and their uses in mathematics. Topics discussed include limits, continuity, derivatives, inverses, bounded variation, and integrals of monotonic functions. In addition, local and global growth properties of monotonic functions including the existence of Pólya peaks are considered.

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INTRODUCTION

One of the most important concepts in mathematics is that of the function, and one of the most important types of functions is the monotonic function. The purpose of this thesis is to collect, organize, and study properties of monotonic functions and to indicate some of their various uses in mathematics.

Topics considered in Chapters I - IV include limits and continuity, derivatives, inverses, and integrals of monotonic functions. These chapters contain definitions, lemmas, and examples, as well as the statements and proofs of a number of important theorems. Chapter V is the study of some relatively new and useful results on the local and global growth properties of monotonic functions including rates of growth and the existence of Pólya Peaks.

With few exceptions, the material presented is in order of increasing difficulty both throughout the thesis and within each chapter. In addition, definitions and explanations of concepts with which the reader may not be familiar are included where appropriate. Since many of the results presented are well-known, we have not attempted to document every theorem and proof; however, references are provided for those which closely follow earlier presentations. Absence of a reference does not imply any claim to originality on our part.

CHAPTER I

BASIC DEFINITIONS AND THEOREMS

In this chapter, we present several definitions and theorems which are fundamental in the study of monotonic functions. We begin with the definition of such a function.

DEFINITION 1.1. A function f defined on a set D is said to be nondecreasing (strictly nondecreasing) if, for every x and y in D, $x < y$ implies $f(x) \leq f(y)$ $(f(x) < f(y))$. The function f is said to be nonincreasing (strictly nonincreasing) if, for every x and y in D, $x < y$ implies $f(x) \geq f(y)$ $(f(x) > f(y))$. A function is said to be monotonic if it is nondecreasing, nonincreasing, strictly nondecreasing, or strictly nonincreasing.

The terms increasing and decreasing are often used for what we have called nondecreasing and nonincreasing, respectively. We will use these expressions interchangeably in the remainder of this thesis.

Three simple but basic theorems concerning the properties of monotonic functions can be proved using only the definition given above. They are stated here without proof.

THEOREM 1.2. If f is a nondecreasing function, then $-f$ is a nonincreasing function. If f is nonincreasing, then $-f$ is nondecreasing.

Because of Theorem 1.2, the great majority of theorems and examples in the sequel will be concerned only with nondecreasing or strictly nondecreasing functions. Corresponding proofs can be supplied for nonincreasing or strictly nonincreasing functions simply by considering $-f$. We begin this practice in the next theorem.

THEOREM 1.3. Let $a < b < c$. If a function f is nondecreasing on the closed interval $[a,b]$ and on the closed interval $[b,c]$, then f is nondecreasing on the closed interval $[a,c]$.

A function f defined on an interval $[-a,a]$ is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. If f is monotonic as well as even or odd, we have the following theorem.

THEOREM 1.4. If a function f is an even function and if f is nondecreasing for $x \geq 0$, then f is nonincreasing for $x \leq 0$. If f is an odd function and is nondecreasing for $x \geq 0$, then f is nondecreasing for all x.

Let us now consider what it means for a function to be monotonic at a point rather than on an entire interval or set.

DEFINITION 1.5. A function f is said to be increasing on the right at the point x if for all y of some interval to the right of x, $f(y) \geq f(x)$. The function f is said to be increasing on the left at the point x if for all y in some interval to the left of x, $f(y) \leq f(x)$. If f is increasing both on the right and on the left at x, then f is said to be increasing at the point x.

Similar definitions can be stated for a function f decreasing on the right, decreasing on the left, or decreasing at a point.

In the following theorem, we see that a sufficient condition for a function to be increasing in an interval is that the function be increasing at every point in the interval.

THEOREM 1.6. If a function f is increasing at every point in an open interval (a,b), then f is an increasing function on (a,b).

PROOF. [Hardy, 1963, p. 208] It will suffice to prove that if $a < x < y < b$, then $f(x) \leq f(y)$. Divide the points y of (x,b) into two classes, L and R, such that $y \in L$ if $f(x_0) \geq f(x)$ for all x_0 in (x,y) and $y \in R$ otherwise. Since f is increasing at x, L is nonempty. Let $s = \sup \{y | y \in L\}$. If $s = b$, then R is empty and the conclusion follows. If $s < b$ and $f(s) \geq f(x)$, we can find an interval to the right of s, (s,y), such that for all y_0 in the interval,

$f(y_o) \geq f(s) \geq f(x)$. Then $y \in L$ which contradicts the definition of s . Therefore if $s < b$, $f(s) < f(x)$. Again we can find an interval to the left of s such that for all y_o in the interval, $f(y_o) \leq f(s) < f(x)$ which contradicts the definition of s . Hence $s = b$ and the proof is complete.

Notice that we obtain the same conclusion for Theorem 1.6 if we know only that f is increasing on the right at every point in (a,b) but add the condition that f be continuous in (a,b) . We then have that $f(y_o) \leq f(s)$ for values of y_o to the left of but sufficiently close to s .

Obviously the conclusion does not follow if we know only that f is increasing on the right at every point. For example, the function $f(x) = x$ $(0 \leq x < 1)$, $f(x) = x - 1$ $(1 \leq x \leq 2)$ is clearly increasing on the right at every point in $(0,2)$ but is not increasing on the entire interval.

CHAPTER II

LIMITS AND CONTINUITY

The questions of the existence of limits and the continuity of monotonic functions are so closely related that we consider both topics in this chapter. We begin with several simple theorems which require only that the function f be continuous as well as monotonic. Proofs for the obvious results are omitted.

THEOREM 2.1. If a function f is continuous on $[a,b]$ and if f is nondecreasing on (a,b), then f is nondecreasing on $[a,b]$.

PROOF. It will suffice to show that if $a < c < b$, then $f(a) \leq f(c) \leq f(b)$. Let d be a point between a and c. Since f is nondecreasing in (a,b), $f(d) \leq f(c)$. Let $\varepsilon > 0$ be given. Since f is continuous in $[a,b]$, there is some point x_0 between a and d such that $|f(a) - f(x_0)| < \varepsilon$. Now, $f(a) < f(x_0) + \varepsilon \leq f(d) + \varepsilon$. Since ε is arbitrary, $f(a) \leq f(d) \leq f(c)$. A similar argument shows that $f(c) \leq f(b)$ and completes the proof.

The next theorem is a simple consequence of the intermediate value theorem.

THEOREM 2.2. If a function f is continuous and nondecreasing on a closed interval $[a,b]$, then its set of values is the closed interval $[f(a), f(b)]$.

The following related result implies that f is strictly monotonic in $[a,b]$.

THEOREM 2.3. If f is a continuous function on $[a,b]$ and if f assumes each value between $f(a)$ and $f(b)$ exactly once, then f is strictly monotonic on $[a,b]$.

PROOF. Suppose $f(a) < f(b)$. Let $a < x < b$. If $f(x) = f(a)$ or if $f(x) = f(b)$, then f is not $1 - 1$ and we have a contradiction. Suppose $f(x) < f(a)$. Since f is continuous, it follows from $f(a) < f(b)$ and $a < x < b$ that there exists some c in $[x,b]$ such that $f(c) = f(a)$ and we again have a contradiction. A similar contradiction is reached if we suppose $f(x) > f(b)$. Therefore $f(a) < f(x) < f(b)$. Now, if $a < x < y < b$, then $f(x) < f(b)$ and similarly $f(x) < f(y)$ so f is strictly increasing on $[a,b]$. A similar argument shows that if $f(a) > f(b)$, then $f(x)$ is strictly decreasing on $[a,b]$ and the proof is complete.

Let us now consider the question of limits of monotonic functions. From advanced calculus, we know that there are many possibilities for

the behavior of a function as its independent variable tends to infinity. However, if we know that the function is nondecreasing or nonincreasing, these possibilities are greatly narrowed as is seen in the following theorem.

THEOREM 2.4. If f is a nondecreasing function defined on (x_0,∞), then either:

(i) $f(x)$ tends to a finite limit, α, as x tends to infinity, or

(ii) $f(x)$ tends to infinity as x tends to infinity.

PROOF. [Hardy, 1963, p. 137] Divide the real numbers into two sets, L and R, placing each real number, c, in L if $f(x) \geq c$ for some value of x (and thus for all greater values) or in R if $f(x) < c$ for all values of x. Clearly, the set L is nonempty; the set R may or may not be empty. If R is empty, then for any number, M, however large, $f(x) > M$ if x is sufficiently large. Thus, $\lim_{x\to\infty} f(x) = \infty$. If R is nonempty, then the two sets, L and R, form a Dedekind cut of the real number line. Let α be the number corresponding to the Dedekind cut; i.e., all numbers less than α belong to L and all numbers greater than α belong to R. Then $f(x) < \alpha + \varepsilon$ for all x and for any $\varepsilon > 0$. Hence $f(x) \leq \alpha$. Also, $f(x) > \alpha - \varepsilon$ for some value of x and so for all sufficiently large values of x. Therefore $\lim_{x\to\infty} f(x) = \alpha$ and the proof is complete.

Notice that, in general, $f(x) < \alpha$ for all x in Theorem 2.4. For if $f(x) = \alpha$ for some x, it must equal α for all greater x. In this case, α is the largest member of L; otherwise, L has no largest member.

Before considering an application of Theorem 2.4, we record the following simple but useful corollary.

COROLLARY 2.5. Let f be as in Theorem 2.4. If there is a number K such that $f(x) < K$ for all values of x, then the limit of f exists and is less than or equal to K. Otherwise, $\lim_{x\to\infty} f(x) = \infty$.

In practice, Theorem 2.4 is very useful in calculating limits. Consider the function $f(x) = c^x$ for $c > 1$. Clearly, $f(x)$ is an increasing function and by Theorem 2.4 tends to a finite limit (obviously greater than 1) or tends to infinity as x tends to infinity. Suppose $f(x)$ tends to a limit $\ell > 1$. Then $\lim_{x\to\infty} f(x+1) = \lim_{x\to\infty} f(x) = \ell$; but $\lim_{x\to\infty} f(x+1) = \lim_{x\to\infty} cf(x) = c \lim_{x\to\infty} f(x) = c\ell$. Thus $\ell = c\ell$ which is impossible since both c and ℓ are greater than 1. Hence $\lim_{x\to\infty} c^x = \infty$.

Often a function which fails to have a full limit may have a one-sided limit on either or both sides. The following definition clarifies this idea.

DEFINITION 2.6. If the function f has a limit at the point a as x approaches a from the right, then f is said to have a right-hand limit at a, denoted by $f(a+) = \lim_{x \to a+} f(x)$. If f has a limit at a as x approaches a from the left, then f is said to have a left-hand limit at a, denoted by $f(a-) = \lim_{x \to a-} f(a)$.

Even if a function has both a left- and right-hand limit at a point, these limits are not necessarily equal and we have the following special case.

DEFINITION 2.7. A function f is said to have a discontinuity of the first kind at a point a if the limits $f(a+)$ and $f(a-)$ both exist but are unequal.

As the above definitions imply, the left- and right-hand limits of an arbitrary function may not be well-behaved. However, if a function is monotonic, these limits have several specific properties as is seen in the next theorem.

THEOREM 2.8. If f is a nondecreasing function on the open interval (a,b), then for each $x \in (a,b)$, the limits $f(x+)$ and $f(x-)$ exist and $f(x-) \leq f(x) \leq f(x+)$.

