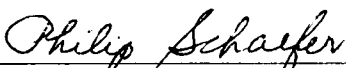


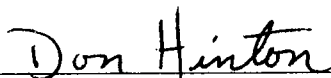
To the Graduate Council:

I am submitting herewith a thesis written by Melissa Ann Smith entitled "On the Linearized Korteweg-deVries Equation." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.



Dr. Philip Schaefer, Major Professor

We have read this thesis
and recommend its acceptance:





Accepted for the Council:



Associate Vice Chancellor and
Dean of The Graduate School

ON THE LINEARIZED KORTEWEG-DEVRIES EQUATION

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Melissa Ann Smith

August 1995

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ABSTRACT

An extensive theory has developed about localized nondissipative waves called *solitons*. In 1895 Korteweg and deVries derived a third order nonlinear partial differential equation that demonstrated solitons exist. In this work, we introduce the concept of solitons and derive one solution of the nonlinear Korteweg-deVries equation. However, the difficulty of the nonlinear term leads us to consider a linearized Korteweg-deVries equation for an infinite space initial-boundary value problem and a finite space initial-boundary value problem. We use methods of Fourier transforms and separation of variables to solve these third order problems. For the infinite case our solution involves the special Airy function. Thus, we derive the Airy function from a second order differential equation and examine some of its properties. However, upon solving the finite initial-boundary value problem by separation of variables, we fail to obtain the usual orthogonality of the eigenfunctions since the eigenvalue problem is not self-adjoint. Therefore, we invoke a theorem which gives us a bi-orthogonality relationship between the eigenfunctions of the eigenvalue problem and the eigenfunctions of the adjoint problem provided the Green's function has simple poles at each eigenvalue of the eigenvalue problem.

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CHAPTER I

INTRODUCTION

Waves are a fundamental part of nature and can travel through many different mediums. Some common examples of waves are water waves, sound waves, and electromagnetic waves. By a wave, we mean an identifiable signal or disturbance in a medium that is propagated in time, carrying energy with it [7]. A simple mathematical model of a wave is

$$u(x,t) = f(x - ct) \quad (1.1)$$

where u represents the strength of the disturbance at the position x and at time t and f is an arbitrary differentiable function of one variable. The constant c is the speed of the wave. It is easy to show that (1.1) is a solution of the linear partial differential equation

$$u_t + cu_x = 0.$$

By linear waves, we mean waves that represent solutions to linear partial differential equations and similarly, nonlinear waves are waves that represent solutions to nonlinear partial differential equations. A feature of linear waves is that they move without distortion, i.e., keep their shape, whereas nonlinear waves may change shape over time.

Another common aspect of different types of waves is dispersion. A typical example of dispersion is the result of dropping a pebble into a still pond. The pebble disturbs the water's surface and creates a series of concentric water waves. At first the waves start strong and tall, but they gradually fade as they move away from the center.

Logan [7] describes a dispersive partial differential equation as one that has solutions of the form

$$u(x,t) = A \exp[i(kx - \omega t)]$$

where ω , the angular frequency, is a definite real function of the wave number k .

In 1834 John Scott Russell, an engineer and naval architect, recorded the first observation of a wave that did not disperse and traveled without changing shape. He called it the "great wave of translation." His report to the British Advancement of Science follows:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped --not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles per hour, preserving its original figure some 30 feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles, I lost it in the windings of the channel" [11].

After some water wave experiments of his own, Russell noticed that these waves could also pass through each other without changing shape or distorting each other. He proposed that these results came "from intrinsic properties of the wave's motion and not from the circumstances of its generation" [11]. His conclusions were not widely accepted, however, until 1895 when D. J. Korteweg and Hendrick deVries gave an analytic treatment of a nonlinear partial differential equation and demonstrated that localized nondissipative waves could exist [11]. The general form of the nonlinear equation derived by Korteweg and deVries, known as the Korteweg-deVries equation or KdV equation, is

$$u_t + au_x + buu_x + ku_{xxx} = 0 \quad (1.2)$$

where a , b , and k are constants.

In 1965 Zabusky and Kruskal [19] observed “solitary-wave pulses” in the numerical solutions of the Korteweg-deVries equation $u_t + uu_x + ku_{xxx} = 0$. For example, “the solution develops a train of eight well-defined waves” which are similar to sech^2 functions, and the tall stronger waves overtake and pass through the shorter slower waves while maintaining their own identity [4]. To indicate this particle-like quality Zabusky and Kruskal named these waves *solitons* after other particles such as photons, protons, and electrons. To give an exact definition of a soliton is not easy; however, we shall apply the term to any solution of a nonlinear equation which satisfies the following properties: (i) the solution represents a wave of permanent form, (ii) it decays or approaches a constant at infinity, i.e. the solution is localized, and (iii) the solution maintains its identity after strong interactions with other solitons [4]. Since 1965 an extensive theory has developed involving solitons, and other equations have been found to have solitons as their solutions (e.g. the Boussinesq equation, the Sine-Gordon equation, the Born-Infeld equation, and nonlinear Schrödinger-type equations) [4]. However, these solutions are not necessarily sech^2 functions like the KdV equation.

We can show the following KdV equation

$$u_t + uu_x + ku_{xxx} = 0, \quad k \geq 0 \quad (1.3)$$

for some constant k , has solutions in terms of sech^2 . We solve equation (1.3) by assuming a solution of the form

$$u = f(s), \quad s = x - ct \quad (1.4)$$

where f and c are to be determined. From (1.4) we have $u_t = -cf'(s)$, $u_x = f'(s)$, and $u_{xxx} = f'''(s)$. Substituting these partial derivatives and (1.4) into the KdV equation, we obtain the differential equation

$$-cf' + ff' + kf''' = 0. \quad (1.5)$$

Then since $ff' = \frac{1}{2} \frac{d}{ds} f^2$, we can rewrite (1.5) as

$$-cf' + \frac{1}{2} \frac{d}{ds} f^2 + kf''' = 0.$$

Integrating this equation, we have

$$-cf + \frac{1}{2} f^2 + kf'' = A \quad (1.6)$$

for some constant A . We multiply (1.6) by f' to obtain

$$-c f f' + \frac{1}{2} f^2 f' + k f'' f' = A f'$$

which can be rewritten as

$$-c \frac{1}{2} \frac{d}{ds} f^2 + \frac{1}{6} \frac{d}{ds} f^3 + k \frac{1}{2} \frac{d}{ds} (f')^2 = A f'$$

and integration results in

$$\frac{-c}{2} f^2 + \frac{1}{6} f^3 + \frac{k}{2} (f')^2 = A f + B$$

where A and B are constants. Solving for $\frac{df}{ds} = f'$, we obtain

$$\frac{df}{ds} = \pm \sqrt{\frac{1}{3k} \phi^{1/2}(f)} \quad (1.7)$$

where $\phi(f) = 6B + 6Af + 3cf^2 - f^3$. Since $\phi(f)$ is a cubic polynomial, we consider the five cases:

- (1) ϕ has one real root α .
- (2) ϕ has three distinct real roots $\gamma < \beta < \alpha$.
- (3) ϕ has three real roots such that $\gamma = \beta < \alpha$.
- (4) ϕ has three real roots such that $\gamma < \alpha = \beta$.
- (5) ϕ has a triple root γ .

