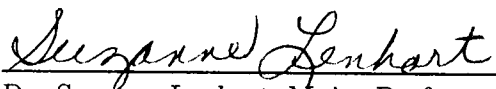
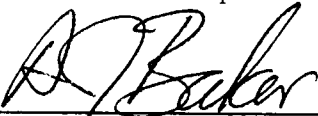


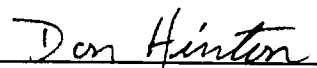
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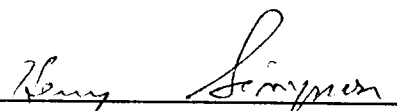
I am submitting herewith a dissertation written by Min Liang entitled "Applications Of Optimal Control Theory To Wave Equations And A Predator-Prey Model ." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.


Dr. Suzanne Lenhart, Major Professor

We have read this dissertation
and recommend its acceptance:


Dr. A.J. Baker


Dr. Don Hinton


Dr. Henry Simpson

Accepted for the Council:


Associate Vice Chancellor and
Dean of the Graduate School

APPLICATIONS OF
OPTIMAL CONTROL THEORY
TO
WAVE EQUATIONS AND
A PREDATOR-PREY MODEL

A Dissertation

Presented for the

Doctor of Philosophy

Degree

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Min Liang

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ABSTRACT

In this dissertation, we investigate optimal control of partial differential equations. We prove the existence of optimal controls for which the objective functional are minimized or maximized. The goal is to characterize the optimal control in terms of the solution of the optimality system. The optimality system consists of the state equations coupled with the adjoint equations. To obtain the optimality system we differentiate the objective functional with respect to the control. This process is applied to studying a predator-prey parabolic system with Robin boundary condition, to control the effects of boundary habitat hostility. In a wave equation problem, a bilinear control is used to bring the state solution close to a desired profile under a quadratic cost control. We also establish the uniqueness of the solution of the optimality system and thus determine the unique optimal control. In the problem of a wave equation with viscous damping, we control the damping term. We seek to minimize the cost functional which measures the closeness of state solutions to a desired profile.

TABLE OF CONTENTS

I	Introduction	1
II	Optimal Control of the Effects of Boundary Habitat Hostility in a Predator-Prey Model	15
1	Introduction to Predator-Prey Problem	16
2	Existence of an Optimal Control	21
3	Derivation of the Optimality System	38
III	Bilinear Optimal Control for a Wave Equation	51
4	Introduction to Wave Equation Problem	52
5	Existence of an Optimal Control	55
6	Characterization of the Optimal Control	68
7	Two Special Cases	80
IV	Bilinear Optimal Control for a Wave Equation with Viscous Damp- ing	103
8	Introduction to Wave Equation with Viscous Damping	104
9	Existence of an Optimal Control	107
10	Characterization of the Optimal Control	126

Bibliography	136
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Vita	142
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Part I

Introduction

The Pontryagin Maximum Principle is a milestone of modern (finite dimensional) optimal control theory for ordinary differential equations [14]. The Pontryagin [29] Maximum Principle generalized the classical calculus of variations to ordinary differential equations. However there is not a complete generalization of the Pontryagin's Maximum Principle to partial differential equations. In 1960's, J.L.Lions started the development of optimal control of partial differential equations [25]. The ideas of studying optimal control of partial differential equations are similar to the ideas of studying ordinary differential equations, but the differentiations are "Gateaux" derivatives in the infinite dimensional spaces. Li and Yong's book [23] gives references and results of the Pontryagin Maximum Principle for partial differential equations.

Control theory as a whole is divided into two main sub-categories: optimal control and control. Controllability and stabilization are the most important areas of control theory [36] [26]. In optimal control theory, the desired performance of the system should be specified. Such a criterion is referred to as a cost functional or objective functional. The goal of an optimal control problem is to characterize the optimal control such that maximizes or minimizes the objective functional over the control space.

Much theoretical work on optimal controls involves linear partial differential equations (linear in state and in control)

$$y' = Ay + Bu,$$

with an objective functional that is quadratic in the control. These "linear-quadratic"

problems have been treated with Riccati equations on Banach Spaces; see the book by Lasiecka and Triggiani [18] for background on this approach. The problems treated in this dissertation do not fall into the Riccati framework, since the state equations are bilinear in the control.

In optimal control problems, variables are divided into two classes, state variables and control variables. The studying of an optimal control problem requires the following initial setting [25]:

1. A control variable f belonging to some set $\mathcal{U}_{ad}$ (the set of 'admissible controls').
2. For a given control f , the state variable $u(f)$ of the system, which is to be controlled, is given by the solution of an equation

$$Au = F(u, f),$$

where A is a known operator. The operator A could be a system of partial differential operators, ordinary differential operators, integral operators, or a "hybrid" system consisting of both PDE's and ODE's. F is a continuous (usually nonlinear) function of u and f . An appropriate solution space for the state equation is chosen.

3. The "objective functional" $J(f)$, the quantity to be maximized or minimized, is defined in terms of $u(f)$, f , or a combination of both terms.

The objective is to characterize optimal controls $f^* \in \mathcal{U}_{ad}$ such that

$$J(f^*) = \sup_{f \in \mathcal{U}_{ad}} J(f),$$

$$\text{or } J(f^*) = \inf_{f \in \mathcal{U}_{ad}} J(f),$$

where the supremum or infimum is taken over the set $\mathcal{U}_{ad}$.

The usual steps in these control problems are :

1. The existence and uniqueness of the solution to the state system are proven; an iteration scheme or semi-group method are frequently used.

2. The existence of an optimal control is proven. A maximizing or minimizing sequence argument with an appropriate priori estimates is used.

3. The necessary conditions for f to be an optimal control are derived. The objective functional is differentiated with respect to the control. We take the Gateaux derivative of J with respect to f in the direction h at f^* , i.e.

$$\lim_{\epsilon \rightarrow 0} \frac{J(f^* + \epsilon h) - J(f^*)}{\epsilon},$$

where h denotes the direction. Since the objective functional may contain the state variable u , we also define the Gateaux derivative of u with respect to f in the direction h , i.e.

$$\lim_{\epsilon \rightarrow 0} \frac{u(f + \epsilon h) - u(f)}{\epsilon},$$

which should be shown to converge to a function in the solution space, ψ . The "sensi-

tivity" function ψ satisfies a PDE linear in ψ ,

$$\mathcal{L}\psi = g(h, f, u).$$

To characterize the optimal control f^* from $\frac{\partial J}{\partial f}$, the adjoint system is introduced with a transversality condition and other appropriate boundary conditions. The adjoint equation involves the adjoint $\mathcal{L}^*$ of the operator $\mathcal{L}$. An explicit representation of an optimal control is determined by standard optimality techniques. The state equation system coupled with the adjoint equation system and the characterization of f^* form the optimality system. This representation characterizes the optimal control in terms of the solutions of optimality system, i.e. the solutions of state equation system and adjoint equation system.

4. The existence and uniqueness of the solution of optimality system is established, which implies the uniqueness of the optimal control.

As a formal example, consider the following linear wave equation problem where f represents a control:

$$\begin{aligned} u_{tt} - \Delta u &= f \text{ in } Q = \Omega \times (0, T) \\ u(x, 0) &= u_0(x), \quad x \in \Omega \\ u_t(x, 0) &= u_1(x), \quad x \in \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial\Omega \times (0, T). \end{aligned} \tag{1}$$

For $f \in U_M = \{f \in L^\infty(Q) \mid -M \leq f \leq M\}$, the set of admissible controls, the

cost functional is

$$J(f) = \frac{1}{2} \left(\int_Q (u - z)^2 dx dt + \int_Q f^2 dx dt \right), \quad (2)$$

where z is the target function. The goal is to find f^* that minimizes the cost functional.

The existence of an optimal control through a minimizing sequence argument can be established.

To derive the optimality system, first analyze $\frac{u(f+\epsilon h)-u(f)}{\epsilon}$ when $\epsilon \rightarrow 0^+$. Let $u^\epsilon = u(f + \epsilon h)$. Then system (1) becomes

$$\begin{aligned} \left(\frac{u^\epsilon - u}{\epsilon} \right)_{tt} - \Delta \left(\frac{u^\epsilon - u}{\epsilon} \right) &= h \text{ in } Q \\ \left(\frac{u^\epsilon - u}{\epsilon} \right)(x, 0) &= 0, \quad x \in \Omega \\ \left(\frac{u^\epsilon - u}{\epsilon} \right)_t(x, 0) &= 0, \quad x \in \Omega \\ \frac{\partial(u^\epsilon - u)}{\partial \nu} &= 0 \text{ on } \partial\Omega \times (0, T). \end{aligned}$$

By a priori estimate, one shows that $\frac{u^\epsilon - u}{\epsilon} \rightharpoonup \psi$ in $L^2([0, T]; H^1(\Omega))$, then ψ satisfies the problem

$$\begin{aligned} \psi_{tt} - \Delta \psi &= h \text{ in } Q \\ \psi(x, 0) &= 0, \quad x \in \Omega \\ \psi_t(x, 0) &= 0, \quad x \in \Omega \\ \frac{\partial \psi}{\partial \nu} &= 0 \text{ on } \partial\Omega \times (0, T). \end{aligned} \quad (3)$$

Next, the adjoint of the operator $\psi_{tt} - \Delta\psi$ is itself. The source of the adjoint partial differential equation equals the derivative of the cost functional integrand with respect to the state. Hence, the adjoint problem is

$$\begin{aligned}
p_{tt} - \Delta p &= u - z \text{ in } Q \\
p(x, T) &= 0, \quad x \in \Omega \\
p_t(x, T) &= 0, \quad x \in \Omega \\
\frac{\partial p}{\partial \nu} &= 0 \text{ on } \partial\Omega \times (0, T),
\end{aligned} \tag{4}$$

where the transversality conditions on the adjoint variable are at $t = T$. Now to determine the representation for f^* , suppose f^* is an optimal control with corresponding solution u^* and consider another control $f^* + \epsilon h$ with associated solution u^ϵ . Since the minimum of the cost functional occurs at f^* , then

$$\begin{aligned}
0 &\leq \lim_{\epsilon \rightarrow 0^+} \frac{J(f^* + \epsilon h) - J(f^*)}{\epsilon} \\
&= \lim_{\epsilon \rightarrow 0^+} \left[\frac{1}{2\epsilon} \int_Q [(u^\epsilon - z)^2 - (u^* - z)^2] dx dt \right] \\
&\quad + \lim_{\epsilon \rightarrow 0^+} \left[\frac{1}{2\epsilon} \int_Q (f^* + \epsilon h)^2 - (f^*)^2 dx dt \right] \\
&= \lim_{\epsilon \rightarrow 0^+} \int_Q \left(\frac{u^\epsilon - u^*}{\epsilon} \right) \left(\frac{u^\epsilon - z + u^* - z}{2} \right) dx dt \\
&\quad + \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\epsilon} \int_Q (2f^* \epsilon h + \epsilon^2 h^2) dx dt \\
&= \int_Q \psi(u^* - z) dx dt + \int_Q f^* h dx dt \\
&= \int_Q [p(\psi_{tt} - \Delta\psi) + f^* h] dx dt
\end{aligned}$$

$$= \int_Q h(f^* + p) dx dt$$

To determine f^* , standard techniques are used. Consider the following three cases.

Case 1: On the set $\{(x, t) \in Q | f^*(x, t) = -M\}$, one can choose nonnegative variations h with the support on the set. Hence, $-M = f^* \geq -p$.

Case 2: On the set $\{(x, t) \in Q | -M < f^*(x, t) < M\}$, one can choose h with arbitrary sign and support on this set. This implies that $-M < f^* = -p < M$.

Case 3: On the set $\{(x, t) \in Q | f^*(x, t) = M\}$, one can choose nonpositive variations h with support on this set. Then $M = f^* \leq -p$.

Combining these three cases, we obtain $f^* = \max(-M, \min(-p, M))$.

Using the relationship between the optimal control and the associated adjoint variable, the optimality system is as follows (formally):

$$\begin{aligned} u_{tt} - \Delta u &= \max(-M, \min(-p, M)) \text{ in } Q \\ p_{tt} - \Delta p &= u - z \text{ in } Q \\ \frac{\partial u}{\partial \nu} &= \frac{\partial p}{\partial \nu} = 0 \text{ in } \partial\Omega \times (0, T) \\ u &= u_t = 0, \quad t = 0, \quad x \in \Omega \\ p &= p_t = 0, \quad t = T, \quad x \in \Omega \end{aligned} \tag{5}$$

One would then show uniqueness of the optimality system to obtain uniqueness of the optimal control.

Now I summarize the various results contained in this dissertation. All the problems

are posed in a multidimensional domain Q with $Q = \Omega \times (0, T)$.

In part II, we consider optimal control of the effects of boundary habitat hostility in a predator-prey parabolic partial differential equations (PDEs) with Robin boundary conditions. Given control β , measuring boundary habitat hostility, the corresponding prey and predator state variables, $u = u(\beta)$ and $v = v(\beta)$ satisfy the state system:

$$\begin{aligned}
u_t - \alpha_1 \Delta u &= (a_1 - du)u - c_1 uv \quad \text{in } Q = \Omega \times (0, T) \\
v_t - \alpha_2 \Delta v &= -a_2 v + c_2 uv \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x) \quad \text{for } x \in \Omega \\
\alpha_1 \frac{\partial u}{\partial \nu} + \beta u &= 0, \quad \alpha_2 \frac{\partial v}{\partial \nu} + \beta v = 0 \quad \text{on } \partial\Omega \times (0, T)
\end{aligned} \tag{6}$$

This system has been used to model the degradation of the matrix habitat caused by some factors such as increased hunting pressure, pesticide contamination, or pollution [17] [10]. Solutions of the system represent population densities of the prey and the predator species. The system has Lotka-Volterra type growth terms with local interaction terms representing the predator-prey situation. The coefficients a_1, a_2, d are the standard Lotka-Volterra (logistic) growth terms. The interaction terms, $-c_1 uv$ and $c_2 uv$, signify decay for the prey population growth and for the predator population respectively.

The control is the convective coefficient of the Robin boundary condition and indicates the hostility of the external environment to the species in question. The extremal cases $\beta = 0$ and $\beta = \infty$ correspond to a completely closed patch (i.e. one with a re-

flecting boundary) and to an immediately lethal exterior, respectively. The $\beta = 0$ case is the most favorable to the species.

The objective is to maximize J with respect to control β where

$$J(\beta) = \int_{\Omega \times (0,T)} (A_1 u + A_2 v) dx dt - \frac{1}{2} \int_{\partial \Omega \times (0,T)} (M - \beta)^2 ds dt$$

with positive constants A_1, A_2 . We establish the existence of an optimal control and then characterize β^* such that

$$J(\beta^*) = \max_{\beta \in U_M} J(\beta),$$

where U_M is our control set. This cost functional combines the control on the boundary and its effect on the populations. It reflects a balancing effects of growth of the species and the cost of making a favorable boundary environment. An unfavorable boundary environment may be profitable to some neighboring industries. This functional gives a balance between preserving the species and cost of making "favorable to the species" boundary. That cost could include a loss of profit to neighboring industries.

The solution to this optimal control problem involves establishing the existence of solutions to the state system (6). This result involves a method of monotone iterations which involves ccc control is shown via a maximizing sequence argument and a priori estimates. The optimality system, which consists of the state system coupled with an adjoint system, is produced by taking the directional derivative of the objective func-

tional J with respect to the controls. The optimality system has opposite orientations due to the forward and backward in time nature of the state and adjoint equations.

The next two parts of this dissertation involve optimal control of wave equation. Russell [32, 33, 34, 35] developed the theory of control of wave equations, including many controllability and stability results. We will treat optimal control problems which are bilinear in the control. Lenhart together with Bradley, Protopopescu, and Yong [19, 20] have studied bilinear control for the Kirchhoff plate equation and acoustic wave equation. Control theory, virtually by definition, is concerned with the presence of a variable control in the state equation. The difficulty in studying bilinear optimal control is that it is hard to get a priori regularity.

In part III, we study an optimal control of a wave equation with Dirichlet boundary condition. We consider a bilinear optimal control problem in which the control acts as a dispersive term. Given the control h in our control set U_M , the corresponding state variable $w = w(h)$ satisfies the state equation

$$\begin{aligned}
w_{tt} &= \Delta w + hw + f & \text{in } Q &= \Omega \times (0, T) \\
w &= 0 & \text{in } \partial\Omega \times (0, T) \\
w &= w_0(x) & t = 0, \quad x \in \Omega \\
w_t &= w_1(x) & t = 0, \quad x \in \Omega
\end{aligned} \tag{7}$$

where $f \in L^2(Q)$, $w_0 \in H_0^1(\Omega)$, $w_1 \in L^2(\Omega)$.

We take as our objective functional:

$$J(h) = \frac{1}{2} \left(\int_Q (w - z)^2 dQ + \beta \int_Q h^2 dQ \right),$$

where z is a given target function. This objective functional combines the cost of control and the measure of the "closeness" to the desired profile z . The objective functional is defined as

$$J(h^*) = \min_{h \in U_M} J(h).$$

The goal of this work is to obtain an unique optimal control in terms of the solution to the optimality system, which will consist of the original wave equation problem coupled with an adjoint problem.

The solution to this optimal control problem also involves establishing the existence of the solution to the above state problem (7). But this existence result involves a semi-

group method, since the maximum principle is not available in wave equations. The existence of an optimal control is shown via a minimizing sequence argument. Then the optimality system is derived by differentiating the objective functional with respect to the control. Finally, for sufficiently small time T , under an assumption of boundedness of state solutions, we prove uniqueness of the optimality system, which characterizes the unique optimal control. We also treat two special cases: $h = h(t)$ and $h = h(x)$. In these two special cases, we use two different methods to obtain the uniqueness of the optimality system without boundedness assumption of solutions. The uniqueness proofs are unusual due to the opposite time orientation of the two equations.

Finally, in part IV, we study optimal control of a wave equation with viscous damping and with Dirichlet boundary condition. We consider a bilinear optimal control problem in which the control acts as a damping term. Given the control k in control set, the corresponding state variable $y = y(k)$ satisfies the state equation

$$\begin{aligned}
y_{tt} &= \Delta y - k(t)y_t - q(x)y + f & \text{in } Q &= \Omega \times (0, T) \\
y &= 0 & \text{in } \partial\Omega \times (0, T) \\
y &= y_0(x) & t = 0, \quad x \in \Omega \\
y_t &= y_1(x) & t = 0, \quad x \in \Omega
\end{aligned} \tag{8}$$

We take as our objective functional:

$$J(k) = \frac{1}{2} \left(\int_Q (y - z)^2 dQ + \beta \int_0^T k^2(t) dt \right).$$

The objective is to minimize the functional over the control set U_M , i.e. to characterize k^* such that

$$J(k^*) = \min_{k \in U_M} J(k).$$

The solution technique to this optimal control problem is similar to part III where the control variable is the coefficient of term hu in the state equation. In this part, the control variable is the coefficient of term ky_t in the state equation.

Part II

Optimal Control of the Effects of Boundary Habitat Hostility in a Predator-Prey Model

Chapter 1

Introduction to Predator-Prey

Problem

We consider optimal control of a parabolic system of partial differential equations (PDEs) with Robin boundary conditions. The parabolic system is posed in a bounded, multi-dimensional domain Q . Solutions of the system represent population densities of the prey and the predator species. The system has Lotka-Volterra type growth terms with local interaction terms representing the predator-prey situation. The control is the convective coefficient of the Robin boundary condition.

Given controls β , the corresponding prey and predator state variables, $u = u(\beta)$ and $v = v(\beta)$ satisfy the state system:

$$\begin{aligned}
u_t - \alpha_1 \Delta u &= (a_1 - du)u - c_1 uv \quad \text{in } Q = \Omega \times (0, T) \\
v_t - \alpha_2 \Delta v &= -a_2 v + c_2 uv \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x) \quad \text{for } x \in \Omega \\
\alpha_1 \frac{\partial u}{\partial \nu} + \beta u &= 0, \quad \alpha_2 \frac{\partial v}{\partial \nu} + \beta v = 0 \quad \text{on } \partial\Omega \times (0, T)
\end{aligned} \tag{1.1}$$

where Ω is a bounded open connected set in R^n . When $n = 2, 3$, Ω represents a habitat for the species whose population densities at location $x \in \Omega$ and time $t \geq 0$ are given by $u = u(x, t)$ and $v = v(x, t)$. Because of the dispersive nature of the predator and prey species, the population dynamics of these species depends on space and time. This dependence is evident by the use of parabolic diffusive Lotka-Volterra PDEs. The functions $u_0(x)$ and $v_0(x)$ give the initial density distributions of the prey and predator species inside our spatial area. The coefficients α_1 and α_2 represent the diffusion rates for the predator and prey within Ω respectively. The coefficients a_1, a_2, d are the standard Lotka-Volterra (logistic) growth terms, with a_1 denoting growth for prey population, $-a_2$ denoting decay for predator population and d signifying crowding. The c_1, c_2 coefficients represent interaction effects. The $-c_1 uv$ term is a decay term for the prey population; whereas, the $c_2 uv$ term is a growth term for the predator population. α_1 and α_2 are the diffusion rates corresponding to the prey population and the predator population respectively within Ω . The convective coefficient β is an indicator of the hostility of the external environment to the species

in question. The extremal cases $\beta = 0$ and $\beta = \infty$ correspond to a completely closed patch (i.e. one with a reflecting boundary) and to an immediately lethal exterior, respectively. Thus the $\beta = 0$ case is the most favorable to the species. The $\beta = 0$ case corresponds to Neumann boundary condition, whereas the $\beta = \infty$ case corresponds to Dirichlet boundary condition.

We define the class of admissible controls,

$$U_M \equiv \left\{ \beta(x, t) \mid \beta \in L^2(\partial\Omega \times (0, T)); 0 \leq \beta \leq M \right\}.$$

Our payoff functional is

$$J(\beta) = \int_{\Omega \times (0, T)} (A_1 u + A_2 v) dx dt - \frac{1}{2} \int_{\partial\Omega \times (0, T)} (M - \beta)^2 ds dt$$

with positive constants A_1, A_2 . This functional combines the control on the boundary and its effect on the populations. It reflects a balancing effects of growth of the species and the cost of making a favorable boundary environment. An unfavorable boundary environment may be profitable to some neighboring industries. This functional gives a balance between preserving the species and cost of making "favorable to the species" boundary. That cost could include a loss of profit to neighboring industries. To make the most favorable boundary environment, we would take $\beta = 0$ and incur the largest cost $(M - 0)^2$. We seek to maximize the functional over the admissible class of control

space, i.e. to characterize β such that

$$J(\beta^*) = \max_{\beta \in U_M} J(\beta).$$

The idea for this problem came from discussion with Steve Cantrell and Chris Corner and from their work with Fagan [8]. Cantrell et al. [8] studied a coupled pair of Lotka-Volterra competition equations with Robin boundary conditions, which models the dynamics of two competing species inhabiting the pristine patch and incorporated matrix degradation through a tunable "hostility" parameter representing species mortality rates in the matrix. Their analysis of steady-state equations shows how a competitor that is inferior in the pristine core habitat when the matrix habitat is of high quality can become dominant over the other species as degradation in the surrounding matrix region proceeds. As an example, species diversity of birds inhabiting remnant woodlots in Ontario, Canada, declines as the extent of housing development outside forest fragments increases [15]. Similarly, Angelstam [1] suggests that the severity of predation impacts on Swedish forest birds by generalist predators (e.g., corvids, foxes) residing in matrix habitat increases as human activities make the habitat patch and its surrounding more and more dissimilar. On a large scale, differential sensitivity to habitat degradation may account for recent increases in black bear populations at the expense of grizzly bears in Yellowstone park [10]. In many ecological systems, progressive deterioration of the surrounding matrix habitat may be accompanied by a predictable pattern of species extinctions in remnant patches [6].

Here we investigate optimally balancing the effects of habitat degradation on species preservation and the cost of a favorable boundary environment. Specifically, we have a patch of favorable habitat that is surrounded by matrix habitat. Populations of predator and prey reside and reproduce in the favorable habitat but can disperse (unknowingly) into the surrounding matrix.

Fister [13] dealt with a similar model with growth predator population and Neumann boundary conditions, and the controls there are a proportion of the species populations to be harvested. For background in control of PDE's, see Lions [25]. In addition, Waltman [38] and Murray [27] provide insight into competitive and predator prey models. Leung [21, 22] considers optimal control of the harvesting of two interacting populations. Cañada, Gámez and Montero [9] analyze the profitability of harvesting species modeled by a diffusive Lotka-Volterra elliptic system.

Chapter 2 gives the existence of solutions of the system (1.1) and the existence of an optimal control. In Chapter 3, an optimal control is characterized in terms of the optimality system, which is the state system coupled with the adjoint system. We derive the optimality system by differentiating the payoff functional with respect to the control and evaluate the result at an optimal control. Uniqueness of solutions to the optimality system is established. Hence, we obtain an explicit characterization of the unique optimal control.

Chapter 2

Existence of an Optimal Control

The following assumptions are made throughout this part:

Ω is a smooth bounded domain in $\mathcal{R}^n$,

$u_0(x), v_0(x) \in L^\infty(\Omega)$,

$0 < u_0(x) < B, 0 < v_0(x) < B$ for some $B \in \mathcal{R}^+$,

$a_i, c_i, d \in L^\infty(Q)$, and a_i, c_i, d are nonnegative functions, $i = 1, 2$.

$\alpha_i > 0, \quad i = 1, 2, \beta > 0$, and are constants .

The normal direction ν is outward.

The underlying state space for system (1.1) is

$$V = L^2((0, T); H^1(\Omega)).$$

Definition: We say a pair of functions

$$u, v \in L^2(0, T; H^1(\Omega)), \text{ with } u_t, v_t \in L^2(0, T; (H^1(\Omega))^*),$$

is a weak solutions of (1.1) provided

(1)

$$\begin{aligned} & \int_0^T (u_t, \phi) dt + \alpha_1 \int_{\Omega \times (0, T)} \nabla u \nabla \phi dx dt \\ &= - \int_{\Omega \times (0, T)} (c_1 uv \phi + du^2 \phi - a_1 u \phi) dx dt - \int_{\partial \Omega \times (0, T)} \beta u \phi ds \end{aligned} \quad (2.1)$$

$$\begin{aligned} & \int_0^T (v_t, \psi) dt + \alpha_2 \int_{\Omega \times (0, T)} \nabla v \nabla \psi dx dt \\ &= \int_{\Omega \times (0, T)} (c_2 uv \psi - a_2 v \psi) dx dt - \int_{\partial \Omega \times (0, T)} \beta v \psi ds \end{aligned}$$

for all $\phi, \psi \in V$ and a.e. $0 \leq t \leq T$, where $(\cdot, \cdot)$ inner product is the duality between $(H^1(\Omega))^*$ and $H^1(\Omega)$, and

$$(2) \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x).$$

Remark (1) By trace theorem [11], $u \in L^2(0, T; H^1(\Omega))$ implies $u \in L^2(\partial \Omega \times (0, T))$.

