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A COMPARATIVE STUDY OF MODEL REDUCTION METHODS

A Thesis

Presented for the

Master of Science

Degree

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ABSTRACT

Model reduction techniques are grouped into four approaches according to their underlying principle, which are (1) Continued fraction expansion approach, (2) Dominant mode approach, (3) Optimum fitting approach, (4) Singular value decomposition approach.

Based on the simulation of the step response and the least sum of the squares of the sample erorrs, the simplified models are evaluated. In practical application, the quality of approximation and the ease of hand calculation are considered. Some complicated techniques can be replaced by a simple calculation in the single variable case so that they are seen to be equivalent.

A new generalized Cauer form of continued fraction expansion method is developed, which provides a general algorithm to match different numbers of time moments or Markov parameters so that different Cauer forms can be a special case of the new method.

For the discrete-time transfer function obtained by the least square method, a new transform approach is presented to find the equivalent continuous-time transfer function. The denominator of the transfer function shows that the simplified low-order models need "optimum poles" instead of "dominant poles" in order to approximate the original model.

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CHAPTER 1

INTRODUCTION

The study of linear dynamic systems behavior consists of analysis, simulation, and design. Often, the order of a physical system is so high that the dynamic behavior is difficult to analyze. The reduction of high-order systems becomes important for the analytical consideration of the systems. For example, the complex circuits, process control systems, and power systems may often be of more than tenth order and their complexity often makes its behaviors difficult to be understood. The lower-order equivalent model can save much analytical work for such large systems.

In general, the reduced-order models can be applied in the following uses.

a. To simulate the dynamic behavior of a system by a simplified model.

b. To design a controller based on reduced models with less computational efforts [1].

c. To obtain a lower dimensional control law or the optimal control policy by using a low order model [2,3].

During the last two decades, a great deal of work has been done on the lowering of the model order. Sinha [4] divided the techniques of the model reduction into three groups which are dominant mode methods, optimum approximation methods, and continued fraction or time-moment matching methods. Gerstle [1] added one technique --

internal balance method and used those reduced models in controller design.

Since the dominant mode method by retaining dominant eigenvalues fails to match with the steady-state response of the original system, a compensative approach should be taken in order to maintain the steady-state gain of the original system. On the other hand, the continued fraction expansion methods can be seen as special cases of matching Markov parameters [4,5]. The optimum approaches are supposed to provide the best result or the reduced model but require more computation time. The internal balance method proposed by Moore [1,6] reorganizes the original model with an internal coordinate transformation and eliminates weak subsystem which contributes little to the impulse response matrix. Therefore, the internal balance method by retaining dominant subsystem has a similar viewpoint to the dominant mode method.

A model reduction approach may yield a good result in a special condition but not for other cases. Especially, the continued fraction expansion method is the simplest method in computational effort, as compared with another methods. Its reduced model always gives good approximation, but sometimes the resulting models may be unstable even though the original system is stable [4]. Therefore, the development of a reliable and accurate approach to model reduction for arbitrary systems is important.

Using sum of the squares of the sample errors as the optimum index, this research will investigate the best approach of model

reduction methods in general cases. Each approach will be illustrated by a numerical example. The properties of different methods will also be compared by simulation of the reduced models.

CHAPTER 2

REVIEW OF THE LITERATURE

2-1. Review of Continued Fraction Methods

Chen and Shieh [7] proposed the continued fraction method [8,9,10] in which the model simplification can be performed by an algebraic method. The result of Chen and Shieh's method gives the correct steady-state reponse, but the initial transient response of the original system may not be matched. Furthermore, the stability of the simplified model is not guaranteed even if the original system is stable. Actually, Chen and Shieh's method is the application of the continued fraction expansion of the Cauer second form. Bosley et al. [11] showed that the continued fraction expansion of the Cauer second form is equivalent to matching time moments with a Taylor series expansion about s=0, and the Cauer first form expansion is equivalent to a Taylor series expansion about $s=\infty$. By mixing the first and the second Cauer form two improved versions of the continued fraction expansion were developed.

One is the so-called "Cauer third method" which was proposed by Shieh and Goldman [12]. Hwang [13] indicated that the Cauer third form is equivalent to mathing the first m time-moments of the impulse response and the first m Markov parameters, where m is the order of the reduced model. The continued fraction expansion of the Cauer third form is also equivalent to a Taylor series expansion

of the transfer function with respect to s=0 and s=∞, respectively.

The other method is "the modified Cauer form method" which is proposed by Chuang [14,15]. Shamash [16] showed that the method is equivalent to the continued fraction expansion about s=0 for m+k terms and about s=∞ for m-k terms, where m is the order of the reduced model and k is an integer satisfying the condition $|k|<m$. The reduced model is to fit the m+k time-moments and m-k Markov parameters. For k=0, the method gives the same reduced model as the results of Chuang's method.

Lucas [5] presented an algorithm in which the combination of retained time-moments and Markov parameters [4,17] could be varied and the methods of Chen and Shieh [7] and Chuang [14,15] could be seen as particular cases of the the algorithm.

2-2. Review of Dominant Mode Methods

The principle of the dominant mode method is to neglect eigenvalues of the original system, which are farthest from the origin, and retain only dominant eigenvalues. This implies that the overall behavior of the approximate system will be very similar to the original system since the contribution of the unretained eigenvalues to the system response are important only at the beginning of the response, whereas the eigenvalues retained are important throughout the whole of the response. Davison [18] first proposed the dominant mode reduction method by retaining dominant eigenvalues. Aoki [19] used an aggregation matrix to obtain the simplified model, which has

been shown as a generalized dominant mode method [20]. Mahapatra [21,22], Elrazaz and Sinha [23,24], and Rao, et al. [25] suggested selecting criteria of the reduced model order. The aggregated model which retained dominant eigenvalues of the original system is stable if the original system is stable. Unfortunately, the steady-state response of the reduced model obtained by dominant mode method can't always match with that of the original system. A compensation method, called partial realization method, can be used by matching time moments and Markov parameters to give a better result [1,26]. This method retains dominant eigenvalues and gives accurate steady state response.

2-3. Review of Optimum Approaches

The optimum reduced models are based on identification techniques which numerically match either the time-domain response or the frequency-domain response of the full model. Anderson's method [27] based on fitting the mean-square error between the response of the full model and the reduced model over a finite interval. Sinha and Pille [28] used the matrix pseudo-inverse method as the same manner as Anderson's method but easier to be implimented without matrix inversion. The suggested continuous model by Sinha and Pille is not a general equation which can be improved to give a better result as shown in Chapter 5. Wilson [29] formulated equations for the optimum reduced model of a multivariable time-invariant linear system by minimizing a function of error between the output vector of the

reduced model and the original model. Sinha and Bereznai [30] developed a more general method which can provide an optimum low-order model with respect to any specified criterion. Riggs and Edgar [31] proposed a least square reduction method by using impulse response. Edgar [32] applied the least square algorithm for model reduction by using step response. But the least square algorithm proposed by Riggs Edgar can only be used in distinct real eigenvalues cases. Chung et al. [33] employed Golub's least squares method [34] for discrete model simplification and identification, which appeared to give a good result but didn't concern with the continuous transfer function.

Using optimum approaches may have following drawbacks.

a. Reducing models by optimization approaches requires that the modeler to judge error criteria.

b. The computer time in calculating the reduced model may be large.

c The optimum model is a function of the input so that different inputs have different optimum approaches and different results.

2-4. Review of Singular Value Decompositions

The singular value decomposition technique is based on the minimal realization theory [35,36]. According to this theory, a lower order model can be obtained by eliminating any weak subsystem which contributes little to the impulse response matrix [6]. Kung [37] used the Hankel matrix formed from a set of Markov parameters to solve the model reduction problems. Moore [6] employed internal

balancing concept to reorganize the full model so that a dominant subsystem can be identified. So far the reduced model by singular value decomposion method can't obtain a good approximation [1,6,38]. Thus, the method can be considered a further research area.

CHAPTER 3

CONTINUED FRACTION EXPANSION METHODS

3-1. Cauer First Form

The Cauer first form was not intended for model reduction originally. This form is introduced here as a reference for generalized Cauer forms. Given the transfer function of nth order,

$$G(s) = \frac{B_{20}s^{n-1}+B_{21}s^{n-2}+ \ldots +B_{2,n-2}s+B_{2,n-1}}{B_{10}s^{n}+B_{11}s^{n-1}+ \ldots .. +B_{1,n-1}s+B_{1n}} \tag{3-1}$$

the continued fraction expansion of the Cauer first form is

$$G(s) = \cfrac{1}{H_1 s + \cfrac{1}{H_2 + \cfrac{1}{H_3 s + \cfrac{1}{H_4 + \cfrac{1}{\ddots}}}}} \tag{3-2}$$

The coefficients of the Cauer first form can be calculated by the following Routh array,

$$\begin{array}{lccccccc} & B_{10} & B_{11} & B_{12} & \cdots & B_{1,n-2} & B_{1,n-1} & B_{1n} \\ H_1 < & & & & & & & \\ & B_{20} & B_{21} & B_{22} & \cdots & B_{2,n-2} & B_{2,n-1} & \\ H_2 < & & & & & & & \\ & B_{30} & B_{31} & B_{32} & \cdots & B_{3,n-2} & B_{3,n-1} & \\ H_3 < & & & & & & & \\ & B_{40} & B_{41} & B_{42} & \cdots & B_{4,n-2} & & \\ H_4 < & & & & & & & \\ & B_{50} & B_{51} & B_{52} & \cdots & B_{5,n-2} & & \end{array} \tag{3-3}$$

or written in computational algorithm

$$H_k = \frac{B_{k0}}{B_{k+1,0}} \tag{3-4}$$

$$B_{ij} = B_{i-2,j+1} - H_{i-2} B_{i-1,j+1} \tag{3-5}$$

$$B_{k+1,0} \neq 0 \tag{3-6}$$

where $k=1,2,3, \ldots ,2n$

$i=3,4,5, \ldots ,2n+1$

$j=0,1,2, \ldots ,n-1.$

For a reduced model of m-th order, the first 2m coefficients must be calculated.

In case of the second order reduced transfer function

$$G^*(s) = \frac{C_{21}s+C_{20}}{C_{12}s^2+C_{11}s+C_{10}} \tag{3-7}$$

the expansion and inversion of the Cauer first form can be obtained as

$$G^*(s) = [H_1 s+[H_2+[H_3 s+[H_4]^{-1}]^{-1}]^{-1}]^{-1} \tag{3-8}$$

or

$$G^*(s) = \frac{H_2H_3H_4s+(H_2+H_4)}{H_1H_2H_3H_4s^2+(H_1H_2+H_1H_4+H_3H_4)s+1} \tag{3-9}$$

The relationships between C's and H's can be obatined from equations (3-7) and (3-9).

The characteristics of the full model, retained by the reduced model may be illustrated by equation (3-9). Applying the final value

theorem to equations (3-1) and (3-9) with a step input yields

$$\lim_{s \to 0} G(s) = \frac{B_{2,n-1}}{B_{1n}} \tag{3-10}$$

$$\lim_{s \to 0} G^*(s) = H_2 + H_4 = \frac{B_{20}}{B_{30}} + \frac{B_{40}}{B_{50}} \tag{3-11}$$

while the initial value for an impulse input gives

$$\lim_{s \to \infty} sG(s) = \frac{B_{20}}{B_{10}} \tag{3-12}$$

$$\lim_{s \to \infty} sG^*(s) = \frac{1}{H_1} = \frac{B_{20}}{B_{10}} \tag{3-13}$$

From equations (3-10) through (3-13) the redeced model of the Cauer first form is seen to maintain the fast components of the system, but the final value theorem shows that the steady state components are not maintained.

Example 3-1: Consider the following transfer function

$$G(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)(s+5)} = \frac{s^2+3s+2}{s^3+12s^2+47s+60} \tag{3-14}$$

The coefficients can be calculated by equations (3-3) through (3-6)

$H_1 = 1$	1	12	47	60
$H_2 = .11111$	1	3	2	
$H_3 = -4.5$	9	45	60	
$H_4 = -.0833$	-2	-4.6667		
	24			

and the reduced model can be obtained by equation (3-9) as shown in equation (3-15).

$$G_1^*(s) = \frac{s+0.6667}{s^2+9.6667s+24} \tag{3-15}$$

3-2. Cauer Second Form

The use of the Cauer second form of continued fraction expansion for model reduction was proposed by Balabanian [8]. The reduced model of the Cauer second form has different characteristics from that of the Cauer first form. To obtian the Cauer second form, one can rearrange equation (3-1) as

$$G(s) = \frac{A_{20}+A_{21}s+ \cdots +A_{2,n-2}s^{n-2}+A_{2n}s^{n-1}}{A_{10}+A_{11}s+ \cdots +A_{1,n-1}s^{n-1}+A_{1n}s^{n}} \tag{3-16}$$

The continued fracation expansion of the Cauer second form is

$$G(s) = \cfrac{1}{h_1+\cfrac{1}{\cfrac{h_2}{s}+\cfrac{1}{h_3+\cfrac{1}{\cfrac{h_4}{s}+\cfrac{1}{\ddots}}}}} \tag{3-17}$$

As in the case of the Cauer first form, the variable h_k can be calculated from the following Routh array

$$
\begin{array}{lccccccc}
 & A_{10} & A_{11} & A_{12} & \cdots & A_{1,n-2} & A_{1,n-1} & A_{1n} \\
h_1 < & & & & & & & \\
 & A_{20} & A_{21} & A_{22} & \cdots & A_{2,n-2} & A_{2,n-1} & \\
h_2 < & & & & & & & \\
 & A_{30} & A_{31} & A_{32} & \cdots & A_{3,n-2} & A_{3,n-1} & \\
h_3 < & & & & & & & \\
 & A_{40} & A_{41} & A_{42} & \cdots & A_{4,n-2} & & \\
h_4 < & & & & & & & \\
 & A_{50} & A_{51} & A_{52} & \cdots & A_{5,n-2} & &
\end{array}
\qquad (3\text{-}18)
$$

or written in computational algorithm

$$h_k = \frac{A_{k0}}{A_{k+1,0}} \qquad (3\text{-}19)$$

$$A_{ij} = A_{i-2,j+1} - h_{i-2}A_{i-1,j+1} \qquad (3\text{-}20)$$

$$A_{k+1,0} \neq 0 \qquad (3\text{-}21)$$

where $k=1,2,3, \ldots ,2n$

$i=3,4,5, \ldots ,2n+1$

$j=0,1,2, \ldots ,n-1$

For a reduced model of m-th order the first 2m coefficients must be calculated.

In case of the second order reduced transfer function

$$G^*(s) = \frac{C_{21}s+C_{20}}{C_{12}s^2+C_{11}s+C_{10}} \qquad (3\text{-}22)$$

the inversion and expansion of the Cauer second form becomes

$$G^*(s) = [h_1+[\frac{h_2}{s}+[h_3+[\frac{h_4}{s}]^{-1}]^{-1}]^{-1}]^{-1} \qquad (3\text{-}23)$$

or

$$G^*(s) = \frac{(h_2+h_4)s+h_2h_3h_4}{s^2+(h_1h_2+h_1h_4+h_3h_4)s+h_1h_2h_3h_4} \tag{3-24}$$

The relationships between C's and h's can be obtained from equations (3-22) and (3-24).

The reduced model of equation (3-24) maintains the steady state components of the original model, which may be shown with application of the final value theorem to equation (3-16) and (3-24) respectively,

$$\lim_{s\to 0} G(s) = \frac{A_{20}}{A_{10}} \tag{3-25}$$

$$\lim_{s\to 0} G^*(s) = \frac{1}{h_1} = \frac{A_{20}}{A_{10}} \tag{3-26}$$

and the initial value for an impulse input gives

$$\lim_{s\to \infty} sG(s) = \frac{A_{2,n-1}}{A_{1n}} \tag{3-27}$$

$$\lim_{s\to \infty} sG^*(s) = h_2+h_4 = \frac{A_{20}}{A_{30}} + \frac{A_{40}}{A_{50}} \tag{3-28}$$

From equations (3-25) through (3-28) one can see the reduced model of the Cauer second form contains the steady state gains of the original model.

