

Optimal Control of Transition Time in Solid-State Lasers

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Abstract

Solid-state and semi-conductor laser arrays have great potential for applications in space communication and internet service. Laser optics for the purposes of communication are most efficient when the array elements are aligned in the in-phase state. In order to counteract the typically unstable nature of the in-phase state, it is advantageous to inject a common driving field into the laser array. This causes the array to become entrained for sufficiently high amplitudes of the injected field. Of great interest is the level of this injected field which transitions the array from out-of-phase to in-phase in minimum time. In order to answer this question, a system of ordinary differential equations modeling the phases of two coupled solid-state lasers is studied. Optimal control and minimum time techniques are employed, treating the amplitude of the injected field as the control and using the array intensity as the measure of phase state. For initial positions not close to 0 or π , and for detunings which are the additive inverses of one another, it is shown that the optimal amplitude is simply the arbitrary upper bound placed on the amplitude. This, in turn, shows that by raising the upper bound, the transition time is lowered. Neither of these results are affected by choice of intensity cutoff. It is also shown that the array intensity is monotonically increasing on the optimal path and that the transition time is decreased by lower detuning values.

Background

Laser arrays hold great promise for applications which require high optical output from compact sources and fast switching times. One such application is the rapidly growing space communication industry. Most research in this area has focused on solid-state and semiconductor arrays, lasers which are optically amplified by solid materials. Most solid-state lasers are excited optically by pulsed discharge lamps, while electron beams are used for semiconductor lasers [1].

Laser communication has evolved dramatically over the last fifteen years. Advances in all branches of technology have increased the need for faster, more efficient data transfer. This, in turn, has led to the growing use and research of optics communication. Radio frequency wavelengths are thousands of times longer than those of optical frequencies, allowing laser communication systems to have smaller antenna and run at much lower power levels [2]. The high-bandwidth and narrow beam characteristics make laser communication ideal for space-based satellite communication [3]. NASA is currently developing an extensive satellite based tracking and data relay network based on laser crosslink technology. The U.S. Military continues to work on their GEO (geosynchronous Earth orbit) satellite network, which can relay information to ground stations, aircraft, and even submarines [2].

One area where laser communication is proving most useful is the Internet. The role of satellites in the Internet had always been as a cost-effective alternative to transoceanic fiber-optic connections. However, this may be changing. The overwhelming world-wide demand for Internet use and the increasing congestion problems have forced telecommunication companies to consider alternatives to traditional radio frequency technology [4]. INTELSAT has organized a Multicast Internet Caching and Replication System (MICRS) which sends frequently accessed Internet material via satellite to kiosks in Egypt, South Africa, Cyprus, and Brazil. In 1998, several Asian ISPs began using a satellite link from Asia to the U.S., tripling their bandwidth [4]. Satellite Internet links have proved to be compatible in performance with a terrestrial fiber-optic T-1 cable network, and superior in terms of configuration and cost. The newly developed Ka-band satellite systems will respond to the increasing global demand for data networking and Internet use with small,

low-cost, portable terminals, which will greatly increase the capacity of the shrinking market [4].

The most efficient mode of operation for space communication is obtained when the elements of the laser array are synchronized to an in-phase state. In this situation, the output of the lasers interfere constructively and the light intensity is maximized [5]. Unfortunately, the in-phase state is typically unstable. Instead, the attracting dynamic is the out-of-phase state, whose destructive interference pattern results in low intensity output [6]. The in-phase state can be stabilized, however, by injecting a common driving laser field into the laser array elements. This is accomplished by a technique called frequency locking, where the laser amplification is driven by the injected field and emission occurs at the frequency of this field. Full entrainment of the array elements occurs when the amplitude of the field is above a threshold determined by the coupling of the array. This will cause the in-phase state to stabilize, maximizing the intensity output [1, 5].

Equally important to the synchronization of the lasers, is the time required to reach the in-phase state from an arbitrary out-of-phase position. For the purposes of satellite communication and high-speed data transfer, it is desirable to minimize the transient time between the out-of-phase and in-phase states. This has only recently become the interest of researchers. The first use of optimal control techniques on this problem was by Jung et al. in 2002 [5].

We begin with the system of equations describing the dynamics of two evanescently coupled solid-state lasers, where the polarization is adiabatically removed [5]:

$$\begin{aligned} E'_j(t) &= (G_j - \alpha_j + \delta_j)E_j + \kappa(E_{j+1} + E_{j-1}) + E_e(t) \\ G'_j(t) &= \frac{\tau_c}{\tau_f}[p_j - (1 + |E_j|^2)G_j], \quad j = 1, 2. \end{aligned} \tag{1}$$

In equations (1), the free end boundary conditions $E_0(t) = E_3(t) = 0$ are imposed. The variables E_j and G_j are the dimensionless complex electric field and gain, respectively, for the j^{th} laser. All time and frequencies are scaled relative to the cavity round trip time, τ_c . τ_f is the fluorescence time of the laser medium. α_j and p_j are the dimensionless cavity decay and pump rates, respectively, for the j^{th} laser. κ is the coupling constant between the two lasers, and $E_e(t)$ is the slowly varying amplitude of the external field. δ_j is the

detuning for laser j [5,7]. The frequency of the external field is tuned to minimize the detunings from the cavity resonances, so we may assume $\delta_1 \approx -\delta_2$ [6].

We assume $\alpha_j = \alpha$, $p_j = p$, $p > \alpha$ [7]. Substituting $E_j(t) = \sqrt{I_j(t)} \exp(ix_j(t))$, where $I_j(t)$ is the intensity and $x_j(t)$ is the phase of laser j . Assuming $E_e(t) = E_e \equiv \sqrt{I_e}$ to be a constant, the model equations for two lasers become:

$$\begin{aligned} I'_j &= 2(G_j - \alpha)I_j + 2\kappa\sqrt{I_1 I_2} \cos(x_2 - x_1) + 2\sqrt{I_e I_j} \cos(x_j) \\ x'_j &= \delta_j + (-1)^j \kappa \frac{\sqrt{I_1 I_2}}{I_j} \sin(x_1 - x_2) - \sqrt{\frac{I_e}{I_j}} \sin(x_j) \\ G'_j &= (p - G_j - G_j I_j)w_0 \end{aligned} \quad (2)$$

where $w_0 = \tau_c/\tau_f$. We denote the dimensionless amplitude of the injected field by $u = \sqrt{I_e/I}$, where $I = p/\alpha - 1$. System (2) has been studied theoretically for N lasers [7] and the condition for full entrainment of the laser array was found to be $u = 4|\kappa|$. This condition assumes small deviations in detunings and small couplings between the lasers [5, 7].

For most dynamical alignments of the laser array, the intensity and gain oscillations are small. Khibnik et al., by solving the complete system (2), verified numerically that the phase equations alone give an accurate description of the laser dynamics [6]. Thus, we consider the “phase model”:

$$\begin{aligned} x'_1(t) &= \delta_1 + \kappa \sin(x_2 - x_1) - u(t) \sin(x_1) \\ x'_2(t) &= \delta_2 + \kappa \sin(x_1 - x_2) - u(t) \sin(x_2) \end{aligned} \quad (3)$$

where δ_1, δ_2 are the detunings, κ is the coupling constant, and $u(t)$ is the amplitude of the injected field at time t . Further, $\delta_1 > 0$, $\delta_1 \approx -\delta_2$, and $\kappa < 0$. For the purposes of driving the array from out-of-phase to in-phase, we consider the initial conditions $0 < x_1(0) < \pi$ and $x_2(0) = x_1(0) - \pi$. The control function $u(t)$ is a Lebesgue integrable function such that $4|\kappa| \leq u(t) \leq M$. The lower bound is set at $4|\kappa|$ as this is the smallest constant amplitude of the external field input which will drive the phases into the in-phase state [7]. The upperbound M , placed on the amplitude, is dependent on the system and application.

