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# Finite Topological Spaces

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I am submitting herewith a thesis written by Jimmy Edward Miller entitled "Finite Topological Spaces." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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# Finite Topological Spaces

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Jimmy Edward Miller

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*I dedicate this paper to my fiance Jillian.*

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I would like to thank God, Jillian, Dr. Conant, Jessie, and my parents for their support and guidance.

*Go down deep enough into anything and you will find mathematics. Dean Schlicter*

# Abstract

A thesis on some fundamental aspects of Algebraic Topology on Finite Topological Spaces, specifically 4 point spaces.

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# Chapter 1

# Introduction

We begin by giving a brief overview of the fundamentals of point-set Topology. A basic understanding of set arithmetic will be assumed. These fundamentals will provide an adequate foundation to understand the topics in Chapters 4 and 5.

**Definition 1.0.1.** *A* **topology** *on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ having the following properties:*

*(i)* $\emptyset, X \in \mathcal{T}$

*(ii) Given a collection* $\{A_\alpha\}_{\alpha\in\Omega}$ *where* $A_\alpha \in \mathcal{T}$*, then* $\bigcup_{\alpha\in\Omega} A_\alpha \in \mathcal{T}$.

*(iii) Given a finite collection* $\{A_i\}_{i=1}^n$ *where* $A_i \in \mathcal{T}$*, then* $\bigcap_{i=1}^{n} A_i \in \mathcal{T}$.

Munkres (2000)

Elements of $\mathcal{T}$ are called **open** sets. The complement of an open set is called a **closed** set. Sets that are both open and closed are called **clopen** sets.

**Definition 1.0.2.** *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The* ***subspace topology*** *of $A$ is the collection $\mathcal{T}_A = \{A \cap U \text{ where } U \in \mathcal{T}\}$.*

Munkres (2000)

Let's verify that $\mathcal{T}_A$ is in fact a topology.

(i) $A = A \cap X \in \mathcal{T}_A$ and $\emptyset = A \cap \emptyset \in \mathcal{T}_A$ since $X, \emptyset \in \mathcal{T}$

(ii) Let $\{W_\alpha\}_{\alpha\in\Omega} \subseteq \mathcal{T}_A$. Then $W_\alpha = A \cap U_\alpha$ where $U_\alpha \in \mathcal{T}$. Then $\bigcup_{\alpha\in\Omega} W_\alpha = \bigcup_{\alpha\in\Omega}(A \cap U_\alpha) = A \cap (\bigcup_{\alpha\in\Omega} U_\alpha) \in \mathcal{T}_A$

(iii) Let $\{W_i\}_{i=1}^n \in \mathcal{T}_A$. Then $W_i = A \cap U_i$ where $U_i \in \mathcal{T}$. Then $\bigcap_{i=1}^n W_i = \bigcap_{i=1}^n (A \cap U_i) = A \cap (\bigcap_{i=1}^n U_i) \in \mathcal{T}_A$.

Hence by Definition 1.0.1, $\mathcal{T}_A$ is a topology.

**Definition 1.0.3.** *The **discrete topology** on a set $X$ is the topology that includes all subsets of $X$, i.e the power set of $X$. The **coarse topology** on $X$ is the topology consisting of only $X$ and $\emptyset$. The coarse topology is also called the trivial topology.*

**Definition 1.0.4.** *Let $X$ be a set. A **basis** for a topology on $X$ is a collection of subsets $\mathcal{B}$ of $X$ called basis elements such that the following properties hold:*

*(i) For every $x \in X$ there exists some $B \in \mathcal{B}$ containing $x$.*

*(ii) If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there exists some $B_3 \in \mathcal{B}$ where $x \in B_3 \subseteq B_1 \cap B_2$.*

Munkres (2000)

If such a collection $\mathcal{B}$ satisfies these two properties, we can define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A set $U$ is open in $X$ if for every $x \in U$ there exists some $B \in \mathcal{B}$ where $x \in B \subseteq U$.

Let's verify that $\mathcal{T}$ is a topology on $X$.

First, for any $x \in X$ we know by (i) that there exists some $B \in \mathcal{B}$ containing $x$; and since $B \subseteq X$ be definition, we know that $X$ is open. Also, $\emptyset$ is open vacuously.

Let $\{U_\alpha\}_{\alpha\in\Omega}$ be open in $X$. Let $U = \bigcup_{\alpha\in\Omega} U_\alpha$. Suppose that $x \in U$. Then $x \in U_\beta$ for some $\beta \in \Omega$. Since $U_\beta$ is open we know there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U_\beta \subseteq U$. Hence $U$ is open.

Let $\{U_i\}_{i=1}^n$ be open in $X$. Let $V = \bigcap_{i=1}^n U_i$. Suppose that $x \in V$. Then $x \in U_i$ for all $i$. Hence there exists $B \in \mathcal{B}$ where $x \in B \in U_i$ for all $i$. Which implies that $B \subseteq V$, hence $V$ is open. Therefore $\mathcal{T}$ is a topology.

