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Elasticity of Krull Domains with Infinite Divisor Class Group

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Dedication

I would like to dedicate this dissertation to my wife, Rachel, and son, James.

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I would like to express my sincere gratitude to my advisor, Dr. David Anderson. I would also like to thank the other professors in the Mathematics Department for their instruction over the last several years. Most importantly I would like to thank my family for their support while writing this dissertation.

Abstract

The elasticity of a Krull domain R is equivalent to the elasticity of the block monoid B(G,S), where G is the divisor class group of R and S is the set of elements of G containing a height-one prime ideal of R. Therefore the elasticity of R can by studied using the divisor class group. In this dissertation, we will study infinite divisor class groups to determine the elasticity of the associated Krull domain. The results will focus on the divisor class groups Z, Z(p infinity), Q, and general infinite groups. For the groups Z and Z(p infinity), it has been determined which distributions of the height-one prime ideals will make R a half-factorial domain (HFD). For the group Q, certain distributions of height-one prime ideals are proven to make R an HFD. Finally, the last chapter studies general infinite groups and groups involving direct sums with Z. If certain conditions are met, then the elasticity of these divisor class groups is the same as the elasticity of simpler divisor class groups.

Contents

Chapter 1

Introduction

Studies of factorization properties usually focus on unique factorization domains (UFD). In a UFD each nonzero nonunit can be factored into a unique (up to associates) product of irreducible elements. The most common factorization problem is factoring integers into prime numbers. The set of integers, denoted $\mathbb{Z}$, is a UFD. The study of nonunique factorization, when an integral domain is not a UFD, began in earnest within the last 50 years. This dissertation will focus on studying the divisor class group of Krull domains to learn about the domain's factorization properties. The distribution of the height-one prime ideals of a Krull domain in the divisor class group determines the factorization properties of the domain. The focus will be on calculating the elasticity of a Krull domain and determining when the Krull domain is a half-factorial domain (HFD). Elasticity is a measure of the difference in length of different factorizations of the same element. A domain is half-factorial if every factorization of an element into irreducibles has the same length. The case when the divisor class group is finite has been studied extensively; so this dissertation will provide results for infinite divisor class groups.

1.1 Notation

For this dissertation, we will use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ to refer to the natural numbers, integers, rational numbers, and real numbers, respectively. We let $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. For any $S \subseteq \mathbb{R}$, we will define $S^- = \{s \in S \mid s < 0\}$ and $S^+ = \{s \in S \mid s > 0\}$.

Whenever a quotient group is used, an overline will be used to represent the cosets for an element or a set of elements. So if $S \subseteq G$, then $\overline{S}$ will represent the cosets in the quotient group G/H that contain elements of S. The same notation will be used for quotients of monoids and quotients of rings. For a ring R, define $U(R) = \{r \in R \mid r \text{ is a unit}\}$.

For $S \subseteq G$, with G a group, define $[S] = \{g \in G \mid g = \sum_{i=}^{k} n_i g_i$ for some $n_i \in \mathbb{N}$ and $g_i \in G\}$. We will say the set $[S]$ is generated by S as a monoid. As usual, define $< S >$ to be the smallest subgroup of G containing S. Then $[S] \subseteq < S >$, and the two may not be equal.

1.2 Factorization

We begin the background on factorization theory with the definitions of an irreducible element and a UFD.

Definition 1.1. A nonzero nonunit element r of an integral domain R is called *irreducible* if whenever $r = r_1 r_2$ for some $r_1, r_2 \in R$, then either r_1 is a unit in R or r_2 is a unit in R.

Definition 1.2. An integral domain R is a *unique factorization domain* (UFD) if every nonzero nonunit of R can be factored into irreducibles, and if $x_1 \cdots x_n = y_1 \cdots y_m$ with each x_i and y_j irreducible elements of R, then

1. $n = m$, and
2. there exists $\sigma \in S_n$ such that x_i and $y_{\sigma(i)}$ are associates for every $1 \leq i \leq n$ (i.e., there exists a unit u_i in R such that $x_i = u_i y_{\sigma(i)}$).

Non-unique factorization is the study of factorization properties of domains that are not UFDs. Most of the research begins with the basic assumption that the domain being studied must be atomic. The definition of atomic is due to Cohn [9]. Atomic domains are integral domains where every nonzero nonunit of the domain can be factored into irreducibles. This is the first assumption in the definition of a UFD. If a domain satisfies the ascending chain condition on principal ideals (ACCP), then the domain is atomic. In [14], Grams provided an example showing the converse does not hold. Once the atomic property is assumed, there are many other factorization properties that can be studied. In [3], many of these properties, such as bounded factorization domains (BFD) and finite factorization domains (FFD), are defined. The paper details the relationships between these factorization properties. The paper also gives a proof that Krull domains, our focus of study, are BFDs. Krull domains are also atomic.

For this dissertation, we will focus on half-factorial domains, which are so named because they fulfill the first "half" of the requirements for a UFD.

Definition 1.3. An atomic domain R is a *half-factorial domain* (HFD) if whenever $x_1 \cdots x_n = y_1 \cdots y_m$ with each x_i and y_j irreducible elements of R, then $n = m$.

The definition of an HFD is due to Zaks [20]. This paper studied properties of Dedekind domains that were HFDs. Then Zaks [21] expanded his study of HFDs to Krull domains. Skula also published a paper with similar results on Dedekind domains that were proved independently [17].

The elasticity of a domain is a measure of the difference in the number of irreducible factors for different factorizations of the same element.

Definition 1.4. Let R be an atomic domain. For all nonzero nonunits $r \in R$, the *elasticity of* r, denoted $\rho(r)$, is defined as

$$\rho(r) = \sup\left\{\frac{m}{n} \mid r = x_1 \cdots x_m = y_1 \cdots y_n \text{ for some irreducibles } x_i, y_j \in R\right\}.$$

The *elasticity of* R, denoted $\rho(R)$, is defined as

$$\rho(R) = \sup\left\{\rho(r) \mid 0 \neq r \in R - U(R)\right\}.$$

The term elasticity was coined by Valenza in his paper from 1990 [19]. Valenza studied elasticity in the context of the ring of integers of an algebraic number field. Valenza proved that the elasticity is bounded above by $h/2$, where h is the class number of the class group of the number field (as long as the class group is nontrivial). The definition of

elasticity (but not the name) first appeared in the literature in a paper by Steffan in 1986 [18]. Steffan proved that a Dedekind domain R with finite divisor class group, $Cl(R)$, has $\rho(R) \leq \max\{|Cl(R)|/2, 1\}$. Curiously, the paper by Valenza was published in 1990, but submitted in 1980; so it actually predates the 1986 paper of Steffan.

It is clear that the elasticity of a domain is always between 1 and ∞. In [2], it was shown that for any $1 \leq r \leq \infty$ there exists a Dedekind domain R with torsion class group such that $\rho(R) = r$. A common question is whether the elasticity of a domain is finite (sometimes called rationally bounded), a rational number, and realizable. An atomic domain R has a *realizable elasticity* if there exist irreducibles $x_1, \ldots, x_m, y_1, \ldots y_n$ in R such that $x_1 \cdots x_m = y_1 \cdots y_n$ and $\rho(R) = \frac{m}{n}$. For Krull domains with only a finite number of divisor classes containing height-one prime ideals, the elasticity is a rational number and realizable [2].

Some factorization work has used commutative, cancellative monoids instead of integral domains. These monoids are defined so that they retain all the properties of an integral domain without having an addition operation. This is a more general approach to factorization than using only integral domains. In this case, HFDs are referred to as being *half-factorial* since they are not domains. The book by Geroldinger and Halter-Koch [13] is a good reference for factorization in monoids, and factorization in general.

1.3 Krull Domains

In 1960, Carlitz [6] proved that the ring of integers of an alegraic number field is an HFD if and only if it has class number 1 or 2. This paper was used as a starting point for the study of nonnunique factorization. It showed that the divisor class group can be studied to learn about a domain's factorization. In UFDs each nonzero nonunit can be factored into prime elements. A similar property exists for principal ideals in Krull domains. In Krull domains each nontrivial principal ideal can be uniquely factored into a v-product of height-one prime ideals. Factorizations of elements in a Krull domain is related to factoring this v-product of height-one prime ideal into sub-v-products of height-one prime ideals that are principal ideals. This factoring of the height-one prime ideals can be studied in the divisor class group.

Krull domains were introduced by Krull in 1931 as finite discrete principal orders [16]. Most of the following definitions and theorems on Krull domains can be found in the book "The divisor class group of a Krull domain" by Fossum [10]. Before the definition of a Krull domain is stated, we will recall some basic ring theory definitions. The height of a prime ideal P is defined as

$$ht(P) = sup\{t \in \mathbb{N} \mid P_1 \subsetneq \cdots \subsetneq P_{t-1} \subsetneq P \text{ for some prime ideals } P_1, \ldots, P_{t-1}\}.$$

For an integral domain R, Define $X^{(1)}(R)$ to be the set of all height-one prime ideals of R, i.e., the nonzero minimal prime ideals of R. The dimension of an integral domain R, dim R, is dim $R = sup\{ht(P) \mid P \text{ a prime ideal of } R\}$. A discrete valuation ring (DVR) is a local PID. So a DVR has only one maximal ideal, and each ideal is principal.

Definition 1.5. An integral domain R is a *Krull domain* if and only if

1. The localization R_P is a DVR for each height-one prime ideal P of R,

2. $R = \cap_{P \in X^{(1)}(R)} R_P$, and

3. each $0 \neq r \in R$ is contained in at most finitely many height-one prime ideals.

Krull domains are a generalization of Dedekind domains. Dedekind domains are integral domains which are noetherian, integrally closed, and dim $R \leq 1$. In fact, a Krull domain R is a Dedekind domain if and only if dim $R \leq 1$. If R is a Krull domain, then the following are also Krull domains: any localization of R, any polynomial ring over a family of indeterminates with coefficients in R, and any power series ring with coefficients in R. So Krull domains behave nicely under most extensions of the domain. Any UFD is also a Krull domain; so is any integrally closed Noetherian domain.

Before the theorem about factoring principal ideals into height-one prime ideals can be stated, some background on fractional ideals must be provided. Let R be an integral domain with quotient field K. A fractional ideal I of a ring R is an R-submodule of K such that there exists a nonzero element $x \in R$ such that xI is an integral ideal of R. The ideals of R are called integral ideals to avoid confusion. Every integral ideal is a fractional ideal (let $x = 1$). For a nonzero fractional ideal I, define $I^{-1} = \{a \in K \mid aI \subseteq R\}$. Then I^{-1} is a fractional ideal of R. Obviously, $II^{-1} \subseteq R$. If $II^{-1} = R$, then the ideal is called *invertible*. Dedekind domains can be defined as a domain where every nonzero ideal is invertible. The v-operation is defined as $I_v = (I^{-1})^{-1}$ for all nonzero fractional ideals of R. If $I_v = I$, then the fractional ideal I is called *divisorial*. The set of divisorial ideals is denoted by $Div(R)$.

We will define an operation on $Div(R)$ so that it is a monoid. For I and J divisorial ideals, their product IJ may not be divisorial. But $(IJ)_v$ is divisorial, so define the operation on $Div(R)$ using the function $f : Div(R) \times Div(R) \to Div(R)$ by $f(I, J) = (IJ)_v$. Then $Div(R)$ is a commutative monoid for any integral domain R. An integral domain R is completely integrally closed if and only if $Div(R)$ is a group. In particular, if R is a Krull domain, then $Div(R)$ is a group. Define $Prin(R)$ to be the group of prinicipal fractional ideals, and note that $Prin(R) \subseteq Div(R)$. Now we can define the divisor class group.

Definition 1.6. The *divisor class group* of a Krull domain is the abelian group $Cl(R) = Div(R)/Prin(R)$.

In a Krull domain, any divisorial ideal I can be factored uniquely into height-one prime ideals under the v-product. This is the key property for studying the divisor class group to learn about factorization of a Krull domain.

Theorem 1.7. *Let I be a divisorial integral ideal of a Krull domain R with $I \neq R$. Then for some height-one prime ideals $P_1, \ldots, P_n$ of R,*

$$I = (P_1 \cdots P_n)_v,$$

and these height-one prime ideals $P_1, \ldots, P_n$ are uniquely determined.

So if a Krull domain is not a UFD, then a nonzero nonunit $a \in R$ may not have a factorization into a unique (up to associates) product of prime elements. However, the principal ideals are divisorial; so aR has a unique factorization into a v-product of height-one prime ideals, $aR = (P_1 \cdots P_n)_v$. This is a generalization of the Dedekind domain property that any nontrivial principal ideal aR can be factored uniquely into a product of prime ideals $aR = P_1 \cdots P_n$. The v-product of height-one prime ideals property is strong enough

to study factorization; so the literature has standardized to studying the more general Krull domains instead of Dedekind domains.

Factoring a nonzero nonunit $a \in R$ into irreducibles corresponds to factoring the v-product of height-one prime ideals $aR = (P_1 \cdots P_n)_v$ into sub-v-products that are principal ideals. The next theorem illustrates this relationship.

Theorem 1.8. *Let R be a Krull domain and $a \in R$ a nonzero nonunit.*

1. *Let $a = xy$ for some nonunits $x, y \in R$. Since R is a Krull domain, there exists height-one prime ideals $P_1, \ldots, P_k, Q_1, \ldots, Q_l$ of R such that $xR = (P_1 \cdots P_k)_v$ and $yR = (Q_1 \cdots Q_l)_v$. Then*
$$aR = (P_1 \cdots P_k Q_1 \cdots Q_l)_v,$$
and this v-product of height-one prime ideals is unique.

2. *Since R is a Krull domain, there exists height-one prime ideals $T_1, \ldots, T_n$ in R such that $aR = (T_1 \cdots T_n)_v$. If $bR = (T_1 \cdots T_m)_v$ for some $m < n$ and $b \in R$, then $(T_{m+1} \cdots T_n)_v = cR$ and $a = bcu$ for some $c \in R$ and $u \in U(R)$. (If any sub-v-product of $\{T_1, \ldots T_n\}$ is principal, then the ideals can be rearranged so these ideals come first.)*

Thus an element of R is irreducible if its corresponding v-product of height-one prime ideals has no proper sub-v-product that is principal. Recall, Krull domains are atomic. Therefore each nonzero nonunit $a \in R$ can be factored into irreducible elements $a = x_1 \cdots x_n$ for some $n \in \mathbb{N}$ and each $x_i \in R$ irreducible. For each of these irreducible elements x_i, $x_i R = (P_1 \cdots P_m)_v$ for some height-one prime ideals $P_1, \ldots, P_m$ of R. Since each x_i is irreducible, then no sub-v-product of $(P_1 \cdots P_m)_v$ is principal. Also, the height-one primes in the v-product of aR must equal the v-product of these $P_1, \ldots, P_m$ for each x_i. Therefore factoring an element $a \in R$ into irreducibles corresponds to grouping its unique v-product of height-one prime ideals into sub-v-products that are principal, and each of these sub-v-products has no proper sub-v-product that is principal.

Theorem 1.9. *Let R be a Krull domain. Let $a \in R$ be a nonzero nonunit with $aR = (P_1 \cdots P_m)_v$, where each P_i is a height-one prime ideal of R. Then $a = x_1 \cdots x_n$, where each x_i is irreducible in R, if and only if the order of the P_i's can be rearranged so that $x_1 R = (P_1 \cdots P_{m_1})_v$, ..., $x_n R = (P_{m_{n-1}+1} \cdots P_{m_n})_v$, and $(P_{m_i+1} \cdots P_{m_{i+1}})_v$ has no proper sub-v-product that is principal for each $0 \leq i < n$.*

The height-one prime ideals of a Krull domain are divisorial. Thus each height-one prime ideal P can be viewed as an element $\overline{P}$ of the divisor class group (for notation purposes an overline will be used to represent the class for a particular element). Then for a nonzero nonunit a in a Krull domain R, $\overline{aR} = 1$ (the identity of the divisor class group). Then $aR = (P_1 \cdots P_n)_v$ for some height-one prime ideals P_i of R, and in the divisor class group $\overline{P_1} \cdots \overline{P_n} = \overline{(P_1 \cdots P_n)_v} = 1$. Then factoring an element $a \in R$ into irreducibles corresponds to grouping $\overline{P_1}, \ldots, \overline{P_n}$ into subsets whose v-product is 1 and no proper sub-v-product is 1 in the divisor class group. Also, if a product of height-one prime ideals is 1 in the divisor class group, then that v-product is principal. So if such a product of height-one prime ideals is 1 in the divisor class group, with chosen subsets whose v-product is 1 with no proper sub-v-product equal to 1, then there is a corresponding factorization of an element into irreducible elements. Thus studying v-products of height-one prime ideals in the divisor

class group yields results about factoring elements into irreducible elements. In general, we will use S to represent the classes of the divisor class group that contain a height-one prime ideal. So for a Krull domain with $Cl(R) = G$, we will study the corresponding pair (G, S) to learn about the Krull domain's factorization. The block monoid introduced in the next section provides a convenient notation to study the pair (G, S).

1.4 Block Monoids

A good reference for block monoids is [13]. Let G be an abelian group and $S \subseteq G$. Let $\mathcal{F}(S)$ be the free abelian monoid with basis S. Then a typical element of $\mathcal{F}(S)$ has the form $B = \prod_{g \in S} g^{v_g(B)}$, where $v_g(B) \in \mathbb{N}_0$ and $v_g(B) = 0$ for all but finitely many $g \in S$. Then we can define a (monoid) homomorphism $\sigma : \mathcal{F}(S) \to G$ by $\sigma(B) = \sum_{g \in S} v_g(B)g$.

Definition 1.10. For an abelian group G and $S \subseteq G$, define the block monoid to be $\mathcal{B}(G, S) = ker(\sigma)$, i.e., $\mathcal{B}(G, S)$ is the set of $B \in \mathcal{F}(S)$ such that $\sigma(B) = 0$.

To clearly distinguish between the block monoid and the original group G, we will use multiplicative notation for the block monoid and additive notation for the group G. Elements of $\mathcal{B}(G, S)$ are referred to as blocks. Then a typical block of the block monoid $\mathcal{B}(G, S)$ is of the form $B = \prod_{g \in S} g^{v_g(B)}$, where $\sum_{g \in G} v_g(B)g = 0$ in the group G. The identity element for the block monoid is the empty product $B = \prod_{g \in S} g^0$; denote the identity by 1.

The one operation of the block monoid will be treated similarly to multiplication in a ring. So we will talk about factoring the blocks in the block monoid. An element $B = \prod_{g \in S} g^{v_g(B)}$ in $\mathcal{B}(G, S)$ is *irreducible* if no proper subsum of $\sum_{g \in S} v_g(B)g$ is 0 in G and B is not the identity. Define $\mathcal{A}(G, S)$ to be the set of irreducible blocks in $\mathcal{B}(G, S)$. Note that for the identity 0 of G, if $0 \in S$, then the block $B = 0^1$ is an irreducible block. Also, the block $B = 0^1$ is not the identity of the block monoid. Factoring a general block $B = \prod_{g \in S} g^{v_g(B)}$ corresponds to grouping the elements into subproducts whose sum is 0 in G with no proper subsum equal to 0. Suppose we split B into $B = B_1B_2$ and then B_1 can be split into $B_1 = C_1C_2$, and we continue this process. The factoring cannot continue indefinitely because B cannot equal a product of more than $\sum_{g \in G} v_g(B)$ blocks. Thus each block must have a factorization into irreducible blocks. So the block monoid is always atomic (extending the definition of atomic to cover monoids).

The definition of elasticity can be expanded to cover factoring in monoids such as the block monoid.

Definition 1.11. For $B \in \mathcal{B}(G, S)$, define

$$\rho(B) = max\left\{\frac{m}{n} \mid B = B_1 \cdots B_m = C_1 \cdots C_n \text{ and each } B_i, C_j \in \mathcal{A}(G, S)\right\},$$

and then

$$\rho(G, S) = sup\{\rho(B) \mid B \in \mathcal{B}(G, S)\}.$$

These are called the *elasticity of a block* and the *elasticity of the block monoid*, respectively.

Using $\rho(G, S)$ and $\mathcal{A}(G, S)$ instead of the somewhat more appropriate $\rho(\mathcal{B}(G, S))$ and $\mathcal{A}(\mathcal{B}(G, S))$, respectively, is an attempt to make the notation slightly less cumbersome.

For a Krull domain R with the corresponding divisor class group pair (G, S), we will study the block monoid $\mathcal{B}(G, S)$. Let $a \in R$ be a nonzero nonunit and $aR = (P_1 \cdots P_n)_v$ its unique v-product of height-one primes ideals. Then $\overline{P_1} \cdots \overline{P_n} = \overline{aR} = 1$ in the divisor class group G (this does not follow the convention mentioned above, since multiplication is being used for the divisor class group and the block monoid). Therefore the product $B = \overline{P_1} \cdots \overline{P_n}$ is a block in the block monoid $\mathcal{B}(G, S)$ because $\overline{P_1} \cdots \overline{P_n} = 1$ in the divisor class group G. Then $\rho(a) = \rho(B)$ since factoring a into irreducibles corresponds to grouping the product $\overline{P_1} \cdots \overline{P_n}$ into irreducible blocks. Also, for any block $B \in \mathcal{B}(G, S)$, the product of height-one prime ideals is principal; so there is a corresponding element $a \in R$ with $\rho(a) = \rho(B)$. Hence we have the following theorem.

Theorem 1.12. *Let R be a Krull domain with $Cl(R) = G$ and $S \subseteq G$ the set of classes containing a height-one prime ideal. Then*

$$\rho(R) = \rho(G, S).$$

Therefore the results of this dissertation will be concerned with finding the elasticity of the block monoid, which then gives the elasticity of the corresponding Krull domain. Similar to expanding the definition of elasticity to cover monoids, the definition of an HFD can be expanded to cover monoids. Since monoids are not domains, we say a monoid is *half-factorial* if the elasticity of the monoid is one. Then a Krull domain is an HFD if and only if the corresponding block monoid is half-factorial.

Every Krull domain has a corresponding pair (G, S) and block monoid $\mathcal{B}(G, S)$. It is of interest to determine when a pair (G, S) is realizable as a Krull domain, i.e., when there exists a Krull domain R with $Cl(R) = G$ and $S \subseteq G$ the set of classes containing a height-one prime ideal. It was originally proved by Claborn that every abelian group is the divisor class group of a Dedekind domain [7]. Claborn then studied what properties S must have if the pair (G, S) has a corresponding Krull domain [8] [10]. Building on this work, Grams showed that if a S generates G as a monoid (elements of S can be added only, not subtracted, to generate G), then a Dedekind domain exists with pair (G, S). Since Dedekind domains are Krull, then a Krull domain exists with pair (G, S).

The converse also holds. Suppose R is a Krull domain with corresponding pair (G, S). Let $g \in G$. Then there is a divisorial ideal J of R such that $\overline{J} = g$. Since J is a fractional ideal, there exists $0 \neq x \in R$ such that $xJ = I$ for some integral ideal I of R. Then I is also divisorial. Therefore $\overline{I} = \overline{xRJ} = (\overline{xR})(\overline{J}) = \overline{J}$ in G since $\overline{xR} = 1$. Since any integral divisorial ideal can be written as a product of height-one prime ideals in a Krull domain, then $g = \overline{I}$ can be written as a v-product of height-one prime ideals. Therefore S generates G as a monoid. So we have the following theorem.

Theorem 1.13. *Let G be an abelian group and $S \subseteq G$. The pair (G, S) has a corresponding Krull domain if and only if S generates G as a monoid.*

The study of factorization then focuses on pairs (G, S) and the block monoid $\mathcal{B}(G, S)$, where S generates G as a monoid. However, some results in this dissertation determine the elasticity of $\mathcal{B}(G, S)$ without requiring that S generates G as a monoid. Later results, such as Proposition 3.3, reduce complicated block monoids to simpler block monoids. These simpler block monoids may not have S generating G as a monoid; so it is convenient if some of the theorems can be applied in these situations. In any of these theorems proving results about a block monoid where S may not generate G as a monoid, a corollary could be stated

for the Krull domain case. For example, if a theorem proves that $\mathcal{B}(G, S)$ is half-factorial, a corollary could be added stating that if a Krull domain has the corresponding pair (G, S), then the Krull domain is an HFD.