Thus the integral over the boundary in (2.1) makes sense.

Remark (2) Since here u, v are continuous in time, i.e. $u, v \in C([0, T]; L^2(\Omega))$ from [11], thus the equalities in (2) make sense.

We will prove an existence and uniqueness result for the state system (1.1) in Theorem 2.1. First, we give a maximum principle.

Lemma 2.1 Let $a \in L^\infty(Q)$, $a \geq 0$, $u_0(x) \geq 0$, $\alpha > 0$, $\beta > 0$. If u satisfies $u \in C^{2,1}(Q) \cap C(\bar{Q})$ and

$$\begin{aligned} u_t - \alpha \Delta u &= au & \text{in } Q \\ \alpha \frac{\partial u}{\partial \nu} + \beta u &= 0, & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) &= u_0(x) & \text{for } x \in \Omega \end{aligned}$$

then

$$0 \leq u \leq (\sup_{\Omega} u_0(x)) e^{T\|a\|_{L^\infty(Q)}} \text{ in } \bar{Q}.$$

Proof: Let $\lambda = \sup_Q a$. By assumption, $\lambda \geq 0$. Let $v = ue^{-\lambda t}$, then v satisfies

$$\begin{aligned} v_t - \alpha \Delta v + (\lambda - a)v &= 0 & \text{in } Q \\ \alpha \frac{\partial v}{\partial \nu} + \beta v &= 0, & \text{on } \partial\Omega \times (0, T) \\ v(x, 0) &= u_0(x) & \text{for } x \in \Omega \end{aligned}$$

By the parabolic maximum principle [30], we have

$$-\sup_{\Omega \times \{t=0\}} v^- \leq v \leq \sup_{\Omega \times \{t=0\}} v^+.$$

where $v^+ = \max(u_0, 0) = u_0$. $v^- = -\min(u_0, 0) = 0$. Thus

$$0 \leq v \leq \max_{\Omega} u_0.$$

i.e. $0 \leq u \leq e^{\lambda T} \max_{\Omega} u_0 = (\max_{\Omega} u_0(x)) e^{T\|a\|_{L^\infty(Q)}}.$

□

Theorem 2.1 *Given $\beta \in U_M$, there exists a unique solution (u, v) in $V \times V$ solving system (1.1).*

Proof: To obtain the existence, we will construct two sequences by means of iteration.

To obtain L^∞ bounds, we need supersolutions and subsolutions for the u, v iterates. Let

$\bar{U}$ and $\bar{V}$ be the classical solutions of the following linear PDE problems respectively.

$$\begin{aligned} u_t - \alpha_1 \Delta u &= a_1 u & \text{in } Q \\ \alpha_1 \frac{\partial u}{\partial \nu} + \beta u &= 0, & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) &= u_0(x) & \text{for } x \in \Omega \end{aligned} \tag{2.2}$$

$$\begin{aligned} v_t - \alpha_2 \Delta v &= c_2 \bar{U} v & \text{in } Q \\ \alpha_2 \frac{\partial v}{\partial \nu} + \beta v &= 0, & \text{on } \partial\Omega \times (0, T) \\ v(x, 0) &= v_0(x) & \text{for } x \in \Omega \end{aligned} \tag{2.3}$$

By Lemma 2.1, $\bar{U}$ and $\bar{V}$ are L^∞ bounded in Q . In fact,

$$0 \leq \bar{U} \leq \|u_0\|_{L^\infty(Q)} e^{\gamma T} \tag{2.4}$$

$$0 \leq \bar{V} \leq \|v_0\|_{L^\infty(Q)} e^{\eta T} \tag{2.5}$$

where $\gamma = \|a_1\|_{L^\infty(\Omega)}$ and $\eta = \|c_2\|_{L^\infty(Q)} \|u_0\|_{L^\infty(\Omega)} e^{\|a_1\|_{L^\infty(Q)} T}$. Also define $u^1 = \bar{U}$,

$v^2 = \bar{V}$, $u^0 = 0$ and $v^1 = 0$.

For $i=2,3,\dots$, we define u^i and v^{i+1} as the solutions of the following problem (2.6) and (2.7) respectively:

$$\begin{aligned} u_t^i - \alpha_1 \Delta u^i + R u^i &= f(u^{i-2}, v^i) \quad \text{in } Q \\ \alpha_1 \frac{\partial u^i}{\partial \nu} + \beta u^i &= 0, \quad \text{on } \partial\Omega \times (0, T) \\ u^i(x, 0) &= u_0(x) \quad \text{for } x \in \Omega \end{aligned} \tag{2.6}$$

$$\begin{aligned} v_t^i - \alpha_2 \Delta v^i + R v^i &= g(u^{i-1}, v^{i-2}) \quad \text{in } Q \\ \alpha_2 \frac{\partial v^i}{\partial \nu} + \beta v^i &= 0, \quad \text{on } \partial\Omega \times (0, T) \\ v^i(x, 0) &= v_0(x) \quad \text{for } x \in \Omega \end{aligned} \tag{2.7}$$

where $f(u^{i-2}, v^i) = R u^{i-2} + u^{i-2}(a_1 - d u^{i-2}) - c_1 u^{i-2} v^i$

$g(u^{i-1}, v^{i-2}) = R v^{i-2} - a_2 v^{i-2} + c_2 u^{i-1} v^{i-2}$.

R is chosen to be a constant satisfying

$$R > \sup_Q \{a_2 + 2d\bar{U} + c_1\bar{V}\}.$$

Both the problems (2.6) and (2.7) are linear PDE problems, so solutions $u^i, v^i, i = 0, 1, 2, \dots$, exist.

Claim 1 For $i = 1, 2, \dots$, $0 \leq u^i \leq \bar{U}$, $0 \leq v^i \leq \bar{V}$ in Q .

Proof of Claim 1: Using induction method, note that the statement is true when $i=1$.

Assume that

$$0 \leq u^j \leq \bar{U}, \quad 1 \leq j \leq i-1$$

$$0 \leq v^j \leq \bar{V}, \quad 1 \leq j \leq i$$

for some $i > 1$. Then for u^i ,

$$\begin{aligned} u_t^i - \Delta u^i + Ru^i &= f(u^{i-2}, v^i) \\ &\geq u^{i-2} \{R - du^{i-2} - c_1 v^i\} \\ &\geq u^{i-2} \{R - d\bar{U} - c_1 \bar{V}\} \\ &\geq 0. \end{aligned}$$

Then we conclude by parabolic maximum principle,

$$u^i \geq 0.$$

Compare

$$u_t^i - \Delta u^i + Ru^i = f(u^{i-2}, v^i)$$

and

$$u_t^1 - \Delta u^1 + Ru^1 = Ru^1 + a_1 u^1,$$

we have

$$\begin{aligned}
& (u^1 - u_t^i) - \Delta(u^1 - u^i) + R(u^1 - u^i) \\
&= R(u^1 - u^{i-2}) + a_1(u^1 - u^{i-2}) + d(u^{i-2})^2 + c_1 u^{i-2} v^i.
\end{aligned}$$

Since $u^{i-2} \leq \bar{U}$, $v^i \leq \bar{V}$ and R, a_1, d, c_1 are all positive numbers, the right hand side of the above equality is positive. By maximum principle, $0 \leq u^i \leq \bar{U}$. It is similar to prove the result about v^i . By induction, the claim holds. $\square$

Claim 2 For $i=1,2,\dots$,

$f(u^{i-2}, v^i)$ is increasing in u^{i-2} , $f(u^{i-2}, v^i)$ is decreasing in v^i .

$g(u^{i-1}, v^{i-2})$ is increasing in u^{i-1} , $g(u^{i-1}, v^{i-2})$ is increasing in v^{i-2} .

Proof of Claim 2: By Lemma 2.2 and the choice of R , we have

$$\begin{aligned}
\frac{\partial f}{\partial u^{i-2}} &= (R + a_1 - 2du^{i-2} - c_1 v^i) \\
&\geq (R + a_1 - 2d\bar{U} - c_1 \bar{V}) \\
&\geq 0.
\end{aligned}$$

It is similar to prove the other inequalities. $\square$

Claim 3 (Alternating Monotone) There exist $\bar{u}, \bar{v}, u, v$ in V , such that the following monotone pointwise convergences hold

$$\begin{aligned}
u^{2i} &\nearrow \bar{u}, & u^{2i+1} &\searrow u, \quad \text{in } Q \\
v^{2i} &\searrow \bar{v}, & v^{2i+1} &\nearrow v \quad \text{in } Q,
\end{aligned}$$

pointwise for $i = 1, 2, \dots$

Proof of Claim 3: We use the induction method. From (2.7) and $v^1 = 0$, we have

$$(v^3 - v^1)_t - \alpha_2 \Delta(v^3 - v^1) + R(v^3 - v^1) = c_2 u^2 v^1 \geq 0,$$

so $v^1 \leq v^3$. Similarly, we have $u^3 \leq u^1$, $v^2 \geq v^4$, $u^0 \leq u^2$. Also $u^2 \leq u^3$, $v^4 \geq v^3$.

Fix i , assume the statement holds for all j , $j \leq i$, such that

$$\begin{array}{cc} u^{2j} \nearrow & u^{2j+1} \searrow, \\ v^{2j} \searrow & v^{2j+1} \nearrow. \end{array}$$

For, $i + 1$, we compare v^{2i+2} and v^{2i} . From (2.7), we have

$$\begin{aligned} & (v^{2i+2} - v^{2i})_t - \alpha_2 \Delta(v^{2i+2} - v^{2i}) + R(v^{2i+2} - v^{2i}) \\ &= g(u^{2i+1}, v^{2i}) - g(u^{2i-1}, v^{2i-2}) \\ &\leq 0 \end{aligned}$$

by Claim 2. Then $v^{2i+2} \leq v^{2i}$. We can prove the other cases similarly. By the boundedness of u^i , v^i from Claim 1 and the monotone properties, we get the pointwise convergence. $\square$

Claim 4 The subsequences of $\{u^i\}, \{v^i\}$ satisfy

$$u^{2j} \rightharpoonup \bar{u}, \quad u^{2j+1} \rightharpoonup u, \quad v^{2j} \rightharpoonup \bar{v}, \quad v^{2j+1} \rightharpoonup v \quad \text{weakly in } V.$$

Proof of Claim 4: Consider the right hand sides of (2.6) and (2.7)

$$\begin{aligned} \|f(u^{i-2}, v^i)\|_{L^\infty(Q)} &= \|Ru^{i-2} + a_1 u^{i-2} - d(u^{i-2})^2 - c_1 u^{i-2} v^i\|_{L^\infty} \\ &\leq \|\bar{U}\|_{L^\infty(Q)} (R + \|a_1\|_{L^\infty(Q)} + \|d\|_{L^\infty(Q)} \|\bar{U}\|_{L^\infty(Q)}) \end{aligned}$$

$$\begin{aligned}
& + \|c_1\|_{L^\infty(Q)} \|\bar{V}\|_{L^\infty(Q)} \\
& \equiv C_1 \\
\|g(u^{i-1}, v^{i-2})\|_{L^\infty(Q)} &= \|Rv^{i-2} - a_2v^{i-2} + c_2u^{i-1}v^{i-2}\|_{L^\infty} \\
&\leq \|\bar{V}\|_{L^\infty(Q)}(R + \|a_2\|_{L^\infty(Q)} + \|c_2\|_{L^\infty(Q)}\|\bar{U}\|_{L^\infty(Q)}) \\
&\equiv C_2
\end{aligned}$$

The right hand sides of (2.6) and (2.7) are bounded in $L^\infty(Q)$. Using a priori estimate in u, u_{x_i} for parabolic equation (2.6), we have

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left(\int_{\Omega} (u^k)^2(x, t) dx \right) + \int_0^T \int_{\Omega} |\nabla u^k|^2 dx dt + \int_{\partial\Omega \times (0, T)} \beta(u^k)^2 ds dt \\
& \leq C_3 \int_{\Omega} u_0^2 dx + C_4 \int_0^T \int_{\Omega} [f(u^{k-2}, v^k)]^2 dx dt \\
& \leq C_3 \int_{\Omega} u_0^2 dx + C_4 C_1^2 T \text{meas}(\Omega) \\
& \equiv C_5
\end{aligned} \tag{2.8}$$

Thus u^{2i}, u^{2i+1} are uniformly bounded in V . It is similar to prove that v^{2i}, v^{2i+1} are uniformly bounded in V . Hence, using weak compactness, we get

$$u^{2j} \rightharpoonup \bar{u}, \quad u^{2j+1} \rightharpoonup u, \quad v^{2j} \rightharpoonup \bar{v}, \quad v^{2j+1} \rightharpoonup v \quad \text{weakly in } V. \quad \square$$

Claim 5 The subsequences of $\{u^i\}, \{v^i\}$ satisfy

$$u^{2j} \rightarrow \bar{u}, \quad u^{2j+1} \rightarrow u, \quad v^{2j} \rightarrow \bar{v}, \quad v^{2j+1} \rightarrow v \quad \text{strongly in } L^2(Q).$$

and $u^{2j} \rightarrow \bar{u}, \quad u^{2j+1} \rightarrow u, \quad v^{2j} \rightarrow \bar{v}, \quad v^{2j+1} \rightarrow v \quad \text{strongly in } L^2(\partial\Omega \times (0, T)).$

Proof of Claim 5: Using the V boundedness on $u^{2i}, u^{2i+1}, v^{2i}, v^{2i+1}$ and the PDE in

(2.6) and (2.7), we get that $\|u_t^{2i}\|, \|u_t^{2i+1}\|, \|v_t^{2i}\|$ and $\|v_t^{2i+1}\|$ are bounded in $L^2((0, T), (H^1(\Omega))^*)$. Hence, using the weak compactness again, we have

$$u_t^{2j} \rightharpoonup \bar{u}_t, u_t^{2j+1} \rightharpoonup u_t, v_t^{2j} \rightharpoonup \bar{v}_t, v_t^{2j+1} \rightharpoonup v_t \text{ weakly in } L^2((0, T), (H^1(\Omega))^*).$$

Since $L^2(0, T; H_0^1(\Omega))$ compactly imbeds into $L^2(0, T; H^{\frac{1}{2}+\epsilon}(\Omega))$, $0 \leq \epsilon \leq \frac{1}{2}$, and using the compactness result by Simon [37], we have

$$u^{2j} \rightarrow \bar{u}, u^{2j+1} \rightarrow u, v^{2j} \rightarrow \bar{v}, v^{2j+1} \rightarrow v \quad \text{strongly in } L^2(0, T; H^{\frac{1}{2}+\epsilon}(\Omega)).$$

Using the continuous mapping from $H^{\frac{1}{2}+\epsilon}(\Omega)$ into $L^2(\partial\Omega)$, we have

$$u^{2j} \rightarrow \bar{u}, u^{2j+1} \rightarrow u, v^{2j} \rightarrow \bar{v}, v^{2j+1} \rightarrow v \quad \text{strongly in } L^2(0, T; L^2(\partial\Omega)).$$

□

Now we finish the proof of the main theorem. Passing the limit in the u^{2i}, u^{2i+1}, v^{2i} and v^{2i+1} PDEs, we obtain

$$\begin{aligned} \bar{u}_t - \alpha_1 \Delta \bar{u} &= a_1 \bar{u} - d(\bar{u})^2 - c_1 \bar{u} \bar{v} \quad \text{in } Q \\ \alpha_1 \frac{\partial \bar{u}}{\partial \nu} + \beta \bar{u} &= 0, \quad \text{on } \partial\Omega \times (0, T) \\ \bar{u}(x, 0) &= u_0(x) \quad \text{for } x \in \Omega \end{aligned} \tag{2.9}$$

$$\begin{aligned} u_t - \alpha_1 \Delta u &= a_1 u - du^2 - c_1 uv \quad \text{in } Q \\ \alpha_1 \frac{\partial u}{\partial \nu} + \beta u &= 0, \quad \text{on } \partial\Omega \times (0, T) \\ u(x, 0) &= u_0(x) \quad \text{for } x \in \Omega \end{aligned} \tag{2.10}$$

$$\begin{aligned}
\bar{v}_t - \alpha_2 \Delta \bar{v} &= -a_2 \bar{v} + c_2 \bar{u} \bar{v} \quad \text{in } Q \\
\alpha_2 \frac{\partial \bar{v}}{\partial \nu} + \beta \bar{v} &= 0, \quad \text{on } \partial\Omega \times (0, T) \\
\bar{v}(x, 0) &= v_0(x) \quad \text{for } x \in \Omega
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
v_t - \alpha_2 \Delta v &= -a_2 v + c_2 uv \quad \text{in } Q \\
\alpha_2 \frac{\partial v}{\partial \nu} + \beta v &= 0, \quad \text{on } \partial\Omega \times (0, T) \\
v(x, 0) &= v_0(x) \quad \text{for } x \in \Omega
\end{aligned} \tag{2.12}$$

To show $\bar{u} = u, \bar{v} = v$, let $u = e^{\lambda t} w, v = e^{\lambda t} z, \bar{u} = e^{\lambda t} \bar{w}, \bar{v} = e^{\lambda t} \bar{z}$, where $\lambda \geq 0$ is to be chosen. We illustrate the transformed PDE for $u = e^{\lambda t} w$:

$$\begin{aligned}
w_t - \alpha_1 \Delta w + \lambda w &= a_1 w - d e^{\lambda t} w^2 - c_1 e^{\lambda t} w z \quad \text{in } Q \\
w(x, 0) &= u_0(x) \quad \text{for } x \in \Omega \\
\alpha_1 \frac{\partial w}{\partial \nu} + \beta w &= 0, \quad \text{for } x \in \partial\Omega \times (0, T)
\end{aligned} \tag{2.13}$$

By the weak formulations utilizing equations for w, z , we have

$$\begin{aligned}
&\int_{\Omega \times (0, T)} \{ (w - \bar{w})_t (w - \bar{w}) + (\alpha_1 |\nabla(w - \bar{w})|)^2 + \lambda (w - \bar{w})^2 \\
&\quad + (z - \bar{z})_t (z - \bar{z}) + (\alpha_2 |\nabla(z - \bar{z})|)^2 + \lambda (z - \bar{z})^2 \} dx dt \\
&\quad + \int_{\partial\Omega \times (0, T)} \beta (w - \bar{w})^2 + \beta (z - \bar{z})^2 ds dt
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega \times (0,T)} \{a_1(w - \bar{w})^2 - de^{\lambda t}(w^2 - \bar{w}^2)(w - \bar{w}) \\
&\quad - c_1 e^{\lambda t}(wz - \bar{w}\bar{z})(w - \bar{w})\} dx dt \\
&\quad + \int_{\Omega \times (0,T)} \{-a_2(z - \bar{z})^2 - c_2 e^{\lambda t}(wz - \bar{w}\bar{z})(z - \bar{z})\} dx dt
\end{aligned}$$

Starting to simplify, we have

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega} \{[w(x, T) - \bar{w}(x, T)]^2 + [z(x, T) - \bar{z}(x, T)]^2\} dx \\
&\quad + \int_{\Omega \times (0,T)} [\alpha_1 |\nabla(w - \bar{w})|^2 + \alpha_2 |\nabla(z - \bar{z})|^2] dx dt \\
&\quad + \int_{\Omega \times (0,T)} [\lambda(w - \bar{w})^2 + \lambda(z - \bar{z})^2] dx dt \\
&\quad + \int_{\partial\Omega \times (0,T)} \beta(w - \bar{w})^2 + \beta(z - \bar{z})^2 ds dt \\
&= \int_{\Omega \times (0,T)} e^{\lambda t}(w - \bar{w})^2 [-d(w + \bar{w}) - c_1 z] dx dt \\
&\quad + \int_{\Omega \times (0,T)} e^{\lambda t}(z - \bar{z})^2 c_2 \bar{w} dx dt \\
&\quad + \int_{\Omega \times (0,T)} e^{\lambda t}(z - \bar{z})(w - \bar{w})(-c_1 \bar{w} + c_2 z) dx dt \\
&\quad + \int_{\Omega \times (0,T)} [a_1(w - \bar{w})^2 - a_2(z - \bar{z})^2] dx dt.
\end{aligned}$$

Since $u, \bar{u}, v, \bar{v}$ and d, c_1, c_2, a_1, a_2 are all L^∞ bounded in Q , using Cauchy inequality yields

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega} [w(x, T) - \bar{w}(x, T)]^2 + [z(x, T) - \bar{z}(x, T)]^2 dx \\
&\quad + \int_{\Omega \times (0,T)} [\alpha_1 |\nabla(w - \bar{w})|^2 + \alpha_2 |\nabla(z - \bar{z})|^2] dx dt \\
&\quad + (\lambda - C) \int_{\Omega \times (0,T)} [(w - \bar{w})^2 + (z - \bar{z})^2] dx dt
\end{aligned}$$

$$\begin{aligned}
& + \int_{\partial\Omega \times (0,T)} \beta(w - \bar{w})^2 + \beta(z - \bar{z})^2 ds dt \\
& \leq 0
\end{aligned} \tag{2.14}$$

where C depends only on the coefficients, T and $\|u^1\|_{L^\infty}$, $\|v^2\|_{L^\infty}$. If $\lambda > C$, then the inequality (2.14) holds if and only if

$$w = \bar{w}, \text{ and } z = \bar{z} \text{ a.e. in } Q.$$

Therefore $u = \bar{u}$ and $v = \bar{v}$ in Q , and u, v solve (1.1). The uniqueness of u, v follows as in the above argument. $\square$

By the above construction the solutions are nonnegative and bounded above, i.e.,

$$0 \leq u \leq \bar{U}, \quad 0 \leq v \leq \bar{V}. \tag{2.15}$$

from claim 1 and claim 3, where $\bar{U}$ and $\bar{V}$ are bounded functions from (2.4) and (2.5).

Next a maximizing sequence argument is used to prove the existence of an optimal control.

Theorem 2.2 *There exists an optimal control in U_M that maximizes the functional $J(\beta)$.*

Proof: Since the state variables and the controls are bounded,

$\sup\{J(\beta) \mid \beta \in U_M\} < \infty$. Then there exists a maximizing sequence $\{\beta_n\} \subset U_M$ such

that

$$\lim_{n \rightarrow \infty} J(\beta_n) = \sup\{J(\beta) | \beta \in U_M\}.$$

By the existence and uniqueness of solutions to the state system (1.1), we define

$$u^n = u(\beta_n), \quad v^n = v(\beta_n) \quad \text{for each } n.$$

We consider the state system (1.1) on $\Omega \times (0, t)$, $0 \leq t \leq T$,

$$\begin{aligned} & \int_{\Omega \times (0, t)} u_t^n u^n dx d\tau + \alpha_1 \int_{\Omega \times (0, t)} u_{x_i}^n u_{x_i}^n dx d\tau - \int_{\Omega \times (0, t)} a_1 (u^n)^2 dx d\tau \\ & + \int_{\partial \Omega \times (0, t)} \beta_n (u^n)^2 ds d\tau + \int_{\Omega \times (0, t)} v_t^n v^n dx d\tau + \alpha_2 \int_{\Omega \times (0, t)} v_{x_i}^n v_{x_i}^n dx d\tau \\ & + \int_{\Omega \times (0, t)} a_2 (v^n)^2 dx d\tau + \int_{\partial \Omega \times (0, t)} \beta_n (v^n)^2 ds d\tau \\ = & - \int_{\Omega \times (0, t)} c_1 (u^n)^2 v^n dx d\tau - \int_{\Omega \times (0, t)} d (u^n)^3 dx d\tau \\ & + \int_{\Omega \times (0, t)} c_2 u^n (v^n)^2 dx d\tau \end{aligned} \tag{2.16}$$

Using Cauchy's inequality and that the state variables are uniformly bounded, we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega \times \{t\}} [(u^n)^2 + (v^n)^2] dx + \int_{\Omega \times (0, t)} [\alpha_1 |\nabla u^n|^2 + \alpha_2 |\nabla v^n|^2] dx d\tau \\ \leq & \frac{1}{2} \int_{\Omega} [(u_0)^2 + (v_0)^2] dx + C \int_{\Omega \times (0, t)} [(u^n)^2 + (v^n)^2] dx d\tau. \end{aligned} \tag{2.17}$$

Since

$$0 \leq u^n \leq \bar{U}, \quad 0 \leq v^n \leq \bar{V}, \tag{2.18}$$

and $\bar{U}, \bar{V}$ are bounded,

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left\{ \int_{\Omega} [u^n(x, t)]^2 dx + \int_{\Omega} [v^n(x, t)]^2 dx \right\} + \int_0^T \int_{\Omega} [|\nabla u^n|^2 + |\nabla v^n|^2] dx dt \\ & \leq C_1 \left(\int_{\Omega} [(u_0)^2 + (v_0)^2] + 1 \right) \end{aligned}$$

where C_1 depends only on α_1 and α_2 . Since $\{\beta_n\}$ are uniformly bounded, using weak compactness on the space $L^2(\partial\Omega \times (0, T))$, there exists a subsequence, $\beta^n \rightharpoonup \beta^*$ weakly in $L^2(Q)$. Since U_M is closed in the weak topology, $\beta^* \in U_M$. As in the proof of the existence of solutions to the state system (1.1), we obtain the existence of u^*, v^* such that on a subsequence

$$u^n \rightarrow u^*, \quad v^n \rightarrow v^* \text{ strongly in } L^2(Q) \text{ and weakly in } V.$$

$$u^n \rightarrow u^*, \quad v^n \rightarrow v^* \text{ strongly in } L^2(0, T; L^2(\partial\Omega))$$

Therefore

$$\beta^n u^n \rightharpoonup \beta^* u^*, \quad \beta^n v^n \rightharpoonup \beta^* v^* \text{ weakly in } L^2(0, T; L^2(\partial\Omega))$$

Passing to the limit in the u^n, v^n PDEs and using the convergences like in Theorem 2.1, we have that u^*, v^* is a weak solution associated with β^* . We have, by definition of

functional $J(\beta)$,

$$J(\beta^*) \leq \sup\{J(\beta) \mid \beta \in U_M\}.$$

Since

$$J(\beta^n) = \int_{\Omega \times (0,T)} (A_1 u^n + A_2 v^n) dx dt - \frac{1}{2} \int_{\partial\Omega \times (0,T)} (M - \beta^n)^2 ds dt,$$

using the upper semi-continuity of L^2 norm with respect to weak convergence [12] , we have

$$\int_{\partial\Omega \times (0,T)} (M - \beta^*)^2 \leq \lim_{n \rightarrow \infty} \int_{\partial\Omega \times (0,T)} (M - \beta^n)^2.$$

So,

$$\begin{aligned} & J(\beta^*) \\ & \geq \lim_{n \rightarrow \infty} \int_{\Omega \times (0,T)} (A_1 u^n + A_2 v^n) dx dt - \frac{1}{2} \lim_{n \rightarrow \infty} \int_{\partial\Omega \times (0,T)} (M - \beta^n)^2 ds dt \\ & = \overline{\lim}_{n \rightarrow \infty} \int_{\Omega \times (0,T)} (A_1 u^n + A_2 v^n) dx dt + \overline{\lim}_{n \rightarrow \infty} \left[-\frac{1}{2} \int_{\partial\Omega \times (0,T)} (M - \beta^n)^2 ds dt \right] \\ & \geq \overline{\lim}_{n \rightarrow \infty} \left[\int_{\Omega \times (0,T)} (A_1 u^n + A_2 v^n) dx dt - \frac{1}{2} \int_{\partial\Omega \times (0,T)} (M - \beta^n)^2 ds dt \right] \\ & \geq \lim_{n \rightarrow \infty} \sup J(\beta^n) \\ & = \sup\{J(\beta) \mid \beta \in U_M\}. \end{aligned}$$

Therefore β^* is an optimal control that maximizes the payoff functional. $\square$

Chapter 3

Derivation of the Optimality System

We now derive the optimality system which consists of the state system coupled with the adjoint system. We differentiate the payoff functional with respect to the controls to obtain necessary conditions for the optimality system. Since the payoff functional contains u and v , we must differentiate u and v with respect to the controls.