**Corollary 1.0.1.** *Every open set is a union of basis elements*

*Proof.* Let $(X, \mathcal{T})$ be a topological space where $\mathcal{B}$ is a basis for $\mathcal{T}$. Let $U \subseteq X$ be open. Then for every $x \in U$ there exists $B_x \in \mathcal{B}$ where $x \in B_x \subseteq U$. Hence $\bigcup_{x \in U} B_x = U$. $\square$

**Lemma 1.0.1.** *A set $U \subseteq X$ is open if and only if for every $x \in U$ there exists an open $V_x \subseteq X$ where $x \in V_x \subseteq U$.*

*Proof.* Let $U \subseteq X$ and $x \in U$. Suppose there exists an open $V_x \subseteq X$ where $x \in V_x \subseteq U$. It is clear that $\bigcup_{x \in U} V_x \subseteq U$ since $V_x \subseteq U$ for all $x \in X$. Also, for any $x \in U$, $x \in V_x$, hence $x \in \bigcup_{x \in U} V_x$. Therefore $U = \bigcup_{x \in U} V_x$. Hence $U$ is a union of open sets, hence open. $\square$

**Definition 1.0.5.** *Let $X$ and $Y$ be sets. Define $X \sqcup Y$ to be the disjoint union of $X$ and $Y$, i.e $X \cap Y = \emptyset$.*

**Definition 1.0.6.** *Let $(X, \mathcal{T})$ and $(Y, \Gamma)$ be topological spaces. Then the topology of $X \sqcup Y$ consists of sets of the form $U \sqcup V$ where $U$ is open in $X$ and $V$ is open in $Y$.*

# Chapter 2

# Continuous Functions and Separation Axioms

Continuous functions play a major part in Topology. We will use continuous functions and their properties extensively in Chapter 5 to prove properties of some finite topological spaces. Separation axioms give a method of classifying topological spaces given certain characteristics.

**Definition 2.0.7.** *Let $(X, \mathcal{T})$ and $(Y, \Gamma)$ be topological spaces.*

*A function $f : X \to Y$ is called **continuous** if for every open $U \subseteq Y$, $f^{-1}(U)$ is open in $X$.*

Munkres (2000)

**Lemma 2.0.2.** *Let $(X, \mathcal{T})$ and $(Y, \Gamma)$ be topological spaces.*

*$f : X \to Y$ is continuous if and only if given open $U \subseteq Y$, for every $f(x) \in U$ there exists open $V \subseteq X$ containing $x$ where $f(V) \subseteq U$.*

*Proof.* ($\Rightarrow$) Let $U \subseteq Y$ be open and $f(x) \in U$. Since $f$ is continuous $f^{-1}(U)$ is open. Hence there exists an open neighborhood $V$ such that $x \in V \subseteq f^{-1}(U)$, i.e $f(V) \subseteq U$.

($\Leftarrow$) Let $U \subseteq Y$ be open. We know that for every $f(x) \in U$ there exists open $V$ containing $x$ such that $f(V) \subseteq U$, i.e $x \in V \subseteq f^{-1}(U)$. Hence $f^{-1}(U)$ is open. Therefore $f$ is continuous. □

**Lemma 2.0.3.** *Let $(X, \mathcal{T})$ and $(Y, \Gamma)$ be topological spaces. A function $f : X \to Y$ is continuous if and only if for closed $C \subseteq Y$, $f^{-1}(C)$ is closed in $X$*

*Proof.* ($\Rightarrow$) Let $C \subseteq Y$ be closed. Then $Y \backslash C$ is open. Hence $f^{-1}(Y \backslash C) = f^{-1}(Y) \backslash f^{-1}(C) = X \backslash f^{-1}(C)$ is open. Therefore $f^{-1}(C)$ is closed.

($\Leftarrow$) Let $U \subseteq Y$ be open. Then $Y \backslash U$ is closed. Hence $f^{-1}(Y \backslash U) = f^{-1}(Y) \backslash f^{-1}(U) = X \backslash f^{-1}(U)$ is closed. Hence $f^{-1}(U)$ is open. Therefore $f$ is continuous. □

**Definition 2.0.8.** *Let $(X, \mathcal{T})$ and $(Y, \Gamma)$ be topological spaces. A function $f : X \to Y$ is called **open** if for any open set $U \subseteq X$, the image $f(U)$ is open in $Y$.*

**Definition 2.0.9.** *Let $(X, \mathcal{T})$ and $(Y, \Gamma)$ be topological spaces. A function $f : X \to Y$ is called a **homeomorphism** if $f$ is continuous, open, injective, and surjective.*

We will now discuss Separation Axioms. The term "separation" comes from the fact that these axioms describe the ability to separate points or closed sets by disjoint open sets.

**Definition 2.0.10.** *Let $(X, \mathcal{T})$ be a topological space. The following are called Separation Axioms.*

*(i) $X$ is called $T_0$ if for every pair of distinct points $x, y \in X$ there exists an open set $U$ containing $x$ which does not contain $y$.*

*(ii) $X$ is called $T_1$ if one point sets are closed*

*(iii) $X$ is called $T_2$ or Hausdorff if $X$ is $T_1$ and for any pair of distinct points $x, y \in X$ there exist disjoint open sets $U$ and $V$ where $x \in U$ and $y \in V$.*

*(iv) $X$ is called $T_3$ or regular if $X$ is $T_1$ and for any point $x \in X$ and any closed set $C$ not containing $x$ there exist open sets $U$ and $V$ where $x \in U$ and $C \subseteq V$.*

*(v) $X$ is called completely regular if for any point $x \in X$ and closed set $C$ not containing $x$ there exists a continuous function $f : X \to [0, 1]$ where $f(x) = 0$ and $f(C) = 1$.*

*(vi) $X$ is called $T_4$ or normal if $X$ is $T_1$ and for any pair of disjoint closed sets $C$ and $D$ there exist disjoint open sets $U$ and $V$ where $C \subseteq U$ and $D \subseteq V$.*

**Lemma 2.0.4.** $T_4 \Rightarrow$ *completely regular* $\Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$

*Proof.* ($T_4 \Rightarrow$ completely regular). Let $X$ be $T_4$. Let $x \in X$ and $C \subseteq X$ be closed. Since $X$ is $T_1$ $\{x\}$ is closed. Hence by Urysohn's Lemma there exists a continuous function $f : X \to [0, 1]$ where $f(x) = 0$ and $f(C) = 1$. Therefore $X$ is completely regular.