PROOF. [Fulks, 1969, p. 51] We shall prove the theorem for the left-hand limit, $f(x-)$. Let $x \in (a,b)$ and let $x_n = x - \frac{1}{n}$. Since f is nondecreasing, $f(x - \frac{1}{n}) \leq f(x - \frac{1}{n+1}) \leq f(x)$ so that the sequence $f(x_n)$ is nondecreasing and bounded. Hence, there is a number A such that $\lim_{n\to\infty} f(x_n) = A \leq f(x)$. Given $\varepsilon > 0$, there is an N such that $0 \leq A - f(x_N) < \varepsilon$. Then for $x_N = x - \frac{1}{N} < x_o < x$, we have $f(x_N) \leq f(x_o) \leq A$ or $\varepsilon > A - f(x_N) \geq A - f(x_o) \geq 0$. Therefore $\lim_{N\to\infty} f(x_N) = \lim_{x\to x-} f(x)$ $= f(x-) = A \leq f(x)$. Similarly, $f(x+)$ exists and $f(x) \leq f(x+)$. This completes the proof.

A related result which identifies the limits $f(x\pm)$ can be proved using the ideas of supremum and infimum of f in the interval (a,b).

THEOREM 2.9. Let f be a nondecreasing function on the open interval $I = (a,b)$. For each $x \in I$, let $I^-(x) = \{y \in I | y < x\}$ and let $I^+(x) = \{y \in I | y > x\}$. Then $\sup_{y\in I^-} f(y) = f(x-)$, $\inf_{y\in I^+} f(y) = f(x+)$ and $f(x-) \leq f(x) \leq f(x+)$.

PROOF. [Sprecher, 1970, p. 174] We shall prove that $f(x-) = \sup_{y\in I^-} f(y)$. Let x be fixed in $(a,b) = I$. Since $I^-(x)$ is necessarily nonempty and bounded above, $\sup_{y\in I^-} f(y)$ exists; denote it by α. Since f is nondecreasing, the definition of $I^-(x)$ implies $\alpha \leq f(x)$. Given $\varepsilon > 0$, the definition of supremum

guarantees the existence of a point $x_0 \in I^-(x)$ such that $\alpha - \varepsilon < f(x_0) \leq \alpha$. Since f is monotonic, $f(x_0) \leq f(y) \leq \alpha$ whenever $x_0 < y < x$ and, thus, $\alpha - \varepsilon < f(y) \leq \alpha$. Hence, $f(x-) = \alpha$. A similar argument shows that $f(x+) = \inf_{y \in I^+} f(y)$. Now, from the fact that f is nondecreasing, we have $f(x-) = \sup_{y \in I^-} f(y) \leq f(x) \leq \inf_{y \in I^+} f(y) = f(x+)$ which completes the proof.

The following corollary, which follows immediately from Theorem 2.9, classifies the discontinuities of a monotonic function.

COROLLARY 2.10. If a function f is nondecreasing or nonincreasing at a point x, then f is either continuous at x or has a discontinuity of the first kind at x.

The next theorem follows almost immediately from Theorem 2.9.

THEOREM 2.11. If f is a nondecreasing function on $(a,b) = I$, and if $a < x < y < b$, then $f(x+) \leq f(y-)$.

PROOF. Let $I^+(x)$ and $I^-(x)$ be defined as in Theorem 2.9, and let $J = (x,y)$. Then J is nonempty and, by Theorem 2.9, $\sup_{z \in J} f(z) = f(y-)$, $\inf_{z \in J} f(z) = f(x+)$. Since $\inf_{z \in J} f(z) \leq \sup_{z \in J} f(z)$ the result follows.

Combining Corollary 2.10 and Theorem 2.11, we can prove the following important theorem on the continuity of monotonic functions. It asserts that such a function cannot have arbitrarily many discontinuities.

THEOREM 2.12. If f is a nondecreasing function defined on an interval I, then the set of discontinuities of f is at most countable.

PROOF. Suppose that x is a point of discontinuity of f. By Corollary 2.10, the discontinuity of f at x must be of the first kind, and since f is nondecreasing, we must have $f(x+) > f(x-)$. Let $r(x)$ be an arbitrary rational number such that $f(x-) < r(x) < f(x+)$. Similarly, associate a rational number with each point of discontinuity of f. If x and y are both points of discontinuity of f such that $x < y$, we know that $r(x) \neq r(y)$ since by Theorem 2.11, $r(x) < f(x+) \leq f(y-) < r(y)$. Hence, there exists a 1 - 1 correspondence between the set of discontinuities of f and a subset of the rational numbers. Since the rationals are countable, so are the points of discontinuity of f and our proof is complete. (Another proof of Theorem 2.12 will be given in the discussion of jump functions later in this chapter.)

It is natural to ask if the set of discontinuities of a monotonic function can be described more specifically. However, as the following

theorem shows, any countable set is the set of discontinuities of some monotonic function [Sprecher, 1970, p. 175].

THEOREM 2.13. Let A be a countable subset of the real numbers, R. Then there is a monotonic function f defined on R which is discontinuous on the set A but continuous on $R - A$.

PROOF. Let A be an arbitrary countable set with members a_n, and let $\sum_{n=1}^{\infty} \varepsilon_n = \alpha$ be an arbitrary convergent series of positive terms. For each value of n, define the function f_n by

$$f_n(x) = \begin{cases} 0 & x < a_n \\ \varepsilon_n & x \geq a_n \end{cases}$$

and pointwise define f by $f(x) = \sum_{n=1}^{\infty} f_n(x)$. We wish to show that

(i) $f(x)$ is nondecreasing on R,

(ii) $f(a_K+) - f(a_K-) = \varepsilon_K$ for each K; i.e., f is discontinuous on A, and

(iii) $f(x)$ is continuous on $R\backslash A$.

It is clear that for each x, $f(x) = \sum_{a_n \leq x} \varepsilon_n$ where the summation is over all values of n for which $a_n \leq x$. It is also clear that $0 \leq f \leq \alpha$. If $x < y$, we have

$$f(x) = \sum_{a_n \leq x} \varepsilon_n = \sum_{a_n \leq y} \varepsilon_n - \sum_{x < a_n \leq y} \varepsilon_n = f(y) - \sum_{x < a_n \leq y} \varepsilon_n \leq f(y),$$

since each term ε_n is positive. Thus, $f(x)$ is nondecreasing on R.

Now, fix a point $a_K \in A$ and let $y = a_K$. As above, for $x < a_K$ we have $f(x) = f(a_K) - \sum_{x<a_n\leq a_K} \varepsilon_n$. Since $\lim_{x\to a_K-} \sum_{x<a_n\leq a_K} \varepsilon_n = \varepsilon_K$, it follows that $f(a_K-) = f(a_K) - \varepsilon_K$. On the other hand, for each $x \geq a_K$, $f(x) = f(a_K) + \sum_{a_K<a_n\leq x} \varepsilon_n$. Since $\lim_{x\to a_K+} \sum_{a_K<a_n\leq x} \varepsilon_n = 0$, $f(a_K+) = f(a_K)$. Combining these results we have $f(a_K+) = f(a_K) = f(a_K-) + \varepsilon_K$. Therefore, f is discontinuous on A and $f(a_K+) - f(a_K-) = \varepsilon_K$.

To show that f is continuous on $R\backslash A$, fix a point y in $R-A$. As before, if $x < y$, $f(x) = f(y) - \sum_{x<a_n\leq y} \varepsilon_n$. Since $y \notin A$, $\lim_{x\to y-} \sum_{x<a_n\leq y} \varepsilon_n = 0$ and we have $f(y-) = f(y)$. If $x \geq y$, $f(x) = f(y) + \sum_{y<a_n\leq x} \varepsilon_n$. Again we have $\lim_{x\to y+} \sum_{y<a_n\leq x} \varepsilon_n = 0$ and $f(y+) = f(y)$. Therefore, f is continuous on $R\backslash A$ and the proof is complete.

As we have seen, the left- and right-hand limits of a function at a point are not necessarily equal to each other or to the value of the function at that point. The following definition assigns special names to the differences between these values.

DEFINITION 2.14. Let f be a function defined on $[a,b]$. Then the differences $f(x) - f(x-)$ and $f(x+) - f(x)$ are called the left and right jumps of f at the point x. The sum of the left and right jumps, $f(x+) - f(x-)$ is called the jump of f at the point x.

If we know that a function f is monotonic on $[a,b]$, then we can show that the sum of a finite number of its jumps cannot exceed the difference of the values of f at the endpoints of the interval, $|f(b) - f(a)|$.

LEMMA 2.15. Let f be a nondecreasing function on $[a,b]$. Let $x_1, x_2, \ldots, x_n$ be arbitrary points in (a,b). Then

$$(1)\quad f(a+) - f(a) + \sum_{K=1}^{n} [f(x_K+) - f(x_K-)] + f(b) - f(b-) \leq f(b) - f(a).$$

PROOF. Assume $a = x_o < x_1 < x_2 < \ldots < x_n < x_{n+1} = b$ and choose points $y_o, y_1, \ldots, y_n$ such that $x_K < y_K < x_{K+1}$ for $K = 0, 1, \ldots, n$. Then

(i) $f(x_K+) - f(x_K-) \leq f(y_K) - f(y_{K-1})$ $(K = 1, 2, \ldots, n)$,

(ii) $f(a+) - f(a) \leq f(y_o) - f(a)$, and

(iii) $f(b) - f(b-) \leq f(b) - f(y_n)$.

Adding the three inequalities, we see that the left side of (1) is bounded above by

$$f(y_o) - f(a) + f(y_1) - f(y_o) + \ldots + f(y_n) - f(y_{n-1}) + f(b) - f(y_n)$$
$$= f(b) - f(a)$$

and the proof is complete.

We can use inequality (1) to give a different proof of Theorem 2.12 which states that the set of discontinuities of a nondecreasing function f is at most countable. By Lemma 2.15, we know that f can have only a finite number of points of discontinuity at which its jump is greater than a given positive number, σ. (Otherwise, we would have $n\sigma \leq f(b) - f(a)$ where n, the number of such jumps, could be arbitrarily large.) If we let H be the set of all points of discontinuity of f and let H_K be the set of all points of discontinuity of f at which the jump is greater than $1/K$, we obviously have $H = H_1 \cup H_2 \cup H_3 \cup \ldots$. Since, by the above observation, each H_K is finite, H must be countable and Theorem 2.12 is proved.

Now that we know that the set of jumps of f is countable, we can take the limit as $n \to \infty$ in inequality (1). We then have

$$f(a+) - f(a) + \sum_{K=1}^{\infty} [f(x_K+) - f(x_K-)] + f(b) - f(b-) \leq f(b) - f(a)$$

which tells us that the sum of <u>all</u> jumps of the nondecreasing function f cannot exceed $f(b) - f(a)$.

We can also use the jumps of a nondecreasing function f to construct a special type of nondecreasing function called a jump function as described in the following definition.

DEFINITION 2.16. If $f(x)$ is a nondecreasing function on $[a,b]$, then the function $s(x)$ defined by

$$s(a) = 0$$

$$s(x) = [f(a+) - f(a)] + \sum_{x_K<x} [f(x_K+) - f(x_K-)] + [f(x) - f(x-)] \text{ for } (a < x \leq b)$$

is called the jump function of $f(x)$.

It is easy to show that the jump function s is nondecreasing.

LEMMA 2.17. If s is the jump function of the nondecreasing function f, then s is nondecreasing.

PROOF. If $x < y$, then

$$s(y) - s(x) = \sum_{x \leq x_K < y} [f(x_K+) - f(x_K-)] + f(y) - f(y-) - f(x) + f(x-)$$

$$= \sum_{x < x_K < y} [f(x_K+) - f(x_K-)] + [f(y) - f(x)] + [f(x+) - f(y-)].$$

It follows from Theorem 2.9, Theorem 2.11, and the fact that f is nondecreasing that $s(y) - s(x) \geq 0$. Hence, s is nondecreasing.

One special relationship between a nondecreasing function f and its jump function s is that their difference $f - s$ is always a

nondecreasing, continuous function. We prove this fact in the following theorem.

THEOREM 2.18. Let f be a nondecreasing function and let s be the jump function of f. Then the difference $g = f - s$ is a nondecreasing, continuous function.