Optimal Control

Before preceeding, we offer a brief review of optimal control theory. For the purposes of this paper, we focus on minimization problems, as this is the type of problem we have. However, the theory is easily converted to maximization problems, and many texts cover this topic. Consider the problem

$$\begin{aligned} \min_u \int_0^T f(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) dt \\ \text{subject to, } x'_i(t) = g_i(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \\ x_i(0), x_i(T) \text{ fixed, } i = 1, \dots, n. \end{aligned} \quad (4)$$

where T fixed, the functions f, g_i are continuously differentiable for all i , and u_j is piecewise continuous for all j . Note that n need not equal m . We call x_i the state variables and u_j the control variables. Let $x = [x_1, \dots, x_n]$, $u = [u_1, \dots, u_m]$, and $g = [g_1, \dots, g_n]$ denote the indicated vector functions. Let u^* be the vector of optimal control functions. Solving simultaneously the system of state variable equations with $u = u^*$ gives us the corresponding optimal state variables x^* . By Pontryagin's Minimum Principle [8], if u^*, x^* solve the above problem, then there exist continuously differentiable functions $\lambda(t) = [\lambda_1(t), \dots, \lambda_n(t)]$ such that, defining

$$H(t, x, u, \lambda) = f(t, x, u) + \sum_{i=1}^n \lambda_i g_i(t, x, u),$$

u^*, x^*, λ together simultaneously satisfy

$$\begin{aligned} x'_i(t) &= \frac{\partial H}{\partial \lambda_i} = g_i(t, x, u), \quad x_i(0), x_i(T) \text{ fixed, } i = 1, \dots, n \\ \lambda'_k(t) &= -\frac{\partial H}{\partial x_k} = -\left(\frac{\partial f}{\partial x_k} + \sum_{i=1}^n \lambda_i \frac{\partial g_i}{\partial x_k}\right), \quad k = 1, \dots, n \\ 0 &= \frac{\partial H}{\partial u_j} = \frac{\partial f}{\partial u_j} + \sum_{i=1}^n \lambda_i \frac{\partial g_i}{\partial u_j}, \quad j = 1, \dots, m. \end{aligned}$$

We call H the Hamiltonian function. u^* minimizes $H(t, x^*(t), u, \lambda(t))$ with respect to u at each t [8].

Now consider a similar optimal control problem where $m = 1$ and our control u is bounded, namely, $a \leq u \leq b$, where $a < b$ real. In this situation, we lose the Hamiltonian condition $0 = \frac{\partial H}{\partial u_j}$. The condition can be recovered using nonlinear programming methods by inserting multipliers w_1, w_2 and creating a new equation called the Lagrangian, defined

$$L(t, x, u, \lambda) = H(t, x, u, \lambda) + w_1(b - u) + w_2(u - a).$$

Using this, we arrive at the following generalization,

$$\begin{aligned} \frac{\partial H}{\partial u}(t) > 0 &\text{ implies } u^*(t) = a \\ \frac{\partial H}{\partial u}(t) = 0 &\text{ implies } a \leq u^*(t) \leq b \\ \frac{\partial H}{\partial u}(t) < 0 &\text{ implies } u^*(t) = b \end{aligned} \tag{5}$$

This is what we would expect considering $u^*(t)$ minimizes the Hamiltonian at each t [8].

We now present an important result from optimal control theory. Again, the result can be extended to a more general case.

DEF: If the functions f, g in (4) have no explicit dependence on t , we say the system is *autonomous*.

THM: If H is autonomous, then it is a constant function of time along the optimal path.

PROOF: As $H(x, u, \lambda)$ has no explicit dependence on t , we must the chain rule to calculate the time derivative. So, $\frac{dH}{dt} = \frac{\partial H}{\partial x}x'(t) + \frac{\partial H}{\partial u}u'(t) + \frac{\partial H}{\partial \lambda}\lambda'(t) = -\lambda'(t)x'(t) + x'(t)\lambda'(t) = 0$. Thus, H is a constant function of time. ■

Finally, we discuss a specific type of optimal control problem, minimum time problems. Suppose we are given a system of differential equations $x'_i(t) = g_i(t, x, u)$, $x_i(0)$ fixed for $i = 1, \dots, n$, and we wish to find the optimal control u which achieves a final time condition $K(T, x(T)) = 0$ in minimal time. We can solve this problem by amending our optimal control problem. Consider (4) with $f(t, x, u) \equiv 1$ and T free. Then we have,

$$\min_u \int_0^T 1 \, dt$$

$$\text{subject to, } x'_i(t) = g'_i(t, x(t), u(t)), x_i(0) \text{ fixed, } i = 1, \dots, n$$

$$K(T, x(T)) = 0$$

$$a \leq u(t) \leq b \text{ for all } t.$$

Notice that by solving this problem with optimal control techniques and minimizing the integral, we minimize T . All necessary conditions for a solution are as before, and we gain the following conditions,

$$\lambda_i(T) = p \frac{\partial K}{\partial x_i}(T, x^*(T)), \text{ where } p \text{ is a real number,}$$

$$H(T, x^*(T), u^*(T), \lambda(T)) + p \frac{\partial K}{\partial T}(T, x^*(T)) = 0.$$

The proof of these conditions rely on Farkas' Lemma [8].

We are now ready to state our problem in formal terms. We will use the intensity of the laser array to measure how close to in-phase the system is at time t .

$$\min_u \int_0^T 1 dt$$

$$\begin{aligned} \text{subject to, } \quad & x'_1(t) = \delta_1 + \kappa \sin(x_2 - x_1) - u \sin(x_1) \\ & x'_2(t) = \delta_2 + \kappa \sin(x_1 - x_2) - u \sin(x_2) \\ & 0 < x_1(0) < \pi \text{ and } x_2(0) = x_1(0) - \pi \\ & I(T) = \cos^2(\tfrac{1}{2}(x_1(T) - x_2(T))) = I_0 \\ & 4|\kappa| \leq u(t) \leq M \text{ for all } t. \end{aligned} \tag{6}$$

where $\delta_1 > 0$, $\delta_1 \approx -\delta_2$, $\kappa < 0$, and M an arbitrary constant. I_0 , where $0 < I_0 \leq 1$, is the desired final intensity specified by the operator and determined by the laser system. We assume that I_0 is an attainable final intensity. Note, an intensity of 0 is the pure out-of-phase state, 1 is the pure in-phase state, and $I(0) = 0$.

Analysis

For the purposes of this paper, we will only consider initial phases which satisfy $\delta_1 < 4|\kappa| \sin(x_1(0))$. As the detunings are small, this will exclude only initial phases which are very close to 0 or π . Further, we will assume $\delta_1 = -\delta_2$. Again, we lose little generality as $\delta_1 \approx -\delta_2$ is guaranteed. We begin by using Pontryagin's Minimum Principle. Note the Hamiltonian for our problem is

$$H = 1 + \lambda_1(\delta_1 + \kappa \sin(x_2 - x_1) - u \sin(x_1)) + \lambda_2(\delta_2 + \kappa \sin(x_1 - x_2) - u \sin(x_2)) \tag{7}$$

where λ_1, λ_2 are continuously differentiable functions. It follows that

$$\lambda'_1(t) = \lambda_1 \kappa \cos(x_2 - x_1) + \lambda_1 u \cos(x_1) - \lambda_2 \kappa \cos(x_1 - x_2)$$

$$\lambda'_2(t) = \lambda_2 \kappa \cos(x_1 - x_2) + \lambda_2 u \cos(x_2) - \lambda_1 \kappa \cos(x_2 - x_1)$$

Further, the system is autonomous and $\frac{\partial H}{\partial T} = 0$, so $H(t) = 0$ for all t . Using this information, we present a result we will use in the calculation of the optimal control.

CLAIM: On the optimal path, $\lambda_1 \neq \lambda_2$ for all $t \in [0, T)$.

PROOF: Suppose not, i.e., there exists $t_0 \in [0, T)$ such that $\lambda_1(t_0) = \lambda_2(t_0)$. Then,

$$\begin{aligned} H(t_0) = 0 &= 1 + \lambda_1(\delta_1 + \kappa \sin(x_2 - x_1) - u \sin(x_1) + \delta_2 + \kappa \sin(x_1 - x_2) - u \sin(x_2)) \\ &= 1 + \lambda_1(-u \sin(x_1) - u \sin(x_2)) \\ &= 1 + \lambda_2(-u \sin(x_1) - u \sin(x_2)) \end{aligned}$$

As $H = 0$ for all t , $\frac{dH}{dt} = 0$ for all t . So,

$$\frac{dH}{dt}(t_0) = 0 = \lambda'_1(-u \sin(x_1) - u \sin(x_2)) + \lambda_1(-u \sin(x_1) - u \sin(x_2))' \quad (8)$$

$$= \lambda'_2(-u \sin(x_1) - u \sin(x_2)) + \lambda_2(-u \sin(x_1) - u \sin(x_2))' \quad (9)$$

Subtracting (9) from (8) gives $(-u \sin(x_1) - u \sin(x_2))(\lambda'_1 - \lambda'_2) = 0$