Example 3-2: Consider the same original model as example 3-1. The coefficients and the reduced model can be obtained by equations

(3-18) through (3-21) and (3-24)

$$
\begin{array}{lll}
h_1 = 30 & 60 & 47 \quad 12 \quad 1 \\
h_2 = -.0465 & 2 & 3 \quad 1 \\
h_3 = -19.88 & -43 & -18 \quad 1 \\
h_4 = .7707 & 2.1628 & 1.0465 \\
 & 2.806 &
\end{array}
$$

$$
G_2^*(s) = \frac{0.7242s+0.7125}{s^2+6.4045s+21.3735} \tag{3-29}
$$

3-3. Cauer Third Form

Shieh and Goldman [12] introduced the Cauer third form or mixed Cauer form which is a combination of the Cauer first and the Cauer second forms. The motivation for this method is that the resulting reduced model maintains both the fast and steady state characteristics of the full model. The continued fraction expansion in the Cauer third form is

$$
G(s) = \cfrac{1}{H_1 s + h_1 + \cfrac{1}{H_2 + \cfrac{h_2}{s} + \cfrac{1}{H_3 s + h_3 + \cfrac{1}{\ddots}}}} \tag{3-30}
$$

The partial fraction coefficients, H_k and h_k, of the Cauer third form are calculated from the following Routh array

$$
\begin{array}{lccccccc}
h_1 < & A_{10} & A_{11} & A_{12} & \cdots & A_{1,n-2} & A_{1,n-1} & A_{1n} \\
h_2 < & A_{20} & A_{21} & A_{22} & \cdots & A_{2,n-2} & A_{2,n-1} & \\
h_3 < & A_{30} & A_{31} & A_{32} & \cdots & A_{3,n-2} & & \\
 & A_{40} & A_{41} & A_{42} & \cdots & & & \\
 & \cdots\cdots\cdots\cdots\cdots\cdots & & & & & & \\
 & \cdots\cdots\cdots\cdots\cdots\cdots & & & & & &
\end{array}
\qquad
\begin{array}{l}
\\
\\
\\
\\
\\
\\

$$G^*(s) = \cfrac{1}{H_1 s + h_1 + \cfrac{1}{H_2 + \cfrac{h_2}{s}}} \tag{3-36}$$

and inverted into

$$G^*(s) = \frac{H_2 s + h_2}{H_1 H_2 s^2 + (1 + h_1 H_2 + H_1 h_2) s + h_1 h_2} \tag{3-37}$$

Applying the final value thoerem to the equations (3-1) and (3-37) with a step input yields

$$\lim_{s \to 0} G(s) = \frac{A_{20}}{A_{10}} \tag{3-38}$$

$$\lim_{s \to 0} G(s) = \frac{1}{h_1} = \frac{A_{20}}{A_{10}} \tag{3-39}$$

and the initial value for an impulse input gives

$$\lim_{s \to \infty} sG(s) = \frac{A_{2,n-1}}{A_{1n}} \tag{3-40}$$

$$\lim_{s \to \infty} sG(s) = \frac{1}{H_1} = \frac{A_{2,n-1}}{A_{1n}} \tag{3-41}$$

Thus, the reduced model maintains the steady state gain and the fast response of the original model.

Example 3-3: Consider the same original model as example 3-1. The coefficients and the reduced model can be obtained as

$$
\begin{array}{lcccccc}
 & 60 & 47 & 12 & 1 & \\
h_1=30 & & & & & H_1=1 \\
 & 2 & 3 & 1 & & \\
h_2=-.0444 & & & & H_2=-.0476 & \\
 & -45 & -21 & & &
\end{array}
$$

$$G_3^*(s) = \frac{s+0.9333}{s^2+9.9333s+28} \tag{3-42}$$

3-4. The Modified Cauer Form

Chuang [14,15] proposed a modified Cauer form for which the continued fraction expansion is

$$G(s) = \cfrac{1}{h_1 + \cfrac{s}{H_1 + \cfrac{1}{h_2 + \cfrac{s}{H_2 + \cfrac{1}{\ddots}}}}} \tag{3-43}$$

The coefficients can be calculated from the following Routh array

$$
\begin{array}{lcccccccc}
 & A_{10} & A_{11} & A_{12} & \cdots & A_{1,n-2} & A_{1,n-1} & A_{1n} & \\
h_1 & & & & & & & & \\
 & A_{20} & A_{21} & A_{22} & \cdots & A_{2,n-2} & A_{2,n-1} & & H_1 \\
 & A_{30} & A_{31} & A_{32} & \cdots & A_{3,n-2} & A_{3,n-1} & & \\
h_2 & & & & & & & & \\
 & A_{40} & A_{41} & A_{42} & \cdots & A_{4,n-2} & & H_2 & \\
 & A_{50} & A_{51} & A_{52} & \cdots & A_{5,n-2} & & & \\
 & \cdots & \cdots & \cdots & \cdots & & & & \\
 & \cdots & \cdots & \cdots & \cdots & & & &
\end{array}
\tag{3-44}
$$

where

$h_1 = A_{10}/A_{20}$, $H_1 = A_{1n}/A_{2,n-1}$, $h_2 = A_{20}/A_{30}$ etc., and

$$A_{ij} = A_{i-2,j+1} - h_{(i-1)/2} A_{i-1,j+1} \quad (i=3,5,\ldots,2n+1,\ j=0,1,\ldots n-1) \tag{3-45}$$

$$A_{ij} = A_{i-2,j} - H_{(i/2)} A_{i-1,j} \quad (i=4,6,\ldots,2n,\ \ j=0,1,\ldots,n-2)$$

The second order transfer function can be expanded by

$$G^*(s) = \cfrac{1}{h_1 + \cfrac{s}{H_1 + \cfrac{1}{h_2 + \cfrac{s}{H_2}}}} \tag{3-46}$$

and inverted to

$$G^*(s) = \frac{H_1 s + (H_2 + H_1 H_2 h_2)}{s^2 + (h_1 H_1 + h_2 H_2)s + h_1 (H_2 + H_1 H_2 h_2)} \tag{3-47}$$

The final value with a step input of the reduced model is

$$\lim_{s \to 0} G^*(s) = \frac{1}{h_1} = \frac{A_{20}}{A_{10}} \tag{3-48}$$

and the initial value for an impulse input is

$$\lim_{s \to \infty} sG^*(s) = H_1 = \frac{A_{2,n-1}}{A_{3,n-1}} = \frac{A_{2,n-1}}{A_{1n}} \tag{3-49}$$

Aa mentioned before, the modified Cauer form expansion maintains the steady state gains and the accurate initial transient response.

Example 3-4: Consider the same original model as example 3-1. The coefficients and the reduced model are obtained as

$$h_1=30 \;<\; \begin{matrix} 60 & 47 & 12 & 1 \\ 2 & 3 & 1 & \end{matrix} \;>\; H_1=1$$

$$h_2=-.9556 \;<\; \begin{matrix} -43 & -18 & 1 & \\ 45 & 21 & & \\ 2.0667 & 1 & & \end{matrix} \;>\; H_2=21$$

$$G_4^*(s) = \frac{s+0.9333}{s^2+9.9333s+28} \qquad (3\text{-}50)$$

3-5. A General Cauer Form

Based on matching the time moments and Markov parameters, Lucas [5] presented a more general Cauer form which can yield the reduced model in Cauer first, Cauer second, and the modified Cauer form by selecting required parameters.

Let m be the order of the reduced model, t be the retained time moments, then 2m-t will be the retained Markov parameters. The continued fraction of the general Cauer form is shown in equation (3-46). In case of t=0, the expansion form will be the same as the Cauer first form. Retaining t=2m gives the Cauer second form. To obtain the modified Cauer form, we need t=m so that the form retains m time moments and m Markov parameters.

$$G(s) = \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{\ddots \cfrac{s}{h_t + \cfrac{1}{sH_1 + \cfrac{1}{H_2 + \cfrac{1}{sH_3 + \cfrac{1}{\ddots}}}}}}}}} \qquad (3\text{-}51)$$

The $h_i (i=1,2,\ldots,t)$ and $H_i (i=1,2,\ldots,2m-t)$ can be obtained from the following Routh array

$$\begin{array}{lccccc}
h_1 < & A_{10} & A_{11} & A_{12} & \cdots & A_{1n} \\
h_2 < & A_{20} & A_{21} & A_{22} & \cdots & A_{2,n-1} \\
 & A_{30} & A_{31} & A_{32} & \cdots & A_{3,n-1} \\
 & \cdots\cdots & \cdots\cdots & & & \\
 & \cdots\cdots & \cdots\cdots & & &
\end{array} \qquad (3\text{-}52)$$

$$\begin{array}{lccc}
h_t < & A_{t0} & A_{t1} \cdots & A_{t,n-t/2} \\
 & A_{t+1,0} & A_{t+1,1} \cdots & A_{t+1,n-(t+1)/2} \\
 & A_{t+2,0} & A_{t+2,1} \cdots & A_{t+2,n-(t+2)/2} \\
H_1 < & B_{10} & B_{11} \cdots & B_{1,n-(t+1)/2} \\
H_2 < & B_{20} & B_{21} \cdots & B_{2,n-(t+2)/2} \\
 & B_{30} & B_{31} \cdots & \\
 & \cdots\cdots & \cdots\cdots & \\
 & \cdots\cdots & \cdots\cdots &
\end{array} \quad \text{reverse}$$

where

$$h_i = A_{i0}/A_{i+1,0} \qquad (i=1,2,\ldots,t) \tag{3-53}$$

$$H_i = B_{i0}/B_{i+1,0} \qquad (i=1,2,\ldots,2m-t) \tag{3-54}$$

$$A_{ij} = A_{i-2,j+1} - h_{i-2}A_{i-1,j+1} \qquad (i=3,4,\ldots,t+2,\ j=0,1,\ldots,n-i/2) \tag{3-55}$$

$$B_{ij} = B_{i-2,j+1} - H_{i-2}B_{i-1,j+1} \qquad (i=3,4,\ldots,2m-t+1,\ j=0,1,\ldots,n-i/2) \tag{3-56}$$

and the first two rows of the B_{ij} are identical to the reversing order of the last two rows of the A_{ij}, or

$$B_{ij} = A_{t+i,n-(t+i)/2-j} \qquad (i=1,2,\quad j=0,1,\ldots,n-(w+i)/2) \tag{3-57}$$

Retaining one time moment and three Markov parameters and wanting a reduced second order system, one has

$$G^*(s) = \cfrac{1}{h_1 + \cfrac{s}{H_1 + \cfrac{1}{sH_2 + \cfrac{1}{H_3}}}} \tag{3-58}$$

Inverting equation (3-58) yields

$$G(s) = \frac{H_1H_2H_3s+(H_1+H_3)}{H_2H_3s^2+(1+h_1H_1H_2H_3)s+h_1(H_1+H_3)} \tag{3-59}$$

The final value with a step input of the reduced model is

$$\lim_{s\to\infty} G^*(s) = \frac{1}{h_1} = \frac{A_{20}}{A_{10}} \tag{3-60}$$

and the initial value for an impulse response is

$$\lim_{s\to 0} sG^*(s) = H_1 = \frac{A_{2n}}{A_{3n}} = \frac{A_{2n}}{A_{1,n+1}} \tag{3-61}$$

As expected, matching the first time moment and the first three Markov parameters maintains the steady state and the accurate initial transient response.

Example 3-5: Consider the same original model as example 3-1, The coefficients and the reduced model are obtained as

h_1=30 < 60 47 12 1

2	3	1
-43	-18	1

reverse

1	3	2
1	-18	-43

H_1=1 <

H_2=0.0476 < 21 45

H_3=-1.0425 < -20.143

$$G_5^*(s) = \frac{s+0.8569}{s^2+10.0464s+25.7056} \tag{3-62}$$

Comparing the step responses of $G_1^*(s)$, $G_2^*(s)$, $G_3^*(s)$, $G_4^*(s)$, $G_5^*(s)$ and the original model G(s) gives results shown in Figure 3-1.

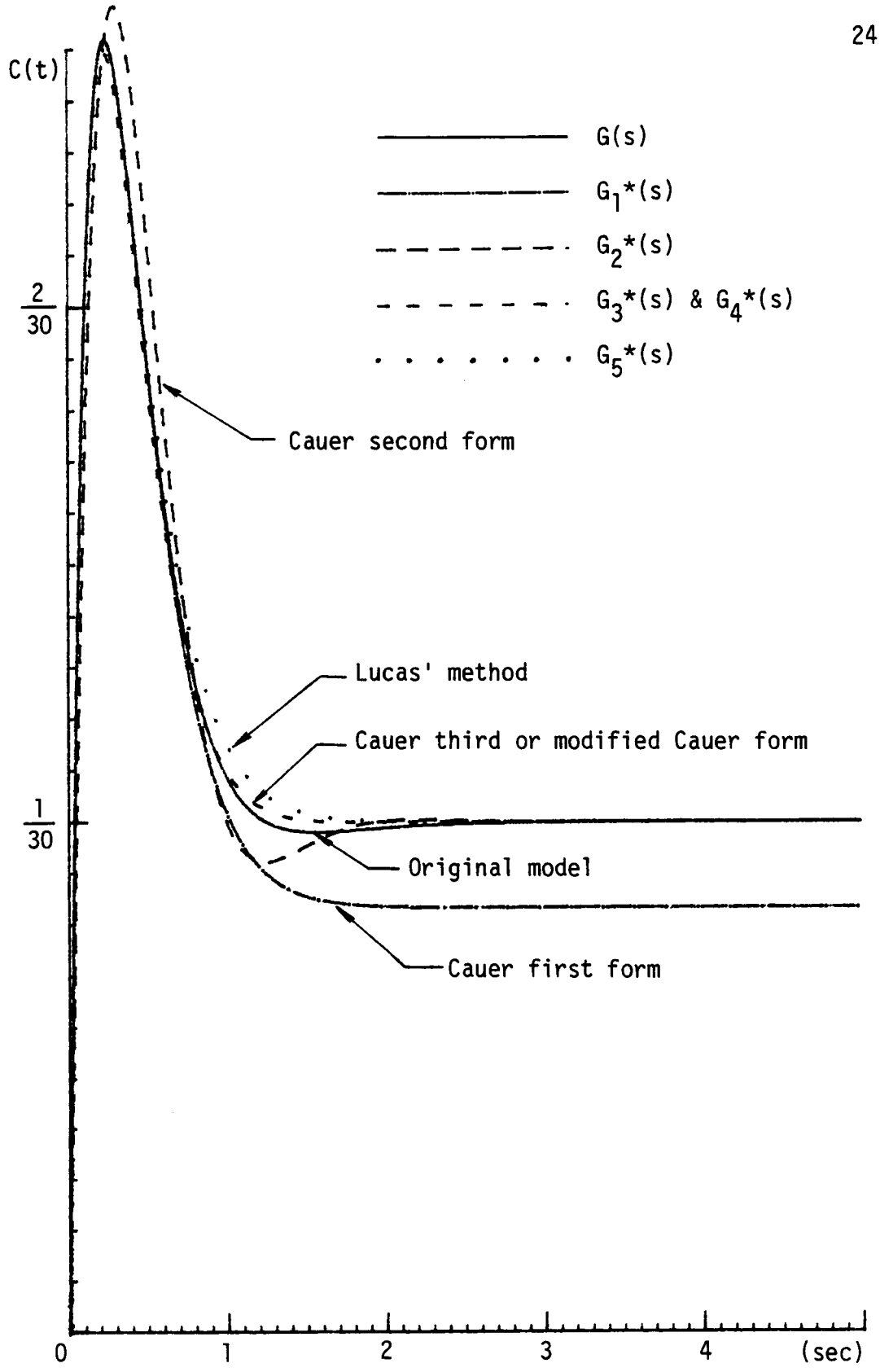

Figure 3-1. Step responses of G(s), $G_1^*(s)$, $G_2^*(s)$, $G_3^*(s)$, $G_4^*(s)$, and $G_5^*(s)$.