It is known that cases (1) and (4) have unbounded solutions [7]. The solutions for case (2) can be expressed in terms of the Jacobian elliptical function cn ; therefore, these solutions were named *cnoidal waves* by Korteweg and deVries [17]. We shall examine case (3) since the solutions in this case are solitons. For more development of the other cases, we refer the reader to Drazin and Johnson [4]. We can solve (1.7) for f since $\phi(f)$ has three real roots such that $\gamma = \beta < \alpha$. Thus, (1.7) can be written as

$$\frac{df}{ds} = \pm \frac{1}{\sqrt{3k}} [(f - \gamma)^2 (\alpha - f)]^{1/2}.$$

We now consider a special case and separate variables, i.e.,

$$\frac{ds}{\sqrt{3k}} = \frac{df}{(f - \gamma) \sqrt{\alpha - f}}. \quad (1.8)$$

Following Logan [7] we use the substitution

$$f = \gamma + (\alpha - \gamma) \text{sech}^2 w.$$

Then $df = -2(\alpha - \gamma) \text{sech}^2 w \tanh w dw$ since $D_x(\text{sech} u) = -\text{sech} u \tanh u D_x u$ for some function u of x . Therefore, (1.8) becomes

$$\begin{aligned}
\frac{ds}{\sqrt{3k}} &= \frac{-2(\alpha - \gamma) \operatorname{sech}^2 w \tanh w dw}{(\alpha - \gamma) \operatorname{sech}^2 w \sqrt{(\alpha - \gamma)} - (\alpha - \gamma) \operatorname{sech}^2 w} \\
&= \frac{-2 \tanh w dw}{\sqrt{(\alpha - \gamma) \tanh^2 w}} \\
&= \frac{-2 dw}{\sqrt{\alpha - \gamma}}.
\end{aligned}$$

Integrating this equation, we have

$$\frac{s}{\sqrt{3k}} = \frac{-2w}{\sqrt{\alpha - \gamma}}$$

where we take the constant of integration to be zero. Thus

$$w = -\sqrt{\frac{\alpha - \gamma}{12k}} s,$$

and a solution of (1.7), case (3), is

$$f(s) = \gamma + (\alpha - \gamma) \operatorname{sech}^2 \left(\sqrt{\frac{\alpha - \gamma}{12k}} s \right). \quad (1.9)$$

We can find the velocity c of the wave using (1.7) and the roots α and γ . We know,

$$\begin{aligned}
\phi(f) &= -f^3 + 3cf^2 + 6Af + 6B \\
&= (f - \gamma)^2 (\alpha - f) \\
&= -f^3 + (\alpha + 2\gamma)f^2 + (-2\gamma\alpha - \gamma^2)f + \gamma^2\alpha.
\end{aligned}$$

Thus, $3c = \alpha + 2\gamma$ and $c = \frac{\alpha + 2\gamma}{3}$. Finally, the solution of the KdV equation (1.3)

can be written in the form

$$\begin{aligned}
u(x,t) &= f(s) = f(x-ct) \\
&= \gamma + (\alpha - \gamma) \operatorname{sech}^2 \left(\sqrt{\frac{\alpha - \gamma}{12k}} (x - ct) \right) \\
&= \gamma + (\alpha - \gamma) \operatorname{sech}^2 \left(\sqrt{\frac{\alpha - \gamma}{12k}} \left[x - \left(\frac{\alpha + 2\gamma}{3} \right) t \right] \right) \\
&= \gamma + a \operatorname{sech}^2 \left(\sqrt{\frac{a}{12k}} \left[x - \left[\gamma + \frac{a}{3} \right] t \right] \right)
\end{aligned}$$

where $a = \alpha - \gamma$.

During the past thirty years, the KdV equation and solitons have found applications in a variety of areas such as shallow water waves [3], ion-acoustic waves of small amplitude [16], collisionless-plasma waves [19], and waves in bubble-liquid mixtures [14]. Thus, other methods have been developed to solve and analyze the nonlinear KdV equation. These methods include the inverse scattering transform, conservation laws and nonlinear transformations, Bäcklund transformations, and the nonlinear superposition principle. For more information see Drazin and Johnson [4], Debnath [3], or Whitham [17]. The difficulty in solving the KdV equation arises from the nonlinear term uu_x . However, if we assume this term is small, i.e., approximately zero, then (1.3) becomes

$$u_t + ku_{xxx} = 0 \quad (1.10)$$

or (1.2) becomes

$$u_t + au_x + ku_{xxx} = 0 \quad (1.11)$$

which are both linearized KdV equations. The remaining part of this work concerns equation (1.10). In Chapter II we obtain a solution for an infinite space initial-boundary

value problem, and in Chapter III we solve a finite space initial-boundary value problem. Our goal is to gain further insight about the nonlinear KdV equation from this linearized version. Also, since third order partial differential equations are little studied, we will gain valuable techniques to solve them which are analogous to solving second order partial differential equations.

CHAPTER II

AN INFINITE SPACE INITIAL-BOUNDARY VALUE PROBLEM

In this chapter we consider the linearized KdV equation (1.10) where $-\infty < x < \infty$. Upon solving this problem, we can write the solution in terms of the Airy function. Thus, we proceed to derive the Airy function as a solution of a second order linear differential equation, and since the Airy integral is a special function, we examine some of its properties.

1. Statement and Solution of the Problem

In this section we consider the linearized KdV equation on the whole real line where the solution is subject to an initial displacement. For physical reasons we assume that the wave's amplitude, velocity, and acceleration approach zero as $|x|$ goes to infinity. Thus, the initial-boundary value problem has the following form:

$$\begin{aligned}\frac{\partial u}{\partial t} &= k \frac{\partial^3 u}{\partial x^3}, & -\infty < x < +\infty, & \quad t > 0 \\ u(x, 0) &= f(x), & -\infty < x < +\infty \\ u \rightarrow 0, u_x \rightarrow 0, u_{xx} &\rightarrow 0 \quad \text{as } |x| \rightarrow +\infty.\end{aligned}\tag{2.1}$$

This problem can be solved using Fourier transforms.

The Fourier transform of $u(x, t)$ is defined by

$$U(\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u(x, t) e^{i\omega x} dx,$$

and we use the notation $U(\omega, t) = \mathcal{F}[u(x, t)]$. We begin by taking the Fourier transform of (2.1), i.e.,

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k \mathcal{F}\left[\frac{\partial^3 u}{\partial x^3}\right]. \quad (2.2)$$

Now on the left hand side, we have

$$\begin{aligned} \mathcal{F}\left[\frac{\partial u}{\partial t}\right] &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\partial u}{\partial t} e^{i\omega x} dx \\ &= \frac{\partial}{\partial t} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} u(x, t) e^{i\omega x} dx \right] = \frac{\partial}{\partial t} U(\omega, t), \end{aligned}$$

where the interchange of the integration and differentiation is valid because u, u_t are continuous and $\phi(t) = \int_{-\infty}^{+\infty} \frac{\partial u}{\partial t}(x, t) dx$ converges uniformly on any finite time interval

([15], Theorem 6.6). The Fourier transform of the right hand side can be found upon integration by parts and using the asymptotic behavior of u, u_x, u_{xx} . Thus,

$$\begin{aligned} \mathcal{F}\left[\frac{\partial u}{\partial x}\right] &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\partial u}{\partial x} e^{i\omega x} dx \\ &= \frac{1}{2\pi} u e^{i\omega x} \Big|_{-\infty}^{+\infty} - i\omega \frac{1}{2\pi} \int_{-\infty}^{+\infty} u e^{i\omega x} dx \\ &= -i\omega U(\omega, t). \end{aligned}$$

More generally, by repeated use of integration by parts, we have

$$\mathcal{F}\left[\frac{\partial^n u}{\partial x^n}\right] = (-i\omega)^n U(\omega, t)$$

for any positive integer n , and in particular for $n = 3$,

$$\mathcal{F}\left[\frac{\partial^3 u}{\partial x^3}\right] = i\omega^3 U(\omega, t).$$

Consequently, (2.2) becomes

$$\frac{\partial}{\partial t} U(\omega, t) = ki\omega^3 U(\omega, t)$$

or

$$\frac{\partial}{\partial t} U - ki\omega^3 U = 0 \quad (2.3)$$

which is a first order linear differential equation in t .

The general solution of (2.3) is $U(\omega, t) = c(\omega) e^{ki\omega^3 t}$ where c is an arbitrary function. We determine the function c from the initial condition $u(x, 0) = f(x)$.