Case 1: $-u \sin(x_1) - u \sin(x_2) = 0$

Case 2: $\lambda'_1 - \lambda'_2 = 0$

$$\lambda'_1 = \kappa \lambda_1 \cos(x_1 - x_2) - \kappa \lambda_2 \cos(x_2 - x_1) + \lambda_1 u \cos(x_1)$$

$$\text{So, } \lambda'_1(t_0) = \lambda_1 u \cos(x_1). \text{ Similarly, } \lambda'_2(t_0) = \lambda_2 u \cos(x_2)$$

$$\text{Thus, } 0 = \lambda'_1 - \lambda'_2 = \lambda_1 u (\cos(x_1) - \cos(x_2)).$$

$$4|\kappa| \leq u \leq M, \text{ so } u \neq 0. \text{ If } \lambda_1 = \lambda_2 = 0, \text{ then } H = 1 \neq 0.$$

$$\text{So, } \cos(x_1) - \cos(x_2) = 0 \Rightarrow \cos(x_1) = \cos(x_2).$$

If $\sin(x_1) = \sin(x_2)$, then $x_1 = x_2 \Rightarrow I(t_0) = 1$. If the intensity were to reach 1, then we would be in the pure in-phase state and choose to end the laser transition. In other words, $t_0 = T$, which contradicts our choice of t_0 . Thus, in this case, $\cos(x_1(t_0)) = \cos(x_2(t_0)) \Rightarrow \sin(x_1) = -\sin(x_2) \Rightarrow -u \sin(x_1) - u \sin(x_2) = 0$.

$$\text{Hence, } -u(t_0) \sin(x_1(t_0)) - u(t_0) \sin(x_2(t_0)) = 0.$$

Substituting back into the Hamiltonian,

$$H(t_0) = 1 + \lambda_1(-u \sin(x_1) - u \sin(x_2)) = 1 \neq 0. \text{ Contradiction.}$$

Therefore, $\lambda_1 \neq \lambda_2$ on $[0, T)$. ■

THM: $u^* \equiv M$.

PROOF: From (5), $u^*(t)$ is equal to the upper bound whenever $\frac{\partial H}{\partial u} < 0$. Thus, it suffices to show $-\lambda_1 \sin(x_1) - \lambda_2 \sin(x_2) < 0$ for all t .

Consider $t = 0$: $x_2(0) = x_1(0) - \pi$. So, $x_1(0) - x_2(0) = \pi$.

$$H(0) = 1 + \lambda_1(\delta_1 + \kappa \sin(x_2 - x_1) - u \sin(x_1)) + \lambda_2(\delta_2 + \kappa \sin(x_1 - x_2) - u \sin(x_2))$$

$$\begin{aligned}
&= 1 + \lambda_1 \delta_1 - \lambda_2 \delta_1 - u \lambda_1 \sin(x_1) - u \lambda_2 \sin(x_1 - \pi) \\
&= 1 + (\delta_1 - u \sin(x_1))(\lambda_1 - \lambda_2) = 0.
\end{aligned}$$

As $u \sin(x_1(0)) \geq 4|\kappa| \sin(x_1(0)) > \delta_1$ by choice, $\lambda_1 - \lambda_2 > 0 \Rightarrow \lambda_1 > \lambda_2$ at $t = 0$. But, $\lambda_1 \neq \lambda_2$, so $\lambda_1 > \lambda_2$ for all $t \in [0, T)$. As λ_1, λ_2 continuous, we have $\lambda_1 \geq \lambda_2$ for all t .

Note, once and for all, that this forces $x_1 - x_2 \leq \pi$ for all t . To see this, observe $(x_1 - x_2)'(0) < 0$, so $(x_1 - x_2) < \pi$ for some small interval following $t = 0$. Now, suppose $(x_1 - x_2)$ moves back up to π . Then, $H = 0 = 1 + (\delta_1 - u \sin(x_1))(\lambda_1 - \lambda_2)$. We know $(\lambda_1 - \lambda_2) > 0$ on $[0, T)$. So, $\delta_1 < u \sin(x_1)$. This begins the cycle over again. Thus, $(x_1 - x_2)$ will never exceed π and $\kappa \sin(x_2 - x_1) \geq 0$ for all t .

$$\begin{aligned}
H = 0 &= 1 + \lambda_1(\delta_1 + \kappa \sin(x_2 - x_1) - u \sin(x_1)) + \lambda_2(\delta_2 + \kappa \sin(x_1 - x_2) - u \sin(x_2)) \Rightarrow \\
&u(\lambda_1 \sin(x_1) + \lambda_2 \sin(x_2)) = 1 + (\delta_1 + \kappa \sin(x_2 - x_1))(\lambda_1 - \lambda_2) > 0.
\end{aligned}$$

As $u > 0$, $\lambda_1 \sin(x_1) + \lambda_2 \sin(x_2) > 0$. ■

Notice this proof was given with no reference to the final time condition. Thus, it follows regardless of choice of I_0 . Also, the remaining analysis is given for phases on the optimal path, but we have shown the optimal path to be the arbitrarily chosen M . Thus, the results hold for all phases with constant control $u = C$, where $C \geq 4|\kappa|$. To verify this, we need only take $M = C$.

CLAIM: On the optimal path, $x'_1 \neq x'_2$ for all $t \in [0, T)$

PROOF: Suppose not, i.e., suppose there exists $t_0 \in [0, T)$ such that $x'_1(t_0) = x'_2(t_0)$.

$$\text{Then, } H(t_0) = 1 + x'_1(\lambda_1 + \lambda_2) = 1 + x'_2(\lambda_1 + \lambda_2)$$

$$H = 0 \text{ for all } t, \text{ so } \frac{dH}{dt} = 0 \text{ for all } t. \text{ Hence,}$$

$$\begin{aligned}
0 &= x''_1(\lambda_1 + \lambda_2) + x'_1(\lambda'_1 + \lambda'_2) = x''_2(\lambda_1 + \lambda_2) + x'_2(\lambda'_1 + \lambda'_2) \Rightarrow \\
&(x''_1 - x''_2)(\lambda_1 + \lambda_2) = 0.
\end{aligned}$$

Case 1: $\lambda_1 + \lambda_2 = 0 \Rightarrow H(t_0) = 1 + x'_1(\lambda_1 + \lambda_2) = 1 \neq 0$. Contradiction.

Case 2: $x''_1 = x''_2 \Rightarrow$

$$\kappa \cos(x_2 - x_1)(x'_2 - x'_1) - u' \sin(x_1) - u \cos(x_1)x'_1 =$$

$$\kappa \cos(x_1 - x_2)(x'_1 - x'_2) - u' \sin(x_2) - u \cos(x_2)x'_2.$$

$u^* \equiv M$, so $u' = 0$. Thus, using $x'_1 = x'_2$, we see

$$-M \cos(x_1)x'_1 = -M \cos(x_2)x'_2 \Rightarrow \cos(x_1) = \cos(x_2)$$

Subcase 1: $\sin(x_1) = -\sin(x_2) \Rightarrow x'_1 = -x'_2 \Rightarrow x'_1 = x'_2 = 0 \Rightarrow H = 1 \neq 0$.

Contradiction.

Subcase 2: $\sin(x_1) = \sin(x_2) \Rightarrow x_1 = x_2 \Rightarrow I = 1$. Again, this contradicts our choice of t_0 .

Hence, $x'_1 \neq x'_2$ for all t . ■

COROLLARY: $(x_1 - x_2)'(t) \leq 0$ for all t .

PROOF: As $\delta_1 < 4|\kappa|\sin(x_1(0))$, $x'_1(0) < 0 < x'_2(0)$ is guaranteed. As $u^* \equiv M$, u continuous, so x'_1, x'_2 are continuous as well. $x'_1 \neq x'_2$ on $[0, T)$, so $(x'_1 - x'_2)(0) < 0 \Rightarrow (x'_1 - x'_2) \leq 0$ for all t . ■

COROLLARY: The intensity function $I(t)$ is monotonically increasing.

PROOF: Follows immediately from above corollary and definition of $I(t)$.

We now consider the fixed points of the system. Let $\bar{x}_1, \bar{x}_2$ be the fixed points of the phases. Then,

$$0 = \delta_1 + \kappa \sin(\bar{x}_2 - \bar{x}_1) - M \sin(\bar{x}_1)$$

$$0 = \delta_2 + \kappa \sin(\bar{x}_1 - \bar{x}_2) - M \sin(\bar{x}_2)$$

Solving for these equations, we see $0 = -M(\sin(\bar{x}_1) + \sin(\bar{x}_2)) \Rightarrow \sin(\bar{x}_1) = -\sin(\bar{x}_2)$. As $0 < x_1 - x_2 < \pi$ guaranteed and $(x_1 - x_2)'(t) \leq 0$, $\bar{x}_1 = -\bar{x}_2$, and, further, $0 = \delta_1 - \kappa \sin(2\bar{x}_1) - M \sin(\bar{x}_1)$. This equation has infinitely many solutions. However, we know $(x_1 - x_2)'(t) \leq 0$ for all t , which forces $\bar{x}_1 - \bar{x}_2 < \pi$. Further, $\bar{x}_1 = -\bar{x}_2$ tells us precisely which solutions are the fixed points, namely $\bar{x}_1 \in [0, \pi)$ and $\bar{x}_2 \in (-\pi, 0]$. So, we can solve for the fixed points given δ_1, κ , and M . $\bar{x}_1, \bar{x}_2$ act as limits for the phases x_1, x_2 respectively as $t \rightarrow \infty$ (see figure 1). It is worth noting that the initial positions of the phases have no effect on $\bar{x}_1, \bar{x}_2$ (see figure 2).