PROOF. Let $a \leq x < y \leq b$. By Lemma 2.17,

$$(2) \quad s(y) - s(x) = [f(x+) - f(x)] + \sum_{x<x_K<y} [f(x_K+)-f(x_K-)]+[f(y)-f(y-)].$$

Then by Lemma 2.15 and inequality (1), we have $s(y) - s(x) \leq f(y) - f(x)$ or $g(x) \leq g(y)$ and g is nondecreasing. If we let y approach x in the last inequality, we obtain $s(x+) - s(x) \leq f(x+) - f(x)$. However, from equation (2) we have $f(x+) - f(x) \leq s(y) - s(x)$. Letting y approach x gives $f(x+) - f(x) \leq s(x+) - s(x)$. Combining these two results we have $f(x+) - f(x) = s(x+) - s(x)$. Therefore, $g(x+) = g(x)$ and, similarly, $g(x-) = g(x)$. Since g is obviously continuous on $R \setminus A$ where A is the set of discontinuities of f, g is a continuous function and the proof is complete.

Since we may write $f = s + f - s$, the following corollary is obvious.

COROLLARY 2.19. If f is any nondecreasing function, then f can be represented as the sum of a continuous, nondecreasing function g and a jump function s.

Our final theorem in this chapter concerns the relationship between absolute continuity and monotonic functions. We first record the following definition.

DEFINITION 2.20. A function f defined on $[a,b]$ is said to be absolutely continuous on $[a,b]$ if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon$ for every finite collection $\{(x_i , y_i)\}$ of nonoverlapping intervals with $\sum_{i=1}^{n} |y_i - x_i| < \delta$.

We can now show that the composite function of two absolutely continuous functions will be absolutely continuous if one of the functions is monotonic.

THEOREM 2.21. Let f be an absolutely continuous function on $[\alpha,\beta]$ and let g be an absolutely continuous function defined on $[a,b]$ with values in $[\alpha,\beta]$. Let h be the composite function $h(x) = f(g(x))$. If g is monotonic, then h is absolutely continuous.

PROOF. [Botts and McShane, 1959, p. 103] Suppose g is decreasing. If $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if (α_1,β_1), ..., (α_n,β_n) are disjoint intervals in $[\alpha,\beta]$ for

which $\sum_{i=1}^{n} |\alpha_i - \beta_i| < \delta_1$, then $\sum_{i=1}^{n} |f(\alpha_i) - f(\beta_i)| < \varepsilon$. Since g is absolutely continuous, there exists $\delta > 0$ such that if (a_1,b_1) , ..., (a_n, b_n) are disjoint intervals in $[a,b]$ for which $\sum_{i=1}^{n} |b_i - a_i| < \delta$, then $\sum_{i=1}^{n} |g(b_i) - g(a_i)| < \delta_1$. But, since g is decreasing, the intervals $(g(b_1), g(a_1))$, ..., $(g(b_n), g(a_n))$ are disjoint subintervals of $[\alpha,\beta]$ (perhaps some are empty) with total length less than δ_1 , so $\sum_{i=1}^{n} |f(g(b_i)) - f(g(a_i))| < \varepsilon$.

We remark that Theorem 2.21 is a special case of the more general result that $f \circ g$ is absolutely continuous if and only if g is of bounded variation. The proof of this more general theorem is considerably difficult and we refer the interested reader to Hewitt and Stromberg [1965, p. 297].

CHAPTER III

DERIVATIVES

In this chapter, we wish to explore the natural relationship between monotonic functions and derivatives. In Section I, we consider several standard results of basic calculus including a few results which use Dini derivatives. Section II is devoted to the Lebesgue theorem and related results, and, in Section III, we see how the concepts of bounded variation and inverse functions are related to monotonic functions and derivatives.

I. BASIC RESULTS

Our first theorem uses the sign of the derivative of a function to provide necessary and sufficient conditions for the function to be monotonic.

THEOREM 3.1. Let a function f be differentiable on an interval I. Then

(i) $f'(x) \geq 0$ on I if and only if f is nondecreasing,

(ii) $f'(x) > 0$ on I implies f is strictly increasing.

PROOF. Suppose f is nondecreasing. Then both numerator and denominator in the difference quotient $[f(x+h) - f(x)]/h$ have the same sign. Taking the limit as $h \to 0$, we have $f'(x) \geq 0$.

Now suppose $f'(x) \geq 0$ on I and let $x_1 > x_2$ be two points in the interval. Then by the mean value theorem, there exists a point x_0 such that $x_1 > x_0 > x_2$ and $f(x_1) - f(x_2) = f'(x_0)(x_1-x_2)$. Since $x_1 - x_2 > 0$, we have $f(x_1) - f(x_2) \geq 0$ if $f'(x_0) \geq 0$ and $f(x_1) - f(x_2) > 0$ if $f'(x_0) > 0$. The theorem follows immediately.

The simple example $f(x) = x^3$ shows that $f'(x)$ need not be positive at all points of an interval for the function f to be strictly increasing on that interval. The important fact is that the points where $f'(x) = 0$ be isolated as we see in the following theorem.

THEOREM 3.2. If a function f is differentiable on an interval (a,b), and if $f'(x) > 0$ on (a,b) except at a single point x_0 where $f'(x_0) = 0$, then f is strictly increasing on (a,b).

PROOF. By Theorem 3.1, f is strictly increasing on the intervals (a, x_0) and (x_0, b). From Theorem 2.9 and the continuity of f, we have that $f(x_0) = \lim_{x \to x_0+} f(x) = \inf_{x > x_0} f(x)$. If $x_0 < x$ and $f(x_0) = f(x)$, we can find a point x_1 such that $x_0 < x_1 < x$ and $f(x_1) < f(x) = f(x_0)$. Since this contradicts the definition of $f(x_0)$, we have $f(x_0) < f(x)$ for all $x > x_0$. Similarly, $f(x_0) > f(x)$ for all $x < x_0$ and the proof is complete.

We remark that Theorem 3.2 can be extended in the obvious way to the case $f'(x_i) = 0$ at a finite number of points x_i.

In order to discuss the derivative of a function f more precisely, we define four quantities called Dini derivatives.

DEFINITION 3.3. If a function f is defined in some neighborhood of the point x, then the upper right, lower right, upper left, and lower left Dini derivatives of f at the point x are, respectively:

$$D^+f(x) = \limsup_{h\to 0+} \frac{f(x+h) - f(x)}{h}$$

$$D_+f(x) = \liminf_{h\to 0+} \frac{f(x+h) - f(x)}{h}$$

$$D^-f(x) = \limsup_{h\to 0-} \frac{f(x+h) - f(x)}{h}$$

$$D_-f(x) = \liminf_{h\to 0-} \frac{f(x+h) - f(x)}{h} \quad .$$

Clearly $D^+f(x) \geq D_+f(x)$ and $D^-f(x) \geq D_-f(x)$. If all four Dini derivatives are equal at a point x, we say that f is differentiable at x and define $f'(x)$ to be their common value.

A generalization of Theorem 3.1, known as Zygmund's Lemma, uses only one of the Dini derivatives to show that a function is monotonic on an interval [Szarski, 1965, p. 10].

LEMMA 3.4. (Zygmund's Lemma) Let f be a continuous function on an interval I and let $Z_+ = \{t \in I : D_+f(t) > 0\}$. If the set $f(I - Z_+)$ does not contain any interval, then f is increasing on I.

PROOF. Suppose that f is not increasing on I. Then there exist two points $t_1 < t_2$ in I such that $f(t_1) > f(t_2)$. Since $f(I - Z_+)$ does not contain the interval $(f(t_1), f(t_2))$, there is a point y_o in $(f(t_1), f(t_2))$ such that $y_o \notin f(I - Z_+)$. Since f is continuous, the set $E = \{t \in (t_1, t_2) : f(t) = y_o\}$ is not empty. Let t_o be the supremum of E. Then $t_o \in (t_1, t_2)$ and by continuity, $f(t_o) = y_o$. Also, $f(t) < y_o$ for $t_o < t < t_2$ and since $y_o \notin f(I - Z_+)$, we must have $t_o \in Z_+$. Hence, $D_+f(t_o) > 0$. On the other hand, $f(t) - f(t_o) < 0$ for $t_o < t < t_2$ and it follows from Definition 3.3 that $D_+f(t_o) \leq 0$. Thus we have a contradiction and f must be increasing on I.

Notice that Lemma 3.4 remains true if we replace Z_+ by $Z^+ = \{t \in I : D^+f(t) > 0\}$ since the fact that $f(t) - f(t_o) < 0$ for $t_o < t < t_2$ also implies $D^+f(t) \leq 0$. We may also replace Z_+ by either of the sets $Z_- = \{t \in I : D_-f(t) > 0\}$ or $Z^- = \{t \in I : D^-f(t) > 0\}$ simply by letting t_o be the infimum of the set E in the above proof.

An important consequence of Zygmund's Lemma is the following theorem which gives both a necessary and sufficient condition for a continuous function to be monotonic [Szarski, 1965, p. 11].

THEOREM 3.5. Let f be a continuous function on an interval I, and let $Q_+ = \{t \in I : D_+f(t) \geq 0\}$. Then f is increasing on I if and only if the set $I - Q_+$ is at most countable.

PROOF. The necessity is obvious since for an increasing function the set $I - Q_+$ is empty. To prove the sufficiency of the condition, let $\varepsilon > 0$ be arbitrary and let $g(t) = f(t) + \varepsilon t$. We then have $D_+g(t) = D_+f(t) + \varepsilon$ and, consequently, $D_+g(t) > 0$ for $t \in Q_+$. It follows that for the set $Z_+ = \{t \in I : D_+g(t) > 0\}$ we have $Q_+ \subset Z_+$ and, thus, $I - Z_+ \subset I - Q_+$. Then, since the set $I - Q_+$ is at most countable, so are the sets $I - Z_+$ and $g(I - Z_+)$. Hence, the set $g(I - Z_+)$ does not contain any interval and, by Zygmund's Lemma, $g(t)$ is increasing. Since $\varepsilon > 0$ is arbitrary, it follows that $f(t)$ is increasing also.

As a simple consequence of Theorem 3.5, we have the following corollary.

COROLLARY 3.6. Let f be as in Theorem 3.5, and let $P_+ = \{t \in I : D_+f(t) > 0\}$. Then f is strictly increasing on I if the set $I - P_+$ is at most countable.

PROOF. Let $I - P_+$ be at most countable. By Theorem 3.5, f is increasing on I. If it were not strictly increasing, we would have $f(t_1) = f(t_2)$ for two points $t_1 < t_2$ in I. Therefore, f would be constant on the interval $[t_1, t_2]$ and, consequently, $f'(t) = 0$ on $[t_1, t_2]$. Then $[t_1, t_2]$ is in $I - P_+$ which contradicts the fact that $I - P_+$ is at most countable. Hence, f is strictly increasing on I.

II. THE LEBESGUE THEOREM AND RELATED RESULTS

In order to prove the main theorem of this section, we need several preliminary definitions concerning measure zero and length of an interval. We begin with a definition of component interval.

DEFINITION 3.7. Let S be an open subset of R. An open interval I is called a component interval of S if $I \subseteq S$ and if there is no open interval $J \neq I$ such that $I \subseteq J \subseteq S$.

We note that any open subset O of R can be written uniquely as the union of a countable collection of disjoint component intervals O_n [Apostle, 1974, p. 51], and we define the length of O, $\ell(O)$, to be $\Sigma\ell(O_n)$. We also remark that if $\{I_n\}$ is any countable collection of (open) intervals which covers O, that is, if $O \subseteq \bigcup I_n$, then $\ell(O) \leq \Sigma\ell(I_n)$. The next definition concerns an arbitrary subset of R.

DEFINITION 3.8. A set S of real numbers is said to have measure zero if, for every $\varepsilon > 0$, there is a countable covering of S by open intervals, the sum of whose lengths is less than ε.

It is a simple exercise to show that the union of any countable collection of sets of measure zero also has measure zero [Apostle, 1974, p. 170].