Theorem 3.1 *The mapping $\beta \in U_M \rightarrow (u, v) = (u(\beta), v(\beta)) \in V \times V$ is differentiable in the following sense:*

$$\begin{aligned} \frac{u(\beta + \epsilon h) - u(\beta)}{\epsilon} &\rightharpoonup \psi \quad \text{weakly in } V \\ \frac{v(\beta + \epsilon h) - v(\beta)}{\epsilon} &\rightharpoonup \phi \quad \text{weakly in } V \end{aligned}$$

as $\epsilon \rightarrow 0$ for any $\beta \in U_M$ and $h \in L^\infty(Q)$ such that $\beta + \epsilon h \in U_M$ for ϵ small. Also

$\psi = \psi(\beta, h), \phi = \phi(\beta, h)$ satisfy

$$\begin{aligned}
(\psi)_t - \alpha_1 \Delta \psi &= a_1 \psi - 2d\psi u - c_1 u \phi - c_1 v \psi \quad \text{in } U_M \\
(\phi)_t - \alpha_2 \Delta \phi &= -a_2 \phi + c_2 u \phi + c_2 v \psi \\
\psi(x, 0) &= 0 = \phi(x, 0) \quad \text{for } x \in \Omega \\
\alpha_1 \frac{\partial \psi}{\partial \nu} &= -\beta \psi - hu \quad \text{on } \partial\Omega \times (0, T) \\
\alpha_2 \frac{\partial \phi}{\partial \nu} &= -\beta \phi - hv
\end{aligned} \tag{3.1}$$

Proof: Define $u^\epsilon = u(\beta + \epsilon h), v^\epsilon = v(\beta + \epsilon h)$. Let $u^\epsilon = e^{\lambda t} w^\epsilon, u = e^{\lambda t} w, v^\epsilon = e^{\lambda t} z^\epsilon, v = e^{\lambda t} z$, where $\lambda > 0$ is to be chosen below. Estimating from the PDE's satisfied by $\frac{w^\epsilon - w}{\epsilon}, \frac{z^\epsilon - z}{\epsilon}$, we obtain

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \left[\left(\frac{w^\epsilon(x, t) - w(x, t)}{\epsilon} \right)^2 + \left(\frac{z^\epsilon(x, t) - z(x, t)}{\epsilon} \right)^2 \right] dx \\
& + \int_{\Omega \times (0, t)} \left[\alpha_1 |\nabla \left(\frac{w^\epsilon - w}{\epsilon} \right)|^2 + \alpha_2 |\nabla \left(\frac{z^\epsilon - z}{\epsilon} \right)|^2 \right] dx dt \\
& + \lambda \int_{\Omega \times (0, t)} \left[\left(\frac{w^\epsilon - w}{\epsilon} \right)^2 + \left(\frac{z^\epsilon - z}{\epsilon} \right)^2 \right] dx dt \\
& = \int_{\Omega \times (0, t)} \left(\frac{w^\epsilon - w}{\epsilon} \right)^2 [a_1 - d(u^\epsilon + u) - c_1 v] dx dt \\
& + \int_{\Omega \times (0, t)} \left(\frac{z^\epsilon - z}{\epsilon} \right)^2 [-a_2 + c_2 u] dx dt \\
& - \int_{\Omega \times (0, t)} (c_1 u + c_2 v^\epsilon) \left(\frac{w^\epsilon - w}{\epsilon} \right) \left(\frac{z^\epsilon - z}{\epsilon} \right) dx dt \\
& - \int_{\partial\Omega \times (0, t)} \left[\beta \left(\frac{w^\epsilon - w}{\epsilon} \right)^2 + h w^\epsilon \left(\frac{w^\epsilon - w}{\epsilon} \right) \right] dx ds
\end{aligned}$$

$$\begin{aligned}
& - \int_{\partial\Omega \times (0,t)} \left[\beta \left(\frac{z^\epsilon - z}{\epsilon} \right)^2 + h z^\epsilon \left(\frac{z^\epsilon - z}{\epsilon} \right) \right] dx ds \\
\leq & C \int_Q \left[\left(\frac{w^\epsilon - w}{\epsilon} \right)^2 + \left(\frac{z^\epsilon - z}{\epsilon} \right)^2 \right] dx dt - \int_{\partial\Omega \times (0,T)} h w^\epsilon \left(\frac{w^\epsilon - w}{\epsilon} \right) dx ds \\
& - \int_{\partial\Omega \times (0,T)} h z^\epsilon \left(\frac{z^\epsilon - z}{\epsilon} \right) dx ds \\
\leq & C \int_Q \left[\left(\frac{w^\epsilon - w}{\epsilon} \right)^2 + \left(\frac{z^\epsilon - z}{\epsilon} \right)^2 \right] dx dt + 2 \int_{\partial\Omega \times (0,T)} \left[(h w^\epsilon)^2 + \left(\frac{w^\epsilon - w}{\epsilon} \right)^2 \right] dx ds \\
& + 2 \int_{\partial\Omega \times (0,T)} \left[(h z^\epsilon)^2 + \left(\frac{z^\epsilon - z}{\epsilon} \right)^2 \right] dx ds \\
\leq & C_1 \int_Q \left[\left(\frac{w^\epsilon - w}{\epsilon} \right)^2 + \left(\frac{z^\epsilon - z}{\epsilon} \right)^2 \right] dx dt + 2 \int_{\partial\Omega \times (0,T)} (h w^\epsilon)^2 dx ds \\
& + 2 \int_{\partial\Omega \times (0,T)} (h z^\epsilon)^2 dx ds + \int_Q \left[\frac{\alpha_1}{2} \left| \nabla \left(\frac{w^\epsilon - w}{\epsilon} \right) \right|^2 + \frac{\alpha_2}{2} \left| \nabla \left(\frac{z^\epsilon - z}{\epsilon} \right) \right|^2 \right] dx dt
\end{aligned}$$

where C_1 depends on the coefficients and the trace theorem [11].

Continuing to estimate, using L^∞ bounds on the coefficients and u, v , we have

$$\begin{aligned}
& \frac{1}{2} \int_\Omega \left[\left(\frac{w^\epsilon(x,t) - w(x,t)}{\epsilon} \right)^2 + \left(\frac{z^\epsilon(x,t) - z(x,t)}{\epsilon} \right)^2 \right] dx dt \\
& + \int_Q \left[\frac{\alpha_1}{2} \left| \nabla \left(\frac{w^\epsilon - w}{\epsilon} \right) \right|^2 + \frac{\alpha_2}{2} \left| \nabla \left(\frac{z^\epsilon - z}{\epsilon} \right) \right|^2 \right] dx dt \\
& + (\lambda - C_1) \int_Q \left[\left(\frac{w^\epsilon - w}{\epsilon} \right)^2 + \left(\frac{z^\epsilon - z}{\epsilon} \right)^2 \right] dx dt \\
\leq & 2 \int_{\partial\Omega \times (0,T)} (h w^\epsilon)^2 ds d\tau + 2 \int_{\partial\Omega \times (0,T)} (h z^\epsilon)^2 ds d\tau \\
\leq & C_2 \int_{\partial\Omega \times (0,T)} h^2 ds d\tau
\end{aligned}$$

For $\lambda > C_1$, we conclude

$$\left\| \frac{z^\epsilon - z}{\epsilon} \right\|_V^2 + \left\| \frac{w^\epsilon - w}{\epsilon} \right\|_V^2 \leq C_2 \int_{\partial\Omega \times (0,T)} h^2.$$

This estimate justifies the convergence of w, z quotients, and hence there exists ψ, ϕ in V , such that

$$\frac{u^\epsilon - u}{\epsilon} \rightharpoonup \psi \quad , \quad \frac{v^\epsilon - v}{\epsilon} \rightharpoonup \phi \quad \text{weakly in } V \quad (3.2)$$

Similarly to Theorem 2.1,

$$\begin{aligned} \left(\frac{u^\epsilon - u}{\epsilon} \right)_t &\rightharpoonup \psi_t \quad \text{and} \quad \left(\frac{v^\epsilon - v}{\epsilon} \right)_t \rightharpoonup \phi_t \\ &\text{weakly in } L^2((0, T), (H^1(\Omega))^*). \end{aligned}$$

So $\frac{u^\epsilon - u}{\epsilon} \rightarrow \psi$ and $(\frac{v^\epsilon - v}{\epsilon}) \rightarrow \phi$ strongly in $L^2(Q)$. Also we have $\frac{u^\epsilon - u}{\epsilon} \rightarrow \psi$ and $(\frac{v^\epsilon - v}{\epsilon}) \rightarrow \phi$ strongly in $L^2(\partial\Omega \times (0, T))$. Passing the limit in the PDEs which are satisfied by $\frac{u^\epsilon - u}{\epsilon}$ and $\frac{v^\epsilon - v}{\epsilon}$ justifies that ψ, ϕ solve system (3.1). $\square$

To derive the optimality system and to characterize the pair of optimal controls, we need to define adjoint variables and the adjoint operator associated with ψ_1, ψ_2 . We rewrite the ψ, ϕ PDEs as

$$\mathcal{L} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\mathcal{L} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} L_1 \psi \\ L_2 \phi \end{pmatrix} + B \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

and

$$\begin{pmatrix} L_1 \psi \\ L_2 \phi \end{pmatrix} = \begin{pmatrix} \psi_t - \alpha_1 \Delta \psi \\ \phi_t - \alpha_2 \Delta \phi \end{pmatrix},$$

$$B = \begin{pmatrix} -a_1 + 2d_1 u + c_1 v & c_1 u \\ -c_2 v & a_2 - c_2 u \end{pmatrix}$$

Note the non-homogeneous terms in the ψ, ϕ system occur in the boundary terms.

We define the adjoint PDEs as

$$\mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

where

$$\mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} L_1^* p \\ L_2^* q \end{pmatrix} + B^T \begin{pmatrix} p \\ q \end{pmatrix}, \quad \begin{pmatrix} L_1^* p \\ L_2^* q \end{pmatrix} = \begin{pmatrix} -p_t - \alpha_1 \Delta p \\ -q_t - \alpha_2 \Delta q \end{pmatrix} \quad (3.3)$$

and B^T is the transpose of B . We illustrate in the appropriate weak sense,

$$\begin{aligned}
& \int_Q (\psi, \phi) \mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} dxdt \\
&= \int_Q (p, q) \mathcal{L} \begin{pmatrix} \psi \\ \phi \end{pmatrix} dxdt \\
&\quad - \int_Q [(\psi p)(x, T) - (\psi p)(x, 0) + (\phi q)(x, T) - (\phi q)(x, 0)] dxdt \\
&\quad + \int_{\partial\Omega \times (0, T)} [-\alpha_1 \psi \frac{\partial p}{\partial \nu} + \alpha_1 p \frac{\partial \psi}{\partial \nu} - \alpha_2 \phi \frac{\partial q}{\partial \nu} + \alpha_2 q \frac{\partial \phi}{\partial \nu}] dsdt.
\end{aligned} \tag{3.4}$$

For the adjoint system, we must attach the appropriate boundary and terminal conditions.

Theorem 3.2 *Given an optimal control β and corresponding solutions u, v , there exists a weak solution $(p, q) \in V \times V$ satisfying the adjoint system*

$$\begin{aligned}
L_1^* p &= A_1 + a_1 p - 2d_1 p u - c_1 v p + c_2 v q & \text{in } Q \\
L_2^* q &= A_2 - a_2 q - c_1 u p + c_2 u q & \text{in } Q \\
\alpha_1 \frac{\partial p}{\partial \nu} + \beta p &= 0 & \text{on } \partial\Omega \times (0, T) \\
\alpha_2 \frac{\partial q}{\partial \nu} + \beta q &= 0 & \text{on } \partial\Omega \times (0, T) \\
p(x, T) = 0 \text{ and } q(x, T) &= 0 & \text{when } t = T, x \in \Omega
\end{aligned} \tag{3.5}$$

(transversality conditions)

Furthermore

$$\beta = \min ((M - pu - qv)^+, M). \quad (3.6)$$

Proof: Let β be an optimal control and (u, v) be its corresponding solution. Let $\beta + \epsilon h \in U_M$ for $\epsilon > 0$ and u^ϵ, v^ϵ be the corresponding weak solution of the state equation (1.1). Since the adjoint equations are linear, there exist weak solutions p, q satisfying (3.5). We compute the directional derivative of the cost functional $J(\beta)$ with respect to β in the direction of h . Since $J(\beta)$ is the maximum value,

$$\begin{aligned} 0 &\geq \lim_{\epsilon \rightarrow 0^+} \frac{J(\beta + \epsilon h) - J(\beta)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0^+} \int_Q \left[A_1 \left(\frac{u^\epsilon - u}{\epsilon} \right) + A_2 \left(\frac{v^\epsilon - v}{\epsilon} \right) \right] dx dt \\ &\quad + \frac{1}{2} \int_{\partial\Omega \times (0, T)} h(2M - 2\beta - \epsilon h) ds dt \\ &= \int_Q [A_1 \psi + A_2 \phi] dx dt + \int_{\partial\Omega \times (0, T)} h(M - \beta) ds dt \\ &= \int_Q (\psi, \phi) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} dx dt + \int_{\partial\Omega \times (0, T)} h(M - \beta) ds dt \\ &= \int_Q \left[-\psi p_t + \alpha_1 \nabla \psi \nabla p - \phi q_t + \alpha_2 \nabla \psi \nabla q + (\psi, \phi) B^T \begin{pmatrix} p \\ q \end{pmatrix} \right] dx dt \\ &\quad + \int_{\partial\Omega \times (0, T)} (\beta p \psi + \beta q \phi) ds dt + \int_{\partial\Omega \times (0, T)} h(M - \beta) \\ &= \int_Q \left[p \psi_t + \alpha_1 \nabla \psi \nabla p + q \phi_t + \alpha_2 \nabla \psi \nabla q + (\psi, \phi) B^T \begin{pmatrix} p \\ q \end{pmatrix} \right] dx dt \end{aligned}$$

$$\begin{aligned}
& + \int_{\partial\Omega \times (0,T)} (\beta p \psi + \beta q \phi) ds dt + \int_{\partial\Omega \times (0,T)} h(M - \beta) ds dt \\
& - \int_Q [(\psi p)(x, T) - (\psi p)(x, 0) + (\phi q)(x, T) - (\phi q)(x, 0)] dx \\
= & \int_Q \left[p \psi_t + \alpha_1 \nabla \psi \nabla p + q \phi_t + \alpha_2 \nabla \phi \nabla q + (p, q) B \begin{pmatrix} \psi \\ \phi \end{pmatrix} \right] dx dt \\
& + \int_{\partial\Omega \times (0,T)} (\beta p \psi + \beta q \phi) ds dt + \int_{\partial\Omega \times (0,T)} h(M - \beta) ds dt \\
= & \int_{\partial\Omega \times (0,T)} h(M - \beta - pu - qv) ds dt
\end{aligned}$$

where we used equations (3.5) and then (3.1). Hence, we conclude

$$\int_{\partial\Omega \times (0,T)} h(M - \beta - pu - qv) ds dt \leq 0.$$

By a standard control argument concerning the sign of the variation h depending on the size of β , we obtain the desired characterization of β ,

$$\beta = \min ((M - pu - qv)^+, M). \quad \square$$

Substituting (3.6) for β into the state system (1.1) and the adjoint equation (3.5), we obtain the optimality system (OS):

$$\begin{aligned}
L_1 u &= (a_1 - du)u - c_1 uv \quad \text{in } Q \\
L_2 v &= -a_2 v + c_2 uv \\
L_1^* p &= A_1 + a_1 p - 2d_1 pu - c_1 vp + c_2 vq
\end{aligned}$$

$$L_2^* q = A_2 - a_2 q + c_1 u p + c_2 u q$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad x \in \Omega$$

$$p(x, T) = 0, \quad q(x, T) = 0 \quad x \in \Omega$$

$$\alpha_1 \frac{\partial u}{\partial \nu} + \min((M - pu - qv)^+, M) u = 0,$$

$$\alpha_2 \frac{\partial v}{\partial \nu} + \min((M - pu - qv)^+, M) v = 0 \quad \text{on } \partial\Omega \times (0, T)$$

$$\alpha_1 \frac{\partial p}{\partial \nu} + \min((M - pu - qv)^+, M) p = 0,$$

$$\alpha_2 \frac{\partial q}{\partial \nu} + \min((M - pu - qv)^+, M) q = 0 \quad \text{on } \partial\Omega \times (0, T)$$

Weak solutions of the optimality system exist by Theorems 2.1 and 3.2. For small final time T , we now prove uniqueness of weak solutions of (OS), which gives the characterization (3.6) of the unique optimal control in terms of the solution of (OS).

Theorem 3.3 *For T sufficiently small, bounded weak solutions of the optimality system (OS) are unique.*

Proof: Suppose (u, v, p, q) and $(\bar{u}, \bar{v}, \bar{p}, \bar{q})$ are two weak solutions, we change variables

$$u = e^{\lambda t} w, \quad \bar{u} = e^{\lambda t} \bar{w}, \quad v = e^{\lambda t} z, \quad \bar{v} = e^{\lambda t} \bar{z}$$

$$p = e^{-\lambda t} y, \quad \bar{p} = e^{-\lambda t} \bar{y}, \quad q = e^{-\lambda t} \xi, \quad \bar{q} = e^{-\lambda t} \bar{\xi}.$$

$$\beta = \min((M - pu - qv)^+, M),$$

$$\bar{\beta} = \min((M - \bar{p}\bar{u} - \bar{q}\bar{v})^+, M),$$

where $\lambda > 0$ is to be chosen.

We subtract the bilinear forms of w and $\bar{w}$, z and $\bar{z}$, y and $\bar{y}$, ξ and $\bar{\xi}$. The form for w is illustrated below

$$\begin{aligned} & \int_Q (w - \bar{w})_t \phi dx dt + \int_Q \alpha_1 \nabla(w - \bar{w}) \nabla \phi dx dt + \lambda \int_Q (w - \bar{w}) \phi dx dt \\ = & \int_Q a_1 (w - \bar{w}) \phi dx dt - \int_Q c_1 e^{\lambda t} (wz - \bar{w}\bar{z}) \phi dx dt - \int_Q d_1 e^{\lambda t} (w^2 - \bar{w}^2) \phi dx dt \\ & - \int_{\partial\Omega \times (0, T)} \beta e^{\lambda t} (w - \bar{w}) \phi ds dt, \end{aligned}$$

where $\phi \in V$. By standard estimation techniques and using test functions $w - \bar{w}$, $z - \bar{z}$, $y - \bar{y}$, $\xi - \bar{\xi}$ in the corresponding equations, we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \{ [w(x, T) - \bar{w}(x, T)]^2 + [z(x, T) - \bar{z}(x, T)]^2 + [y(x, 0) - \bar{y}(x, 0)]^2 \\ & \quad + [\xi(x, 0) - \bar{\xi}(x, 0)]^2 \} dx \\ & + \int_Q \{ \alpha_1 |\nabla(w - \bar{w})|^2 + \alpha_2 |\nabla(z - \bar{z})|^2 + \alpha_1 |\nabla(y - \bar{y})|^2 + \alpha_2 |\nabla(\xi - \bar{\xi})|^2 \\ & \quad + \lambda [(w - \bar{w})^2 + (z - \bar{z})^2 + (y - \bar{y})^2 + (\xi - \bar{\xi})^2] \} dx dt \\ \leq & \int_Q \{ a_1 [(w - \bar{w})^2 + (y - \bar{y})^2] - a_2 [(z - \bar{z})^2 + (\xi - \bar{\xi})^2] \\ & - (w - \bar{w}) c_1 e^{\lambda t} (wz - \bar{w}\bar{z}) \\ & + (y - \bar{y}) [-2d_1 e^{\lambda t} (wy - \bar{w}\bar{y}) - c_1 e^{\lambda t} (zy - \bar{z}\bar{y}) + c_2 e^{\lambda t} (z\xi - \bar{z}\bar{\xi})] \\ & + (z - \bar{z}) c_2 e^{\lambda t} (wz - \bar{w}\bar{z}) \\ & + (\xi - \bar{\xi}) [c_1 e^{\lambda t} (wy - \bar{w}\bar{y}) + c_2 e^{\lambda t} (w\xi - \bar{w}\bar{\xi})] \} dx dt \\ & - \int_{\partial\Omega \times (0, T)} e^{\lambda t} [(\beta w - \bar{\beta}\bar{w})(w - \bar{w}) + (\beta z - \bar{\beta}\bar{z})(z - \bar{z}) \\ & - (\beta y - \bar{\beta}\bar{y})(y - \bar{y}) - (\beta \xi - \bar{\beta}\bar{\xi})(\xi - \bar{\xi})] ds dt. \end{aligned} \tag{3.7}$$

We illustrate the estimates of two specific terms from the right hand side. Using that z, w, y, ξ are L^∞ bounded, we obtain the first estimate

$$\begin{aligned} \int_Q c_1 e^{\lambda t} (wz - \bar{w}\bar{z})(w - \bar{w}) dx dt &\leq C_1 e^{\lambda T} \int_Q |wz - \bar{w}z + \bar{w}z - \bar{w}\bar{z}| |w - \bar{w}| dx dt \\ &\leq C_2 e^{\lambda T} \int_Q [(w - \bar{w})^2 + (z - \bar{z})^2] dx dt. \end{aligned}$$

Next we will estimate the term

$$\int_{\partial\Omega \times (0, T)} e^{\lambda t} ((\beta y - \bar{\beta}\bar{y})(y - \bar{y})) ds dt.$$

One can show

$$|\beta - \bar{\beta}|^2 \leq (\bar{w}\bar{y} - wy + \bar{z}\bar{\xi} - z\xi)^2.$$

Using trace theorem [11], we have

$$\begin{aligned} &\int_{\partial\Omega \times (0, T)} (w - \bar{w})^2 ds dt \\ &\leq \epsilon \int_Q |\nabla(w - \bar{w})|^2 dx dt + C(\epsilon) \int_Q (w - \bar{w})^2 dx dt \end{aligned}$$

where $C(\epsilon)$ depends on ϵ . The above inequality is also true for $y - \bar{y}, z - \bar{z}$ and $\xi - \bar{\xi}$.

Now we have the following estimate

$$\begin{aligned}
& \int_{\partial\Omega \times (0,T)} e^{\lambda t} (\beta y - \bar{\beta} \bar{y})(y - \bar{y}) ds dt \\
& \leq C_3 e^{\lambda T} \int_{\partial\Omega \times (0,T)} [(\beta - \bar{\beta})^2 + (y - \bar{y})^2] ds dt \\
& \leq C_4 e^{\lambda T} \int_{\partial\Omega \times (0,T)} [(w - \bar{w})^2 + (y - \bar{y})^2 + (z - \bar{z})^2 + (\xi - \bar{\xi})^2] ds dt \\
& \leq C_4 \epsilon e^{\lambda T} \int_Q [|\nabla(w - \bar{w})|^2 + |\nabla(y - \bar{y})|^2 + |\nabla(z - \bar{z})|^2 + |\nabla(\xi - \bar{\xi})|^2] dx dt \\
& \quad + C_4 C(\epsilon) e^{\lambda T} \int_Q [(w - \bar{w})^2 + (y - \bar{y})^2 + (z - \bar{z})^2 + (\xi - \bar{\xi})^2] dx dt.
\end{aligned}$$

The inequality (3.7) becomes

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \left\{ [w(x, T) - \bar{w}(x, T)]^2 + [z(x, T) - \bar{z}(x, T)]^2 \right. \\
& \quad \left. + [y(x, 0) - \bar{y}(x, 0)]^2 + [\xi(x, 0) - \bar{\xi}(x, 0)]^2 \right\} dx \\
& + \int_Q \left\{ (\alpha_1 - C_5 \epsilon e^{\lambda T}) |\nabla(w - \bar{w})|^2 + (\alpha_2 - C_5 \epsilon e^{\lambda T}) |\nabla(z - \bar{z})|^2 \right. \\
& \quad \left. + (\alpha_1 - C_5 \epsilon e^{\lambda T}) |\nabla(y - \bar{y})|^2 + (\alpha_2 - C_5 \epsilon e^{\lambda T}) |\nabla(\xi - \bar{\xi})|^2 \right\} dx dt \\
& + (\lambda - C_6 e^{\lambda T} - C_7 C(\epsilon) e^{\lambda T}) \int_Q \left\{ (w - \bar{w})^2 + (z - \bar{z})^2 + (y - \bar{y})^2 + (\xi - \bar{\xi})^2 \right\} dx dt \\
& \leq 0
\end{aligned}$$

where $C_i, i = 1, 2, 3, 4, 5, 6, 7$ depend only on the the coefficients and the L^∞ bounds of z, w, y , and ξ .