(completely regular $\Rightarrow T_3$). Let $x \in X$ and closed $C \subseteq X$ where $x \notin C$. Since $X$ is completely regular there exists a continuous $f : X \to [0, 1]$ where $f(x) = 0$ and $f(C) = 1$. Note that $(\frac{1}{2}, 1]$ and $[0, \frac{1}{2})$ are disjoint, open in $[0, 1]$, and contain 1 and 0 respectively. Hence $f^{-1}((\frac{1}{2}, 1])$ and $f^{-1}([0, \frac{1}{2}])$ are disjoint, open in $X$, and contain $x$ and $C$ respectively. Hence $X$ is $T_3$.

($T_3 \Rightarrow T_2$). Let distinct $x, y \in X$. Since $X$ is $T_1$ we know that $\{x\}$ is closed. Since $X$ is $T_3$ there exist disjoint open sets $U$ and $V$ where $\{x\} \subseteq U$ and $y \in V$, i.e $x \in U$ and $y \in V$. Therefore $X$ is $T_2$.

($T_2 \Rightarrow T_1$). Let $x \in X$ and $y \in \{x\}^c$. Since $X$ is $T_2$ there exist disjoint open sets $U$ and $V$ where $x \in U$ and $y \in V$. Note that $U$ and $V$ disjoint implies that $V \subseteq \{x\}^c$. Hence $\{x\}^c$ is open, i.e $\{x\}$ is closed. Hence $X$ is $T_1$.

($T_1 \Rightarrow T_0$). Let distinct $x, y \in X$. Since $X$ is $T_1$ we know that $\{x\}$ is closed. Hence $y \in \{x\}^c$ which is open. Therefore $X$ is $T_0$. □

**Definition 2.0.11.** *A topological space having the property that any intersection of open sets is open is called an $\mathcal{A}$ space.*

May (2003b)

Most of the theorems in Chapter 5 will apply to $\mathcal{A}$ spaces as well as finite topological spaces.

**Lemma 2.0.5.** *Every finite topological space is an $\mathcal{A}$ space.*

*Proof.* If $X$ is a finite topological space then any intersection of open sets of $X$ must be a finite intersection. Hence by (iii) $X$ is an $\mathcal{A}$ space. □

An interesting consequence of a topological space being finite is that for each point there exists a smallest open set containing that point.

**Definition 2.0.12.** *Let $X$ be an $\mathcal{A}$ space. For each $x \in X$ define $U_x$ to be the intersection of all open sets containing $x$.*

*Let $x, y \in X$. Define a relation $\leq$ by $x \leq y$ if and only if $x \in U_y$, i.e $U_x \subseteq U_y$.*

May (2003b)

**Lemma 2.0.6.** *$\leq$ is reflexive and transitive*

*Proof.* reflexive: clear. transitive: Let $x, y, z \in X$. Assume that $x \leq y$ and $y \leq z$. Then $U_x \subseteq U_y$ and $U_y \subseteq U_z$. Hence $U_x \subseteq U_z$. Therefore $x \leq z$. □

**Theorem 2.1.** *Let $(X, \mathcal{T})$ and $(Y, \Gamma)$ be topological spaces.*

*A function $f : X \to Y$ is continuous if and only if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in X$.*

May (2003b)

*Proof.* ($\Rightarrow$) Let $x, y \in X$ where $x \leq y$. Then by definition $x \in U_y \subseteq f^{-1}(U_{f(y)})$. Hence $f(x) \in f(f^{-1}(U_{f(y)}) \subseteq U_{f(y)}$, which gives $f(x) \leq f(y)$.

($\Leftarrow$) Let $V \subseteq Y$ be open. Claim: $f^{-1}(V) = \bigcup_{y \in f^{-1}(V)} U_y$.

$\subseteq$ Let $x \in f^{-1}(V)$. By definition $x \in U_x$. Hence $x \in \bigcup_{y \in f^{-1}(V)} U_y$.

$\supseteq$ Let $x \in \bigcup_{y \in f^{-1}(V)} U_y$. Then $x \in U_y$ for some $y \in f^{-1}(V)$, i.e for some $f(y) \in V$. Therefore $x \leq y$, hence $f(x) \leq f(y)$. Hence $f(x) \in U_{f(y)} \subseteq V$. Hence $x \in f^{-1}(V)$.

Therefore $f^{-1}(V)$ is open being a union of open sets, hence $f$ is continuous. □

We can see that there are several equivalent conditions and definitions of continuity. This can be very useful given the context in which we are working. Definition 2.0.7 is considered to be the primary definition of continuity. It will be used extensively in Chapter 5.

# Chapter 3

# Connected and Path Connected Spaces

Connectedness and Path Connectedness are characteristics that describe the structure of a topological space. Informally, a connected space is in one "piece" while a disconnected space may be broken into several pieces. In Chapter 5 we will break disconnected spaces into their "pieces" and classify each piece, therefore classifying the entire space.