We are now prepared to prove the Lebesgue Theorem which states that any nondecreasing function defined on a bounded interval will be differentiable almost everywhere (a.e.); that is, the function will have a derivative except possibly on a set of measure zero. This result was first proved for continuous functions by H. Lebesgue and, later, was extended to include all nondecreasing functions by W. H. Young. The following proof which includes four preliminary lemmas is due to F. Riesz [Taylor, 1965, p. 404].

In the first lemma, we consider a function $g : D \to R$ and introduce an auxilliary function $G : \overline{D} \to R$ defined as follows: Fix $x \in \overline{D}$, the closure of D, and let X be the class of all sets $N = D \cap U$ where U can be any neighborhood of x. Then $G(x) \equiv \inf_{N \in X} (\sup \{g(t) : t \in N\})$, $x \in \overline{D}$. We note that the definition of G is different from that of the lim sup function in that the neighborhoods of x are allowed to contain x.

LEMMA 3.9. Let g be a function defined on (a,b) and let $G(x) = \inf_{N \in X} (\sup \{g(t) : t \in N\})$. Let subsets E and F of (a,b) be defined as follows:

$$E = \{x : G(x) < g(t) \text{ for some } t \in (x,b)\},$$

$$F = \{x : G(x) < g(t) \text{ for some } t \in (a,x)\}.$$

Then E and F are open sets. Also if E is nonempty and if $E = \bigcup_n I_n$ where $I_1, I_2, \ldots$ are disjoint open intervals with

$I_n = (a_n, b_n)$, then $g(x) \leq G(b_n)$ when $x \in I_n$. If F is nonempty and if $F = \bigcup_n J_n$ where J_1 , J_2 , ... are disjoint open intervals with $J_n = (c_n, d_n)$, then $g(x) \leq G(c_n)$ when $x \in J_n$.

PROOF. It will suffice to prove the lemma for E . We can then prove the result for F by letting $g_1(x) = g(-x)$ for $-b < x < -a$ and letting $G_1(x) = \inf_{N \in X} (\sup \{g_1(t) : t \in N\})$. Applying the result about E to g_1 with $(-b, -a)$ in place of (a,b) leads to the assertion about F in relation to g .

Suppose E is not empty. The fact that E is open follows from the fact that G is upper semicontinuous; that is, $G(x) = \inf_{N \in X} \sup \{G(t) : t \in N\}$. To see this, let x_o be in E ; then there is some $t_o \in (x_o, b)$ such that $G(x_o) < g(t_o)$. Then the set $\{x : G(x) < g(t_o)\}$ is open because G is upper semicontinuous. This set contains x_o and therefore contains an open interval $(x_o - \delta, x_o + \delta)$ where δ may be chosen so small that $x_o + \delta < t_o$ and $x_o - \delta > a$. Thus, this open interval lies in E and E is open.

Since E is open, we can write $E = \bigcup_n I_n$ where the I_n are disjoint open intervals with $I_n = (a_n, b_n)$. Fix any $x \in I_n$ and let $C = \{t : x \leq t \leq b_n$ and $g(x) \leq G(t)\}$. Then $x \in C$ since $g(x) \leq G(x)$. Let $M = \sup C$. Then $x \leq M$ and $G(M) \geq g(x)$; for if $G(M) < g(x)$, the upper semicontinuity of G would imply $G(t) < g(x)$ for all t in $[a,b]$ sufficiently near M and thus for some t in C which contradicts the definition of C . Now,

if $M = b_n$, the proof is complete. If $M < b_n$, then M is in I_n so there is some t such that $M < t < b$ and $G(M) < g(t)$. Then $g(x) \leq G(M) < g(t) \leq G(t)$. But t is not in C since $M < t$, so $b_n < t$. If $G(b_n) < g(t)$, the definition of E would show that b_n is in E. Since b_n is not in E, we must have $g(t) \leq G(b_n)$ which implies $g(x) < G(b_n)$ and completes the proof.

LEMMA 3.10. Let f be a bounded nondecreasing function defined on (a,b). Let $c > 0$ and define $g(x) = f(x) - cx$. Define E, F, and G as in Lemma 3.9. Then if E and F are nonempty, we have

(i) $c\Sigma(b_n - a_n) \leq f(b-) - f(a+)$ and

(ii) $f(d_n-) - f(c_n+) \leq c(d_n - c_n)$

where a_n, b_n, c_n, and d_n are as in Lemma 3.9.

PROOF. Here, $G(x) = \inf_{N \in X} (\sup \{f(t) - ct : t \in N\})$. To estimate $G(a)$, we let $a < t < b$ and deduce from $f(t) - ct < f(t) - ca$ that $G(a) \leq f(a+) - ca$. Similarly, to estimate $G(b)$, we let $a < t < b$ and deduce from $f(t) - ct \leq f(b-) - ct$ that $G(b) \leq f(b-) - cb$. (Since f is bounded, $f(a+)$ and $f(b-)$ are finite.) When $a < x < b$, we have $f(t) - ct < f(t) - cx$ if $x < t < b$ and $f(t) - ct \leq f(x+) - ct$ if $a < t \leq x$. Hence

$G(x) \leq f(x+) - cx$.

Now suppose $E = \bigcup_n I_n$ as in Lemma 3.9. Then $g(x) = f(x) - cx \leq G(b_n)$ if $a_n < x < b_n$. Hence $f(a_n+) - ca_n \leq G(b_n)$. Combining this with results from the above paragraph we have that

$$c(b_n - a_n) \leq f(b_n+) - f(a_n+) \quad \text{if} \quad b_n < b \quad \text{and}$$

$$c(b_n - a_n) \leq f(b_n-) - f(a_n+) \quad \text{if} \quad b_n = b \ .$$

Since the open intervals I_1 , I_2 , ... are nonoverlapping, and since f is nondecreasing, we obtain inequality (i) from these two inequalities.

Now suppose that $F = \bigcup_n J_n$ where $J_n = (c_n, d_n)$. Then $f(x) - cx \leq G(c_n)$ if $c_n < x < d_n$ so that $f(d_n-) - cd_n \leq G(c_n)$. Since we know from the first paragraph of the proof that $G(c_n) \leq f(c_n+) - cc_n$, we can obtain inequality (ii) by the same type of argument used above and the lemma is proved.

We now come to two lemmas on derivatives.

LEMMA 3.11. Let f be a bounded, nondecreasing function on (a,b) . Then the set $B = \{x : D^+f(x) = +\infty\}$ has measure zero.

PROOF. Let S be the set of points in (a,b) at which f is discontinuous. By Theorem 2.12, S is countable and therefore of measure zero. For each $c > 0$, let $B(c) = \{x : D^+f(x) > c\}$.

Let g, G, and E be as in Lemma 3.10. If $x \in B(c)$, there exists arbitrarily small $h > 0$ such that $f(x+h) - f(x) > ch$ or $f(x+h) - c(x+h) > f(x) - cx$. Moreover, if $x \in B(c) - S$, then by the continuity of $g(x) = f(x) - cx$, we have $g(x) = G(x)$ and, hence, the last inequality can be written as $g(x+h) > G(x)$. It follows that $x \in E$ and $B(c) - S \subset E$. Since $D^+f(x) = +\infty$ implies $x \in B(c)$, we have $B \subset (S \cup E)$. Now, let $E = \bigcup I_n$ where I_1, I_2, ... are disjoint open intervals with $I_n = (a_n, b_n)$. Then by Lemma 3.10, $\Sigma(b_n - a_n) \leq [f(b-) - f(a+)]/c$. Since c may be arbitrarily large, the set E has measure zero. Finally, since the sets S and E have measure zero and since $B \subset (S \cup E)$, the set $B = \{x : D^+(x) = +\infty\}$ has measure zero.

LEMMA 3.12. Let f be a bounded, nondecreasing function on (a,b). Suppose $0 < \alpha < \beta$ and let $A(\alpha) = \{x : D_-f(x) < \alpha\}$, $B(\beta) = \{x : D^+(x) > \beta\}$. Then $A(\alpha) \cap B(\beta)$ has measure zero.

PROOF. Let S be the countable set of points of discontinuity of f in (a,b). We will show that $A(\alpha) \cap B(\beta) - S$ has measure zero. Since we need to apply Lemma 3.10 with α or β in place of c and with a subinterval of (a,b) in place of (a,b), we introduce some further notation. For any $\alpha > 0$, define $g_\alpha(x) = f(x) - \alpha x$ and $G_\alpha(x) = \inf_{N \in X} (\sup \{g_\alpha(t) : t \in N\})$. Regarding a and b as variables with $a < b$, define $E(a,b;\alpha) = \{x : a < x < b$, $G_\alpha(x) < g_\alpha(t)$ for some $t \in (x,b)\}$ and define

$F(a,b;\alpha) = \{x : a < x < b,\ G_\alpha(x) < g_\alpha(t) \text{ for some } t \in (a,x)\}$.
As in Lemma 3.11, $B(\beta) - S \subset E(a,b;\beta)$. Also, if $x \in A(\alpha)$, there exists arbitrarily small $h > 0$ such that $f(x-h) - \alpha(x-h) > f(x) - \alpha x$, and it follows that $A(\alpha) - S \subset F(a,b;\alpha)$.

Let $T = A(\alpha) \cap B(\beta) - S$ and suppose T is nonempty. Then $E(a,b;\beta)$ is nonempty and $E(a,b;\beta)$ can be expressed as a countable union of disjoint open intervals $I_n = (a_n, b_n)$. Also since $[A(\alpha) - S] \cap I_n \subset F(a_n, b_n;\alpha)$, we have $T \subset \bigcup_n \{[A(\alpha) - S] \cap I_n\} \subset \bigcup_n F(a_n, b_n;\alpha)$. Consider only those $F(a_n, b_n;\alpha)$ which are nonempty, and write $F(a_n, b_n;\alpha) = \bigcup_i J_{ni}$, where the intervals $J_{ni} = (c_{ni}, d_{ni})$ are disjoint. Now $[B(\beta) - S] \cap J_{ni} \subset E(c_{ni}, d_{ni};\beta)$, and therefore $T = T \cap [B(\beta) - S] \subset \bigcup_n \bigcup_i E(c_{ni}, d_{ni};\beta) \equiv V_1$. Using Lemma 3.10 as in Lemma 3.11, we obtain $\Sigma(d_{ni} - c_{ni}) \leq [f(d_{ni}-) - f(c_{ni}+)]/\beta \leq (\alpha/\beta)(d_{ni} - c_{ni})$. Therefore, $E(c_{ni}, d_{ni};\beta)$ is the union of a collection of open intervals of total length less than or equal to $(\alpha/\beta)(d_{ni} - c_{ni})$, and V_1 is covered by a collection of open intervals of total length less than or equal to $(\alpha/\beta)(b - a)$. This same reasoning can be applied to investigate the part of T contained in any open subinterval of (a,b) . The result is that if I is an open subinterval of (a,b) , then there is an open subset V of I such that $I \cap T \subset V$ and V is the union of a collection of open intervals of total length less than or equal to $(\alpha/\beta)(\ell(I))$.

We now wish to construct a sequence $\{V_n\}$ of open subsets of (a,b) such that

(i) $T \subset V_n$ and

(ii) V_n is the union of a collection of open intervals of total length less than or equal to $(\alpha/\beta)^n(b - a)$.

Since $0 < \alpha < \beta$, we can then conclude that T has measure zero and the proof will be complete. We construct the sequence $\{V_n\}$ by induction. The case for $n = 1$ has already been treated. Assuming that (i) and (ii) are true for $n \leq m$, we express V_m as the union of disjoint open intervals I_{m1} , I_{m2} , ... There exists an open subset V_{mi} of I_{mi} such that $T \cap I_{mi} \subset V_{mi}$ and V_{mi} is the union of a collection of open intervals of total length less than or equal to $(\alpha/\beta)(\ell(I_{mi}))$. Let $V_{m+1} = \bigcup_i V_{mi}$. Then $T = T \cap V_m = \bigcup_i (T \cap I_{mi}) \subset \bigcup_i V_{mi} = V_{m+1}$. Therefore, V_{m+1} can be covered by a collection of open subintervals of total length less than or equal to $(\alpha/\beta) \sum_i \ell(I_{mi})$. Since $\sum_i \ell(I_{mi}) = \ell(V_m)$, we have V_{m+1} covered by a collection of intervals of total length less than or equal to $(\alpha/\beta)^{m+1}(b - a)$ and the proof is complete.