3-6. A New Generalized Form of Continued Fraction Expansion

Observing the results shown in Figure 3-1, one sees that the settling time of the reduced model will closer to the original model in case of matching more time moments. Therefore, one can assume to match three time moments and one Markov parameter in order to get a reduced model whose time response is located between that of $G_2^*(s)$ which matches four time moments and $G_3^*(s)$ or $G_4^*(s)$ which matches two time moments and two Markov parameters.

The general Cauer form proposed by Lucas [5] fails to get a reduced model of matching three time moments and one Markov parameter. A new generalized Cauer form of continued fraction expansion is proposed here as shown in equation (3-63). It is similar to Lucas' method and is effective for obtaining a reduced model of matching three time moments and one Markov parameter. The simulation result in this example will be consistant to the assumption.

$$G(s) = \cfrac{1}{sH_1 + \cfrac{1}{H_2 + \cfrac{1}{sH_3 + \cfrac{1}{\ddots + \cfrac{1}{H_w + \cfrac{s}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3 + \ddots}}}}}}}} \qquad (3\text{-}63)$$

Let m be the order of the reduced model, w be the retained Markov parameters, then 2m-w will be the retained time moments. The continued fraction expansion of the suggested form from equation (3-1) is then shown in equation (3-63), and the parameters can be obtained from the following Routh array.

$$
\begin{array}{lllll}
H_1 < & B_{10} & B_{11} & B_{12} \cdots B_{1,n-1} & B_{1n} \\
H_2 < & B_{20} & B_{21} & B_{22} \cdots B_{2,n-1} & \\
 & B_{30} & B_{31} & B_{32} \quad B_{3,n-1} & \\
 & \cdots\cdots\cdots\cdots\cdots\cdots & & & \\
H_w < & B_{w0} & B_{w1} \cdots B_{w,n-w/2} & & \\
 & B_{w+1,0} & B_{w+1,1} \cdots B_{w+1,n-(w+1)/2} & & \\
 & B_{w+2,0} & B_{w+2,1} \cdots B_{w+2,n-(w+2)/2} & & \\
h_1 < & A_{10} & A_{11} \cdots A_{1,n-(w+1)/2} & & \\
h_2 < & A_{20} & A_{21} \cdots A_{2,n-(w+2)/2} & & \\
 & A_{30} & A_{31} \cdots & & \\
 & \cdots\cdots\cdots\cdots & & &
\end{array}
\qquad \text{reverse} \qquad (3\text{-}64)
$$

where

$$H_k = \frac{B_{k0}}{B_{k+1,0}} \qquad (k=1,2,\ldots,w) \qquad (3\text{-}65)$$

$$h_k = \frac{A_{k0}}{A_{k+1,0}} \qquad (k=1,2,\ldots,2m-w) \qquad (3\text{-}66)$$

$$B_{ij} = B_{i-2,j+1} - H_{i-2}B_{i-1,j+1} \qquad (3\text{-}67)$$

$$(i=3,4,\ldots,w+2,\ j=0,1,\ldots,n-i/2)$$

$$A_{ij} = A_{i-2,j+1} - h_{i-2}A_{i-1,j+1} \qquad (3\text{-}68)$$

$$(i=3,4,\ldots,2m-w+1,\ j=0,1,\ldots n-i/2)$$

and the first two rows of A_{ij} are identical to the reversing order of the last two rows of B_{ij} or

$$A_{ij} = B_{w+i,n-(w+i)/2-j} \qquad (3\text{-}69)$$
$$(i=1,2,\ j=0,1,\ \ldots\ n-(w+i)/2)$$

When w=0, the method will be equivalent to the Cauer second form. Similarily, w=m means that the form retains m Markov parameters and m time moments. To calculate a reduced second order system by retaining three time moments and one Markov parameter, one can obtain

$$G^*(s) = \cfrac{1}{sH_1 + \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3}}}} \qquad (3\text{-}70)$$

The inversion of equation (3-64) gives

$$G^*(s) = \frac{(h_1 + h_3)s + h_1 h_2 h_3}{H_1(h_1 + h_3)s^2 + (1+h_1 h_2 h_3 H_1)s + h_2 h_3} \qquad (3\text{-}71)$$

The final value with a step input of the reduced model is

$$\lim_{s\to 0} G^*(s) = h_1 = \frac{B_{20}}{B_{30}} = \frac{B_{20}}{B_{10}} \qquad (3\text{-}72)$$

and the initial value for an impulse response is

$$\lim_{s\to\infty} sG^*(s) = \frac{1}{H_1} = \frac{B_{2,n-1}}{B_{1,n-1}} \tag{3-73}$$

Equations (3-72) and (3-73) show that matching one Markov parameter and three time moments maintains the steady state and the initial transient response.

Example 3-6: To compare the new algorithm and the other forms of continue fraction expansion, one can consider the same original model as example 3-1 and simulate the step response of the reduced model. The parameters and the reduced model are calculated as follows.

$H_1 = 1$

1	12	47	60
1	3	2	
9	45	60	

reverse

2	3	1
60	45	9

$h_1 = .03333$

$h_2 = 40$

1.5	0.7

$h_3 = .08824$

17

$$G_6^*(s) = \frac{s+0.967742}{s^2+9.19355s+29.03226} \tag{3-74}$$

Figure 3-2 shows the step responses of $G_6^*(s)$ and $G(s)$. As expected, the curve of $G_6^*(s)$ locates between the curves of the Cauer second form—$G_2^*(s)$ and the modified Cauer form—$G_4^*(s)$, which are shown in Figure 3-1.

In order to compare the performances of all continued fraction

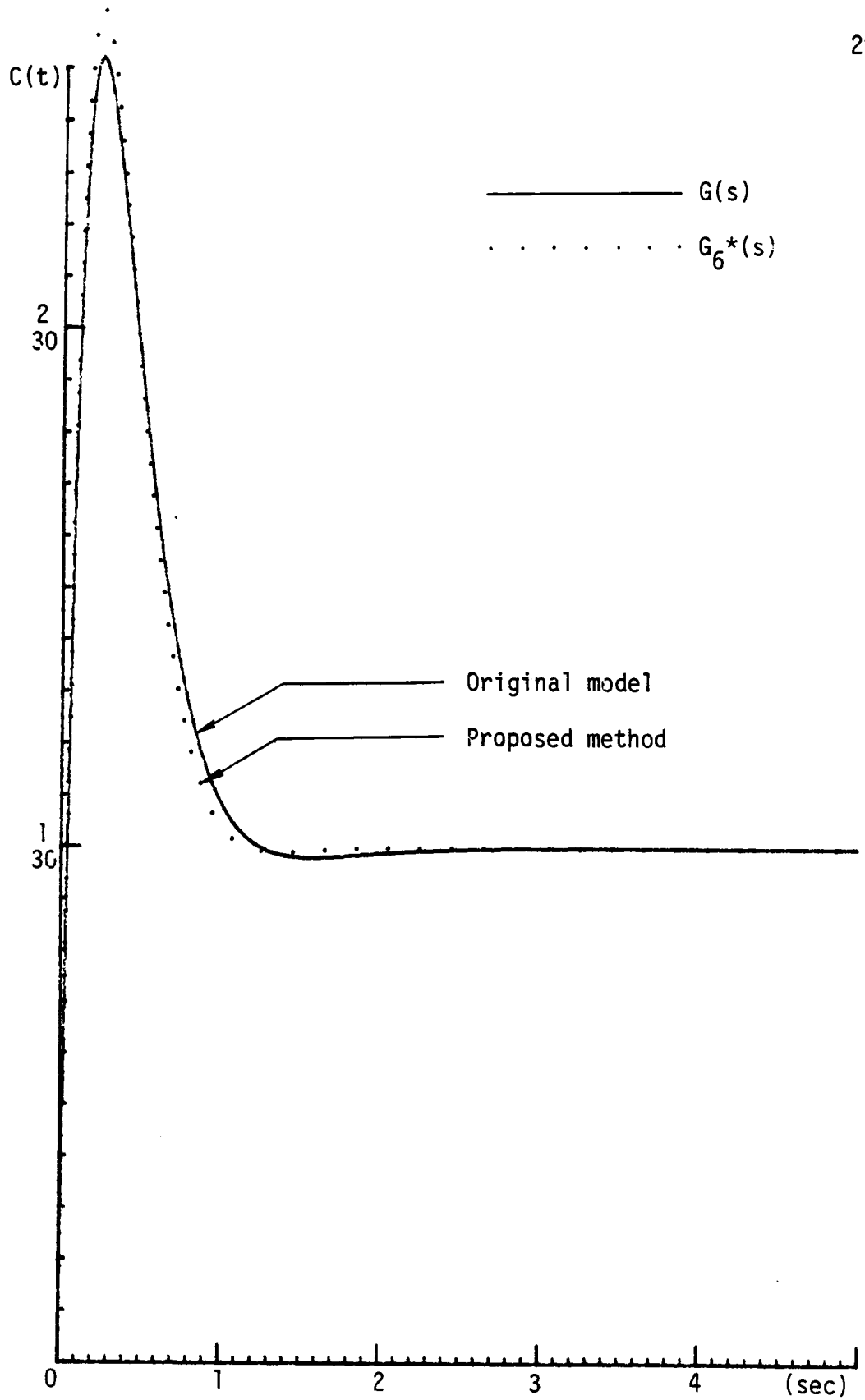

Figure 3-2. Step responses of G(s) and G_6*(s)

expansion methods, one can use sum of the squares of the sample errors as the error index, which is convinient to be compared with the discrete-time model. Table 3-1 shows the error index with the sampling periods .05 second and .10 second and 500 sampling points for above reduced order systems. The Cauer third form has the least sum of the error squares in this case.

Table 3-1. Comparison of the error index for different continued fraction expansion methods.

Model	Σe^2 (.05 sec., 500 points)	Σe^2 (.10 sec., 500 points)
$G_1^*(s)$	0.014525	0.01498
$G_2^*(s)$	6.497×10^{-4}	3.160×10^{-4}
$G_3^*(s)$	3.024×10^{-5}	1.512×10^{-5}
$G_5^*(s)$	5.746×10^{-5}	2.873×10^{-5}
$G_6^*(s)$	9.367×10^{-5}	4.690×10^{-5}

To illustrate the instability of the reduced system obtained by the Cauer second form of continued fraction expansion, and to compare the performances of different continued fraction expansion methods, one can consider another numerical example as below.

Example 3-7: Consider the following transfer function

$$G(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)(s+20)} = \frac{s^2+7s+12}{s^3+23s^2+62s+40} \qquad (3\text{-}75)$$

The reduced order systems obtained by different forms of continued

fraction expansion methods are

(1). Cauer first form (matching four Markov parameters)

$$G_1^*(s) = \frac{s+4.33325}{s^2+18.33274s+12.66624} \tag{3-76}$$

(2). Cauer second form (matching four time moments)

$$G_2^*(s) = \frac{0.22137s-1.19083}{s^2-3.0992s-3.96943} \tag{3-77}$$

(3). Cauer third form or Modified Cauer form (Matching two time moments and two Markov parameters)

$$G_3^*(s) = \frac{s+5.7}{s^2+21.7s+19} \tag{3-78}$$

(4). Matching one time moment and three Markov parameters (Lucas' method)

$$G_4^*(s) = \frac{s+4.894734}{s^2+20.894772s+16.31578} \tag{3-79}$$

(5). Matching one Markov parameter and three time moments (proposed method)

$$G_5^*(s) = \frac{s+9.230766}{s^2+33.076915s+30.76922} \tag{3-80}$$

Figure 3-3 shows the comparison of the step responses of above reduced

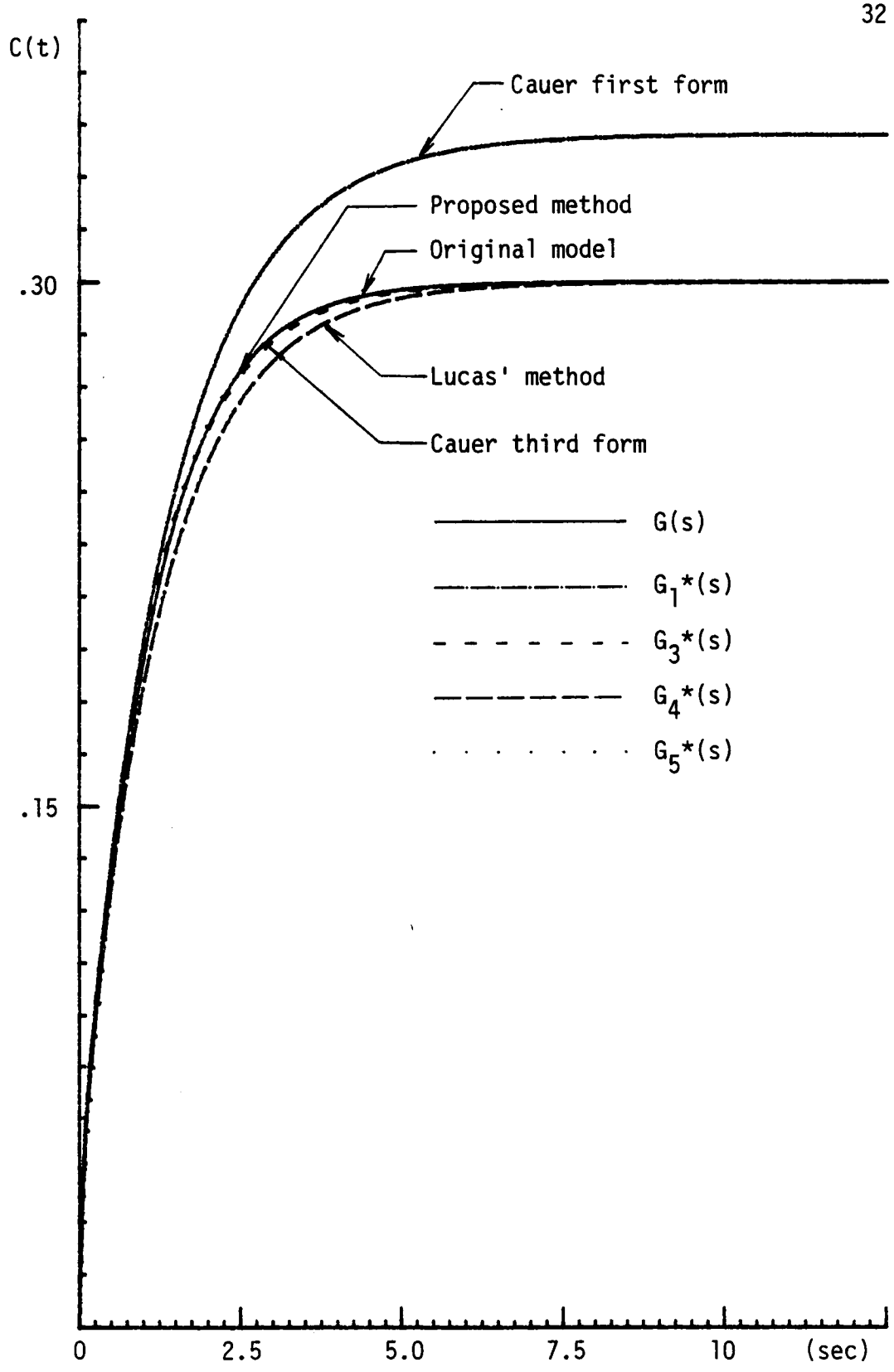

Figure 3-3. Comparison of the step responses for example 3-7.

order systems and the original system for example 3-7.