1.5 Block Monoid Lemmas

The following lemmas are easy results concerning block monoids that will be useful later in the dissertation. Many of the papers on block monoids contain similar lemmas, or these lemmas in particular. These results are fairly simple and repeated so frequently in the literature that it is hard to give proper credit to the first author who proved them. The first part of the next lemma concluding $\rho(G_1, S_1) \leq \rho(G_2, S_2)$ can be found in [2]. This lemma will provide a convenient standard method to show one block monoid has a smaller elasticity or to show that two block monoids have the same elasticity.

Lemma 1.14. *Let $f : \mathcal{B}(G_1, S_1) \to \mathcal{B}(G_2, S_2)$ be a homomorphism such that $f(B)$ is irreducible whenever B is irreducible. Then $\rho(G_1, S_1) \leq \rho(G_2, S_2)$. Also, $\rho(G_1, S_1) = \rho(G_2, S_2)$ if the following hold:*

1. *$ker f = \{1\}$.*
2. *There exists a homomorphism $g : \mathcal{B}(G_2, S_2) \to \mathcal{B}(G_1, S_1)$ such that $fg(B) = B$ for all $B \in \mathcal{B}(G_2, S_2)$.*

In particular, if f is an injective homomorphism such that $f(B)$ is irreducible whenever B is irreducible and for all $B' \in \mathcal{A}(G_2, S_2)$ there exists a $B \in \mathcal{A}(G_1, S_1)$ such that $f(B) = B'$, then $\rho(G_1, S_1) = \rho(G_2, S_2)$.

Proof. Suppose $B_1 \cdots B_m = C_1 \cdots C_n$ are two factorizations of irreducible elements in $\mathcal{B}(G_1, S_1)$. Then $f(B_1) \cdots f(B_m) = f(C_1) \cdots f(C_n)$ since f is a homomorphism. Since each $f(B_i)$ and $f(C_i)$ is irreducible, then we have that $\rho(G_2, S_2) \geq \frac{m}{n}$. Thus $\rho(G_1, S_1) \leq \rho(G_2, S_2)$.

Now suppose that $ker f = \{1\}$ and such a g exists. Let $A \in \mathcal{B}(G_2, S_2)$ be irreducible, and assume $g(A)$ is not irreducible. Then there exists some $A_1, A_2 \in \mathcal{B}(G_1, S_1) - \{1\}$ such that $g(A) = A_1 A_2$. But then $A = fg(A) = f(A_1 A_2) = f(A_1) f(A_2)$. Since $A_1, A_2 \neq 1$, then $f(A_1), f(A_2) \neq 1$. Thus, we have that A is not irreducible, which is a contradiction. So if $A \in \mathcal{B}(G_2, S_2)$ is irreducible, then $g(A)$ is irreducible. Let $D_1 \cdots D_h = E_1 \cdots E_k$ in $\mathcal{B}(G_2, S_2)$ with each D_i, E_j an irreducible block. Since g is a homomorphism, then $g(D_1) \cdots g(D_h) = g(E_1) \cdots g(E_k)$. Since each D_i and E_j is irreducible, then each $g(D_i)$ and $g(E_j)$ is irreducible. Then $g(D_1) \cdots g(D_h) = g(E_1) \cdots g(E_k)$ are two factorizations into irreducibles in $\mathcal{B}(G_1, S_1)$. Then $\rho(G_1, S_1) \geq \frac{h}{k}$; so $\rho(G_1, S_1) \geq \rho(G_2, S_2)$. Combining this with the first part of the proof, we have shown that $\rho(G_1, S_1) = \rho(G_2, S_2)$.

For the "in particular" statement, suppose f is injective with the properties listed above. Recall, $\mathcal{B}(G_2, S_2)$ is atomic; so each element of $\mathcal{B}(G_2, S_2)$ can be written as a product of elements in $\mathcal{A}(G_2, S_2)$, the irreducible blocks of $\mathcal{B}(G_2, S_2)$. But by assumption, $\mathcal{A}(G_2, S_2) \subseteq im f$. Then each element of $\mathcal{B}(G_2, S_2)$ can be written as a product of elements in $\mathcal{A}(G_2, S_2) \subseteq im f$. Therefore, f is surjective, and thus an isomorphism. If f is an isomophism of monoids, then it must have an inverse g, and that inverse must be a homomorphism. Also, since f is an isomorphism, then $ker f = \{1\}$. Therefore, the conditions are satisfied, and $\rho(G_1, S_1) = \rho(G_2, S_2)$.

□

Note that the above proof is not stated for general monoids, but only block monoids that have been constructed from an abelian group G and a subset S. The fact that $\mathcal{B}(G,S)$ is a block monoid was used twice in the proof. First, the only unit of the block monoid is the identity, 1, so in the proof $A_1, A_2 \neq 1$ guarantees that A_1, A_2 are not units. Thus, $A = A_1A_2$ shows that A is not irreducible. If $f : M_1 \to M_2$ were a homomorphism between general monoids, the first condition would need to state that the preimage of the units of M_2 equals the units of M_1. Secondly, for the"in particular" statement, we used the fact that $\mathcal{B}(G_2, S_2)$ is atomic to show that f is a surjection. We would only want to apply the theorem to the case M_1 and M_2 are atomic, so we would need to add this assumption.

Since we are dealing with monoids, the conditions that $kerf = \{1\}$ and f is injective are not equivalent. The proof that these are equivalent for groups uses the fact that each element has an inverse which is not true for monoids. For the "in particular" statement, since f was shown to be an isomorphism, then $kerf = \{1\}$.

On first glance, it seems the condition that for all $B' \in \mathcal{A}(G_2, S_2)$, there exists a $B \in \mathcal{A}(G_1, S_1)$ such that $f(B) = B'$ should be enough to obtain equality among the elasticities, without requiring that f is injective. However, this is not true. Suppose we have $B'_1 \cdots B'_h = C'_1 \cdots C'_k$ for some irreducible B'_i, C'_j in $\mathcal{B}(G_2, S_2)$. Then this condition says that there exists B_i, C_j irreducible blocks in $\mathcal{B}(G_1, S_1)$ such that $f(B_i) = B'_i$ and $f(C_j) = C'_j$. Then f is a homomorphism, so

$$\begin{aligned} f(B_1 \cdots B_h) &= f(B_1) \cdots f(B_h) \\ &= B'_1 \cdots B'_h \\ &= C'_1 \cdots C'_k \\ &= f(C_1) \cdots f(C_k) \\ &= f(C_1 \cdots C_k). \end{aligned}$$

But there is no reason to assume that $B_1 \cdots B_h = C_1 \cdots C_k$ unless f is injective. In many of the applications of this theorem, the function f will not be injective. The conditions that $kerf = \{1\}$ and the right inverse homomorphism g exist provide a solution to this problem. If these conditions are satisfied, it guarantees that there does exist a choice for each B_i and C_j using g such that $B_1 \cdots B_h = C_1 \cdots C_k$.

Lemma 1.15. *Let G be a group, and $S', S \subseteq G$. If $S' \subseteq S$, then $\rho(G, S') \leq \rho(G, S)$.*

Proof. Define a function $f : \mathcal{B}(G, S') \to \mathcal{B}(G, S)$ by the identity map on elements of G. Thus

$$f\left(\prod_{s \in S'} s^{u_s}\right) = \prod_{s \in S'} s^{u_s},$$

which is in $\mathcal{B}(G, S)$ since $S' \subseteq S$. It is clear that if a block B is irreducible in $\mathcal{B}(G, S')$, then $f(B)$ is irreducible. This is clearly a homorphism; so Lemma 1.14 gives that $\rho(G, S') \leq \rho(G, S)$. □

The following result is similar to the previous lemma, and will be useful in the sections on $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$. In these cases, it is easier to break the set S into convenient subsets S_i that are finite. The lemma then connects results about these subsets S_i to the original set S.

Lemma 1.16. *Let G be a group and $S_i \subseteq G$ such that $S_i \subseteq S_{i+1}$ for each $i \in \mathbb{N}$. If $S = \cup_{i\in\mathbb{N}} S_i$, then*

$$\rho(G, S) = \lim_{i\to\infty} \rho(G, S_i).$$

In particular,

$$\rho(G, S) = \sup_{i\in\mathbb{N}} \rho(G, S_i).$$

Proof. Since $S_i \subseteq S$, then $\rho(G, S) \geq \rho(G, S_i)$ for all $i \in \mathbb{N}$ by Lemma 1.15. Therefore the sequence $\{\rho(G, S_i)\}_{i=1}^{\infty}$ is increasing as i increases. Then the sequence $\rho(G, S_i)$ must either converge to a finite number or increase without bound. Suppose that it increases without bound, i.e., $lim_{n\to\infty}\rho(G, S_n) = \infty$. Then for all $m \in \mathbb{N}$, there exists an $i \in \mathbb{N}$ such that $\rho(G, S_i) > m$. By Lemma 1.14, $\rho(G, S) \geq \rho(G, S_i)$ for all $i \in \mathbb{N}$. Then $\rho(G, S) > m$ for all $m \in \mathbb{N}$, so $\rho(G, S) = \infty$.

Now suppose that $lim_{n\to\infty}\rho(G, S_n) = r$ for some $1 \leq r < \infty$. Since $\rho(G, S) \geq \rho(G, S_i)$ for all $i \in \mathbb{N}$, then $\rho(G, S) \geq \lim_{i\to\infty} \rho(G, S_i)$. Suppose that $\rho(G, S) > r$. Then there exists two factorizations of irreducible blocks in $\mathcal{B}(G, S)$, $B_1 \cdots B_m = C_1 \cdots C_n$, such that $\frac{m}{n} > r$. Then for each B_i, there exists an $a_i \in \mathbb{N}$ such that the elements of B_i are all in S_{a_i}. Similarly, for each C_j, there exists a $b_j \in \mathbb{N}$ such that the elements of C_j are all in S_{b_j}. Let c be the maximum of the a_i and b_j. Then each $B_i, C_j \in \mathcal{B}(G, S_c)$; so $\rho(G, S_c) > r$. This is a contradiction. So $\rho(G, S) = r$.

The "in particular" statement follows because the sequence $\rho(G, S_i)$ is increasing; so its limit and its supremum are equal. □

Semi-length functions are useful for determining when a domain R or block monoid is half-factorial. We will state the definition of semi-length functions for general monoids. The definition was originally for domains R in [1], but this way it will apply to block monoids as well.

Definition 1.17. Let M be an atomic monoid. A function $f : M \to \mathbb{R}^+ \cup \{0\}$ is a *semi-length function* if it satisfies:

- $f(xy) = f(x) + f(y)$ for all $x, y \in R$ with x and y nonunits.
- $f(x) = 0$ for all units $x \in R$.

In a block monoid, the condition that $f(B) = 0$ for all units B is usually trivial since the only unit is the empty product. Therefore checking this condition is often omitted for block monoids. This is a generalization of a length function, which has the same definition except the range of f is $\mathbb{Z}^+ \cup \{0\}$. The following theorem uses semi-length functions to provide an easy method for proving monoids are half-factorial. The "⇐" direction was proven in [21], and the "⇒" direction is simple. In the result, semi-length function can be replaced with a length function and it will still hold.

Lemma 1.18. *Let M be a monoid. Then the elasticity of M is one if and only there exists a semi-length function $f : M \to \mathbb{R}^+ \cup \{0\}$ such that $f(m) = 1$ for all irreducible elements $m \in M$.*

1.6 Splittable Sets

When dealing with block monoids, the definition of a splittable set is often important. Splittable sets were introduced by Zaks in [20].

Definition 1.19. A set $M \subseteq \mathbb{N}$ is called *splittable* if whenever $\sum_{i=1}^{k} r_i \frac{1}{m_i} \in \mathbb{N}$ for some $m_i \in M$ and $r_i \in \mathbb{N}$, there exists $0 \leq r_i' \leq r_i$ for each $1 \leq i \leq k$ such that $\sum_{i=1}^{k} r_i' \frac{1}{m_i} = 1$.

We next introduce a similar definition that will be pertinent to the later material.

Definition 1.20. A set $\{m_1, \ldots, m_k\} \subset \mathbb{N}$ is called *m-split* if whenever $\sum_{i=1}^{k} r_i m_i \in \mathbb{N}m$, there exists $0 \leq r_i' \leq r_i$ for each $1 \leq i \leq k$ such that $\sum_{i=1}^{k} r_i' m_i = m$.

Note that the set $\{m_1, \ldots, m_k\}$ is m-split implies that $m_i | m$ for all $1 \leq i \leq k$.

Lemma 1.21. *Suppose that $\{m_1, \ldots, m_k\} \subset \mathbb{N}$ and each $m_i | m$. Then $\{m_1, \ldots, m_k\}$ is m-split if and only if $\{ord_{\mathbb{Z}/m\mathbb{Z}}(\overline{m_i}) \mid 1 \leq i \leq k\}$ is a splittable set.*

Proof. The key to this relationship is that $ord_{\mathbb{Z}/m\mathbb{Z}}(\overline{m_i}) = \frac{m}{m_i}$ since $m_i | m$. Thus $\sum_{i=1}^{k} r_i m_i \in \mathbb{N}m$ if and only if $\sum_{i=1}^{k} r_i \frac{1}{ord(\overline{m_i})} \in \mathbb{N}$. Hence there exists $0 \leq r_i' \leq r_i$ for each $1 \leq i \leq k$ such that $\sum_{i=1}^{k} r_i' m_i = m$ if and only if there exists $0 \leq r_i' \leq r_i$ for each $1 \leq i \leq k$ such that $\sum_{i=1}^{k} r_i' \frac{1}{ord(\overline{m_i})} = 1$. □

Geroldinger and Wao [11] proved the following theorem showing when a block monoid is half-factorial for the group $\mathbb{Z}/m\mathbb{Z}$. This result will be used extensively throughout the dissertation. Block monoids for many of the infinite groups that will be studied have the same elasticity as block monoids with a group of the form $\mathbb{Z}/m\mathbb{Z}$.

Theorem 1.22. *Let $S \subseteq \mathbb{Z}/m\mathbb{Z}$. Then $\rho(\mathbb{Z}/m\mathbb{Z}, S) = 1$ if and only if there exists $\varphi \in Aut(\mathbb{Z}/m\mathbb{Z})$ such that $\varphi(S) \subseteq \{d + m\mathbb{Z} \mid d \in \mathbb{N} \text{ and } d | m\}$ and $\{ord(s) \mid s \in S\}$ is a splittable set.*

Corollary 1.23. *Suppose $S = \{m_1, \ldots, m_k\} \subseteq \mathbb{N}$ and each $m_i | m$. Then $\rho(\mathbb{Z}/m\mathbb{Z}, \overline{S}) = 1$ if and only if S is m-split.*

Proof. ($\Rightarrow$) If $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$, then Theorem 1.22 gives that the set $\{ord(\overline{m_i}) \mid 1 \leq i \leq k\}$ is splittable. Thus $\{m_1, \ldots, m_k\}$ is m-split by Lemma 1.21.

($\Leftarrow$) Let $\varphi \in \text{Aut}(\mathbb{Z}/m\mathbb{Z})$ be the identity automorphism. Then $\varphi(\overline{m_i}) \subseteq \{d + m\mathbb{Z} \mid d \in \mathbb{N} \text{ and } d | m\}$ for each m_i. Since $\{m_1, \ldots, m_k\}$ is m-split, then $\{ord(\overline{m_i}) \mid 1 \leq i \leq k\}$ is splittable by Lemma 1.21. Thus $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$ by Theorem 1.22. □

The following is a technical lemma for sets that are m-split. It shows that if a sum of these elements equals tm for some $t \in \mathbb{N}$, then the coefficients can be reduced so that the sum is equal to am for any $1 \leq a \leq t$.

Lemma 1.24. *Suppose $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$, where each $m_i | m$. If $\sum_{i=1}^{k} s_i m_i = tm$ for some $t \in \mathbb{N}$, then there exists $0 \leq s_i' \leq s_i$ for $1 \leq i \leq k$ such that $\sum_{i=1}^{k} s_i' m_i = m$. Also, for all $a \in \mathbb{N}$ with $a \leq t$, there exists $0 \leq a_i \leq s_i$ for $1 \leq i \leq k$ such that $\sum_{i=1}^{k} a_i m_i = am$.*

Proof. Since $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$, then by Corollary 1.23 the set $\{\overline{m_1}, \ldots, \overline{m_k}\}$ is m-split. So if $\sum_{i=1}^{k} s_i m_i = tm$ for some $t \in \mathbb{N}$, then there exists $0 \leq s_i' \leq s_i$ for $1 \leq i \leq k$ such that $\sum_{i=1}^{k} s_i' m_i = m$. Now we can repeat the process on $\sum_{i=1}^{k}(s_i - s_i')m_i = (t-1)m$ choosing $0 \leq d_i \leq (s_i - s_i')$ such that $\sum_{i=1}^{k}(s_i' + d_i)m_i = 2m$. We can repeat this process until we have a subsum equal to am for all $1 \leq a \leq t$. □

Chapter 2

Common Infinite Groups

In this chapter, the elasticity of several common infinite groups will be studied. The focus will be on determining which subsets S of G will give a half-factorial block monoid. Then any Krull domain with divisor class group G and such an S as its subset of height-one prime ideals will be a half-factorial domain.

2.1 $\mathbb{Z}$

The first infinite abelian group we will study is $\mathbb{Z}$. Let R be a Krull domain with $Cl(R) = \mathbb{Z}$ and S the classes of $Cl(R)$ containing a height-one prime ideal of R. If the set S is finite, i.e., only a finite number of classes of $Cl(G)$ contain height-one primes, then the elasticity of R is rational and realizable [2]. So the elasticity is a rational number, and there exists some irreducibles $x_i, y_j \in R$ with $x_1 \cdots x_m = y_1 \cdots y_n$ such that $\rho(R) = \frac{m}{n}$. For this section, we will focus on the case when S is an infinite set, which is not as simple. If S is infinite, then [5] provides an example where the elasticity of R is rational, but not realizable. The same paper proves the following theorem about S.

Theorem 2.1. *Let R be a Krull domain with $Cl(R) = \mathbb{Z}$ and S the classes of $Cl(R)$ containing a height-one prime ideal of R. If S^- and S^+ are both infinite sets, then $\rho(R) = \infty$.*

Therefore, the elasticity when S^- and S^+ are both infinite has been solved; so we will restrict to the case that one of these sets is finite. We will assume without lose of generality that $|S^-| < \infty$ and $|S^+| = \infty$. So $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\}$ for some $m_i, n_j \in \mathbb{N}$ for all $i, j \in \mathbb{N}$. By the simple automorphism of multiplication by -1, these results also apply to the case $|S^+| < \infty$ and $|S^-| = \infty$. The following definition from [4] will be important.

Definition 2.2. Let $\{m_1, \ldots, m_k\} \subset \mathbb{N}$ and $p_1, p_2, ..., p_h$ be distinct primes such that each $m_i = p_1^{x_{i,1}} p_2^{x_{i,2}} \ldots p_h^{x_{i,h}}$. Let $I = \{i \mid x_{j,i} \neq x_{l,i} \text{ for some } 1 \leq j, l \leq k\}$, and define

$$\ll m_1, \ldots, m_k \gg = \prod_{i \in I} p_i^{max\{x_{1,i}, x_{2,i}, \ldots, x_{k,i}\}}.$$

In short, if a prime number appears in two m_i's a different number of times, then its index is in I. Then the product contains the primes with index in I to the maximum power that prime has in the m_i's. The following lemma will be useful in the proofs.

Lemma 2.3. *Let* $\{m_1, \ldots, m_k\} \subset \mathbb{N}$, $c = \ll m_1, \ldots, m_k \gg$, *and* $m = lcm\{m_1, \ldots, m_k\}$. *Then* $m = dc$ *for some* $d \in \mathbb{N}$, *where* $(c, d) = 1$ *and* $d | m_j$ *for all* $1 \leq j \leq k$.

Proof. Using the notation from Definition 2.2 of $c = \ll m_1, \ldots, m_k \gg$,

$$m = lcm\{m_1, \ldots, m_k\} = \prod_{i=1}^{h} p_i^{max\{x_{1,i}, x_{2,i}, \ldots, x_{k,i}\}} = c \cdot \prod_{i \notin I} p_i^{max\{x_{1,i}, x_{2,i}, \ldots, x_{k,i}\}} = c \cdot \prod_{i \notin I} p_i^{x_{1,i}}$$

since $i \notin I$ means $x_{1,i} = \cdots = x_{k,i}$. Let $d = \prod_{i \notin I} p_i^{x_{1,i}}$, and thus, $m = dc$. Also, $d | m_j$ for each $1 \leq j \leq k$. It is clear that $(c, d) = 1$ since they have no common prime factors. □

Let $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\}$ and $c = \ll m_1, \ldots, m_k \gg$. If $c | n_j$ for all $j \in \mathbb{N}$, then we can establish a relationship between $\rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$ and $\rho(\mathbb{Z}, S)$. The next lemma will be our first such result.

Lemma 2.4. *Let* $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\} \subseteq \mathbb{Z}$ *with each* $m_i, n_j > 0$. *Define* $c = \ll m_1, \ldots, m_k \gg$ *and* $m = lcm(m_1, \ldots, m_k)$. *Suppose* $c | n_i$ *for all* $i \in \mathbb{N}$. *Then*

$$\rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}) \leq \rho(\mathbb{Z}, S).$$

Proof. Let $B \in \mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$. Then $B = \prod_{j=1}^{a} t_j^{s_j}$ for some $t_j \in \overline{S^+}$ and $s_j \in \mathbb{N}_0$ with $\sum_{j=1}^{a} s_j t_j = 0$ in $\mathbb{Z}/m\mathbb{Z}$. For each $t_j \in \overline{S^+}$, choose n_{i_j} to be the smallest integer in S^+, that is equivalent to t_j in $\overline{S^+}$. Then $\sum_{j=1}^{a} s_j n_{i_j} \equiv 0 (\text{mod } m)$. Therefore $\sum_{j=1}^{a} s_j n_{i_j} = tm$ for some $t \in \mathbb{Z}$. Since each $n_{i_j} > 0$ and $s_j \geq 0$, then $t \geq 0$. Then $sm_1 = tm$ for some $s \in \mathbb{N}_0$ since $m_1 | m$. Hence $s(-m_1) + \sum_{j=1}^{a} s_j n_{i_j} = 0$. Therefore $C = (-m_1)^s \cdot \prod_{j=1}^{a} n_{i_j}^{s_j}$ is a block in $\mathcal{B}(\mathbb{Z}, S)$. Now we can use this construction to create the following function.

Define $f : \mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}) \to \mathcal{B}(\mathbb{Z}, S)$ by

$$f\left(\prod_{j=1}^{a} t_j^{s_j}\right) = (-m_1)^s \cdot \prod_{j=1}^{a} n_{i_j}^{s_j},$$

where n_{i_j} and s are defined as above. It is clear that f is a homomorphism. Suppose that $B = \prod_{j=1}^{a} t_j^{s_j}$ is irreducible and $f(B) = (-m_1)^s \cdot \prod_{j=1}^{a} n_{i_j}^{s_j}$ is not irreducible. Since B is irreducible, then $\sum_{j=1}^{a} s_j t_j = 0$ with no proper subsum equal to 0 in $\mathbb{Z}/m\mathbb{Z}$. Since $f(B)$ is not irreducible, there exists an irreducible block C' in $\mathcal{B}(\mathbb{Z}, S)$ dividing $f(B)$, and $C' \neq f(B)$. Therefore $C' = (-m_1)^{s'} \cdot \prod_{j=1}^{a} n_{i_j}^{s'_j}$ for some $s', s'_j \in \mathbb{N}_0$ with $0 \leq s' \leq s$ and $0 \leq s'_j \leq s_j$ for all $1 \leq j \leq a$. Suppose $s'_j = s_j$ for all $1 \leq j \leq a$. Then $s'm_1 = \sum_{j=1}^{a} s'_j n_{i_j}$ since C' is a block. But then $\sum_{j=1}^{a} s'_j n_{i_j} = \sum_{j=1}^{a} s_j n_{i_j} = sm_1$ since $f(B)$ is a block. Therefore $s'm_1 = sm_1$, and thus $s' = s$. Then $C' = C$, which is a contradiction. Hence $s'_j < s_j$ for some $1 \leq j \leq a$.