We are now ready to prove the Lebesgue Theorem.

THEOREM 3.13. Let f be a nondecreasing function defined on (a,b) . Then f is differentiable almost everywhere on (a,b) .

PROOF. We may assume that f is bounded since (a,b) is a countable union of open subintervals on each of which f is bounded. Since f is nondecreasing, each of the Dini derivatives of f is

nonnegative at each point of (a,b). We claim that

(i) $\{x : D^+f(x) > D_-f(x)\}$ has measure zero,

(ii) $\{x : D_+f(x) < D^-f(x)\}$ has measure zero.

To see (i), let $\{(\alpha_n, \beta_n)\}$ be the collection of all pairs of rational numbers such that $0 < \alpha_n < \beta_n$. Then $\{x : D^+f(x) > D_-f(x)\} = \bigcup_n \{A(\alpha_n) \cap B(\beta_n)\}$ and (i) follows from Lemma 3.12.

To prove (ii), let $t = a + b - x$ so that t decreases from b to a as x increases from a to b. Let $F(t) = -f(x)$. Then F is nondecreasing on (a,b), and we see by easy calculations that $D^+F(t) = D^-f(x)$ and $D_-F(t) = D_+f(x)$. Since the mapping $t = a + b - x$ preserves the lengths of intervals, and since $D_+f(x) < D^-f(x)$ is equivalent to $D_-F(t) < D^+F(t)$, we may apply (i) to F to obtain (ii). Now using (ii) and the definition of Dini derivatives, we have $D_-f \leq D^-f \leq D_+f \leq D^+f$ almost everywhere. By (i), $D_-f < D^+f$ only on a set of measure zero so that the equality signs in the above inequality must hold almost everywhere. Lemma 3.11 now implies that f is differentiable almost everywhere.

Before concluding this section with an unusual example, we present a final theorem on derivatives which allows us to differentiate a convergent series of monotonic functions term by term.

THEOREM 3.14. (Fubini's Theorem) Let $\sum_{n=1}^{\infty} F_n(x)$ be a series of monotonic (nondecreasing) functions defined on $[a,b]$ which converges everywhere to $F(x)$. Then $\sum_{n=1}^{\infty} F_n(x)$ admits term by term differentiation almost everywhere; i.e., $\sum_{n=1}^{\infty} F_n'(x) = F'(x)$ a.e.

PROOF. [Shilov, 1965, p. 291] Suppose without loss of generality that all $F_n(x) \geq 0$ and vanish at $x = a$. Otherwise, we could replace $F_n(x)$ by $F_n(x) - F_n(a)$. Since the sum of a series of nondecreasing functions is a nondecreasing function, we may consider the set E on which all the derivatives $F_n'(x)$ and $F'(x)$ exist. Then the set $[a,b] - E$ has measure zero. For $x \in E$ and any $y > x$ we have

$$\frac{\sum_{n=1}^{\infty}[F_n(y) - F_n(x)]}{y - x} = \frac{F(y) - F(x)}{y - x} .$$

Since the terms on the left are nonnegative, we have for any N,

$$\frac{\sum_{n=1}^{N}[F_n(y) - F_n(x)]}{y - x} \leq \frac{F(y) - F(x)}{y - x} .$$

Taking the limit as y approaches x yields $\sum_{n=1}^{N} F_n'(x) \leq F'(x)$; letting N tend to infinity, and remembering $F_n'(x) \geq 0$, we find

(i) $\sum_{n=1}^{\infty} F_n'(x) \leq F'(x)$ $(x \in E)$.

We must show that the equality sign holds for almost all x.

For a given K, there is a partial sum $S_{n_K}(x)$ of the series $\sum_{n=1}^{\infty} F_n(x) = F(x)$ for which $0 \leq F(b) - S_{n_K}(b) < 1/2^K$ for $K = 1, 2, 3, \ldots$. Since the difference $F(x) - S_{n_K}(x) = \sum_{j>n_K} F_j(x)$, it follows that for all x, $0 \leq F(x) - S_{n_K}(x) < 1/2^K$, and hence, $\sum_{K=1}^{\infty} [F(x) - S_{n_K}(x)]$ converges on the closed interval $a \leq x \leq b$. Moreover, by what we have proved in the first paragraph, the series of derivatives converges almost everywhere. The general term $F'(x) - S'_{n_K}(x)$ of this series tends almost everywhere to zero, which means that $S'_{n_K} \to F'(x)$ almost everywhere. Now, if the strict inequality occurs in (i), no sequence of partial sums could have the limit $F'(x)$. Therefore, the equality sign must hold for almost all x and the proof is complete.

So that the reader will not think that all nondecreasing functions and their derivatives behave in an expected manner, we conclude this section with a curious example. We shall construct a function $f : [0,1] \to R$ such that $f(0) = 0$, $f(1) = 1$ and $f'(x) = 0$ almost everywhere.

Let $I = [0,1]$ and let $A_1 = I - (\frac{1}{3}, \frac{2}{3})$ be the subset of I obtained by removing the open middle third of I. Let A_2 be the subset of I obtained by removing the open middle thirds of $[0, \frac{1}{3}]$ and of $[\frac{2}{3}, 1]$. Continuing this process, we can define a set $S = \bigcap_{n=1}^{\infty} A_n$ called the Cantor set. The complement of S, denoted by $C(S)$, is an open set whose component intervals we

shall denote by $I(\frac{m}{2^n})$ $(1 \leq m < 2^n)$ where $I(\frac{1}{2}) = (\frac{1}{3}, \frac{2}{3})$, $I(\frac{1}{4}) = (\frac{1}{3^2}, \frac{2}{3^2})$, $I(\frac{3}{4}) = (\frac{2}{3} + \frac{1}{3^2}, \frac{2}{3} + \frac{2}{3^2})$, $I(\frac{1}{8}) = (\frac{1}{3^3}, \frac{2}{3^3})$, etc. We now wish to define a function f on $[0,1]$. Let $f(0) = 0$, $f(1) = 1$, and let $f(x) = \frac{m}{2^n}$ for every m and n if $x \in I(\frac{m}{2^n})$. If $0 < x < 1$ and $x \in S$, define $f(x) = \sup \{f(x_0) \mid x_0 \in C(S), x_0 < x\}$.

We shall show that f is nondecreasing. Suppose $x < y$. If $x, y \in C(S)$, then, obviously, $f(x) \leq f(y)$. If $x \in C(S)$ and $y \in S$, then $f(y) = \sup \{f(x_0) \mid x_0 \in C(S), x_0 < y\} \geq f(x)$. If $x \in S$ and $y \in C(S)$, then for $z < x$ and $z \in C(S)$, $f(z) < f(y)$. It follows that $f(y) \geq \sup \{f(z) \mid z \in C(S), z < x\} = f(x)$. Finally, if x and y are in S, there is some $z \in C(S)$ such that $x < z < y$. But then $f(x) \leq f(z)$ and $f(z) \leq f(y)$ imply $f(x) \leq f(y)$.

We next show that f is continuous. Let $\varepsilon > 0$. There is some n such that $\frac{1}{2^n} < \frac{\varepsilon}{2}$. For every $m < 2^n$, let $x_m \in I(\frac{m}{2^n})$. Moreover, let $x_0 = 0$, $x_{2^n} = 1$. Then $f(x_0) = 0$, $f(x_1) = \frac{1}{2^n}$, ..., $f(x_{2^n-1}) = \frac{2^n-1}{2^n}$, $f(x_{2^n}) = 1$. Let $\delta = \min [|x_i - x_{i-1}|, i = 0, 1, \ldots, 2^n]$. If $0 < y - x < \delta$, there is at most one x_i such that $x_{i-1} \leq x \leq x_i \leq y \leq x_{i+1}$. But then $0 \leq f(y) - f(x) = f(y) - f(x_i) + f(x_i) - f(x) \leq f(x_{i+1}) - f(x_i) + f(x_i) - f(x_{i-1}) = \frac{1}{2^n} + \frac{1}{2^n} < \varepsilon$. Thus, f is continuous.

Finally, we show that $f'(x) = 0$ a.e. Since the Cantor set has measure zero, it will suffice to show that $f'(x) = 0$ for $x \in C(S)$. Let $x \in C(S)$. Then there exists $\varepsilon > 0$

such that $(x - \varepsilon, x + \varepsilon) \subseteq C(S)$ and f is constant on $(x - \varepsilon, x + \varepsilon)$, say $f(x) = c$ there. Then $f'(x) = 0$ for all $x \in C(S)$. This completes the example.

Frielich [1973] constructs a strictly increasing function F such that F is continuous and has derivative equal to zero almost everywhere. Extend the function f in the above example by letting $f(x) = 1$ for $x \geq 1$ and letting $f(x) = 0$ for $x \leq 0$. Let $A = \{a_1, a_2, \ldots\}$ be any countable set that is dense in the reals and define $F(x) = \sum_{n=1}^{\infty} 2^{-n} f(2^n(x-a_n))$. The uniform convergence of the sum defining F implies that F is continuous. If $x_2 < x_1$, choose a_n so that $x_2 < a_n < x_1$. Then $f(2^n(x_1 - a_n)) > 0 = f(2^n(x_2 - a_n))$ and $f(2^m(x_1 - a_m)) \geq f(2^m(x_2 - a_m))$ for $m \neq n$ so that $F(x_1) > F(x_2)$. Finally, Fubini's Theorem gives $F'(x) = \Sigma f'(2^n(x - a_n)) = 0$ for almost all x.

III. BOUNDED VARIATION AND INVERSES

In this section, we wish to show how the concepts of bounded variation and inverse functions are related to monotonic functions and derivatives. We begin with a definition of bounded variation.

DEFINITION 3.15. Let f be a function defined on $[a,b]$, and let $a = x_0 < x_1 < x_2 < \ldots < x_K = b$ be any subdivision of $[a,b]$. Define $p = \sum_{i=1}^{K} [f(x_i) - f(x_{i-1})]^+$,

$n = \sum_{i=1}^{K} [f(x_i) - f(x_{i-1})]^-$ and $t = n + p = \sum_{i=1}^{K} |f(x_i) - f(x_{i-1})|$, where $r^+ = \max\{r,0\}$ and $r^- = |r| - r^+$. Set $P = \sup p$, $N = \sup n$, and $T = \sup t$, where each supremum is over all possible subdivisions of $[a,b]$. Then P, N, and T are the positive, negative, and total variations of f over $[a,b]$, respectively. If $T < \infty$, we say that f is of bounded variation over $[a,b]$.

Henceforth, we will delete the limits of summation and will assume that suprema are taken over all possible subdivisions of the interval. When misunderstanding could occur, we will use the notation T_a^b, $T_a^b(f)$, etc., to denote the dependence of the variation on the interval $[a,b]$ or on the function f.

Notice that if f is an increasing function on $[a,b]$, then $T = P = \sup [f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots + f(x_K) - f(x_{K-1})] = f(b) - f(a) < \infty$, so that f is of bounded variation with total variation equal to $f(b) - f(a)$. Similarly, a decreasing function f on $[a,b]$ is of bounded variation with total variation equal to $f(a) - f(b)$.

We shall now prove two lemmas concerning several important properties of the variations of f which will be needed in later theorems.

LEMMA 3.16. If f is of bounded variation on $[a,b]$, then $T = P + N$ and $f(b) - f(a) = P - N$.