Let $\alpha = \min(\alpha_1, \alpha_2)$, If we choose ϵ, λ and T such that

$$\epsilon < \frac{\alpha}{C_5},$$

$$\lambda > C_6 + C_7 C(\epsilon),$$

$$T < \frac{1}{\lambda} \min \left\{ \ln \left(\frac{\lambda}{C_6 + C_7 C(\epsilon)} \right), \ln \left(\frac{\alpha}{C_5 \epsilon} \right) \right\}.$$

then $w = \bar{w}, z = \bar{z}, y = \bar{y}$ and $\xi = \bar{\xi}$. Therefore $u = \bar{u}, v = \bar{v}, p = \bar{p}$ and $q = \bar{q}$. $\square$

This T sufficiently small condition in the uniqueness result is due to opposite time orientation of the state and adjoint equations.

In conclusion, we have that the optimal control

$$\beta = \min ((M - pu - qv)^+, M)$$

is characterized in term of (u, v, p, q) , the unique solution of the optimality system for T sufficiently small.

Part III

Bilinear Optimal Control for a Wave Equation

Chapter 4

Introduction to Wave Equation Problem

In this part we study an optimal control of a wave equation with Dirichlet boundary condition. The wave equation problem is posed in a bounded multidimensional domain Q . We consider a bilinear optimal control problem in which the control acts as a dispersive term.

Given the control set

$$U_M = \{h \in L^\infty(Q) \mid -M \leq h \leq M\},$$

where $Q = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^n$, the corresponding state variable $w = w(h)$ satisfies

the state equation

$$\begin{aligned}
w_{tt} &= \Delta w + hw + f & \text{in } Q &= \Omega \times (0, T) \\
w &= 0 & \text{in } \partial\Omega \times (0, T) \\
w &= w_0(x) & t = 0, \quad x \in \Omega \\
w_t &= w_1(x) & t = 0, \quad x \in \Omega
\end{aligned} \tag{4.1}$$

where $f \in L^2(Q)$, $w_0 \in H_0^1(\Omega)$, $w_1 \in L^2(\Omega)$.

We take as our objective functional:

$$J(h) = \frac{1}{2} \left(\int_Q (w - z)^2 dQ + \beta \int_Q h^2 dQ \right)$$

where z is a given target function. This objective functional combines the cost of control and the measure of the "closeness" to the desired profile z . We want to minimize the functional over the control set U_M , i.e. to characterize h^* such that

$$J(h^*) = \min_{h \in U_M} J(h).$$

For the background in control of PDE's, see Lions [25]. Russell [32, 33, 34, 35] developed controllability and stability theory for wave equation. Note that Ball, Marsden and Slemrod [2, 3, 4] consider stabilization and controllability for this type of problem, when $h = h(t)$ is only a function of time. Bradley and Lenhart [19] treated this type of bilinear control for the Kirchhoff plate equation. Belinskiy et. al. [5] recently proved

controllability results for a damped wave equation.

The bilinear control problem presented here is new for wave equations. Due to the bilinear feature, this problem does not fit into the standard Riccati framework [18] .

The goal of this work is to obtain a unique optimal control in terms of the solution to the optimality system, which will consist of the original wave problem coupled with an adjoint problem. In chapter 5, we show the well-posedness of our state problem in an appropriate solution space. Then we show the existence of an optimal control by a minimizing sequence argument. In chapter 6, the optimality system is derived by differentiating the objective functional with respect to the control. The solution map $h \rightarrow w(h)$ is differentiated which is used in the differentiation of our objective functional. Then for sufficiently small time T , under some boundedness assumption, we prove uniqueness of the optimality system, which characterizes the unique optimal control. Finally, in chapter 7, we treat two special cases: $h = h(t)$ and $h = h(x)$. We use two different ways to obtain the uniqueness of the optimality system without boundedness assumption of solutions. The uniqueness proof is unusual due to the opposite time orientation of the two equations.

Chapter 5

Existence of an Optimal Control

The following assumptions are made through this part:

Ω is a bounded space in $\mathcal{R}^n$ and $\partial\Omega$ is C^2 smooth,

$Q = \Omega \times (0, T)$,

$f, f_t \in L^2(Q)$.

To define our notion of weak solution, we first define the bilinear form

$$B[u, v; t] = \int_{\Omega} \nabla u \nabla v dx - \int_{\Omega} h u v dx.$$

We present our definition of weak solution:

Definition (Weak Solution) Given $h \in U_M$, a function $w \in C([0, T]; H_0^1(\Omega))$, with $w_t \in C([0, T]; L^2(\Omega))$, $w_{tt} \in C([0, T]; H^{-1}(\Omega))$ is a weak solution of the problem (4.1) provided

(1)

$$\langle w_{tt}, \phi \rangle + \int_{\Omega} \nabla w \nabla \phi dx = \int_{\Omega} h w \phi dx + \int_{\Omega} f \phi dx \quad (5.1)$$

for any $\phi \in H_0^1(\Omega)$, and a.e. $0 \leq t \leq T$;

(2) $w(0) = w_0$;

(3) $w_t(0) = w_1$.

For notational convenience, set

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega),$$

$$w = w(h), \quad \tilde{w} = (w, w_t).$$

Lemma 5.1 (Well-posedness) *Let $\tilde{w}_0 = (w_0, w_1) \in \mathcal{H}$ and $h \in U_M$, then the problem*

(4.1) has a unique weak solution w .

Proof We write (4.1) in semigroup formulation

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} w \\ w_t \end{pmatrix} &= \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \begin{pmatrix} w \\ w_t \end{pmatrix} + \begin{pmatrix} 0 \\ h w + f \end{pmatrix} \\ \tilde{w}(0) &= \tilde{w}_0 = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \end{aligned} \quad (5.2)$$

Define the operator $A: [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \longrightarrow \mathcal{H}$

$$A\tilde{w} = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \tilde{w},$$

Its domain, $D(A) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, is clearly dense in $\mathcal{H}$.

Formulation (5.2) may be written as

$$\frac{d}{dt}\tilde{w}(t) = A\tilde{w}(t) + B_h(\tilde{w})(t) \quad (5.3)$$

where

$$B_h(\tilde{w}) = \begin{pmatrix} 0 \\ hw + f \end{pmatrix}$$

Note that A is skew adjoint [28], i.e., $A^* = -A$

$$A^*\tilde{w} = \begin{bmatrix} 0 & -I \\ -\Delta & 0 \end{bmatrix} \tilde{w},$$

and thus $D(A) = D(A^*)$. Thus we have that A generates a unitary group on $\mathcal{H}$ [28].

Motivated by the semigroup formulation of (5.3), we seek a solution of the form:

$$\tilde{w}(t) = e^{At}\tilde{w}_0 + \int_0^t e^{A(t-\tau)} B_h(\tilde{w})(\cdot, \tau) d\tau.$$

We will prove that the map T_h ,

$$T_h \tilde{w}(t) = e^{At} \tilde{w}_0 + \int_0^t e^{A(t-\tau)} B_h(\tilde{w})(\cdot, \tau) d\tau.$$

has a unique fixed point in $C([0, T_0]; \mathcal{H})$.

Step 1:

We will prove: If T_0 is small enough, then there exists a unique fixed point such that

$$T_h(\tilde{w}(t)) = \tilde{w}(t) \quad \text{in} \quad C([0, T_0]; \mathcal{H}).$$

To use the contraction mapping theorem, we need to show that T_h is bounded and contractive.

Boundedness:

$$\begin{aligned} & \|T_h \tilde{w}\|_{C([0, T_0]; \mathcal{H})} \\ & \leq \|e^{At} \tilde{w}_0\|_{C([0, T_0]; \mathcal{H})} + \sup_{0 \leq t \leq T_0} \int_0^t \|e^{A(t-\tau)} B_h(\tilde{w})(\cdot, \tau)\|_{\mathcal{H}} d\tau \\ & \leq \|e^{At} \tilde{w}_0\|_{C([0, T_0]; \mathcal{H})} + \sup_{0 \leq t \leq T_0} \int_0^t \|e^{A(t-\tau)} \begin{bmatrix} 0 \\ h(\cdot, \tau) w(\cdot, \tau) + f(\cdot, \tau) \end{bmatrix}\|_{L^2(\Omega)} d\tau \\ & \leq \|\tilde{w}_0\|_{\mathcal{H}} + \sup_{0 \leq t \leq T_0} \int_0^t \|h(\cdot, \tau) w(\cdot, \tau) + f(\cdot, \tau)\|_{L^2(\Omega)} d\tau \end{aligned} \tag{5.4}$$

since A generates a unitary group and that $\|e^{At}\| = 1$. Since $\|h\|_\infty \leq M$, we obtain

$$\begin{aligned} & \|T_h \tilde{w}\|_{C([0, T_0]; \mathcal{H})} \\ & \leq \|\tilde{w}_0\|_{\mathcal{H}} + MT_0(\|\tilde{w}\|_{C([0, T_0]; \mathcal{H})}) + T_0\|f\|_{C([0, T_0]; \mathcal{H})} , \end{aligned} \quad (5.5)$$

and hence T_h is bounded.

Contractive: Similarly, for any $\tilde{v}, \tilde{w} \in C([0, T_0]; \mathcal{H})$,

$$\|T_h \tilde{v} - T_h \tilde{w}\|_{C([0, T_0]; \mathcal{H})} \leq MT_0\|\tilde{v} - \tilde{w}\|_{C([0, T_0]; \mathcal{H})}$$

By choosing $T_0 < \frac{1}{M}$, we have T_h is contractive for $t \leq T_0$. Thus, we have the existence of a unique fixed point on $C([0, T_0]; \mathcal{H})$.

Step 2:

Extend above result to a solution on $[T_0, 2T_0]$ by selecting a 'new initial data' as

$$\tilde{w}_{T_0} = \tilde{w}(T_0) \in \mathcal{H}.$$

By a second contraction argument, we have a unique solution on $C([T_0, 2T_0]; \mathcal{H})$. Repeating above process a finite number of times, we obtain the existence of a unique weak solution to (4.1), with $\tilde{w} \in C([0, T]; \mathcal{H})$. $\square$

We need regularity on the state solution to derive the necessary conditions that an optimal control and corresponding state must satisfy. At first, we introduce Evans result

[11] which gave the regularity of the weak solution of problem (4.1) when control h is smooth.

Lemma 5.2 (Regularity) *Assume $h \in C^1(Q) \cap U_M$, $w_0 \in H^2(\Omega)$ and $w_1 \in H_0^1(\Omega)$, then the weak solution $w = w(h)$ satisfies $w \in C([0, T]; H^2(\Omega))$, $w_t \in C([0, T]; H^1(\Omega))$ and $w_{tt} \in C([0, T]; L^2(\Omega))$.*

Next we generalize the above regularity result without the smoothness assumption on control h .

Lemma 5.3 (Apriori Estimate) *Given $w_0 \in H_0^1(\Omega)$, $w_1 \in L^2(\Omega)$, $f \in L^2(Q_T)$, $f_t \in L^2(Q_T)$, $\partial\Omega$ is C^2 , $h \in U_M$, then the weak solution $w = w(h)$ satisfies*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\|w(t)\|_{H_0^1(\Omega)} + \|w_t(t)\|_{L^2(\Omega)} + \|w_{tt}\|_{L^2([0, T]; H^{-1}(\Omega))} \right) \\ & \leq C(\|f\|_{L^2([0, T]; L^2(\Omega))} + \|w_0\|_{H_0^1(\Omega)} + \|w_1\|_{L^2(\Omega)}) \end{aligned}$$

where C is a constant which depends only on Ω and T .

Proof: There exist sequences $\{w_{0n}\} \subset H^2(\Omega)$, $\{w_{1n}\} \subset H_0^1(\Omega)$ and $\{h_n\} \subset C^1(Q) \cap U_M$ such that

$$\begin{aligned} w_{0n} &\longrightarrow w_0, \quad \text{strongly in } H_0^1(\Omega) \\ w_{1n} &\longrightarrow w_1, \quad \text{strongly in } L^2(\Omega) \\ h_n &\longrightarrow h, \quad \text{strongly in } L^2(Q). \end{aligned} \tag{5.6}$$

Denote the weak solution of (4.1) corresponding to the initial data w_{0n}, w_{1n} with control h_n by w_n , then w_n satisfies the regularity of Lemma 5.2. Multiply the PDE (4.1) by $(w_n)_t$ and integrating over $Q_t = \Omega \times (0, t)$, we obtain

$$\begin{aligned} 0 &= \int_{Q_t} [(w_n)_{tt}(w_n)_t - (\Delta w_n)(w_n)_t - h_n w_n (w_n)_t - f(w_n)_t] dx d\tau \\ &= \int_{Q_t} \left[\frac{1}{2} \frac{d}{dt} ((w_n)_t)^2 + |Dw_n|^2 - h_n w_n (w_n)_t - f(w_n)_t \right] dx d\tau \\ &\quad - \int_{\partial\Omega \times (0, t)} \sum_{i=1}^n (w_n)_{x_i} (w_n)_t n_i ds d\tau, \end{aligned}$$

where Dw_n denotes the spatial gradient of w_n . Since $w_n = 0$, $(w_n)_t = 0$ on $\partial\Omega \times (0, T)$, the boundary term vanishes.

$$\begin{aligned} 0 &= \int_{Q_t} \left\{ \frac{1}{2} \frac{d}{dt} [(w_n)_t)^2 + |Dw_n|^2 - h_n w_n (w_n)_t - f(w_n)_t \right\} dx d\tau \\ &= \int_{\Omega \times \{t\}} \frac{1}{2} [(w_n)_t)^2 + |Dw_n|^2] dx - \int_{\Omega} \frac{1}{2} [(w_{1n})^2 + |Dw_{0n}|^2] dx \\ &\quad - \int_{Q_t} [h_n w_n (w_n)_t + f(w_n)_t] dx d\tau \end{aligned} \tag{5.7}$$

so

$$\begin{aligned} &\int_{\Omega \times \{t\}} [(w_n)_t)^2 + |Dw_n|^2] dx \\ &= \int_{\Omega} [(w_{1n})^2 + |Dw_{0n}|^2] dx + 2 \int_{Q_t} [h_n w_n (w_n)_t + f(w_n)_t] dx d\tau \\ &\leq \|w_{1n}\|_{L^2(\Omega)}^2 + \|Dw_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_t)}^2 + \|(w_n)_t\|_{L^2(Q_t)}^2 \\ &\quad + M(\|w_n\|_{L^2(Q_t)} + \|(w_n)_t\|_{L^2(Q_t)}). \end{aligned} \tag{5.8}$$

Using Poincaré's Inequality

$$\int_{\Omega \times \{t\}} |Dw_n(x, t)|^2 dx \geq C_1 \int_{\Omega \times \{t\}} w_n^2(x, t) dx,$$

where C_1 is some constant independent of w_n , we obtain

$$\begin{aligned} & \int_{\Omega \times \{t\}} [((w_n)_t)^2 + w_n^2] dx \\ \leq & C_2 \int_{Q_t} [((w_n)_t)^2 + w_n^2] dx d\tau \\ & + C_2 (\|w_{1n}\|_{L^2(\Omega)}^2 + \|Dw_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2), \end{aligned} \quad (5.9)$$

where C_2 is independent of w_n .

Let

$$\begin{aligned} \eta(t) &= \int_{\Omega \times \{t\}} [w_n^2 + ((w_n)_t)^2] dx \\ \xi &= C_2 (\|w_{1n}\|_{L^2(\Omega)}^2 + \|Dw_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2). \end{aligned}$$

Then the inequality (5.9) becomes

$$\eta(t) \leq \xi + C_2 \int_0^t \eta(t) dt. \quad (5.10)$$

Using Gronwall's inequality, we obtain

$$\eta(t) \leq \xi(1 + C_2 T e^{C_2 T}),$$

i.e.,

$$\begin{aligned} & \int_{\Omega \times \{t\}} [w_n^2 + ((w_n)_t)^2] dx \\ & \leq (1 + C_2 T e^{C_2 T}) C_2 (\|w_{1n}\|_{L^2(\Omega)}^2 + \|Dw_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_T)}^2). \end{aligned} \quad (5.11)$$

Combining (5.6) and (5.11) and letting $n \rightarrow \infty$, we get at time t

$$\begin{aligned} & \|w_t(t)\|_{L^2(\Omega)}^2 + \|Dw(t)\|_{L^2(\Omega)}^2 + \|w(t)\|_{L^2(\Omega)}^2 \\ & \leq C_3 (\|w_1\|_{L^2(\Omega)}^2 + \|Dw_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_T)}^2) \end{aligned} \quad (5.12)$$

where C_3 depends only on Ω and T . Taking the supremum, gives

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|w\|_{H_0^1(\Omega)}^2 + \|w_t\|_{L^2(\Omega)}^2) \\ & \leq C_3 (\|w_1\|_{L^2(\Omega)}^2 + \|w_0\|_{H_0^1(\Omega)}^2 + \|f\|_{L^2([0,T],L^2(\Omega))}^2). \end{aligned} \quad (5.13)$$

Next, we are going to estimate w_{tt} . Since

$$\begin{aligned}
& \langle w_{tt}, \phi \rangle(t) \\
&= - \int_{\Omega} [\nabla w \nabla \phi - h w \phi - f \phi](t) dx
\end{aligned}$$

for any $\phi \in H_0^1(\Omega)$. Let $\|\phi\|_{H_0^1(\Omega)} \leq 1$, then

$$\begin{aligned}
& |\langle w_{tt}, \phi \rangle(t)| \\
&\leq \|f(t)\|_{L^2(\Omega)} + C_4 \|w\|_{H_0^1(\Omega)}
\end{aligned} \tag{5.14}$$

where C_4 depends on M . Using (5.12), we obtain

$$\begin{aligned}
& \|w_{tt}\|_{L^2([0,T],H^{-1}(\Omega))}^2 \\
&\leq C_5 \int_0^T (\|f(t)\|_{L^2(\Omega)}^2 + \|w\|_{H_0^1(\Omega)}^2) dt \\
&\leq C_6 T (\|w_1\|_{L^2(\Omega)}^2 + \|w_0\|_{H_0^1(\Omega)}^2 + \|f\|_{L^2([0,T],L^2(\Omega))}^2).
\end{aligned}$$

where C_5, C_6 are constants independent of h . □

Now we obtain the existence of an optimal control.

Theorem 5.1 (Existence of Optimal Control) *There exists an optimal control $h^* \in U_M$, which minimizes the objective functional $J(h)$ for $h \in U_M$.*

Proof: Let $\{h^n\} \in U_M$ be a minimizing sequence such that

$$\lim_{n \rightarrow \infty} J(h^n) = \inf_{h \in U_M} J(h). \quad (5.15)$$

Denote $w^n = w(h^n)$. By priori estimate Lemma 5.3 we have

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|w^n\|_{H_0^1(\Omega)} + \|(w^n)_t\|_{L^2(\Omega)} + \|(w^n)_{tt}\|_{L^2([0,T];H^{-1}(\Omega))}) \\ & \leq C(\|f\|_{L^2([0,T];L^2(\Omega))} + \|w_0\|_{H_0^1(\Omega)} + \|w_1\|_{L^2(\Omega)}). \end{aligned}$$

On a subsequence, by weak compactness, there exists w^* in $C([0, T]; H_0^1(\Omega))$ such that

$$\begin{aligned} w^n & \rightharpoonup w^* \text{ weak } * \text{ in } L^\infty([0, T]; H_0^1(\Omega)), \\ (w^n)_t & \rightharpoonup (w^*)_t \text{ weak } * \text{ in } L^\infty([0, T]; L^2(\Omega)), \\ h^n & \rightharpoonup h^* \text{ weakly } L^2(\Omega), \\ (w^n)_{tt} & \rightharpoonup (w^*)_{tt} \text{ weakly in } L^2([0, T]; H^{-1}(\Omega)). \end{aligned} \quad (5.16)$$

By using a compactness result from [37], we have

$$w^n \rightarrow w^* \text{ strongly in } L^\infty([0, T]; L^2(\Omega)).$$

By the definition of weak solution, we have

$$\langle w_{tt}^n, \phi \rangle = - \int_{\Omega} [\nabla w^n \nabla \phi - h^n w^n \phi - f \phi] dx$$

for any $\phi \in H_0^1(\Omega)$ and a.e. $0 \leq t \leq T$. Since

$$w^n \rightarrow w^* \text{ strongly in } L^2(Q),$$

$$h^n \rightharpoonup h^* \text{ weakly in } L^2(Q),$$

and $\{h^n\}$ is L^∞ bounded, then

$$w^n h^n \rightharpoonup w^* h^* \text{ weakly in } L^2(Q).$$

Passing to the limit as $n \rightarrow \infty$ in the weak formulation of w^n , we obtain

$$\langle w_{tt}^*, \phi \rangle = - \int_{\Omega} [\nabla w^* \nabla \phi - h^* w^* \phi - f \phi] dx.$$

Thus $w^* = w(h^*)$ is the solution of state equation (4.1) with control h^* . Since

$$J(h^*) = \frac{1}{2} \int_Q (w^* - z)^2 dQ + \frac{\beta}{2} \int_Q (h^*)^2 dQ,$$

using lower-semicontinuity of L^2 norm with respect to weak convergence, we have

$$\begin{aligned} J(h^*) &\leq \lim_{n \rightarrow \infty} \frac{1}{2} \int_Q (w^n - z)^2 dQ + \liminf_{n \rightarrow \infty} \frac{\beta}{2} \int_Q (h^n)^2 dQ \\ &\leq \liminf_{n \rightarrow \infty} J(h^n) \\ &= \inf_{h \in U_M} J(h). \end{aligned}$$

Finally, we conclude h^* is an optimal control.

□

Chapter 6

Characterization of the Optimal Control

We now derive the optimality system by differentiating the objective functional $J(h)$ with respect to the control h . Since $w = w(h)$ is involved in $J(h)$, we first must prove appropriate differentiability of the mapping

$$h \rightarrow (w(h), w_t(h)) = \tilde{w}(h).$$

Lemma 6.1 *The mapping*

$$h \in U_M \longrightarrow \tilde{w}(h) = (w(h), w_t(h)) \in L^2([0, T]; \mathcal{H})$$

is differentiable in the following sense

$$\frac{\tilde{w}(h + \varepsilon \ell) - \tilde{w}(h)}{\varepsilon} \rightharpoonup \tilde{\Psi}$$

weakly in $L^2([0, T]; \mathcal{H})$ as $\varepsilon \rightarrow 0$, for any h satisfying $h + \varepsilon \ell \in U_M$ for ε small.

Moreover $\tilde{\Psi} = (\Psi, \Psi_t)$ in $L^2([0, T]; \mathcal{H})$ is a weak solution of the following problem:

$$\begin{aligned} \Psi_{tt} &= \Delta \Psi + h\Psi + \ell w, \text{ in } Q \\ \Psi &= 0 \quad \partial\Omega \times (0, T) \\ \Psi &= 0 \quad t = 0, x \in \Omega \\ \Psi_t &= 0 \quad t = 0, x \in \Omega \end{aligned} \tag{6.1}$$

where $w = w(h)$.

Proof: Denote $w^\varepsilon = w(h + \varepsilon \ell)$ and $w = w(h)$, then $(\frac{w^\varepsilon - w}{\varepsilon})$ is a weak solution of

$$\begin{aligned} (\frac{w^\varepsilon - w}{\varepsilon})_{tt} &= \Delta(\frac{w^\varepsilon - w}{\varepsilon}) + h(\frac{w^\varepsilon - w}{\varepsilon}) + \ell w^\varepsilon \text{ in } Q \\ \frac{w^\varepsilon - w}{\varepsilon} &= 0 \quad \text{in } \partial\Omega \times (0, T) \\ \frac{w^\varepsilon - w}{\varepsilon} &= 0 \quad t = 0, x \in \Omega \\ (\frac{w^\varepsilon - w}{\varepsilon})_{tt} &= 0 \quad t = 0, x \in \Omega \end{aligned}$$

Using priori estimate Lemma 5.3 , we get

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left\| \frac{(w^\varepsilon - w)(t)}{\varepsilon} \right\|_{H_0^1(\Omega)} + \left\| \frac{w_{tt}^\varepsilon - w_{tt}}{\varepsilon} \right\|_{L^2([0,T]; H^{-1}(\Omega))} \\
& \leq C \|\ell w^\varepsilon\|_{L^2([0,T]; L^2(\Omega))} \\
& \leq C \|\ell\|_{L^\infty} \sup_{0 \leq t \leq T} \|w^\varepsilon\|_{L^2(\Omega)}. \\
& \leq C_2 \|\ell\|_{L^\infty}
\end{aligned}$$

where C depends on the L^∞ bound on ℓ , but is independent of ε . Hence on a subsequence, by weak compactness, we have

$$\begin{aligned}
\frac{w^\varepsilon - w}{\varepsilon} & \rightharpoonup \Psi \text{ weak }^* \text{ in } L^\infty([0, T]; H_0^1(\Omega)), \\
\frac{w_{tt}^\varepsilon - w_{tt}}{\varepsilon} & \rightharpoonup \Psi_{tt} \text{ weak }^* \text{ in } L^\infty([0, T]; H^{-1}(\Omega))
\end{aligned}$$

By the definition of weak solution, we have

$$\begin{aligned}
& \left\langle \left(\frac{w^\varepsilon - w}{\varepsilon} \right)_{tt}, \phi \right\rangle + \int_{\Omega} \nabla \left(\frac{w^\varepsilon - w}{\varepsilon} \right) \nabla \phi dx \\
& = \int_{\Omega} h \left(\frac{w^\varepsilon - w}{\varepsilon} \right) \phi dx + \int_{\Omega} \ell w^\varepsilon \phi dx
\end{aligned}$$

for any $\phi \in H_0^1(\Omega)$, and a.e. $0 \leq t \leq T$.

Letting $\varepsilon \rightarrow 0$ and using that $w^\varepsilon \rightarrow w$ weakly in $L^2(\Omega)$, we obtain Ψ is the weak solution of problem (6.1). $\square$

Now we are ready to derive the necessary conditions that characterize an optimal control.