Since C' is a block, then $s'm_1 = \sum_{j=1}^{a} s'_j n_{i_j}$. By Lemma 2.3, there exist a $d \in \mathbb{N}$ such that $m = cd$ with $(c, d) = 1$ and $d | m_1$. Then d divides $s'm_1$. Since c divides each n_{i_j}, then c divides $\sum_{j=1}^{a} s'_j n_{i_j}$. Therefore m divides $s'm_1 = \sum_{j=1}^{a} s'_j n_{i_j}$ since $(c, d) = 1$. Hence $\sum_{j=1}^{a} s'_j n_{i_j} \in \mathbb{N}m$. But then $\sum_{j=1}^{a} s'_j n_{i_j} \equiv 0 (\text{mod } m)$. Since each n_{i_j} is equivalent to t_j in $\mathbb{Z}/m\mathbb{Z}$, then $\sum_{j=1}^{a} s'_j t_j = 0$ in $\mathbb{Z}/m\mathbb{Z}$. Then $B' = \prod_{j=1}^{a} t_j^{s'_j}$ is a block in $\mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S_+})$, and B' divides B. Since $s'_j < s_j$ for at least on $1 \leq j \leq a$, then $B' \neq B$. Hence B

is not irreducible, a contradiction. Thus $f(B)$ must be irreducible, and by Lemma 1.14, $\rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}) \leq \rho(\mathbb{Z}, S)$. □

The next theorem builds on the above lemma by additionally assuming a condition about irreducible blocks in $\mathcal{B}(\mathbb{Z}, S)$. The theorem assumes that

$$\prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{A}(\mathbb{Z}, S) \Rightarrow \prod_{j=1}^{\infty} \overline{n_j}^{r_j} \in \mathcal{A}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}).$$

This assumption shows that we can define a homomorphism, seen in the proof, from $\mathcal{B}(\mathbb{Z}, S)$ to $\mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$, that will preserve irreducible blocks. Then $\rho(\mathbb{Z}, S) \leq \rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$ by Lemma 1.14. Then by Lemma 2.4, these two elasticities are equivalent.

Theorem 2.5. *Let $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\} \subseteq \mathbb{Z}$ with each $m_i, n_j > 0$. Define $c = \ll m_1, \ldots, m_k \gg$ and $m = lcm(m_1, \ldots, m_k)$. Suppose that $c|n_i$ for all $i \in \mathbb{N}$ and*

$$\prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{A}(\mathbb{Z}, S) \Rightarrow \prod_{j=1}^{\infty} \overline{n_j}^{r_j} \in \mathcal{A}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}).$$

Then $\rho(\mathbb{Z}, S) = \rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$.

Proof. Define $f : \mathcal{B}(\mathbb{Z}, S) \to \mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$ by

$$f(\prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j}) = \prod_{j=1}^{\infty} \overline{n_j}^{r_j}.$$

Suppose that $B = \prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{B}(\mathbb{Z}, S)$. Then $\sum_{i=1}^{k} s_i m_i = \sum_{j=1}^{\infty} r_j n_j$. By Lemma 2.3, there exist a $d \in \mathbb{N}$ such that $m = cd$ with $(c, d) = 1$ and $d|m_i$ for all $1 \leq i \leq k$. Therefore d divides $\sum_{i=1}^{k} s_i m_i$. Each n_j is divisible by c; so c divides $\sum_{j=1}^{\infty} r_j n_j$. Since $(c, d) = 1$, then m divides $\sum_{i=1}^{k} s_i m_i = \sum_{j=1}^{\infty} r_j n_j$. Thus $f(B) = \prod_{j=1}^{\infty} \overline{n_j}^{r_j} \in \mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$ so f is well-defined. This definition is clearly a homomorphism. If B is irreducible, then $f(B)$ is irreducible by assumption. Thus, by Lemma 1.14, $\rho(\mathbb{Z}, S) \leq \rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$. This combined with Lemma 2.4 concludes the proof. □

So for the block monoid $\mathcal{B}(\mathbb{Z}, S)$, if these two conditions are met, then we can factor out by $\mathbb{Z}m$ without changing the elasticity. The next theorem will use this result to determine exactly when $\mathcal{B}(\mathbb{Z}, S)$ is half-factorial. If $\rho(\mathbb{Z}, S) = 1$, then the conditions to Theorem 2.5 can be proved to hold. So $\mathcal{B}(\mathbb{Z}, S)$ is half-factorial if and only if those two conditions hold and $\mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$ is half-factorial.

Theorem 2.6. *Let $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\} \subseteq \mathbb{Z}$ with each $m_i, n_j > 0$. Define $c = \ll m_1, \ldots, m_k \gg$ and $m = lcm(m_1, \ldots, m_k)$. Then $\rho(\mathbb{Z}, S) = 1$ if and only if the following three conditions are true:*

1. *$c|n_i$ for all $i \in \mathbb{N}$.*
2. *$\rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}) = 1$.*

3.

$$\prod_{i=1}^{k}(-m_i)^{s_i}\cdot\prod_{j=1}^{\infty}n_j^{r_j}\in\mathcal{A}(\mathbb{Z},S)\Rightarrow\prod_{j=1}^{\infty}\overline{n_j}^{r_j}\in\mathcal{A}(\mathbb{Z}/m\mathbb{Z},\overline{S^+}).$$

Proof. ($\Leftarrow$) Suppose all three conditions are met. Then by the first and third condition and using Theorem 2.5, we have $\rho(\mathbb{Z},S)=\rho(\mathbb{Z}/m\mathbb{Z},\overline{S^+})$. But then by condition 2, we know that $\rho(\mathbb{Z},S)=1$.

($\Rightarrow$) Suppose that $\rho(\mathbb{Z},S)=1$.

1. This was proved in [4].
2. Since $c|n_i$ for all $i\in\mathbb{N}$, then Theorem 2.5 proves that $\rho(\mathbb{Z}/m\mathbb{Z},\overline{S^+})=\rho(\mathbb{Z},S)=1$.
3. Let $B\in\mathcal{B}(\mathbb{Z},S)$. Then
$$B=\prod_{i=1}^{k}(-m_i)^{s_i}\cdot\prod_{j=1}^{\infty}n_j^{r_j}$$
for some $s_i,r_j\in\mathbb{N}_0$ with only finitely many of the r_j nonzero. Since B is a block, then
$$\sum_{i=1}^{k}s_im_i=\sum_{j=1}^{\infty}r_jn_j.$$
Note that c divides the right hand side of this equation because $c|n_j$ for all $j\in\mathbb{N}$. By Lemma 2.3, $m=dc$ for some d with $(d,c)=1$, and $d|m_i$ for all $1\le i\le k$. Then d divides the left hand side of the equation. Since $(c,d)=1$, then m must divide both sides of the equation. Therefore we have
$$\sum_{i=1}^{k}s_im_i=\sum_{j=1}^{\infty}r_jn_j\in\mathbb{N}_0m.$$
Hence
$$B'=\prod_{j=1}^{\infty}\overline{n}_j^{r_j}$$
is a block in $\mathcal{B}(\mathbb{Z}/m\mathbb{Z},\overline{S^+})$. Suppose that B' is not irreducible. Since block monoids are atomic, then for some $h\in\mathbb{N}$ with $h>1$, there exist $C_1',\dots,C_h'\in\mathcal{A}(\mathbb{Z}/m\mathbb{Z},\overline{S^+})$, with
$$B'=C_1'\cdots C_h'.$$
Then for each $1\le a\le h$,
$$C_a'=\prod_{j=1}^{\infty}\overline{n}_j^{(r_j^a)},$$
for some $r_j^a\in\mathbb{N}_0$ (note the a is an indexing superscript, not a power) and
$$r_j^1+r_j^2+\cdots+r_j^h=r_j$$
for each $j\in\mathbb{N}$. Now we want to consider m copies of the irreducible block B in $\mathcal{B}(\mathbb{Z},S)$, and show this can be factored into hm irreducible blocks which means $\rho(\mathbb{Z},S)>\frac{hm}{m}=$

h, a contradiction. Notice that

$$\begin{aligned}
B^m &= \left(\prod_{i=1}^{k}(-m_i)^{s_i}\cdot\prod_{j=1}^{\infty}n_j^{r_j}\right)^m \\
&= \prod_{i=1}^{k}(-m_i)^{ms_i}\cdot\left(\prod_{j=1}^{\infty}n_j^{r_j}\right)^m \\
&= (-m_1)^{ms_1}\cdots(-m_k)^{ms_k}\cdot\left(\prod_{j=1}^{\infty}n_j^{r_j}\right)^m \\
&= \left((-m_1)^{m/m_1}\right)^{m_1s_1}\cdots\left((-m_k)^{m/m_k}\right)^{m_ks_k}\cdot\left(\prod_{j=1}^{\infty}n_j^{r_j}\right)^m.
\end{aligned}$$

Note $\frac{m}{m_i}\cdot(-m_i) = -m$ for each $1 \le i \le k$. So we can split the product

$$\left((-m_1)^{m/m_1}\right)^{m_1s_1}\cdots\left((-m_k)^{m/m_k}\right)^{m_ks_k}$$

into $x = m_1s_1 + \cdots + m_ks_k$ sub-products that when summed equal m. Label these as α_1 to α_x. So we have

$$\alpha_1 = m_1^{m/m_1}, \ldots, \alpha_{m_1s_1} = m_1^{m/m_1},$$

$$\alpha_{m_1s_1+1} = m_2^{m/m_2}, \ldots, \alpha_{m_1s_1+m_2s_2} = m_2^{m/m_2},$$

$$\cdots$$

$$\alpha_{x-m_ks_k} = m_h^{m/m_k}, \ldots, \alpha_x = m_k^{m/m_k}.$$

Then we can rewrite B^m as

$$B^m = \alpha_1\cdots\alpha_x\cdot\left(\prod_{j=1}^{\infty}n_j^{r_j}\right)^m.$$

Now we will consider the second half of this equation: $\left(\prod_{j=1}^{\infty}n_j^{r_j}\right)^m$. Define $C_1, \ldots, C_h$ by

$$C_a = \prod_{j=1}^{\infty}n_j^{(r_j^a)}$$

for each $1 \le a \le h$. Note that the C_a and the α_i are products, but not blocks. But then

$$C_1\ldots C_h = \prod_{a=1}^{h}\left(\prod_{j=1}^{\infty}n_j^{(r_j^a)}\right) = \prod_{j=1}^{\infty}n_j^{r_j}$$

since $r_j^1 + r_j^2 + \cdots + r_j^h = r_j$ for all $j \in \mathbb{N}$ (we can freely interchange any of these sums

and products because only finitely many of the r_j's are nonzero). So we have that

$$C_1^m \dots C_h^m = \left(\prod_{j=1}^{\infty} n_j^{r_j} \right)^m .$$

So our block B^m is equivalent to

$$B^m = \alpha_1 \cdots \alpha_x \cdot C_1^m \cdots C_h^m.$$

Now we would like to group the right hand side of this equation into irreducible blocks. Since C_j' is an irreducible block in $\mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$ for $1 \leq j \leq h$, then

$$\sum_{j=1}^{\infty} r_j^a n_j = t_a m$$

for some $t_a \in \mathbb{N}$ with no proper subsum in $\mathbb{N}m$. Thus,

$$\alpha_1 \cdots \alpha_{t_1} \cdot C_1$$

is an irreducible block since treating it as a sum we have each α_y equals $-m$ and C_1 sums to $t_1 m$. Call this irreducible block D_1. Then we can create an irreducible block D_2 by the same method; so

$$D_2 = \alpha_{t_1+1} \cdots \alpha_{2t_1} \cdot C_1.$$

Continuing, we can create up to

$$D_m = \alpha_{(m-1)t_1+1} \cdots \alpha_{mt_1} \cdot C_1.$$

We can repeat this process for the copies of C_2 creating D_{m+1} to D_{2m}:

$$D_{m+1} = \alpha_{mt_1+1} \cdots \alpha_{mt_1+t_2} \cdot C_1,$$

$$\vdots$$

$$D_{2m} = \alpha_{mt_1+(m-1)t_2+1} \cdots \alpha_{mt_1+mt_2} \cdot C_1.$$

We would like to continue this process up through all m copies of each C_a creating irreducible blocks D_1 up to D_{hm}. We need to show that we have the correct number of α_j's to cancel exactly with the m copies of each C_a. Then we need to have exactly $mt_1 + mt_2 + \cdots + mt_h$ such α_j's to cancel the m copies of each C_a. But we have $x = m_1 s_1 + m_2 s_2 + \cdots m_k s_k$ such α_j's. So we need to show that

$$m_1 s_1 + m_2 s_2 + \cdots m_k s_k = mt_1 + mt_2 + \cdots mt_h.$$

Recall that B is an irreducible block and

$$B = \prod_{i=1}^{k} (-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j},$$

so we know that

$$\sum_{i=1}^{k} s_i m_i = \sum_{j=1}^{\infty} r_j n_j.$$

Recall that $\sum_{a=1}^{h} r_j^a = r_j$ by definition of each C_a. Thus, the right hand side of the equation above can be rewritten as

$$\sum_{i=1}^{k} s_i m_i = \sum_{j=1}^{\infty} \left(\sum_{a=1}^{h} r_j^a \right) n_j.$$

Once again we can change the order of summation on the right hand side since only a finite number of these terms is nonzero to get:

$$\sum_{i=1}^{k} s_i m_i = \sum_{a=1}^{h} \left(\sum_{j=1}^{\infty} r_j^a n_j \right).$$

Each $\sum_{j=1}^{\infty} r_j^a n_j$ corresponds to the sum of C_a and equals $t_a m$. Then the equation becomes

$$\sum_{i=1}^{k} s_i m_i = \sum_{a=1}^{h} t_a m.$$

Thus we have shown that we have the correct number of α_j's. Therefore we have shown that

$$B^m = D_1 \cdots D_{mh}$$

are two equivalent factorizations of irreducible blocks in $\mathcal{B}(\mathbb{Z}, S)$. Since $h > 1$, then $\rho(\mathbb{Z}, S) \geq \frac{hm}{m} > 1$, a contradiction. Thus

$$B' = \prod_{j=1}^{\infty} \overline{n}_j^{r_j}$$

must be an irreducible block in $\mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$.

□

Note that condition 3 in the above theorem can also be stated that if $B = \prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j}$ is irreducible in $\mathcal{B}(\mathbb{Z}, S)$, then no proper subsum of $\sum_{j=1}^{\infty} r_j n_j$ is in $\mathbb{N}m$. Condition 3 will always hold if $\{m_1, \ldots, m_k\}$ is m-split.

Lemma 2.7. *Let $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\} \subseteq \mathbb{Z}$ with each $m_i, n_j > 0$. Define $c = \ll m_1, \ldots, m_k \gg$ and $m = lcm(m_1, \ldots, m_k)$. Let $\{m_1, \ldots, m_k\}$ be m-split. Then*

$$\prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{A}(\mathbb{Z}, S) \Rightarrow \prod_{j=1}^{\infty} \overline{n_j}^{r_j} \in \mathcal{A}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}).$$

Proof. Let $B = \prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{A}(\mathbb{Z}, S)$. In the proof of Theorem 2.5, we already proved that $\prod_{j=1}^{\infty} \overline{n_j}^{r_j} \in \mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$. Suppose it is not an irreducible block.

Then there exists $0 \leq r'_j \leq r_j$ for all $j \in \mathbb{N}$ (where at least one such $r'_j \neq r_j$ and at least one $r'_j > 0$), such that $\sum_{j=1}^{\infty} r'_j n_j = am$ for some $a \in \mathbb{N}$. Since $\{m_1, \ldots, m_k\}$ is m-split, Lemma 1.21 proves there exists $0 \leq s'_i \leq s_i$ such that $\sum_{i=1}^{k} s'_i m_i = am$. Therefore $B' = \prod_{i=1}^{k}(-m_i)^{s'_i} \cdot \prod_{j=1}^{\infty} n_j^{r'_j} \in \mathcal{B}(\mathbb{Z}, S)$. But B' divides B and $B' \neq B$, a contradiction since B is irreducible. Thus $\prod_{j=1}^{\infty} \overline{n_j}^{r_j}$ is an irreducible block. □

Since m in the previous theorems is defined as the least common multiple of the m_i's, then each $m_i | m$. Then by Corollary 1.23, $\{m_1, \ldots, m_k\}$ is m-split is equivalent to $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$. Therefore if $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$, then the condition in the preceding theorems about irreducible blocks will be satisfied. So Theorems 2.5 and 2.6, can be repeated without this condition. Notice that $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$ is not part of the if and only if in the following theorem. It is possible to have $\rho(\mathbb{Z}, S) = 1$ and $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) \neq 1$.

Corollary 2.8. *Let $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\} \subseteq \mathbb{Z}$ with each $m_i, n_j > 0$. Define $c = \ll m_1, \ldots, m_k \gg$ and $m = lcm(m_1, \ldots, m_k)$. Suppose that $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$. If $c | n_i$ for all $i \in \mathbb{N}$, then $\rho(\mathbb{Z}, S) = \rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$. In particular, the following are equivalent:*

1. *$\rho(\mathbb{Z}, S) = 1$.*
2. *$c | n_i$ for all $i \in \mathbb{N}$ and $\rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}) = 1$.*

Proof. Since $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$, then Corollary 1.23 gives that $\{m_1, \ldots, m_k\}$ is m-split. Then Lemma 2.7 proves that

$$\prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{A}(\mathbb{Z}, S) \Rightarrow \prod_{j=1}^{\infty} \overline{n_j}^{r_j} \in \mathcal{A}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}).$$

Then by Theorem 2.5, if $c | n_i$ for all $i \in \mathbb{N}$, then $\rho(\mathbb{Z}, S) = \rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$. Now the "if and only if" statement will be proved. Note that $c | n_i$ for all $i \in \mathbb{N}$ is being proven, not assumed, for the "if and only if" statement.

($\Rightarrow$) This was proven in Theorem 2.6.

($\Leftarrow$) Suppose that $c | n_i$ for all $i \in \mathbb{N}$ and $\rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}) = 1$. It has already been shown that

$$\prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{A}(\mathbb{Z}, S) \Rightarrow \prod_{j=1}^{\infty} \overline{n_j}^{r_j} \in \mathcal{A}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}).$$

Then the three conditions from Theorem 2.6 are satisfied so $\rho(\mathbb{Z}, S) = 1$. □

When $k = 1$ or $k = 2$ the theorems and corollaries can be simplified considerably. For any two-element set $\{a, b\}$, if $a | c$ and $b | c$, the set is always c-split, proved in the following lemma.

Lemma 2.9. *If $a | c$ and $b | c$, then $\{a, b\}$ is c-split.*

Proof. Suppose that $ra + sb = tc$, where $r, s, t \in \mathbb{N}_0$. Suppose $r \geq \frac{c}{a}$. Define $r' = \frac{c}{a}$ and $s' = 0$. Then $r' < r$ and $s' < s$ with $r'a + s'b = c$. Similary, if $s \geq \frac{c}{b}$ let $r' = 0$ and $s' = \frac{c}{b}$, and we have $r'a + s'b = c$. Now suppose that both $r < \frac{c}{a}$ and $s < \frac{c}{b}$. Then

$$tc = ra + sb < \frac{c}{a} \cdot a + \frac{c}{b} \cdot b = 2c.$$

Therefore, we must have $t = 1$. These cases are exhaustive, so we have that $\{a, b\}$ is c-split. □

Then if $S = \{-m_1, -m_2, n_1, n_2, n_3, \ldots\}$, the lemma proves that $\{m_1, m_2\}$ is always m-split. Then by Corollary 1.23, $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \overline{m_2}\}) = 1$ always holds. Thus, when $k = 2$, Corollary 2.8 can be rewritten without the assumption that $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \overline{m_2}\}) = 1$. So for the case $k = 2$, we have the following theorem about the elasticity of $\mathcal{B}(\mathbb{Z}, S)$. The "in particular" part of the theorem was proven in [4].

Theorem 2.10. *Let $S = \{-m_1, -m_2, n_1, n_2, n_3, \ldots\} \subseteq \mathbb{Z}$ with each $m_i, n_j > 0$. Define $c = \ll m_1, m_2 \gg$ and $m = lcm(m_1, m_2)$. If $c|n_i$ for all $i \in \mathbb{N}$, then $\rho(\mathbb{Z}, S) = \rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$. In particular, the following are equivalent:*

1. *$\rho(\mathbb{Z}, S) = 1$.*
2. *$c|n_i$ for all $i \in \mathbb{N}$ and $\rho(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}) = 1$.*

The case $k = 1$ is even simpler than $k = 2$. In this case $c = \ll m_1 \gg = 1$ by definition. But the requirement that $c|d_i$ for all $i \in \mathbb{N}$ is then trivially satisfied. Also, $m = m_1$ so it is trivial that the set $\{m_1\}$ is m-split. So we can again repeat Corollary 2.8, but can now also drop the condition that $c|d_i$ for all $i \in \mathbb{N}$. This case does not actually require that S^+ is infinite like all the previous theorems in this section. The following theorem was proved in [5].

Theorem 2.11. *Let $S = \{-m_1\} \cup A$ with $m_1 \in \mathbb{N}$ and $A \subseteq \mathbb{N}$. Then $\rho(\mathbb{Z}, S) = \rho(\mathbb{Z}/m_1\mathbb{Z}, \overline{A})$. In particular, $\rho(\mathbb{Z}, S) = 1$ if and only if $\rho(\mathbb{Z}/m_1\mathbb{Z}, \overline{A}) = 1$.*

So we can clearly see the conditions that need to be considered as we generalize from 1, to 2, to finitely many negative elements in S. As we move from 1 to 2, we need to consider whether $c|n_i$ for all $i \in \mathbb{N}$. As we move to finitely many negative elements, we then need to consider whether the following condition holds:

$$\prod_{i=1}^{k}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{A}(\mathbb{Z}, S) \Rightarrow \prod_{j=1}^{\infty} \overline{n_j}^{r_j} \in \mathcal{A}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+}).$$

If $\rho(\mathbb{Z}/m\mathbb{Z}, \{\overline{m_1}, \ldots, \overline{m_k}\}) = 1$ (which is equivalent to $\{m_1, \ldots, m_k\}$ is m-split), then this condition will be true.

For a torsion group G, define $k : \mathcal{B}(G, S) \to \mathbb{Q}^+ \cup \{0\}$ by

$$k\left(\prod_{s \in S} s_s^r\right) = \sum_{s \in S} \frac{r_i}{ord(s)}.$$

If $B \in \mathcal{B}(G, S)$, then $k(B)$ is called the *cross number* of B. Note that k is a semi-length function. It was proven by Zaks [20] and Skula [17] independently that $\mathcal{B}(G, S)$ has elasticity one if and only if the cross number is 1 for each irreducible block. Theorem 2.6 shows that if the block monoid $\mathcal{B}(\mathbb{Z}, S)$ has elasticity one, then the irreducible blocks exactly correspond to irreducible blocks in the block monoid $\mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$. This group is torsion; so the cross number can be applied. Then we can translate this cross number calculation back to the group $\mathbb{Z}$ to get an analog of the cross number when $G = \mathbb{Z}$. This was already done for

$k = 2$ in [4]. That paper also listed $c|n_i$ as a condition in the if and only if statement, but that is unnecessary since it can be proven. This paper also refers to the function as the Zaks-Skula function, which is another commonly used name. Then we can define a cross number function for block monoids with class group $\mathbb{Z}$.