We can also show $x_1(t) > 0$ for all t . To see this, consider a point in time t where $x_1(t) = 0$. Then, $x'_1(t) \leq 0$ in order for x_1 to reach 0. But, if $x_1(t) = 0$, then $x'_1(t) = \delta_1 + \kappa \sin(x_2) \leq 0 \iff x_2 > 0 \iff x_2$ has already crossed 0. A similar argument shows $x_2(t) = 0 \iff x_1$ has already crossed 0. This is clearly impossible. Thus, $-\pi < x_2(t) < 0 < x_1(t) < \pi$ for all time t . It directly follows that $\sin(x_2) < 0 < \sin(x_1)$.

Define $\bar{I} = \cos^2(\frac{1}{2}(\bar{x}_1 - \bar{x}_2))$. So, $I(0) = 0$ and $I(t) \rightarrow \bar{I}$ as $t \rightarrow \infty$, and does so

monotonically (see figure 3). Thus, we must choose $I_0 < \bar{I}$ for the system to have a valid final time condition. Also, notice that an intensity of 1, pure in-phase, is impossible to attain. In fact, $I(t) \rightarrow 1$ only when $x_1, x_2 \rightarrow 0$. This occurs if and only if $\delta_1 = \delta_2 = 0$ (see figure 4). In our earlier proofs, we were forced to omit the point $t = T$ to accomodate the possibility of $I(T) = 1$. However, we now know this to be unnecessary, so we strengthen those results by noting $\lambda_1 > \lambda_2$ for all t and $(x_1 - x_2)'(t) < 0$ for all t .

We now discuss “phase alignment.” Suppose for some time t that $x_1(t) = -x_2(t)$. Then, by (3), $x_1'(t) = -x_2'(t)$. So, if we choose $h \neq 0$ small, $\frac{x_1(t+h)-x_1(t)}{h} \approx x_1'(t) = -x_2'(t) \approx -\frac{x_2(t+h)-x_2(t)}{h}$. As $x_1(t) = -x_2(t)$, we get $x_1(t+h) = -x_2(t+h)$. Taking $h > 0$, $x_1(t) = -x_2(t) \Rightarrow x_1 = -x_2$ for some small interval after t . But, the same argument can be made for each point in this interval, extending the interval to T . Similarly, considering $h < 0$, we create a small interval before t where $x_1 = -x_2$. Again, we can use the same argument to extend this interval to 0. Thus, $x_1(t) = -x_2(t)$ for some time $t \iff x_1(t) = -x_2(t)$ for all time t . This directly implies

$$x_1(0) = \frac{\pi}{2}, x_2(0) = -\frac{\pi}{2} \Rightarrow x_1 = -x_2 \forall t \Rightarrow \sin(x_1) = -\sin(x_2) \forall t \in [0, T]$$

$$x_1(0) < \frac{\pi}{2}, x_2(0) < -\frac{\pi}{2} \Rightarrow x_1 < -x_2 \forall t \Rightarrow \sin(x_1) < -\sin(x_2) \forall t \in (0, T]$$

$$x_1(0) > \frac{\pi}{2}, x_2(0) > -\frac{\pi}{2} \Rightarrow x_1 > -x_2 \forall t \Rightarrow \sin(x_1) > -\sin(x_2) \forall t \in (0, T]$$

It is also of interest if and when the phases are individually monotonic. We know from the previous results there could not exist a point in time where $x_1'(t) > 0$ and $x_2'(t) < 0$, but it could be possible for these to occur at different times. Consider a point in time t where $x_1'(t), x_2'(t) > 0$. Then, $M \sin(x_1(t)) < \delta_1 + \kappa \sin(x_2(t) - x_1(t)) < -M \sin(x_2(t)) \Rightarrow \sin(x_1(t)) < -\sin(x_2(t))$. As discussed earlier, this implies $\sin(x_1) < -\sin(x_2)$ for all $t \in (0, T]$, which only occurs when $x_1(0) < \frac{\pi}{2}$. Similarly, we can show $x_1'(t), x_2'(t) < 0$ for some time t only when $x_1(0) > \frac{\pi}{2}$. Hence, at least one phase is monotonic, if not both, and if a phase is not monotonic, it must be the phase which starts closer to 0 (see figure 5).

We have shown what effect M has on the fixed points of the phases. Let us now examine how it affects the transition time T .

THM: Let u be any admissible control. If there exists an interval $[t_0, T]$ where $u < M$,

then u is not optimal.

PROOF: Fix $\delta_1, \delta_2, \kappa, I_0$, and $x_1(0)$. Let u^* be an optimal control with associated final time T^* . Let u be an admissible control with above condition and have associated time T and phases x_1, x_2 . By definition, $T^* \leq T$. We wish to show $T^* < T$. Introduce third control $\tilde{u}$ with final time $\tilde{T}$ and phases $\tilde{x}_1, \tilde{x}_2$. Define

$$\tilde{u} = \begin{cases} u, & 0 \leq t < t_1 \\ M, & t_1 \leq t \leq \tilde{T} \end{cases}$$

where t_1 is a point in time chosen so that $(\tilde{x}_1 - \tilde{x}_2)(t_1) < \frac{\pi}{2}$, $(x_1 - x_2)(t_1) < \frac{\pi}{2}$ and, $(\tilde{x}_1 - \tilde{x}_2)' \leq 0$ on $[t_1, \tilde{T}]$. Such a point must exist, or $I = I_0$ would never be achieved. At t_1 , $x_1 = \tilde{x}_1$, $x_2 = \tilde{x}_2$. But,

$$x_1' - \tilde{x}_1' = -u \sin(x_1) + M \sin(\tilde{x}_1) = \sin(x_1)(M - u) > 0 \Rightarrow x_1' > \tilde{x}_1' \quad \text{and}$$

$$x_2' - \tilde{x}_2' = -u \sin(x_2) + M \sin(\tilde{x}_2) = \sin(x_2)(M - u) < 0 \Rightarrow x_2' < \tilde{x}_2'$$

Hence, $x_1 > \tilde{x}_1 > 0$, $x_2 < \tilde{x}_2 < 0$ for some interval which may or may not include $\tilde{T}$. Suppose it does not include $\tilde{T}$. Suppose either $x_1 = \tilde{x}_1$ or $x_2 = \tilde{x}_2$ for some point after t_1 . First consider $x_1(t_1) = \tilde{x}_1(t_1)$. Clearly, $x_1' \leq \tilde{x}_1'$ so that x_1 can “catch up” to $\tilde{x}_1$. So, $0 \leq \tilde{x}_1' - x_1' = \kappa(\sin(\tilde{x}_2 - \tilde{x}_1) - \sin(x_2 - x_1)) - M \sin(\tilde{x}_1) + u \sin(x_1) = \kappa(\sin(x_1 - x_2) - \sin(\tilde{x}_1 - \tilde{x}_2)) + \sin(x_1)(u - M)$. Since $\sin(x_1)(u - M) < 0$ and $\kappa < 0$, $\sin(x_1 - x_2) - \sin(\tilde{x}_1 - \tilde{x}_2) < 0$. [Note: This shows $x_1 = \tilde{x}_1$, $x_2 = \tilde{x}_2$ could not occur at the same point in time]. By choice of t_1 , $\tilde{x}_1 - \tilde{x}_2 < \frac{\pi}{2}$, $x_1 - x_2 < \frac{\pi}{2}$, so $\sin(x_1 - x_2) < \sin(\tilde{x}_1 - \tilde{x}_2) \Rightarrow \tilde{x}_1 - \tilde{x}_2 > x_1 - x_2 \Rightarrow \tilde{x}_2 < x_2$. So, x_2 has already “passed” $\tilde{x}_2$. Now, suppose $x_2 = \tilde{x}_2$. By precisely the same argument as above, we can show $\tilde{x}_1 < x_1$. In other words, x_2 cannot catch up to $\tilde{x}_2$ unless x_1 has already passed $\tilde{x}_1$. But, the converse is also true. Hence, $x_1 > \tilde{x}_1$, $x_2 < \tilde{x}_2$ for all $t \in (t_1, \tilde{T}]$. Thus, $\tilde{x}_1 - \tilde{x}_2 < x_1 - x_2 \Rightarrow \tilde{I} > I \Rightarrow \tilde{T} < T$. By definition of optimality, $\tilde{T} \geq T^*$. Therefore, $T > \tilde{T} \geq T^*$. ■

This fully describes the effects of M on the transition time. Suppose our system has upper bound M_1 . Then, our optimal control is $u^* \equiv M_1$ and we have an associated final time T_1 . If we raise the upper bound to M_2 , where $M_2 > M_1$, then our new optimal control becomes $u^* \equiv M_2$, and we have a new final time T_2 . However, we have made no claims of uniqueness of our optimal control, so it is still possible that $T_1 = T_2$. Using the

above result, though, we see the control $u \equiv M_1$ has an interval where $u < M_2$, namely $[0, T_1]$, thus $u \equiv M_1$ is not optimal and $T_1 > T_2$. So, we have the general result that raising the upper bound lowers the transition time (see figure 6). We can show a similar result involving the detunings.