Therefore,

$$\begin{aligned} c(\omega) &= U(\omega, 0) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} u(x, 0) e^{i\omega x} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx. \end{aligned} \quad (2.4)$$

We can now solve for $u(x, t)$ by using the inverse Fourier transform, i.e.,

$$u(x, t) = \int_{-\infty}^{+\infty} U(\omega, t) e^{-i\omega x} d\omega. \quad (2.5)$$

By substituting $U(\omega, t)$ into (2.5), we obtain

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{+\infty} c(\omega) e^{ki\omega^3 t} e^{-i\omega x} d\omega \\ &= \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(s) e^{i\omega s} ds \right] e^{-i\omega x} e^{ki\omega^3 t} d\omega \\ &= \int_{-\infty}^{+\infty} f(s) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ki\omega^3 t} e^{-i\omega(x-s)} d\omega \right] ds \end{aligned} \quad (2.6)$$

where the order of integration can be changed as a result of Fubini's theorem.

We would like to simplify this integral representation for $u(x,t)$. First, we define a function g by

$$g(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega^3 t} e^{-i\omega x} d\omega$$

such that the bracket in (2.6) is $g(x-s, kt)$.

By using Euler's formula and a trigonometric addition formula, we can determine an alternate expression for g , namely,

$$\begin{aligned} g(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega^3 t} e^{-i\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\cos \omega^3 t + i \sin \omega^3 t] [\cos \omega x - i \sin \omega x] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\cos \omega^3 t \cos \omega x + \sin \omega^3 t \sin \omega x - i \cos \omega^3 t \sin \omega x + i \sin \omega^3 t \cos \omega x] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\cos \omega^3 t \cos \omega x + \sin \omega^3 t \sin \omega x] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(\omega^3 t - \omega x) d\omega \end{aligned} \tag{2.7}$$

where the imaginary part vanishes due to the oddness of the integrand. We now

introduce two substitutions into (2.7). First, let $v = \omega (3t)^{1/3}$. Then we write

$$g(x,t) = \frac{1}{2\pi (3t)^{1/3}} \int_{-\infty}^{+\infty} \cos\left(\frac{v^3}{3} - \frac{v}{(3t)^{1/3}} x\right) dv.$$

Now, we let $y = \frac{-x}{(3t)^{1/3}}$ so that

$$\begin{aligned}
 g(y,t) &= \frac{1}{2\pi (3t)^{1/3}} \int_{-\infty}^{+\infty} \cos\left(\frac{v^3}{3} + vy\right) dv \\
 &= \frac{1}{\pi (3t)^{1/3}} \int_0^{\infty} \cos\left(\frac{v^3}{3} + vy\right) dv .
 \end{aligned}$$

We define $A(y) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{v^3}{3} + vy\right) dv$ which is a special function named for Sir

George Airy. Thus,

$$g(y,t) = \frac{1}{(3t)^{1/3}} A(y) ,$$

and the solution of the initial value problem (2.1) can be written as

$$u(x,t) = \int_{-\infty}^{+\infty} f(s) g(x-s, kt) ds$$

where g is defined above.

2. Development of the Airy Function

Sir George Biddell Airy (1801 - 1892) was an English astronomer and mathematician who served as the Director of the Greenwich Observatory from 1835 to 1881. His early work involved the theory of light, and he discovered a certain special function that has applications to the theory of diffraction. Later the mathematician Stokes realized that this special function, the Airy function, is a solution of the differential equation $y'' - xy = 0$.

The Airy function can be represented by an infinite series or an improper integral.

In this section we consider the integral representation since this form appears in the

solution of (2.1). For a development of the Airy function in series form see Boyce and DiPrima [1].

To obtain the Airy function in integral form, we solve the overdetermined boundary value problem

$$\begin{aligned}\frac{d^2 y}{dx^2} - xy &= 0, & -\infty < x < +\infty \\ y(0) &= \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{\omega^3}{3}\right) d\omega \\ \lim_{|x| \rightarrow \infty} y &= 0.\end{aligned}\tag{2.8}$$

Again, we can use Fourier transforms. Thus, taking transforms of the differential equation in (2.8), we have

$$\mathcal{F}\left[\frac{d^2 y}{dx^2}\right] - \mathcal{F}[xy] = 0.\tag{2.9}$$

We let $y = f(x)$, and then (2.9) becomes

$$\mathcal{F}\left[\frac{d^2 f}{dx^2}\right] - \mathcal{F}[xf(x)] = 0.\tag{2.10}$$

We define the Fourier transform of $f(x)$ by

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx.$$

Integrating by parts and using the asymptotic behavior of $y = f(x)$, we get

$$\begin{aligned}\mathcal{F}\left[\frac{df}{dx}\right] &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{df}{dx} e^{i\omega x} dx \\ &= \frac{1}{2\pi} f(x) e^{i\omega x} \Big|_{-\infty}^{+\infty} - i\omega \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx\end{aligned}$$

$$\begin{aligned}
&= -i\omega \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx \\
&= -i\omega F(\omega).
\end{aligned}$$

Then,

$$\mathcal{F}\left[\frac{d^2 f}{dx^2}\right] = -i\omega \mathcal{F}\left[\frac{df}{dx}\right] = (-i\omega)^2 F(\omega) = -\omega^2 F(\omega).$$

The Fourier transform of $xf(x)$ is $-i \frac{dF}{d\omega}$. To show this we differentiate $F(\omega)$ and

obtain

$$\begin{aligned}
\frac{dF}{d\omega} &= \frac{d}{d\omega} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx \right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d}{d\omega} (f(x) e^{i\omega x}) dx \\
&= \frac{i}{2\pi} \int_{-\infty}^{+\infty} x f(x) e^{i\omega x} dx
\end{aligned}$$

where the interchange of integration and differentiation is valid. Thus,

$$-i \frac{dF}{d\omega} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} x f(x) e^{i\omega x} dx = \mathcal{F}[xf(x)].$$

The differential equation (2.10) then becomes

$$-\omega^2 F(\omega) + i \frac{dF}{d\omega} = 0$$

or

$$\frac{dF}{d\omega} + i\omega^2 F = 0$$

which is a first order linear differential equation. The general solution is

$F(\omega) = c_0 e^{-(\omega^3/3)i}$, where c_0 is some constant. We solve for $f(x)$ by using the inverse transform, i.e.,

$$\begin{aligned}
 f(x) &= \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega x} d\omega \\
 &= \int_{-\infty}^{+\infty} c_0 e^{-\frac{\omega^3}{3}i} e^{-i\omega x} d\omega \\
 &= \int_{-\infty}^{+\infty} c_0 e^{-\left(\frac{\omega^3}{3} + \omega x\right)i} d\omega \\
 &= \int_{-\infty}^{+\infty} c_0 \left[\cos\left(\frac{\omega^3}{3} + \omega x\right) - i \sin\left(\frac{\omega^3}{3} + \omega x\right) \right] d\omega \\
 &= c_0 \int_{-\infty}^{+\infty} \cos\left(\frac{\omega^3}{3} + \omega x\right) d\omega \\
 &= 2c_0 \int_0^{\infty} \cos\left(\frac{\omega^3}{3} + \omega x\right) d\omega .
 \end{aligned}$$

We can determine c_0 from the condition on $y(0)$. Thus,

$$f(0) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{\omega^3}{3}\right) d\omega = 2c_0 \int_0^{\infty} \cos\left(\frac{\omega^3}{3} + \omega \cdot 0\right) d\omega$$

which implies $c_0 = \frac{1}{2\pi}$. Thus,

$$y = f(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{\omega^3}{3} + \omega x\right) d\omega \quad (2.11)$$

which is the integral representation of the Airy function that solves (2.8).