PROOF. For any subdivision of $[a,b]$, $p = n + f(b) - f(a) \leq N + f(b) - f(a)$ and, hence, $P \leq N + f(b) - f(a)$. Since $N \leq T < \infty$, $P - N \leq f(b) - f(a)$. Similarly, $N - P \leq f(a) - f(b)$, so that $P - N = f(b) - f(a)$. Thus, $T \geq p + n = p + p - \{f(b) - f(a)\} = 2p + N - P$ and $T \geq 2P + N - P = P + N$. Since $T \leq P + N$, we have the desired result.

LEMMA 3.17. Let f be of bounded variation on $[a,b]$ and let $a < c < b$. Then $T_a^b = T_a^c + T_c^b$.

PROOF. [Apostle, 1974, p. 130] Let P_1 be a partition of $[a,c]$, and let P_2 be a partition of $[c,b]$. Then $P_o = P_1 \cup P_2$ is a partition of $[a,b]$. If we let $\Sigma(P)$ denote the sum $\Sigma|f(x_i) - f(x_{i-1})|$ corresponding to the partition $P = \{x_o, x_1, \ldots, x_K\}$, we can write $\Sigma(P_1) + \Sigma(P_2) = \Sigma(P_o) \leq T_a^b$. Therefore, $T_a^c + T_c^b \leq T_a^b$.

Now let $P = \{x_o, x_1, \ldots, x_K\}$ be a partition of $[a,b]$ and let $P_o = P \cup \{c\}$ be the partition obtained by adjoining the point c. If $c \in [x_{i-1}, x_i]$ we have $|f(x_i) - f(x_{i-1})| \leq |f(x_i) - f(c)| + |f(c) - f(x_{i-1})|$ and, hence, $\Sigma(P) \leq \Sigma(P_o)$. Now, the points of P_o in $[a,c]$ determine a partition P_1 of $[a,c]$ and those in $[c,b]$ determine a partition P_2 of $[c,b]$. The corresponding sums for all these partitions are connected by the relation

$$\Sigma(P) \leq \Sigma(P_0) = \Sigma(P_1) + \Sigma(P_2) \leq T_a^c + T_c^b \quad .$$

Therefore, $T_a^c + T_c^b$ is an upper bound for every sum $\Sigma(P)$. Since this cannot be smaller than the least upper bound, we must have $T_a^b \leq T_a^c + T_a^b$ and the proof is complete.

Now, if we keep the function f and the left endpoint of the interval fixed, we can study the variations of f as functions of the right endpoint.

THEOREM 3.18. If f is of bounded variation on $[a,b]$, then the functions P_a^x, N_a^x, and T_a^x are nondecreasing functions of x for $a \leq x \leq b$.

PROOF. If $a \leq x < y \leq b$, we have $T_a^y = T_a^x + T_x^y$ by Lemma 3.17. Writing this equality as $T_a^y - T_a^x = T_x^y \geq 0$ gives the desired result for T_a^x. Now, let $g(x) = T_a^x(f) - f(x)$ and $h(x) = T_a^x(f) + f(x)$ for $a \leq x \leq b$. We first show that g is nondecreasing. If $a \leq x < y \leq b$, we have $f(y) - f(x) \leq |f(y) - f(x)| \leq T_x^y(f)$, and hence $g(y) - g(x) = T_a^y(f) - f(y) - T_a^x(f) + f(x) = T_x^y(f) - [f(y) - f(x)] \geq 0$. Thus, g is nondecreasing; a similar argument shows that h is nondecreasing. Now, if we add the two equations in Lemma 3.16 but with b replaced with x, we have $2P_a^x(f) = T_a^x(f) + [f(x)-f(a)]$ or $P_a^x(f) = \frac{1}{2}[h(x) - f(a)]$. Similarly, taking the difference of

the two equations in Lemma 3.16, leads to $N_a^x(f) = \frac{1}{2}[g(x) + f(a)]$. Therefore, $P_a^x(f)$ and $N_a^x(f)$ are nondecreasing.

The following theorem characterizes functions of bounded variation in terms of nondecreasing functions. Its proof is based on Lemma 3.16 and Theorem 3.18.

THEOREM 3.19. Let f be a function on $[a,b]$. Then f is of bounded variation if and only if f can be written as the difference of two nondecreasing functions.

PROOF. Let f be of bounded variation and let $g(x) = P_a^x$ and $h(x) = N_a^x$. By Theorem 3.18, g and h are nondecreasing functions, and by Lemma 3.16, $f(x) = g(x) - h(x) + f(a)$. Since $h - f(a)$ is nondecreasing, we have f expressed as the difference of two nondecreasing functions. On the other hand, if $f = g - h$ on $[a,b]$ with g and h nondecreasing, then for any subdivision of $[a,b]$, we have $\Sigma|f(x_i) - f(x_{i-1})| \leq \Sigma[g(x_i) - g(x_{i-1})]$ $+ \Sigma[h(x_i) - h(x_{i-1})] = g(b) - g(a) + h(b) - h(a)$. Therefore, $T_a^b(f) \leq g(b) + h(b) - g(a) - h(a) < \infty$ and the proof is complete.

As an immediate consequence of Theorem 3.19, we have the following corollary.

COROLLARY 3.20. If f is of bounded variation on $[a,b]$, then $f'(x)$ exists for almost all x in $[a,b]$.

We remark that the representation of a function of bounded variation as a difference of two increasing functions is by no means unique. If $f = f_1 - f_2$ where f_1 and f_2 are increasing, then $f = (f_1 + g) - (f_2 + g)$, where g is an arbitrary increasing function, is another representation of f.

We now turn to inverses and their relationship with monotonic functions and derivatives. We shall make use of the following definition.

DEFINITION 3.21. Let $f = \{(a,b)\}$ be a function. Then the relation $\breve{f}$ defined by $\breve{f} = \{(b,a) \mid (a,b) \in f)\}$ is called the converse of f. If $\breve{f}$ is also a function, then $\breve{f}$ is called the inverse of f and is denoted by f^{-1}.

The following theorem shows that if a function f is continuous and strictly monotonic, then the converse of f will be a strictly monotonic, continuous function.

THEOREM 3.22. Let f be a continuous and strictly increasing function defined on $[a,b]$. Then the inverse of f exists on the interval $[f(a), f(b)]$ and is also continuous and strictly increasing.

PROOF. [Fulks, 1969, p. 61] Let $y \in [f(a), f(b)]$. Since f is continuous, there is at least one x such that $a \leq x \leq b$

and $f(x) = y$. If there were two such x's , say $x_2 > x_1$, we would have $f(x_2) = f(x_1) = y$. Since this contradicts the fact that f is strictly increasing, there is only one such x and f^{-1} must be a function.

To see that f^{-1} is strictly increasing, we choose y_1 and y_2 from $[f(a), f(b)]$ such that $y_1 > y_2$ and let x_1 and x_2 be their corresponding x values. Then suppose $x_1 = f^{-1}(y_1) \leq f^{-1}(y_2) = x_2$. Applying f , we have $y_1 = f(x_1) \leq f(x_2) = y_2$ which contradicts our original assumption and proves that f^{-1} is strictly increasing.

To show that f^{-1} is continuous, we again use a proof by contradiction. Assume that f^{-1} is discontinuous at some point y in $[f(a), f(b)]$. Then either

(i) $f^{-1}(y-) < f^{-1}(y)$ or

(ii) $f^{-1}(y) < f^{-1}(y+)$.

Assume (ii) holds. Let $f^{-1}(y+) = f^{-1}(y) + d$ where $d > 0$, and let $f^{-1}(y) = x$. For $\delta > 0$, we have $f^{-1}(y + \delta) > f^{-1}(y + \delta/2) > f^{-1}(y+) = f^{-1}(y) + d = x + d$. Applying f and using the fact that f is strictly increasing, gives $y + \delta > f(x+d) > f(x)$. Letting $\delta \to 0$, we have $f(x) \geq f(x+d) > f(x)$ which is a contradiction and establishes the fact that f^{-1} is continuous.

We note that the above theorem can be extended to include unbounded functions and infinite or semi-infinite intervals. One simply writes the unbounded interval as the union of an increasing

sequence of bounded intervals on each of which f is bounded and applies Theorem 3.22.

We now have the following relationship between f^{-1} and the derivative of f.

THEOREM 3.23. Let f be a continuous and strictly increasing function on an interval, and let $f'(x_0) > 0$ for some x_0 in the interval. Then f^{-1} is differentiable at the point $y_0 = f(x_0)$, and $(f^{-1})'(y_0) = 1/f'(x_0)$.

PROOF. Let x be a point in the interval, and let $y = f(x)$. Then $f^{-1}(y) = x$ and

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \left[\frac{f(x) - f(x_0)}{x - x_0}\right]^{-1}.$$

Taking the limit as $x \to x_0$, we have the desired result.

CHAPTER IV

INTEGRATION

In this short chapter, we present, first, a theorem on the Riemann integrability of a function which is bounded and monotonic on a compact interval and, second, a theorem on the Lebesgue integrability of the derivative of a function which is monotonic on a compact interval. In addition, we briefly indicate the role of monotonic functions in the theory of Riemann-Stieltjes integration. We begin with a brief discussion of the Riemann integral.

We define a partition P of the interval $[a,b]$ to be a finite sequence of numbers $x_0, x_1, \ldots, x_n$, such that $a = x_0 < x_1 < \ldots < x_n = b$. The K^{th} subinterval, $[x_{K-1}, x_K]$, determined by P will be denoted by I_K and its length, $x_{K-1} - x_K$, will be denoted by $\Delta_K x$. If f is any bounded function on $[a,b]$, we define $M_K = \sup_{I_K} f(x)$ and $m_K = \inf_{I_K} f(x)$. The upper sum for f over the partition P is $\overline{S}(f ; P) = \sum_P M_K \Delta_K x$ and the lower sum is $\underline{S}(f ; P) = \sum_P m_K \Delta_K x$ where the sums are over all subintervals formed by the partition P.

The upper Riemann-Darboux integral of f over $[a,b]$ is $\overline{\int_a^b} f(x)dx = \inf \overline{S}(f; P)$ and the lower Riemann-Darboux integral is $\underline{\int_a^b} f(x)dx = \sup \underline{S}(f; P)$ where the supremum and infimum are taken over all partitions of $[a,b]$. If $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$, we say that f is Riemann integrable on $[a,b]$ and call their common value the Riemann integral of f over $[a,b]$, denoted

by $\int_a^b f(x)dx$.

We now show that any monotonic function f defined on a compact interval is Riemann integrable.

THEOREM 4.1. If f is a nondecreasing function defined on $[a,b]$, then f is Riemann integrable on $[a,b]$.

PROOF. Let P be any partition of $[a,b]$ and let $||P||$ be the maximum of the lengths of the subintervals formed by the partition P . Then we have

$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq \overline{S}(f; P) - \underline{S}(f; P)$$

$$= \sum_P (M_K - m_K)\Delta_K x \leq ||P|| \sum_P (M_K - m_K) .$$

Since f is nondecreasing on each subinterval I_K , $M_K = f(x_K)$ and $m_K = f(x_{K-1})$ so that

$$\sum_P (M_K - m_K) = \sum_P [f(x_K) - f(x_{K-1})] = f(b) - f(a) .$$

If $f(b) = f(a)$, then f is constant and obviously integrable. If $f(a) < f(b)$, then for $||P|| < \varepsilon/(f(b) - f(a)$, we have $0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, f is Riemann integrable, and our proof is complete.

If f is piecewise monotonic on $[a,b]$, that is, if f is monotonic on each of a finite number of subintervals of $[a,b]$, then we can apply Theorem 4.1 on each of the subintervals to show that f is Riemann integrable on the entire interval.

Before proving our next result, we present a brief description of the Lebesgue integral. For a more complete discussion we refer the reader to Apostle [1974, p. 252] or Riesz and Nagy [1955, p. 29].