Definition 2.12. Let $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\} \subseteq \mathbb{Z}$ with each $m_i, n_j > 0$. Define $m = lcm(m_1, \ldots, m_k)$. Define the function $k_m : \mathcal{B}(\mathbb{Z}, S) \to \mathbb{Q}^+$ by

$$k_m(B) = \sum_{j=1}^{\infty} \frac{r_j n_j}{lcm(n_j, m)}$$

for all blocks $B = \prod_{i=1}^{h}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{B}(\mathbb{Z}, S)$.

Theorem 2.13. *Let $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\} \subseteq \mathbb{Z}$ with each $m_i, n_j > 0$. Define $c = \ll m_1, \ldots, m_k \gg$ and $m = lcm(m_1, \ldots, m_k)$. Then $\rho(\mathbb{Z}, S) = 1$ if and only if $k_m(B) = 1$ for all irreducible blocks $B \in \mathcal{B}(\mathbb{Z}, S)$.*

Proof. ($\Rightarrow$) Suppose $\rho(\mathbb{Z}, S) = 1$. Let

$$B = \prod_{i=1}^{h}(-m_i)^{s_i} \cdot \prod_{j=1}^{\infty} n_j^{r_j} \in \mathcal{B}(\mathbb{Z}, S)$$

be an irreducible block. Then by Theorem 2.6 we know that

$$B' = \prod_{j=1}^{\infty} \overline{n}_j^{r_j}$$

is an irreducible block in $\mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$. But Theorem 2.6 tells us that this block monoid must have elasticity one. Since $\mathcal{B}(\mathbb{Z}/m\mathbb{Z}, \overline{S^+})$ has a torsion group, then

$$1 = k(B') = \sum_{j \in I}^{\infty} \frac{r_j}{ord_{\mathbb{Z}/m\mathbb{Z}}(\overline{n_i})}$$

Thus

$$k_m(B) = \sum_{j=1}^{\infty} \frac{r_j n_j}{lcm(n_j, m)} = 1$$

since

$$ord_{\mathbb{Z}/m\mathbb{Z}}(\overline{n_i}) = \frac{lcm(n_j, m)}{n_j}.$$

($\Leftarrow$) k_m as defined is a semi-length function. Since we have a semi-length function and each irreducible block is sent to 1, then $\rho(\mathbb{Z}, S) = 1$ by Lemma 1.18. □

2.2 $\mathbb{Z}(p^\infty)$

Now we will turn our attention to another infinite abelian group, $\mathbb{Z}(p^\infty)$. This group is the subset of $\mathbb{Q}/\mathbb{Z}$ defined for a prime number p as $\mathbb{Z}(p^\infty) := \{\frac{m}{p^n} + \mathbb{Z} \mid m \in \mathbb{Z}, n \geq 0\}$. We

will assume throughout this section that $p \in \mathbb{N}$ is prime. Define $H_{n,p} := H_n =< \frac{1}{p^n} + \mathbb{Z} >$. Then

$$\{0\} = H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq H_3 \subsetneq H_4 \cdots$$

and $\cup_{n=1}^{\infty} H_n = \mathbb{Z}(p^\infty)$. We will study $S \cap H_n$ to learn about the factorization of the block monoid $\mathcal{B}(G, S)$. Define $S_n = S \cap H_n$ for all $n \in \mathbb{N}_0$. Since $H_n \subsetneq H_{n+1}$, then $S_n \subset S_{n+1}$ for all $n \in \mathbb{N}_0$. So we can get the following lemma.

Lemma 2.14. *Let $S \subseteq \mathbb{Z}(p^\infty)$, and define $S_n = S \cap H_n$ for all $n \in \mathbb{N}_0$. If $m < n$, then $\rho(\mathbb{Z}(p^\infty), S_m) \leq \rho(\mathbb{Z}(p^\infty), S_n)$.*

Proof. This follows directly from Lemma 1.15 since $S_n \subseteq S_{n+1} \subseteq \cdots \subseteq S_m$. □

Since $\mathbb{Z}(p^\infty) = \cup_{n=0}^{\infty} H_0$, then $S = \cup_{i \in \mathbb{N}_0} S_i$. Then we can compare the elasticity for the block monoid with S to the block monoid for each S_n using Lemma 1.16.

Lemma 2.15. *Let $S \subseteq \mathbb{Z}(p^\infty)$, and define $S_n = S \cap H_n$ for all $n \in \mathbb{N}_0$. Then*

$$\rho(\mathbb{Z}(p^\infty), S) = \lim_{n \to \infty} \rho(\mathbb{Z}(p^\infty), S_n) = \sup_{n \in \mathbb{N}} \rho(G, S_n).$$

In particular, $\rho(\mathbb{Z}(p^\infty), S) = 1$ if and only if $\rho(\mathbb{Z}(p^\infty), S_n)$ for all $n \in \mathbb{N}_0$.

Proof. Since $S_n \subseteq S_{n+1}$ for all $n \in \mathbb{N}$ and $S = \cup_{n \in \mathbb{N}} S_n$, then

$$\rho(\mathbb{Z}(p^\infty), S) = \lim_{n \to \infty} \rho(\mathbb{Z}(p^\infty), S_n) = \sup_{i \in \mathbb{N}_0} \rho(G, S_i).$$

follows from Lemma 1.16.

The "in particular" statement is simple. Each $\rho(\mathbb{Z}(p^\infty), S_n) \geq 1$. Then $\rho(\mathbb{Z}(p^\infty), S) = \sup_{i \in \mathbb{N}_0} \rho(G, S_i) = 1$ if and only if $\rho(\mathbb{Z}(p^\infty), S_n) \geq 1$ for all $n \in \mathbb{N}_0$. □

Therefore if $\mathcal{B}(\mathbb{Z}(p^\infty), S)$ has elasticity one, then each block monoid $\mathcal{B}(\mathbb{Z}(p^\infty), S_n)$ must have elasticity one. But $S_n \subseteq H_n$, a subgroup of $(\mathbb{Z}(p^\infty)$. Then $\mathcal{B}(\mathbb{Z}(p^\infty), S_n)$ and $\mathcal{B}(H_n, S_n)$ are essentially the same block monoid. Notice that each H_n is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. The isomorphism is obviously multiplication by p^n. So $\frac{a}{p^n} + \mathbb{Z} \mapsto a + p^n\mathbb{Z}$. Now using this isomorphism, define T_n to be the image of S_n:

$$T_n = \left\{ a + p^n\mathbb{Z} \mid \frac{a}{p^n} + \mathbb{Z} \in S_n \right\}.$$

Then the following lemma is easy since the groups are isomorphic and that isomorphism sends S_n to T_n.

Lemma 2.16. *Let $S \subseteq \mathbb{Z}(p^\infty)$, and define $S_n = S \cap H_n$ for all $n \in \mathbb{N}_0$. The map $\varphi : H_n \to \mathbb{Z}/p^n\mathbb{Z}$ defined by $\varphi(h + \mathbb{Z}) = p^n h$ is an isomorphism. As above, define $T_n = \varphi(S_n)$, and therefore $\rho(\mathbb{Z}(p^\infty), S_n) = \rho(\mathbb{Z}/p^n\mathbb{Z}, T_n)$.*

Combining Lemma 2.16 with Lemma 2.15, the elasticity of $\rho(\mathbb{Z}(p^\infty), S)$ is 1 if and only if the elasticity of each block monoid $\rho(\mathbb{Z}/p^n\mathbb{Z}, T_n)$ is 1. So we will turn our attention to determining when a block monoid with divisor class group $\mathbb{Z}/p^n\mathbb{Z}$ is half-factorial. If a Krull domain has divisor class group $\mathbb{Z}/p^n\mathbb{Z}$, then the set of classes containing a height-one prime ideal must contain an element that is relatively prime to p. Otherwise, the set of classes

containing a height-one prime ideal will not generate the group as a monoid. So we will include this assumption in the theorem.

Theorem 2.17. *Let $n \in \mathbb{N}$ and $S \subset \mathbb{Z}/p^n\mathbb{Z}$ such that $\overline{a} \in S$ with $(a,p) = 1$ for some $a \in \mathbb{N}$. Then $\rho(\mathbb{Z}/p^n\mathbb{Z}, S) = 1$ if and only if $S \subseteq \{0, a, pa, p^2a, ..., p^{n-1}a\}$.*

Proof. ($\Leftarrow$) Suppose B is an irreducible block in $\mathcal{B}(\mathbb{Z}/p^n\mathbb{Z}, S)$. Then $B = \prod_{i=0}^{n-1}(p^i a)^{m_i}$ for some $m_i \in \mathbb{N}_0$. Then

$$\sum_{i=0}^{n-1} m_i p^i a = rp^n$$

for some $r \geq 0$. Since $(a,p) = 1$ and $a\sum_{i=0}^{n-1} m_i p^i = rp^n$, then p^n must divide $\sum_{i=0}^{n-1} m_i p^i$. Then

$$m_0 + m_1 p + \cdots + m_{n-1}p^{n-1} = tp^n$$

where $t = \frac{r}{a}$. We will show that $m_0 + m_1 p + \cdots + m_{n-1}p^{n-1} \leq p^n$; so t must be 1. Suppose $m_{n-1}p^{n-1} > p^n$, and thus $m_{n-1} > p$. Then $C = (p^{n-1}a)^p$ is a block dividing B, and not equal to B. But B is irreducible, so this is a contradiction. Therefore, $m_{n-1}p^{n-1} \leq p^n$.

Suppose

$$\sum_{i=k}^{n-1} m_i p^i \leq p^n$$

and

$$\sum_{i=k-1}^{n-1} m_i p^i > p^n.$$

Then

$$\begin{aligned} m_{k-1}p^{k-1} &> p^n - \sum_{i=k}^{n-1} m_i p^i \\ &= p^{k-1}(p^{n-k+1} - \sum_{i=k}^{n-1} m_i p^{i-k+1}) \\ &= p^{k-1}s \end{aligned}$$

for some integer s. Since $p^n - \sum_{i=k}^{n-1} m_i p^i \geq 0$, then $s \geq 0$. So $m_{k-1} \geq s$, and we have that

$$sp^{k-1} + \sum_{i=k}^{n-1} m_i p^i = p^n.$$

Therefore, $s * p^{k-1}a + \sum_{i=k}^{n-1} m_i p^i a \in \mathbb{Z}p^n$. Then $D = (p^{k-1}a)^s \cdot \prod_{i=k}^{n-1}(p^i a)^{m_i}$ is a block dividing, but not equal to, B. But B is irreducible, so this is a contradiction. So

$$\sum_{i=k-1}^{n-1} m_i p^i \leq p^n.$$

By induction we have that

$$\sum_{i=1}^{n-1} m_i p^i \leq p^n.$$

Then $t = 1$ (If $t = 0$, we would have an empty sum since the m_i's are nonnegative.) Therefore,

$$1 = \sum_{i=1}^{n-1} \frac{m_i p^i}{p^n} = \sum_{i=1}^{n-1} \frac{m_i}{p^{n-i}} = \sum_{i=1}^{n-1} \frac{m_i}{ord(p^i a)} = k(B),$$

the cross number on B. Since the cross number is one for each irreducible block, then $\rho(\mathbb{Z}/p^n\mathbb{Z}, S) = 1$.

($\Rightarrow$) Let $0 \neq \overline{b} \in S$. Then $-b \equiv ra (\text{mod } n)$ for some $r \in \{1, 2, 3, ..., p^n - 1\}$ since $\overline{a}$ generates the group $\mathbb{Z}/p^n\mathbb{Z}$. Therefore, $B = \overline{b} \cdot \overline{a}^r$ is a block in $\mathcal{B}(\mathbb{Z}/p^n\mathbb{Z})$ since $\overline{b} + r\overline{a} = 0$. This block is irreducible because the r chosen above is unique. So no proper subproduct containing b will sum to 0. Also, $r < p^{n-1}$; so no subproduct containing only copies of a can have a sum that is equal to 0 since the order of $\overline{a}$ in $\mathbb{Z}/p^n\mathbb{Z}$ is p^n.

Therefore, $B = \overline{b} \cdot \overline{a}^r$ is a irreducible block. Note $|\overline{b}| = p^i$ for some $1 \leq i \leq n$. Also, $r = p^j r'$ for some $1 \leq j < p^n$ and (r', p)=1. Then $C = \overline{b}^{p^i}$ and $D = \overline{a}^{p^n}$ are irreducible blocks. Then $B^{p^{i+n-j}} = C^{p^{n-j}} D^{p^i r'}$ are two equivalent factorizations into irreducible blocks. Then we must have $p^{i+n-j} = p^{n-j} + p^i r'$.

Suppose $i > n - j$, then

$$p^{i+n-j} = p^{n-j}(1 + p^{i-n+j} r') \Leftrightarrow p^i = (1 + p^{i-n+j} r'),$$

and thus

$$1 = p^i - p^{i-n+j} r' = p^{i-n+j}(p^{n-j} - r').$$

Since $i > n - j$, then $p|1$, which is a contradiction.

Suppose that $i \leq n - j$. Then

$$\begin{aligned} p^{i+n-j} - p^{n-j} = p^i r' &\Leftrightarrow p^{n-j}(p^i - 1) = p^i r' \\ &\Leftrightarrow p^{n-j-i}(p^i - 1) = r'. \end{aligned}$$

If $i < n - j$, then $p|r'$, which is a contradiction. If $i = n - j$, then $p^i - 1 = r'$. Then $-b \equiv ra \equiv p^j(p^i - 1)a (\text{mod } p^n)$. Since $|b| = p^i$, then $b + (p^i - 1)b \equiv p^i b \equiv 0 (\text{mod } p^n)$. Thus $(p^i - 1)b \equiv -b \equiv p^j(p^i - 1)a (\text{mod} p^n)$. Then $(p^i - 1)b - p^j(p^i - 1)a = cp^n$ for some $c \in \mathbb{Z}$. Then $(p^i - 1)(b - p^j a) = cp^n$. Therefore, $p^n|(p^i - 1)(b - p^j a)$. Then $p^n|(b - p^j a)$ since p is prime and does not divide $p^i - 1$. Therefore $b \equiv p^j a (\text{mod } p^n)$. Thus we have shown that $S \subseteq \{0, a, pa, p^2 a, ..., p^{n-1} a\}$. □

Now we are able to state the theorem showing when a Krull domain with divisor class group $(\mathbb{Z}(p^\infty), S)$ is an HFD.

Theorem 2.18. *Let R be a Krull domain with divisor class group $\mathbb{Z}(p^\infty)$ and $S \subseteq \mathbb{Z}(p^\infty)$ the set of classes containing a height-one prime ideal of R. Then R is an HFD if and only if S is infinite and for all $\frac{a}{p^n} + \mathbb{Z}, \frac{b}{p^m} + \mathbb{Z} \in S$ with $(a, p) = 1 = (b, p)$ and $n \leq m$, then $b \equiv a (mod\ p^n)$.*

Proof. ($\Rightarrow$) Suppose S is finite. Then clearly it cannot generate $(\mathbb{Z}(p^\infty), S)$ as a monoid, so we have a contradiction. Thus S must be infinite. Define $S_n = S \cap H_n$ for all $n \in \mathbb{N}_0$. Suppose there exists such an $\frac{a}{p^n} + \mathbb{Z}, \frac{b}{p^m} + \mathbb{Z} \in S$ with $n \leq m$ and $(a, p) = 1$ and $(b, p) = 1$. Then $\frac{a}{p^n} + \mathbb{Z}, \frac{b}{p^m} + \mathbb{Z} \in S_m$, and $p^{m-n}a + p^n\mathbb{Z}, b + p^n\mathbb{Z} \in T_m$. But

$$1 = \rho(\mathbb{Z}(p^\infty), S) \geq \rho(\mathbb{Z}(p^\infty), S_m) = \rho(\mathbb{Z}/p^n\mathbb{Z}, T_m);$$

so $\rho(\mathbb{Z}/p^n\mathbb{Z}, T_m) = 1$. Thus $T_m \subseteq \{0, c, pc, p^2c, ..., p^{n-1}c\}$ for some $(c, p) = 1$ with $c \in T_n$ by Theorem 2.17. Since $b \in T_m \subseteq \{0, c, pc, p^2c, ..., p^{n-1}c\}$ and $(b, p) = 1$, then $b \equiv c (\text{mod } p^n)$. So $T_m \subseteq \{0, b, pb, p^2b, ..., p^{n-1}b\}$. Then $p^{m-n} \equiv p^j b (\text{mod } p^m)$. Since $(a, p) = 1$ and $(b, p) = 1$, then $p^{m-n}a$ has order p^n and $p^j b$ has order p^{m-j} in $\mathbb{Z}/p^n\mathbb{Z}$. Thus $p^n = p^{m-j}$, so $n = m-j$ which is equivalent to $j = m - n$. So $p^{m-n}a - p^{m-n}b \in p^m\mathbb{Z}$. Therefore, $a - b \in p^n\mathbb{Z}$, and $a \equiv b (\text{mod } n)$.

($\Leftarrow$) First we need to show S generates $\mathbb{Z}(p^\infty)$ as a monoid. Let $\frac{d}{p^n} + \mathbb{Z} \in \mathbb{Z}(p^\infty)$. Suppose there is no $\frac{e}{p^m} + \mathbb{Z} \in S$ with $(e, p) = 1$ and $m \geq n$. Then when all the fractions of S are reduced, none of the elements have a denominator with a power of p greater than or equal to n. But this means that S must be finite since we are in $\mathbb{Z}(p^\infty)$, which is a contradiction. Thus, such an $\frac{e}{p^m} \in S$ must exist. Then $d \equiv re (\text{mod } p^m)$ for some r since $(e, p) = 1$. Thus, $d - re \in p^m\mathbb{Z}$; so $\frac{d}{p^m} - r\frac{e}{p^m} \in \mathbb{Z}$. Thus, $\frac{d}{p^n} - p^{n-m}r\frac{e}{p^m} \in \mathbb{Z}$. Therefore, in $\mathbb{Z}(p^\infty)$, $\frac{d}{p^n}$ is a positive multiple of an element of S, and we have that S generates $\mathbb{Z}(p^\infty)$ as a monoid.

If we show that $\rho(\mathbb{Z}(p^\infty), S^n) = 1$ for all $n > 0$, then R must be an HFD by 2.15. Let $n > 0$. When all the fractions in S are reduced, one of the elements must have a denominator with a higher power than n as above. So we must have some $\frac{b}{p^m} + \mathbb{Z} \in S_m$ for some $(b, p) = 1$ and $m > n$. Now for all the elements $0 \neq \frac{a}{p^k} + \mathbb{Z} \in S$ with $(a, p) = 1$ and $1 \leq k \leq m$, we know that $b \equiv a (\text{mod } p^k)$. But this means that $b - a = tp^k$ for some t, and thus $\frac{b}{p^k} - \frac{a}{p^k} \in \mathbb{Z}$. Hence, $\frac{b}{p^k} + \mathbb{Z} = \frac{a}{p^k} + \mathbb{Z}$ in $\mathbb{Z}(p^\infty)$. So we have shown that all the elements of S_m are either 0 or have the form $\frac{b}{p^k} + \mathbb{Z}$ for $1 < k \leq m$. Therefore, $\frac{b}{p^m} \in S_m$ and $S_n \subseteq \{0, \frac{b}{p}, \frac{b}{p^2}, \ldots, \frac{b}{p^m}\}$. Hence $b \in T_m$, and $T_m \subseteq \{0, b, pb, \ldots, p^{m-1}b\}$. Therefore, $\rho(\mathbb{Z}/p^n\mathbb{Z}, T^m) = 1$ by Theorem 2.17. Then using Lemma 2.16, $\rho(\mathbb{Z}(p^\infty), S^m) = 1$. Since $S_n \subset S_m$, then $\rho(\mathbb{Z}(p^\infty), S^n) = 1$, by Lemma 2.14. □

2.3 $\mathbb{Q}$

In this section, we will attempt to characterize the distributions of height-one prime ideals in divisor class group $\mathbb{Q}$ that result in the corresponding Krull domain being an HFD. First it is proved which subsets $S \subseteq \mathbb{Q}$ will have a corresponding Krull domain R with divisor class group $\mathbb{Q}$ and S is the set of classes containing a height-one prime ideal of R. Recall, when this is true we say the pair $(\mathbb{Q}, S)$ is realizable. The following lemma will be useful. Recall $[S]$ is the set generated by S as a monoid, i.e., $[S]$ is the set containing all sums of elements of S with positive coefficients only. In contrast, $< S >$ is the group generated by the elements of S, so elements of S can be subtracted. The first lemma is a technical condition to be used in Theorem 2.20.

Lemma 2.19. *Let $S \subseteq G$ where G is an abelian group. Suppose S generates G as a group (i.e., $< S >= G$) and for each $0 \neq s \in S$ there exists some $c \in \mathbb{N}$ such that $-cs$ is generated*

by S as a monoid (i.e., $-cs \in [S]$). Then S generates G as a monoid (i.e., $[S] = G$).

Proof. Suppose the conditions are satisfied. Let $g \in G$ and suppose that $g = r_1x_1 + \cdots + r_nx_n$, where each $x_i \in S$ and $r_i \in \mathbb{Z}$. If any of the r_i or x_i are 0, then drop this term from the sum and renumber. Now suppose some $r_i < 0$, call it r_1 without loss of generality. Then there exists $c_1 \in \mathbb{Z}^+$ such that $-c_1x_1 = a_1y_1 + \cdots + a_ky_k$, where each $y_i \in S$ and $a_i \in \mathbb{N}$. Then choose some $b_1 \in \mathbb{Z}$ such that $b_1c_1 > -r_1$. Then $g = (r_1 + bc_1)x_1 + \cdots + r_nx_n + ba_1y_1 + \cdots + ba_ky_k$. Note the coefficient of x_1 and the coefficients of each y_j are positive. This method can be repeated so the coefficient of each x_i is also positive. Thus, $g \in [S]$, and G can be generated by S as a monoid. □

The above lemma gives a simple condition to show that S generates G as a monoid. The condition in the lemma is actually close to Grams' original condition in [15] guaranteeing that a pair (G, S) has a corresponding Dedekind domain. The fact that S generates G as a monoid is an equivalent statement to Grams' condition. Now we will use Lemma 2.19 to prove a result when the pair $(\mathbb{Q}, S)$ is realizable by a Krull domain. The conditions in the following theorem are analagous to the conditions that Grams showed that the pair $(\mathbb{Z}, S)$ must satisfy to be realizable by a Dedekind domain [15].

Theorem 2.20. *Let $S \subseteq \mathbb{Q}$. Then $(\mathbb{Q}, S)$ is realizable by a Krull domain if and only if S generates $\mathbb{Q}$ as a group (i.e. $< S >= \mathbb{Q}$) and S contains both positive and negative elements.*

Proof. ($\Rightarrow$) $(\mathbb{Q}, S)$ is realizable as a Krull domain implies that S generates G as a monoid by Theorem 1.13. Then clearly S generates G as a group. Also, suppose S had only positive elements. Then each $s \in [S]$ is greater than 0. So $[S] \neq \mathbb{Q}$, which is a contradiction. Similarly, we get a contradiction if S contains only negative elements, since then each $s \in [S]$ is less than 0. So S contains both positive and negative elements.