3. Properties of the Airy Function

Since the Airy function is a special function and not as well-known as others, we consider some of its properties: convergence, graphical representation, and oscillatory behavior. We first consider the convergence of the Airy function given by (2.11). By definition,

$$\int_0^{\infty} \cos\left(\frac{\omega^3}{3} + \omega x\right) d\omega = \lim_{b \rightarrow \infty} \int_0^b \cos\left(\frac{\omega^3}{3} + \omega x\right) d\omega \quad (2.12)$$

for any $x \in (-\infty, \infty)$. To evaluate the latter integral, we integrate by parts. Let

$$u = \frac{1}{\omega^2 + x} \quad \text{and} \quad dv = (\omega^2 + x) \cos\left(\frac{\omega^3}{3} + \omega x\right) d\omega.$$

Then

$$du = \frac{-2\omega d\omega}{(\omega^2 + x)^2} \quad \text{and} \quad v = \sin\left(\frac{\omega^3}{3} + \omega x\right),$$

and

$$\int_0^b \cos\left(\frac{\omega^3}{3} + \omega x\right) d\omega = \frac{\sin\left(\frac{b^3}{3} + bx\right)}{b^2 + x} + 2 \int_0^b \frac{\omega}{(\omega^2 + x)^2} \cdot \sin\left(\frac{\omega^3}{3} + \omega x\right) d\omega. \quad (2.13)$$

The first term on the right hand side of (2.13) converges to zero as $b \rightarrow \infty$ since

$$0 \leq \left| \frac{\sin\left(\frac{b^3}{3} + bx\right)}{b^2 + x} \right| \leq \frac{1}{|b^2 + x|} \quad (2.14)$$

for any $x \in (-\infty, \infty)$.

We consider the second term on the right hand side of (2.13) when $x \neq 0$ and when $x = 0$. If $x \neq 0$, we have

$$\begin{aligned} \int_0^b \left| \frac{\omega}{(\omega^2 + x)^2} \cdot \sin\left(\frac{\omega^3}{3} + \omega x\right) \right| d\omega &\leq \int_0^b \frac{\omega}{(\omega^2 + x)^2} d\omega \\ &= \frac{-1}{2(b^2 + x)} + \frac{1}{2x} \end{aligned}$$

which is finite as $b \rightarrow \infty$. Thus, since $\frac{\omega}{(\omega^2 + x)^2}$ is improperly integrable on $(0, \infty)$

by the Comparison Theorem for improper integrals ([15], Theorem 3.17), we know

$\frac{\omega}{(\omega^2 + x)^2} \cdot \sin\left(\frac{\omega^3}{3} + \omega x\right)$ is absolutely and improperly integrable on $(0, \infty)$.

When $x = 0$, (2.13) becomes

$$\int_0^b \cos\left(\frac{\omega^3}{3}\right) d\omega = \frac{\sin\left(\frac{b^3}{3}\right)}{b^2} + 2 \int_0^b \frac{1}{\omega^3} \sin\left(\frac{\omega^3}{3}\right) d\omega \quad (2.15)$$

We know the first term on the right hand side vanishes as $b \rightarrow \infty$. The second term on the right hand side is not improper at $\omega = 0$ since

$$\lim_{\omega \rightarrow 0} \left| \frac{\sin\left(\frac{\omega^3}{3}\right)}{3\omega^3/3} \right| = \frac{1}{3}.$$

Hence, the integral exists on $[0, 1]$. Now, since

$$\begin{aligned} \int_1^b \left| \frac{1}{\omega^3} \sin\left(\frac{\omega^3}{3}\right) \right| d\omega &\leq \int_1^b \frac{1}{\omega^3} d\omega \\ &= \left. \frac{-1}{2} \cdot \frac{1}{\omega^2} \right|_1^b \\ &= \frac{1}{2} - \frac{1}{2b^2} \end{aligned}$$

is finite as $b \rightarrow \infty$, we conclude that $\frac{1}{\omega} \sin(\frac{\omega^3}{3})$ is improperly integrable on $(0, \infty)$.

Therefore, the limit of the integral term on the right hand side of (2.13) as $b \rightarrow \infty$ exists when $x = 0$ or $x \neq 0$. Thus, we conclude that

$$\int_0^{\infty} \cos\left(\frac{\omega^3}{3} + \omega x\right) d\omega = \lim_{b \rightarrow \infty} \int_0^b \cos\left(\frac{\omega^3}{3} + \omega x\right) d\omega$$

exists for all $x \in (-\infty, \infty)$. However, the Airy integral diverges when x is not real ([10], p. 53), but for the solution of (2.1), x is always in the interval $(-\infty, \infty)$.

The graphical representation of the Airy function [10] that appears below in Figure 2.1 was generated by *Mathematica*TM.

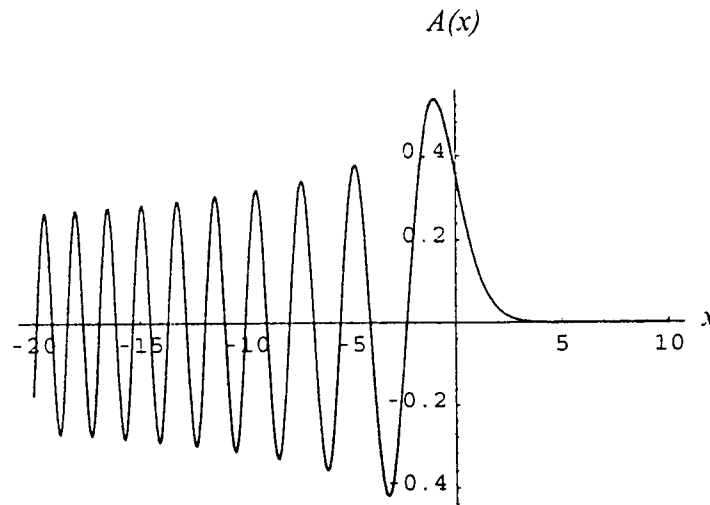


Figure 2.1 The Airy function

From the graph we notice that for x positive $A(x)$ is positive while $A'(x)$ is negative, and for x negative the function features rapid oscillations. We also notice that there are infinitely many zeros for x negative and none for x positive.

4. Discussion

In this chapter we have been able to solve the initial-boundary value problem for the linearized KdV equation and Airy's equation by using Fourier transforms. This technique could also be applied to solve other linearized KdV equations such as (1.11). Our solution to the initial-boundary value problem on an infinite space interval was formulated in terms of the special Airy function (2.11) which has interesting properties of its own.

However, Drazin and Johnson [4] approach the initial value problem for the nonlinear equation $u_t - 6uu_x + u_{xxx} = 0$ by the inverse scattering method. With an initial condition of $u(x,0) = -U_0 \delta(x)$, where U_0 is a constant and $\delta(x)$ is the Dirac delta function, they obtain a solution that is an oscillatory dispersive wave and indicate that the relationship between this wave and the oscillatory part of the Airy function is quite close. In particular, this is true when the wave has "vanishingly small amplitude" since in this case the solution must satisfy our linearized equation (2.1).

CHAPTER III

A FINITE SPACE INITIAL-BOUNDARY VALUE PROBLEM

1. Statement and Solution of the Problem

We now consider the linearized KdV equation on a finite interval. Consequently, we must impose boundary conditions at the ends of the interval, and since we have a third order derivative in the equation, we need three boundary conditions. Thus, an initial-boundary value problem with initial wave displacement can be formulated as

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^3 u}{\partial x^3}, & 0 < x < l, \quad t > 0 \\ u(0,t) &= 0 = u(l,t), & t > 0 \\ \frac{\partial u}{\partial x}(0,t) &= 0, & t > 0 \\ u(x,0) &= f(x), & 0 < x < l.\end{aligned}\tag{3.1}$$

We can solve (3.1) by using the separation of variables method.