THM: Fix $x_1(0) = y_1(0)$, κ , I_0 and M . Let x_1, x_2 be the phases of a system with detunings $\delta_1 = -\delta_2$ and final time T_x . Let y_1, y_2 be the phases of a system with detunings $\bar{\delta}_1 = -\bar{\delta}_2$ and final time T_y . If $\delta_1 < \bar{\delta}_1$, then $T_x < T_y$.

PROOF: We know the optimal control of both systems is $u^* \equiv M$. Introduce a third phase system z_1, z_2 with same κ, M, I_0 , and $z_1(0) = x_1(0) = y_1(0)$, where

$$z_1'(t) = \delta_1 + \kappa \sin(z_2(t) - z_1(t)) - \bar{u}(t) \sin(z_1(t))$$

$$z_2'(t) = \delta_2 + \kappa \sin(z_1(t) - z_2(t)) - \bar{u}(t) \sin(z_2(t))$$

Partition $[0, T_y]$ into N equal intervals $[t_i, t_{i+1}]$, $0 \leq i \leq N-1$. Define $\bar{u}(t_i)$ such that $y_1'(t_i) = \bar{\delta}_1 + \kappa \sin(y_2(t_i) - y_1(t_i)) - M \sin(y_1(t_i)) = \delta_1 + \kappa \sin(z_2(t_i) - z_1(t_i)) - \bar{u}(t_i) \sin(z_1(t_i)) = z_1'(t_i)$ for each i . If we let $N \rightarrow \infty$, $\bar{u}$ becomes a continuous function and we force $y_1' = z_1'$ and $y_1 = z_1$ for all t .

Case 1: $x_1(0) = y_1(0) = z_1(0) = \frac{\pi}{2}$

Then $y_1 = -y_2$, $y_1' = -y_2'$, $z_1 = -z_2$, and $z_1' = -z_2'$ for all t . Thus, $y_1 - y_2 = z_1 - z_2 \Rightarrow I_y = I_z$ for all $t \Rightarrow T_y = T_z$. Note, $z_1' = y_1' \Rightarrow \delta_1 + \kappa \sin(z_2 - z_1) - \bar{u} \sin(z_1) = \bar{\delta}_1 + \kappa \sin(y_2 - y_1) - M \sin(y_1) \Rightarrow \bar{\delta}_1 - \delta_1 = \sin(y_1)(M - \bar{u})$. As $\bar{\delta}_1 > \delta_1$ and $\sin(y_1) > 0$ for all t , $M - \bar{u} > 0$ for all t . Hence, $\bar{u}$ is not optimal. Observe that the x system and z system differ only in control. So, $T_y = T_z > T_x$.

Case 2: $x_1(0) = y_1(0) = z_1(0) > \frac{\pi}{2}$

As $y_1(0) > \frac{\pi}{2}$, we know $\sin(y_1) > -\sin(y_2)$ for all t . Consider a point in time where $y_2 = z_2$. Then, $z_1' = y_1' \Rightarrow \delta_1 - \bar{u} \sin(z_1) = \bar{\delta}_1 - M \sin(y_1) \Rightarrow (\bar{\delta}_1 - \delta_1) = \sin(y_1)(M - \bar{u})$. So, $z_2' - y_2' = \delta_2 - \bar{u} \sin(z_2) - \bar{\delta}_2 + M \sin(y_2) = (\bar{\delta}_1 - \delta_1) + \sin(y_2)(M - \bar{u}) = \sin(y_1)(M - \bar{u}) + \sin(y_2)(M - \bar{u}) = (\sin(y_1) + \sin(y_2))(M - \bar{u}) > 0$. $y_2(0) = z_2(0)$, so $(z_2 - y_2)'(0) > 0 \Rightarrow z_2 > y_2$ for some interval. Suppose $y_2 = z_2$ for some time $t > 0$. Then, $y_2'(t) \geq z_2'(t)$ in order for y_2 to catch up to z_2 . But, we have just shown that if $y_2(t) = z_2(t) \Rightarrow y_2'(t) > z_2'(t)$. Thus, $z_2 > y_2$ for all

$t \in (0, T_y] \Rightarrow z_1 - z_2 > y_1 - y_2 \Rightarrow I_z > I_y \Rightarrow T_z > T_y$. By definition of optimality, $T_x \leq T_z < T_y$.

Case 3: $x_1(0) = y_1(0) = z_1(0) < \frac{\pi}{2}$

Redefine $\bar{u}$ so that $y'_2(t_i) = z'_2(t_i)$, $0 \leq i \leq N - 1$. Let $N \rightarrow \infty$. Then $y_2 = z_2$ for all t . By symmetry of Case 2, we can show $z_1 < y_1$ for all $t \in (0, T_y]$, so $T_x \leq T_z < T_y$. ■

Conclusions

The importance of these results could be large. We have shown the precise value of the optimal amplitude of the injected field which will drive the laser array from out-of-phase to in-phase in minimum time. We showed the impact of the detunings and the injected field upperbound on the transient time. We have also explicitly found the limits of the phases on the optimal path and proven the monotonic behavior of the intensity function. In conclusion, we make a few remarks.

When minimizing transition time, it seems logical that one goal might be to find an explicit expression of the this time. However, such a formula was not available in this problem. Generally, in optimal control applications, final time can be determined using the constant p found in the necessary conditions. However, because of the nature of the intensity function, not enough information could be generated to find p other than $p \neq 0$. Nevertheless, given the detuning values, coupling constant, upperbound, initial conditions, and I_0 , we can find T using a numerical differential equation package, such as MATLAB.

Second, we have assumed the detunings to be precisely the negatives of each other, i.e., $\delta_1 = -\delta_2$. While it has proven difficult to duplicate our results analytically with the slightly weaker condition of approximate negativity, numeric work has shown that if the δ_1 differs from $-\delta_2$ by only a few orders of magnitude of κ , then the previous results do still hold. In fact, with these “close” detuning levels, the phase models are almost identical. Further, most of the proofs contain some level of “spare room.” For example, in our proof that $u^* \equiv M$, we showed the coefficient of u in the Hamiltonian was negative using the fact that $u(\lambda_1 \sin(x_1) + \lambda_2 \sin(x_2)) = 1 + (\delta_1 + \kappa \sin(x_2 - x_1))(\lambda_1 - \lambda_2) > 0$. If the detunings are only approximately opposite, then we cannot make this equality. However, we can

introduce ϵ as some small error term, specifically, $\delta_1 + \delta_2 = \epsilon$. Then we have,

$$u(\lambda_1 \sin(x_1) + \lambda_2 \sin(x_2)) = 1 + (\delta_1 + \kappa \sin(x_2 - x_1))(\lambda_1 - \lambda_2) - \epsilon(\lambda_1 - \lambda_2),$$