Table 3-2 shows the comparison of the error index of different reduced order models for example 3-7. Obviously, the Cauer second form of the continued fraction expansion method fails to obtain a stable reduced order model and will gives an infinitive error index in this example, while matching one Markov parameter and three time moments by proposed method has the least error index in this example.

Table 3-2. Comparison of the error index of different reduced order models for example 3-7.

Model	Σe^2 (.05 sec., 500 points)	Σe^2 (.10 sec., 500 points)
$G_1^*(s)$	.742453	.8150
$G_2^*(s)$	-	-
$G_3^*(s)$	3.531×10^{-4}	1.765×10^{-4}
$G_4^*(s)$	4.376×10^{-3}	2.188×10^{-3}
$G_5^*(s)$	3.454×10^{-4}	1.746×10^{-4}

CHAPTER 4

DOMINANT MODE METHODS

4-1. Davison-Chidambara's Method

Davison [18,39] proposed a reduction method by retaining dominant eigenvalues and eigenvectors. Chidambara [40,41] presented an improved version which considered the system has repeated eigenvalues and degenerate eigenvectors and used the neglected system to improve the steady state response of the reduced order system. Therefore, Davison's method can be seen as a special case of Chidambara's method.

Suppose the original system model is given by

$$\begin{aligned} \underline{\dot{x}} &= A\,\underline{x} + B\,\underline{u} \\ \underline{y} &= C\,\underline{x} \end{aligned} \tag{4-1}$$

where A is n x n matrix, $\underline{x}$ is an n-dimensional state vector, $\underline{u}$ is an p-dimensional input vector, $\underline{y}$ is q-dimensional output vector. B and C are constant matrices of appropriate dimension.

Require that the reduced system model be

$$\begin{aligned} \underline{\dot{x}}^* &= A^*\underline{x}^* + B^*\underline{u} \\ \underline{y} &= C\,\underline{x} \end{aligned} \tag{4-2}$$

where A* is m x m matrix ($m < n$).

To obtain the reduced model, the coefficient matrices A and B and the relationship between $\underline{x}$ and $\underline{x}^*$ must be determined. Assume that the characteristic equation associated with the matrix A is

$$(\lambda - \lambda_1)^{j_1}(\lambda - \lambda_2)^{j_2} \ldots\ (\lambda - \lambda_k)^{j_k} = 0 \qquad (4\text{-}3)$$

where $|\lambda_1| < |\lambda_2| < \ldots < |\lambda_k|$, and j_i is the order of multiplicity of eigenvalue $_i$, which has the relation as

$$\sum_{i=1}^{k} j_i = n \qquad (4\text{-}4)$$

If the eigenvalues give degenerate eigenvectors, the eigenvalues are repeated. There exists a nonsingular matrix Q such that the matrix A can be transformed into Jordan canonical form.

$$J = Q^{-1}AQ = \begin{bmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & J_k \end{bmatrix} \qquad (4\text{-}5)$$

where J_i are $j_i \times j_i$ Jordan block. If the column of Q are denoted by q_i, then

$$Q = [\, q_1 \;\; q_2 \;\; \ldots \;\; q_n \,] \qquad (4\text{-}6)$$

where q_i can be seen as eigenvectors and their generalized eigenvectors or degenerate eigenvectors.

Define a vector $\underline{z}$ such that

$$\underline{x} = Q\underline{z} \qquad (4\text{-}7)$$

From equations (4-5), (4-7) and (4-1) one can derive

$$\dot{\underline{z}} = J\underline{z} + Q^{-1}B\underline{u} \tag{4-8}$$

Partition $\underline{z}$ into two vectors $\underline{z}$ and $\underline{z}$ of order m x 1 and (n-m) x 1, respectively. Since the eigenvalues of matrix A have been arranged in a proper order, one can select Q such that the dominant eigenvalues of A are present in J_1, J_2, ..., J_d, and the others are present in J_{d+1}, J_{d+2}, J_{d+3}, ..., J_k. Rewrite equation (4-8) as

$$\begin{bmatrix} \dot{\underline{z}}_1 \\ \dot{\underline{z}}_2 \end{bmatrix} = \begin{bmatrix} J_M & 0 \\ 0 & J_N \end{bmatrix} \begin{bmatrix} \underline{z}_1 \\ \underline{z}_2 \end{bmatrix} + \begin{bmatrix} B_M \\ B_N \end{bmatrix} \underline{u} \tag{4-9}$$

where J_M is an m x m matrix which consists of submatrices J_1, J_2, ..., J_d. J_N is an (n-m) x (n-m) matrix which consists of submatrices J_{d+1}, J_{d+2}, ..., J_k. B_M is the first m rows of $Q^{-1}B$ and B_N is the remained rows of $Q^{-1}B$. The eigenvalues of the matrix J_M are seen as the dominant eigenvalues so that the eigenvalues of J_N are assumed to have little significant effect to the system and can be neglected. Thus, the reduced order system is now obtained as

$$\dot{\underline{z}}_1 = J_M\underline{z}_1 + B_M\underline{u} \tag{4-10}$$

Comparing equations (4-10) and (4-2) gives the relationships as

$$A^* = J_M \tag{4-11}$$

$$B^* = B_M \tag{4-12}$$

$$\underline{x}^* = \underline{z}_1 \tag{4-13}$$

The resulting reduced order model obtained by Davison's method

[18] will be

$$\dot{\underline{x}}^* = J_M \underline{x}^* + B_M \underline{u}$$

$$\underline{x} = \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} \underline{x}^* \tag{4-14}$$

$$\underline{y} = C\underline{x}$$

From equation (4-9) one has the form

$$\dot{\underline{z}}_2 = J_N \underline{z}_2 + B_N \underline{u} \tag{4-15}$$

By the method of Laplace transform $Z_2(s)$ can be obtained from equation (4-15).

$$Z_2(s) = G_2(s)U(s) \tag{4-16}$$

where

$$G_2(s) = [sI - J_N]^{-1} B_N \tag{4-17}$$

The reduced order model obtained by Chidambara's method [39] can be expressed as

$$\underline{x}^* = J_M \underline{x}^* + B_M \underline{u}$$

$$\underline{x} = \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} \underline{x}^* + \begin{bmatrix} Q_{12} G_2(0) \\ Q_{22} G_2(0) \end{bmatrix} \underline{u} \tag{4-18}$$

$$\underline{y} = C\underline{x}$$

If $G_2(0)=0$, then the model (4-18) is identical to (4-14).

Example 4-1: Consider the transfer function

$$G(s) = \frac{s^2+3s+2}{s^3+12s^2+47s+60} \tag{4-19}$$

which is the same system as example 3-1 and can be rearranged into column companion form

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 0 & -60 \\ 1 & 0 & -47 \\ 0 & 1 & -12 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -9 & 63 \end{bmatrix} \underline{x} \tag{4-20}$$

Using Davison-Chidambara's method, one can solve the reduced second order models by following steps.

$$Q = \begin{bmatrix} 20 & 15 & 12 \\ 9 & 8 & 7 \\ 1 & 1 & 1 \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} 0.5 & -1.5 & 4.5 \\ -1 & 4.0 & -16.0 \\ 0.5 & -2.5 & -12.5 \end{bmatrix}$$

$$A^* = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}$$

$$B^* = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$$

$$G_2(s) = \frac{0.5}{s+5}$$

$$G_2(0) = 0.1$$

$$\underline{x} = \begin{bmatrix} 20 & 15 \\ 9 & 8 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 1.2 \\ 0.7 \\ 0.1 \end{bmatrix} u$$

$$y = [1 \quad -9 \quad 63]\ \underline{x}$$

The resulting transfer function obtained by Davison's method (G (0)=0) will be

$$G_a^*(s) = \frac{-5s-14}{s^2+7s+12} \tag{4-21}$$

while Chidambara's method gives result

$$G_b^*(s) = \frac{-5s-12.8}{s^2+7s+12} \tag{4-22}$$

Obviously, the resulting transfer functions which have a negative steady-state gain are not expected. The reason may be that the sample system doesn't have dominant eigenvalues.

Actually, in single-input single-output systems Davison's method can be simplified by direct calculated from the original transfer function without any matrix operation. That is, taking partial fraction expansion of the original transfer function and retaining the fast responses which corresponds to the dominant eigenvalues.

In example 4.1 the partial fraction expansion of the original system is

$$G(s) = \frac{1}{s+3} - \frac{6}{s+4} + \frac{6}{s+5} \tag{4-23}$$

Neglecting the last part of equation (4-23) yields a reduced second order transfer function

$$G_c^*(s) = \frac{1}{s+3} - \frac{6}{s+4} = \frac{-5s-14}{s^2+7s+12} \tag{4-24}$$

which is exactly the same as equation (4-21).

4-2. The Selecting Criterion for the Order of the Reduced Model

For an nth order system the dominant eigenvalues can be roughly defined [21] as $\lambda_1, \lambda_2, \ldots, \lambda_m$, with the condition

$$|\lambda_{m+1}| > \sqrt{n-m} \tag{4-25}$$

where $1<m<n$.

Elrazaz and Sinha [23] proposed a criterion for reducing a model by retaining the system's dominant eigenvalues. They also showed the method can be an extended criterion for the system with complex eigenvalues [24]. For this method the state equation (4-1) with zero initial conditions can be rewritten in the partitioned form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \underline{u} \tag{4-26}$$

where $\underline{x}_1$=m x1 vector, $\underline{u}$=p x 1 vector, m is the order of the reduced model, and p is the number of inputs. The vector $\underline{x}_1$ also represents the state to be retained in the low-order model.

Consider the linear transformation

$$\underline{x} = P\underline{z} \tag{4-27}$$

where P is the modal matrix of A. Then the normal form of the state equation can be written

$$\underline{z} = D\underline{z} + P^{-1}B\underline{u} \tag{4-28}$$

and

$$D = P^{-1}AP \tag{4-29}$$

Assume that all the eigenvalues of A are distinct, D will be a diagonal matrix. Let

$$D = \text{Diag}\,[\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_n] \tag{4-30}$$

where the λ_i are complex eigenvalues, the real part of λ_i is greater than or equal to the real part of λ_{i+1}, for i=1,2, ..., n-1.

The criterion function [23,24] can be written as

$$U_m = \frac{\sqrt{n-m}}{\min\limits_{m+1 \le i \le n} |\lambda_i|}[2\sqrt{n-m}+1] \tag{4-31}$$

U_m can be seen as an error function. A smaller U_m will give a better approximation for the reduced model. Following Rao et al. [25], one

can define the ratio V_m.

$$V_m = \frac{U_{m-1}}{U_m} \tag{4-32}$$

which represents a measure of the improvement achieve by increasing the model order from m-1 to m. In this case we prefer a larger V_m.

Example 4-2: Considering a sample power system example [23] whose eigenvalues are -0.2605, -0.601+j0.898, -13.54+j376.0, -37.46, and -45.84, select a reduced order which is consistent to above criteria.

Using equation (4-31) and (4-32), one can solve the criterion functions U_m and V_m as shown in Table 4-1. Apparently, a good choice of the reduced order is 3, which has the maximum V_m value although its U_m value is not minimum.

Table 4-1 Selection Criteria of the Reduced Model Order

Model Order m	Criterion Function U_m	$V_m = U_{m-1}/U_m$
1	1.324	-
3	0.267	4.959
5	0.145	1.841
6	0.065	2.231

4-3. Aggregation Method

The aggregation method proposed by Aoki [19] has been shown as

the general reduction method of dominant mode methods [4,42]. This method is based on determining an aggregation matrix which gives the relationship between the state variables of the full and the reduced models. Consider the linear system as shown in equation (4-1). The objective of model reduction is of the form

$$\begin{aligned} \dot{\underline{x}}^* &= F\underline{x}^* + G\underline{u} \\ \underline{y}^* &= H\underline{x}^* \end{aligned} \tag{4-33}$$

where $\underline{x}^*$ is m x 1 vector, $\underline{u}$ is p x 1 vector, $\underline{y}^*$ is the approximation of $\underline{y}$. Intuitively, let

$$\underline{x}^* = K\underline{x} \tag{4-34}$$

where K is an m x n constant projection matrix, and is called the "aggregation matrix." Sinha [4] defined equation (4-34) as "the aggregation law."

Differentiating equation (4-34) and substituting for $\underline{x}$ in equation (4-1) give

$$\dot{\underline{x}}^* = KA\underline{x} + KB\underline{u} \tag{4-35}$$

Comparing equations (4-32) and (4-34) gives

$$\begin{aligned} KA &= FK \\ G &= KB \\ C &= HK \end{aligned} \tag{4-36}$$

Equation (4-36) is considered an approximate equality since $\underline{y}$ is said to be a "close approximation" of the full model response. The

matrices F, G, and H may be calculated from (4-36) by utilizing the psudo-inverse of K. Note that K is a rectangular matrix.

$$K^{\dagger} = K^{T} (KK^{T})^{-1} \tag{4-37}$$

then

$$\begin{aligned} F &= KAK^{\dagger} \\ G &= KB \\ H &= CK^{\dagger} \end{aligned} \tag{4-38}$$

Aoki's method [19] of choosing K is based on considering the controllability matrix of equation (4-1),

$$W_A = [B \quad AB \quad .... \quad A^{n-1}B] \tag{4-39}$$

Note that KB=G and KA=FK. Premultiplying equation (4-39) by K yields

$$KW_A = [G \quad FG \quad .... \quad F^{n-1}G] \tag{4-40}$$

The right-hand side of equation (4-39) is known. If F and G are arranged in column companion form, W_A will be the identity matrix for single-input-single-output system. In this case

$$K = [G \quad FG \quad ... \quad F^{n-1}G] \tag{4-41}$$

The aggregation matrix is to be chosen in such a way that the error in modeling the system (4-1) is minimized. The reduced model can then be determined by equation (4-38).

Example 4-3: Consider the same system as example 4-1 which has been

arranged into column companion form. The reduced model by the aggregation method can be obtained by following steps.

$$F = \begin{bmatrix} 0 & -12 \\ 1 & -7 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & -7 \end{bmatrix}$$

$$H = [\ 0.2865 \qquad -9.415\ ]$$

The reduced model then can be obtained by equation (4-38).

$$G_d^*(s) = \frac{0.2865s-7.41}{s^2+7s+12} \tag{4-42}$$

4-4. Partial Realization Method

Model reduction by retaining the dominant eigenvalues of the original system can guarantee the stability if the system is stable, but the steady-state response of the reduced model can't always be the same as that of the original system. Since the time-moment J_{-1} is related to the steady-state response and the Markov parameter J_0 is related to the initial transcient response of the system, one can use the "generalized Markov parameters", see Appendix A, to compensate the reduced model of the dominant mode.

Assume the reduced model of the system (4-1) is shown in (4-33). To obtain a reduced model of the dominant mode of the single-input single-output system, one can arrange the model in the column companion form, that is

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & \dots & -F_{1m} \\ 1 & 0 & 0 & \dots & -F_{2m} \\ 0 & 1 & 0 & \dots & -F_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -F_{mm} \end{bmatrix}$$

$$\mathbf{G} = [1 \quad 0 \quad 0 \quad \dots \quad 0] \tag{4-43}$$

$$\mathbf{H} = [H_1 \quad H_2 \quad H_3 \quad \dots \quad H_m]$$

The corresponding transfer function of the reduced model is

$$G^*(s) = \frac{H_1 s^{m-1} + H_2 s^{m-2} \dots + H_m}{(s-\lambda_1)(s-\lambda_2) \dots (s-\lambda_m)}$$

or

$$G^*(s) = \frac{H_1 s^{m-1} + H_2 s^{m-2} + \dots + H_m}{s^m + F_{mm} s^{m-1} \dots + F_{2m} s + F_{1m}} \tag{4-44}$$

where λ_1, λ_2, ..., λ_m are dominant eigenvalues so that F and G can easily be obtained by the assumed eigenvalues of the reduced model. The H_i (i=1,2, ...,m) parameters are needed to be matched with some Markov parameters and time moments.