Theorem 6.1 *Given an optimal control h and corresponding $\tilde{w} = \tilde{w}(h) = (w, w_t)$, there exists a weak solution $\tilde{p} = (p, p_t)$ in $\mathcal{H}$ to the adjoint problem:*

$$\begin{aligned} p_{tt} &= \Delta p + hp + w - z, \quad \text{in } Q \\ p &= 0, \quad \partial\Omega \times (0, T) \\ p &= p_t = 0, \quad t = T, x \in \Omega \\ &\quad (\text{transversality condition}) \end{aligned} \tag{6.2}$$

Furthermore, h satisfies

$$h = \max(-M, \min(-\frac{wp}{\beta}, M)). \tag{6.3}$$

Proof: Let $h \in U_M$ be an optimal control and $\tilde{w} = \tilde{w}(h)$ be the corresponding optimal solution. Let $h + \varepsilon\ell \in U_M$ for $\varepsilon > 0$ and $\tilde{w}^\varepsilon = \tilde{w}(h + \varepsilon\ell)$ be the corresponding weak solution of the equation (4.1). We compute the directional derivative of the objective functional $J(h)$ with respect to h in the direction of ℓ . Since $J(h)$ is a minimal value,

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \rightarrow 0^+} \frac{J(h + \varepsilon\ell) - J(h)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} \left\{ \frac{1}{2\varepsilon} \int_Q [(w^\varepsilon - w)^2 - (w - z)^2] dQ + \frac{\beta}{2\varepsilon} \int_Q [(h + \varepsilon\ell)^2 - h^2] dQ \right\} \end{aligned}$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_Q \frac{(w^\varepsilon - w)(w^\varepsilon + w - 2z)}{2\varepsilon} dQ + \frac{\beta}{2} \int_Q (2h\ell + \varepsilon\ell^2) dQ \right\} \\
&= \int_Q \Psi(w - z) dQ + \beta \int_Q h\ell dQ,
\end{aligned} \tag{6.4}$$

where Ψ is defined in Lemma 6.1.

Consider the adjoint problem (6.2), let $\tau = T - t$, then we get a problem (3.5)

$$\begin{aligned}
p_{tt} &= \Delta p + hp + w - z, \quad \text{in } Q \\
p &= 0, \quad \partial\Omega \times (0, T) \\
p &= p_t = 0, \quad t = 0, x \in \Omega
\end{aligned} \tag{6.5}$$

Taking $w - z$ as source like " f " and using the similar argument in Lemma (5.1), we get the existence and uniqueness of solution $\tilde{p} = (p, p_t)$. So we obtain the existence and uniqueness of solution of problem (6.2).

Assume p is the solution of problem (6.2), i.e., p satisfies

$$\langle p_{tt}, \phi \rangle(t) + \int_{\Omega} \nabla p \nabla \phi(t) dx = \int_{\Omega} hp\phi(t) dx + \int_{\Omega} f\phi(t) dx \tag{6.6}$$

for any $\phi \in H_0^1(\Omega)$, and a.e. $0 \leq t \leq T$. Then

$$\begin{aligned}
0 &\leq \int_Q \Psi(w - z) dQ + \beta \int_Q h\ell dQ \\
&= \int_0^T \left\{ \langle p_{tt}, \Psi \rangle + \int_{\Omega} \nabla p \nabla \Psi dx - \int_{\Omega} hw\Psi dx \right\} dt + \beta \int_Q h\ell dQ \\
&= \int_0^T \left\{ \langle \Psi_{tt}, p \rangle + \int_{\Omega} \nabla p \nabla \Psi dx - \int_{\Omega} hw\Psi dx \right\} dt + \beta \int_Q h\ell dQ
\end{aligned}$$

by integration by parts twice in time and using $\Psi = \Psi_t = 0$ on $t = 0$ and $p = p_t = 0$ on $t = T$. Then from equation (6.1) for Ψ , we obtain

$$0 \leq \int_Q \ell(pw + \beta h) dQ.$$

Note $\ell = \ell(x, t)$ is an arbitrary function with $h + \varepsilon \ell \in U_M$ for small ε .

Considering three cases involving the sign of the variation ℓ depending on the size of h , we obtain the desired characterization of h .

Case 1:

Let $A_1 = \{(x, t) \in Q \mid -M < h(x, t) < M\}$. Taking support of $\ell(x, t) \subset A_1$ with arbitrary sign, we obtain

$$pw + \beta h = 0, \text{ in } A_1$$

So in this case, we conclude

$$-M < h = -\frac{wp}{\beta} < M.$$

Case 2:

Let $A_2 = \{(x, t) \in Q \mid h(x, t) = M\}$. Taking support of $\ell(x, t) \subset A_2$ and $\ell(x, t) \leq 0$, we obtain

$$pw + \beta h \leq 0,$$

So in this case, we conclude

$$M = h \leq -\frac{wp}{\beta}.$$

Case 3;

Let $A_3 = \{(x, t) \in Q \mid h(x, t) = -M\}$. Taking support of $\ell(x, t) \subset A_3$ and $\ell(x, t) \geq 0$, then we have

$$-M = h \geq -\frac{wp}{\beta}.$$

Combinning the above three cases, we conclude

$$h = \max(-M, \min(-\frac{wp}{\beta}, M)). \quad \square$$

Substituting (6.3) for h into the state equation (4.1) and the adjoint equation (6.2), we obtain the optimality system (OS).

$$\begin{aligned} w_{tt} &= \Delta w + \max(-M, \min(-\frac{wp}{\beta}, M))w + f \text{ in } Q \\ p_{tt} &= \Delta p + \max(-M, \min(-\frac{wp}{\beta}, M))p + w - z \text{ in } Q \\ w &= p = 0 & \partial\Omega \times (0, T) \\ w &= 0 & t = 0, x \in \Omega \\ p &= p_t = 0 & t = T, x \in \Omega \end{aligned} \tag{6.7}$$

The weak solutions of the optimality system exist by Theorems 5.1 and 6.1; The existence of an optimal control h^* and corresponding w^* implies the existence of p .

Suppose the weak solution of (OS) is bounded, for small time T , we now prove the uniqueness of weak solutions of (OS), which gives the characterization (6.3) of the unique optimal control in terms of the solutions of (OS)

Theorem 6.2 *For T sufficiently small, bounded weak solutions of the optimality system (OS) are unique.*

Proof: Suppose we have two weak solutions,

$$\tilde{w} = (w, w_t), \quad \tilde{p} = (p, p_t), \quad \hat{w} = (\bar{w}, \bar{w}_t), \quad \hat{p} = (\bar{p}, \bar{p}_t).$$

Assume $w, p, \bar{w}$ and $\bar{p}$ all are bounded in Q . Let

$$w = e^{\lambda t} u, \quad p = e^{-\lambda t} q, \quad \bar{w} = e^{\lambda t} \bar{u}, \quad \bar{p} = e^{-\lambda t} \bar{q}.$$

then u, q satisfy in a weak sense

$$u_{tt} + 2\lambda u_t + \lambda^2 u = \Delta u + \max(-M, \min(-\frac{uq}{\beta}, M))u + fe^{-\lambda t}, \quad \text{in } Q$$

$$q_{tt} + 2\lambda q_t + \lambda^2 q = \Delta q + \max(-M, \min(-\frac{uq}{\beta}, M))q + e^{2\lambda t}u - e^{\lambda t}z \quad \text{in } Q$$

$$u = q = 0 \quad \text{in } \partial\Omega \times (0, T)$$

$$u = u_t = 0, \quad t = 0, \quad x \in \Omega$$

$$q = q_t = 0 \quad t = T, \quad x \in \Omega$$

Also $\bar{u}$ and $\bar{q}$ satisfy a similar system.

We denote

$$h = \max(-M, \min(-\frac{uq}{\beta}, M)),$$

$$\bar{h} = \max(-M, \min(-\frac{\bar{u}\bar{q}}{\beta}, M)).$$

Using multiplier $(u - \bar{u})_t$ on the $u - \bar{u}$ equation and multiplier $(q - \bar{q})_t$ on the $q - \bar{q}$ equation and combining, we have the following estimate

$$\begin{aligned} 0 &= \int_Q \left[(u - \bar{u})_{tt}(u - \bar{u})_t + 2\lambda(u - \bar{u})_t^2 + \lambda^2(u - \bar{u})(u - \bar{u})_t \right. \\ &\quad \left. + \nabla(u - \bar{u})\nabla(u - \bar{u})_t - (hu - \bar{h}\bar{u})(u - \bar{u})_t \right] dx dt \\ 0 &= \int_Q \left[-(q - \bar{q})_{tt}(q - \bar{q})_t - 2\lambda(q - \bar{q})_t^2 - \lambda^2(q - \bar{q})(q - \bar{q})_t \right. \\ &\quad \left. - \nabla(q - \bar{q})\nabla(q - \bar{q})_t + (hq - \bar{h}\bar{q})(q - \bar{q})_t + e^{2\lambda t}(u - \bar{u})(q - \bar{q})_t \right] dx dt. \end{aligned}$$

Combining the above two equations and the initial and boundary conditions, we get

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} [((u - \bar{u})_t)^2(x, T) + ((q - \bar{q})_t)^2(x, 0)] dx \\ &\quad + \frac{\lambda^2}{2} \int_{\Omega} [(u - \bar{u})^2(x, T) + (q - \bar{q})^2(x, 0)] dx \\ &\quad + \frac{1}{2} \int_{\Omega} |\nabla(u - \bar{u})|^2(x, T) + |\nabla(q - \bar{q})|^2(x, 0)] dx \\ &\quad + 2\lambda \int_{\Omega} [((q - \bar{q})_t)^2 + ((u - \bar{u})_t)^2] dx dt \\ &= \int_Q [(hu - \bar{h}\bar{u})(u - \bar{u})_t - (hq - \bar{h}\bar{q})(q - \bar{q})_t \\ &\quad - e^{2\lambda t}(u - \bar{u})(q - \bar{q})_t] dx dt \end{aligned} \tag{6.8}$$

Rewriting,

$$hu - \bar{h}\bar{u} = h(u - \bar{u}) + \bar{u}(h - \bar{h}),$$

$$hq - \bar{h}\bar{q} = h(q - \bar{q}) + \bar{q}(h - \bar{h}).$$

It is easy to verify the following inequality

$$|h - \bar{h}| \leq \frac{1}{\beta} |\bar{u}\bar{q} - uq| \leq \frac{1}{\beta} [|u - \bar{u}||\bar{q}| + |q - \bar{q}||u|].$$

Estimating the last term in the LHS of (6.8), we obtain

$$\begin{aligned} & 2\lambda \int_{\Omega} [((q - \bar{q})_t)^2 + ((u - \bar{u})_t)^2] dQ \\ & \leq \int_Q [h(u - \bar{u})(u - \bar{u})_t + \bar{u}(h - \bar{h})(u - \bar{u})_t - h(q - \bar{q})(q - \bar{q})_t \\ & \quad \bar{q}(h - \bar{h})(q - \bar{q})_t - e^{2\lambda t}(u - \bar{u})(q - \bar{q})_t] dQ \\ & \leq 3 \int_Q [(u - \bar{u})_t^2 + (q - \bar{q})_t^2] dQ + Ce^{4\lambda T} \int_Q [(u - \bar{u})^2 + (q - \bar{q})^2] dQ \end{aligned} \tag{6.9}$$

where C depends only on L^∞ bound of $w, \bar{w}, p, \bar{p}$ and M . Since

$$\begin{aligned} \int_Q (u - \bar{u})^2 dQ &= \int_{\Omega} \int_0^T \left(\int_0^t (u - \bar{u})_s(x, s) ds \right)^2 dx dt \\ &\leq \int_{\Omega} \int_0^T t \left(\int_0^t ((u - \bar{u})_s)^2(x, s) ds \right) dx dt \\ &= \int_0^T t dt \int_Q ((u - \bar{u})_s)^2 ds dx \\ &\leq \frac{T^2}{2} \int_Q ((u - \bar{u})_s)^2 ds dx. \end{aligned}$$

Similarly, we have

$$\int_Q (q - \bar{q})^2 dQ \leq \frac{T^2}{2} \int_Q ((q - \bar{q})_t)^2 dQ$$

Thus the inequality (6.9) implies

$$\begin{aligned} & (2\lambda - 3) \int_Q [((q - \bar{q})_t)^2 + ((u - \bar{u})_t)^2] dQ \\ & \leq T^2 C_1 e^{4\lambda T} \int_Q [((q - \bar{q})_t)^2 + ((u - \bar{u})_t)^2] dQ, \end{aligned}$$

where C_1 is a constant. Now fix λ such that $2\lambda - 3 > 0$. Choose T small enough, so that

$$2\lambda - 3 > C_1 T^2 e^{4\lambda T}$$

and thus

$$(q - \bar{q})_t = (u - \bar{u})_t = 0 \quad \text{in } Q$$

Due to the agreement of $q, \bar{q}, u$ and $\bar{u}$ at top and bottom of the cylinder respectively, $q = \bar{q}, u = \bar{u}$. □

In R^1 space, by using Sobolev Inequality [16], weak solutions of the optimality system in $H_0^1(\Omega)$ are in the space $C(\bar{\Omega})$. Therefore the weak solutions are always bounded when $\Omega \subset R^1$. We have the following result.

Corollary 6.1 *In the case $n = 1$, for T sufficiently small, the weak solutions of the optimality system (OS) are unique.*

Remark: In general, our solutions of the optimality system may not be bounded, and thus we don't have uniqueness of the optimal control in this case. But if the controls are restricted to only be a function of time, $h = h(t)$, or a function of space, $h = h(x)$, then we can get uniqueness in certain dimensions. We will show these cases in the next chapter.

Chapter 7

Two Special Cases

In this chapter, we will treat two special cases that the controls are only functions of time or space. At first, we deal with time dependent case.

CASE I $h = h(t)$:

We take for our control set

$$U_M = \{h \in L^\infty(0, T) \mid -M \leq h(t) \leq M\}, \quad (7.1)$$

and for our functional

$$J(h) = \frac{1}{2} \left(\int_Q (w - z)^2 dQ + \beta \int_0^T h^2 dt \right). \quad (7.2)$$

Lemma 7.1 Under conditions (7.1) and (7.2), an optimal control $h = h(t)$ satisfies

$$h(t) = \max \left(-M, \min \left(-\frac{\int_{\Omega} wp(x, t) dx}{\beta}, M \right) \right) \quad (7.3)$$

where $w = w(h)$ and p satisfies (6.2).

Proof: Use the same notation as in the proof of Theorem 6.1. Let $\ell = \ell(t)$ is an arbitrary function with $h + \varepsilon \ell \in U_M$ for small ε . We have

$$\int_0^T \ell(t) \left(\int_{\Omega} p(x, t) w(x, t) dx + \beta h(t) \right) dt \geq 0.$$

We still need to consider three cases in order to derive the characterization of $h(t)$.

Case 1:

Let $A_1 = \{0 \leq t \leq T \mid -M < h(t) < M\}$. Taking support of $\ell(t) \subset A_1$ with arbitrary sign, we obtain

$$\int_{\Omega} pw(x, t) dx + \beta h = 0, \text{ in } A_1$$

So in this case, we conclude

$$-M < h = -\frac{\int_{\Omega} wp(x, t) dx}{\beta} < M$$

Case 2:

Let $A_2 = \{0 \leq t \leq T \mid h(t) = M\}$. Taking support of $\ell(t) \subset A_2$ and $\ell(t) \leq 0$,

then we obtain

$$\int_{\Omega} pw(x, t)dx + \beta h \leq 0,$$

So in this case, we conclude

$$M = h \leq -\frac{\int_{\Omega} wp(x, t)dx}{\beta}$$

Case 3;

Let $A_3 = \{0 \leq t \leq T \mid h(t) = -M\}$. Taking support of $\ell(t) \subset A_3$ and $\ell(t) \geq 0$, then we have

$$-M = h \geq -\frac{\int_{\Omega} wp(x, t)dx}{\beta}$$

Combining the above three cases, we get

$$h = \max\left(-M, \min\left(-\frac{\int_{\Omega} wp(x, t)dx}{\beta}, M\right)\right)$$

□

Theorem 7.1 *Under conditions (7.1) and (7.2), for T sufficiently small, the weak solutions of the optimality system (OS) are unique.*

Proof: We use the notation of the beginning of Theorem 6.2. As in the proof of

Theorem 6.2, we have

$$\begin{aligned}
& 2\lambda \int_Q [(q - \bar{q})_t^2 + (u - \bar{u})_t^2] dx dt \\
& \leq \int_Q ((hu - \bar{h}\bar{u})(u - \bar{u})_t - (hq - \bar{h}\bar{q})(q - \bar{q})_t \\
& \quad - e^{2\lambda t}(u - \bar{u})(q - \bar{q})_t) dx dt.
\end{aligned} \tag{7.4}$$

Rewriting,

$$hu - \bar{h}\bar{u} = h(u - \bar{u}) + \bar{u}(h - \bar{h}),$$

$$hq - \bar{h}\bar{q} = h(q - \bar{q}) + \bar{q}(h - \bar{h}).$$

It is easy to verify the following inequality

$$|h - \bar{h}| \leq \frac{1}{\beta} \int_{\Omega} |\bar{u}\bar{q} - uq| dx \leq \frac{1}{\beta} \left(\int_{\Omega} |u - \bar{u}||\bar{q}| + |q - \bar{q}||u| \right) dx.$$

Estimating the LHS of (7.4), we obtain

$$\begin{aligned}
& 2\lambda \int_Q [(q - \bar{q})_t^2 + (u - \bar{u})_t^2] dx dt \\
& \leq \int_Q [h(u - \bar{u})(u - \bar{u})_t + \bar{u}(h - \bar{h})(u - \bar{u})_t - h(q - \bar{q})(q - \bar{q})_t \\
& \quad - \bar{q}(h - \bar{h})(q - \bar{q})_t - e^{2\lambda t}(u - \bar{u})(q - \bar{q})_t] dx dt.
\end{aligned}$$

To illustrate the difference in the estimate here, we estimate one representative term

$$\begin{aligned}
& \int_Q |(h - \bar{h})(u - \bar{u})_t \bar{u}| dx dt \\
& \leq \int_0^T \int_\Omega \left[\frac{1}{\beta} \left(\int_\Omega (|u - \bar{u}| |\bar{q}| + |q - \bar{q}| |u|) dy \right) |(u - \bar{u})_t \bar{u}| dx dt \right. \\
& = \frac{1}{\beta} \int_0^T \left(\int_\Omega |(u - \bar{u}) \bar{q}| dy \right) \left(\int_\Omega |\bar{u}(u - \bar{u})_t| dx \right) dt \\
& \quad + \frac{1}{\beta} \int_0^T \left(\int_\Omega |(q - \bar{q}) u| dy \right) \left(\int_\Omega |\bar{u}(u - \bar{u})_t| dx \right) dt \\
& \leq \frac{1}{\beta} \left[\int_0^T \left(\int_\Omega |u - \bar{u}| |\bar{q}| dy \right)^2 dt \right]^{\frac{1}{2}} \left[\int_0^T \left(\int_\Omega |\bar{u}| |(u - \bar{u})_t| dx \right)^2 dt \right]^{\frac{1}{2}} \\
& \quad + \frac{1}{\beta} \left[\int_0^T \left(\int_\Omega |q - \bar{q}| |u| dy \right)^2 dt \right]^{\frac{1}{2}} \left[\int_0^T \left(\int_\Omega |\bar{u}| |(u - \bar{u})_t| dx \right)^2 dt \right]^{\frac{1}{2}} \\
& \leq \frac{1}{\beta} \left[\int_0^T \left(\int_\Omega |u - \bar{u}|^2 dy \int_\Omega |\bar{q}|^2 dy \right) dt \right]^{\frac{1}{2}} \left[\int_0^T \left(\int_\Omega |\bar{u}|^2 dx \int_\Omega |(u - \bar{u})_t|^2 dx \right) dt \right]^{\frac{1}{2}} \\
& \quad + \frac{1}{\beta} \left[\int_0^T \left(\int_\Omega |q - \bar{q}|^2 dy \int_\Omega |u|^2 dy \right) dt \right]^{\frac{1}{2}} \left[\int_0^T \left(\int_\Omega |\bar{u}|^2 dx \int_\Omega |(u - \bar{u})_t|^2 dx \right) dt \right]^{\frac{1}{2}}.
\end{aligned}$$

By using priori estimate Lemma 5.3 ,

$$\sup_{0 \leq t \leq T} \|\bar{q}(t)\|_{L^2(\Omega)} \leq e^{\lambda T} C, \quad \sup_{0 \leq t \leq T} \|\bar{u}(t)\|_{L^2(\Omega)} \leq e^{\lambda T} C;$$

where C depends only on f, w_0, w_1 , we obtain

$$\begin{aligned}
& \int_\Omega |(h - \bar{h})(u - \bar{u})_t \bar{u}| dx dt \\
& \leq \frac{C_4}{\beta} \left[\int_0^T \left(\int_\Omega |e^{2\lambda T} (u - \bar{u})|^2 dy \right) dt \right]^{\frac{1}{2}} \left[\int_0^T \left(\int_\Omega |(u - \bar{u})_t|^2 dx \right) dt \right]^{\frac{1}{2}} \\
& \quad + \frac{C_4}{\beta} \left[\int_0^T \left(\int_\Omega |e^{2\lambda T} (q - \bar{q})|^2 dy \right) dt \right]^{\frac{1}{2}} \left[\int_0^T \left(\int_\Omega |(u - \bar{u})_t|^2 dx \right) dt \right]^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{C_4}{\beta} \left[\int_0^T \left(\int_{\Omega} |e^{2\lambda T}(u - \bar{u})|^2 dy \right) dt + \int_0^T \left(\int_{\Omega} |(u - \bar{u})_t|^2 dx \right) dt \right. \\
&\quad \left. + \int_0^T \left(\int_{\Omega} |e^{2\lambda T}(q - \bar{q})|^2 dy \right) dt + \int_0^T \left(\int_{\Omega} |(q - \bar{q})_t|^2 dx \right) dt \right] \\
&= \frac{C_4}{\beta} \left(\|e^{2\lambda T}(u - \bar{u})\|_{L^2(Q)}^2 + 2\|(u - \bar{u})_t\|_{L^2(Q)}^2 + \|e^{2\lambda T}(q - \bar{q})\|_{L^2(Q)}^2 \right).
\end{aligned}$$

Similarly estimating the other $h - \bar{h}$ terms on RHS, we have

$$\begin{aligned}
&2\lambda \left(\|(u - \bar{u})_t\|_{L^2(Q)}^2 + \|(q - \bar{q})_t\|_{L^2(Q)}^2 \right) \\
&\leq \frac{C_4}{\beta} \left(\|e^{2\lambda T}(u - \bar{u})\|_{L^2(Q)}^2 + 2\|(u - \bar{u})_t\|_{L^2(Q)}^2 + \|e^{2\lambda T}(q - \bar{q})\|_{L^2(Q)}^2 \right) \\
&\quad + M \left(\|(u - \bar{u})\|_{L^2(Q)}^2 + \|(u - \bar{u})_t\|_{L^2(Q)}^2 + \|(q - \bar{q})\|_{L^2(Q)}^2 + \|(q - \bar{q})_t\|_{L^2(Q)}^2 \right) \\
&\quad + \frac{C_4}{\beta} \left(\|e^{2\lambda T}(u - \bar{u})\|_{L^2(Q)}^2 + 2\|(q - \bar{q})_t\|_{L^2(Q)}^2 + \|e^{2\lambda T}(q - \bar{q})\|_{L^2(Q)}^2 \right) \\
&\quad + \left(\|e^{2\lambda T}(u - \bar{u})\|_{L^2(Q)}^2 + \|(q - \bar{q})_t\|_{L^2(Q)}^2 \right) \\
&\leq (C_5 + M)e^{2\lambda T} (\|(u - \bar{u})\|_{L^2(Q)}^2 + \|(q - \bar{q})\|_{L^2(Q)}^2) \\
&\quad + (C_5 + M) (\|(u - \bar{u})_t\|_{L^2(Q)}^2 + \|(q - \bar{q})_t\|_{L^2(Q)}^2).
\end{aligned}$$

Since

$$\begin{aligned}
\|(u - \bar{u})\|_{L^2(Q)}^2 &\leq \frac{T^2}{2} \|(u - \bar{u})_t\|_{L^2(Q)}^2 \\
\|(q - \bar{q})\|_{L^2(Q)}^2 &\leq \frac{T^2}{2} \|(q - \bar{q})_t\|_{L^2(Q)}^2,
\end{aligned}$$

finally, we obtain

$$\begin{aligned}
& 2\lambda \left(\|(u - \bar{u})_t\|_{L^2(q)}^2 + \|(q - \bar{q})_t\|_{L^2(q)}^2 \right) \\
& \leq \left[(C_5 + M) \frac{T^2}{2} e^{2\lambda T} + (C_5 + M) \right] \left(\|(u - \bar{u})_t\|_{L^2(q)}^2 + \|(q - \bar{q})_t\|_{L^2(q)}^2 \right).
\end{aligned}$$

Now fix λ such that

$$2\lambda - (C_5 + M) \geq 0,$$

Choose T small enough , such that

$$2\lambda - (C_5 + M) \geq (C_5 + M) \frac{T^2}{2} e^{2\lambda T}.$$

and thus

$$(q - \bar{q})_t = (u - \bar{u})_t = 0 \quad \text{in } Q.$$

Due to the agreement of $q, \bar{q}, u$ and $\bar{u}$ at top and bottom of the cylinder respectively,

$$q = \bar{q}, \quad u = \bar{u}.$$

□

CASE II $h=h(x)$:

Now we are going to deal with space dependent case, i.e. $h = h(x)$, $x \in R^n$, $n = 2, 3, 4$.

We take for our control set

$$U_M = \{h \in L^\infty(\Omega) | -M \leq h(x) \leq M\}, \quad (7.5)$$

and for our functional

$$J(h) = \frac{1}{2} \left(\int_Q (w - z)^2 dQ + \beta \int_\Omega h^2(x) dx \right) \quad (7.6)$$

Lemma 7.2 *Under conditions (7.5) and (7.6), an optimal control $h = h(t)$ satisfies*

$$h(x) = \max \left(-M, \min \left(-\frac{\int_0^T w p dt}{\beta}, M \right) \right) \quad (7.7)$$

where $w = w(h)$ and p satisfies (6.2).