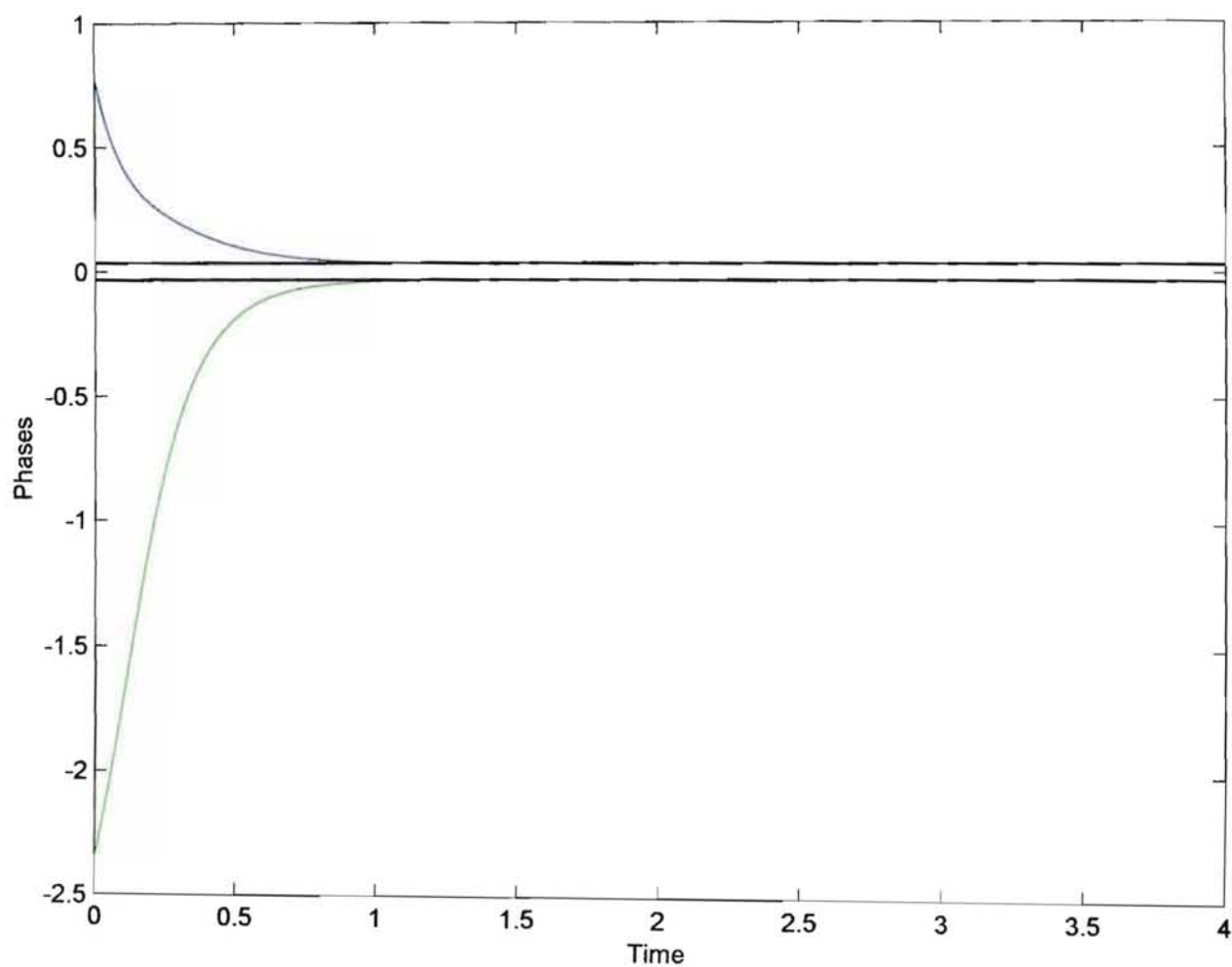
but ϵ is small. So, even if $\epsilon < 0$, we can assume $1 > -\epsilon(\lambda_1 - \lambda_2)$. Thus, both sides of the equality are positive as before, and the proof still holds. In this case, the 1 in the Hamiltonian absorbs the error of the difference in the detunings.

Finally, only certain initial phase positions were considered. Specifically, we required $\delta_1 < 4|\kappa| \sin(x_1(0))$. As previously stated, because the detunings are small, this will include almost all of the possible initial conditions. However, we cannot assume the same results will hold on these alternate starting positions. In fact, we know they will not. Consider starting positions $\delta_1 > M \sin(x_1(0))$. Then, by the first proof, $\lambda_1(0) < \lambda_2(0)$. From this, $\lambda_1(0) \sin(x_1(0)) + \lambda_2(0) \sin(x_2(0)) = \lambda_1(0) \sin(x_1(0)) - \lambda_2(0) \sin(x_1(0)) = \sin(x_1(0))(\lambda_1(0) - \lambda_2(0)) < 0$. As $x_1, x_2, \lambda_1, \lambda_2$ all continuous, this must hold for some interval. Thus, $u(t) = 4|\kappa|$ on some interval including 0. On the other hand, it can be shown there exists an interval where $u(t) = M$. Numeric work suggests the control to be bang-bang, i.e., the control is a piecewise constant function switching between the upper and lower bounds only. In fact, it appears the control is $4|\kappa|$ for a small interval at the beginning, then switches to M for the remainder of the transition time. This may seem to discredit the remaining results, but, in fact, it does not. If this bang-bang control were analytically proven to be the optimal control, then the monotonicity of $I(t)$ would follow directly. Further, numeric work has shown the remaining results about the upper bound and the detunings to also be valid in this case.

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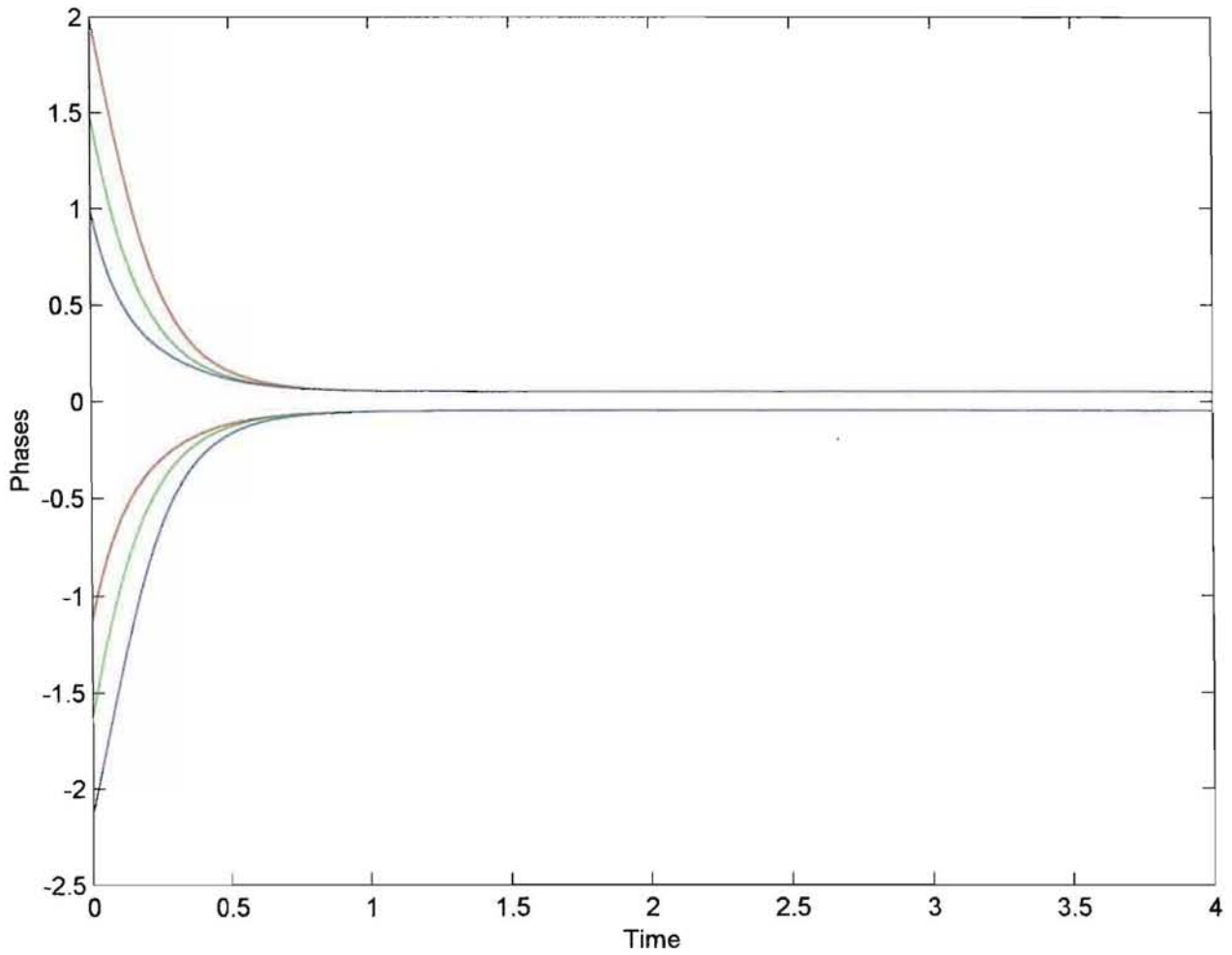
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Figure 1