The procedures of finding H vector proposed by Rozsa and

Sinha [4,43] are described as follows.

Define a Hankel metrix of order (u,v) and index i as

$$S_{uv}(i) = \begin{bmatrix} J_{-i} & J_{-i+1} & \cdots & J_{-i+v-1} \\ J_{-i+1} & J_{-i+2} & \cdots & J_{-i+v} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ J_{-i+u-1} & J_{-i+u} & \cdots & J_{-i+u+v-2} \end{bmatrix} \tag{4-45}$$

or

$$S_{uv}(i) = \begin{bmatrix} CA^{-i}B & CA^{-i+1}B & \cdots & CA^{-i+v-1}B \\ CA^{-i+1}B & CA^{-i+2}B & \cdots & CA^{-i+v}B \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ CA^{-i+u-1}B & CA^{-i+u}B & \cdots & CA^{-i+u+v-2}B \end{bmatrix} \tag{4-46}$$

To match the moments, one needs that

$$HF^{-i}G = J_{-i} = CA^{-i}B \tag{4-47}$$

A second order reduced system needs to match

$$\begin{aligned} HF^{-1}G &= J_{-1} = CA^{-1}B \\ HF^{0}G &= J_{0} = CB \end{aligned} \tag{4-48}$$

Solving the Hankel matrix [43] will give H and G matrices. A special case in obtaining second order reduced system by matching J_{-1} and J_0 will have the form

$$H = [J_{-1} \quad J_0]F \tag{4-49}$$

where F can be obtained from the retained eigenvalues and should be

arranged in observable matrix form as shown in equation (4-43).

Example 4-4: Considering the same model as example 4-1, the reduced model by partial realization method can be obtained as follows. Finding the generalized Markov parameters gives

$$J_0 = 1$$

$$J_{-1} = -1/30$$

Then H matrix can be found by using equation (4-49)

$$H = [\ 1 \quad -6.6]$$

$$G_e^*(s) = \frac{s+0.4}{s^2+7s+12} \tag{4-50}$$

$G_a^*(s)$ $G_b^*(s)$, and $G_d^*(s)$ fail to get a positive final value and their step responses are meaningless; therefore, let us compare the step responses of $G_e^*(s)$—equation (4-50) and G(s)—equation (4-19). The results are shown in Figure 4-1.

If the original system has dominant eigenvalues, then the dominant mode methods may give better results but the steady-state matching is still a problem as shown in following example.

Example 4-5: Consider the following transfer function

$$G(s) = \frac{s^2+7s+12}{s^3+23s^2+62s+40} = \frac{.31579}{s+1} + \frac{-.11111}{s+2} + \frac{.79532}{s+20} \tag{4-51}$$

The reduced order systems obtained by different dominant mode methods

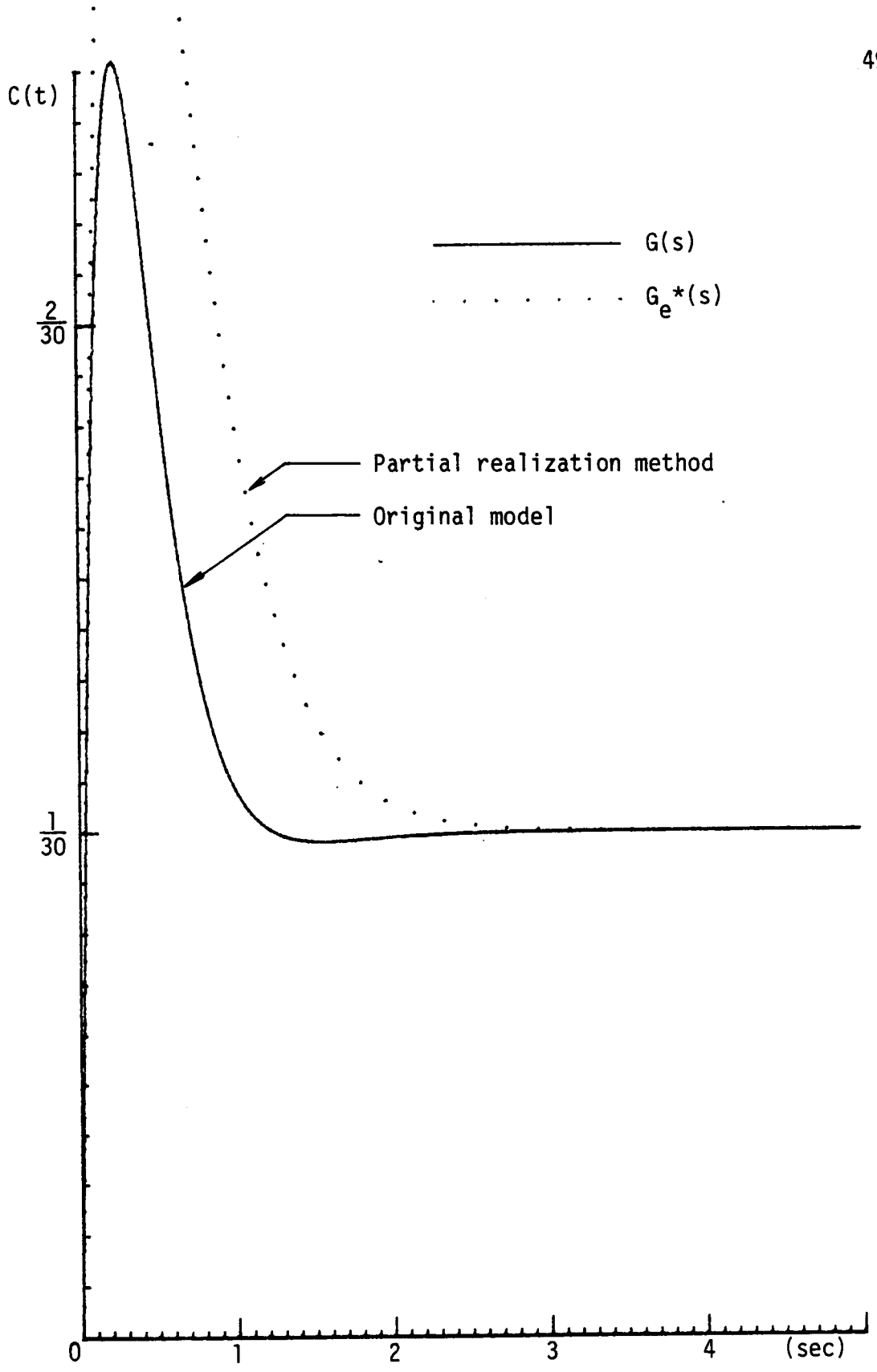

Figure 4-1. Comparison of step responses for G(s) and $G_e^*(s)$.

are

(1). Davison's method

$$G_a^*(s) = \frac{.20468s+.52047}{s^2+3s+2} \tag{4-52}$$

(2). Chidambara's method

$$G_b^*(s) = \frac{.20468s+.56036}{s^2+3s+2} \tag{4-53}$$

(3). Aggregation method by using Aoki's aggregation matrix

$$G_c^*(s) = \frac{-37.8575-113.571}{s^2+3s+2} \tag{4-54}$$

(4). Partial realization method

$$G_d^*(s) = \frac{s+6}{s^2+3s+2} \tag{4-55}$$

Figure 4-2 shows the comparison of the step responses of above reduced order system and the original system for example 4-5. To retain dominant eigenvalues, the dominant mode methods always give a poor result. Thus, it is not necessary to retain the dominant eigenvalues of the original system in order to maintain the stability. Since the reduced order systems of the dominant mode methods do provide a good approximation with respect to the orignal system, the error indexes are not compared here.

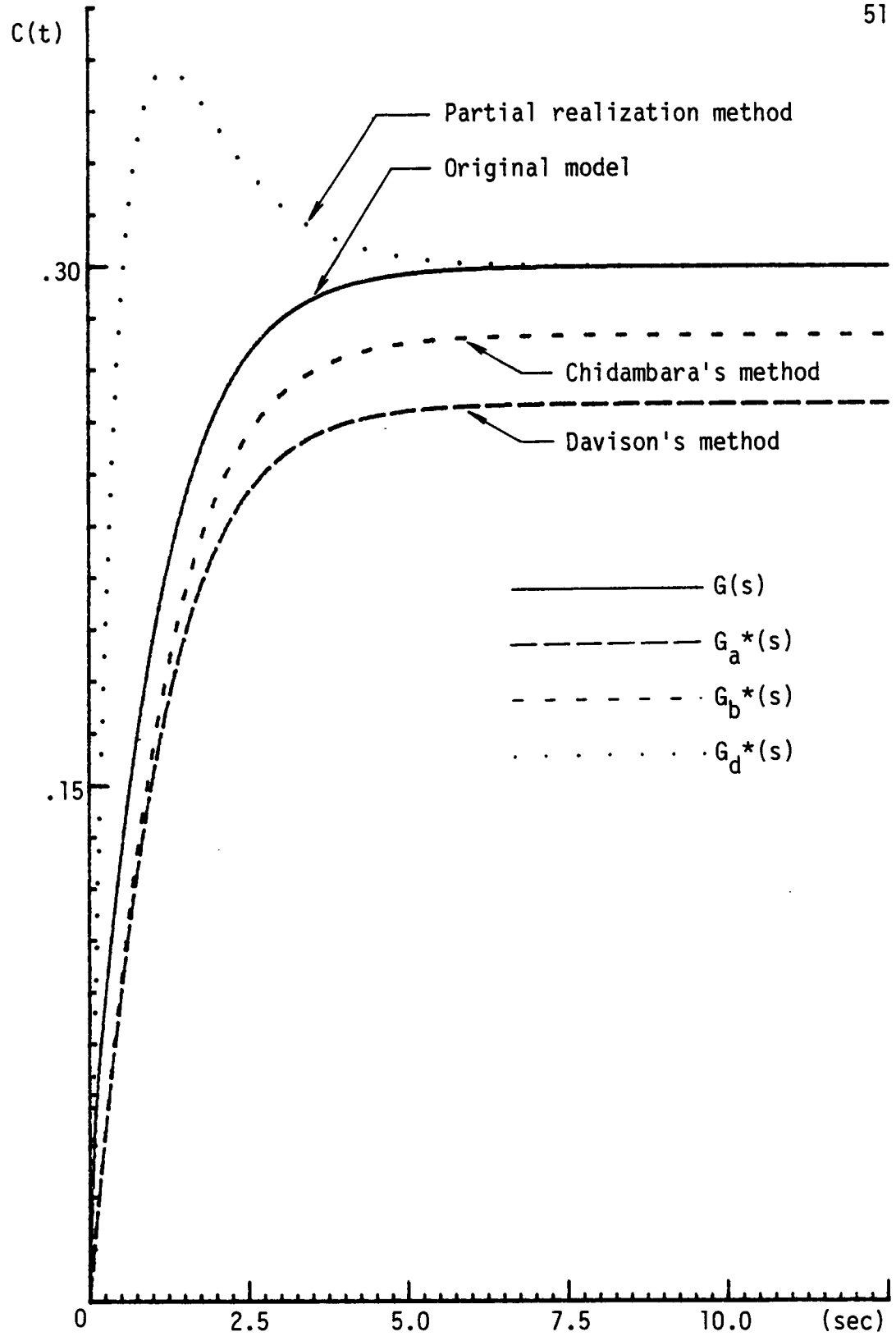

Figure 4-2. Comparison of the step responses for example 4-5.

CHAPTER 5

OPTIMUM APPROACHES

5-1. Estimation from the Samples of the Impulse Response

Assume that the samples of the impulse response are known and are not biased with noise. The transfer function of a discrete-time system, H(z), can be easily estimated [4]. The corresponding continuous-time transfer function, G(s), can then be derived from an inverse z-transform method.

5-1-1. Estimation of the Discrete-Time Transfer Function

Consider a discrete-time transfer function H(z) of order n

$$H(z) = \frac{d_0+d_1z^{-1}+d_2z^{-2}+ \ldots +d_nz^{-n}}{1+c_1z^{-1}+c_2z^{-2}+ \ldots . +c_nz^{-n}} \tag{5-1}$$

Let w_i (i=0,1,2,...) be the impulse response of system at t=iT, where T is the sampling interval, or

$$w_i = w(iT) \tag{5-2}$$

then H(z) has the form

$$H(z) = w_0+w_1z^{-1}+w_2z^{-2}+ \ldots . \tag{5-3}$$

Sustituting equation (5-3) into equation (5-1) gives

$$d_0+d_1z^{-1}+\ldots+d_nz^{-n}=w_0+(w_1+c_1w_0)z^{-1}+\ldots+\sum_{i=0}^{n} c_iw_{n-i}z^{-n}+ \ldots \tag{5-4}$$

Note that $c_0=1$. Equating the coefficients of the same powers of z^{-1} from both sides and letting the coefficients of the higher powers of z^{-1} at the right hand side be zero yield

$$d_j = \sum_{i=1}^{j} c_i w_{n-i} \qquad (j=0,1,\ldots n) \tag{5-5}$$

$$w_{n+k} + \sum_{i=1}^{n} c_i w_{n+k-i} = 0 \qquad (k=1,2,\ldots) \tag{5-6}$$

Since equation (5-6) does not contain d_i, one can condiser terms containing from $z^{-(n+1)}$ through z^{2n} to determine b_i. Rearranging equation (5-6) gives

$$\begin{bmatrix} w_1 & w_2 & \cdots & w_n \\ w_2 & w_3 & \cdots & w_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_n & w_{n+1} & \cdots & w_{2n-1} \end{bmatrix} \begin{bmatrix} c_n \\ c_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ c_1 \end{bmatrix} = \begin{bmatrix} -w_{n+1} \\ -w_{n+2} \\ \cdot \\ \cdot \\ \cdot \\ -w_{2n} \end{bmatrix} \tag{5-7}$$

After solving c one can determine d from equation (5-5), that is

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ c_n & c_{n-1} & c_{n-2} & & c_1 & 1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} \tag{5-8}$$

5-1-2. Determination of the Continuous-Time Transfer Function

To determine the continuous-time transfer function G(s) which has

the same impulse response as the discrete-time transfer function H(z) at the sampling instants, one can employ the z-transform technique [44] to find poles and residues of G(s) by performing the partial fraction expansion of H(z)/z.