($\Leftarrow$) Suppose $< S >= \mathbb{Q}$ and S contains both positive and negative elements. We will use Lemma 2.19 to show that S generates G as a monoid, and then Theorem 1.13 says that $(\mathbb{Q}, S)$ is realizable as a Krull domain. It is assumed that S generates G as a group. Let $a \in S$ with $a > 0$. Then $a = \frac{s}{r}$ for some $s, r \in \mathbb{N}$. Then there exists $b \in S$ such that $b < 0$. We can write b as $b = \frac{-n}{m}$ for some $n, m \in \mathbb{N}$. Then $-(nr)a = -(nr) \cdot \frac{s}{r} = (ms) \cdot \frac{-n}{m} = (ms)b \in [S]$. The case when $a < 0$ is done similarly. Thus the conditions of Lemma 2.19 are satisfied. □

Therefore we will focus on the case where $S^+ = \{q \in S \mid q > 0\}$ is infinite and $S^- = \{q \in S \mid q < 0\}$ contains at least one element. By a simple automorphism switching positives and negatives, the results for this case can also be applied to the case $|S^-| = \infty$ and S^+ has at least one element. The following theorem gives a nice way to construct an HFD with divisor class group $(\mathbb{Q}, S)$ where $|S^+| = \infty$ and $|S^-| = \infty$. This is in sharp contrast to the divisor class group $\mathbb{Z}$. Theorem 2.1 from [5], proved that for $T \subseteq \mathbb{Z}$ if $|T^+| = \infty$ and $|T^-| = \infty$, then the elasticity of $\mathcal{B}(\mathbb{Z}, T)$ is infinite, .

Theorem 2.21. *Let $\{n_1, n_2, \ldots\} \subseteq \mathbb{N}$ be a splittable set. Then the block monoid $\mathcal{B}(\mathbb{Q}, S)$, where $S^+ = \{\frac{1}{n_1}, \frac{1}{n_2}, \ldots\}$ and $S^- \subseteq \mathbb{Z}^-$, has elasticity one. If $(\mathbb{Q}, S)$ is realizable by a Krull domain, then any such Krull domain is an HFD.*

Proof. Define a semi-length function $f : \mathcal{B}(\mathbb{Q}, S) \to \mathbb{N}$, where each block is sent to the number of elements in S^- that it contains. It is clear that f is a homomorphism. Suppose

that $B \in \mathcal{B}(\mathbb{Q}, S)$ is irreducible. Then

$$B = \prod_{i=1}^{\infty} \left(\frac{1}{n_i}\right)^{a_i} \cdot \prod_{j=1}^{\infty} (-j)^{b_j}$$

for some $a_i, b_j \in \mathbb{N}_0$ with only finitely many of the a_i, b_j nonzero. Then

$$\sum_{i=1}^{\infty} a_i \frac{1}{n_i} = \sum_{j=1}^{\infty} b_j(j).$$

Suppose that $b_k \geq 1$ for some $k \geq 1$. Note that $\sum_{i=1}^{\infty} a_i \frac{1}{n_i} = \sum_{j=1}^{\infty} b_j(j) \in \mathbb{N}$, $b_k j \in \mathbb{N}$, and $\sum_{j=1}^{\infty} b_j(j) > b_k j$. Since S^+ is splittable, there exists $0 \leq a_i' \leq a_i$ for each $i \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} a_i' \cdot \frac{1}{n_i} = k$. Then $B' = (-k)^1 \cdot \prod_{i=1}^{\infty} a_i' \cdot \frac{1}{n_i}$ is a block dividing B. But B is irreducible; so we must have $B' = B$. Thus $f(B) = 1$. Since $f(B) = 1$ for all irreducible blocks $B \in \mathcal{B}(\mathbb{Q}, S)$, then $\mathcal{B}(\mathbb{Q}, S)$ is an HFD by Lemma 1.18. □

Before we use this theorem to generate examples, we need the following lemma that was proved by Geroldinger and Wao in [11].

Lemma 2.22. *Let $M \subset \mathbb{N}$ be a chain of divisors (i.e., $m, m' \in M$ with $m' < m$ implies $m' | m$). Then M is splittable.*

Example 2.23. The set $\{\frac{1}{n!} \mid n \in \mathbb{N}\}$ is a splittable set by Lemma 2.22. Then the block monoid $\mathcal{B}(\mathbb{Q}, S)$, where $S^+ = \{\frac{1}{n!} \mid n \in \mathbb{N}\}$ and $S^- = \mathbb{Z}^-$ has elasticity one by Theorem 2.21. Also, this pair is realizable as a Krull domain, and thus, any corresponding Krull domain is an HFD. This is an example where a Krull domain is an HFD and has divisor class group $(\mathbb{Q}, S)$ and S has infinitely many positive and negative elements.

Example 2.24. Notice that the condition in Theorem 2.21 that $\{n_1, n_2, \ldots\} \subseteq \mathbb{Z}^+$ is a splittable set guarantees (so is a stricter condition than) that $\rho(\mathbb{Q}, S^+ \cup \{-1\}) = 1$. However, we cannot relax the assumption in the theorem. The following is an example of a pair $(\mathbb{Q}, S)$ with $\rho(\mathbb{Q}, S^+ \cup \{-1\}) = 1$, and $\rho(\mathbb{Q}, S^+ \cup \mathbb{Z}^-) > 1$. Consider $(\mathbb{Q}, S)$, where $S^+ = \{\frac{1}{n!} | n \in \mathbb{Z}^+\} \cup \{5/2\}$ and $S^- = \mathbb{Z}^-$. Using Theorem 2.25, $\rho(\mathbb{Q}, S^+ \cup \{-1\}) = \rho(\mathbb{Q}, \{\frac{1}{n!} | n \in \mathbb{Z}^+\} \cup \{-1\})$ since both of these S sets are equivalent modulo $\mathbb{Z}$. By Theorem 2.21 this second pair has elasticity one. Therefore $\rho(\mathbb{Q}, S^+ \cup \{-1\}) > 1$.

Also, by the calculation $\{(-2)^2 \cdot \frac{5}{2} \cdot (1/2)^3\}\{-1 \cdot (1/2)^2\}^3 = \{-2 \cdot (1/2)^4\}^2\{(-1)^3 \cdot \frac{5}{2} \cdot (1/2)\}$, we have 4 irreducible blocks equivalent to 3 irreducible blocks in $\mathcal{B}(\mathbb{Q}, S)$ (the braces are used to separate the irreducible blocks). Thus, the elasticity of $(\mathbb{Q}, S)$ is at least 4/3. Note that this example is also realizable as a Krull domain by Theorem 1.13.

The following theorem is analogous to Theorem 2.11 for $\mathbb{Z}$. If S has only one negative element, then we can factor out by $a\mathbb{Z}$, and the elasticity will remain the same.

Theorem 2.25. *Let $S \subseteq \mathbb{Q}$ such that $S^- = \{-a\}$ for some $a > 0$ in $\mathbb{Q}$. Then $\rho(\mathbb{Q}, S) = \rho(\mathbb{Q}/a\mathbb{Z}, \overline{S^+})$.*

Proof. Then $S = S^+ \cup \{-a\}$. Define $f : \mathcal{B}(\mathbb{Q}, S) \to \mathcal{B}(\mathbb{Q}/a\mathbb{Z}, \overline{S^+})$ by

$$f\left(a^{u_a} \cdot \prod_{s \in S^+} s^{u_s}\right) = \prod_{s \in S^+} \overline{s}^{u_s},$$

where $\pi : \mathbb{Q} \to \mathbb{Q}/a\mathbb{Z}$ is the canonical projection. First we need to show this function actually maps into $\mathcal{B}(\mathbb{Q}/a\mathbb{Z}, \overline{S^+})$. Suppose $B \in \mathcal{B}(\mathbb{Q}, S)$. Then $B = (-a)^{u_a} \cdot \prod_{s \in S^+} s^{u_s}$ for some $u_s \in \mathbb{N}_0$, where all but finitely many of the $u_s = 0$, and $u_a(-a) + \sum_{s \in S^+} u_s s = 0$. Then $\sum_{s \in S^+} u_s s \in a\mathbb{Z}$; so $f(B) = \prod_{s \in S^+} \overline{s}^{u_s}$ is a block in $\mathcal{B}(\mathbb{Q}/a\mathbb{Z}, \overline{S})$.

Now we want to use this function f along with Lemma 1.14 to show that the elasticities are equivalent. Thus, we need to show three things: if B is irreducible, then $f(B)$ is irreducible; $kerf = \{1\}$; and there exists a homomorphism $g : \mathcal{B}(\mathbb{Q}/a\mathbb{Z}, \overline{S^+}) \to \mathcal{B}(\mathbb{Q}, S)$ such that fg is the identity homomorphism on $\mathcal{B}(\mathbb{Q}/a\mathbb{Z}, \overline{S^+})$. It is clear that $kerf = \{1\}$; so we only need to check the other two conditions.

Now suppose B is irreducible, and $f(B)$ is not irreducible. Then there exists $0 \leq u'_s \leq u_s$ for each $s \in S^+$ such that

$$\sum_{s \in S^+} u'_s \overline{s} = 0$$

in $\mathbb{Q}/a\mathbb{Z}$ and $u'_s < u_s$ for at least one $s \in S$. Then $\sum_{s \in S^+} u'_s s = ta$ for some $t \in \mathbb{Z}$. Since each $u'_s, s > 0$, then $t \geq 0$. Since B is a block, then

$$0 = u_a(-a) + \sum_{s \in S^+} u_s s \geq u_a(-a) + \sum_{s \in S^+} u'_s s = u_a(-a) + ta.$$

Thus $u_a a \geq ta$; so $u_a \geq t$. Therefore

$$(-a)^t \cdot \prod_{s \in S^+} s^{u'_s}$$

is a block dividing B, but not equal to B since $u'_s < u_s$ for at least one $s \in S$. But this is a contradiction since B is irreducible. Thus, $f(B)$ is irreducible.

Let $C \in \mathcal{B}(\mathbb{Q}/a\mathbb{Z}, \pi(S))$ be a block. Then $C = \prod_{d \in \overline{S^+}} d^{v_d}$ for some $v_d \in \mathbb{N}_0$ and all but finitely many of the v_d are zero. For each $d \in \pi(S)$, choose a $s_d \in S$ such that $\overline{s_d} = d$. Then $C = \prod_{d \in \overline{S}} \overline{s_d}^{v_d}$. Therefore

$$\sum_{d \in \pi(S)} v_d s_d = ra$$

for some $r \geq 0$ since C is a block and each $v_d, s_d > 0$. Then $D = (-a)^r \cdot \prod_{d \in \pi(S)} s_d^{v_d}$ is a block in $\mathcal{B}(\mathbb{Q}, S)$ with $f(D) = C$. Using this construction, define $g : \mathcal{B}(\mathbb{Q}/a\mathbb{Z}, \overline{S^+}) \to \mathcal{B}(\mathbb{Q}, S)$ by $g(C) = D$ as constructed above. The construction clearly makes g a homomorphism, and gf is the identity homomorphism. □

The following corollary gives a useful result for the case when S^- has only one element. The corollary says that every element can be replaced with an element that is in the interval $[0, a)$ without changing the elasticity. This can drastically reduce the size of S^+ for the purposes of calculating the elasticity. For example, let $S = \{-5, 2, 7, 12, 17, \ldots\} \subseteq S^+$. Then we can calculate the elasticity of $\mathcal{B}(\mathbb{Q}, \{-5, 2\})$, and it will be the same as $\mathcal{B}(\mathbb{Q}, S)$. Note this simple example is not realizable by a Krull domain.

Corollary 2.26. *Let $S \subseteq \mathbb{Q}$ such that $S^- = \{-a\}$ for some $a > 0$ in $\mathbb{Q}$. Define $T \subseteq \mathbb{Q}$ by $T = \{-a\} \cup \{t \in \mathbb{Q} \mid 0 \leq t < a$ and $t - s \in a\mathbb{Z}$ for some $s \in S\}$. Then $\rho(\mathbb{Q}, S) = \rho(\mathbb{Q}, T)$ (i.e., each element of S^+ can be replaced with an element in $[0, a)$ without changing the elasticity).*

Proof. By Theorem 2.25, $\rho(\mathbb{Q}, S) = \rho(\mathbb{Q}/a\mathbb{Z}, \overline{S^+})$ and $\rho(\mathbb{Q}, T) = \rho(\mathbb{Q}/a\mathbb{Z}, \overline{T^+})$. By definition of T, $\overline{S^+} = \overline{T^+}$ in $\mathbb{Q}/a\mathbb{Z}$. Therefore, $\rho(\mathbb{Q}/a\mathbb{Z}, \overline{S^+}) = \rho(\mathbb{Q}/a\mathbb{Z}, \overline{T^+})$. □

Now we would like to prove a theorem for testing exactly when $\mathcal{B}(\mathbb{Q}, S)$, where S has only one negative element, is an HFD. We will begin with $S = \{-1, \frac{b_1}{d_1}, \frac{b_2}{d_2}, \ldots\}$, where each $b_i, d_i \in \mathbb{N}$ in the following theorem, and the corollary will apply it to a general negative element instead of -1.

Theorem 2.27. *Let $S = \{-1, \frac{b_1}{d_1}, \frac{b_2}{d_2}, \ldots\} \subseteq \mathbb{Q}$, where each $b_i, d_i > 0$ and $(b_i, d_i) = 1$. Then $\rho(\mathbb{Q}, S) = 1$ if and only if there exists an $x_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ such that the following hold:*

1. *$x_n \equiv b_i^{\phi(d_i)-1} (mod\ d_i)$ for all $1 \leq i \leq n$, where ϕ is the Euler phi function.*
2. *$\{d_i \mid 1 \leq i \leq n\}$ is a splittable set.*

Proof. Define $S_n = \{-1, \frac{b_1}{d_1}, \frac{b_2}{d_2}, ..., \frac{b_n}{d_n}\}$ for all $n \in \mathbb{Z}_+$. Then $S = \cup_{i \in I} S_n$ and $S_n \subseteq S_{n+1}$. Then by Lemma 1.16, $\rho(\mathbb{Q}, S) = sup_{n \in \mathbb{N}} \rho(\mathbb{Q}, S_n)$. Therefore, $\rho(\mathbb{Q}, S) = 1$ if and only if $\rho(\mathbb{Q}, S_n) = 1$ for all $n \in \mathbb{Z}$.

Let $n \in \mathbb{N}$. Define $d = d_1 d_2 \cdots d_n$ and $dS_n = \{-d, \hat{d}_1 b_1, \hat{d}_2 b_2, \ldots, \hat{d}_n b_n\}$ using the notation $\hat{d}^i = d/d_i$. Then $\mathcal{B}(\mathbb{Q}, S_n)$ is isomorphic to $\mathcal{B}(\mathbb{Q}, dS_n)$ since multiplication by d is an automorphism of $\mathbb{Q}$. But $dS_n \in \mathbb{Z}$ so $\mathcal{B}(\mathbb{Q}, dS_n)$ is the same as $\mathcal{B}(\mathbb{Z}, dS_n)$. Then $\rho(\mathbb{Q}, S_n) = 1$ if and only if $\rho(\mathbb{Z}, dS_n) = 1$. Since there is only one negative element in dS_n, we can use Theorem 2.11 that gives $\rho(\mathbb{Z}, dS_n) = 1$ if and only if $\rho(\mathbb{Z}/d\mathbb{Z}, \overline{dS_N}) = 1$. By Theorem 1.22, $\rho(\mathbb{Z}/d\mathbb{Z}, \overline{dS_N}) = 1$ if and only if there exists a $\sigma \in Aut(\mathbb{Z}/d\mathbb{Z})$ such that $\sigma(dS_n) \subseteq \{\overline{s} \mid s \in \mathbb{N} \text{ and } s|d\}$ and $\{ord_{\mathbb{Z}/d\mathbb{Z}} \sigma(\hat{d}_j b_j) \mid 1 \leq j \leq n\}$ is a splittable set.

Therefore, $\rho(\mathbb{Q}, S) = 1$ if and only if for all $n \in \mathbb{N}$ there exists a $\sigma \in Aut(\mathbb{Z}/d\mathbb{Z})$ such that $\sigma(dS_n) \subseteq \{\overline{s} \mid s \in \mathbb{N} \text{ and } s|d\}$ and $\{ord_{\mathbb{Z}/d\mathbb{Z}} \sigma(\hat{d}_j b_j) \mid 1 \leq j \leq n\}$ is a splittable set. So if we prove these two conditions are equivalent to the two conditions in the theorem, then the proof is finished.

Now for each $1 \leq i \leq n$,

$$ord_{\mathbb{Z}/d\mathbb{Z}}(b_j \hat{d}_j) = \frac{lcm(d, b_j \hat{d}_j)}{b_j \hat{d}_j} = \frac{lcm(\hat{d}_j d_j, b_j \hat{d}_j)}{b_j} = \frac{\hat{d}_j lcm(d_j, b_j)}{b_j \hat{d}_j} = \frac{\hat{d}_j d_j b_j}{\hat{d}_j b_j} = d_j.$$

So $\{ord_{\mathbb{Z}/d\mathbb{Z}} \sigma(\hat{d}_j b_j) \mid 1 \leq j \leq n\}$ is splittable if and only if $\{d_i \mid 1 \leq i \leq n\}$ is a splittable set.

Now we would like to prove there exists a $\sigma \in Aut(\mathbb{Z}/d\mathbb{Z})$ such that $\sigma(dS_n) \subseteq \{\overline{s} \mid s \in \mathbb{N} \text{ and } s|d\}$ if and only if there exists an $x_n \in \mathbb{N}$ such that $x_n \equiv b_i^{\phi(d_i)-1} (\text{mod } d_i)$ for all $1 \leq i \leq n$.

($\Rightarrow$) Suppose that there exists a $\sigma \in Aut(\mathbb{Z}/d\mathbb{Z})$ such that $\sigma(dS_n) \subseteq \{\overline{s} \mid s \in \mathbb{N} \text{ and } s|d\}$. Then σ is multiplication by an element $1 \leq x_n \leq n$ such that $(x_n, d) = 1$ (in particular, $x_n = \sigma(1)$). Let $1 \leq j \leq n$. Recall from above that

$$ord_{\mathbb{Z}/d\mathbb{Z}}(b_j \hat{d}_j) = d_j.$$

Since σ is an automorphism, then it must be order preserving. Also, the only number between 1 and d that divides d and has order d_j is $\hat{d}_j$. Thus,

$$x_n b_j \hat{d}_j \equiv \sigma(b_j \hat{d}_j) \equiv \hat{d}_j (\text{mod } d).$$

Dividing this equation by $\hat{d}_i$, shows that

$$x_n b_j \equiv 1 (\text{mod } d_j).$$

The elements of $\mathbb{Z}/d_i\mathbb{Z}$ that are relatively prime to d_i form a multiplicative group of order $\phi(d_i)$, where ϕ is the Euler phi function. Then in this multiplicative group, x_n is equivalent to the multiplicative inverse of b, which is equivalent to $b^{\phi(d_i)-1}$. Thus, we have

$$x_n \equiv b^{\phi(d_i)-1} (\text{mod } d_i).$$

($\Leftarrow$) Suppose that for all $n \in \mathbb{N}$ there exists an x_n such that $x_n \equiv b_j^{\phi(d_i)-1} (\text{mod } d_i)$ and $\{d_i \mid i \in \mathbb{N}\}$ is a splittable set. We can define a function $\sigma \in Aut(\mathbb{Z}/d\mathbb{Z})$ by $\sigma(a) = x_n a$. We need to show that $(x_n, d) = 1$ to prove this is actually an automorphism of $\mathbb{Z}/m\mathbb{Z}$. By assumption $x_n \equiv b_i^{\phi(d_i)-1} (\text{mod } d_i)$ for all $1 \leq i \leq n$ and $(b_i, d_i) = 1$. Therefore, $b_i^{\phi(d_i)-1}$ is relatively prime to d_i, and x_n is then relatively prime to d_i since $x_n \equiv b_i^{\phi(d_i)-1} (\text{mod } d_i)$. Then $(x_n, d) = 1$ since $(x_n, d_i) = 1$ for all $1 \leq i \leq n$ and $d = d_1 \cdots d_n$. So σ is an automorphism.

Let $a \in dS_n$. Then $a = b_j \hat{d}_j$ for some $1 \leq j \leq n$. So

$$\sigma(a) \equiv \sigma(b_j \hat{d}_j) \equiv x_n b_j \hat{d}_j (\text{mod } d).$$

But

$$x_n b_j \equiv b_j^{\phi(d_j)} \equiv 1 (\text{mod } d_j).$$

Then $x_n b_j = 1 + r d_j$ for some $r \in \mathbb{Z}$. Thus $x_n b_j \hat{d}_j = \hat{d}_j + rd$. Therefore

$$x_n b_j \hat{d}_j \equiv \hat{d}_j (\text{mod } d).$$

So the automorphism σ sends each element of dS_n to a divisor of d. □

Corollary 2.28. *Let $m, n \in \mathbb{N}$ with $(m, n) = 1$. Let $S = \{-\frac{m}{n}, \frac{b_1}{d_1}, \frac{b_2}{d_2}, \ldots\}$, where each $b_i, d_i > 0$ and each $\frac{b_i}{d_i}$ is reduced. Then $\rho(\mathbb{Q}, S) = 1$ if and only if there exists an $x_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ such that the following hold:*

1. *$x_n \equiv (n_i b_i)^{\phi(m_i d_i)-1} (mod\ m_i d_i)$, where $n_i = \frac{n}{(n,d_i)}$ and $m_i = \frac{m}{(m,b_i)}$ for all $i \leq n$, and ϕ is the Euler phi function.*
2. *$\{m_i d_i \mid 1 \leq i \leq n\}$ is a splittable set.*

Proof. Multiplying by $\frac{n}{m}$, which is an autmorphism of the group $\mathbb{Q}$, this is equivalent to the group with $S = \{-1, \frac{n_1 b_1}{m_1 d_1}, \frac{n_2 b_2}{m_2 d_2}, \ldots\}$. Then by Theorem 2.27 the corollary holds. □

Corollary 2.29. *Let $S = \{-1, \frac{b_1}{d_1}, \frac{b_2}{d_2}, \ldots\}$, where each $b_i, d_i > 0$ and each $\frac{b_i}{d_i}$ is reduced. Then $\rho(\mathbb{Q}, S) = 1$ if and only if for all $n \in \mathbb{Z}_+$*

1. *$\rho(\mathbb{Z}/d\mathbb{Z}, dS_n) = 1$ where $d = d_1 \cdots d_n$, and*
2. *$\rho(\mathbb{Z}/d\mathbb{Z}, \{d_1, \ldots, d_n\}) = 1$.*

Proof. This follows directly from the proof of Theorem 2.27. □

Chapter 3

General Infinite Groups

3.1 Subsets of Independent Elements of Infinite Order

In this section, general infinite abelian groups will be studied by finding simpler groups that have the same or a related elasticity. This study was motivated in part by a proposition of Geroldinger and Göbel [12, Proposition 3.7]. In this proposition, a maximal subset of independent elements of infinite order, denoted X, is used to study the group. A set of elements is independent if for any $x_1, \ldots, x_n \in X$ and $r_1, \ldots r_n \in \mathbb{Z}$ with $r_1x_1 + \cdots + r_nx_n = 0$, then each $r_i = 0$. The same approach is taken here, but the maximal condition is left off when it is unnecessary. Proposition 3.1 shows that, under certain conditions, the group can be modded out by multiples of the elements of X without changing the elasticity. When the set X is assumed to be maximal, then a group with non-torsion elements will be reduced to a torsion group. Then the cross number can be applied to this torsion group, and thus the original group. This means the cross number has been extended to cover some groups which are not torsion. The result on cross numbers is a corollary to this theorem, Corollary 3.2. The proposition by Geroldinger will be generalized in Proposition 3.3.

Note the last condition in the following theorem, can be replaced with $\{m_1^i, \ldots, m_{k_i}^i\}$ is l_i-split for all $i \in I$. Also, for the elements m_j^i in the theorem, the i is an index, not a power.