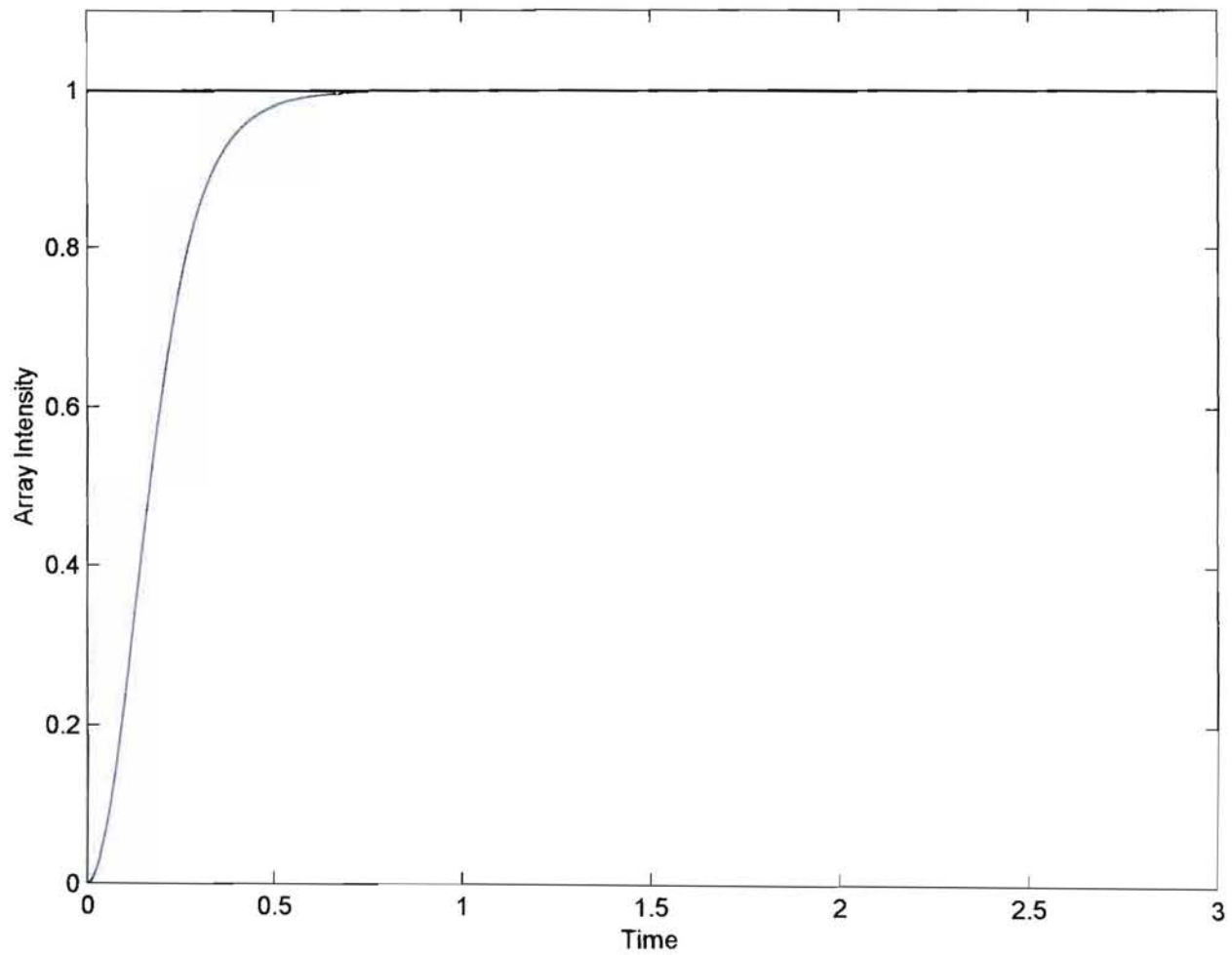
Parameters are $M = 8$, $\delta_1 = 0.2$, $\kappa = -1$, $x_1(0) = \frac{\pi}{4}$. Blue line represents phase path of x_1 , and the green line is x_2 . Black lines are fixed points and asymptotic limits $x_1 = 0.033333$, $x_2 = -0.0333333$.

Figure 2



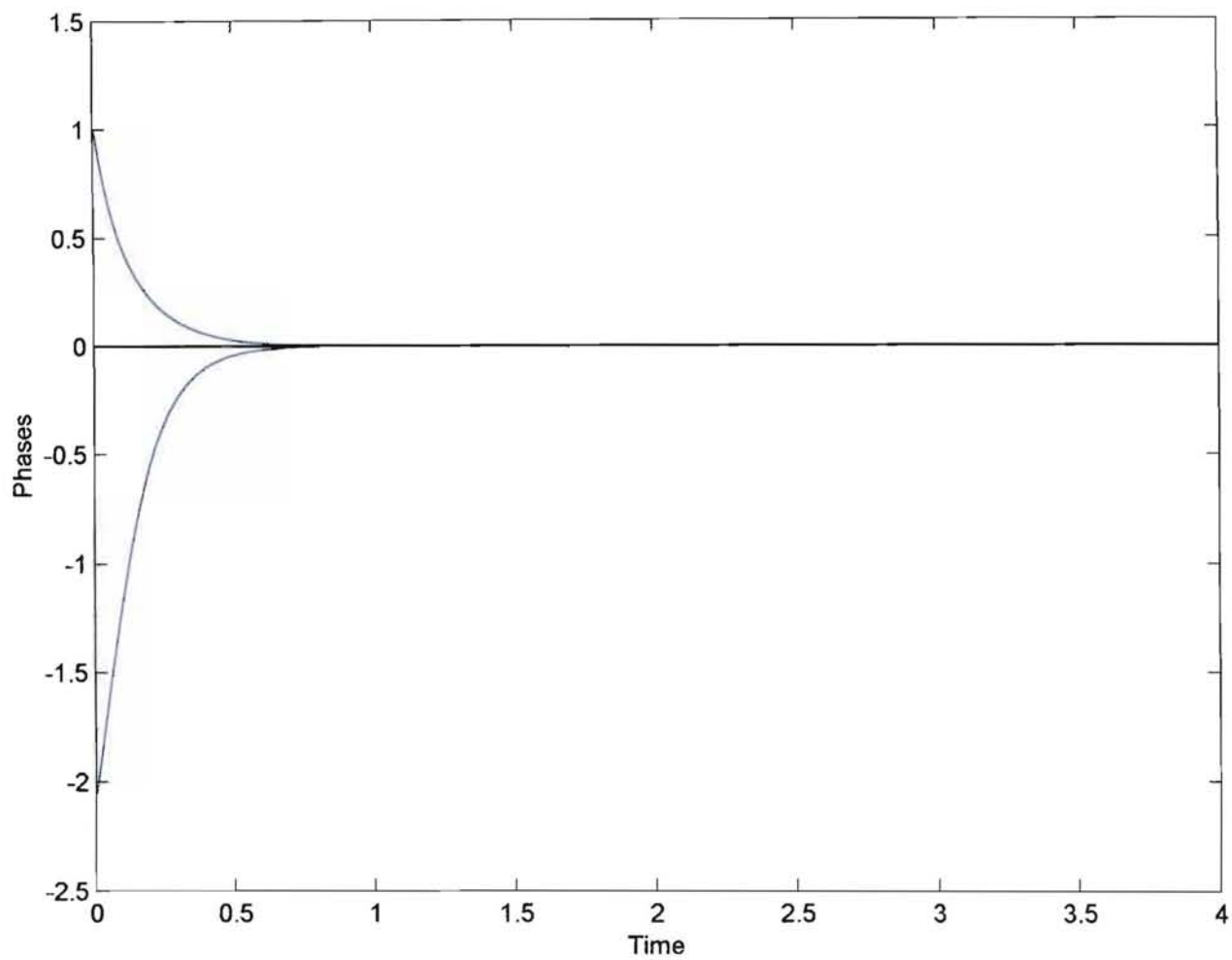
Parameters are $M = 8$, $\delta_1 = 0.3$, $\kappa = -1$. Red lines represent the phases when $x_1(0) = 2$, green lines when $x_1(0) = 1.5$, and blue lines when $x_1(0) = 1$. We can see initial position has no effect on limit points.