Example 5-1: For the third-order transfer function

$$G(s) = \frac{s^2+3s+2}{s^3+12s^2+47s+60} \tag{5-9}$$

which was considered in example 3.1, the impulse responses at intervals of 0.1 second are

t	0	0.1	0.2	0.3	0.4
w(t)	1	0.3580819	0.0601145	-0.061847	-0.0981732

To identify a second-order system with the discrete-time transfer function

$$H(z) = \frac{d_0+d_1z^{-1}+d_2z^{-2}}{1+c_1z^{-1}+c_2z^{-2}} = \frac{d_0z^2+d_1z+d_2}{z^2+c_1z+c_2} \tag{5-10}$$

one can obtain the coefficients of the denominator which are calculated from equation (5-7).

$$\begin{bmatrix} 0.3580819 & 0.0601145 \\ 0.0601145 & -0.061847 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0.061847 \\ 0.0981732 \end{bmatrix} \tag{5-11}$$

Solving equation (5-11) gives c_1=-1.22096 and c_2=0.377602. Then Substituting these values into equation (5-8) yields

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.22096 & 1 & 0 \\ .377602 & -1.22096 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ .3580819 \\ .0601145 \end{bmatrix} = \begin{bmatrix} 1 \\ -.8628781 \\ 0 \end{bmatrix} \quad (5\text{-}12)$$

The resulting discrete-time transfer function becomes

$$H_1^*(z) = \frac{z^2-0.8628781z}{z^2-1.22096z+0.377602} \quad (5\text{-}13)$$

which can be arranged into the form

$$\frac{H_1^*(z)}{z} = \frac{(z-e^{-4.8695T}\cos\sqrt{1.3076}T)-3.5998e^{-4.8695T}\sin\sqrt{1.3076}T}{z^2-2(e^{-4.8695T})\cos\sqrt{1.3076}\,T)z+e^{-9.739T}} \quad (5\text{-}14)$$

Then the resulting Laplace transfer function can be obtained as

$$H_1^*(s) = \frac{s+0.75315}{s^2+9.739s+25.0196} \quad (5\text{-}15)$$

Different sampling intervals will give similar results. As T=0.05sec, then

$$H_2^*(s) = \frac{s+0.717455}{s^2+9.716s+24.52} \quad (5\text{-}16)$$

As T=0.02sec, then

$$H_3^*(s) = \frac{s+0.76185}{s^2+9.5776s+25.552} \quad (5\text{-}17)$$

Figure 5-1 compares the step responses of $H_1^*(s)$, $H_2^*(s)$, $H_3^*(s)$, and

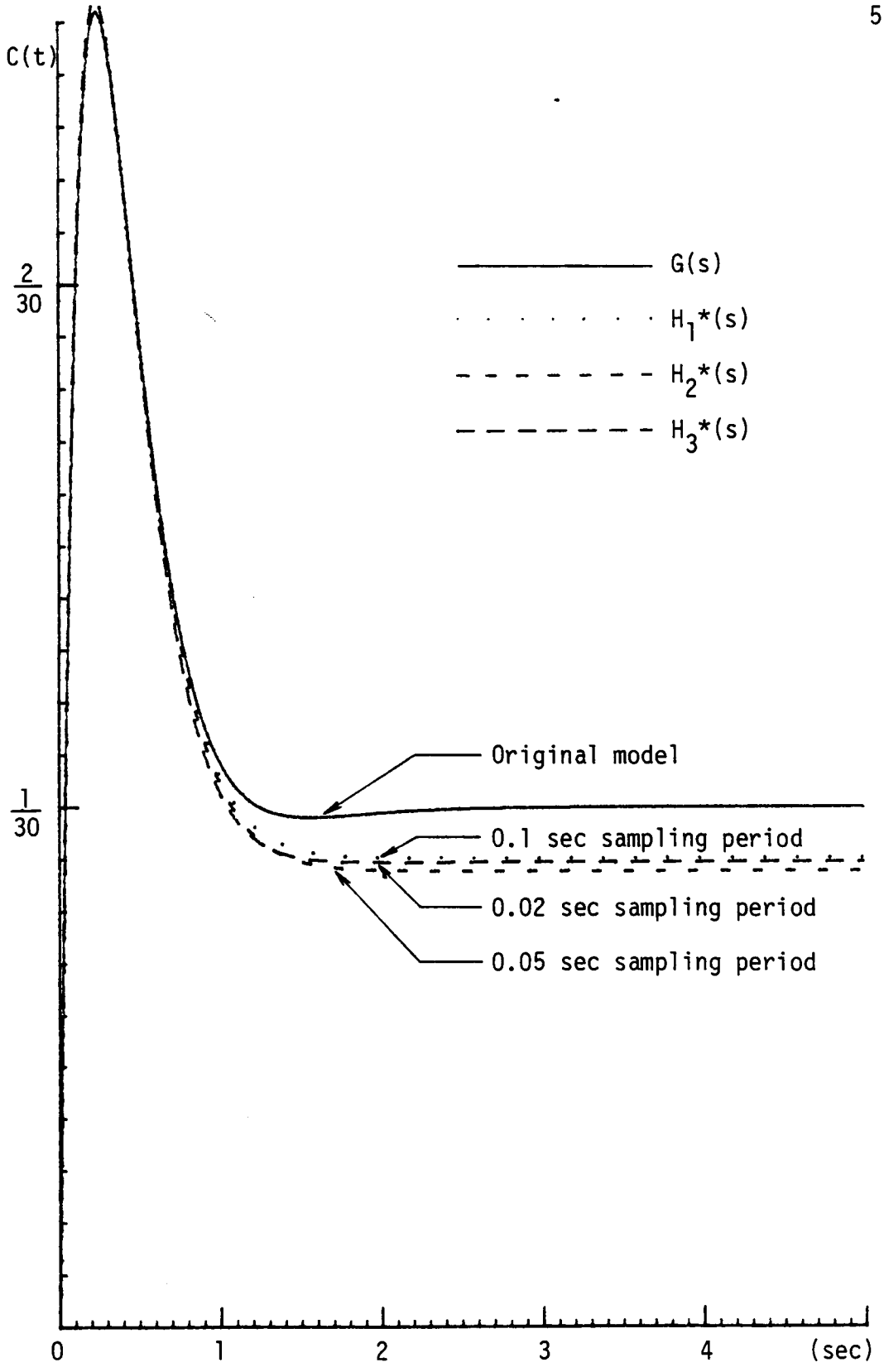

Figure 5-1. Step responses of G(s), H_1*(s), H_2*(s), and H_3*(s).

the original system G(s). Although the results do not retain the steady-state gain since limited sampling points, the identification method can be further considered if it can match more sampling points. Equations (5-16), (5-17), and (5-18) show that the resulting poles of the identification methods with different sampling time are very close. Thus, there may exist some optimum poles which will give the least error between the optimum result and the original system.

5-2. Least Square Method

Sinha and Pille [28] proposed the application of the matrix pseuso-inverse method for the determination of the discrete model. But they used Smith's method [44] to derive a continuous model whose numerator wasn't assumed to be a general case. Therefore, the continuous model needs to be determined in a proper way which gives a good approximation with respect to the discrete model.

5-2-1 Matrix Pseudo-Inverse Algorithm

Consider the discrete model of the single-input single-output system

$$H(z) = \frac{X(z)}{U(z)} = \frac{d_0+d_1z^{-1}+d_2z^{-2}+ \ldots +d_mz^{-m}}{1+c_1z^{-1}+c_2z^{-2}+ \ldots +c_nz^{-n}} \qquad (5\text{-}18)$$

The equivalent difference equation has the form

$$x_k = \sum_{i=0}^{m} d_i u_{k-i} - \sum_{i=1}^{n} c_i x_{k-i} \qquad (5\text{-}19)$$

where T is sampling interval, $x_i=x(iT)$, and $u_i=u(iT)$. Thus, the problem is to determine the parameters c_1, c_2, ...,c_n and d_1,d_2, ..., d_m.

Define

$$\underline{\theta} = [\ d_0 \quad d_1 \quad \cdots \quad d_m \quad c_1 \quad c_2 \quad \cdots \quad c_n\ ]^T \tag{5-20}$$

The dimension of $\underline{\theta}$ is p=m+n+1. Equation (5-19) may be represented by matrix equation

$$\underline{x}_k = A_k\underline{\theta} \tag{5-21}$$

where $k \geq p$ and

$$A_k = \begin{bmatrix} u_0 & u_{-1} & \cdots & u_{-m} & -x_{-1} & -x_{-2} & \cdots & -x_{-n} \\ u_1 & u_0 & \cdots & u_{1-m} & -x_0 & -x_{-1} & \cdots & -x_{1-n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ u_k & u_{k-1} & \cdots & u_{k-m} & -x_{k-1} & -x_{k-2} & \cdots & -x_{k-n} \end{bmatrix} \tag{5-22}$$

and

$$\underline{x}_k = [\ x_1 \quad x_2 \quad \cdots \quad x_k\ ]^T \tag{5-23}$$

The determination of the vector $\underline{\theta}$ is from the input-output data at sampling instants. By minimizing the sum of the square of the errors which are defined to be the differences between the step responses of the original system and the reduced model [4,28], one has

$$\underline{\theta} = A_k^{\dagger}\underline{x}_k \tag{5-24}$$

where

$$A_k^{\dagger} = [\, A_k^T A_k \,]^{-1} A_k^T \qquad (5\text{-}25)$$

To assure that the pseudo-inverse matrix, A_k , exists and is unique, the input sequence u_i must be satisfied any of following conditions [28].

(a). u_i is a random sequence;

(b). $u_i = 0 \quad$ for $i < 0$
$\quad\;\; = 1 \quad$ for $i \geq 0$; (5-26)

(c). u_i is a sequence of n discrete Fourier components.

Once the input of the sampling data satisfies one of above conditions, the solution of parameters $\underline{\theta}$ can then be obtained by applying equations (5-24) and (5-25). Appendix B shows the least square program solving a second order identified model with above condition(b).

5-2-2 Determination of the Equivalent Continuous Model

After solving , one needs to determine the equivalent continuous model. Sinha and Pille [28] suggested Smith's method [45] which determines continuous model by discrete model straightforwardly but gives poor fast component matching [28]. A new approach is proposed here which will be able to obtain a well matched continuous model.

For the second-order discrete model shown in equation (5-18) one can assume the equivalent continuous model H(s) has the form

$$H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \qquad (5\text{-}27)$$

where a_0 and a_1 can be obtained by Smith's method or by following equations

$$a_1 = -(\ln c_2)/T \tag{5-28}$$

and

$$a_0 = \left(\frac{\ln c_2}{2T}\right)^2 + \left(\frac{\cos^{-1}(-c_1/2e^{-a_1 t/2})}{T}\right)^2 \tag{5-29}$$

Note that T is the sampling period. Since the discrete parameter d_0 may not be zero, one can't employ inverse z-transform of H(z)/z, which was mentioned in section 5-1-2, to obtain the transfer function of Laplace-transform. In order to match the fast component and steady-state gain of the discrete model, one can use the concept of partial realization to match some time moments and Markov parameters but retain the so-called "optimum poles", not dominant poles.

Define that the poles which can be found by parameters a_1 and a_0 calaulated from equations (5-28) and (5-29) are optimum poles for a second order system. The unknown coefficients of numerator, b_1 and b_0 can be obtained by matching Markov parameters J_{-1} and J_0 so that the system will give accurate fast component and steady-state gain.

Example 5-2: Consider that the transfer function given in equation (5-9) can be identified as a second-order discrete transfer function shown in equation (5-10).

By using the conditon (5-26), the matrix A can be written as equation (5-30) shown. Different sampling time will have different x_i

value.

$$A_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -x_1 & 0 \\ 1 & 1 & 1 & -x_2 & -x_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (5\text{-}30)$$

Let sampling period T be 0.05sec and the total sampling time be 25sec. The resulting discrete transfer function can be calculated as

$$H_4^*(z) = \frac{0.0399647z-0.0382033}{z^2-1.56005z+0.612958} \quad (5\text{-}31)$$

To observe the performence of the reduced model, one can simulate the step response of equation (5-31) and check the sum of the squares of the sample errors. The error index can be expressed by

$$\Sigma e^2 = 1.436 \times 10^{-5} \quad (5\text{-}32)$$

Comparing the same error index obtained by the Cauer third form with $\Sigma e^2 = 3.024 \times 10^{-5}$. Obviously, the optimum approach gives better result than the Cauer third method of continued fraction expansion method.

The coefficients of denominator of the equivalent continuous model can be obtained from equations (5-28) and (5-29) as

$$a_1 = 9.789177$$

$$a_0 = 26.910981$$

Establishing the observable matrix for the reduced second order system obtained from equation (4-43) yields

$$F = \begin{bmatrix} 0 & -9.778917 \\ 1 & -26.910981 \end{bmatrix}$$

$$G = [\ 1 \qquad 0\ ]$$

To match $J_0=1$ and $J_{-1}=-1/30$, one can calculate the coefficients of numerator by using equation (4-49). The resulting continuous model then becomes

$$H_4^*(s) = \frac{s+0.8959137}{s^2+9.789177s+26.910981} \tag{5-33}$$

The sum of the squares of the sample errors will be

$$\Sigma e^2 = 1.5587 \times 10^{-5} \tag{5-34}$$

which shows that the performance of the equivalent continuous model is close to the discrete model. The same approach applied to different sampling period will give similar result as expected. Letting T be 0.1sec and the total sampling point be 500, the discrete model will be

$$H_5^*(z) = \frac{0.0640253z-0.0584164}{z^2-1.20919z+0.377859} \tag{5-35}$$

which has the error index

$$\Sigma e^2 = 6.13 \times 10^{-6} \tag{5-36}$$

The equivalent continuous model and its error index can then be obtained

$$H_5^*(s) = \frac{s+0.8992}{s^2+9.7323417s+26.9760113} \tag{5-37}$$

$$\Sigma e^2 = 7.0 \times 10^{-6} \tag{5-38}$$

Figure 5-2 and Figure 5-3 compare the step responses of $H_4^*(s)$, $H_5^*(s)$ and the original model G(s), respectively.

Both results of continuous model by proposed method has similar performances with its discrete model.

To compare the error index with continued fraction expansion methods (or matching time moments and Markov parameters), one can consider the same model as the transfer function of example 3-7.

Example 5-3: Consider the transfer function

$$G(s) = \frac{s^2+7s+12}{s^3+23s^2+62s+40} \tag{5-39}$$

Equations (5-40) and (5-41) are the resulting second-order discrete transfer functions obtained by using the identification method and the least square method, respectively. Note that the sampling period is taken as 0.1 second and the total sampling points are 500.

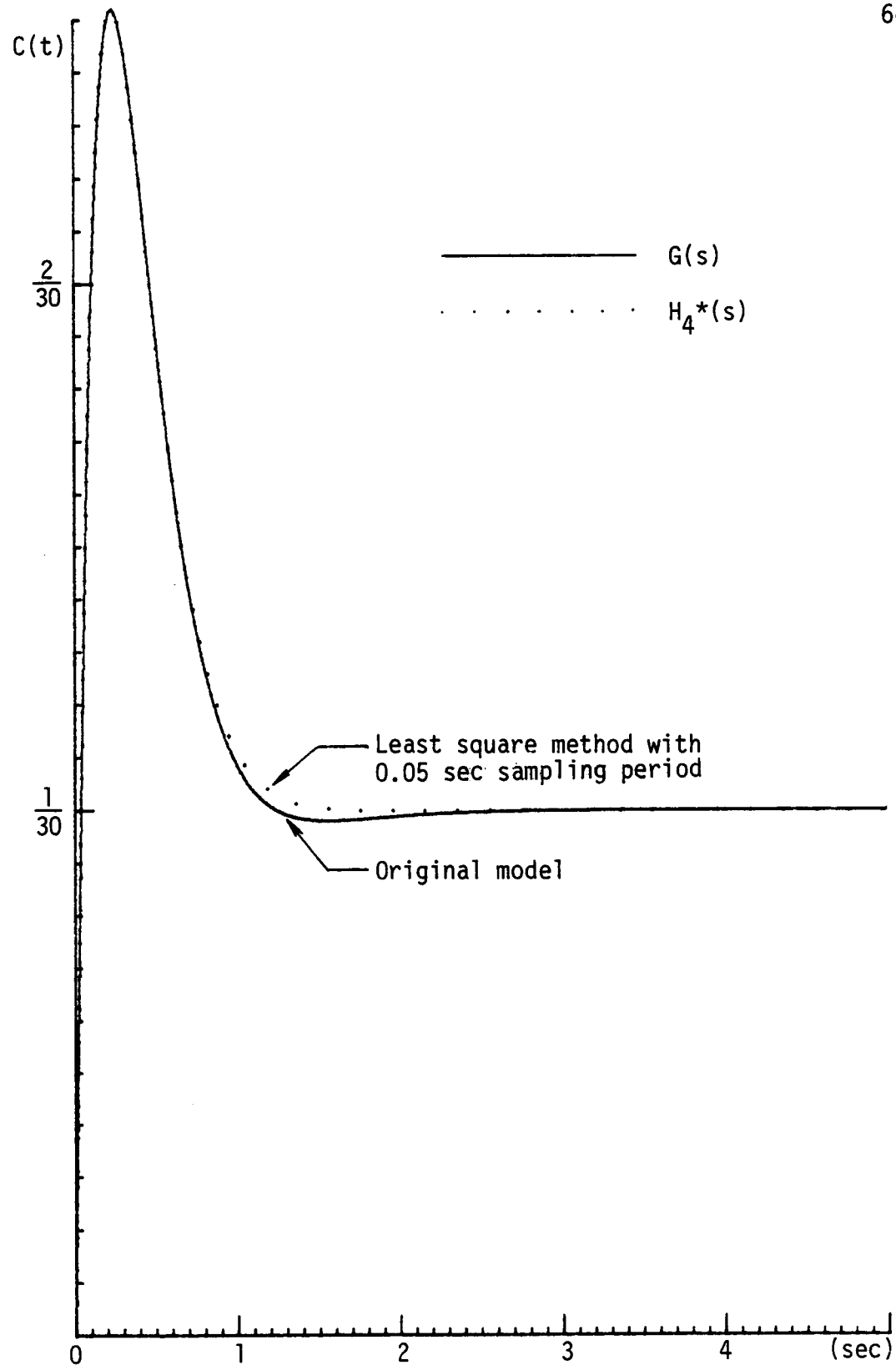

Figure 5-2. Step responses of G(s) and H_4*(s).