We assume $u(x, t) = X(x)T(t)$ so that

$$\frac{\partial u}{\partial t} = X(x)\dot{T}(t) \quad \text{and} \quad \frac{\partial^3 u}{\partial x^3} = X'''(x)T(t),$$

where the dot and prime notation represent differentiation with respect to t and x respectively. Thus, the differential equation in (3.1) becomes

$$X(x)\dot{T}(t) = X'''(x)T(t),$$

and we can write

$$\frac{\dot{T}(t)}{T(t)} = \frac{X'''(x)}{X(x)} = \lambda, \tag{3.2}$$

where λ is a constant. This is valid since the left hand side is a function of t only, and the right hand side is a function of x only. Thus, we obtain two ordinary differential equations from (3.2), namely,

$$\dot{T}(t) = \lambda T(t) \quad \text{and} \quad X'''(x) = \lambda X(x)$$

or simply

$$\dot{T} - \lambda T = 0 \quad \text{and} \quad X''' - \lambda X = 0.$$

A solution of the first differential equation is

$$T(t) = c e^{\lambda t} \quad (3.3)$$

where c is an arbitrary constant. From the second differential equation and the boundary conditions in (3.1), we obtain the following eigenvalue problem:

$$\begin{aligned} X'''(x) - \lambda X(x) &= 0, & 0 < x < l \\ X(0) &= 0 = X(l) \\ X'(0) &= 0. \end{aligned} \quad (3.4)$$

Thus, $u(x, t) = c e^{\lambda t} X(x)$ solves the partial differential equation and the boundary conditions in (3.1), where $X(x)$ is a nontrivial solution of (3.4).

We need to find the eigenvalues and eigenfunctions of (3.4). We assume that λ is real. We first show that $\lambda = 0$ is not an eigenvalue of (3.4). If we assume $\lambda = 0$, then the differential equation becomes $X''' = 0$, and the general solution is $X(x) = c_1 x^2 + c_2 x + c_3$. However, the boundary conditions imply $c_1 = c_2 = c_3 = 0$ so that $X(x)$ is the trivial solution. Therefore, $\lambda = 0$ is not an eigenvalue.

Now consider the case where $\lambda \neq 0$. We assume a solution of the form

$X(x) = e^{\alpha x}$. Then

$$X''' - \lambda X = 0 \Rightarrow \alpha^3 e^{\alpha x} - \lambda e^{\alpha x} = 0,$$

and consequently,

$$\alpha^3 - \lambda = 0$$

since $e^{\alpha x} \neq 0$. For convenience we let $\lambda = (2\omega)^3$. The roots of the characteristic equation

$$\alpha^3 - 8\omega^3 = 0$$

are $\alpha_1 = 2\omega$ and $\alpha_{2,3} = -\omega \pm i\omega\sqrt{3}$. Thus, the general solution of the differential equation in (3.4) is

$$X(x) = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + c_3 e^{\alpha_3 x}$$

for constants c_1, c_2 , and c_3 , and from Euler's formula we have

$$X(x) = c_1 e^{2\omega x} + e^{-\omega x} (c_2 \cos\sqrt{3}\omega x + c_3 \sin\sqrt{3}\omega x).$$

The boundary conditions $X(0) = 0$ and $X'(0) = 0$ imply

$$c_1 = -c_2 \quad \text{and} \quad c_2 = c_3 \frac{\sqrt{3}}{3}.$$

Thus, we can write $X(x)$ in terms of one constant k , i.e.,

$$X(x) = -\frac{\sqrt{3}}{3} k e^{2\omega x} + e^{-\omega x} \left(\frac{\sqrt{3}}{3} k \cos\sqrt{3}\omega x + k \sin\sqrt{3}\omega x \right).$$

While we cannot solve for ω explicitly, we use the remaining boundary condition

$X(l) = 0$ to derive a transcendental equation that ω must satisfy. For example,

$X(l) = 0$ implies

$$k \left[-\frac{\sqrt{3}}{3} e^{2\omega l} + e^{-\omega l} \left(\frac{\sqrt{3}}{3} \cos \sqrt{3} \omega l + \sin \sqrt{3} \omega l \right) \right] = 0.$$

If k is an arbitrary constant, then

$$\frac{\sqrt{3}}{3} e^{2\omega l} = e^{-\omega l} \left(\frac{\sqrt{3}}{3} \cos \sqrt{3} \omega l + \sin \sqrt{3} \omega l \right).$$

Now, multiplying both sides by $\frac{3}{\sqrt{3}} e^{\omega l}$ and dividing by 2, we obtain

$$\begin{aligned} \frac{e^{3\omega l}}{2} &= \frac{1}{2} \cos \sqrt{3} \omega l + \frac{\sqrt{3}}{2} \sin \sqrt{3} \omega l \\ &= \sin \frac{\pi}{6} \cos \sqrt{3} \omega l + \cos \frac{\pi}{6} \sin \sqrt{3} \omega l. \end{aligned}$$

Thus,

$$\frac{e^{3\omega l}}{2} = \sin \left(\frac{\pi}{6} + \sqrt{3} \omega l \right) \quad (3.5)$$

is the eigenvalue equation for (3.4).

We solve graphically for ω in Figure 3.1.

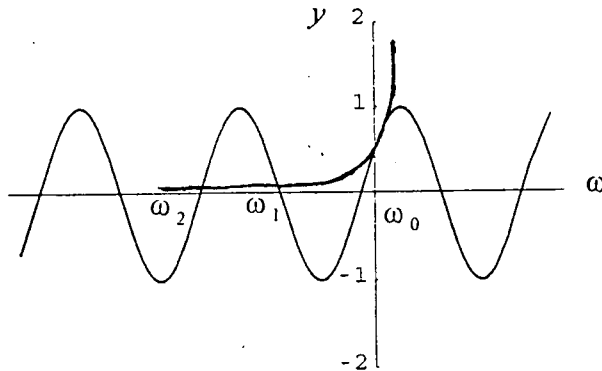


Figure 3.1 Eigenvalues for (3.4)

We label the intersection points $\omega_0, \omega_1, \omega_2, \dots$. From Figure 3.1 and analysis of (3.5) we see the ω_n are less than zero for $n = 1, 2, 3, \dots$, and $\omega_0 = 0$. However, since $\lambda_0 = 8\omega_0^3 = 0$ is not an eigenvalue, this solution of (3.5) is extraneous. Therefore, the eigenvalues of (3.4) are $\lambda_i = 8\omega_i^3$ for $i = 1, 2, 3, \dots$, and the corresponding eigenfunctions are

$$X_i(x) = -\frac{\sqrt{3}}{3}e^{2\omega_i x} + e^{-\omega_i x} \left(\frac{\sqrt{3}}{3} \cos \sqrt{3}\omega_i x + \sin \sqrt{3}\omega_i x \right). \quad (3.6)$$

By the extended superposition principle, we can now write the formal solution $u(x, t)$ of (3.1) as an infinite sum,

$$u(x, t) = \sum_{i=1}^{\infty} c_i e^{(8\omega_i^3)t} X_i(x), \quad (3.7)$$

where the c_i are arbitrary constants. In the classical theory of regular Sturm-Liouville eigenvalue problems, we could determine the constants c_i by employing the orthogonality of the eigenfunctions and invoking the initial condition. However, the eigenfunctions are only guaranteed to be orthogonal if the eigenvalue problem is self-adjoint. Unfortunately, (3.4) is not self-adjoint.

To obtain the adjoint problem, we seek an operator M such that the integral

$$\int_0^l [Y L(X) - X M(Y)] dx$$

is zero. To determine M we consider

$$\begin{aligned} YL(X) &= X'''Y \\ &= (X''Y)' - X''Y' \\ &= (X''Y)' - [(X'Y')' - X'Y''] \end{aligned}$$

$$\begin{aligned}
&= (X''Y - X'Y')' + [(XY'')' - XY'''] \\
&= (X''Y - X'Y' + XY'')' - XY'''.
\end{aligned}$$

Thus, we define $M(Y) = -Y'''$ so that the Lagrange identity

$$YL(X) - XM(Y) = (X''Y - X'Y' + XY'')',$$

and we can write (Green's formula)

$$\begin{aligned}
\int_0^l [YL(X) - XM(Y)] dx &= (X''Y - X'Y' + XY'') \Big|_0^l \\
&= X''(l)Y(l) - X''(0)Y(0) - X'(l)Y'(l).
\end{aligned}$$