**Definition 3.0.1.** *Let $(X, \mathcal{T})$ be a topological space. A **separation** of $X$ is a nonempty disjoint pair of open sets $\{U, V\}$ where $X = U \cup V$. We call $X$ **connected** if no such separation exists for $X$. If a separation exists we call $X$ disconnected.*

**Lemma 3.0.1.** *The elements of a separation are clopen sets.*

*Proof.* Let $(X, \mathcal{T})$ be a topological space and $\{U, V\}$ be a separation of $X$. Since $X = U \cup V$ and $U \cap V = \emptyset$, $U = V^c$ hence $U$ is closed. Likewise $V = U^c$ hence $V$ is closed. Therefore $U$ and $V$ are both clopen. □

**Corollary 3.0.1.** *$X$ is connected if and only if the only clopen sets of $X$ are $X$ and $\emptyset$.*

*Proof.* ($\Rightarrow$) Assume there exists clopen $A \subseteq X$ where $A \neq X$ and $A \neq \emptyset$. Since $A$ is clopen and non-empty, $A^c$ is also clopen and non-empty. Which makes $\{A, A^c\}$ a separation of $X$. $\Rightarrow\Leftarrow$

($\Leftarrow$) Assume to the contrary that $\{U, V\}$ forms a separation of $X$. But then by Lemma 3.0.1 $U$ is clopen a set of $X$. $\Rightarrow\Leftarrow$. □

**Lemma 3.0.2.** *If $(X, \mathcal{T})$ is a disconnected topological space with separation $\{U, V\}$, and $A \subseteq X$ is connected, then $A \subseteq U$ or $A \subseteq V$.*

*Proof.* Note that $A \cap U$ and $A \cap V$ are disjoint and open in the subspace topology of $A$. One of these sets must be empty, otherwise $\{A \cap U, A \cap V\}$ would constitute a separation of $A$. WLOG assume that $A \cap U = \emptyset$. Then $A \subseteq A \cap V$, hence $A \subseteq V$. □

**Theorem 3.1.** *Let $(X, \mathcal{T})$ and $(Y, \Gamma)$ be topological spaces where $X$ is connected. If $f : X \to Y$ is continuous then $f(X)$ is connected.*

*Proof.* Suppose to the contrary that $\{U, V\}$ is a separation of $f(X) = Z$. Then $U$ and $V$ are each open in the subspace topology of $Z$. Hence $U = Z \cap U_z$ and $V = Z \cap V_z$ where $U_z$ and $V_z$ are open in $Y$. Then $f^{-1}(Z \cap U_z) = f^{-1}(Z) \cap f^{-1}(U_z) = f^{-1}(U_z)$ and $f^{-1}(V) = f^{-1}(Z \cap V_z) = f^{-1}(Z) \cap f^{-1}(V_z) = f^{-1}(V_z)$. Hence $\{f^{-1}(U_z), f^{-1}(V_z)\}$ are disjoint sets whose union is $X$. They are open in $X$ because $f$ is continuous and nonempty since $f$ is onto. Hence this is a separation of $X$. This is a contradiction since $X$ is connected. □

**Lemma 3.1.1.** *If $X$ and $Y$ are two connected topological spaces where $X \cap Y \neq \emptyset$ then $X \cup Y$ is connected.*

Suppose to the contrary that $\{U, V\}$ is a separation of $X \cup Y$. Since $X \subseteq U \cup V$, by Lemma 3.0.2 we know WLOG that $X \subseteq U$. Since $Y \subseteq U \cup V$ and $U \cap V = \emptyset$ by Lemma 3.0.2 we know that $Y \subseteq V$. Hence $X \cap Y = \emptyset$. This is a contradiction. Hence $X \cup Y$ must be connected.

**Definition 3.1.1.** *Let $(X, \mathcal{T})$ be a topological space. Let $x, y \in X$. Let $\sim$ be a relation on $X$ defined by $x \sim y$ if and only if there exists a connected subspace of $X$*

*which contains both $x$ and $y$. $X/\sim$ is the set of equivalence classes called* ***connected components*** *of $X$.*

**Corollary 3.1.1.** *$\sim$ is an equivalence relation*

*Proof.* $\sim$ is reflexive Let $x \in X$. It is clear that $\{x\}$ is a connected set containing $x$. Hence $x \sim x$.

$\sim$ is symmetric Let $x, y \in X$ where $x \sim y$. Then there exists a connected $A \subseteq X$ where $x, y \in A$, i.e. $y, x \in A$. Therefore $y \sim x$.

$\sim$ is transitive Let $x, y, z \in X$ where $x \sim y$ and $y \sim z$. Then there exists connected subsets $A$ and $B$ where $x, y \in A$ and $y, z \in B$. Note that $y \in A \cap B$. Since $A$ and $B$ are connected we know by Lemma 3.1.1 that $A \cup B$ is connected. Since $x, z \in A \cup B$ we now have that $x \sim z$.

□

Let $C = \{C_x\}_{x \in X}$ be the set of connected components of $X$. Since $\sim$ is an equivalence relation, we now know that $C$ forms a partition of X. This means that $X = \bigcup_{x \in X} C_x$ where $C_x \cap C_y = \emptyset$ when $x \notin C_y$.

**Definition 3.1.2.** *Let $(X, \mathcal{T})$ be a topological space. Let $x, y \in X$. A* ***path*** *in $X$ from $x$ to $y$ is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. $X$ is called* ***path connected*** *if there exists a path between any two points in $X$.*

**Corollary 3.1.2.** *A path connected space is connected.*

*Proof.* Let $(X, \mathcal{T})$ be a path connected topological space. Suppose that $\{U, V\}$ is a separation of $X$ where $x \in U$ and $y \in V$. Since $X$ is path connected there exists a path $f : [0, 1] \to X$ between $x$ and $y$. By Theorem 3.1, $f([0, 1])$ is connected. By Lemma 3.0.2, $f([0, 1]) \subseteq U$ WLOG. This is a contradiction since $f(1) = y \in V$. Hence $X$ must be connected. □

**Definition 3.1.3.** *Let $(X, \mathcal{T})$ be a topological space. Let $x, y \in X$. Let $\sim$ be a relation of $X$ defined by $x \sim y$ if and only if there exists a path in $X$ from $x$ to $y$. $X/\sim$ is the set of equivalence classes called* ***path components*** *of $X$.*

**Corollary 3.1.3.** *$\sim$ is an equivalence relation.*

*Proof.* $\sim$ is reflexive Let $x \in X$. The the function $f : [0,1] \to X$ where $f(c) = x$ for all $c \in [0,1]$ is a path from $x$ to itself.