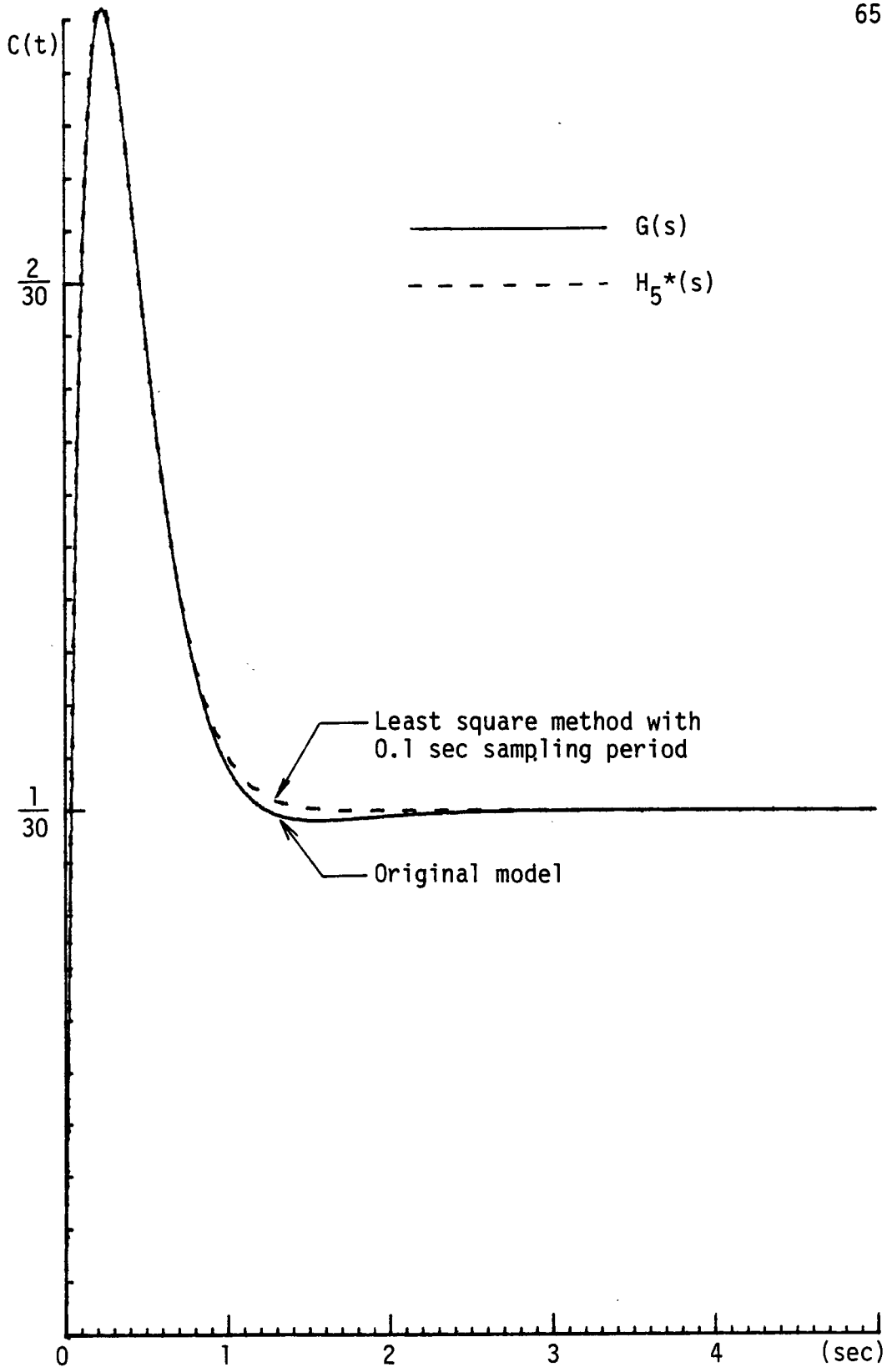

Figure 5-3. Step responses of G(s) and $H_5^*(s)$.

$$H_1^*(z) = \frac{z - .75726z}{z^2 - 1.05966z + .117556} \tag{5-40}$$

$$H_2^*(z) = \frac{.0543671z - .0305961}{z^2 - .994336 + .0735375} \tag{5-41}$$

The equivalent continuous models are

$$H_1^*(s) = \frac{s + 5.06324}{s^2 + 21.4084s + 14.2016} \tag{5-42}$$

$$H_2^*(s) = \frac{s + 6.80901}{s^2 + 26.0996s + 22.6967} \tag{5-43}$$

Figure 5-4 shows the comparison of step responses between the reduced order models and the original model. As expected, the identification method by using impulse response doesn't match the steady state gain, but the least square method by using step response gives a well matched result. The error index of 500 sampling points for the simplified model of equation (5-41) is 1.308×10^{-4}, which shows the optimum approach by the least square algorithm is better than any kind of the continued fraction expansion methods.

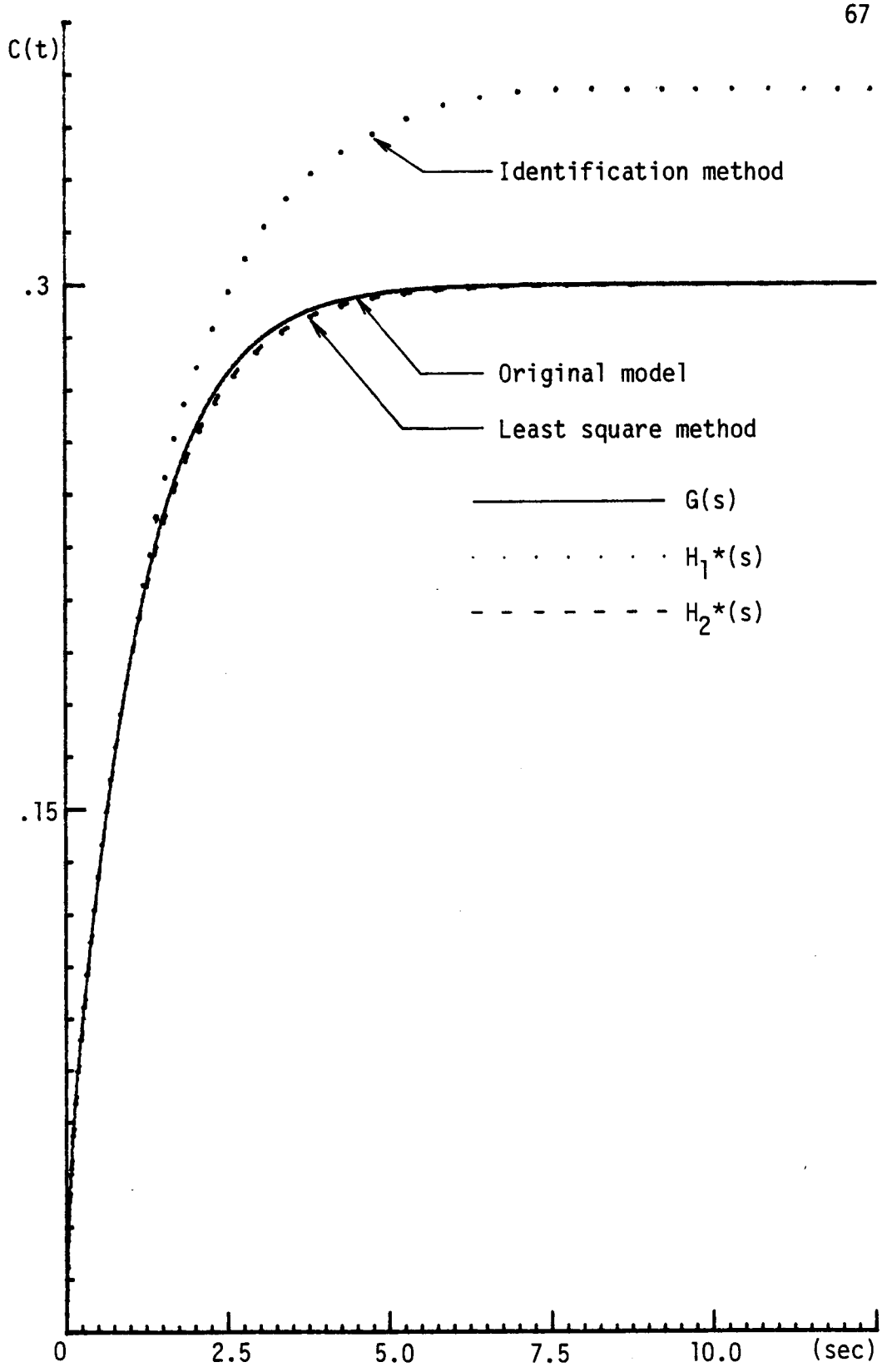

Figure 5-4. Step responses of G(s), H_1*(s), and H_2*(s) for example 5-3.

CHAPTER 6

SINGULAR VALUE DECOMPOSITION METHOD

6-1 Description of the Singular Value Decomposition Theorem

For an m x n matrix A the strictly positive square roots σ_i of the nonzero eigenvalues of $A^H A$(and AA^H, equivalently) are called the singular values of A [34]. If m=n, then the singular values are identical to the absolute eigenvalues of A.

Theorem: Let the m x n matrix A has rank r. Then there exists numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, the singular values of A, an m x m unitary matrix $U=[u_1,\ldots,u_m]$ and an n x n unitary matrix $V=[v_1,\ldots,v_n]$, such that

$$A = U\Sigma V^H \tag{6-1}$$

where

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \tag{6-2}$$

D is a r x r diagonal matrix with ith diagonal elements

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \tag{6-3}$$

where V^H means the Hermitian transpose of V. If A is real then U and V can be taken to be real matrices and $V^H = V^T$. The columns of U are

called the left singular vector of A while the columns of V are called the right singular vectors of A.

6-2 Discrete-time Markov Parameters and Hankel Matrices

The discrete-time state-space model with p input and q output is given by

$$\begin{aligned} \underline{x}(k+1) &= A\underline{x}(k)+B\underline{u}(k) \\ \underline{y}(k) &= C\underline{x}(k) \end{aligned} \tag{6-4}$$

where A is an n x n constant matrix, B is a p x n constant matrix, C is an n x p constant matrix, $\underline{x}$ is an n-vector, $\underline{u}$ is a p-vector, and $\underline{y}$ is a q-vector. The discrete-time Markov parameters [17] are defined by the quantities

$$J(k) = CA^{k-1}B \qquad (k=1,2,\ldots) \tag{6-5}$$

Obviously, the discrete-time Markov parameters are defined by the impulse responses of the system. If the given system is a continuous model represented by the triple [A,B,C], then the discrete-time Markov parameters can be obtained by using the sample of the impulse responses, i.e.,

$$J(k) = Ce^{AkT}B \qquad (k=1,2,\ldots) \tag{6-6}$$

where T is the sampling interval.

The Markov parameters can be used to define Hankel matrices [17]. Denote a Hankel matrix of order (u,v) and index i as

$$S_{uv}(i) = \begin{bmatrix} J_i & J_{i+1} & \cdots & J_v \\ J_{i+1} & J_{i+2} & \cdots & J_{v+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ J_u & J_{u+1} & & J_{u+v-i} \end{bmatrix} \tag{6-7}$$

6-3 Model Reduction Algorithm

The rank of the Hankel matrix will reflect the order of the corresponding system [34]. This rank-order relationship can be used in the realization problems.

In order to deal with the approximation problem, one needs a further information which shows the closeness of the Hankel matix to a lower rank matrix. Mathematically, the number of nonzero singular value is corresponding to the rank of the matrix. Therefore, applying singular value decomposition technique to find a lower rank Hankel matrix (so as to correspond to a lower order system) is an reasonable approach for model reduction.

Consider a multi-input-multi-output system with p input and q output. The impulse response sequence is given by J_i (i=1,2,...2N+1) which is a q x p block matrix.

Using equation (6-7), we have a q(N+1) by p(N+1) Hankel matrix.

$$H = S_{N+1,N+1}(1) \tag{6-8}$$

The singular value decomposition of the Hankel matrix is obtained as

$$H = U\Sigma V = \hat{U}\hat{V} \tag{6-9}$$

where U and V are unitary matrices, and

$$\Sigma = \text{Diag}[\sigma_1, \sigma_2, \ldots, \sigma_r, \ldots, \sigma_{N+1}] \tag{6-10}$$

with $\sigma_1 > \sigma_2 > \ldots > \sigma_r > \ldots > \sigma_{N+1} > 0$; and

$$\begin{aligned} \hat{U} &= U\Sigma^{\frac{1}{2}} \\ \hat{V} &= \Sigma^{\frac{1}{2}} V \end{aligned} \tag{6-11}$$

If the rank of H is r, then the singular values $\sigma_{r+1}, \ldots, \sigma_{N+1}$ will be zero. When they are not exactly zero but very small, then the matrix can be recognized an approximated r-rank matrix. To realize a r-th order reduced system, Kung [29] developed a triple result [A,B,C] which are related to equation (6-11). The following denotation is first adopted.