Hence, the appropriate boundary conditions on Y are $Y(0) = Y(l) = Y'(l) = 0$, and the operator L is not self-adjoint since L does not equal M . Thus, the adjoint problem of (3.4) is

$$\begin{aligned}
Y'''(x) + \mu Y(x) &= 0, \quad 0 < x < l \\
Y(0) &= 0 = Y(l) \\
Y'(l) &= 0.
\end{aligned} \tag{3.8}$$

We can determine the eigenvalues and eigenfunctions $Y_j(x)$ for the adjoint eigenvalue problem (3.8). First, assume $Y(x) = e^{\beta x}$ to be a solution of the differential equation in (3.8). Then the characteristic equation is

$$\beta^3 + \mu = 0.$$

For convenience, we let $\mu = (2\theta)^3$. Then the roots of the characteristic equation are

$$\beta_1 = -2\theta \quad \text{and} \quad \beta_{2,3} = \theta \pm i\theta\sqrt{3}.$$

Consequently,

$$Y(x) = c_1 e^{-2\theta x} + e^{\theta x} (c_2 \cos \theta \sqrt{3} x + c_3 \sin \theta \sqrt{3} x)$$

is the general solution for the differential equation in (3.8). From the boundary conditions $Y(0) = 0$ and $Y(l) = 0$, we can write this general solution in terms of one constant k , i.e.,

$$Y(x) = k[e^{-2\theta x} - e^{\theta x} \cos \theta \sqrt{3}x + e^{\theta x} B \sin \theta \sqrt{3}x]$$

where $B = \frac{\cos \theta \sqrt{3}l - e^{-3\theta l}}{\sin \theta \sqrt{3}l}$ provided $\theta \neq \frac{n\pi}{\sqrt{3}l}$, $n = 0, \pm 1, \pm 2, \dots$. This restriction

on θ is established in section 3. In a manner analogous to the derivation of (3.5), the third boundary condition $Y'(l) = 0$ results in the transcendental equation

$$\frac{e^{3\theta l}}{2} = \sin\left(\frac{\pi}{6} + \theta \sqrt{3}l\right)$$

which θ must satisfy if k is arbitrary. This eigenvalue equation is equivalent to (3.5); therefore, $\theta_i = \omega_i$ for $i = 1, 2, 3, \dots$. Thus, the eigenvalues of (3.8) are $\mu_i = 8\theta_i^3$ for $i = 1, 2, 3, \dots$, and their corresponding eigenvectors are

$$Y_i(x) = e^{-2\theta_i x} + e^{\theta_i x} (B_i \sin \theta_i \sqrt{3}x - \cos \theta_i \sqrt{3}x) \quad (3.9)$$

where $B_i = \frac{\cos \theta_i \sqrt{3}l - e^{-3\theta_i l}}{\sin \theta_i \sqrt{3}l}$.

Since the operator L is not self-adjoint, then the eigenvalue problem (3.4) is not self-adjoint. Therefore, we cannot be certain that the eigenfunctions of (3.4) are orthogonal. However, from Theorem 5.2 of Coddington and Levinson [2], there is a bi-orthogonality relationship between the eigenfunctions of (3.4) and the eigenfunctions of the adjoint problem (3.8) if the Green's function for the operator in (3.4) has only simple

poles at each eigenvalue of (3.4). We shall find the Green's function in section 2 and show it has only simple poles in section 3. The relationship between X_i and Y_j is

$$\langle X_i, Y_j \rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker delta symbol. Thus

$$\int_0^l X_i(x) Y_j(x) dx = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

As a result of the bi-orthogonality of the sets $\{X_i\}$ and $\{Y_j\}$, we can solve for the constants in (3.7) by using the initial condition, $u(x,0) = f(x)$. Using the initial condition in (3.7), we obtain

$$u(x,0) = f(x) = \sum_{i=1}^{\infty} c_i X_i(x).$$

Multiplying by Y_j and integrating from 0 to l , we have

$$\begin{aligned} \int_0^l f(x) Y_j(x) dx &= \sum_{i=1}^{\infty} c_i \int_0^l X_i(x) Y_j(x) dx \\ &= c_j. \end{aligned}$$

Thus,

$$c_i = \int_0^l f(x) Y_i(x) dx. \quad (3.10)$$

In conclusion, the formal solution of (3.1) is given by (3.7) where the c_i are determined by (3.10).

2. Derivation Of The Green's Function

In order to show the hypothesis of Theorem 5.2 [2] is satisfied, we must derive the Green's function for the nonhomogeneous problem

$$\begin{aligned} X'''(x) - \lambda X(x) &= f(x), & 0 < x < l \\ X(0) &= 0 = X(l) \\ X'(0) &= 0. \end{aligned} \tag{3.11}$$

We can write (3.11) as a system

$$\begin{pmatrix} X \\ X' \\ X'' \end{pmatrix}'(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ X' \\ X'' \end{pmatrix}(x) + \begin{pmatrix} 0 \\ 0 \\ f(x) \end{pmatrix},$$

and we indicate the boundary conditions by the matrices

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

such that

$$\Gamma \begin{pmatrix} X(0) \\ X'(0) \\ X''(0) \end{pmatrix} + \Psi \begin{pmatrix} X(l) \\ X'(l) \\ X''(l) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we assume λ is not an eigenvalue of the associated homogeneous problem, then we can write as in [8]

$$\begin{pmatrix} X \\ X' \\ X'' \end{pmatrix}(x) = \int_0^l \mathcal{G}(x, s, \lambda) \begin{pmatrix} 0 \\ 0 \\ f(s) \end{pmatrix} ds$$

where $\mathcal{G}$ is a 3x3 matrix such that if $\Phi(x, \lambda)$ is a fundamental matrix for the associated homogeneous problem with $\Phi(0, \lambda) = I$, the identity matrix, then

$$\mathcal{G}(x, s, \lambda) = \begin{cases} \Phi(x, \lambda) \mathcal{L}^{-1}(\lambda) \Gamma \Phi^{-1}(s, \lambda) & 0 < s \leq x < l \\ -\Phi(x, \lambda) \mathcal{L}^{-1}(\lambda) \Psi \Phi(l, \lambda) \Phi^{-1}(s, \lambda) & 0 < x < s < l \end{cases} \quad (3.12)$$

and $\mathcal{L}(\lambda) = \Gamma + \Psi \Phi(l, \lambda)$. The Green's function G is defined as an element of $\mathcal{G}$ such that

$$X(x) = \int_0^l \mathcal{G}_{1,3}(x, s, \lambda) f(s) ds = \int_0^l G(x, s, \lambda) f(s) ds \quad (3.13)$$

where λ is not an eigenvalue of the associated homogeneous problem. We find G for the cases when $\lambda = 0$ and $\lambda \neq 0$ since the fundamental matrix differs in these two cases.

For the first case, $\lambda = 0$ is not an eigenvalue. The basis elements for our fundamental matrix are $x^2, x, 1$ since $X''' = 0$ implies $X(x) = c_1 x^2 + c_2 x + c_3$. Therefore,

$$\Phi(x, 0) = \begin{pmatrix} 1 & x & \frac{1}{2}x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \Phi^{-1}(x, 0) = \begin{pmatrix} 1 & -x & \frac{1}{2}x^2 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{L}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & l & \frac{1}{2}l^2 \\ 0 & 1 & l \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & l & \frac{1}{2}l^2 \end{pmatrix}$$

so that

$$\mathcal{L}^{-1}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{l^2} & -\frac{2}{l} & \frac{2}{l^2} \end{pmatrix}.$$

Now, substituting the appropriate matrices into (3.12) and calculating the matrix products, we have the Green's function

$$\begin{aligned} G(x, s, 0) &= \mathcal{G}_{1,3}(x, s, 0) \\ &= \begin{cases} \frac{1}{2}s^2 - xs + \frac{s}{l}\left(1 - \frac{s}{2l}\right)x^2 & 0 < s \leq x < l \\ -\frac{1}{2}x^2\left(\frac{s}{l} - 1\right)^2 & 0 < x < s < l. \end{cases} \end{aligned}$$