Proposition 3.1. *Let G be an additive abelian group and $X = \{x_i \mid i \in I\} \subseteq G$ a set of independent elements of infinite order.*

- *Let $S_i = \{-m_1^i, \ldots, -m_{k_i}^i\} \subset \mathbb{Z}$ with each $m_j^i \in \mathbb{N}$ for all $i \in I$.*
- *Define $c_i = \ll m_1^i, \ldots, m_{k_i}^i \gg$ and $l_i = lcm\{m_1^i, \ldots, m_{k_i}^i\}$ for all $i \in I$.*
- *Let $G_0 \subseteq G$ such that $[G_0] \cap < X > \subseteq [\{c_i x_i \mid i \in I\}]$.*
- *Let $S = G_0 \cup (\cup_{i \in I} S_i x_i)$.*
- *Suppose that whenever*

$$\prod_{i \in I} \left(\prod_{s_i \in S_i} (s_i x_i)^{u_{s_i}} \right) \cdot \prod_{g \in G_0} g^{v_g}$$

is an irreducible block in $\mathcal{B}(G,S)$*, then*

$$\prod_{g\in G_0} \overline{g}^{v_g}$$

is an irreducible block in $\mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$.

Then $\rho(G,S) = \rho(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$.

Proof. Define

$$f : \mathcal{B}(G,S) \to \mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$$

by

$$f\left(\prod_{i\in I}\left(\prod_{s_i\in S_i}(s_i x_i)^{u_{s_i}}\right)\cdot\prod_{g\in G_0} g^{v_g}\right) = \prod_{g\in G_0}\overline{g}^{v_g}.$$

First, we need to show that this function actually maps into $\mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$. Suppose $B \in \mathcal{B}(G,S)$, and then for some $v_g, u_{s_i} \in \mathbb{N}_0$

$$B = \prod_{i\in I}\left(\prod_{s_i\in S_i}(s_i x_i)^{u_{s_i}}\right)\cdot\prod_{g\in G_0} g^{v_g}.$$

Since B is a block, then

$$\sum_{i\in I}\left(\sum_{s_i\in S_i} u_{s_i}(s_i x_i)\right) + \sum_{g\in G_0} v_g g = 0.$$

Therefore $\sum_{g\in G_0} v_g g \in < X >$; so by the assumption the sum is in $[\{c_i x_i \mid i \in I\}]$. Then for some $r_i \in \mathbb{N}_0$,

$$\sum_{g\in G_0} v_g g = \sum_{i\in I} r_i c_i x_i.$$

Then we can replace $\sum_{g\in G_0} v_g g$ in the earlier equation; so it becomes:

$$\sum_{i\in I}\left(\sum_{s_i\in S_i} u_{s_i}(s_i x_i)\right) + \sum_{i\in I} r_i c_i x_i = 0.$$

Since the x_i's are independent, then

$$\sum_{s_i\in S_i} u_{s_i}(s_i x_i) + r_i c_i x_i = 0 \text{ for all } i \in I.$$

Because each x_i has infinite order, then

$$\sum_{s_i\in S_i} u_{s_i}(s_i) + r_i c_i = 0 \text{ for all } i \in I,$$

which can be rewritten as

$$\sum_{j=1}^{k_i} u_{(-m_j^i)}(-m_j^i) + r_i c_i = 0.$$

Therefore

$$\sum_{j=1}^{k_i} u_{(-m_j^i)}(m_j^i) = r_i c_i.$$

Now $l_i = c_i d_i$ for some $d_i \in \mathbb{N}$ with $(c_i, d_i) = 1$ by Lemma 2.3. Also, by Lemma 2.3, $d_i | m_j^i$ for all $m_j^i \in -S_i^-$, and therefore, d_i divides the left hand side of the equation. The right hand side is a multiple of c_i and since $(c_i, d_i) = 1$, then l_i divides both sides of the equation. Thus

$$r_i c_i \in \mathbb{N}_0 l_i.$$

So

$$r_i c_i x_i \in < l_i x_i > \text{ for all } i \in I.$$

Therefore

$$\sum_{i \in I} r_i c_i x_i \in < \{l_i x_i \mid i \in I\} > .$$

But $\sum_{i \in I} r_i c_i x_i = \sum_{g \in G_0} v_g g$; so

$$\sum_{g \in G_0} v_g g \in < \{l_i x_i \mid i \in I\} > .$$

Then

$$f(B) = \prod_{g \in G_0} \overline{g}^{v_g}$$

is a block in $\mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >), \overline{G_0})$; so the homomorphism f has the correct range. If $B \in \mathcal{B}(G, S)$ is irreducible, then by assumption, $f(B)$ is also irreducible.

Then by Lemma 1.14 it remains to show that $kerf = \{1\}$, and there exists a homomorphism $g : \mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{T}) \to \mathcal{B}(G, S)$ such that $fg(B) = B$ for all $B \in \mathcal{B}(G, S)$. Let $C \in \mathcal{B}(G, S)$ and $f(C) = 0$. Then by the definition of f,

$$C = \prod_{i \in I} \left(\prod_{j=1}^{k_i} (-m_j^i x_i)^{w_i(j)} \right)$$

for some $w_i(j) \in \mathbb{N}_0$. But C is a block in $\mathcal{B}(G, S)$; so

$$\sum_{i \in I} \left(\sum_{j=1}^{k_i} w_j(-m_j^i x_i) \right) = 0.$$

Since the x_i's are independent, then $\sum_{j=1}^{k_i} w_i(j)(-m_j^i x_i) = 0$ for all $i \in I$. Therefore $\sum_{j=1}^{k_i} w_i(j)(-m_j^i) = 0$ for all $i \in I$ because each x_i has infinite order. But, each $-m_j^i < 0$ and each $w_i(j) \geq 0$; so each term $w_i(j)(-m_j^i) \leq 0$. Therefore we must have each $w_i(j) = 0$. So C is the empty product which is the identity in $\mathcal{B}(G, S)$. Therefore $kerf = \{1\}$.

Now we need to create a right inverse homomorphism g to the homomorphism f. Let $D \in \mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$. Then $D = \prod_{a \in \overline{G_0}} a^{v_a}$ for some $v_a \in \mathbb{N}_0$. For each $a \in \overline{G_0}$, choose $g_a \in G_0$ such that $\overline{g_a} = a$ in $G/ < \{l_i x_i \mid i \in I\} >$. Therefore

$$D = \prod_{a \in \overline{G_0}} \overline{g_a}^{v_a}.$$

Since D is a block, then

$$\sum_{a \in \overline{G_0}} v_a \overline{g_a} = 0$$

in $G/ < \{l_i x_i \mid i \in I\} >$. So there exists $t_i \in \mathbb{N}_0$ for each $i \in I$ such that

$$\sum_{a \in \overline{G_0}} v_a g_a = \sum_{i \in I} t_i l_i x_i.$$

Note that for the block D, the choice of each t_i is unique since the x_i's are independent and have infinite order. Also, it was assumed that positive sums of elements in G_0 that are in $< \{l_i x_i \mid i \in I\} >$ must be in $[\{l_i x_i \mid i \in I\}]$. Then each $t_i \geq 0$ since the t_i's are unique. For each $i \in I$, $m_1^i | l_i$, hence there exists a $w_i \in \mathbb{N}_0$ such that $t_i l_i = w_i m_1^i$. Thus the above equation can be rewritten as

$$\sum_{a \in \overline{G_0}} v_a g_a = \sum_{i \in I} w_i m_1^i x_i.$$

Therefore

$$E = \prod_{i \in I} \left((m_1^i x_i)^{w_i} \cdot \prod_{s_i \in S_i^+} (s_i x_i)^{u_{s_i}} \right) \cdot \prod_{a \in \overline{G_0}} g_a^{v_a}$$

is a block in $\mathcal{B}(G, S)$ and the powers w_i are unique for each block D since each t_i is unique. So this construction provides a unique block E for each block D. Also, notice that it is clear using this construction of E that $f(E) = D$.

Define

$$g : \mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{T}) \to \mathcal{B}(G, S)$$

by $g(D) = E$, where each E is constructed as above. It is clear that g is a homomorphism. Also, since $f(E) = D$, then fg is the identity function on $\mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$. Therefore, using Lemma 1.14, $\rho(G, S) = \rho(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$. □

For the next corollary, we will add the assumption that X is a maximal set of independent elements of infinite order. Then the group $G/ < X >$ is a torsion group. A block monoid with a torsion divisor class group has elasticity one if and only if the cross number for each irreducible block is one. Then this calculation for each irreducible block in $\mathcal{B}(G/ < X >, \overline{S})$ can be applied to the original block monoid $\mathcal{B}(G, S)$. This will give us a cross number for general abelian groups that are not torsion if group satisfies the conditions. The assumptions are the same as Theorem 3.1 except X is now a maximal set of independent elements of infinite order, and a function k is defined.

Corollary 3.2. *Let G be an additive abelian group and $X = \{x_i \mid i \in I\} \subseteq G$ a maximal set of independent elements of infinite order.*

- *Let* $S_i = \{-m_1^i, \ldots, -m_{k_i}^i\} \subset \mathbb{Z}$ *with each* $m_j^i \in \mathbb{N}$ *for all* $i \in I$.
- *Define* $c_i = \ll m_1^i, \ldots, m_{k_i}^i \gg$ *and* $l_i = lcm\{m_1^i, \ldots, m_{k_i}^i\}$ *for all* $i \in I$.
- *Let* $G_0 \subseteq G$ *such that* $[G_0] \cap < X > \subseteq [\{c_i x_i \mid i \in I\}]$.
- *Let* $S = G_0 \cup (\cup_{i \in I} S_i x_i)$.
- *Suppose that whenever*

$$\prod_{i \in I} \left(\prod_{s_i \in S_i} (s_i x_i)^{u_{s_i}} \right) \cdot \prod_{g \in G_0} g^{v_g}$$

is an irreducible block in $\mathcal{B}(G, S)$, *then*

$$\prod_{g \in G_0} \overline{g}^{v_g}$$

is an irreducible block in $\mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$.

- *For every block*

$$B = \prod_{i \in I} \left(\prod_{s_i \in S_i} (s_i x_i)^{u_{s_i}} \right) \cdot \prod_{g \in G_0} g^{v_g}$$

in $\mathcal{B}(G, S)$, *define the function* $k : \mathcal{B}(G, S) \to \mathbb{Q}^+$ *by*

$$k(B) := \sum_{g \in G_0} \frac{v_g}{ord(\bar{g})},$$

where $ord(\bar{g})$ *is the order of* $\overline{g} \in G/ < \{l_i x_i \mid i \in I\} >$.

Then $\rho(G, S) = 1$ *if and only if* $k(B) = 1$ *for every irreducible block* $B \in \mathcal{B}(G, S)$.

Proof. Before the "if and only if" is proven, we need to show that $G/ < \{l_i x_i \mid i \in I\} >$ is torsion to show that $ord(\bar{g})$ is well-defined. Suppose $g \in G$ has infinite order. Since X is a maximal set of independent elements of infinite order, then we must have

$$tg = \sum_{i \in I} r_i x_i$$

for some $t \in \mathbb{N}$ and $r_i \in \mathbb{Z}$, with only finitely many of the r_i nonzero. If we define $s = lcm\{l_i \mid i \in I \text{ such that } r_i \neq 0\}$, then

$$stg = \sum_{i \in I} s r_i x_i \in < \{l_i x_i \mid i \in I\} > .$$

Therefore $\bar{g}$ has order less than or equal to st in $G/ < \{l_i x_i \mid i \in I\} >$. If $g \in G$ has finite order, then clearly $\overline{g}$ has finite order in $G/ < \{l_i x_i \mid i \in I\} >$. Hence, $G/ < \{l_i x_i \mid i \in I\} >$ is torsion.

($\Leftarrow$) Suppose that $k(B) = 1$ for all irreducible blocks $B \in \mathcal{B}(G, S)$. Since k is a semi-length function, then $\rho(G, S) = 1$ by Lemma 1.18.

($\Rightarrow$) By Proposition 3.1, $\rho(G,S) = \rho(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$. Since $G/ < \{l_i x_i \mid i \in I\} >$ is a torsion group, then $\rho(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0}) = 1$ if and only if the cross number of each irreducible block in $\mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$ is 1.

Suppose $B \in \mathcal{B}(G,S)$ is irreducible. Then

$$B = \prod_{i \in I} \left(\prod_{s_i \in S_i} (s_i x_i)^{u_{s_i}} \right) \cdot \prod_{g \in G_0} g^{v_g}$$

for some $u_{s_i}, v_g \in \mathbb{N}_0$. Then using the function f from Proposition 3.1,

$$f(B) = \prod_{g \in G_0} \overline{g}^{v_g}$$

is an irreducible block of $\mathcal{B}(G/ < \{l_i x_i \mid i \in I\} >, \overline{G_0})$. Therefore,

$$1 = k(f(B)) = \sum_{g \in G_0} \frac{v_g}{ord(\overline{g})} = k(B).$$

In this equation, $k(f(B))$ is the standard cross number defined on a torsion group. □

The next proposition is similar to Proposition 3.1, however, the condition on G_0 has been strengthened. It will be assumed that $[G_0] \cap < X > \subseteq [\{l_i x_i \mid i \in I\}]$, replacing the c_i with l_i. However, if an element of G_0 was a positive power of some x_i, we still only require that it is divisible by $c_i x_i$. These positive multiples of some x_i will then be grouped in with the sets S_i, instead of G_0, so they only need satisfy the weaker condition. Under these assumptions, each subgroup $\mathbb{Z}x_i$ will act independently from each other, as far as factorization is concerned. Also, the blocks containing elements of G_0 will only contain multiples of x_i's whose sum is divisible by $-l_i x_i$. It will not matter which negative multiples of x_i add up to this multiple of $-l_i x_i$. Then it is conjectured that the elasticity of $\mathbb{B}(G,S)$ will be the same as $sup\left(\{\rho(G/ < X >, \overline{G_0})\} \cup \{\rho(\mathbb{Z}, S_i) \mid i \in I\}\right)$. Again, note that the superscript i in m_j^i, n_j^i is an index, not a power.

Proposition 3.3. *Let G be an additive abelian group and $X = \{x_i \mid i \in I\} \subset G$ a maximal set of independent elements of infinite order.*

- *Let $S_i = \{-m_1^i, \ldots, -m_{k_i}^i, n_1^i, n_2^i, n_3^i, \ldots\} \subset \mathbb{Z}$ with each $m_j^i, n_j^i \in \mathbb{N}$ for all $i \in I$.*
- *Define $c_i = \ll m_1^i, \ldots, m_{k_i}^i \gg$ and $l_i = lcm\{m_1^i, \ldots, m_{k_i}^i\}$ for all $i \in I$.*
- *Suppose each $s_i \in S_i^+$ is divisible by c_i.*
- *Let $G_0 \subseteq G$ such that $[G_0] \cap < X > \subseteq [\{l_i x_i \mid i \in I\}]$*
- *Suppose $\{m_1^i, \ldots, m_{k_i}^i\}$ is l_i-split.*

Let $S = G_0 \cup (\cup_{i \in I} S_i x_i)$. Then the following hold for the block monoid $\mathcal{B}(G,S)$:

1. $\mathcal{B}(G,S) = (\cup_{i \in I} \mathcal{B}(G, S_i x_i)) \cup \mathcal{B}\left(G, G_0 \cup \left(\cup_{i \in I} S_i^- x_i\right)\right)$.
2. $\rho\left(G, G_0 \cup \left(\cup_{i \in I} S_i^- x_i\right)\right) = \rho(G/ < X >, \overline{G_0})$.

3. *(Conjecture)* $\rho(G,S) = sup\left(\{\rho(G/ < X >, \overline{G_0})\} \cup \{\rho(\mathbb{Z}, S_i) \mid i \in I\}\right)$.

4. *(Conjecture)* $\mathcal{B}(G,S)$ *is half-factorial if and only if* $(G/ < X >, \overline{G_0})$ *is half-factorial and* $\mathcal{B}(\mathbb{Z}, S_i)$ *is half-factorial for all* $i \in I$. *Also, if* $\mathcal{B}(G,S)$ *is half-factorial, then the requirement that each* $s_i \in S_i^+$ *is divisible by* c_i *for all* $i \in I$ *can be proved without being assumed.*

Proof. 1. Suppose that $B \in \mathcal{B}(G,S)$. Then

$$B = \prod_{i \in I} \left(\prod_{s_i \in S_i^-} (s_i x_i)^{u_{s_i}} \cdot \prod_{s_i \in S_i^+} (s_i x_i)^{u_{s_i}} \right) \cdot \prod_{g \in G_0} g^{v_g}$$

for some $v_g, u_{s_i} \in \mathbb{N}_0$ for all $g \in G_0$ and $u_{s_i} \in S_i$ for all $i \in I$, and only finitely many of the v_g and u_{s_i} are nonzero.

Suppose that $v_g = 0$ for all $g \in G_0$. Then

$$B = \prod_{i \in I} \left(\prod_{s_i \in S_i^-} (s_i x_i)^{u_{s_i}} \cdot \prod_{s_i \in S_i^+} (s_i x_i)^{u_{s_i}} \right) \in \mathcal{B}(G, \cup_{i \in I} S_i x).$$

This implies that

$$\sum_{i \in I} \left(\sum_{s_i \in S_i^-} u_{s_i}(s_i x_i) + \sum_{s_i \in S_i^+} u_{s_i}(s_i x_i) \right) = 0.$$

Since the x_i's are independent from each other, this implies

$$\sum_{s_i \in S_i^-} u_{s_i}(s_i x_i) + \sum_{s_i \in S_i^+} u_{s_i}(s_i x_i) = 0$$

for each $i \in I$. Then

$$B_i = \prod_{s_i \in S_i^-} (s_i x_i)^{u_{s_i}} \cdot \prod_{s_i \in S_i^+} (s_i x_i)^{u_{s_i}}$$

would be an irreducible block for each $i \in I$. But B is irreducible, so we must have B_i is an empty product for all $i \in I$ except for one $j \in I$. Thus, $B = B_j$ for some $j \in I$. Hence, $B \in \mathcal{B}(G, S_j x_j) \subseteq \cup_{i \in I} \mathcal{B}(G, S_i x_i)$.

Suppose $v_g > 0$ for at least one $g \in G_0$. Then we know that

$$\sum_{i \in I} \left(\sum_{s_i \in S_i^-} u_{s_i}(s_i x_i) + \sum_{s_i \in S_i^+} u_{s_i}(s_i x_i) \right) + \sum_{g \in G_0} v_g g = 0.$$

Then

$$\sum_{g \in G_0} v_g g = \sum_{i \in I} \left(\sum_{s_i \in S_i^-} -u_{s_i}(s_i x_i) + \sum_{s_i \in S_i^+} -u_{s_i}(s_i x_i) \right) \in < X > .$$

Thus, $\sum_{g\in G_0} v_g g \in [\{l_i x_i \mid i \in I\}]$ by assumption; so $\sum_{g\in G_0} v_g g = \sum_{i\in I} a_i l_i x_i$ for some $a_i \in \mathbb{N}_0$ for each $i \in I$.

Then

$$\sum_{i\in I} a_i l_i x_i + \sum_{i\in I}\left(\sum_{s_i\in S_i^+} u_{s_i}(s_i x_i)\right) = \sum_{i\in I}\left(\sum_{s_i\in S_i^-} -u_{s_i}(s_i x_i)\right)$$

Since the x_i's are independent, then for each $i \in I$ we have

$$a_i l_i x_i + \sum_{s_i\in S_i^+} u_{s_i}(s_i x_i) = \sum_{s_i\in S_i^-} -u_{s_i}(s_i x_i).$$

Since each x_i has infinite order, then the coefficients above must be equal:

$$a_i l_i + \sum_{s_i\in S_i^+} u_{s_i}(s_i) = \sum_{s_i\in S_i^-} u_{s_i}(-s_i).$$

Recall from Lemma 2.3 that $l_i = c_i d_i$ for some $d_i \in \mathbb{N}$ with $(c_i, d_i) = 1$. Also by Lemma 2.3, $d_i|(-s_i)$ for all $s_i \in S_i^-$; so d_i divides the right-hand side of the equation. Since $c_i|l_i$ and, by assumption, $c_i|s_i$ for all $s_i \in S_i^+$, then c_i divides the left-hand side of the equation. Since $(c_i, d_i) = 1$, then l_i divides both sides of the equation. Thus $\sum_{s_i\in S_i^-} u_{s_i}(-s_i) = t_i l_i$ for some $t_i \in \mathbb{N}_0$ (for some $i \in I$ the sum $\sum_{s\in S^-} u_{s_i} s_i$ may equal 0, but they will not all be 0). So the previous equation becomes

$$a_i l_i + \sum_{s_i\in S_i^+} u_{s_i}(s_i) = t_i l_i.$$

Then we must have $t_i \geq a_i$ since $\sum_{s\in S^+} u_{s_i} s_i \geq 0$. Therefore, $\sum_{s_i\in S_i^-} u_{s_i} s_i = t_i l_i > a_i l_i$. Thus, since $-S_i^- = \{m_1^i, \ldots, m_{k_i}^i\}$ is l_i-split, there exists $0 \leq b_{s_i} \leq u_{s_i}$ for each $s_i \in S_i^-$ such that $\sum_{s_i\in S_i^-} b_{s_i}(-s_i) = a_i l_i$ by Lemma 1.24.

Since $\sum_{g\in G_0} v_g g = \sum_{i\in I} a_i l_i x_i$, then

$$\sum_{i\in I}\left(\sum_{s_i\in S_i^-} b_{s_i} s_i x_i\right) + \sum_{g\in G_0} v_g g = 0.$$

Hence

$$B' = \prod_{i\in I}\left(\prod_{s\in S^-} (s_i x_i)^{b_{s_i}}\right) + \prod_{g\in G_0} g^{v_g}$$

is a block dividing B. Since B is irreducible, we must have $B' = B$. Therefore $b_{s_i} = u_{s_i}$ for all $s_i \in S_i^-$ for each $i \in I$, and $u_{s_i} = 0$ for all $s_i \in S_i^+$ for each $i \in I$. Thus

$$B = \prod_{i\in I}\left(\prod_{s_i\in S_i^-} (s_i x_i)^{u_{s_i}}\right)\prod_{g\in G_0} g^{v_g} \in \mathcal{B}(G, G_0 \cup \left(\cup_{i\in I} S_i^- x_i\right)).$$

Therefore $\mathcal{B}(G,S) \subseteq (\cup_{i\in I}\mathcal{B}(G,S_ix)) \cup \mathcal{B}\left(G, G_0 \cup \left(\cup_{i\in I} S_i^- x_i\right)\right)$. It is clear that the reverse inclusion holds, so these are actually equal.

2. Define
$$f : \mathcal{B}(G, G_0 \cup \left(\cup_{i\in I} S_i^- x_i\right)) \to \mathcal{B}(G/ < X >, \overline{G_0})$$
by
$$f\left(\prod_{g\in G_0} g^{v_g} \cdot \prod_{i\in I}\left(\prod_{s_i\in S_i^-}(s_ix_i)^{u_{s_i}}\right)\right) = \prod_{g\in G_0}\overline{g}^{v_g}.$$
This is clearly a homomorphism.

Suppose that $B = \prod_{g\in G_0} g^{v_g}\cdot\prod_{i\in I}\left(\prod_{s_i\in S_i^-}(s_ix_i)^{u_{s_i}}\right)$ is irreducible, with $u_{s_i}, v_g \in \mathbb{N}_0$ for all $g \in G_0$ and $s_i \in S_i^-$ for all $i \in I$. Suppose that $f(B) = \prod_{g\in G_0}\overline{g}^{v_g}$ is not irreducible. Then there exists $0 \le v'_g \le v_g$ for each $g \in G_0$ with at least one $v'_g < v_g$ such that $\sum_{g\in G_0} v'_g g \in < X >$. But then by assumption, $\sum_{g\in G_0} v'_g g \in [\{l_ix_i \mid i \in I\}]$. Therefore $\sum_{g\in G_0} v'_g g = \sum_{i\in I} t_il_ix_i$ for some $t_i \in \mathbb{N}_0$ for each $i \in I$.