Figure 3



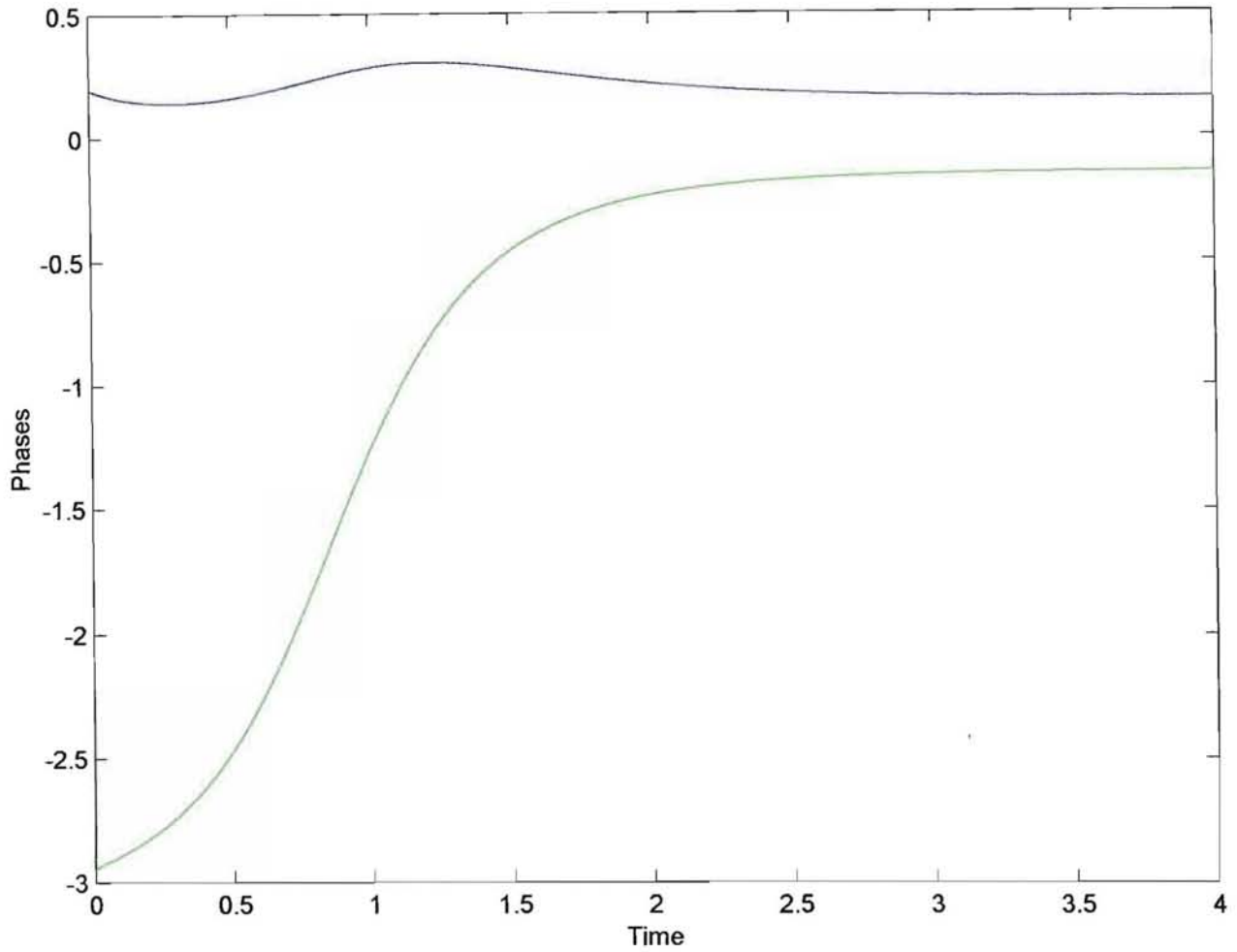
Parameters are $M = 8$, $\delta_1 = 0.2$, $\kappa = -1$, $x_1(0) = \frac{\pi}{4}$. Blue line is the array intensity. Black line is asymptotic limit determined by fixed point $I = 0.9988893$.

Figure 4



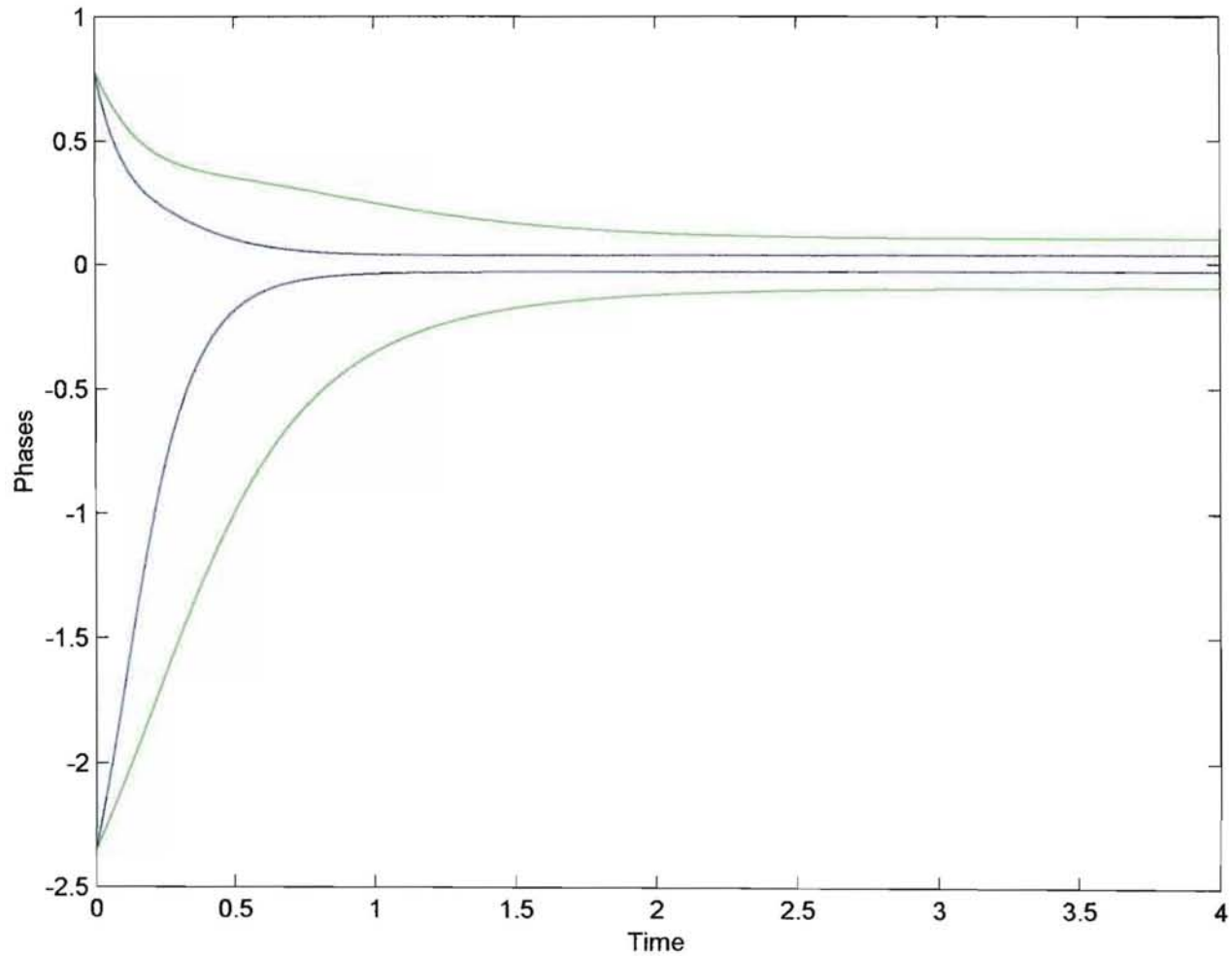
Parameters are $M = 10$, $\delta_1 = 0$, $\kappa = -1$, $x_1(0) = \frac{\pi}{3}$. As detuning is zero, both phases approach 0 and intensity approaches 1.

Figure 5



Parameters are $M = 4$, $\delta_1 = 0.3$, $\kappa = -1$, $x_1(0) = \frac{\pi}{16}$. We can see x_1 is not monotonically decreasing. This can happen only because x_1 begins closer to 0 than x_2 . We can see x_2 is monotonically increasing.

Figure 6



Parameters are $\delta_1 = 0.2$, $\kappa = -1$, $x_1(0) = \frac{\pi}{4}$. Blue lines represent phases of x_1, x_2 where $M = 8$. Green lines are when $M = 4$. Observe the blue lines are closer together for all time. Thus, $I_4 < I_8$ for all time, and so $T_4 > T_8$ regardless of choice of I_0 .

UNIVERSITY HONORS PROGRAM

SENIOR PROJECT - APPROVAL

Name: John Workmen

College: Arts and Sciences Department: Mathematics

Faculty Mentor: Dr. Suzanne Lenhart

PROJECT TITLE: Optimal Control of Transition Time in Solid-State Lasers

I have reviewed this completed senior honors thesis with this student and certify that it is a project commensurate with honors level undergraduate research in this field.

Signed: Suzanne Lenhart, Faculty Mentor

Date: Apr 30, 2003

Comments (Optional):