$\sim$ is symmetric Let $x, y \in X$ where $x \sim y$. Then there exists a path $f : [0,1] \to X$ where $f(0) = x$ and $f(1) = y$. Note the function $g : [0,1] \to X$ defined by $g(x) = f(1-x)$ is a path from $y$ to $x$. Note $g$ is continuous be a composition of continuous function. Hence $y \sim x$.

$\sim$ is transitive Let $x, y, z \in X$ where $x \sim y$ and $y \sim z$. Then there is a path $f : [0,1] \to X$ where $f(0) = x$ and $f(1) = y$ Also, there is a path $g : [0,1] \to X$ where $g(0) = y$ and $g(1) = z$. The the function $h : [0,1] \to X$ defined by

$$h(x) = \begin{cases} f(2x) & \text{for } x \in [0, \frac{1}{2}] \\ g(2x-1) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

is a path from $x$ to $z$ hence $x \sim z$.

□

**Definition 3.1.4.** *Let $X$ be a finite topological space. Define a relation $\sim$ on $X$ by $x \sim y$ if and only if there is a chain $x \leq x_1 \leq x_2 \cdots \leq y$ or a chain $y \leq \cdots \leq x_1 \leq x$.*

May (2003b)

**Lemma 3.1.2.** *The equivalence classes of $\sim$ are open.*

*Proof.* Let $\mathcal{E}$ be an equivalence class. If $x \in \mathcal{E}$ then $U_x \subseteq \mathcal{E}$. Hence $\bigcup_{x \in \mathcal{E}} U_x = \mathcal{E}$. □

Hence if $X$ is connected, every pair of points can be connected by a chain.

**Lemma 3.1.3.** *If $x \leq y$ then there is a path $p$ from $x$ to $y$.*

May (2003b)

*Proof.* Define $p(t) = \begin{cases} x & \text{if } t < 1 \\ y & \text{if } t = 1 \end{cases}$

We claim that $p$ is continuous. Let $V \subseteq X$ be open. If $x \in V$ and $y \notin V$ then $p^{-1}(V) = [0, 1)$. If $x \in V$ and $y \in V$ then $p^{-1} = [0, 1]$. If $y \in V$ then $x \in U_y \subseteq V$ since $x \leq y$. Therefore $p^{-1}(V) = I$. Hence in any case $p^{-1}(V)$ is open. □

**Theorem 3.2.** *A finite space is connected if and only if it is path connected.*

May (2003b)

*Proof.* ($\Rightarrow$) By the previous lemmas, $X$ connected implies that $x \leq y$ for all $x, y \in X$, which implies that there is a path $p$ from $x$ to $y$. Hence $X$ is path connected.

($\Leftarrow$) Corollary 3.1.2 □

Theorem 3.2 will play a role in Chapter 5 with finite connected topological spaces and universal covers. We will be considering finite connected topological spaces, hence finite path connected topological spaces.

# Chapter 4

# Four Point Spaces

In this chapter we will spend time characterizing all the 33 four point topological spaces up to homeomorphism. The following definitions will provide the criteria for the characterization.

**Definition 4.0.1.** *For any topological space $X$ in this chapter, define* ***proper open set*** *to be open sets not equal to $\emptyset$ or $X$*

**Definition 4.0.2.** *Let $C_n$ and $D_n$ denote the coarse and discrete topology on an $n$ element set respectively.*

**Definition 4.0.3.** *For an $n$ element set, let $P_n = P_{1,n}$ be the space which has only one proper open set, containing only one point; for $1 < m < n$, let $P_{m,n}$ be the space whose proper open subsets are the non-empty subsets of a given subset with $m$ elements.*

May (2003b)

**Definition 4.0.4.** *For a space $X$ define the* ***non-Hausdorff cone,*** *$\mathbb{C}X$, by adjoining a new point * and letting the proper open subsets of $\mathbb{C}X$ be the non-empty open subsets of $X$.*

May (2003b)

**Definition 4.0.5.** *Let $(X, \mathcal{T})$ be a topological space. Define the* ***double cone,*** $\mathbb{C}^2 X$ *by adjoining two new points denoted $+$ and $-$, and letting the proper open subsets be the open sets in $X$ and $X \cup \{+, -\}$*

May (2003a)

**Definition 4.0.6.** *Let $(X, \mathcal{T})$ be a topological space. Let $z$ be a point not in $X$. Define a new space called $(\theta(X), \theta(\mathcal{T}))$, where $\theta(X) = X \cup \{z\}$, with open sets of the form $\emptyset$ or $U \cup \{z\}$ where $U$ is open in $X$.*

**Corollary 4.0.1.** *$\theta(\mathcal{T})$ is a topology.*

*Proof.* First $\emptyset \in \theta(\mathcal{T})$ by definition. Also, $\theta(X) = X \cup \{z\} \in \theta(\mathcal{T})$ since $X \in \mathcal{T}$.