Since B is a block, then
$$\sum_{g\in G_0} v_g g = \sum_{i\in I}\left(\sum_{s_i\in S_i^-} u_{s_i}(-s_ix_i)\right) \in < X > .$$
Thus $\sum_{g\in G_0} v_g g = \sum_{i\in I} a_il_ix_i$ for some $a_i \in \mathbb{N}_0$ for each $i \in I$, and
$$\sum_{i\in I}\sum_{s_i\in S_i^-} u_{s_i}(-s_ix_i) = \sum_{i\in I} a_il_ix_i.$$
We need to establish that $a_i \ge t_i$ for each $i \in I$. Notice that
$$\sum_{g\in G_0}(v_g - v'_g)g = \sum_{g\in G_0}(v_g)g - \sum_{g\in G_0}(v'_g)g = \sum_{i\in I} a_il_ix_i - \sum_{i\in I} t_il_ix_i = \sum_{i\in I}(a_i - t_i)l_ix_i.$$
Since $\sum_{g\in G_0}(v_g - v'_g)g \in < X >$, then $\sum_{g\in G_0}(v_g - v'_g)g \in [\{l_ix_i \mid i \in I\}]$ by assumption. Thus $\sum_{i\in I}(a_i - t_i)l_ix_i \in [\{l_ix_i \mid i \in I\}]$, and since each x_i has infinite order, then $a_i - t_i \ge 0$.

Now the x_i's are independent; so
$$\sum_{i\in I}\left(\sum_{s_i\in S_i^-} u_{s_i}(-s_ix_i)\right) = \sum_{i\in I} a_il_ix_i$$
means that for each $i \in I$,
$$\sum_{s_i\in S_i^-} u_{s_i}(-s_ix_i) = a_il_ix_i.$$
Since each x_i has infinite order, then $\sum_{s_i\in S_i^-} u_{s_i}(-s_i) = a_il_i$ for each $i \in I$. Now $-(S_i^-) = \{m_1^i, \ldots, m_{k_i}^i\}$ is l_i-split and $a_i \ge t_i$; so there exist $0 \le u'_{s_i} \le u_{s_i}$ for each

$s_i \in S_i^-$ such that $\sum_{s_i \in S_i^-} u'_{s_i}(-s_i) = t_i l_i$. But then

$$\sum_{g \in G_0} v'_g g + \sum_{i \in I} \left(\sum_{s_i \in S_i^-} u'_{s_i}(s_i x_i) \right) = 0.$$

Therefore

$$\prod_{g \in G_0} g^{v'_g} \cdot \prod_{i \in I} \left(\prod_{s_i \in S_i^-} (s_i x_i)^{u'_{s_i}} \right)$$

is a block dividing B. This block does not equal B since at least one $v'_g < v_g$. Therefore we have a contradiction. So $f(B)$ must be irreducible. Hence $\rho\left(G, G_0 \cup \left(\cup_{i \in I} S_i^- x\right)\right) \leq \rho(G/ < X >, \overline{G_0})$ by Lemma 1.14.

Let $C = \prod_{(h \in \overline{G_0})} h^{w_h} \in \mathcal{B}(G/ < X >, \overline{G_0})$ be an irreducible block. We need to find a block D such that $f(D) = C$. For each $h \in \overline{G_0}$, choose a $g_h \in G$ such that $\overline{g_h} = h$. Then $C = \prod_{h \in \overline{G_0}} \overline{g_h}^{w_h}$, so $\sum_{(h \in \overline{G_0})} w_h \overline{g_h} = 0$ in $G/ < X >$. Therefore, $\sum_{h \in \overline{G_0}} w_h g_h \in < X >$ so $\sum_{h \in \overline{G_0}} w_h g_h \in [\{l_i x_i \mid i \in I\}]$. Then $\sum_{h \in \overline{G_0}} w_h g_h = \sum_{i \in I} a_i m_1^i x_i$ for some $a_i \in \mathbb{N}_0$ for each $i \in I$ since $m_1^i | l_i$. Thus we can define a block

$$D = \prod_{h \in \overline{G_0}} g_h^{w_h} \cdot \left(\prod_{i \in I} (-m_1^i x_i)^{a_i} \right) \in \mathcal{B}(G, G_0 \cup \left(\cup_{i \in I} (S_i^-) x_i\right),$$

and $f(D) = C$.

Now define

$$g : \mathcal{B}(G/ < X >, \overline{G_0}) \to \mathcal{B}(G, G_0 \cup \left(\cup_{i \in I} S_i^- x_i\right))$$

defined by $g(C) = D$ using the construction above. It is clear that g is a homomorphism. Then by Lemma 1.14, $\rho(G/ < X >, \overline{G_0}) \leq \rho\left(G, G_0 \cup \left(\cup_{i \in I} S_i^- x\right)\right)$. Therefore $\rho\left(G, G_0 \cup \left(\cup_{i \in I} S_i^- x\right)\right) = \rho(G/ < X >, \overline{G_0})$.

3. This is still a conjecture.

4. This is still a conjecture.

□

If we let each $S_i = \{-1, 1\}$ in the previous theorem, then we get the case proved by Geroldinger and Göbel. In this simpler case, each $c_i = 1$ and $S_i^+ = \{1\}$; so the condition that c_1 divides each element of S_i^+ is always satisfied. Also, each $l_i = 1$; so the condition for G_0 becomes, if $G_0 \in < X >$, then $G_0 \in [X]$. They also restrict G_0 so that no two elements in G_0 are in the same coset after modding out by $< X >=< \{l_i x_i \mid i \in i\} >$. This makes the function in the proof of part 2 an isomorphism of the block monoids. Lastly, they assume that X is a maximal set of independent elements of infinite order. Then $G/ < X >$ is a torsion group, so $ord_{G/<X>}(\overline{g})$ is finite for each $g \in G_0$. Then the condition that if $G_0 \in < X >$, then $G_0 \in [X]$, is actually equivalent to $ord_{G/<X>}(\overline{g}) \cdot g \in [X]$ for each $g \in G_0$.

The obvious question about Propositions 3.1 and 3.3 is how reasonable are the assumptions on the set S? For Proposition 3.3, if $\mathcal{B}(G, S)$ has elasticity one, then $\mathcal{B}(G, S_i x_i)$ must have elasticity one for each $i \in I$ by Lemma 1.15. But this is equivalent to $\mathcal{B}(\mathbb{Z}, S_i)$ having

elasticity one. Then by Theorem 2.6, the positive elements of S_i must be divisible by c_i, assuming $|S_i^+| = \infty$. So if it was assumed that $\rho(G, S) = 1$, then it can be proved that each $s_i \in S_i^+$ divisible by c_i. The question is still open on whether any element in $[X]$ must be in $[\{c_i x_i \mid i \in I\}]$. Also, in both of these theorems it was assumed that the only elements in $[\{-x_i \mid i \in I\}]$ are the elements of the form $-m_j^i x_i$. So elements in S can be a negative multiple of one x_i, but not the sum of negative multiples of several different x_i. Also, G_0 has no elements in $[\{-x_i \mid i \in I\}]$. It has not been proved that this must be true to have $\rho(G, S) = 1$. However, Theorem 3.9, which deals with the simpler case of $G = \mathbb{Z} \oplus \mathbb{Z}$ shows that if there is an element $(a, b) \in S$ with $a, b < 0$, then the elasticity is infinite. To get this result, it must be assumed that there are infinitely many elements of S of the form $(c, 0)$ for $c > 0$ and $(0, d)$ for $d > 0$. Theorem 3.9 suggests our assumption about negative multiples of the x_i's is reasonable, but the assumption has not been proven.

3.2 Direct Sums of $\mathbb{Z}$

In Section 3.1, a group G was studied using a set $X \subseteq G$ of independent elements of infinite order. So for each element $x \in X$, G has a subgroup isomorphic to $\mathbb{Z}$. Now the group splits over the subgroups generated by these independent elements if and only if the group G can be written as a direct sum of copies of $\mathbb{Z}$ with a group H. This group H would be isomorphic to $G/ < X >$. The number of copies of $\mathbb{Z}$ is the same as the number of x_i's which were indexed by the set I. Therefore, we can record the theorems of the previous section for this special case without proof.

Note that the last condition in the following theorem, can be replaced with $\{m_1^i, \ldots, m_{k_i}^i\}$ is l_i-split. This theorem is the same as Proposition 3.1.

Proposition 3.4. *Let I be a nonempty set, H a group, and $G = H \oplus (\oplus_{i \in I} \mathbb{Z})$. Label the canonical projections and injections as π_H, π_i, ι_H, and ι_i, respectively.*

- *Let $S_i = \{-m_1^i, \ldots, -m_{k_i}^i\} \subset \mathbb{Z}$ with each $m_j^i \in \mathbb{N}$ for all $i \in I$.*
- *Define $c_i = \ll m_1^i, \ldots, m_{k_i}^i \gg$ and $l_i = lcm\{m_1^i, \ldots, m_{k_i}^i\}$ for all $i \in I$.*
- *Let $A \subseteq H \oplus (\oplus_{i \in I} \mathbb{N}_0 c_i)$.*
- *Let $S = A \cup (\cup_{i \in I} \iota_i(S_i))$.*
- *Define $\bar{G} = H \oplus (\oplus_{i \in I} \mathbb{Z}/l_i\mathbb{Z})$.*
- *Suppose that if*

$$\prod_{i \in I} \left(\prod_{s_i \in S_i} (\iota_i(s_i))^{u_{s_i}} \right) \cdot \prod_{a \in A} a^{v_a}$$

is an irreducible block in $\mathcal{B}(G, S)$, then

$$\prod_{a \in A} \overline{a}^{v_a}$$

is an irreducible block in $\mathcal{B}(\overline{G}, \overline{A})$.

Then

$$\rho(G,S) = \rho(\overline{G},\overline{A}).$$

Now we can repeat Corollary 3.2 for this case. The corollary required that X was a maximal set of independent elements of infinite order. The corresponding assumption for this case is that H must be torsion. Then the generator for each copy of $\mathbb{Z}$ will represent a a maximal set of independent elements of infinite order.

Corollary 3.5. *Let I be a nonempty set, H a torsion group, and $G = H \oplus (\oplus_{i\in I}\mathbb{Z})$. Label the canonical projections and injections as π_H, π_i, ι_H, and ι_i, respectively.*

- *Let $S_i = \{-m_1^i, \ldots, -m_{k_i}^i\} \subset \mathbb{Z}$ with each $m_j^i \in \mathbb{N}$ for all $i \in I$.*
- *Define $c_i = \ll m_1^i, \ldots, m_{k_i}^i \gg$ and $l_i = lcm\{m_1^i, \ldots, m_{k_i}^i\}$ for all $i \in I$.*
- *Let $A \subseteq H \oplus (\oplus_{i\in I}\mathbb{N}_0 c_i)$.*
- *Let $S = A \cup (\cup_{i\in I}\iota_i(S_i))$.*
- *Define $\overline{G} = H \oplus (\oplus_{i\in I}\mathbb{Z}/l_i\mathbb{Z})$.*
- *Suppose that if*

$$\prod_{i\in I}\left(\prod_{s_i\in S_i}(\iota_i(s_i))^{u_{s_i}}\right)\cdot\prod_{a\in A}a^{v_a}$$

is an irreducible block in $\mathcal{B}(G,S)$, then

$$\prod_{a\in A}\overline{a}^{v_a}$$

is an irreducible block in $\mathcal{B}(\overline{G},\overline{A})$.

- *For every block*

$$B = \prod_{i\in I}\left(\prod_{s_i\in S_i}(\iota_i(s_i))^{u_{s_i}}\right)\cdot\prod_{a\in A}a^{v_a}$$

in $\mathcal{B}(G,S)$, define the function $k : \mathcal{B}(G,S) \to \mathbb{Q}^+$ by

$$k(B) := \sum_{a\in A}\frac{v_a}{ord(\bar{a})},$$

where $ord(\bar{a})$ is the order of $\overline{a} \in \overline{G}$.

Then $\rho(G,S) = 1$ if and only if $k(B) = 1$ for every irreducible block $B \in \mathcal{B}(G,S)$.

For the case in this section where $G = H \oplus (\oplus_{i\in I}\mathbb{Z})$, we can add a second corollary to Proposition 3.4. Using Corollary 3.5, we can replace each element of S with an element that is less than or equal to l_i in each slot. This is similar to Corollary 2.26 for $\mathbb{Q}$.

Corollary 3.6. *Let I be a nonempty set, H a torsion group, and $G = H \oplus (\oplus_{i\in I}\mathbb{Z})$. Label the canonical projections and injections as π_H, π_i, ι_H, and ι_i, respectively.*

- *Let* $S_i = \{-m_1^i, \ldots, -m_{k_i}^i\} \subset \mathbb{Z}$ *with each* $m_j^i \in \mathbb{N}$.
- *Define* $c_i = \ll m_1^i, \ldots, m_{k_i}^i \gg$ *and* $l_i = lcm\{m_1^i, \ldots, m_{k_i}^i\}$ *for all* $i \in I$.
- *Let* $A \subseteq H \oplus (\oplus_{i\in I}\mathbb{N}_0 c_i)$.
- *Let* $S = A \cup (\cup_{i\in I}\iota_i(S_i))$.
- *Define* $\bar{G} = H \oplus (\oplus_{i\in I}\mathbb{Z}/l_i\mathbb{Z})$.
- *Suppose that if*

$$\prod_{i\in I}\left(\prod_{s_i\in S_i}(\iota_i(s_i))^{u_{s_i}}\right)\cdot\prod_{a\in A}a^{v_a}$$

is an irreducible block in $\mathcal{B}(G,S)$, *then*

$$\prod_{a\in A}\overline{a}^{v_a}$$

is an irreducible block in $\mathcal{B}(\overline{G},\overline{A})$.

- $-S_i = \{m_1^i, \ldots, m_{k_i}^i\}$ *is* $l_i -$ *split for all* $i \in I$ *(this assumption can be proven to imply the previous assumption about irreducible blocks in* $\mathcal{B}(G,S)$ *having a corresponding irreducible block in* $\mathcal{B}(\overline{G},\overline{A})$*).*
- *Define* A^* *by*

$$A^* = \{g \in G \mid \text{for some } a \in A, \pi_h(g) = \pi_h(a) \text{ and } \pi_i(g) \equiv \pi_i(a)(mod\ l_i) \\ \text{with } 0 \leq \pi_i(g) < l_i \text{ for all } i \in I\}$$

(i.e., for each element in A, *put the same element element in* A^*, *but for each* $i \in I$ *replace the ith coordinate with the remainder when divided by* l_i*).*

- *Define* $S^* := A^* \cup (\cup_{i\in I}\iota_i(S_i))$.

Then $\rho(G,S) = \rho(G,S^*)$.

Proof. First we will show that $\mathcal{B}(G,S^*)$ satisfies the requirements of Proposition 3.4. We will use the same notation as Proposition 3.4, but will use $*$ to denote the notation for the set S^*. Then $S_i^* = S_i$ and $c_i^* = << S_i^* >>= c_i$. Also $l_i^* = lcmS_i = l_i$.

Then we need to show that $A^* \subseteq H \oplus (\oplus_{i\in I}\mathbb{N}_0 c_i)$. Let $a \in A^*$. Then there exists $a \in A$ such that $\pi_i(a) \equiv \pi_i(a^*)(\text{mod } l_i)$ for all $i \in I$. It was assumed that $A \subseteq H \oplus (\oplus_{i\in I}\mathbb{N}_0 c_i)$. Then $c_i|\pi_i(a)$. Recall that Lemma 2.3 shows that $c_i|l_i$. Then $c_i|\pi_i(a^*)$ for all $i \in I$. Now $\pi_i(a) \geq 0$ for all $i \in I$. Thus $\pi_i(a) \in \mathbb{N}_0 c_i$. Therefore $a^* \in H \oplus (\oplus_{i\in I}\mathbb{N}_0 c_i)$ and $A^* \subseteq H \oplus (\oplus_{i\in I}\mathbb{N}_0 c_i)$.

Define $\bar{G} = H \oplus (\oplus_{i\in I}\mathbb{Z}/l_i\mathbb{Z})$. This will be the same factor group in the proposition that we use for S^* and S. The only other thing we need to show is that if

$$\prod_{i\in I}\left(\prod_{s_i\in S_i}\iota_i(s_i)^{v_{s_i}}\right)\cdot\prod_{a\in A^*}a^{v_a}$$

is an irreducible block in $\mathcal{B}(G, S^*)$, then

$$\prod_{a\in A^*} \overline{a}^{v_a}$$

is an irreducible block in $\mathcal{B}(\overline{G}, \overline{A^*})$.

Suppose $B = \prod_{i\in I} \left(\prod_{s_i\in S_i} \iota_i(s_i)^{u_{s_i}}\right) \cdot \prod_{a\in A^*} a^{v_a}$ is an irreducible block in $\mathcal{B}(G, S^*)$. Then

$$\sum_{i\in I} \left(\sum_{s_i\in S_i} u_{s_i}\iota_i(s_i) \right) + \sum_{a\in A^*} v_a a = 0$$

in G. Let $i \in I$. Then

$$0 = \pi_i \left(\sum_{i\in I} \left(\sum_{s_i\in S_i} u_{s_i}\iota_i(s_i) \right) + \sum_{a\in A^*} v_a a \right) = \sum_{s_i\in S_i} u_{s_i} s_i + \pi_i \left(\sum_{a\in A^*} v_a a \right).$$

Therefore $\sum_{s_i\in S_i} u_{s_i}(-s_i) = \pi_i \left(\sum_{a\in A^*} v_a a\right) \in \mathbb{N}_0 c_i$. If the sum is equal to 0, then $\pi_i \left(\sum_{a\in A^*} v_a a\right) \in \mathbb{N}_0 l_i$. Otherwise, assume it is not zero. Lemma 2.3 proved that $l_i = c_i d_i$ for some $d_i \in \mathbb{N}$ with $(d_i, c_i) = 1$. Also, $d_i|(-s_i)$ for all $s_i \in S_i$ (each s_i is negative). Then $\sum_{s_i\in S_i} u_{s_i}(-s_i)$ is divisible by d_i. It was already shown to be divisible by d_i so it is divisible by l_i. Therefore $\sum_{s_i\in S_i} u_{s_i}(-s_i) = \pi_i \left(\sum_{a\in A^*} v_a a\right) \in \mathbb{N} l_i$.

Also,

$$\begin{aligned} 0 = \pi_h(0) &= \left(\sum_{i\in I} \left(\sum_{s_i\in S_i} u_{s_i}\iota_i(s_i) \right) + \sum_{a\in A^*} v_a a \right) \\ &= \pi_h \left(sum_{a\in A^*} v_a a \right). \end{aligned}$$

Then $\sum_{a\in A^*} v_a a \in 0 \oplus (\oplus_{i\in I} \mathbb{N}_0 l_i)$. Therefore $\sum_{a\in A^*} v_a \overline{a} = 0$ in $\overline{G}$. Thus $B' = \prod_{a\in A*} \overline{a}^{v_a}$ is a block in $\mathcal{B}(\overline{G}, \overline{S^*})$.

Suppose that $B' = \prod_{a\in A*} \overline{a}^{v_a}$ is not an irreducible block. Then there exist $1 \le v'_a \le v_a$ for each $a \in A^*$ such that $\sum_{a\in A^*} v'_a \overline{a} = 0$ (and $v'_a < v_a$ for at least one $a \in A^*$). Then $\sum_{a\in A^*} v'_a a = 0 \oplus (\oplus_{i\in I} r_i l_i)$ for some $r_i \in \mathbb{Z}$ for each $i \in I$. By definition of A^*, $\pi_i(a) \ge 0$ for all $a \in A^*$, and $i \in I$. Therefore, each $r_i > 0$. We already showed that $\sum_{s_i\in S_i} u_{s_i}(-s_i) = \pi_i \left(\sum_{a\in A^*} v_a a\right) \in \mathbb{N}_0 l_i$ for all $i \in I$. So there exists $t_i \in \mathbb{N}_0$ for all $i \in I$ such that $\pi_i \left(\sum_{a\in A^*} v_a a\right) = t_i l_i$. Then

$$\begin{aligned} t_i l_i - r_i l_i &= \pi_i \left(\sum_{a\in A^*} v_a a \right) - \pi_i \left(\sum_{a\in A^*} v'_a a \right) \\ &= \left(\sum_{a\in A^*} v_a \pi_i(a) \right) - \left(\sum_{a\in A^*} v'_a \pi_i(a) \right) \\ &= \left(\sum_{a\in A^*} (v_a - v'_a) \pi_i(a) \right) \\ &\ge 0 \end{aligned}$$

for each $i \in I$ since $v_a \geq v'_a$ and $\pi_i(a) \geq 0$ for all $a \in A^*$. Thus $t_i > r_i$ for all $i \in I$.

Then $\sum_{s_i \in S_i} u_{s_i}(-s_i) = \pi_i\left(\sum_{a \in A^*} v_a a\right) = t_i l_i > r_i l_i$. Then since the set $-S$ is l_i-split, there exists $u'_{s_i} \leq u_{s_i}$ such that $\sum_{s_i \in S_i} u'_{s_i} s_i = r_i l_i$. Then

$$C := \prod_{i \in I} \left(\sum_{s_i \in S_i} \iota(s_i)^{u'_{s_i}} \right) \cdot \prod_{a \in A^*} a^{v'_a}$$

is a block dividing B. And $C' \neq B$ since $v'_a < v_a$ for at least one $a \in A^*$. This is a contradiction since B is irreducible. Thus B' must be irreducible.

Then by Theorem 3.4, $\rho(G, S^*) = \rho(\overline{G}, \overline{A^*})$. Also, by the theorem $\rho(G, S) = \rho(\overline{G}, \overline{A})$. By definition of A^*, $\overline{A} = \overline{A^*}$. Therefore $\rho(G, S^*) = \rho(G, S)$. □

Proposition 3.3 can also be restated for this case.

Proposition 3.7. *Let I be a nonempty set, H a torsion group, and $G = H \oplus (\oplus_{i \in I} \mathbb{Z})$. Label the canonical projections and injections as π_H, π_i, ι_H, and ι_i, respectively.*

- *Let $S_i = \{-m_1^i, \ldots, -m_{k_i}^i, n_1^i, n_2^i, n_3^i, \ldots\} \subset \mathbb{Z}$ with each $m_i^i, n_j^i \in \mathbb{N}$.*
- *Define $c_i = \ll m_1^i, \ldots, m_{k_i}^i \gg$ and $l_i = lcm\{m_1^i, \ldots, m_{k_i}^i\}$ for all $i \in I$.*
- *Suppose each $s_i \in S_i^+$ is divisible by c_i.*
- *Let $A \subseteq H \oplus (\oplus_{i \in I} \mathbb{N}_0 l_i)$.*
- *Suppose $\{m_1^i, \ldots, m_{k_i}^i\}$ is l_i-split.*

Define $S = A \cup (\cup_{i \in I} \iota(S_i))$. Then for the block monoid $\mathcal{B}(G, S)$ we can conclude the following:

1. $\mathcal{B}(G, S) = (\cup_{i \in I} \mathcal{B}(\mathbb{Z}, S_i)) \cup \mathcal{B}\left(G, A \cup \left(\cup_{i \in I} S_i^-\right)\right)$.
2. $\rho\left(G, A \cup \left(\cup_{i \in I} S_i^-\right)\right) = \rho(H, \pi_H(A))$, *where π_H is the canonical projection on the first coordinate.*
3. *(Conjecture)* $\rho(G, S) = sup\{\rho(H, \varphi(A))\} \cup \{\rho(\mathbb{Z}, S_i) \mid i \in I\}$.
4. *(Conjecture) $\mathcal{B}(G, S)$ is an HFD if and only if $(H, \pi_H(A))$ is an HFD and $\mathcal{B}(\mathbb{Z}, S_i)$ is an HFD for all $i \in I$.*

3.3 $\mathbb{Z} \oplus \mathbb{Z}$

Now we will turn our attention to an example from [4]. The paper studied Krull domains with divisor class group $\mathbb{Z}$. Example 2.6 dealt with the case where the divisor class group is $\mathbb{Z} \oplus \mathbb{Z}$. The example is presented here.