We shall say that a function s defined on a compact interval $[a,b]$ is a step function if there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that s is constant on every open subinterval (x_{K-1}, x_K), say $s(x) = c_K$ if $x \in (x_{K-1}, x_K)$. The Riemann integral of s over $[a,b]$ is defined by $\int_a^b s(x)dx = \sum_{K=1}^{n} c_K(x_K - x_{K-1})$. If I is any interval, we say that s is a step function on I if there is a compact subinterval $[a,b]$ of I such that s is a step function on $[a,b]$ and $s(x) = 0$ if $x \in I - [a,b]$. The integral of s over I, denoted by $\int_I s(x)dx$, is defined to be the integral of s over $[a,b]$ as given above.

A function f defined on an interval I is called an upper function on I, and we write $f \in U(I)$, if there exists an increasing sequence of step functions $\{s_n\}$ such that $s_n \to f$ almost everywhere on I and $\lim_{n\to\infty} \int_I s_n$ is finite. The integral of f over I is defined by the equation $\int_I f = \lim_{n\to\infty} \int_I s_n$.

Finally, we denote by $L(I)$ the set of all functions f of the form $f = u - v$ where $u, v \in U(I)$. Each function f in $L(I)$ is said to be Lebesgue-integrable on I and its integral is defined by the equation $\int_I f = \int_I u - \int_I v$. We will make use of the fact that all Riemann-integrable functions on a compact interval are also upper functions and therefore are Lebesgue-integrable.

In our next theorem, we use a result called Fatou's Lemma. Since a proof of this lemma would lead us too far astray, we simply state it here and refer the interested reader to Apostle [1974, p. 299].

LEMMA 4.2. (Fatou's Lemma) Let $\{f_n\}$ be a sequence of nonnegative functions in $L(I)$ such that $\{f_n\}$ converges almost everywhere on I to a limit function f and $\int_I f_n \leq A$ for some $A > 0$ and all $n \geq 1$. Then the limit function $f \in L(I)$ and $\int_I f \leq A$.

We now use Fatou's Lemma to show that the derivative of a nondecreasing function f is always Lebesgue-integrable.

THEOREM 4.3. Let f be a nondecreasing function defined on $[a,b]$. Then the derivative f' of f is Lebesgue-integrable and $\int_a^b f'(x)dx \leq f(b) - f(a)$.

PROOF. By Theorem 3.13, $f'(x)$ exists almost everywhere on $[a,b]$. Set $f(x) = f(b)$ for $x \geq b$, and let $g_n(x) = n[f(x + \frac{1}{n}) - f(x)]$. Clearly $g_n \to f'$ a.e. on $[a,b]$

as $n \to \infty$. Since f is Riemann-integrable by Theorem 4.1, $g_n(x)$ is Riemann-integrable and therefore Lebesgue-integrable. Also, since f is nondecreasing, $g_n(x) \geq 0$, and

$$\int_a^b g_n(x)dx = n \int_a^b [f(x + \tfrac{1}{n}) - f(x)]dx$$

$$= n \left[\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx \right]$$

$$= n \int_b^{b+\frac{1}{n}} f(x)dx - n \int_a^{a+\frac{1}{n}} f(x)dx$$

$$= f(b) - n \int_a^{a+\frac{1}{n}} f(x)dx$$

$$\leq f(b) - f(a+) \leq f(b) - f(a) .$$

Therefore, since $g_n(x) \to f'(x)$ $(n \to \infty)$ almost everywhere, Fatou's Lemma with $A = f(b) - f(a)$ implies that $f' \in L(I)$ and $\int_a^b f'(x)dx \leq f(b) - f(a)$. This completes the proof.

Our final goal for this chapter is to briefly indicate the role of monotonic functions in the theory of Riemann-Stieltjes integration.

Let f and α be functions defined and bounded on $[a,b]$ and let $P = \{x_0 , x_1 , \ldots, x_n\}$ be a partition of $[a,b]$. Let $\Delta\alpha_K = \alpha(x_K) - \alpha(x_{K-1})$ and let t_K be a point in the subinterval $[x_{K-1}, x_K]$. Then a sum of the form $S(P, f, \alpha) = \sum_{K=1}^{n} f(t_K)\Delta\alpha_K$ is called a Riemann-Stieltjes sum of f with respect to the

integrator α . We say that f is Riemann-Stieltjes integrable with respect to the integrator α on $[a,b]$, and we write $f \in R(\alpha)$ on $[a,b]$, if there exists a number A having the following property: For every $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ such that for every partition P finer than P_ε - i.e., such that $P \supseteq P_\varepsilon$ - and for every choice of the points t_K in $[x_{K-1}, x_K]$, we have $|S(P, f, \alpha) - A| < \varepsilon$. When such a number A exists, it is uniquely determined and is denoted by $\int_a^b f(x)d\alpha(x)$.

We now give without proof some standard results of Riemann-Stieltjes integration which will enable us to show the relationship between monotonic functions and the Riemann-Stieltjes integral. For proofs and a more detailed discussion see Apostol [1974, p. 140].

(i) If f is continuous on $[a,b]$ and if α is of bounded variation on $[a,b]$, then $f \in R(\alpha)$ on $[a,b]$.

(ii) If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a,b]$, then $f \in R(\alpha+\beta)$ on $[a,b]$ and $\int_a^b fd(\alpha+\beta) = \int_a^b fd\alpha + \int_a^b fd\beta$.

(iii) Let α be of bounded variation on $[a,b]$. Let $T_a^x(\alpha)$ denote the total variation of α on $[a,x]$ if $a < x \leq b$ and define $T_a^a(\alpha) = 0$. Let f be defined and bounded on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then $f \in R(T)$ on $[a,b]$.

Now if α is of bounded variation, we can write $\alpha(x) = T_a^x(\alpha) - [T_a^x(\alpha) - \alpha]$ where by Lemma 3.18, $T_a^x(\alpha)$ and $T_a^x(\alpha) - \alpha$ are nondecreasing functions. Using this fact and

(i), (ii), (iii) above, enables one to reduce the theory of Riemann-Stieltjes integration for integrators of bounded variation to the simpler case of increasing integrators.

CHAPTER V

RATES OF GROWTH AND THE EXISTENCE OF PÓLYA PEAKS

This final chapter is concerned with local and global growth properties of nondecreasing functions defined on an unbounded interval $[t_0,\infty)$. The concepts of order, lower order, and type are defined, and the existence of Pólya peaks is established for monotonic functions. We conclude the chapter with a theorem whose proof shows the usefulness of these ideas.

DEFINITION 5.1. Let $f(t)$ be a positive, continuous, nondecreasing, and unbounded function defined for $t \geq t_0 > 0$. Then the order λ and the lower order μ of $f(t)$ are defined by

$$\lambda = \limsup_{t\to+\infty} \frac{\log f(t)}{\log t} \quad \text{and} \quad \mu = \liminf_{t\to+\infty} \frac{\log f(t)}{\log t}.$$

If $\mu < \lambda$, then f is said to have irregular growth; if $\mu = \lambda$, then f has regular growth. Notice that if $f(t)$ has order λ where $0 < \lambda < \infty$ and $\varepsilon > 0$, then by the definition of limit superior, $f(t) < t^{\lambda+\varepsilon}$ for all large t and $f(t) > t^{\lambda-\varepsilon}$ for some arbitrarily large t.

If $0 < \lambda < \infty$, we set $C = \limsup_{t\to\infty} \frac{f(t)}{t^{\lambda}}$ and distinguish the following cases:

(a) $f(t)$ has maximal type if $C = +\infty$;

(b) $f(t)$ has mean type if $0 < C < +\infty$;

(c) $f(t)$ has minimal type if $C = 0$;

(d) $f(t)$ has convergence class if $\int_{t_0}^{\infty} \frac{f(r)dr}{r^{\lambda+1}}$ converges.

It is not hard to show that case (d) implies case (c). If $f(t)$ is of order λ and has convergence class, then $\int_{t_0}^{\infty} \frac{f(t)dt}{t^{\lambda+1}} < \varepsilon$ if $t_0 > r(\varepsilon)$. Then $\int_{t_0}^{2t_0} \frac{f(t)dt}{t^{\lambda+1}} < \varepsilon$ and since $f(t)$ increases with t , $\int_{t_0}^{2t_0} \frac{f(t)dt}{t^{\lambda+1}} \geq \frac{f(t_0)}{(2t_0)^{\lambda+1}}(t_0)$. Therefore, $f(t_0)2^{-(\lambda+1)}/t_0^{\lambda} < \varepsilon$ for $t_0 > r(\varepsilon)$ and $\limsup_{t\to\infty} \frac{f(t)}{t^{\lambda}} = 0$. Thus, (d) implies (c). Although we shall not use the types in theorems presented here, we remark that they are of considerable use in studying the growth of entire functions in complex variables.

In order to study local behavior of f , that is, behavior of f in small neighborhoods, we introduce the following definition.

DEFINITION 5.2. Let f be as above. An increasing sequence r_1 , r_2 , ..., r_m , ... is called a sequence of Pólya peaks of the second kind of order $\rho (0 \leq \rho < +\infty)$ for f if there exists a pair of associated sequences $\{s_m\}$, $\{S_m\}$ such that $s_m \to +\infty$, $\frac{r_m}{s_m} \to +\infty$, $\frac{S_m}{r_m} \to +\infty$ and such that

$$f(t) \geq [1 + o(1)] \frac{f(r_m)}{r_m^{\rho}} t^{\rho} \quad (m \to +\infty , s_m \leq t \leq S_m)$$

where $o(1)$ is a sequence which tends to 0 as $m \to \infty$.

We remark that Pólya peaks of the first kind are defined in the same way but with the last inequalities reversed.

Shea [1966] shows that Pólya peaks of the second kind always exist for functions of finite lower order. To prove this fact, he makes use of two lemmas; the first is a result of Edrei [1964] on functions of irregular growth, and the second, which concerns the ratio of two functions, is due to Polya [1925]. We present these lemmas before stating and proving the existence theorem.

LEMMA 5.3. Let $f(t)$ be a positive, continuous, nondecreasing function defined for $t \geq t_0 > 0$. Let the order λ and the lower order μ of $f(t)$ be such that $\mu < \lambda \leq \infty$. Let σ , τ be given such that $\mu < \sigma < \tau < \lambda$. Then there exist arbitrarily large r such that

$$\frac{f(r)}{r^{\sigma}} \leq \frac{f(t)}{t^{\sigma}} \quad (t_0 < t \leq r^{\tau/\sigma}) .$$

PROOF. Let $B > 1$ be given and let $0 < \varepsilon < \sigma - \mu$. By the definition of lower order, there exists a sequence $t_n \to \infty$ such that $\log f(t_n)/\log t_n < \mu + \varepsilon$. It follows that $f(t_n) < t_n^{\mu+\varepsilon}$ and, therefore, $0 \leq \dfrac{f(t_n)}{t_n^{\sigma}} < t_n^{\mu+\varepsilon-\sigma}$. Since $\mu + \varepsilon - \sigma < 0$, we have $\liminf_{t\to+\infty} f(t)/t^{\sigma} = 0$. Thus, there exists

an $r_1 > B$ such that

(i) $f(r_1)/r_1^\sigma \leq f(t)/t^\sigma$ $(t_o \leq t \leq r_1)$ and

(ii) $f(r_1)/r_1^\sigma < 1$.

Similarly, since $\lambda > \tau$, the definition of the order of $f(t)$ implies that $\limsup_{t\to\infty} f(t)/t^\tau = \infty$, so that there exists $r_2 > r_1 > B$ such that $f(r_2)/r_2^\tau \geq f(t)/t^\tau$ $(t_o \leq t \leq r_1 < r_2)$ and

(iii) $f(r_2)/r_2^\tau > 1$.

Since $f(t)/t^\sigma$ is continuous, we can choose r such that $r_1 \leq r \leq r_2$ and

(iv) $f(r)/r^\sigma = \inf_{r_1 \leq t \leq r_2} f(t)/t^\sigma$.

Combining (i) and (iv), we have

(v) $f(r)/r^\sigma \leq f(r_1)/r_1^\sigma \leq f(t)/t^\sigma$ $(t_o \leq t \leq r_2)$.