Suppose that $\{U_\alpha\}_{\alpha \in \Omega} \in \theta(\mathcal{T})$. Therefore $U_\alpha = W_\alpha \cup \{z\}$ where $W_\alpha \in \mathcal{T}$ for all $\alpha \in \Omega$. Then $\bigcup_{\alpha \in \Omega} U_\alpha = \bigcup_{\alpha \in \Omega} (W_\alpha \cup \{z\}) = (\bigcup_{\alpha \in \Omega} W_\alpha) \cup \{z\} \in \theta(X)$ since $\bigcup_{\alpha \in \Omega} W_\alpha \in \mathcal{T}$.

Suppose that $\{U_i\}_{i=1}^n \in \theta(X)$ Therefore $U_i = W_i \cap \{z\}$ where $W_i \in \mathcal{T}$ for all $i$. Then $\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (W_i \cup \{z\}) = (\bigcap_{i=1}^n W_i) \cup \{z\} \in \theta(X)$ since $\bigcap_{i=1}^n W_i \in \mathcal{T}$. Therefore $\theta(X)$ is a topology. □

**Definition 4.0.7.** *Let $(X, \mathcal{T})$ be a finite topological space. Define the* ***non Hausdorff supsension,*** $\sum X$*, by adjoining two new points denoted $+$ and $-$, and letting the proper open subsets be the open sets of $X$ and $X \cup \{+\}$ and $X \cup \{-\}$.*

May (2003a)

The following table is a classification of the 33 topologies on a 4 element set up to homeomorphism. The names were determined by breaking up each space into its connected components and classifying each disjoint piece with the above definitions.

| | | |
|---|---|---|
| 1 | all | $D_4$ |
| 2 | $a, b, c, (a, b), (a, c), (b, c), (a, b, c)$ | $\mathbb{C}D_3$ |
| 3 | $a, b, c, (a, b), (a, c), (b, c), (a, b, c), (a, b, d)$ | $D_1 \sqcup \mathbb{C}D_2$ |
| 4 | $a, b, c, (a, b), (a, c), (b, c), (a, d), (a, b, c), (a, b, d), (a, c, d)$ | $D_2 \sqcup \mathbb{C}D_1$ |
| 5 | $a, b, (a, b)$ | $P_{2,4}$ |
| 6 | $a, b, (a, b), (a, b, c)$ | $\mathbb{C}\mathbb{C}D_2$ |
| 7 | $a, b, (a, b), (a, c, d)$ | $D_1 \sqcup C_3$ |
| 8 | $a, b, (a, b), (a, b, c), (a, b, d)$ | $\sum D_2$ |
| 9 | $a, b, (a, b), (a, c), (a, b, c)$ | $\mathbb{C}(D_1 \sqcup C_2)$ |
| 10 | $a, b, (a, b), (a, c), (a, b, c), (a, c, d)$ | $D_1 \sqcup \mathbb{C}\mathbb{C}C_1$ |
| 11 | $a, b, (a, b), (a, c), (a, b, c), (a, b, d)$ | Name |
| 12 | $a, b, (a, b), (c, d), (a, c, d), (b, c, d)$ | $D_2 \sqcup C_2$ |
| 13 | $a, b, (a, b), (a, c), (a, d), (a, b, c), (a, b, d)$ | $D_1 \sqcup \sum D_1$ |
| 14 | $a, b, (a, b), (a, c), (a, d), (a, b, c), (a, b, d)$ | Name |
| 15 | $a$ | $P_4$ |
| 16 | $a, (a, b)$ | $\mathbb{C}^2\mathbb{C}D_1$ |
| 17 | $a, (a, b), (a, b, c)$ | $\mathbb{C}\mathbb{C}\mathbb{C}C_2$ |
| 18 | $a, (b, c), (a, b, c)$ | $\mathbb{C}\mathbb{C}(D_1 \sqcup C_2)$ |
| 19 | $a, (a, b), (a, c, d)$ | $\theta(D_1 \sqcup C_2)$ |
| 20 | $a, (a, b), (a, b, c), (a, b, d)$ | $\sum \mathbb{C}D_1$ |
| 21 | $a, (b, c), (a, b, c), (b, c, d)$ | $D_1 \sqcup \mathbb{C}C_2$ |
| 22 | $a, (a, b), (a, c), (a, b, c)$ | $\theta(D_2)$ |
| 23 | $a, (a, b), (a, c), (a, b, c), (a, b, d)$ | $\theta(D_1 \sqcup \mathbb{C}D_1)$ |
| 24 | $a, (c, d), (a, b), (a, c, d)$ | $P_2 \sqcup C_2$ |
| 25 | $a, (a, b), (a, c), (a, d), (a, b, c), (a, b, d), (a, c, d)$ | $\theta(D_1 \sqcup D_2)$ |
| 26 | $a, (a, b, c)$ | $\mathbb{C}\mathbb{C}^2D_1$ |
| 27 | $a, (b, c, d)$ | $D_1 \sqcup C_3$ |
| 28 | $(a, b)$ | $\mathbb{C}^2\mathbb{C}D_1$ |
| 29 | $(a, b), (c, d)$ | $D_2 \sqcup D_2$ |
| 30 | $(a, b), (a, b, c)$ | $\mathbb{C}\mathbb{C}C_2$ |
| 31 | $(a, b), (a, b, c), (a, b, d)$ | $\theta(\sum D_1)$ |
| 32 | $(a, b, c)$ | $\mathbb{C}C_3$ |
| 33 | none | $C_4$ |

# Chapter 5

# Algebraic Topology on Finite Spaces

This chapter is the heart of the paper. Here we will prove that every finite connected topological space has a universal cover. We will also construct a condition by which a finite topological space is contractible. Again we will begin with several fundamental definitions.