U_1 : the first N block rows and the first r columns of $\hat{U}$;
U_2 : the last N block rows and the first r columns of $\hat{U}$;
U^i : the i-th block row and the first r columns of $\hat{U}$;
V_1 : the first r rows and the first N block columns of $\hat{V}$;
V_2 : the first r rows and the last N block columns of $\hat{V}$;
V^i : the first r rows and the i-th block column of $\hat{V}$;

Then the triples can be represented by

$$\begin{aligned} B &= V^1 \\ C &= U^1 \\ A &= U_1^{\dagger} U_2 \\ &= V_2 V_1^{\dagger} \end{aligned} \tag{6-12}$$

Example 6-1: Consider the third order transfer function

$$G(s) = \frac{s^2+3s+2}{s^3+12s^2+47s+60} \tag{6-13}$$

which was consider in example 3-1. The impulse response at intervals of 0.05 second are

t	0	.05	.10	.15	.20	.25	.30	.35
J(t)	1	.62113	.35808	.17896	.06011	-.01588	-.06187	-.0870

The H matrix can be obtained as

$$H = \begin{bmatrix} .62113 & .35808 & .17896 & .06011 \\ .35808 & .017896 & .06011 & -.01588 \\ .17896 & .06011 & -.01588 & -.06187 \\ .06011 & -.01588 & -.06187 & -.08700 \end{bmatrix} \tag{6-14}$$

In this case the singular values are the same as the absolute eigenvales of H, which are

$$\begin{aligned} \sigma_1 &= .861310 \\ \sigma_2 &= .1643524 \\ \sigma_3 &= 2.33572 \times 10^{-4} \\ \sigma_4 &= 1.23106 \times 10^{-5} \end{aligned} \tag{6-15}$$

The first two corresponding eigenvectors are

$$\begin{aligned} v_1 &= u_1^T = [.858962 \quad .468046 \quad .204970 \quad .033240] \\ v_1 &= u_1^T = [-.296656 \quad .244627 \quad .563296 \quad .731335] \end{aligned} \tag{6-16}$$

Then one has

$$V^T = U = \begin{bmatrix} .7971750 & -.1202655 & - & - \\ .4343785 & .0991727 & - & - \\ .1902257 & .2283626 & - & - \\ .0308483 & .2964860 & - & - \end{bmatrix} \tag{6-17}$$

$$B^T = C = [\ .7971750 \quad -.1202655\ ] \tag{6-18}$$

$$A = \begin{bmatrix} .502264 & .283620 \\ -.283620 & 1.055750 \end{bmatrix} \tag{6-19}$$

The reduced order discrete-time transfer function can be obtained as

$$H^*(z) = \frac{.64995z-.672666}{z^2-1.558014z+.6107} \tag{6-20}$$

From the final value of the step response of (6-20), one can find that the reduced system doesn't retain steady-state value of the original system.

$$\lim_{z\to 1} H^*(z) = -.43116 \tag{6-21}$$

Apparantly, the reduced order model fails to be used. Note that the denominator of equation (6-20) can be transformed into s-domain by equations (5-28) and (5-29), which yields $s^2+9.862993s+26.3312$. The poles are located at

$$s_{1,2} = -4.931495 \pm j1.4182885 \tag{6-22}$$

which are close to the optimum poles obtained by the least square method as

$$s_{1,2} = -4.9845885 \pm j1.7187159 \tag{6-23}$$

The problem is occurred in numerator that means B and C matrix may be obtained by another improved version. The following example will yield the same situation.

Example 6-2: Consider the same transfer function as example 3-7

$$G(s) = \frac{s^2+7s+12}{s^3+23s^2+62s+40} \tag{6-24}$$

Using 0.1 second sampling period, the singular value decomposition method yields the second order discrete transfer function as

$$H^*(z) = \frac{.48787z-.142277}{z^2-1.030337z+.102492} \tag{6-25}$$

The final value of the step response will be

$$\lim_{z \to 1} H^*(z) = 4.7896 \tag{6-26}$$

It's much larger than the true value — 0.3; therefore, the resulting simplified model fails to match the original model.

Observing the denominator of equation (6-25), the s-domain poles can be calculated as

$$s_{1,2} = -.847026;\ -21.9327 \tag{6-27}$$

which are close to the poles obtained by optimum method

$$s_{1,2} = -.9007;\ -25.1989 \qquad (6\text{-}28)$$

Therefore, the singular value decompostion method may be improved by choosing some other B and C matrix to find a suitable numerator.

CHAPTER 7

CONCLUSION

Several model reduction techniques have been presented. Four groups are divided according to the general underlying principle used. The model reduction methods are evaluated on the basis of the least sum of the squares of the sample errors.

From the simulation of the step response, some model reduction methods have following drawbacks.

(a). Steady state errors exist. In this case the total errors during infinite period will be infinity. The Cauer first form of continued fraction expansion method, dominant mode methods without any compensation, and the identification method by using insufficient impluse response data are in such manner.

(b). The simplified model is unstable but the original model is stable. The Cauer second form of continued fraction expansion method and the singular value decomposition method may have this phenomenon in some cases.

From the point of view of practical applications, some methods are recommended here. Consider the quality of approximation. The optimum approach by using step responses appears to be the best method since it gives the least error index. The main drawback of this method is that the computation time will become excessive if the matching data are increased.

For ease of hand calculation, the continued fraction expansion methods matching some time moments and some Markov parameters are preferable. The problem is how to decide on matching numbers of time moments and Markov parameters. To reduce an nth order original model into an mth order model, matching m time moments and m Markov parameters, which can be seen as the same as the Cauer third form of continued fraction expansion method, always gives a satisfactory result. The new proposed general form of continued fraction expansion method provides a complete moment matching configuration.

In single-input single-output systems the proposed partial fraction expansion method is equivalent to Davison's method [18]. Thus, a low-order transfer function can be obtained by simple calculation without any matrix computation.

The interested viewpoint is that the denominators of all reduced models have closed values except dominant mode methods. The results of continued fraction expansion methods, optimum approaches, and the singular value decompositon method show this characteristic. Therefore, the singular perturbation method can be seen as an improvement of the dominant mode method and will be a further work, which was mentioned in some literatues [1,4,46,47,48] and was based on the dominant mode methods but changed the retained eigenvalues.

If the original system doesn't have dominant eigenvalues, the dominant mode method will not provide a good approximation for the reduced model. The partial realization method maintains the initial transient response and the steady state gain by matching some Markov

parameters and some time moments but doesn't provide a good approximation even though it retains the dominant eigenvalues. Therefore, to compensate dominant mode methods, the dominant eigenvalues should not be retained.

The singular value decomposition method using impulse responses is similar to the identification method using impulse response data. To retain some larger singular values, the singular value decomposition method has a similar simplification idea to the dominant mode methods which retain dominant eigenvalues. Since the singular value decomposition method can be considered to match more impulse response data or step response data, a future exploitation of this method to obtain an improved version is a deserving work.

Without any modification, the dominant mode methods and the singular value decomposition method can be applied for multi-input multi-output systems as mentioned in Chapter 4 and Chapter 6. The continued fraction expansion methods had been applied to multi-variable systems with similar manner to the single-input single-output case [1,49,50,51]. Consequently, the multi-variable method of optimum approach might be extended as a future work.

The simulation of step response and the least sum of the squares of the sample errors are observed here for the evaluation index. There are still some other indexes can be used such as frequency domain criteria and the integral of square error, which may provide a further information for the selection of the best model reduction method.

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APPENDICES

APPENDIX A

MARKOV PARAMETERS

A-1. Definition of Markov Parameters

Expanding G(s) in a Laurent series about s=0 gives

$$G(s) = J_0 s^{-1} + J_1 s^{-2} + J_2 s^{-3} + \ldots \qquad \text{(A-1a)}$$

or

$$G(s) = \sum_{i=0}^{\infty} J_i s^{-(i+1)} \qquad \text{(A-1b)}$$

where

$$J_i = CA^i B, \qquad i=0,1,2,..... \qquad \text{(A-2)}$$

are called the "Markov parameters".

A-2. Time-Moments of the Impulse Response

If G(s) has no poles at s=0, then the Taylor series expansion about s=0 can be obtained as

$$G(s) = -T_0 s^0 - T_1 s^1 - T_2 s^2 - \ldots \qquad \text{(A-3a)}$$

or

$$G(s) = -\sum_{i=0}^{\infty} T_i s^i \qquad \text{(A-3b)}$$

where

$$T_i = CA^{-(i+1)}B \qquad (i=0,1,2,\ldots) \qquad \text{(A-4)}$$

are called "the modified time-moments of the impulse response" of the

system. To show that T_i are related to time-moments of the impulse response, one can rewrite equation(A-3a) as

$$G(s) = G(0)+G'(0)s^1+\frac{1}{2!}G''(0)s^2+... \qquad (A-5)$$

Let the inverse Laplace transform of G(s) be g(t), then

$$T_0 = -\int_0^\infty g(t)dt = -\int_0^\infty g(t)e^{-st}dt\Big|_{s=0} = -G(0) \qquad (A-6)$$

$$T_1 = \int_0^\infty tg(t)dt = \int_0^\infty tg(t)e^{-st}dt\Big|_{s=0} = -G'(0) \qquad (A-7)$$

$$T_2 = -\frac{1}{2!}\int_0^\infty t^2g(t)dt = -\frac{1}{2!}\int_0^\infty t^2g(t)e^{-st}dt\Big|_{s=0} = -\frac{1}{2!}G''(0) \qquad (A-8)$$

in general

$$T_i = -\frac{1}{i!}\int_0 t^ig(t)dt = -\frac{1}{i!}G^i(0), \qquad (i=0,1,2,...) \qquad (A-9)$$

Since the ith time-moment about the origin of the impulse response is

$$\alpha_i = \int_0^\infty t^ig(t)dt, \qquad (i=0,1,2,...) \qquad (A-10)$$

the modified time-moments T_i then become

$$T_i = -\frac{(-1)^{i+1}}{i!}\alpha_i \qquad (A-11)$$

equation (A-11) shows the relationship between the modified time-moments and the time-moments of the impulse response of the system.

A-3. Generalized Markov Parameters

The modified time-moments T_i can be seen as an extension of the Markov parameters. Comparing equations (A-2) and (A-4) gives

$$T_{i-1} = J_{-i} \qquad \text{for } i \geq 1 \tag{A-12}$$

Therefore, the term "generalized Markov parameters"[4] will be

$$J_i = CA^i B, \qquad i=0,\pm 1,\pm 2,\ldots \tag{A-13}$$

which include the modified time-moments of the impulse response ($i \leq -1$) and the defined Markov parameters.

APPENDIX B

BASIC PROGRAM FOR LEAST SQUARE METHOD

```
1000 REM ************************************************************
1010 REM ************************************************************
1020 REM ***                                                      ***
1030 REM ***    SOLVING IDENTIFICATION OF A DYNAMIC SYSTEM        ***
1040 REM ***            USING LEAST SQUARE METHOD                 ***
1050 REM ***                                                      ***
1060 REM ***                        Design by Jih-Sheng Lai       ***
1070 REM ***                                  Dec. 1, 1984        ***
1080 REM ***                                                      ***
1090 REM ************************************************************
1100 REM ************************************************************
1110 DIM A(5,6)
1120 DIM Z(511,5),U(511),Y(511)
1130 DT=10
1140 PRINT"LEAST-SQUARE SOLVING DYNAMIC SYSTEM IDENTIFICATION"
1150 READ MM,NN,KN : DATA 2,3,500
1160 N=NN+MM
1170 FOR I=1 TO 10 : Y(I)=0 : U(I)=0 : NEXT I
1180 FOR I=11 TO KN+11
1190 T=(I-11)/DT
1200 Y(I)=.3-6/19*EXP(-T)+1/18*EXP(-2*T)-34/855*EXP(-20*T)
1210 U(I)=1
1220 NEXT I
1230 REM FORMULATING Z-MATRIX
1240 FOR I=1 TO KN : FOR J=1 TO MM
1250 Z(I,J)=-Y(I+10-J)
1260 NEXT J
1270 FOR J=MM+1 TO N
1280 Z(I,J)=U(I+11-J+MM)
1290 NEXT J,I
1300 REM MULTIPLICATION PROCESS
1310 FOR I=1 TO N:FOR J=1 TO N
1320 A(I,J)=0
1330 FOR K=1 TO KN
1340 A(I,J)=A(I,J)+Z(K,I)*Z(K,J)
1350 NEXT K,J,I
1360 PRINT : PRINT
1370 PRINT "Time interval= ";1/DT
1380 PRINT "Total point=   ";KN
1390 PRINT : PRINT
1400 REM Formulating A(I,N+1) column
1410 FOR I=1 TO N : A(I,N+1)=0
1420 FOR J=1 TO KN
1430 A(I,N+1)=A(I,N+1)+Z(J,I)*Y(J+10)
1440 NEXT J,I
1450 GOSUB 5000
```

```
1460 REM Print out discrete transfer function coefficients
1470 PRINT "Discrete-time transfer function"
1480 PRINT : PRINT "Denominator -- ";
1490 PRINT "z^2+(";A(1,N+1);")z+(";A(2,N+1);")"
1500 PRINT : PRINT "Numerator -- ";
1510 PRINT A(3,N+1);"z^2+(";A(4,N+1);")z+(";A(5,N+1);")"
1520 PRINT : PRINT
2000 REM ******* ESTIMATION OF LAPLACE TRANSFORM *******
2010 C1=A(1,N+1) : C2=A(2,N+1)
2020 D1=A(4,N+1) : D2=A(5,N+1)
2030 REM ******* For a second order system *******
2040 A1=-LOG(C2)*DT
2050 QQ=EXP(-A1/DT/2)
2060 X=-C1/QQ/2
2070 IF ABS(X)>1 THEN 2140
2080 W0=(-ATN(X/SQR(-X*X+1))+1.5707633#)*DT
2090 TH=ATN(-A1/2/W0)
2100 A0=(A1/2)*(A1/2)+W0*W0
2110 PRINT "Continuous-time transfer function"
2120 PRINT : PRINT "Denominator --s^2+";A1;"s+";A0
2130 END
2140 REM In case of two real poles
2150 BB=SQR(C1*C1-4*C2)
2160 X1=(C1+BB)/2
2170 X2=(C1-BB)/2
2180 Z1=-LOG(-X1)*DT
2190 Z2=-LOG(-X2)*DT
2200 A1=Z1+Z2
2210 A0=Z1*Z2
2220 GOTO 2120
5000 REM ******* GAUSS-JORDAN ELIMINATION ROUTINE *******
5010 EP=.00001
5020 REM ** Select a pivot element **
5030 FOR I=1 TO N : PI=A(I,I)
5040 IF ABS(PI) > EP THEN 5170
5050 FOR J=I+1 TO N
5060 PI=A(J,I) : K=J
5070 IF ABS(PI)>EP THEN 5100
5080 NEXT J
5090 GOTO 5310
5100 REM ******* Substitution *******
5110 FOR J=I TO N+1
5120 T=A(I,J)
5130 A(I,J)=A(K,J)
5140 A(K,J)=T
5150 NEXT J
```

```
5160 REM ******* Calculation  *******
5170 FOR J=I+1 TO N+1
5180 A(I,J)=A(I,J)/PI
5190 NEXT J
5200 FOR J=I+1 TO N
5210 MULT=A(J,I)
5220 FOR L=I+1 TO N+1
5230 A(J,L)=A(J,L)-MULT*A(I,L)
5240 NEXT L,J
5250 NEXT I
5260 FOR I=N TO 2 STEP -1
5270 FOR J=1 TO I-1
5280 A(J,N+1)=A(J,N+1)-A(J,I)*A(I,N+1)
5290 NEXT J,I
5300 GOTO 5340
5310 REM In case of olution
5320 PRINT "This is a set of linear dependent equations."
5330 PRINT "The program can't find solutions."
5340 RETURN
```

VITA

Jih-Sheng Lai was born in Taichung, Taiwan, Republic of China on October 6, 1953. He attended elementary schools in that city and was graduated from Provincial Taichung Industrial High school in June 1971. The following September he entered National Taiwan Normal University, and in June 1975 he received a Bachelor degree in Industrial Education. Between the fall of 1975 and the summer of 1976 he returned to Provincial Taichung Industrial High school and served as a practice teacher. Then he began the two-year military duty in the Air Force.

After military service, in the fall of 1978 he re-entered the Graduate Institute of National Taiwan Normal University. He received the Master degree in Industrial Education in June 1980. From July 1980 he was employed by Mingchi Institute of Technology and served as an instructor and the chairman of Electrical Engineering Department. After three years service, he accepted a two-year fellowship provided by National Science Council of his country to enter The University of Tennessee, Knoxville and study in Electrical Engineering Department. The Master's degree was awarded in June 1985.

The author is a student member of The Institute of Electrical and Electronics Engineers. Mr. Lai will study in the Ph. D. program of The University of Tennessee with a major in Electrical Engineering after graduation.