Proof: Use the same notation as in the proof of Theorem 6.1. Let $\ell = \ell(x)$ is an arbitrary function with $h + \varepsilon \ell \in U_M$ for small ε . We have

$$\int_\Omega \ell(x) \left(\int_0^T p(x, t) w(x, t) dt + \beta h(x) \right) dx \geq 0.$$

We still need to consider three cases in order to derive the characterization of $h(x)$.

Case 1:

Let $A_1 = \{x \in \Omega \mid -M < h(x) < M\}$. Taking support of $\ell(x) \subset A_1$ with arbitrary sign, we obtain

$$\int_0^T pw(x, t)dt + \beta h = 0, \text{ in } A_1$$

So in this case, we conclude

$$-M < h = -\frac{\int_0^T wp(x, t)dt}{\beta} < M$$

Case 2:

Let $A_2 = \{0 \leq t \leq T \mid h(x) = M\}$. Taking support of $\ell(x) \subset A_2$ and $\ell(x) \leq 0$, then we obtain

$$\int_0^T pw(x, t)dt + \beta h \leq 0,$$

So in this case, we conclude

$$M = h \leq -\frac{\int_0^T wp(x, t)dt}{\beta}$$

Case 3;

Let $A_3 = \{0 \leq t \leq T \mid h(x) = -M\}$. Taking support of $\ell(x) \subset A_3$ and $\ell(x) \geq 0$,

then we have

$$-M = h \geq -\frac{\int_0^T wp(x, t)dt}{\beta}$$

Combinning the above three cases, we get

$$h = \max \left(-M, \min \left(-\frac{\int_0^T wp(x, t)dt}{\beta}, M \right) \right)$$

□

In order to obtain the uniqueness of weak solutions of the Optimality System, we need to get higher order regularity of the weak solution of problem (4.1). Since the weak solution of problem (4.1) is unique from chapter 5, we also can use Galerkin approximation method to get the same solution in this special case , i.e. $h = h(x)$. The Galerkin approximation method clearly displays the higher order regularity of the weak solutions. We will use the similar idea as in [20] to obtain the regularity.

Theorem 7.2 (Improved Regularity) *Assume w is the weak solution of problem*

(4.1), If $h = h(x) \in U_M$, $w_0 \in H^2(\Omega)$, $w_1 \in H_0^1(\Omega)$, $f \in L^2(Q)$, $f_t \in L^2(Q)$,

then

$$w_t \in L^\infty(0, T; H^1(\Omega)), w_{tt} \in L^\infty(0, T; L^2(\Omega))$$

and we have the estimate:

$$\sup_{0 \leq t \leq T} (\|w_t\|_{H^1(\Omega)} + \|w_{tt}\|_{L^2(\Omega)}) \leq C \quad (7.8)$$

with C depending only on $\|f\|_{L^2(Q)}$, $\|f_t\|_{L^2(Q)}$, $\|w_0\|_{H^2(\Omega)}$, $\|w_1\|_{H_0^1(\Omega)}$.

Proof: Construct a sequence of approximations by selecting smooth functions $\sigma_k = \sigma_k(x)$, $(k = 1, 2, \dots)$ such that

$$\{\sigma_k\}_{k=1}^{\infty} \text{ is a basis of } H_0^1(\Omega)$$

and

$$\{\sigma_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^2(\Omega).$$

where $\{\sigma_k\}$ are eigenvectors for $-\Delta$ on $H_0^1(\Omega)$ (corresponding to the eigenvalues λ_k).

For $m \in N$, write

$$u_m(x, t) \equiv \sum_{k=1}^m d_k^m(t) \sigma_k(x) \quad (7.9)$$

where $d_k^m(t)$ satisfy

$$d_k^m(0) = (w_0(x), \sigma_k)_{L^2}, \quad (d_k^m)_t(0) = (w_1(x), \sigma_k)_{L^2} \quad (k = 1, 2, 3, \dots, m) \quad (7.10)$$

and

$$\int_{\Omega} (u_m)_{tt}(t) \sigma_k d\Omega + B[u_m, \sigma_k; t] = \int_{\Omega} f(x, t) \sigma_k d\Omega \quad (7.11)$$

Using the orthogonality of $\{\sigma_k\}$ in $L^2(\Omega)$ and substituting the sum for u_m from (7.9) into (7.11), equation (7.11) becomes a system of ordinary differential equations (ODE)

$$(d_k^m)_{tt}(t) + \sum_{\ell=1}^m B[\sigma_{\ell}, \sigma_k; t] d_{\ell}^m(t) = f_k(t) \quad (0 \leq t \leq T, \quad k = 1, 2, 3, \dots) \quad (7.12)$$

with initial conditions (7.10), where

$$f_k(t) = \int_{\Omega} f(x, t) \sigma_k d\Omega. \quad (7.13)$$

Since h does not depend on time, $B[\sigma_{\ell}, \sigma_k; t]$ does not depend on time also.

By standard existence theorem for ODE, there exists an unique absolutely continuous solution satisfying (7.10) and (7.12). Note that each $d_k^m(t)$ has more regularity since the coefficients are time independent and $f_t \in L^2(Q)$, and we will use that $(d_k^m)_{ttt}(t) \in L^2(0, T)$. Multiply (7.12) by $(d_k^m)_t(t)$ and then sum over k to obtain

$$\int_{\Omega} (u_m)_{tt}(t) (u_m)_t(t) d\Omega + B[u_m, (u_m)_t; t] = \int_{\Omega} f(x, t) (u_m)_t(t) d\Omega \quad (7.14)$$

Rewrite the above equality as

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left[\int_{\Omega} ((u_m)_t)^2(t) d\Omega + \int_{\Omega} \nabla u_m \nabla u_m(t) d\Omega \right] \\
&= \int_{\Omega} h u_m(t) (u_m)_t(t) d\Omega + \int_{\Omega} f(x, t) (u_m)_t(t) d\Omega \\
&\leq \frac{M^2}{2} \int_{\Omega} u_m^2(t) d\Omega + \frac{M^2}{2} \int_{\Omega} ((u_m)_t)^2(t) d\Omega + \frac{1}{2} \int_{\Omega} ((u_m)_t)^2(t) d\Omega + \frac{1}{2} \int_{\Omega} f^2(t) d\Omega \\
&\leq C_0 \int_{\Omega} \nabla u_m \nabla u_m(t) d\Omega + C_0 \int_{\Omega} ((u_m)_t)^2(t) d\Omega + \frac{1}{2} \int_{\Omega} f^2(t) d\Omega \tag{7.15}
\end{aligned}$$

where we use Poincaré's Inequality in the above inequality and C_0 is a constant independent of m . Gronwall's inequality [11] yields the estimate

$$\begin{aligned}
& \max_{0 \leq t \leq T} \left[\int_{\Omega} (((u_m)_t)^2(t) + |\nabla u_m|^2(t)) d\Omega \right] \\
&\leq C_1 \left[\int_{\Omega} ((u_m)_t)^2(0) d\Omega + \int_{\Omega} |\nabla u_m(0)|^2 d\Omega + \int_Q f^2 dQ \right]
\end{aligned}$$

Now we need to estimate

$$\int_{\Omega} ((u_m)_t)^2(0) d\Omega + \int_{\Omega} |\nabla u_m(0)|^2 d\Omega.$$

By the definition of u_m and using the properties of σ_k , we have

$$\begin{aligned}
(u_m)_t(0) &= \sum_{k=1}^m (d_k^m)_t(0) \sigma_k(x) \\
&= \sum_{k=1}^m (w_1, \sigma_k)_{L^2(\Omega)} \sigma_k(x),
\end{aligned}$$

$$\begin{aligned}
\|(u_m)_t(0)\|_{L^2(\Omega)}^2 &= \left\| \sum_{k=1}^m (w_1, \sigma_k)_{L^2(\Omega)} \sigma_k(x) \right\|_{L^2(\Omega)}^2 \\
&\leq \left| \sum_{k=1}^m (w_1, \sigma_k)_{L^2(\Omega)}^2 \right| \\
&\leq \|w_1\|_{L^2(\Omega)}^2;
\end{aligned}$$

and

$$\begin{aligned}
\nabla u_m(0) &= \sum_{k=1}^m d_k^m(0) \nabla \sigma_k(x) \\
\int_{\Omega} \nabla u_m(0) \nabla u_m(0) d\Omega &= - \int_{\Omega} u_m \Delta u_m(0) d\Omega + \int_{\partial\Omega} \frac{\partial u_m}{\partial \nu} u_m ds \\
&= - \int_{\Omega} u_m \Delta u_m(0) d\Omega \\
&= - \int_{\Omega} \left(\sum_{k=1}^m d_k^m(0) \sigma_k(x) \right) \left(\sum_{l=1}^m d_l^m(0) \Delta \sigma_l(0) \right) d\Omega \\
&= \int_{\Omega} \left(\sum_{k=1}^m d_k^m(0) \sigma_k(x) \right) \left(\sum_{l=1}^m d_l^m(0) \lambda_l \sigma_l(0) \right) d\Omega \\
&= \int_{\Omega} \sum_{k=1}^m \lambda_k (d_k^m(0))^2 \sigma_k(x)^2 d\Omega \\
&= \sum_{k=1}^m \lambda_k (d_k^m(0))^2 \\
&= \sum_{k=1}^m \lambda_k (w_0, \sigma_k)^2 \\
&\leq \|w_0\|_{H^1(\Omega)}^2.
\end{aligned}$$

For the last inequality above, see [31]. Note $\lambda_k > 0$, since

$$-\Delta \sigma_k = \lambda_k \sigma_k, \text{ and } \sigma_k|_{\partial\Omega} = 0,$$

$$\begin{aligned}
\|\nabla \sigma_k\|^2 &= - \int_{\Omega} \sigma_k \Delta \sigma_k d\Omega \\
&= \lambda_k \int_{\Omega} \sigma_k^2 d\Omega \\
\lambda_k &= \frac{\|\nabla \sigma_k\|^2}{\|\sigma_k\|^2} > 0.
\end{aligned}$$

Therefore

$$\int_{\Omega} ((u_m)_t)^2(0) d\Omega + \int_{\Omega} |\nabla u_m(0)|^2 d\Omega \leq C_2 (\|w_1\|_{L^2(\Omega)}^2 + \|w_0\|_{H^1(\Omega)}^2).$$

We conclude

$$\begin{aligned}
&\max_{0 \leq t \leq T} \left[\int_{\Omega} (((u_m)_t)^2(t) + |\nabla u_m|^2(t)) d\Omega \right] \\
&\leq C_3 \left[\|w_1\|_{L^2(\Omega)}^2 + \|w_0\|_{H^1(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 \right]
\end{aligned}$$

From (7.11) and [11], we have

$$\|(u_m)_{tt}\|_{L^2(0,T;(H^1)^*(\Omega))} \leq C_4 \left[\|f\|_{L^2(Q)} + \|u_m\|_{L^2(0,T;H^1(\Omega))} \right]$$

Passing to the limit on a subsequence as $m \rightarrow \infty$, and obtain $w \in L^2(0,T;H^1(\Omega))$

such that

$$u_m \rightharpoonup w \text{ in } L^2(0,T;H^1(\Omega))$$

$$(u_m)_t \rightharpoonup w_t \text{ in } L^2(Q)$$

$$(u_m)_{tt} \rightharpoonup w_{tt} \text{ in } L^2(0, T; (H^1)^*(\Omega))$$

In order to verify $w(0) = w_0$, $w_t(0) = w_1$, one chooses any function $v \in C^2([0, T]; H_0^1(\Omega))$, with $v(T) = v_t(T) = 0$, in the weak form of (4.1), we get

$$\begin{aligned} \int_0^T (\langle v_{tt}, w \rangle + B[u, v; t]) dt &= \int_Q f v - \int_{\Omega} w_0 v_t(0) + \int_{\Omega} w_1 v(0) \\ &= \int_Q f v - \int_{\Omega} w v_t(0) + \int_{\Omega} w_t v(0) \end{aligned}$$

Since $v(0), v_t(0)$ are arbitrary, conclude $w(0) = w_0$, $w_t(0) = w_1$.

By Lemma 5.1, w constructed above is the unique solution of problem (4.1). Now, we are going to prove the additional regularity. Fix m , using $(d_k^m)_{ttt}(t) \in L^2(0, T)$, differentiate (7.12) with respect to t , multiply by $(d_k^m)_{tt}(t)$, and sum over $k = 1, \dots, m$ to obtain

$$\int_{\Omega} (u_m)_{ttt} (u_m)_{tt}(t) d\Omega + B[(u_m)_t, (u_m)_{tt}; t] = \int_{\Omega} f_t (u_m)_{tt}(t) d\Omega$$

From the above equation, using Poincaré's inequality, we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\int_{\Omega} ((u_m)_{tt})^2 d\Omega + \int_{\Omega} |\nabla((u_m)_t)|^2 dx \right] \\ &\leq C_5 \left[\int_{\Omega} |\nabla((u_m)_t)|^2 + \int_{\Omega} (f_t)^2(t) d\Omega + \int_{\Omega} ((u_m)_{tt})^2(t) d\Omega \right] \end{aligned}$$

Gronwall's Inequality implies

$$\begin{aligned} & \left[\int_{\Omega} ((u_m)_{tt})^2(t) d\Omega + \int_{\Omega} |\nabla(u_m)_t|^2(t) d\Omega \right] \\ \leq & C_6 \left[\int_Q (f_t)^2(t) d\Omega + \int_{\Omega} ((u_m)_{tt})^2(0) d\Omega + \int_{\Omega} |\nabla(u_m)_t|^2(0) d\Omega \right] \end{aligned}$$

Now we must estimate $|\nabla(u_m)_t(0)|$ and $(u_m)_{tt}(0)$ terms

$$\begin{aligned} & \int_{\Omega} |\nabla((u_m)_t)(0)|^2 d\Omega \\ = & - \int (u_m)_t(0) \Delta(u_m)_t(0) d\Omega + \int_{\Gamma} \frac{\partial((u_m)_t(0))}{\partial \nu} (u_m)_t(0) ds \\ = & - \int (u_m)_t(0) \Delta(u_m)_t(0) d\Omega \\ = & - \int_{\Omega} \left(\sum_{k=1}^m (d_k^m(0))_t \sigma_k \right) \left(\sum_{l=1}^m (d_l^m(0))_t \Delta \sigma_l \right) d\Omega \\ = & \int_{\Omega} \left(\sum_{k=1}^m (w_1, \sigma_k) \sigma_k \right) \left(\sum_{l=1}^m (w_1, \sigma_l) \lambda_l \sigma_l \right) d\Omega \\ = & \int_{\Omega} \sum_{k=1}^m \lambda_k (w_1, \sigma_k)^2 \sigma_k^2 d\Omega \\ = & \sum_{k=1}^m \lambda_k (w_1, \sigma_k)^2 \\ \leq & \|w_1\|_{H^1(\Omega)}^2. \end{aligned}$$

and

$$\begin{aligned} (u_m)_{tt}(0, x) &= \sum_{k=1}^m (d_k^m)_{tt}(0) \sigma_k(x) \\ &= \sum_{k=1}^m ((u_m(0))_{tt}, \sigma_k) \sigma_k \\ (f, \sigma_k) &= ((u_m)_{tt}, \sigma_k) - (\nabla u_m, \nabla \sigma_k) + (h u_m, \sigma_k) \end{aligned}$$

$$\begin{aligned}
((u_m(0))_{tt}, \sigma_k) &= (f(0), \sigma_k) + (\nabla u_m(0), \nabla \sigma_k) - (h u_m(0), \sigma_k) \\
(u_m(0))_{tt} &= \sum_{k=1}^m (f(0), \sigma_k) \sigma_k + \sum_{k=1}^m (\Delta u_m(0), \sigma_k) \sigma_k - \sum_{k=1}^m (h u_m(0), \sigma_k) \sigma_k \\
\|(u_m(0))_{tt}\|_{L^2(\Omega)} &\leq \|f(0)\|_{L^2(\Omega)} + \|(\Delta w_0)\|_{L^2(\Omega)} + M \|w_0\|_{L^2(\Omega)} \\
&\leq \|f(0)\|_{L^2(\Omega)} + C \|w_0\|_{H^2(\Omega)}.
\end{aligned}$$

By using Poincaré's Inequality

$$\int_{\Omega} ((u_m)_t)^2 d\Omega \leq C_7 \int_{\Omega} |\nabla (u_m)_t|^2 d\Omega$$

and passing to limits as $m \rightarrow \infty$, we get the desired result. $\square$

Theorem 7.3 *In $R^n, n = 2, 3, 4$, assume $w_0(x) \in H^2(\Omega)$, $w_1(x) \in H_0^1(\Omega)$. Under conditions (7.5) and (7.6), for T sufficiently small, the weak solution of the optimality system (OS) is unique.*

Proof: Suppose we have two weak solutions,

$$\tilde{w} = (w, w_t), \quad \tilde{p} = (p, p_t), \quad \hat{w} = (\bar{w}, \bar{w}_t), \quad \hat{p} = (\bar{p}, \bar{p}_t).$$

we denote

$$\begin{aligned}
h(x) &= \max \left(-M, \min \left(-\frac{\int_0^T w p dt}{\beta}, M \right) \right) \\
\bar{h}(x) &= \max \left(-M, \min \left(-\frac{\int_0^T \bar{w} \bar{p} dt}{\beta}, M \right) \right)
\end{aligned}$$

It is easy to verify the following inequality

$$|h - \bar{h}| \leq \frac{1}{\beta} \int_0^T |\bar{w}\bar{p} - wp| dt \leq \frac{1}{\beta} \left(\int_0^T |p - \bar{p}||\bar{w}| + |w - \bar{w}||p| \right) dt \quad (7.16)$$

Using multiplier $(w - \bar{w})_t$ on the $w - \bar{w}$ equation and integrate over $(0, t)$ and multiplier $(p - \bar{p})_t$ on the $p - \bar{p}$ equation and integrate over (t, T) for $t \in (0, T)$ and combining , we have the following estimate for $0 \leq t \leq T$,

$$\begin{aligned} & \frac{1}{2} \int_{\Omega \times \{t\}} [(w - \bar{w})_t^2 + |\nabla(w - \bar{w})|^2] dx \\ &= \int_{\Omega \times (0, t)} (hw - \bar{h}\bar{w})(w - \bar{w})_\tau dx d\tau \\ & \frac{1}{2} \int_{\Omega \times \{t\}} [(p - \bar{p})_t^2 + |\nabla(p - \bar{p})|^2] dx \\ &= - \int_{\Omega \times (t, T)} [(hp - \bar{h}\bar{p})(p - \bar{p})_\tau + (w - \bar{w})(p - \bar{p})_\tau] dx d\tau \end{aligned} \quad (7.17)$$

To estimate the RHS of (7.17) by using the inequality (7.16) and integration by parts, we obtain

$$\begin{aligned} & \int_{\Omega \times (0, t)} (hw - \bar{h}\bar{w})(w - \bar{w})_\tau dx d\tau \\ &= \int_{\Omega \times (0, t)} h(w - \bar{w})(w - \bar{w})_\tau dx d\tau + \int_{\Omega \times (0, t)} (h - \bar{h})\bar{w}(w - \bar{w})_\tau dx d\tau \\ &= \int_{\Omega \times (0, t)} h(w - \bar{w})(w - \bar{w})_\tau dx d\tau - \int_{\Omega \times (0, t)} (h - \bar{h})(w - \bar{w})\bar{w}_\tau dx d\tau \\ & \quad + \int_{\Omega \times \{t\}} (h - \bar{h})(w - \bar{w})\bar{w} dx \\ &\leq M \left(\int_{\Omega \times (0, T)} (w - \bar{w})^2 dx d\tau + \int_{\Omega \times (0, T)} (w - \bar{w})_\tau^2 dx d\tau \right) \\ & \quad + \frac{1}{\beta} \int_{\Omega} \int_0^T \left(\int_0^T |(w - \bar{w})p(x, s)| ds \right) |(w - \bar{w})\bar{w}_\tau(x, \tau)| dx d\tau \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\beta} \int_{\Omega} \int_0^T \left(\int_0^T |(p - \bar{p}) \bar{w}(x, s)| ds \right) |(w - \bar{w}) \bar{w}_{\tau}(x, \tau)| dx d\tau \\
& + \frac{1}{\beta} \int_{\Omega \times \{t\}} \left(\int_0^T |(w - \bar{w}) p(x, s)| ds \right) |(w - \bar{w}) \bar{w}(x, t)| dx \\
& + \frac{1}{\beta} \int_{\Omega \times \{t\}} \left(\int_0^T |(p - \bar{p}) \bar{w}(x, s)| ds \right) |(w - \bar{w}) \bar{w}(x, t)| dx \\
\leq & M \left(\int_{\Omega \times (0, T)} (w - \bar{w})^2 dx d\tau + \int_{\Omega \times (0, T)} ((w - \bar{w})_{\tau})^2 dx d\tau \right) \\
& + \frac{1}{2\beta} \int_{\Omega} \int_0^T \int_0^T (w - \bar{w})^2(x, s) \bar{w}_{\tau}^2(x, \tau) dx ds d\tau \\
& + \frac{1}{2\beta} \int_{\Omega} \int_0^T \int_0^T (w - \bar{w})^2(x, \tau) p^2(x, s) dx ds d\tau \\
& + \frac{1}{2\beta} \int_{\Omega} \int_0^T \int_0^T (p - \bar{p})^2(x, s) \bar{w}_{\tau}^2(x, \tau) dx ds d\tau \\
& + \frac{1}{2\beta} \int_{\Omega} \int_0^T \int_0^T (w - \bar{w})^2(x, \tau) \bar{w}^2(x, s) dx ds d\tau \\
& + \frac{1}{2\beta} \int_{\Omega \times \{t\}} \int_0^T (w - \bar{w})^2(x, s) \bar{w}^2(x, t) dx ds \\
& + \frac{1}{2\beta} \int_{\Omega \times \{t\}} \int_0^T (w - \bar{w})^2(x, t) p^2(x, s) dx ds \\
& + \frac{1}{2\beta} \int_{\Omega \times \{t\}} \int_0^T (p - \bar{p})^2(x, s) \bar{w}^2(x, t) dx ds \\
& + \frac{1}{2\beta} \int_{\Omega \times \{t\}} \int_0^T (w - \bar{w})^2(x, t) \bar{w}^2(x, s) dx ds
\end{aligned}$$

By using Hölder's inequality two times and Sobolev Imbedding Inequality

$$H^1(\Omega) \hookrightarrow L^4(\Omega), \quad n = 2, 3, 4,$$

we obtain

$$\int_{\Omega \times (0, t)} (hw - \bar{h}\bar{w})(w - \bar{w})_{\tau} dx d\tau$$

$$\begin{aligned}
&\leq MT \left(\sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|(w - \bar{w})_t\|_{L^2(\Omega)}^2 \right) \\
&\quad + \frac{1}{2\beta} T^2 \sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^4(\Omega)}^2 \sup_{0 \leq t \leq T} \|\bar{w}_t\|_{L^4(\Omega)}^2 \\
&\quad + \frac{1}{2\beta} T^2 \sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^4(\Omega)}^2 \sup_{0 \leq t \leq T} \|p\|_{L^4(\Omega)}^2 \\
&\quad + \frac{1}{2\beta} T^2 \sup_{0 \leq t \leq T} \|p - \bar{p}\|_{L^4(\Omega)}^2 \sup_{0 \leq t \leq T} \|\bar{w}_t\|_{L^4(\Omega)}^2 \\
&\quad + \frac{1}{2\beta} T^2 \sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^4(\Omega)}^2 \sup_{0 \leq t \leq T} \|\bar{w}\|_{L^4(\Omega)}^2 \\
&\quad + \frac{1}{2\beta} T \sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^4(\Omega)}^2 \sup_{0 \leq t \leq T} \|\bar{w}\|_{L^4(\Omega)}^2 \\
&\quad + \frac{1}{2\beta} T \sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^4(\Omega)}^2 \sup_{0 \leq t \leq T} \|p\|_{L^4(\Omega)}^2 \\
&\quad + \frac{1}{2\beta} T \sup_{0 \leq t \leq T} \|p - \bar{p}\|_{L^4(\Omega)}^2 \sup_{0 \leq t \leq T} \|\bar{w}\|_{L^4(\Omega)}^2 \\
&\quad + \frac{1}{2\beta} T \sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^4(\Omega)}^2 \sup_{0 \leq t \leq T} \|\bar{w}\|_{L^4(\Omega)}^2 \\
&= MT \left(\sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|(w - \bar{w})_t\|_{L^2(\Omega)}^2 \right) \\
&\quad + \frac{1}{2\beta} \sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^4(\Omega)}^2 \left(T^2 \sup_{0 \leq t \leq T} \|\bar{w}_t\|_{L^4(\Omega)}^2 + T^2 \sup_{0 \leq t \leq T} \|p\|_{L^4(\Omega)}^2 \right. \\
&\quad \left. + T^2 \sup_{0 \leq t \leq T} \|\bar{w}\|_{L^4(\Omega)}^2 + 2T \sup_{0 \leq t \leq T} \|\bar{w}\|_{L^4(\Omega)}^2 + T \sup_{0 \leq t \leq T} \|p\|_{L^4(\Omega)}^2 \right) \\
&\quad + \frac{1}{2\beta} \sup_{0 \leq t \leq T} \|p - \bar{p}\|_{L^4(\Omega)}^2 \left(T^2 \sup_{0 \leq t \leq T} \|\bar{w}_t\|_{L^4(\Omega)}^2 + T \sup_{0 \leq t \leq T} \|\bar{w}\|_{L^4(\Omega)}^2 \right) \\
&\leq MT \left(\sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|(w - \bar{w})_t\|_{L^2(\Omega)}^2 \right) \\
&\quad + \frac{C_1}{2\beta} \sup_{0 \leq t \leq T} \|w - \bar{w}\|_{H^1(\Omega)}^2 \left(T^2 \sup_{0 \leq t \leq T} \|\bar{w}_t\|_{H^1(\Omega)}^2 + T^2 \sup_{0 \leq t \leq T} \|p\|_{H^1(\Omega)}^2 \right. \\
&\quad \left. + T^2 \sup_{0 \leq t \leq T} \|\bar{w}\|_{H^1(\Omega)}^2 + 2T \sup_{0 \leq t \leq T} \|\bar{w}\|_{H^1(\Omega)}^2 + T \sup_{0 \leq t \leq T} \|p\|_{H^1(\Omega)}^2 \right) \\
&\quad + \frac{C_1}{2\beta} \sup_{0 \leq t \leq T} \|p - \bar{p}\|_{H^1(\Omega)}^2 \left(T^2 \sup_{0 \leq t \leq T} \|\bar{w}_t\|_{H^1(\Omega)}^2 + T \sup_{0 \leq t \leq T} \|\bar{w}\|_{H^1(\Omega)}^2 \right)
\end{aligned}$$