If $r_2 \leq t \leq r_2^{\tau/\sigma}$, then $t^\sigma \leq r_2^\tau$ and, since $f(t)$ is nondecreasing, $f(t)/t^\sigma \geq f(r_2)/r_2^\tau$. Combining this inequality with (i), (ii), and (iv), we have

(vi) $f(t)/t^\sigma \geq f(r_2)/r_2^\tau > 1 > f(r_1)/r_1^\sigma \geq f(r)/r^\sigma$ $(r_2 \leq t \leq r_2^{\tau/\sigma})$.

Finally, combining (v) and (vi) yields $f(r)/r^\sigma \leq f(t)/t^\sigma$ for $t_o \leq t \leq r_2^{\tau/\sigma}$. Since $B > 1$ is arbitrary, the inequalities $B < r_1 \leq r \leq r_2$ imply that r may be arbitrarily large, and the proof is complete.

We now come to the result of Pólya.

LEMMA 5.4. Let $h(t)$ and $g(t)$ be continuous, positive functions defined for $t \geq t_0$. Let $g(t)$ be nondecreasing and let $\limsup_{t\to\infty} h(t) = +\infty$ and $\lim_{t\to\infty} h(t)/g(t) = 0$. Then there exist arbitrarily large values of r such that $h(t) \leq h(r)$ $(t_0 \leq t < r)$ and $h(t)/g(t) \leq h(r)/g(r)$ for $r \leq t$.

PROOF. Suppose $r_0 > t_0$. Since $\limsup_{t\to\infty} h(t) = +\infty$, there exists $v_0 > r_0$ such that $h(v_0) = \sup_{t_0 \leq t \leq v_0} h(t)$. Because $h(t)/g(t)$ is positive and continuous for $t \geq t_0$ and tends to 0 as $t \to \infty$, there exists $v_1 \geq v_0$ such that $h(v_1)/g(v_1) = \sup_{v_0 \leq t} [h(t)/g(t)]$. The continuity of $h(t)$ now implies that there exists an r such that $v_0 \leq r \leq v_1$ and $h(r) = \sup_{v_0 \leq t \leq v_1} h(t) = \sup_{t_0 \leq t \leq v_1} h(t)$. Therefore, $h(v_1)/g(v_1) \geq h(t)/g(t)$ for $t \geq v_1$ and because $g(t)$ is nondecreasing $h(v_1)/g(v_1) \leq h(r)/g(r)$. Thus, since r may be arbitrarily large, the result follows.

The next theorem gives sufficient conditions for the existence of Pólya peaks.

THEOREM 5.5. Let $f(t)$ be a positive, continuous, nondecreasing and unbounded function defined for $t \geq t_0 > 0$ and let the lower order μ of $f(t)$ be finite. Then corresponding to each finite number ρ such that $\mu \leq \rho \leq \lambda$ there exist sequences of Pólya peaks of the first and second kinds of order ρ for $f(t)$.

PROOF. We prove the theorem for Pólya peaks of the second kind. A similar proof can be used to show the existence of Pólya peaks of the first kind.

Let $\{r_m\}$ be a sequence such that

(vii) $r_m > m^m$ for $m \geq m_0$.

Let $s_m = (1/m)r_m$ and $S_m = mr_m$. Then obviously, $s_m \to +\infty$, $r_m/s_m \to +\infty$, and $S_m/r_m \to +\infty$ as $m \to +\infty$ so that the first set of conditions in Definition 5.2 are satisfied. We also have $(t/r_m)^{1/m} = 1 + o(1)$ $(m \to +\infty$, $s_m \leq t \leq S_m)$. To see this, note that $(t/r_m)^{1/m} \leq (S_m/r_m)^{1/m} = m^{1/m} \to 1$ as $m \to \infty$. Thus $(t/r_m)^{1/m} \leq 1 + o(1)$ $(m \to \infty$, $s_m \leq t \leq S_m)$. On the other hand, $(t/r_m)^{1/m} \geq (s_m/r_m)^{1/m} = (1/m)^{1/m} \to 1$ as $m \to \infty$. Therefore, $(t/r_m)^{1/m} \geq 1 + o(1)$ $(m \to \infty$, $s_m \leq t \leq S_m)$ and, hence, $(t/r_m)^{1/m} = 1 + o(1)$ $(m \to +\infty$, $s_m \leq t \leq S)$.

Now assume $\mu < \lambda$. For ρ such that $\mu < \rho < \lambda$, choose numbers σ and τ such that $\sigma = \rho < \tau < \lambda$. Then by Lemma 5.3, we can find $r = r_m$ arbitrarily large such that $r_m > m^m$ and $f(r_m)/r_m^{\sigma} \leq f(t)/t^{\sigma}$ for $t_0 \leq t \leq r_m^{\tau/\sigma}$. We therefore have $f(t) \geq f(r_m)(t/r_m)^{\rho}$ for $t_0 \leq t \leq r_m^{\tau/\sigma}$. Since $r_m > m^m$ and $(\tau/\sigma) - 1 > 0$, we also have $r_m^{(\tau/\sigma)-1} > m^{m[(\tau/\sigma)-1]} > m$ for sufficiently large m , and, therefore, $S_m < r_m^{\tau/\sigma}$. Since $s_m > t_0$ for sufficiently large m , we have $f(t) \geq f(r_m)(t/r_m)^{\rho}$ $(m \to \infty$, $s_m \leq t \leq S_m)$.

For ρ such that $\mu = \rho < \lambda$, choose $\sigma = \sigma_m = \rho + (1/m)$ and $\tau = \tau_m = \rho + (2/m)$. Then $\mu < \sigma_m < \tau_m < \lambda$ $(m \geq m_0)$ and by

Lemma 5.3, we can find r_m for each m such that $r_m > m^m$ and $f(t) \geq f(r_m)(t/r_m)^{\rho+1/m} = f(r_m)(t/r_m)^{\rho}\ (1 + o(1))$ $(m \to \infty\ ,\ t_0 \leq t \leq r_m^{\tau/\sigma})$ .By an argument similar to that in the above paragraph, we can show that the inequality holds for $s_m \leq t \leq S_m$, $m \to \infty$.

If ρ satisfies $\mu < \rho = \lambda$, take $\sigma = \sigma_m = \rho - (2/m)$ and $\tau = \tau_m = \rho - (1/m)$. Then $\mu < \sigma_m < \tau_m < \lambda$ for $m \geq m_0$ and by Lemma 5.3, we can find $r_m > m^m$ for each m such that $f(t) \geq f(r_m)(t/r_m)^{\rho-2/m} = f(r_m)(t/r_m)^{\rho}(t/r_m)^{-2/m}$ for $s_m \leq t \leq S_m$. But for $s_m \leq t \leq S_m$, we have $(t/r_m)^{-2/m} = (t/r_m)^{-1/m}(t/r_m)^{-1/m} = [1/(t/r_m)^{1/m}]^2 = (1/[1+o(1)])^2$ $= 1 + o(1)$ and the conclusion follows. Therefore in each of the above cases, the sequence $\{r_m\}$ is a sequence of Pólya peaks of the second kind of order ρ .

If $f(t)$ has regular growth, that is, if $\mu = \rho = \lambda$, set $\varepsilon = 1/m$ $(m \geq 1)$, $h(t) = t^{\rho+\varepsilon}/f(t)$, and $g(t) = t^{2\varepsilon}$. Then by Lemma 5.4, we can find $r_m > m^m$ for each m such that $t^{\rho+\varepsilon}/f(t) \leq r_m^{\rho+\varepsilon}/f(r_m)$ for $t_0 \leq t < r_m$ and $t^{\rho-\varepsilon}/f(t) \leq r_m^{\rho-\varepsilon}/f(r_m)$ for $t \geq r_m$. Therefore,

$$f(t) \geq f(r_m)(t/r_m)^{\rho}(1 + o(1))\ (m \to \infty\ ,\ s_m \leq t \leq S_m)$$

and $\{r_m\}$ is a sequence of Pólya peaks of the second kind. This completes the proof.

More recently, Drasin and Shea [1972, p. 403] have determined both necessary and sufficient conditions for the existence of Pólya peaks for more general functions. Since the proof of their result is quite technical and since our purpose is to discuss properties of monotonic functions, we refer the reader to their paper for the more general case.

We conclude this chapter by using the idea of Pólya peaks to prove an interesting result.

THEOREM 5.6. Let $f > 0$ be a continuous, increasing function on $(0,\infty)$ with order $\lambda \in (0,\infty)$. If $F(x) = 1/x \int_0^\infty e^{-t/x} f(t)dt$ and if $\alpha = \lim_{x\to\infty} \frac{F(x)}{f(x)}$ exists, then $\alpha = \Gamma(\lambda+1)$ where $\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du$ is the usual Gamma function.

PROOF. We first let $\{r_m\}$ be a sequence of Pólya peaks of the second kind for f of order λ. Then

$$F(r_m) \geq \frac{1}{r_m} \int_{r_m/m}^{mr_m} e^{-t/r_m} f(t)dt$$

$$\geq (1 + o(1)) \frac{1}{r_m} \int_{r_m/m}^{mr_m} e^{-t/r_m} f(r_m)(t/r_m)^\lambda dt$$

$$= (1 + o(1)) f(r_m) \int_{1/m}^{m} e^{-u} u^\lambda du$$

where we let $t = r_m u$ in the last equality. Therefore

$$\alpha = \lim_{m\to\infty} \frac{F(r_m)}{f(r_m)} \geq \Gamma(\lambda+1) .$$

Now, let $\{r_m\}$ be a sequence of Pólya peaks of the first kind of order λ for f. Then

$$F(r_m) = \frac{1}{r_m} \int_0^{r_m/m} e^{-t/r_m} f(t)dt + \frac{1}{r_m} \int_{r_m/m}^{mr_m} e^{-t/r_m} f(t)dt$$

$$+ \frac{1}{r_m} \int_{mr_m}^{\infty} e^{-t/r_m} f(t)dt$$

$$= I_1 + I_2 + I_3 .$$

On $[0, r_m/m]$, $f(t) \leq f(r_m)$. Therefore, $I_1 \leq f(r_m) \int_0^{1/m} e^{-u}du$. I_2 can be handled as in the first paragraph remembering that $\{r_m\}$ is now a sequence of Pólya peaks of the first kind.

For I_3, fix $p > \lambda$ and note that there exists $B < \infty$ such that $(1/u)e^{-1/u} \leq B(u/w)^p(1/w)e^{-1/w}$ for $0 < u \leq w < 1$. We will use this fact with $u = r_m/t$ and $w = mr_m/t$. Now,

$$\frac{1}{r_m} \int_{mr_m}^{\infty} e^{-t/r_m} f(t)dt = \int_{mr_m}^{\infty} \left(\frac{1}{r_m/t}\right) e^{-\frac{1}{r_m/t}} f(t)\frac{dt}{t}$$

$$\leq B \int_{mr_m}^{\infty} \left(\frac{1}{mr_m/t}\right) e^{-\frac{1}{mr_m/t}} f(t)\frac{dt}{t}\left(\frac{1}{m}\right)^p$$

$$= \frac{B}{mr_m} \int_{mr_m}^{\infty} e^{-t/mr_m} f(t)dt \left(\frac{1}{m}\right)^p$$

$$\leq F(mr_m)\left(\frac{1}{m}\right)^p B .$$

Therefore, $F(r_m)/f(r_m) \leq \int_0^{1/m} e^{-u}du + (1 + 0(1)) \int_{1/m}^{m} e^{-u}u^{\lambda}du$ $+ [F(mr_m)/f(r_m)](1/m)^p B$. But as $m \to \infty$, $\int_0^{1/m} e^{-u}du$ and $[F(mr_m)/f(r_m)](1/m)^p$ approach zero so that $\alpha = \lim_{m\to\infty} F(r_m)/f(r_m) \leq 0 + \Gamma(\lambda+1) + 0$. Combining this result with the inequality in the first paragraph gives the desired result.

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