For the second case, we assume $\lambda \neq 0$ and λ is not an eigenvalue for the associated homogeneous problem. Recall the general solution of the homogeneous differential equation in (3.11) is

$$X(x) = c_1 e^{2\omega x} + c_2 e^{-\omega x} \cos \sqrt{3}\omega x + c_3 e^{-\omega x} \sin \sqrt{3}\omega x$$

where $\lambda = (2\omega)^3$. Thus, the basis elements for a fundamental matrix of the homogeneous problem are

$$e^{2\omega x}, \quad e^{-\omega x} \cos \sqrt{3}\omega x, \quad e^{-\omega x} \sin \sqrt{3}\omega x,$$

and the fundamental matrix is

$$\Phi(x, \omega) = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{pmatrix}(x)$$

where $\phi_1 = e^{2\omega x}$, $\phi_2 = e^{-\omega x} \cos \sqrt{3} \omega x$, $\phi_3 = e^{-\omega x} \sin \sqrt{3} \omega x$. We denote the inverse of $\Phi(x, \omega)$ by

$$\Phi^{-1}(x, \omega) = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}(x).$$

At this point we could find G by computing $\mathcal{G}$ as in (3.12). However, the matrix multiplication would be quite tedious. Instead, we use the variation of constants

formula [8]

$$\begin{pmatrix} X \\ X' \\ X'' \end{pmatrix}(x) = \Phi(x, \omega) \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \int_0^x \Phi^{-1}(s, \omega) \begin{pmatrix} 0 \\ 0 \\ f(s) \end{pmatrix} ds \right\} \quad (3.14)$$

to find the Green's function since by definition (3.13)

$$X(x) = \int_0^l \mathcal{G}_{1,3}(x, s, \lambda) f(s) ds = \int_0^l G(x, s, \lambda) f(s) ds$$

Thus, from (3.14)

$$\begin{aligned}
X(x) &= c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) \\
&+ \int_0^x [\phi_1(s)\beta_{13}(x) + \phi_2(s)\beta_{23}(x) + \phi_3(s)\beta_{33}(x)]f(s)ds,
\end{aligned} \tag{3.15}$$

and our objective is to write (3.15) as an integral from 0 to l .

From the boundary conditions $X(0) = 0$ and $X'(0) = 0$, we determine the constants c_2 and c_3 in terms of the constant c_1 . Specifically, $X(0) = 0$ implies $c_1 + c_2 = 0$. Therefore, $c_2 = -c_1$. Next, consider the derivative of (3.15), i.e.,

$$\begin{aligned}
X'(x) &= c_1\phi_1'(x) + c_2\phi_2'(x) + c_3\phi_3'(x) \\
&+ \frac{d}{dx} \int_0^x [\phi_1(s)\beta_{13}(x) + \phi_2(s)\beta_{23}(x) + \phi_3(s)\beta_{33}(x)]f(s)ds \\
&= c_1\phi_1'(x) + c_2\phi_2'(x) + c_3\phi_3'(x) \\
&+ [\phi_1(x)\beta_{13}(x) + \phi_2(x)\beta_{23}(x) + \phi_3(x)\beta_{33}(x)]f(x) \\
&+ \int_0^x [\phi_1'(s)\beta_{13}(x) + \phi_2'(s)\beta_{23}(x) + \phi_3'(s)\beta_{33}(x)]f(s)ds.
\end{aligned}$$

Then $X'(0) = 0$ implies

$$\begin{aligned}
0 &= c_1\phi_1'(0) - c_1\phi_2'(0) + c_3\phi_3'(0) \\
&+ [\phi_1(0)\beta_{13}(0) + \phi_2(0)\beta_{23}(0) + \phi_3(0)\beta_{33}(0)]f(0).
\end{aligned} \tag{3.16}$$

Evaluating ϕ_1, ϕ_2, ϕ_3 at zero, we have

$$\phi_1(0) = 1, \phi_2(0) = 1, \phi_3(0) = 0.$$

Further, the inverse of the fundamental matrix, $\Phi^{-1}(x, \omega)$, can be found by using the computer package *Mathematica*TM. Thus, we determine that

$$\beta_{13}(x) = \frac{1}{12\omega^2 e^{2\omega x}},$$

$$\beta_{23}(x) = e^{\omega x} \left(\frac{\sqrt{3} \sin \sqrt{3} \omega x - \cos \sqrt{3} \omega x}{12\omega^2} \right),$$

$$\beta_{33}(x) = -e^{\omega x} \left(\frac{\sin \sqrt{3} \omega x + \sqrt{3} \cos \sqrt{3} \omega x}{12\omega^2} \right).$$

Therefore, (3.16) becomes

$$c_1 \phi'_1(0) - c_1 \phi'_2(0) + c_3 \phi'_3(0) = 0 \quad (3.17)$$

since $\beta_{13}(0) + \beta_{23}(0) = 0$ and $\phi_3(0) = 0$. Since

$$\begin{aligned} \phi'_1(x) &= 2\omega e^{2\omega x}, \\ \phi'_2(x) &= -\omega e^{-\omega x} \cos \sqrt{3} \omega x - \omega \sqrt{3} e^{-\omega x} \sin \sqrt{3} \omega x, \\ \phi'_3(x) &= -\omega e^{-\omega x} \sin \sqrt{3} \omega x + \omega \sqrt{3} e^{-\omega x} \cos \sqrt{3} \omega x, \end{aligned}$$

from (3.17), we have $c_3 = -\sqrt{3} c_1$.

The last boundary condition $X(l) = 0$ allows us to determine the constant c_1 in terms of l . Thus

$$\begin{aligned}
0 &= c_1 \phi_1(l) - c_1 \phi_2(l) - c_1 \sqrt{3} \phi_3(l) \\
&+ \int_0^l [\phi_1(l) \beta_{13}(s) + \phi_2(l) \beta_{23}(s) + \phi_3(l) \beta_{33}(s)] f(s) ds,
\end{aligned}$$

that is,

$$c_1 = -\frac{1}{h(l)} \int_0^l p(l, s) f(s) ds \quad (3.18)$$

where for convenience we define the two functions

$$\begin{aligned}
h(x) &= \phi_1(x) - \phi_2(x) - \sqrt{3} \phi_3(x), \\
p(x, s) &= \phi_1(x) \beta_{13}(s) + \phi_2(x) \beta_{23}(s) + \phi_3(x) \beta_{33}(s).
\end{aligned}$$

Returning to (3.15), we can rewrite X

$$X(x) = c_1 h(x) + \int_0^x p(x, s) f(s) ds. \quad (3.19)$$

Substituting (3.18) into (3.19), we obtain

$$\begin{aligned}
X(x) &= h(x) \left[-\frac{1}{h(l)} \int_0^l p(l, s) f(s) ds \right] + \int_0^x p(x, s) f(s) ds \\
&= -\frac{h(x)}{h(l)} \left[\int_0^x p(l, s) f(s) ds + \int_x^l p(l, s) f(s) ds \right] + \int_0^x p(x, s) f(s) ds \\
&= \int_0^x \left[p(x, s) - \frac{h(x)}{h(l)} p(l, s) \right] f(s) ds - \int_x^l \frac{h(x)}{h(l)} p(l, s) f(s) ds.
\end{aligned}$$

Therefore, the Green's function of (3.11) for $\lambda = 8\omega^3 \neq 0$ and λ not an eigenvalue of the associated homogeneous problem is

$$G(x, s, \omega) = \begin{cases} \frac{h(l)p(x, s) - h(x)p(l, s)}{h(l)} & \text{for } 0 < s \leq x < l \\ -\frac{h(x)}{h(l)}p(l, s) & \text{for } 0 < x < s < l. \end{cases} \quad (3.20)$$

We notice that for both cases the Green's function $G(x, s, \omega) \neq G(s, x, \omega)$, which is another indication that the eigenvalue problem (3.4) is not self-adjoint. It can also be shown by a straightforward but tedious computation that the Green's function satisfies the appropriate four properties;

(1) G solves the differential equation

$$G''' - \lambda G = 0 \text{ on } 0 < x < s \text{ and } s < x < l,$$

(2) G satisfies the boundary conditions,

$$G(0, s, \omega) = G(l, s, \omega) = \frac{\partial G}{\partial x}(0, s, \omega) = 0,$$

(3) G and G' are continuous at $x = s$,

$$(4) \frac{\partial^2}{\partial x^2} G(s^+, s, \omega) - \frac{\partial^2}{\partial x^2} G(s^-, s, \omega) = 1,$$

that are alternatively used to define this special function.