Since $w, \bar{w}, p, \bar{p}, w_t, \bar{w}_t$ are bounded in $H^1(\Omega)$ by Theorem 7.2, assuming they are all bounded by a constant C_2 , we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega \times \{t\}} \left[((w - \bar{w})_t)^2 + |\nabla(w - \bar{w})|^2 \right] dx \\ \leq & MT \left(\sup_{0 \leq t \leq T} \|w - \bar{w}\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|(w - \bar{w})_t\|_{L^2(\Omega)}^2 \right) \\ & + \frac{3C_1C_2}{2\beta} (T^2 + T) \left(\sup_{0 \leq t \leq T} \|w - \bar{w}\|_{H^1(\Omega)}^2 + \sup_{0 \leq t \leq T} \|p - \bar{p}\|_{H^1(\Omega)}^2 \right) \end{aligned}$$

Similarly, one obtains

$$\begin{aligned} & \frac{1}{2} \int_{\Omega \times \{t\}} \left[((p - \bar{p})_t)^2 + |\nabla(p - \bar{p})|^2 \right] dx \\ \leq & (M + 1)T \left(\sup_{0 \leq t \leq T} \|p - \bar{p}\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|(p - \bar{p})_t\|_{L^2(\Omega)}^2 \right) \\ & + \left(\frac{3C_1C_2}{2\beta} + 1 \right) (T^2 + T) \left(\sup_{0 \leq t \leq T} \|p - \bar{p}\|_{H^1(\Omega)}^2 + \sup_{0 \leq t \leq T} \|w - \bar{w}\|_{H^1(\Omega)}^2 \right) \end{aligned}$$

Combining the above inequalities, and using Poincaré's Inequality

$$\|u\|_{L^2(\Omega)}^2 \leq C_3 \|\nabla u\|_{L^2(\Omega)}^2$$

we obtain

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|(w - \bar{w})_t\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|\nabla(w - \bar{w})\|_{L^2(\Omega)}^2 \\ & + \sup_{0 \leq t \leq T} \|(p - \bar{p})_t\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|\nabla(p - \bar{p})\|_{L^2(\Omega)}^2 \end{aligned}$$

$$\begin{aligned} \leq & 2(C_3 + 1)T \left[M + 1 + \left(\frac{3C_1C_2}{\beta} + 1 \right)(T + 1) \right] \left[\sup_{0 \leq t \leq T} \|(w - \bar{w})_t\|_{L^2(\Omega)}^2 \right. \\ & \left. + \sup_{0 \leq t \leq T} \|\nabla(w - \bar{w})\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|(p - \bar{p})_t\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|\nabla(p - \bar{p})\|_{L^2(\Omega)}^2 \right]. \end{aligned}$$

If we choose T small such that

$$2(C_3 + 1)T \left[M + 1 + \left(\frac{3C_1C_2}{\beta} + 1 \right)(T + 1) \right] < 1,$$

then we obtain

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|(w - \bar{w})_t\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|\nabla(w - \bar{w})\|_{L^2(\Omega)}^2 \\ & + \sup_{0 \leq t \leq T} \|(p - \bar{p})_t\|_{L^2(\Omega)}^2 + \sup_{0 \leq t \leq T} \|\nabla(p - \bar{p})\|_{L^2(\Omega)}^2 \\ & \leq 0. \end{aligned}$$

Therefore we have proved the uniqueness of weak solution of the optimality systems. $\square$

Part IV

Bilinear Optimal Control for a Wave Equation with Viscous Damping

Chapter 8

Introduction to Wave Equation with Viscous Damping

In this part we study optimal control of a wave equation with viscous damping and with Dirichlet boundary condition. The wave equation problem is posed in a bounded multidimensional domain Q . We consider a bilinear optimal control problem in which the control acts as a damping term.

Given the control set

$$U_M = \{k \in L^\infty(0, T) | 0 \leq k \leq M\},$$

where $Q = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^n$, the corresponding state variable $y = y(k)$ satisfies

the state equation

$$\begin{aligned}
 y_{tt} &= \Delta y - k(t)y_t - q(x)y + f & \text{in } Q &= \Omega \times (0, T) \\
 y &= 0 & \text{in } \partial\Omega \times (0, T) \\
 y &= v_0(x) & t = 0, \ x \in \Omega \\
 y_t &= v_1(x) & t = 0, \ x \in \Omega
 \end{aligned} \tag{8.1}$$

where $q(x) \in L^\infty(\Omega)$ and $0 \leq q(x) \leq q_0$, q_0 is a constant, $f \in L^2(Q)$, $v_0 \in H_0^1(\Omega)$, and $v_1 \in L^2(\Omega)$.

We take as our objective functional:

$$J(k) = \frac{1}{2} \left(\int_Q (y - z)^2 dQ + \beta \int_0^T k^2(t) dt \right)$$

where z is a given target function. This objective functional combines the cost of control and the measure of the "closeness" to the desired profile z . We want to minimize the functional over the control set U_M , i.e. to characterize k^* such that

$$J(k^*) = \min_{k \in U_M} J(k)$$

For the background in control of PDE's, see Lions [25]. Russell [32, 33, 34, 35] developed controllability and stability theory for wave equation. Bradley and Lenhart [19] treated bilinear optimal control without the velocity term for the Kirchhoff plate equation. Liang [24] treated this type of bilinear control without velocity term for wave

equation. Bradley, Lenhart and Yong [7] recently treated this type of bilinear control with the velocity term for the Kirchhoff plate equation. Belinskiy et. al. [5] proved controllability results for a damped wave equation.

The bilinear control problem presented here is new for wave equations. Due to the bilinear feature, this problem does not fit into the standard Riccati framework [18] .

The goal of this part is to obtain an optimal control in terms of the solution to the optimality system, which will consist of the original state problem coupled with an adjoint problem. In chapter 9, we show the well-posedness of our state problem in an appropriate solution space. Then we show the existence of an optimal control by a minimizing sequence argument. In chapter 10, the optimality system and a characterization of an optimal control are derived by differentiating the objective functional with respect to the control. The solution map $k \rightarrow y(k)$ is differentiated which is used in the differentiation of our objective functional. Thus necessary conditions, that an optimal control must satisfy, are derived.

Chapter 9

Existence of an Optimal Control

The following assumptions are made through this part:

Ω is a bounded space in $\mathcal{R}^n$ and $\partial\Omega$ is C^2 smooth.

$Q = \Omega \times (0, T)$.

$f, f_t \in L^2(Q)$.

$q(x) \in L^\infty(\Omega)$ and $0 \leq q(x) \leq q_0$, q_0 is a constant.

To define our notion of weak solution, we first define the bilinear form

$$B[u, v; t] = \int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} q u v dx$$

We present our definition of weak solution:

Definition (Weak Solution) Given $k \in U_M$, a function $y \in C([0, T]; H_0^1(\Omega))$, with $y_t \in C([0, T]; L^2(\Omega))$, $y_{tt} \in C([0, T]; H^{-1}(\Omega))$ is a weak solution of the problem (8.1) provided

(1)

$$\int_0^T \langle y_{tt}, \phi \rangle dt + \int_Q k(t) y_t \phi dx dt + \int_0^T B[y, \phi; t] dt = \int_Q f \phi dx dt \quad (9.1)$$

for any $\phi \in H_0^1(\Omega)$, and a.e. $0 \leq t \leq T$;

(2) $y(0) = v_0$;

(3) $y_t(0) = v_1$. where $\langle, \rangle$ denotes the duality pairing of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

For notational convenience, set

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega),$$

$$y = y(k), \quad \tilde{y} = (y, y_t).$$

Theorem 9.1 (Well-posedness) *Let $\tilde{y}_0 = (v_0, v_1) \in \mathcal{H}$ and $k \in U_M$, then the problem (8.1) has a unique weak solution y .*

Proof: We write (8.1) in semigroup formulation

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} y \\ y_t \end{pmatrix} &= \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \begin{pmatrix} y \\ y_t \end{pmatrix} + \begin{pmatrix} 0 \\ -ky_t - qy + f \end{pmatrix} \\ \tilde{y}(0) &= \tilde{y}_0 = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \end{aligned} \quad (9.2)$$

Define the operator $A: [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \longrightarrow \mathcal{H}$

$$A\tilde{y} = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \tilde{y},$$

Its domain, $D(A) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, is clearly dense in $\mathcal{H}$.

Formulation (9.2) may be written as

$$\frac{d}{dt}\tilde{y}(t) = A\tilde{y}(t) + B_k(\tilde{y})(t) \quad (9.3)$$

where

$$B_k(\tilde{y}) = \begin{pmatrix} 0 \\ -ky_t - qy + f \end{pmatrix}$$

Note that A is skew adjoint [28] , i.e., $A^* = -A$

$$A^*\tilde{y} = \begin{bmatrix} 0 & -I \\ -\Delta & 0 \end{bmatrix} \tilde{y},$$

and thus $D(A) = D(A^*)$. Thus we have that A generates a unitary group on $\mathcal{H}$ [28].

Motivated by the semigroup formulation of (9.3), we seek a solution of the form:

$$\tilde{y}(t) = e^{At}\tilde{y}_0 + \int_0^t e^{A(t-\tau)} B_k(\tilde{y})(x, \tau) d\tau.$$

where e^{At} represents the semigroup generated by A . We will prove that the map T_k ,

$$T_k \tilde{y}(t) = e^{At} \tilde{y}_0 + \int_0^t e^{A(t-\tau)} B_k(\tilde{y})(x, \tau) d\tau.$$

has a unique fixed point in $C([0, T_0]; \mathcal{H})$.

Step 1:

We will prove: If T_0 is small enough, then there exists a unique fixed point such that

$$T_k(\tilde{y}(t)) = \tilde{y}(t) \quad \text{in} \quad C([0, T_0]; \mathcal{H})$$

To use the contraction mapping theorem, we need to show that T_k is bounded and contractive.

Boundedness:

$$\begin{aligned} & \|T_k \tilde{y}\|_{C([0, T_0]; \mathcal{H})} \\ & \leq \|e^{At} \tilde{y}_0\|_{C([0, T_0]; \mathcal{H})} + \sup_{0 \leq t \leq T_0} \int_0^t \|e^{A(t-\tau)} B_k(\tilde{y})(\cdot, \tau)\|_{\mathcal{H}} d\tau \\ & \leq \|e^{At} \tilde{y}_0\|_{C([0, T_0]; \mathcal{H})} + \sup_{0 \leq t \leq T_0} \int_0^t \|e^{A(t-\tau)} [-k(\cdot, \tau) y_\tau(\cdot, \tau) \\ & \quad - q(\tau) y(\cdot, \tau) + f(\cdot, \tau)]\|_{L^2(\Omega)} d\tau \\ & \leq \|\tilde{y}_0\|_{\mathcal{H}} + \sup_{0 \leq t \leq T_0} \int_0^t \|-k(\cdot, \tau) y_\tau(\cdot, \tau) - q(\tau) y(\cdot, \tau) + f(\cdot, \tau)\|_{L^2(\Omega)} d\tau \end{aligned}$$

since A generates a unitary group and that $\|e^{At}\| = 1$. Since $\|k\|_\infty \leq M$ and $q \leq q_0$, we

obtain

$$\begin{aligned} & \|T_k \tilde{y}\|_{C([0, T_0]; \mathcal{H})} \\ \leq & \|\tilde{y}_0\|_{\mathcal{H}} + CT_0(\|\tilde{y}\|_{C([0, T_0]; \mathcal{H})}) + T_0\|f\|_{C([0, T_0]; \mathcal{H})} , \end{aligned} \quad (9.4)$$

where C depends only on q_0 and M , hence T_k is bounded.

Contractive: Similarly, for any $\tilde{v}, \tilde{w} \in C([0, T_0]; \mathcal{H})$,

$$\|T_k \tilde{v} - T_k \tilde{w}\|_{C([0, T_0]; \mathcal{H})} \leq CT_0\|\tilde{v} - \tilde{w}\|_{C([0, T_0]; \mathcal{H})}$$

By choosing $T_0 < \frac{1}{M}$, we have T_k is contractive for $t \leq T_0$. Thus, we have the existence of a unique fixed point on $C([0, T_0]; \mathcal{H})$.

Step 2:

Extend above result to a solution on $[T_0, 2T_0]$ by selecting a 'new initial data' as

$$\tilde{y}_{T_0} = \tilde{y}(T_0) \in \mathcal{H}$$

By a second contraction argument, we have a unique solution on $C([T_0, 2T_0]; \mathcal{H})$. Repeating above process a finite number of times, we obtain the existence of a unique weak solution to (8.1), with $\tilde{y} \in C([0, T]; \mathcal{H})$. $\square$

We need regularity on the state solution to derive the necessary conditions that an optimal control and corresponding state must satisfy. At first, we obtain the regularity

of the weak solution of problem (8.1), by using the similar idea from Evans [11], when control k is smooth.

Theorem 9.2 (Regularity) *Assume Ω is a bounded space, $\partial\Omega$ is C^2 smooth, $k(t) \in C^1[0, T] \cap U_M$, $v_0 \in H^2(\Omega)$, $v_1 \in H_0^1(\Omega)$ and $f_t \in L^2(Q)$, then the weak solution $y = y(k)$ satisfies $y \in C([0, T]; H^2(\Omega))$, $y_t \in C([0, T]; H^1(\Omega))$ and $y_{tt} \in C([0, T]; L^2(\Omega))$.*

Proof: Construct a sequence of approximations by selecting smooth functions $\sigma_j = \sigma_j(x)$, ($j = 1, 2, \dots$) such that

$$\{\sigma_j\}_{j=1}^{\infty} \text{ is a basis of } H_0^1(\Omega)$$

and

$$\{\sigma_j\}_{j=1}^{\infty} \text{ is an orthonormal basis of } L^2(\Omega),$$

where $\{\sigma_j\}$ are eigenvectors for $-\Delta$ on $H_0^1(\Omega)$ (corresponding to the eigenvalues λ_j).

For integer m , write

$$y_m(t) \equiv \sum_{j=1}^m d_j^m(t) \sigma_j(x) \quad (9.5)$$

where $d_j^m(t)$ satisfy

$$d_j^m(0) = (v_0(x), \sigma_j)_{L^2}, \quad (d_j^m)_t(0) = (v_1(x), \sigma_j)_{L^2} \quad (j = 1, 2, 3, \dots, m) \quad (9.6)$$

and

$$\int_{\Omega} (y_m)_{tt}(t) \sigma_j d\Omega + \int_{\Omega} k(t) (y_m)_t \sigma_j d\Omega + B[y_m, \sigma_j; t] = \int_{\Omega} f(x, t) \sigma_j d\Omega. \quad (9.7)$$

Using the orthogonality of $\{\sigma_j\}$ in $L^2(\Omega)$ and substituting the sum for y_m from (9.5) into (9.7), equation (9.7) becomes a system of ordinary differential equations (ODE)

$$\begin{aligned} (d_j^m)_{tt}(t) + k(t) (d_j^m)_t(t) + \sum_{\ell=1}^m B[\sigma_{\ell}, \sigma_j; t] d_{\ell}^m(t) &= f_j(t) \\ (0 \leq t \leq T, j = 1, 2, 3, \dots) \end{aligned}$$

with initial conditions (9.6), where

$$\begin{aligned} B[\sigma_{\ell}, \sigma_j; t] &= \int_{\Omega} \nabla \sigma_{\ell} \nabla \sigma_j d\Omega + \int_{\Omega} q(x) \sigma_{\ell} \sigma_j d\Omega \\ f_j(t) &= \int_{\Omega} f(x, t) \sigma_j d\Omega. \end{aligned} \quad (9.8)$$

Since q does not depend on time, $B[\sigma_{\ell}, \sigma_j; t]$ does not depend on time also. Since $k(t) \in C^1[0, T]$, by standard existence theorem for ODE, there exists an unique absolutely continuous solution satisfying (9.6) and (9.8). Multiply (9.8) by $(d_j^m)_t(t)$ and then sum over j to obtain

$$\begin{aligned} &\int_{\Omega} (y_m)_{tt}(t) (y_m)_t(t) d\Omega + k(t) \int_{\Omega} ((y_m)_t)^2 d\Omega + B[y_m, (y_m)_t; t] \\ &= \int_{\Omega} f(x, t) (y_m)_t(t) d\Omega. \end{aligned} \quad (9.9)$$

Rewrite the above equality as

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left[\int_{\Omega} ((y_m)_t)^2(t) d\Omega + \int_{\Omega} \nabla y_m \nabla y_m(t) d\Omega \right] + k(t) \int_{\Omega} ((y_m)_t)^2 d\Omega \\
&= \int_{\Omega} -q(x) y_m(t) (y_m)_t(t) d\Omega + \int_{\Omega} f(x, t) (y_m)_t(t) d\Omega \\
&\leq \frac{q_0}{2} \int_{\Omega} y_m^2(t) d\Omega + \frac{q_0}{2} \int_{\Omega} ((y_m)_t)^2(t) d\Omega + \frac{1}{2} \int_{\Omega} ((y_m)_t)^2(t) d\Omega + \frac{1}{2} \int_{\Omega} f^2(t) d\Omega \\
&\leq C_0 \int_{\Omega} \nabla y_m \nabla y_m(t) d\Omega + C_0 \int_{\Omega} ((y_m)_t)^2(t) d\Omega + \frac{1}{2} \int_{\Omega} f^2(t) d\Omega \tag{9.10}
\end{aligned}$$

where we use Poincaré's Inequality in the above inequality and C_0 is a constant independent of m . Gronwall's inequality [11] yields the estimate

$$\begin{aligned}
& \max_{0 \leq t \leq T} \left[\int_{\Omega} (((y_m)_t)^2(t) + |\nabla y_m|^2(t)) d\Omega \right] \\
&\leq C_1 \left[\int_{\Omega} ((y_m)_t)^2(0) d\Omega + \int_{\Omega} |\nabla y_m(0)|^2 d\Omega + \int_Q f^2 dQ \right].
\end{aligned}$$

Now we need to estimate

$$\int_{\Omega} ((y_m)_t)^2(0) d\Omega + \int_{\Omega} |\nabla y_m(0)|^2 d\Omega.$$

By the definition of y_m and using the properties of σ_j , we have

$$\begin{aligned}
(y_m)_t(0) &= \sum_{j=1}^m (d_j^m)_t(0) \sigma_j(x) \\
&= \sum_{j=1}^m (w_1, \sigma_j)_{L^2(\Omega)} \sigma_j(x),
\end{aligned}$$

$$\begin{aligned}
\|(y_m)_t(0)\|_{L^2(\Omega)}^2 &= \left\| \sum_{j=1}^m (v_1, \sigma_j)_{L^2(\Omega)} \sigma_j(x) \right\|_{L^2(\Omega)}^2 \\
&\leq \left| \sum_{j=1}^m (v_1, \sigma_j)_{L^2(\Omega)}^2 \right| \\
&\leq \|v_1\|_{L^2(\Omega)}^2;
\end{aligned}$$

and

$$\begin{aligned}
\nabla y_m(0) &= \sum_{j=1}^m d_j^m(0) \nabla \sigma_j(x) \\
\int_{\Omega} \nabla y_m(0) \nabla y_m(0) d\Omega &= - \int_{\Omega} y_m \Delta y_m(0) d\Omega + \int_{\partial\Omega} \frac{\partial y_m}{\partial \nu} y_m ds \\
&= - \int_{\Omega} y_m \Delta y_m(0) d\Omega \\
&= - \int_{\Omega} \left(\sum_{j=1}^m d_j^m(0) \sigma_j(x) \right) \left(\sum_{j=1}^m d_j^m(0) \Delta \sigma_j(0) \right) d\Omega \\
&= \int_{\Omega} \left(\sum_{j=1}^m d_j^m(0) \sigma_j(x) \right) \left(\sum_{j=1}^m d_j^m(0) \lambda_j \sigma_j(0) \right) d\Omega \\
&= \int_{\Omega} \sum_{j=1}^m \lambda_j (d_j^m(0))^2 \sigma_j(x)^2 d\Omega \\
&= \sum_{j=1}^m \lambda_j (d_j^m(0))^2 \\
&= \sum_{j=1}^m \lambda_j (v_0, \sigma_j)^2 \\
&\leq \|v_0\|_{H^1(\Omega)}^2.
\end{aligned}$$

For the last inequality above, see [12]. Note $\lambda_j > 0$, since

$$-\Delta \sigma_j = \lambda_j \sigma_j, \text{ and } \sigma_j|_{\Gamma} = 0,$$

$$\begin{aligned}
\|\nabla \sigma_j\|^2 &= - \int_{\Omega} \sigma_j \Delta \sigma_j d\Omega \\
&= \lambda_j \int_{\Omega} \sigma_j^2 d\Omega \\
\lambda_j &= \frac{\|\nabla \sigma_j\|^2}{\|\sigma_j\|^2} > 0.
\end{aligned}$$

Therefore

$$\int_{\Omega} ((y_m)_t)^2(0) d\Omega + \int_{\Omega} |\nabla y_m(0)|^2 d\Omega \leq C_2(\|v_1\|_{L^2(\Omega)}^2 + \|v_0\|_{H^1(\Omega)}^2).$$

We conclude

$$\begin{aligned}
&\max_{0 \leq t \leq T} \left[\int_{\Omega} (((y_m)_t)^2(t) + |\nabla y_m|^2(t)) d\Omega \right] \\
&\leq C_3 \left[\|v_1\|_{L^2(\Omega)}^2 + \|v_0\|_{H^1(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 \right]
\end{aligned} \tag{9.11}$$

From (9.9), we have

$$\int_{\Omega} (y_m)_{tt}(t) \phi d\Omega + \int_{\Omega} k(t) (y_m)_t \phi d\Omega + B[y_m, \phi; t] = \int_{\Omega} f(x, t) \phi d\Omega \tag{9.12}$$

for any $\phi \in H_0^1(\Omega)$. Therefore, we have

$$\begin{aligned}
& \left| \int_{\Omega} (y_m)_{tt}(t) \phi d\Omega \right| \\
& \leq \left| \int_{\Omega} k(t) (y_m)_t \phi d\Omega \right| + |B[y_m, \phi; t]| + \left| \int_{\Omega} f(x, t) \phi d\Omega \right|.
\end{aligned} \tag{9.13}$$

Using the estimate (9.13) and Poincare's inequality, we conclude

$$|\int_{\Omega} (y_m)_{tt}(t) \phi d\Omega| \leq C_4 \|\phi\|_{H_0^1(\Omega)},$$

for any $\phi \in H_0^1(\Omega)$. Passing to the limit on a subsequence as $m \rightarrow \infty$, and obtain $y \in L^2([0, T]; H^1(\Omega))$ such that

$$y_m \rightharpoonup y \text{ in } L^2(0, T; H^1(\Omega)),$$

$$(y_m)_t \rightharpoonup y_t \text{ in } L^2(Q),$$

$$(y_m)_{tt} \rightharpoonup y_{tt} \text{ in } L^2(0, T; (H^1)^*(\Omega)).$$

To verify $y(0) = v_0$, $y_t(0) = v_1$, one choose any function $v \in C^2([0, T]; H_0^1(\Omega))$, with $v(T) = v_t(T) = 0$, in the weak form of (8.1), we get

$$\begin{aligned} \int_0^T \left(\langle v_{tt}, y \rangle + \int_{\Omega} k(t) y_t v d\Omega + B[y, v; t] \right) dt &= \int_Q f v - \int_{\Omega} v_0 v_t(0) + \int_{\Omega} v_1 v(0) \\ &= \int_Q f v - \int_{\Omega} y v_t(0) + \int_{\Omega} y_t v(0). \end{aligned}$$

Since $v(0), v_t(0)$ are arbitrary, conclude $y(0) = v_0$, $y_t(0) = v_1$. $\square$

Next we generalize the above regularity result without the smoothness assumption of control h , and obtain a priori estimate.

Theorem 9.3 (Apriori Estimate) Given $v_0 \in H_0^1(\Omega)$, $v_1 \in L^2(\Omega)$, $f \in L^2(Q_T)$, $f_t \in L^2(Q_T)$, $\partial\Omega$ is C^2 , and $k \in U_M$, then the weak solution $y = y(k)$ satisfies

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\|y(t)\|_{H_0^1(\Omega)} + \|y_t(t)\|_{L^2(\Omega)} + \|y_{tt}\|_{L^2([0,T];H^{-1}(\Omega))} \right) \\ & \leq C(\|f\|_{L^2([0,T];L^2(\Omega))} + \|v_0\|_{H_0^1(\Omega)} + \|v_1\|_{L^2(\Omega)}) \end{aligned}$$

where C is a constant which depends only on Ω and T .