**Definition 5.0.8.** *Let $(X, \mathcal{T})$ and $(Y, \Gamma)$ be a topological spaces. Let $f : X \to Y$ and $g : X \to Y$ be continuous. Let $I = [0, 1]$. $f$ is **homotopic** to $g$, denoted $f \simeq g$, if there exists a continuous function $F : X \times I \to Y$ where $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.*

*The function $F$ is called a homotopy between $f$ and $g$. If $g$ is a constant function, then $f$ is called **nulhomotopic** and $F$ is called a **nulhomotopy**.*

**Definition 5.0.9.** *Let $(X, \mathcal{T})$ be a topological space. $X$ is called **contractible** if the identity map $i_X : X \to X$ is nulhomotopic.*

Contractible finite topological spaces will be discussed at length in this chapter

**Theorem 5.1.** *Let $(X, \mathcal{T})$ be a finite topological space. Then every point of $X$ has a contractible neighborhood.*

We will prove the following lemmas and corollaries first. The proof of Theorem 5.1 will easily follow.

**Lemma 5.1.1.** *If $X$ and $Y$ are finite spaces and $f, g : X \to Y$ are continuous, then $f(x) \leq g(x)$ for all $x \in X$ implies that $f \simeq g$*

*Proof.* We must find a continuous $F : X \times I \to Y$ where $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

Define $F(x,t) = \begin{cases} f(x) & \text{for } t < 1 \\ g(x) & \text{for } t = 1 \end{cases}$

Let $V \subseteq Y$ be open. Show that $F^{-1}(V)$ is open. Let $(x, t) \in F^{-1}(V)$.

If $t = 1$ then $g(x) \in V$. Hence $U_{g(x)} \subseteq V$, but $f(x) \leq g(x)$. Hence $U_{f(x)} \subseteq U_{g(x)} \subseteq V$. Hence $f(x) \in V$. Therefore for all $t \in [0, 1]$, $F(x, t) \in V$. Let $W = f^{-1}(U_{f(x)}) \cap g^{-1}(U_{g(x)})$ and note the following: If $(w, t) \in W \times I$ then,

(a) $t < 1$ implies $F(w, t) = f(w) \in U_{f(x)} \subseteq V$.

(b) $t = 1$ implies $F(w, t) = g(w) \in U_{g(x)} \subseteq V$. Hence $F(W \times I) \subseteq V$, i.e $W \times I \subseteq F^{-1}(V)$. Hence $F^{-1}(V)$ is open.

If $t \neq 1$ set $\tilde{W} = f^{-1}(U_{f(x)}) \times (0, 1)$. Then $F(\tilde{W}) = U_{f(x)} \subseteq V$. Hence $\tilde{W} \subseteq F^{-1}(V)$ and again $F^{-1}(V)$ is open. Therefore $F$ is continuous. Hence $f \simeq g$. □

**Corollary 5.1.1.** *Let $(X, \mathcal{T})$ be a finite topological space. Suppose there exists $x_0 \in X$ such that $x \leq x_0$ for all $x \in X$. Then $X$ is contractible. ($x_0$ is called a **maximal point** of $X$.)*

*Proof.* Let $f : X \to X$ where $f(x) = x_0$ for all $x \in X$. Let $g = i_X : X \to X$. Then by Lemma 5.1.1, $g \simeq f$ making $g$ nulhomotopic. Hence $X$ is contractible. □

**Corollary 5.1.2.** *Each $U_x$ has a maximal point.*

*Proof.* Note that if $z \in U_x$ then $z \leq x$ by definition. Hence $x$ is a maximal point of $U_x$. □

Therefore given $x \in X$, $U_x$ is a contractible neighborhood of $x$.

Note this implies that $\pi_1(U_x, x)$ is trivial. Since this is true for every $x \in X$ we know that $X$ is semilocally semi connected. If we assume that $X$ is connected then being finite $X$ is also path connected.

$U_x$ is always connected hence path connected, and is contained in every neighborhood of $x$, making $X$ locally path connected. Therefore we know that every finite connected topological space has a universal cover.

The following results will show that the union of two spaces, which are both contractible to the same point, intersecting at that point, is contractible.

**Lemma 5.1.2.** *If $(X, \mathcal{T})$ is finite and connected then for $x, y \in X$, $x \leq y$ implies that $k_x \simeq k_y$.*

*Proof.* Let $X$ be connected and $x, y \in X$. Then by Lemma 3.1.2 we know that $x$ and $y$ are connected by a chain, hence $x \leq y$. We know that $k_x$ and $k_y$ are continuous. Hence by Lemma 5.1.1 $k_x \simeq k_y$. □

**Corollary 5.1.3.** *Let $(X, \mathcal{T})$ be a finite topological space where $i_X \simeq k_{x_0}$ where $x_0 \in X$. Then $i_X \simeq k_y$ for all $y \in X$.*

*Proof.* We know that $x_0 \leq y$. Hence $k_{x_0} \simeq k_y$. Therefore $i_X \simeq k_{x_0} \simeq k_y$. □

**Theorem 5.2.** *Suppose that $X$ and $Y$ are two finite topological spaces open in $X \cup Y$ where $X \cap Y = \{a\}$, where $i_X \simeq k_a$ and $i_Y \simeq k_a$. Then $i_{X \cup Y} \simeq k_a$.*

*Proof.* Since $i_X \simeq k_a$ and $i_Y \simeq k_a$, there exist nulhomotopies $F$ which contracts $Y$ to the point $a$ and $G$ which contracts $X$ to the point $a$.