3. Determination of the Simple Poles of G

The bi-orthogonality result, Theorem 5.2 of [2], for problem (3.4) can be stated as follows:

Theorem: If all the poles of the Green's function G of (3.11) as a function of λ are simple and if the eigenfunctions of (3.4) are $\{X_i\}$, then the eigenfunctions of the adjoint problem (3.8) can be arranged in a sequence $\{Y_j\}$ so that

$$\langle X_i, Y_j \rangle = \delta_{ij}.$$

If $\lambda_p = 8\omega_p^3$ is an eigenvalue for (3.4) with eigenfunction X_p , then $\mu = 8\theta_p^3$ is an eigenvalue of (3.8) with eigenfunction Y_p where $\theta_p = \omega_p$.

Thus, for each ω_i determined graphically in Figure 3.1, we must show the Green's function $G(x, s, \omega)$ has a simple pole at ω_i . We do not need to consider the case $G(x, s, 0)$ since $\lambda_i = 8\omega_i^3 = 0$ is not an eigenvalue of (3.4).

We shall approach this problem by considering the function $\Delta(\omega)$ in the denominator of the Green's function and showing that $\Delta(\omega)$ has only simple zeros. Thus, the Green's function will have only simple poles. For a formal definition of $\Delta(\omega)$ or $\Delta(\lambda)$ see Miller [8]. In our case

$$\begin{aligned} \Delta(\omega) &= \det \begin{pmatrix} 1 & 1 & 0 \\ e^{2\omega l} & e^{-\omega l} \cos \sqrt{3} \omega l & e^{-\omega l} \sin \sqrt{3} \omega l \\ 2\omega & -\omega & \omega \sqrt{3} \end{pmatrix} \\ &= -\omega \sqrt{3} e^{2\omega l} + \omega \sqrt{3} e^{-\omega l} \cos \sqrt{3} \omega l + 3\omega e^{-\omega l} \sin \sqrt{3} \omega l \\ &= -\omega \sqrt{3} (e^{2\omega l} - e^{-\omega l} \cos \sqrt{3} \omega l - \sqrt{3} e^{-\omega l} \sin \sqrt{3} \omega l) \quad (3.21) \end{aligned}$$

and $\Delta(\omega) = -\omega \sqrt{3} h(l)$. Therefore, we can write (3.20) as

$$G(x,s,\omega) = \begin{cases} \frac{-\omega\sqrt{3} [h(l)p(x,s) - h(x)p(l,s)]}{\Delta(\omega)} & \text{for } 0 < s \leq x < l \\ \frac{\omega\sqrt{3}}{\Delta(\omega)} [h(x)p(l,s)] & \text{for } 0 < x < s < l. \end{cases}$$

Now, to show $\Delta(\omega)$ has only simple zeros, we write (3.21) as

$$\begin{aligned} \Delta(\omega) &= -\omega\sqrt{3} e^{-\omega l} (e^{3\omega l} - \cos\sqrt{3}\omega l - \sqrt{3} \sin\sqrt{3}\omega l) \\ &= -\omega\sqrt{3} e^{-\omega l} \left[e^{3\omega l} - 2 \sin\left(\frac{\pi}{6} + \sqrt{3}\omega l\right) \right]. \end{aligned}$$

Notice if we set the bracket equal to zero, then we have the eigenvalue equation (3.5).

Thus, for a particular eigenvalue ω_j , $g(\omega_j) = 0$ where

$$g(\omega) = e^{3\omega l} - 2 \sin\left(\frac{\pi}{6} + \sqrt{3}\omega l\right).$$

To show the derivative

$$g'(\omega_j) = 3l e^{3\omega_j l} - 2l\sqrt{3} \cos\left(\frac{\pi}{6} + \sqrt{3}\omega_j l\right) \neq 0$$

we assume the contrary. If $g(\omega_j) = 0 = g'(\omega_j)$, then $\omega_j = \frac{n\pi}{\sqrt{3}l}$ for some $n = -1, -2, \dots$

However, from the boundary conditions of (3.4) $X_j(l)$ must vanish, i.e.,

$$\begin{aligned}
0 &= X_j(l) \\
&= -\frac{\sqrt{3}}{3} e^{2\omega_j l} + \frac{\sqrt{3}}{3} e^{-\omega_j l} \cos n\pi + e^{-\omega_j l} \sin n\pi \\
&= -\frac{\sqrt{3}}{3} e^{2\omega_j l} + \frac{\sqrt{3}}{3} e^{-\omega_j l} \cos n\pi.
\end{aligned}$$

If n is odd, then we obtain $e^{3\omega_j l} = -1$, and if n is even, then $e^{3\omega_j l} = 1$. In either case we have a contradiction which implies $g'(\omega_j) \neq 0$. Therefore, $\Delta(\omega)$ has only simple zeros for each ω_j , $j = 1, 2, 3, \dots$ which is an eigenvalue of (3.5), and we can conclude that the Green's function $G(x, s, \omega)$ has only simple poles for each ω_j . Thus, the hypothesis of Theorem 5.2 is satisfied, and we have bi-orthogonality between the eigenfunctions $\{X_i\}$ of (3.4) and the eigenfunctions $\{Y_j\}$ of the adjoint problem (3.8).

4. Conclusions

In section 1 we were able to use the separation of variables method for an initial-boundary value problem for a third order partial differential equation, in particular, the linearized KdV equation. However, since our problem was non-self-adjoint, we could not guarantee the eigenfunctions of the eigenvalue problem were orthogonal. Thus, a consideration of the adjoint problem was needed. We obtained a bi-orthogonality relationship between the eigenfunctions of the eigenvalue problem (3.4) and its adjoint (3.8) since the Green's function for the nonhomogeneous problem (3.11) has simple poles at each eigenvalue of (3.4). Consequently, we were able to determine a formal solution (series expansion) of problem (3.1).

For our KdV initial-boundary value problem (3.1), we determined that the Green's function had simple poles since the ω_j satisfied a relatively simple transcendental equation (3.5). If we change the boundary conditions in (3.1) to $u(0,t) = u_x(0,t) = u_x(l,t) = 0$, for $t > 0$, then the eigenvalue equation would be $e^{3\omega l} = 2 \cos(\frac{\pi}{3} + \sqrt{3}\omega l)$. However, if we consider other boundary conditions and use the separation of variables method, we may not be able to determine the transcendental equation and/or the eigenvalues of the relevant eigenvalue problem. Consequently, determining whether the Green's function has simple poles at each eigenvalue might be impossible. Thus, a project for future study would be to show conclusively that the eigenvalues of (3.4) cannot be complex values and to determine what other set of boundary conditions would be amenable to the analysis developed in this thesis.

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VITA

Melissa Smith was born in Columbus, Ohio and later attended the public schools of Westerville, Ohio where she graduated from Westerville North High School in 1989. The following year she started her undergraduate studies at John Carroll University in Cleveland, Ohio where she majored in Mathematics. After four years, she graduated from John Carroll University. In the fall of 1993, she began graduate school at the University of Tennessee with an interest in applied mathematics which developed from participating in two summer internships. In the summer of 1991 she worked at Argonne National Laboratory in Argonne, Illinois. Her project was entitled "A Parametric Study of Air Dispersed Radionuclides." The following summers, 1992 and 1993, she worked in Columbus, Ohio at Battelle Memorial Institute as a research intern. While at the University of Tennessee, she has had the opportunity to teach Intermediate Algebra and Calculus for Business Majors and play rugby. She will receive her Master of Science degree with a major in Mathematics from the University of Tennessee in August 1995.