Proof: There exist sequences $\{y_{0n}\} \subset H^2(\Omega)$, $\{y_{1n}\} \subset H_0^1(\Omega)$, and $\{k_n\} \subset C^1[0, T] \cap U_M$ such that

$$\begin{aligned} y_{0n} &\longrightarrow v_0, \text{ strongly in } H_0^1(\Omega) \\ y_{1n} &\longrightarrow v_1, \text{ strongly in } L^2(\Omega) \\ k_n &\longrightarrow k, \text{ strongly in } L^2(Q). \end{aligned} \tag{9.14}$$

Denote the weak solution of (8.1) corresponding to the initial data y_{0n}, y_{1n} with control k_n by y_n , then y_n satisfies the regularity of Lemma 9.2. Multiply the PDE (8.1) by $(y_n)_t$ and integrating over $Q_t = \Omega \times (0, t)$ with $0 \leq t \leq T$, we obtain

$$\begin{aligned} 0 &= \int_{Q_t} \left[(y_n)_{tt}(y_n)_t + k_n(t)((y_n)_t)^2 - (\Delta y_n)(y_n)_t + q y_n (y_n)_t - f(y_n)_t \right] dx d\tau \\ &= \int_{Q_t} \left[\frac{1}{2} \frac{d}{dt} ((y_n)_t)^2 + |\nabla y_n|^2 + k_n(t)((y_n)_t)^2 + q y_n (y_n)_t - f(y_n)_t \right] dx d\tau \\ &\quad - \int_{\partial\Omega \times (0,t)} \sum_{i=1}^n (y_n)_{x_i} (y_n)_t n_i ds d\tau, \end{aligned}$$

where ∇y_n denotes the spatial gradient of y_n . Since $y_n = 0$, $(y_n)_t = 0$ in $\partial\Omega \times (0, T)$, the boundary term vanishes, and we have

$$\begin{aligned}
0 &= \int_{Q_t} \left\{ \frac{1}{2} \frac{d}{dt} [(y_n)_t^2 + |\nabla y_n|^2] + k_n(t) ((y_n)_t)^2 + q y_n (y_n)_t - f(y_n)_t \right\} dx d\tau \\
&= \int_{\Omega \times \{t\}} \frac{1}{2} [(y_n)_t^2 + |\nabla y_n|^2] dx - \int_{\Omega} \frac{1}{2} [(y_{1n})^2 + |\nabla y_{0n}|^2] dx \\
&\quad - \int_{Q_t} [-k_n ((y_n)_t)^2 - q y_n (y_n)_t + f(y_n)_t] dx d\tau
\end{aligned} \tag{9.15}$$

and

$$\begin{aligned}
&\int_{\Omega \times \{t\}} [(y_n)_t^2 + |\nabla y_n|^2] dx \\
&= \int_{\Omega} [(y_{1n})^2 + |Dy_{0n}|^2] dx + 2 \int_{Q_t} [-k_n ((y_n)_t)^2 - q y_n (y_n)_t + f(y_n)_t] dx d\tau \\
&\leq \|y_{1n}\|_{L^2(\Omega)}^2 + \|\nabla y_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_t)}^2 + \|(y_n)_t\|_{L^2(Q_t)} \\
&\quad + M_1 (\|y_n\|_{L^2(Q_t)} + \|(y_n)_t\|_{L^2(Q_t)})
\end{aligned} \tag{9.16}$$

where constant M_1 depends only on M and q_0 . Using Poincaré's Inequality

$$\int_{\Omega \times \{t\}} |\nabla y_n(x, t)|^2 dx \geq C_1 \int_{\Omega \times \{t\}} y_n^2(x, t) dx$$

where C_1 is some constant independent of y_n , we obtain

$$\begin{aligned}
& \int_{\Omega \times \{t\}} [((y_n)_t)^2 + y_n^2] dx \\
\leq & C_2 \int_{Q_t} [((y_n)_t)^2 + y_n^2] dx d\tau \\
& + C_2 (\|y_{1n}\|_{L^2(\Omega)}^2 + \|\nabla y_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2)
\end{aligned} \tag{9.17}$$

where C_2 is independent of y_n .

Let

$$\begin{aligned}
\eta(t) &= \int_{\Omega \times \{t\}} [y_n^2 + ((y_n)_t)^2] dx \\
\xi &= C_2 (\|y_{1n}\|_{L^2(\Omega)}^2 + \|\nabla y_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2) .
\end{aligned}$$

Then the inequality (9.17) becomes

$$\eta(t) \leq \xi + C_2 \int_0^t \eta(t) dt. \tag{9.18}$$

Using Gronwall's inequality, we obtain

$$\eta(t) \leq \xi(1 + C_2 T e^{C_2 T}),$$

i.e.,

$$\begin{aligned} & \int_{\Omega \times \{t\}} [y_n^2 + ((y_n)_t)^2] dx \\ & \leq (1 + C_2 T e^{C_2 T}) C_2 (\|y_{1n}\|_{L^2(\Omega)}^2 + \|\nabla y_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_T)}^2). \end{aligned} \quad (9.19)$$

Combining (9.16) and (9.19) and letting $n \rightarrow \infty$, we get at time t

$$\begin{aligned} & \|y_t(t)\|_{L^2(\Omega)}^2 + \|\nabla y(t)\|_{L^2(\Omega)}^2 + \|y(t)\|_{L^2(\Omega)}^2 \\ & \leq C_3 (\|y_1\|_{L^2(\Omega)}^2 + \|\nabla y_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_T)}^2) \end{aligned} \quad (9.20)$$

where C_3 depends only on Ω and T . Taking the supremum, gives

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|y\|_{H_0^1(\Omega)}^2 + \|y_t\|_{L^2(\Omega)}^2) \\ & \leq C_3 (\|y_1\|_{L^2(\Omega)}^2 + \|y_0\|_{H_0^1(\Omega)}^2 + \|f\|_{L^2([0,T],L^2(\Omega))}^2). \end{aligned} \quad (9.21)$$

Next, we are going to estimate y_{tt} . Since

$$\begin{aligned} & \langle y_{tt}, \phi \rangle (t) \\ & = - \int_{\Omega} [\nabla y \nabla \phi + k y_t \phi + q y \phi - f \phi] (t) dx \end{aligned}$$

for any $\phi \in H_0^1(\Omega)$. Let $\|\phi\|_{H_0^1(\Omega)} = 1$, then

$$|\langle y_{tt}, \phi \rangle (t)|$$

$$\leq \|f(t)\|_{L^2(\Omega)}^2 + C_4[\|y_t\|_{L^2(\Omega)}^2 + \|y\|_{H_0^1(\Omega)}^2] \quad (9.22)$$

where C_4 depends on M . Using (9.22), we obtain

$$\begin{aligned} & \|y_{tt}\|_{L^2([0,T],H^{-1}(\Omega))}^2 \\ & \leq C_5 \int_0^T (\|f(t)\|_{L^2(\Omega)}^2 + \|y_t\|_{L^2(\Omega)}^2 + \|y\|_{H_0^1(\Omega)}^2) dt \\ & \leq C_6 T (\|y_1\|_{L^2(\Omega)}^2 + \|v_0\|_{H_0^1(\Omega)}^2 + \|f\|_{L^2([0,T],L^2(\Omega))}^2). \end{aligned}$$

where C_5, C_6 are constants independent of k . $\square$

Now we obtain the existence of an optimal control.

Theorem 9.4 (Existence of Optimal Control) *There exists an optimal control $k^* \in U_M$, which minimizes the objective functional $J(k)$ for $k \in U_M$.*

Proof: Let $\{k^n\} \in U_M$ be a minimizing sequence in U_M such that

$$\lim_{n \rightarrow \infty} J(k^n) = \inf_{k \in U_M} J(k). \quad (9.23)$$

Denote $y^n = y(k^n)$. By Lemma 9.3 we have

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|y^n\|_{H_0^1(\Omega)} + \|(y^n)_t\|_{L^2(\Omega)} + \|(y^n)_{tt}\|_{L^2([0,T];H^{-1}(\Omega))}) \\ & \leq C(\|f\|_{L^2([0,T];L^2(\Omega))} + \|v_0\|_{H_0^1(\Omega)} + \|v_1\|_{L^2(\Omega)}). \end{aligned}$$

On a subsequence, by weak compactness, there exists y^* in $C([0, T]; H_0^1(\Omega))$ such that

$$\begin{aligned} y^n &\rightharpoonup y^* \text{ weak } * \text{ in } L^\infty([0, T]; H_0^1(\Omega)), \\ (y^n)_t &\rightharpoonup (y^*)_t \text{ weak } * \text{ in } L^\infty([0, T]; L^2(\Omega)), \\ (y^n)_{tt} &\rightharpoonup (y^*)_{tt} \text{ weakly in } L^2([0, T]; H^{-1}(\Omega)) \end{aligned} \quad (9.24)$$

and

$$k^n \rightharpoonup k^* \text{ weakly in } L^2(\Omega). \quad (9.25)$$

By the definition of weak solution, we have

$$\int_0^T \langle y_{tt}^n, \phi \rangle = - \int_Q (\nabla y^n \nabla \phi + k^n ((y^n)_t) \phi + q y^n \phi - f \phi) dx dt. \quad (9.26)$$

In the convergence as $n \rightarrow \infty$, the only difficult term is the term with k^n on the RHS of (9.26). We now show convergence of the RHS. Define the sequence of function $v^n(t)$ by

$$v^n(t) = \int_\Omega y_t^n(x, t) \phi(x) dx.$$

We have

$$\int_Q k^n ((y^n)_t) \phi dx dt = \int_0^T k^n(t) v^n(t) dt.$$

using the weak convergence of $(y^n)_t$ on space and the continuity of y_t^n in time into $L^2(\Omega)$

, for each fixed t ,

$$v^n(t) \rightarrow v(t) = \int_{\Omega} y_t^*(x, t) \phi(x) dx \text{ pointwise as } n \rightarrow \infty,$$

By Egorof's Theorem [31], for any $\varepsilon > 0$, there exists a set $E \subset [0, T]$ such that $m(E) < \varepsilon$ and

$$v^n(t) \rightarrow v(t) \text{ uniformly on } [0, T] \setminus E.$$

Then we obtain

$$\int_0^T |k^n v^n - k^* v| dt \leq \int_0^T |(k^n v^n - k^* v) \chi_E| dt + \int_0^T |(k^n v^n - k^* v) \chi_{[0, T] \setminus E}| dt.$$

The integral term on $[0, T] \setminus E$ approaches 0 as $n \rightarrow \infty$ by the uniform convergence of $v^n \rightarrow v$ on $[0, T] \setminus E$. The integral term on E can be estimated,

$$\begin{aligned} \int_0^T |(k^n v^n - k^* v) \chi_E| dt &\leq M \int_0^T (|v^n| + |v|) \chi_E dt \\ &\leq M (\|v^n\|_{L^2(0, T)} + \|v\|_{L^2(0, T)}) m(E), \end{aligned}$$

which is small due to L^2 bounds on v^n and $m(E) < \varepsilon$. Hence we obtain the desired convergence:

$$\int_Q k^n y_t^n \phi dQ \rightarrow \int_Q k^* y_t^* \phi dQ.$$

We can pass to the limit in the y^n PDE and obtain $y^* = y(k^*)$, solution of (8.1). Thus

$y^* = y(k^*)$ is the solution of state equation (8.1) with control k^* . Since

$$J(k^*) = \frac{1}{2} \int_Q (y^* - z)^2 dQ + \frac{\beta}{2} \int_Q (k^*)^2 dQ,$$

using lower-semicontinuity of L^2 norm with respect to weak convergence, we have

$$\begin{aligned} J(k^*) &\leq \lim_{n \rightarrow \infty} \frac{1}{2} \int_Q (y^n - z)^2 dQ + \liminf_{n \rightarrow \infty} \frac{\beta}{2} \int_Q (k^n)^2 dQ \\ &\leq \liminf_{n \rightarrow \infty} J(k^n) \\ &= \inf_{k \in U_M} J(k). \end{aligned}$$

Finally, we conclude k^* is an optimal control. □

Chapter 10

Characterization of the Optimal Control

We now derive the optimality system by differentiating the objective functional $J(k)$ with respect to the control k . Since $y = y(h)$ is involved in $J(k)$, we first must prove appropriate differentiability of the mapping

$$k \rightarrow (y(k), y_t(k)) = \tilde{y}(k).$$

Lemma 10.1 *The mapping*

$$k \in U_M \longrightarrow \tilde{y}(k) = (y(k), y_t(k)) \in L^2([0, T]; \mathcal{H})$$

is differentiable in the following sense

$$\frac{\tilde{y}(k + \varepsilon \ell) - \tilde{y}(k)}{\varepsilon} \rightharpoonup \tilde{\Psi}$$

weakly in $L^2([0, T]; \mathcal{H})$ as $\varepsilon \rightarrow 0$, for any k satisfying $k + \varepsilon \ell \in U_M$ for ε small and $\ell \in L^\infty(0, T)$. Moreover $\tilde{\Psi} = (\Psi, \Psi_t)$ in $L^2([0, T]; \mathcal{H})$ is a weak solution of the following problem:

$$\begin{aligned} \Psi_{tt} &= \Delta \Psi - k \Psi_t - q \Psi - \ell y_t, \quad \text{in } Q \\ \Psi &= 0 \quad \partial \Omega \times (0, T) \\ \Psi &= 0 \quad t = 0, x \in \Omega \\ \Psi_t &= 0 \quad t = 0, x \in \Omega \end{aligned} \tag{10.1}$$

where $y = y(k)$.

Proof: Denote $y^\varepsilon = y(k + \varepsilon \ell)$ and $y = y(k)$, then $(\frac{y^\varepsilon - y}{\varepsilon})$ is a weak solution of

$$\begin{aligned} \left(\frac{y^\varepsilon - y}{\varepsilon}\right)_{tt} &= \Delta\left(\frac{y^\varepsilon - y}{\varepsilon}\right) - k\left(\frac{y^\varepsilon - y}{\varepsilon}\right)_t - q\left(\frac{y^\varepsilon - y}{\varepsilon}\right) + \ell y_t^\varepsilon \quad \text{in } Q \\ \frac{y^\varepsilon - y}{\varepsilon} &= 0 \quad \text{in } \partial \Omega \times (0, T) \\ \frac{y^\varepsilon - y}{\varepsilon} &= 0 \quad t = 0, x \in \Omega \\ \left(\frac{y^\varepsilon - y}{\varepsilon}\right)_t &= 0 \quad t = 0, x \in \Omega \end{aligned}$$

Using Theorem 9.3 , we get

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left\| \frac{(y^\varepsilon - y)(t)}{\varepsilon} \right\|_{H_0^1(\Omega)} + \left\| \frac{y_{tt}^\varepsilon - y_{tt}}{\varepsilon} \right\|_{L^2([0,T]; H^{-1}(\Omega))} \\
& \leq C \| \ell y_t^\varepsilon \|_{L^2([0,T]; L^2(\Omega))} \\
& \leq C \| \ell \|_{L^\infty} \sup_{0 \leq t \leq T} \| y_t^\varepsilon \|_{L^2(\Omega)}. \\
& \leq C_2 \| \ell \|_{L^\infty}
\end{aligned}$$

where C depends on the L^∞ bound on ℓ , but is independent of ε . Hence on a subsequence, by weak compactness, we have

$$\begin{aligned}
\frac{y^\varepsilon - y}{\varepsilon} & \rightharpoonup \Psi \text{ weak }^* \text{ in } L^\infty([0, T]; H_0^1(\Omega)), \\
\frac{y_{tt}^\varepsilon - y_{tt}}{\varepsilon} & \rightharpoonup \Psi_{tt} \text{ weak }^* \text{ in } L^\infty([0, T]; H^{-1}(\Omega))
\end{aligned}$$

By the definition of weak solution, we have

$$\begin{aligned}
& \left\langle \frac{y^\varepsilon - y}{\varepsilon}, \phi \right\rangle_{tt} + \int_{\Omega} \nabla \left(\frac{y^\varepsilon - y}{\varepsilon} \right) \nabla \phi dx \\
& = - \int_{\Omega} k \left(\frac{y^\varepsilon - y}{\varepsilon} \right)_t \phi dx - \int_{\Omega} q \left(\frac{y^\varepsilon - y}{\varepsilon} \right) \phi dx - \int_{\Omega} \ell y_t^\varepsilon \phi dx
\end{aligned}$$

for any $\phi \in H_0^1(\Omega)$, and a.e. $0 \leq t \leq T$.

Letting $\varepsilon \rightarrow 0$ and using that $y_t^\varepsilon \rightarrow y_t$ weakly in $L^2(\Omega)$, we obtain Ψ is the weak solution of problem (10.1). $\square$

Now we are ready to derive the necessary conditions that characterize an optimal control. Note that the adjoint problem is unusual due to $(kp)_t$ term.

Theorem 10.1 *Given an optimal control k^* and corresponding $\tilde{y}^* = \tilde{y}(k^*) = (y^*, y_t^*)$, there exists a weak solution $\tilde{p} = (p, p_t)$ in $L^\infty(0, T; \mathcal{H})$ to the adjoint problem:*

$$\begin{aligned} p_{tt} &= \Delta p - (kp)_t - qp + y^* - z, \quad \text{in } Q \\ p &= 0, \quad \partial\Omega \times (0, T) \\ p &= p_t = 0, \quad t = T, x \in \Omega \end{aligned} \tag{10.2}$$

(transversality condition)

Furthermore, k satisfies

$$k^*(t) = \max \left(-M, \min \left(\frac{1}{\beta} \int_{\Omega} (py_t^*)(t, x) dx, M \right) \right) \tag{10.3}$$

Proof: Let $k^* \in U_M$ be an optimal control and $\tilde{y}^* = \tilde{y}(k^*)$ be the corresponding optimal state solution. Let $k^* + \varepsilon \ell \in U_M$ for $\varepsilon > 0$ and $\tilde{y}^\varepsilon = \tilde{y}(k^* + \varepsilon \ell)$ be the corresponding weak solution of the equation (8.1). We compute the directional derivative of the objective functional $J(k)$ with respect to k in the direction of ℓ . Since $J(k)$ is a minimal value,

$$0 \leq \lim_{\varepsilon \rightarrow 0^+} \frac{J(k^* + \varepsilon \ell) - J(k^*)}{\varepsilon}$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0^+} \left\{ \frac{1}{2\varepsilon} \int_Q [(y^\varepsilon - z)^2 - (y^* - z)^2] dx dt + \frac{\beta}{2\varepsilon} \int_0^T [(k^* + \varepsilon \ell)^2 - (k^*)^2] dt \right\} \\
&= \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_Q \frac{(y^\varepsilon - y^*)(y^\varepsilon + y^* - 2z)}{2\varepsilon} dx dt + \frac{\beta}{2} \int_0^T (2k^* \ell + \varepsilon \ell^2) dt \right\} \\
&= \int_Q \Psi(y^* - z) dx dt + \beta \int_0^T k^* \ell dt
\end{aligned} \tag{10.4}$$

where Ψ is defined in Lemma 10.1.

We need to prove the existence of the solution to the above adjoint equation. This proof differs from the existence of solution of the state equation due to the $(kp)_t$ term with k not necessarily weakly differentiable and the source term. In the system formulation using the notation from Theorem 2.1, the adjoint equation becomes

$$\frac{d}{dt} \tilde{p}(t) = A \tilde{p}(t) + \begin{pmatrix} 0 \\ -(kp)_t - qp + y - z \end{pmatrix}, \tag{10.5}$$

where A generates a unitary group on $\mathcal{H}$.

We "formally" seek a solution of the form

$$\tilde{p}(t) = \int_t^T e^{A(t-s)} \begin{bmatrix} 0 \\ -(kp)_t - qp + y - z \end{bmatrix} ds,$$

To make sense of the $(kp)_t$, we integrate by parts,

$$\tilde{p} = \int_t^T \left(-A e^{A(t-s)} \begin{bmatrix} 0 \\ kp \end{bmatrix} + e^{A(t-s)} \begin{bmatrix} 0 \\ -qp + y - z \end{bmatrix} \right) ds + \begin{bmatrix} 0 \\ kp(t) \end{bmatrix}.$$

To find a solution $\tilde{p}$ satisfying (10.5), we prove the existence of a unique fixed point in $L^\infty([0, T_0]; \mathcal{H})$ of the map

$$F\tilde{p}(t) = \int_t^T \left(-Ae^{A(t-s)} \begin{bmatrix} 0 \\ kp(s) \end{bmatrix} + e^{A(t-s)} \begin{bmatrix} 0 \\ -qp + y - z \end{bmatrix} \right) ds + \begin{bmatrix} 0 \\ kp(t) \end{bmatrix}. \quad (10.6)$$

One first step is to prove the fixed point result on $[T - T_0, T]$ for T_0 sufficiently small.

We show

$$F : L^\infty([T - T_0, T], \mathcal{H}) \rightarrow L^\infty([T - T_0, T], \mathcal{H})$$

is bounded and contractive.

To show boundedness and contractive, consider the $kp(t)$ term,

$$\begin{aligned} & \left\| \begin{bmatrix} 0 \\ kp(t) \end{bmatrix} \right\|_{L^\infty([T-T_0, T], \mathcal{H})} \\ &= \|kp\|_{L^\infty([T-T_0, T], L^2(\Omega))} \\ &\leq M \sup_t \left(\int_\Omega (p(x, t))^2 dx \right)^{\frac{1}{2}} \\ &= M \sup_t \left(\int_\Omega \left(- \int_t^T p_t(x, s) ds \right)^2 dx \right)^{\frac{1}{2}} \\ &\leq M \sup_t \left(\int_\Omega T_0 \int_t^T p_t^2(x, s) ds dx \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq MT_0^{\frac{1}{2}} \left(\int_{T-T_0}^T \int_{\Omega} p_t^2(x, s) ds dx \right)^{\frac{1}{2}} \\
&\leq MT_0 \|p_t\|_{L^\infty([T-T_0, T], L^2(\Omega))},
\end{aligned}$$

where "sup" means essential supremum for L^∞ . To complete the boundedness property, using the property that e^{At} and A commute [28], we obtain

$$\begin{aligned}
&\|F\tilde{p}\|_{L^\infty([T-T_0, T], \mathcal{H})} \\
&\leq MT_0 \|p_t\|_{L^\infty([T-T_0, T], L^2(\Omega))} + \sup_t \int_t^T \left\| A \begin{pmatrix} 0 \\ kp(t) \end{pmatrix} \right\|_{\mathcal{H}} ds \\
&\quad + T_0 \| -qp + y - z \|_{L^\infty([T-T_0, T], L^2(\Omega))} \\
&\leq \sup_t \int_t^T \|kp\|_{H^1}(s) ds + MT_0 \|p_t\|_{L^\infty([T-T_0, T], L^2(\Omega))} \\
&\quad + T_0 \| -qp + y - z \|_{L^\infty([T-T_0, T], L^2(\Omega))}. \\
&\leq 2MT_0 C_1 \|\tilde{p}\|_{L^\infty([T-T_0, T], \mathcal{H})} + T_0 \| -qp + y - z \|_{L^\infty([T-T_0, T], \mathcal{H})} \\
&\leq 2MT_0 C_1 \|\tilde{p}\|_{L^\infty([T-T_0, T], \mathcal{H})} + T_0 C_2 \|\tilde{p}\|_{L^\infty([T-T_0, T], \mathcal{H})} \\
&\quad + T_0 \|y - z\|_{L^\infty([T-T_0, T], \mathcal{H})} \\
&= (2MC_1 + C_2)T_0 \|\tilde{p}\|_{L^\infty([T-T_0, T], \mathcal{H})} + T_0 \|y - z\|_{L^\infty([T-T_0, T], \mathcal{H})}.
\end{aligned}$$

For contraction property, we have

$$\|F\tilde{p}_1 - F\tilde{p}_2\|_{L^\infty([T-T_0, T], \mathcal{H})} \leq (2MC_1 + C_2)T_0 \|\tilde{p}_1 - \tilde{p}_2\|_{L^\infty([T-T_0, T], \mathcal{H})},$$

where C_1, C_2 and C_3 are constants depending only on the coefficients of state equation. The above inequality gives the contraction for T_0 small. Thus we obtain our unique fixed point on $[T - T_0, T]$ and then apply the argument on $[T - 2T_0, T - T_0]$. Continuing gives the existence of a unique solution to the adjoint equation.

Then substituting the adjoint problem into (10.4), we obtain

$$\begin{aligned} 0 &\leq \int_Q \Psi(y^* - z) dx dt + \beta \int_0^T k^* \ell dt \\ &\leq \int_0^T \left\{ \langle \Psi_{tt}, p \rangle + \int_{\Omega} \nabla p \nabla \Psi dx + \int_{\Omega} k^* p(\Psi)_t dx + \int_{\Omega} qp \Psi dx \right\} dt \\ &\quad + \beta \int_0^T k^* \ell dt \end{aligned}$$

by using $\Psi = \Psi_t = 0$ on $t = 0$ and $p = p_t = 0$ on $t = T$. Then from equation (10.1) for Ψ , we obtain

$$0 \leq \int_0^T \ell \left(\beta k^* - \int_{\Omega} p y_t^* dx \right) dt.$$

Note $\ell = \ell(x, t)$ is an arbitrary function with $k^* + \varepsilon \ell \in U_M$ for small ε .

Considering three cases involving the sign of the variation ℓ depending on the size of k^* , we obtain the desired characterization of k^* .

Case 1:

Let $A_1 = \{(x, t) \in Q \mid 0 < k^*(x, t) < M\}$. Taking support of $\ell(x, t) \subset A_1$ with

arbitrary sign, we obtain

$$\beta k^* - \int_{\Omega} p y_t^* dx = 0, \text{ in } A_1$$

So in this case, we conclude

$$-M < k^* = \frac{1}{\beta} \int_{\Omega} p y_t^* dx < M.$$

Case 2:

Let $A_2 = \{(x, t) \in Q \mid k^*(x, t) = M\}$. Taking support of $\ell(x, t) \subset A_2$ and $\ell(x, t) \leq 0$, we obtain

$$\beta k^* - \int_{\Omega} p y_t^* dx \leq 0,$$

So in this case, we conclude

$$M = k^* \leq \frac{1}{\beta} \int_{\Omega} p^* y_t dx.$$

Case 3;

Let $A_3 = \{(x, t) \in Q \mid k^*(x, t) = 0\}$. Taking support of $\ell(x, t) \subset A_3$ and $\ell(x, t) \geq 0$, then we have

$$0 = k^* \geq \frac{1}{\beta} \int_{\Omega} p y_t^* dx.$$

Combinning the above three cases, we conclude

$$k^* = \max \left(-M, \min \left(\frac{1}{\beta} \int_{\Omega} p y_t^* dx, M \right) \right) \quad \square$$

Substituting (10.3) for k into the state equation (8.1) and the adjoint equation (10.2), we obtain the optimality system (OS)

$$\begin{aligned} y_{tt} &= \Delta y - \max \left(-M, \min \left(\frac{1}{\beta} \int_{\Omega} p y_t dx, M \right) \right) y_t - q(x)y + f \text{ in } Q \\ p_{tt} &= \Delta p - \left[\max \left(-M, \min \left(\frac{1}{\beta} \int_{\Omega} p y_t dx, M \right) \right) p \right]_t + y - z \text{ in } Q \\ y &= p = 0 & \partial\Omega \times (0, T) \\ y &= 0 & t = 0, x \in \Omega \\ p &= p_t = 0 & t = T, x \in \Omega \end{aligned} \tag{10.7}$$

The weak solutions of the optimality system exist by Theorems 9.1 and 10.1; The existence of an optimal control k^* and corresponding y^* implies the existence of p .

Note we have derived necessary conditions for an optimal control. The uniqueness of the optimal control is not known at this time.

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