Define $\tilde{F}(x,t) = \begin{cases} F(x,t) & \text{if } x \neq a \\ a & x = a \end{cases}$

Show that $\tilde{F}$ is continuous:

First note that $\tilde{F}^{-1}(\{a\}) = \{a\} \times I \cup F^{-1}(\{a\})$ which is clearly open.

Let $V$ be open in $Y$. If $a \notin V$ then $\tilde{F}^{-1}(V) = F^{-1}(V)$ which is open.

Note that $V = V \setminus \{a\} \cup \{a\}$. If $a \in V$ then $\tilde{F}^{-1}(V) = \tilde{F}^{-1}(V \setminus \{a\}) \cup \tilde{F}^{-1}(\{a\}) = F^{-1}(V \setminus \{a\}) \cup \tilde{F}^{-1}(\{a\})$ which is open. Hence $\tilde{F}$ is continuous. Define $\tilde{G}$ similarly.

Finally define $H_Y : Y \times I \to Y$ where $H_Y(z,t) = \begin{cases} \tilde{F}(z,2t) & (z,t) \in Y \times [0, \frac{1}{2}] \\ a & (z,t) \in Y \times [\frac{1}{2}, 1] \end{cases}$

And define $H_X : X \times I \to X$ where $H_X(z,t) = \begin{cases} z & (z,t) \in X \times [0, \frac{1}{2}] \\ \tilde{G}(z,2t-1) & (z,t) \in X \times [\frac{1}{2}, 1] \end{cases}$

$H_Y$ and $H_X$ are each continuous and agree on $X \cap Y$. Hence by the pasting lemma, $H_Y$ and $H_X$ combine to give a continuous $K : X \cup Y \times I \to a$ which gives the result. □

We now know that the union of two finite connected contractible spaces intersecting at a point is contractible. This gives us a technique to show that some of the 4 point spaces are contractible. Referring to the table in Chapter 4:

$\mathbb{C}D_3$ is contractible since $(a,b,c,d) = U_d$ is contractible by Cor 5.1.2.

$P_{2,4}$ is contractible since $(a,b,c,d) = U_d$ is contractible by Cor 5.1.2.

$\mathbb{C}\mathbb{C}D_2$ is contractible since $(a,b,c,d) = U_d$ is contractible by Cor 5.1.2.

$\mathbb{C}(D_1 \sqcup C_2)$ is contractible since $(a,b,c,d) = U_d$ is contractible by Cor 5.1.2.

11. is contractible. Note that $U_c = (a,c)$ and $U_d = (a,b,d)$ are contractible and $U_c \cap U_d = \{a\}$ hence 11=$U_c \cup U_d$ is contractible.

$P_4$ is contractible since $U_d = (a,b,c,d)$ is contractible by Cor 5.1.2.

$\mathbb{C}^2(\mathbb{C}D_1)$ is contractible since $U_d = (a,b,c,d)$ is contractible by Cor 5.1.2.

$\mathbb{C}\mathbb{C}\mathbb{C}_1$ is contractible since $U_d = (a,b,c,d)$ is contractible by Cor 5.1.2

$\mathbb{C}\mathbb{C}(D_1 \sqcup C_2)$ is contractible since $U_d = (a,b,c,d)$ is contractible by Cor 5.1.2

$\theta(D_1 \sqcup C_2)$ is contractible. Note that $U_d = (a,c,d)$ and $U_b = (a,b)$. Both are contractible by Cor 5.1.2. Also $U_d \cap U_b = \{a\}$. Hence by Theorem 5.2 $\theta(D_1 \sqcup C_2)$ is contractible .

$\theta(D_2)$ is contractible since $U_d = (a,b,c,d)$ is contractible.

$\theta(D_1 \sqcup \mathbb{C}D_1)$ is contractible. Note that $U_c = (a, c)$ and $U_d = (a, b, d)$. Both are contractible by Cor 5.1.2. Note $U_c \cap U_d = \{a\}$. Hence $\theta(D_1 \sqcup \mathbb{C}D_1) = U_c \cup U_d$ is contractible by Thm 5.2

$P_2 \cup C_2$ is contractible. Note $U_b = (a, b)$ and $U_d = (a, c, d)$ are both contractible by Cor 5.1.2. Note $U_b \cap U_d = \{a\}$. Hence $P_2 \cup C_2$ is contractible by Thm 5.2

$\theta(D_1 \sqcup D_2)$ is contractible. Note that $U_b = (a, b)$, $U_c = (a, c)$, and $U_d = (a, d)$ are all contractible by Cor 5.1.2. Note that $U_b \cap U_c = \{a\}$. Hence $U_b \cup U_c$ is contractible by Thm 5.1.2. Note that $(U_b \cup U_c) \cap U_d = \{a\}$. Hence $\theta(D_1 \sqcup D_2) = (U_b \cup U_c) \cup U_d$ is contractible by Thm 5.2.

$\mathbb{C}\mathbb{C}^2 D_1$ is contractible since $U_d = (a, b, c, d)$ is contractible by Cor 5.1.2.

$\mathbb{C}^2\mathbb{C}D_1$ is contractible since $U_d = (a, b, c, d)$ is contractible by Cor 5.1.2.

$\mathbb{C}\mathbb{C}C_2$ is contractible since $U_d = (a, b, c, d)$ is contractible by Cor 5.1.2.

$\mathbb{C}C_3$ is contractible since $U_d = (a, b, c, d)$ is contractible by Cor 5.1.2.

$C_4$ is contractible since $U_d = (a, b, c, d)$ is contractible by Cor 5.1.2.

As we can see, the application of this result on a space $X$ is easily done by taking various unions and intersections of $U_x$.

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# Vita

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