Example 3.8. Let $G = \mathbb{Z} \oplus \mathbb{Z}$ and

$$S = \{(-1, 0), (1, 0), (2, 0), (3, 0), \ldots\} \cup \{(0, -1), (0, 1), (0, 2), (0, 3), \ldots\}.$$

In this example, S generates G as a monoid so the pair (G, S) is realizable as a Krull domain by Theorem 1.13. The only irreducible blocks are of the form $B_n = (-1, 0)^n \cdot (n, 0)$ and $C_m = (0, -1)^m \cdot (0, m)$ for some $n, m \in \mathbb{N}$. Using Corollary 3.5, $k(B_n) = 1$ and $k(C_m) = 1$ for all $n, m \in \mathbb{N}$; so $\rho(G, S) = 1$. It is easy to prove this without the corollary, which the paper did not have.

Let $S' = \{(-1, -1)\} \cup S$. The paper then uses the following proof to show that $\rho(G, S') = \infty$. Define $A_n = (-1, -1)^n \cdot (1, 0)^n \cdot (0, n)^1$, $B_n = (-1, -1)^n \cdot (n, 0)^1 \cdot (0, 1)^n$, $C_n = (-1, -1)^n \cdot (n, 0)^1 \cdot (0, n)^1$, and $D = (-1, -1)^1 \cdot (1, 0)^1 \cdot (0, 1)^1$ for all $n \in \mathbb{N}$. Then each A_n, B_n, C_n, and D is clearly an irreducible block in $\mathcal{B}(G, S')$. Notice that

$$A_n B_n = C_n D^n$$

are two equivalent factorizations into irreducible blocks for all $n \in \mathbb{N}$. Then $\rho(G, S') > \frac{n+1}{2}$ for all $n \in \mathbb{N}$. Therefore $\rho(G, S') = \infty$. So in this case, adding one element to the set of classes containing a height-one prime ideal changed the elasticity from one to infinity.

The next theorem will generalize the above example to an arbitrary group $S \subseteq \mathbb{Z} \oplus \mathbb{Z}$. In the theorem it will be assumed (changing the notation from the example) that $S, S' \subseteq \mathbb{Z}$ with $|S^+| = \infty$ and $|S^+| = \infty$. So S and S' play the roles of $\{(-1, 0), (1, 0), (2, 0), (3, 0), \ldots\}$ and $\{(0, -1), (0, 1), (0, 2), (0, 3), \ldots\}$, respectively. Also, we place no restrictions on S^- and S'^-. Notice that $(-1, 0)$ and $(0, -1)$ were never used to show the elasticity was infinite. Then we assume one class of the form $(-a, -b)$ where $a, b > 0$ contains a height-one prime ideal. These assumptions are enough to prove the elasticity is infinite in general. The proof will follow the same basic outline as the example above. However, since $S \neq S'$ and $a, b \neq 1$ in general the proof is much more tedious and involves many least common multiples to create irreducible blocks.

Theorem 3.9. *Let $S, S' \subseteq \mathbb{Z}$ with $S_+ = \{n_1, n_2, ...\}$ and $S'_+ = \{m_1, m_2, ...\}$. Let R be a Krull domain corresponding to $(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(-a, -b)\})$, where $a, b > 0$. Then $\rho(R) = \infty$.*

Proof. Let $l_{x,y} = lcm(x, y)$ for any $x, y \in \mathbb{Z}$. Define

$$A = (-a, -b)^f \cdot (n_1, 0)^{x_1} \cdot (0, m_j)^{y_1}$$

and

$$B = (-a, -b)^g \cdot (0, m_1)^{x_2} \cdot (n_i, 0)^{y_2},$$

where

$$f = lcm\left(\frac{l_{a,n_1}}{a}, \frac{l_{b,m_j}}{b}\right)$$

$$g = lcm\left(\frac{l_{a,n_i}}{a}, \frac{l_{b,m_1}}{b}\right),$$

and $x_1 = \frac{af}{c}$, $y_1 = \frac{bf}{m_j}$, $x_2 = \frac{bg}{d}$, and $y_2 = \frac{ag}{n_i}$. Now we need to show that A and B are irreducible blocks. A is a block since, by definition of x_1 and y_1, $f \cdot (-a, -b) + x_1 \cdot (n_1, 0) + y_1 \cdot (0, m_j) = (0, 0)$. Suppose there exists some $f' \leq f$, $x'_1 \leq x_1$ and $y'_1 \leq y_1$ such that $f' \cdot (-a, -b) + x'_1 \cdot (n_1, 0) + y'_1 \cdot (0, m_j) = (0, 0)$. Then $f'a = x'_1 n_1$. so $l_{a,n_1} | (f'a)$. Therefore, $\left(\frac{l_{a,n_1}}{a}\right) | f'$. Similarly, $\left(\frac{l_{b,m_j}}{b}\right) | f'$. But then by definition of f, we have $f | f'$. Thus, we must

have $f = f'$, and hence $x_1' = x_1$ and $y_1' = y_1$. Thus, A is irreducible. By the same argument, B is irreducible. Using the same idea, we can define 2 more irreducible blocks

$$C = h(-a, -b) \cdot z_1(n_i, 0) \cdot w_i(0, m_j)$$

and

$$D = k(a, b) \cdot z_2(n_1, 0) \cdot w_2(0, m_1),$$

where

$$h = lcm\left(\frac{l_{a,n_i}}{a}, \frac{l_{b,m_j}}{b}\right),$$

$$k = lcm\left(\frac{l_{a,n_1}}{a}, \frac{l_{b,m_1}}{b}\right),$$

and $z_1 = \frac{ah}{n_i}$, $w_1 = \frac{bh}{m_j}$, $z_2 = \frac{ak}{n_1}$, and $w_2 = \frac{bk}{m_1}$. Thus, we have

$$A^{khg}B^{kfh} = C^{fkg}D^{fhg}.$$

So the elasticity is greater than

$$\frac{fkg + fhg}{khg + kfh} > \frac{fhg}{hk(g+f)} = \frac{fg}{k(g+f)}.$$

Note that k is fixed with regard to the choice of i and j. So if $\frac{fg}{g+f}$ is unbounded as i and j increase, then we are done. The sets $\{n_i\}$ and $\{m_j\}$ are unbounded above; so f and g are unbounded above, as we let i and j vary. Let M be a positive integer. Then there exists i and j such that $f > 2M$ and $g > 2M$. Then $fg = \frac{1}{2}fg + \frac{1}{2}fg > Mf + Mg$. So $\frac{fg}{f+g} > M$, and we are done. □

The concept for the above proof is the same as the example. We let n_i and m_j become large, but n_1 and m_1 remain fixed (small). So in some sense the irreducibles A and B are hiding many copies of $(n_1, 0)$ and $(0, m_1)$ underneath the larger elements $(0, m_j)$ and $(n_i, 0)$. Thus, there is a medium number of these irreducibles in the factorization on the left-hand side of

$$A^{khg}B^{kfh} = C^{fkg}D^{fhg}.$$

Then when we refactor, we group $(n_i, 0)$ and $(0, m_j)$ to create the block corresponding to C. But D has $(n_1, 0)$ and $(0, m_1)$ grouped together. We think of n_1 and m_1 as being small; so D has few copies of $(n_1, 0)$ and $(0, m_1)$ inside it compared to A and B. Thus, we end up with a very large number of $D's$ on the right side to balance the equation. Note that f, g, h all increase to infinity while k is fixed; so the exponent of D increases to ∞ much faster than all the other exponents. As i and j increase, this gives us an infinite elasticity. The theorem gives rise to the following corollary.

Corollary 3.10. *Let I be a nonempty set, H a group, and $G = H \oplus (\oplus_{i \in I}\mathbb{Z})$. Label the canonical projections and injections as π_H, π_i, ι_H, and ι_i, respectively.*

- *Let $S \subseteq G$.*
- *Define $T_i = (\{s \in S \mid \pi_h(s) = 0$ and $\pi_j(s) = 0$ for all $i \neq j \in I\})$ for all $i \in I$.*

- *Suppose that* $(\pi_i(T_i))^+$ *and* $(\pi_j(T_j))^+$ *are infinite sets for some* $i, j \in I$ *with* $i \neq j$.
- *Suppose* $c \in S$, *where* $c = \iota_i(-a) + \iota_j(-b)$ *for some* $a, b \in \mathbb{N}$.

Then

$$\rho(G, S) = \infty.$$

Proof. By Lemma 1.15, $\rho(G, S) > \rho(G, T_i \cup T_j \cup \{c\})$. Define $f := (\pi_i, \pi_i) : G \to \mathbb{Z} \oplus \mathbb{Z}$ by $f(g) = (\pi_i(g), \pi_i(g))$. Then it is clear that by Lemma 1.14, that $\rho(G, T_i \cup T_j \cup \{c\}) = \rho(\mathbb{Z} \oplus \mathbb{Z}, f(T_i) \cup f(T_j) \cup \{(a, b)\}$. But $f(T_i)$ has infinitely many elements of the form $(x, 0)$, where $x > 0$, and $f(T_j)$ has infinitely many elements of the form $(0, y)$, where $y > 0$. Then by Theorem 3.9, $\rho(\mathbb{Z} \oplus \mathbb{Z}, f(T_i) \cup f(T_j) \cup \{(a, b)\} = \infty$. □

Theorem 3.9 shows that if we want a Krull domain with divisor class group $\mathbb{Z} \oplus \mathbb{Z}$ to be an HFD, or even have finite elasticity, then no divisor class of the form $(-a, -b)$ where $a, b > 0$ should contain a height-one prime ideal (supposing that there are infinitely many classes of the form $(c, 0)$ and $(0, d)$, where $c, d > 0$ containing height-one prime ideals). This corollary applies the result to an arbitrary group $G = H \oplus (\oplus_{i \in I} \mathbb{Z})$. If we want the elasticity to be one, or just finite, then no divisor class containing a height-one prime ideal can contain an element negative in two slots and zero everywhere else (assuming there are infinitely many positive elements in those slots). This suggest that similar results should hold for an element that is negative in more than two slots, and there are infinitely many positive elements in the corresponding slots. So in Sections 1 and 2 of chapter 3, we have used this assumption that an element should be negative in one slot only, and zero everywhere else. However little is known for these cases where an element is negative in more than two slots. It certainly has not been proven this must be the case for the Krull domain to be an HFD.

The rest of the section will be concerned with divisor class group $\oplus_{i \in I} \mathbb{Z}$. We will use $\iota_i : \mathbb{Z} \to \oplus_{i \in I} \mathbb{Z}$ and $\pi_i : \oplus_{i \in I} \mathbb{Z} \to \mathbb{Z}$ to be inclusion into the *ith* slot and projection out of the *ith* slot respectively. In light of the previous lemma, we will restrict to the case that if an element of S is negative in one slot, then then it must be zero in every other slot. In other words, if $s \in S$ with $\pi_j(s) < 0$, then $\pi_i(s) = 0$ for all $i \in I$ with $i \neq j$. We will also assume that each slot has exactly one element that is negative in that slot. For a subset $S \subseteq \oplus_{i \in I} \mathbb{Z}$, define $S^+ := \{s \in S \mid \pi_i(s) \geq 0 \text{ for all } i \in I\}$. Then our assumptions about S translate to $S - S^+ = \{\iota_i(-x_i) \mid i \in I\}$, where each $x_i \in \mathbb{N}$.

The following proposition will be the basis for the rest of the section. It is the generalization to a direct sum of copies of $\mathbb{Z}$ of a proof in [5] that concerned having only one copy of $\mathbb{Z}$.

Proposition 3.11. *Let* T *be a subset of* $G = \oplus_{i \in I} \mathbb{Z}$. *Suppose that* $\{x_i \mid i \in I\} \subset \mathbb{N}_0$ *and* $T - T^+ = \{\iota_i(-x_i) \mid i \in I\}$. *Then*

$$\rho(G, T) = \rho\left(\oplus_{i \in I} \left(\mathbb{Z}/x_i\mathbb{Z}\right), \overline{T}\right).$$

Proof. This is a trivial corollary to Proposition 3.1. Note that each $c_i = 1$; so the conditions relative to c_i are obviously satisfied. □

Now we can list several simple corollaries to Proposition 3.11. Their proofs are easy and will be omitted.

Corollary 3.12. *Let $T \subseteq \mathbb{Z} \oplus \mathbb{Z}$ and $(-x, 0)$ and $(0, -y)$ in T with $x, y > 0$. Define $T^+ = \{t \in T \mid \pi_1(t) > 0 \text{ and } \pi_2(t) > 0\}$. If $\mathcal{B}(\mathbb{Z} \oplus \mathbb{Z}, T)$ is half-factorial, then $\rho(\mathbb{Z}/(x\mathbb{Z}) \oplus \mathbb{Z}/(y\mathbb{Z}), \overline{T^+}) = 1$.*

Corollary 3.13. *Let $S = \{-x, n_1, n_2, ...\}$ and $S' = \{-y, m_1, m_2, ...\}$, with $x, y, n_i, m_j > 0$. Let $a, b > 0$. Then the following are true:*

1. *If $x|a$ and $y|b$, then*

$$\rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(a, b)\}) = \rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S')).$$

2. *If $y|b$, then*

$$\rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(a, b)\}) = \rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(a, 0)\}).$$

3. *If $x|a$, then*

$$\rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(a, b)\}) = \rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(0, b)\}).$$

4. $\rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S')) = max(\rho(\mathbb{Z}, S), \rho(\mathbb{Z}, S'))$.

5. *If $y|b$, then*

$$\rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(a, b)\}) = max(\rho(\mathbb{Z}, S \cup \{a\}), \rho(\mathbb{Z}, S')).$$

What if x does not divide a, and y does not divide b? Do we get a relationship between $\rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup (a, b))$ and $\rho(\mathbb{Z} \oplus \mathbb{Z}, ((S \cup \{a\}) \oplus 0) \cup (0 \oplus (S' \cup \{b\})))$ (which is equivalent to $max(\rho(\mathbb{Z}, S \cup \{a\}), \rho(\mathbb{Z}, S' \cup \{b\})))$? The following two examples will prove in general there is no relationship.

Example 3.14. Consider $\mathcal{B}(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S) \cup \{(2, 2)\})$ with $S = \{-3, 1, 3, 6, 9, ...\}$. Using Corollary 3.12, this has the same elasticity as $(\mathbb{Z}/(3\mathbb{Z}) \oplus \mathbb{Z}/(3\mathbb{Z}), \{(1, 0), (0, 1), (2, 2)\})$. There are only four irreducible blocks: $(2, 2)^3$, $(1, 0)^3$,$(0, 1)^3$, and $(2, 2) \cdots (1, 0) \cdot (0, 1)$. It is clear the the cross number is one for each irreducible block so it is half-factorial. Thus $\mathcal{B}(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S) \cup \{(2, 2)\}) = 1$.

Now consider $(\mathbb{Z}, (S \cup \{2\}))$. Define $B = -3 \cdot 3$, $C = (-3)^2 \cdot (2)^3$, and $D = 3 \cdot 1 \cdot 2$ which are clearly irreducible blocks in $(\mathbb{Z}, (S \cup \{2\}))$. Then $BC = D^3$. Then $(\mathbb{Z}, (S \cup \{2\})) \geq \frac{3}{2}$ and it is actually equal to $\frac{3}{2}$.

Therefore, $\rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S) \cup \{(a, b)\}) < max(\rho(\mathbb{Z} \oplus \mathbb{Z}, S \cup \{a\}), \rho(\mathbb{Z}, S \cup \{b\}))$. So in this case $(a, b) = (2, 2)$ and $x = y = 3$.

Example 3.15. Now consider $\mathcal{B}(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(1, 1)\})$ with $S = \{-2, 1, 2, 4, 6, ...\}$ and $S' = \{-3, 1, 3, 6, 9, ...\}$. Then using the notation above $(a, b) = (1, 1)$, $x = 2$, and $y = 3$. Clearly, $\rho(\mathbb{Z}, S \cup \{a\}) = \rho(\mathbb{Z}, S) = 1$ and $\rho(\mathbb{Z}, S' \cup \{b\}) = \rho(\mathbb{Z}, S') = 1$. However, in $(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(1, 1)\})$ we have the following irreducible blocks: $B = (-2, 0)^3 \cdot (0, -3)^2 \cdot 6 * (1, 1)$, $C = (-2, 0) \cdot (1, 0)^2$, $D = (0, -3) \cdot (0, 1)^3$, and $E = (-2, 0) \cdot (0, -3) \cdot (1, 0) \cdot (1, 1) \cdot (0, 1)^2$. Then $BC^3D^4 = E^6$.

Thus we have 8 irreducible blocks factoring into 6 irreducible blocks so $\rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(1, 1)\}) \geq \frac{4}{3}$. It is obvious that $\rho(\mathbb{Z}, S) = \rho(\mathbb{Z}, S') = 1$. So $\rho(\mathbb{Z} \oplus \mathbb{Z}, (S \oplus 0) \cup (0 \oplus S') \cup \{(a, b)\}) > max(\rho(\mathbb{Z}, S \cup \{a\}), \rho(\mathbb{Z}, S' \cup \{b\}))$.

Chapter 4

Summary and Future Directions

In this dissertation we have studied three specific divisor class groups and general infinite divisor class groups. For a Krull domain R with divisor class group $\mathbb{Z}(p^\infty)$, Theorem 2.18 characterizes exactly when R is an HFD. Future research could attempt to calculate the elasticity of a Krull domain with divisor class group $\mathbb{Z}(p^\infty)$, but the HFD case is finished.

For a Krull domain with divisor class group $\mathbb{Z}$ and $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots\} \subseteq \mathbb{Z}$ the set of classes containing a height-one prime ideal, Theorem 2.6 completely characterizes when R is an HFD. Theorem 2.13 provides another characterization using a generalization of the cross number. There are three more cases for the set S. The case where S^- is infinite and S^+ is finite is the same as the case above by a simple automorphism argument. If S^+ and S^- are both infinite, then Theorem 2.1 showed that the elasticity of the domain is infinite [5].

The last case is S^+ and S^- are both finite. This is equivalent to S being finite since S must contain both positive and negative elements. In this case, the elasticty is rational and realizable [2]. However, when R is an HFD has not been completely characterized. Let $S = \{-m_1, \ldots, -m_k, n_1, n_2, n_3, \ldots, n_l\} \subseteq \mathbb{Z}$ with each $m_i, n_j > 0$. Define $c = \ll m_1, \ldots, m_k \gg$. If $\rho(\mathbb{Z}, S) = 1$, then c may not divide each n_j. The proof in [4] that $c|n_i$ for all $i \in \mathbb{N}$ required that S^+ is infinite. Let $S' = \{-5, -4, -3, -2, -1, 1\}$. Then $\mathcal{B}(\mathbb{Z}, S')$ is an HFD using a simple automorphism argument and Theorem 2.11. Notice that $\ll 5, 4, 3, 2, 1 \gg = 60$ which does not divide 1. However $\ll 1 \gg = 1$ and this does divide $1, 2, 3, 4$ and 5. So when S^+ and S^- are both finite, the value c may need to be calculated using the positive or negative elements of S. We conjecture here that if $\mathcal{B}(G, S)$ is half-factorial then either $c^- = \ll m_1, \ldots, m_k \gg$ divides each n_j or $c^+ = \ll n_1, \ldots, n_l \gg$ divides each m_i. This has not been proven. In the special case when $k = 1$, Theorem 2.11 does characterize the HFDs without needing any assumptions about c.

For a Krull domain R with divisor class group $\mathbb{Q}$, Corollary 2.29 characterizes the HFDs, but only if S contains exactly one negative element. The proof required studying a related block monoid $\mathcal{B}(\mathbb{Z}, T)$ where T contains one negative element and finitely many positive elements. Theorem 2.11 was then applied to the block monoid $\mathcal{B}(\mathbb{Z}, T)$. But this theorem requires that T contains only one negative element. The corresponding theorem when $|T^-| > 1$, required that $|T^+| = \infty$. This leaves us with the same problem discussed in the previous paragraph: determining when $\mathcal{B}(\mathbb{Z}, T)$, where T is finite, is half-factorial. Therefore solving this problem will allow us to prove results about Krull domains with divisor class group $\mathbb{Q}$, where S contains more than one negative element.

As mentioned in Chapter 3, future areas of research could study when the conditions

of Proposition 3.1 and 3.3 hold. Let G be a group, $S \subseteq G$, and $X \subseteq G$ a maximal set of independent elements of infinite order. If $\mathcal{B}(G, S)$ is half-factorial, what properties must S satisfy and how does this relate to X? Also, we could study what properties S must satisfy for the pair (G, S) to be realizable as a Krull domain.

Bibliography

[1] D. D. Anderson and D. F. Anderson. Elasticity of factorizations in integral domains. *J. Pure Appl. Algebra*, 80(3):217–235, 1992.

[2] D. D. Anderson, D. F. Anderson, S. T. Chapman, and W. W. Smith. Rational elasticity of factorizations in Krull domains. *Proc. Amer. Math. Soc.*, 117(1):37–43, 1993.

[3] D. D. Anderson, D. F. Anderson, and M. Zafrullah. Factorization in integral domains. *J. Pure Appl. Algebra*, 69(1):1–19, 1990.

[4] D. F. Anderson, S. T. Chapman, and W. W. Smith. On Krull half-factorial domains with infinite cyclic divisor class group. *Houston J. Math.*, 20(4):561–570, 1994.

[5] D. F. Anderson, S. T. Chapman, and W. W. Smith. Some factorization properties of Krull domains with infinite cyclic divisor class group. *J. Pure Appl. Algebra*, 96(2):97–112, 1994.

[6] L. Carlitz. A characterization of algebraic number fields with class number two. *Proc. Amer. Math. Soc.*, 11:391–392, 1960.

[7] L. Claborn. Every abelian group is a class group. *Pacific J. Math.*, 18:219–222, 1966.

[8] L. Claborn. Specified relations in the ideal group. *Michigan Math. J.*, 15:249–255, 1968.

[9] P. M. Cohn. Bezout rings and their subrings. *Proc. Cambridge Philos. Soc.*, 64:251–264, 1968.

[10] R. M. Fossum. *The divisor class group of a Krull domain*. Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74.

[11] W. Gao and A. Geroldinger. Half-factorial domains and half-factorial subsets of abelian groups. *Houston J. Math.*, 24(4):593–611, 1998.

[12] A. Geroldinger and R. Göbel. Half-factorial subsets in infinite abelian groups. *Houston J. Math.*, 29(4):841–858 (electronic), 2003.

[13] A. Geroldinger and F. Halter-Koch. *Non-unique factorizations*, volume 278 of *Pure and Applied Mathematics (Boca Raton)*. Chapman & Hall/CRC, Boca Raton, FL, 2006. Algebraic, combinatorial and analytic theory.

[14] A. P. Grams. Atomic rings and the ascending chain condition for principal ideals. *Proc. Cambridge Philos. Soc.*, 75:321–329, 1974.

[15] A. P. Grams. The distribution of prime ideals of a Dedekind domain. *Bull. Austral. Math. Soc.*, 11:429–441, 1974.

[16] W. Krull. Über die Zerlegung der Hauptideale in allgemeinen Ringen. *Math. Ann.*, 105(1):1–14, 1931.

[17] L. Skula. On c-semigroups. *Acta Arith.*, 31(3):247–257, 1976.

[18] J. Steffan. Longueurs des décompositions en produits d'éléments irréductibles dans un anneau de Dedekind. *J. Algebra*, 102(1):229–236, 1986.

[19] R. J. Valenza. Elasticity of factorization in number fields. *J. Number Theory*, 36(2):212–218, 1990.

[20] A. Zaks. Half factorial domains. *Bull. Amer. Math. Soc.*, 82(5):721–723, 1976.

[21] A. Zaks. Half-factorial-domains. *Israel J. Math.*, 37(4):281–302, 1980.

Vita

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