To the Graduate Council:

I am submitting herewith a dissertation written by Georges Halim Guirguis entitled "On the Existence, Uniqueness, Regularity and Approximation of the Exterior Stokes Problem in R^3". I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Max D. Gunzburger

Max D. Gunzburger, Major Professor

We have read this dissertation
and recommend its acceptance:

Steven M. Serbin

Vassilios Dougalis

Ohannes Karakashian

Robert Todd Gregory

Vasilios Alexiades

Accepted for the council:

C. W. Minkel

The Graduate School

ON THE EXISTENCE UNIQUENESS REGULARITY AND APPROXIMATION OF THE EXTERIOR STOKES PROBLEM IN R^3

A Dissertation

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Degree

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Georges Halim Guirguis

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ABSTRACT

Conditions under which a solution exists and is unique for the stationary Stokes problem in exterior domains in R^3 are discussed. The domains considered are the complements in R^3 of bounded star shaped sets and the class of functions for which existence is displayed have "*physically reasonable*" behavior at infinity. This class is characterized by a finite Dirichlet integral. We use a method that employs a variational formulation of the problem and avoids the use of a Helmholtz decomposition of vector fields in unbounded domains.

Then we show that a problem on a finite domain denoted by Ω_R approximates the problem on the unbounded domain Ω. An artificial boundary $\delta\Omega_R$ is introduced and different boundary conditions are imposed. We study two cases of artificial boundary conditions, namely the zero velocity boundary condition and the zero stress boundary condition. Approximation results are dicussed. In particular, we emphasize the case where the body force belong to $L^2(\Omega)$ and has compact support in Ω.

Next, we treat the finite element approximation of the truncated problem. The case of zero velocity artificial boundary condition is treated separately from the case of zero stress boundary condition. Error estimates for the different cases are derived. The final estimates between the problem on the unbounded domain and the discrete problem reduce to a two parameter approximation. To balance the truncation error with the discretization error, we consider a mesh grading technique.

Finally, we treat a truncated problem in which we impose exact boundary condtions at the artificial boundary $\delta\Omega_R$. This is achieved by studying the physical conditions associated with the hydrodynamic equilibrium of the fluid patches inside

and outside Ω_R. Actually, the solution, outside the support of the body forces, can be written explicitly in terms of a boundary integral involving the velocity and the stresses at the boundary. Existence and uniqueness of the coupled problem is studied as well as finite element approximations which turn out to be one parameter approximations.

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CHAPTER 1
INTRODUCTION

The equations describing the natural laws of conservation of mass and linear momentum for the steady incompressible viscous fluid are the Navier-Stokes equations. These constitute a general model which is valid for low as well high fluid viscosities. The equations are a non-linear system of partial differential equations. There is also the linearized model, known as the Stokes equations which provides a good mathematical description of the motion of a fluid at low Reynolds number [17].

Answering the basic mathematical questions concerning the existence, uniqueness, regularity and approximation of the solution of the Navier-Stokes (or even Stokes) problem has been of substantial interest to mathematicians and has led to important mathematical achievments; in particular, we mention variational techniques of Brezzi [4] and Nicolaides [21]. Also, we mention the mathematical development of the finite element approximation to the Stokes and Navier-Stokes problem, e.g. [2], [9], [6], [24], [14], [10] and [15].

So far, the study of the Navier-Stokes (or Stokes) system for incompressible viscous flows has been largely restricted to the case of bounded domains. The rigorous mathematical study of these systems, even in bounded domains, is not an easy task to achieve. The difficulty arises from the combinations of results needed from the fields of functional analysis as well as approximation theory. First, one has to choose a function space in which to pose the problem. Then, using the available techniques from functional analysis, one has to determine and verify the conditions under which existence and uniqueness of the solution is guaranteed in that function space. Finally, approximating the soltuion by either finite differences or finite elements may be considered.

If the problem on a bounded domain presents interesting mathematical questions, the problem on the unbounded domain, in particular, an exterior domain, is even more challenging due to the fact that we lack some of the important mathematical tools available in the case of bounded domains, e.g., compactness results of imbeddings of Sobolev spaces. The study could introduce new prospects in the theory of function spaces as well as numerical approximation. For the latter we mention in particular [11], [20], [19] and [22] where different techniques in the approximation were used for the exterior Helmholtz equation and Laplace equations. In our work, we study the linear model known as the Stokes problem in three dimensions. So far, mostly classical studies have been carried on the model and therefore it is our purpose to try and answer some of the mathematical questions related to the different topics related to the problem.

In this work, conditions under which a solution exists for the stationary Stokes problem in exterior domains in R^3 are discussed. The domains considered are the complements in R^3 of bounded star shaped sets and the class of functions for which existence is displayed have "*physically reasonable*" behavior at infinity. This class is characterized by a finite Dirichlet integral. This study is carried in Chapter 2. We use a method that employs a variational formulation of the problem and avoids the use of a Helmholtz decomposition of vector fields in unbounded domains.

In Chapter 3, we show that a problem on a finite domain denoted Ω_R approximates the problem on the unbounded domain Ω. An artificial boundary $\delta\Omega_R$ is introduced and different boundary conditions are imposed. We study two cases of artificial boundary conditions namely the zero velocity boundary condition and the zero stress boundary condition. Approximation results are discussed. In particular, we stress the case where $\underline{f}$, the body force, is in $L^2(\Omega)$ and the support of f is compact in Ω.

Then, in Chapter 4, we treat the finite element approximation of the truncated problem. The case of zero velocity artificial boundary condition is treated separately from the case of zero stress boundary condition. Error estimates for the different cases are derived. The final estimates between the problem on the unbounded domain and the discrete problem reduce to a two parameter approximation. To balance the truncation error with the discretization error, we consider a mesh grading technique.

Finally, in Chapter 5, we consider eliminating the truncation error. In this approach, we treat a truncated problem in which we impose exact boundary condtions at the artificial boundary $\delta\Omega_R$. This is achieved by studying the physical conditions associated with the hydrodynamic equilibrium of the fluid patches in Ω_R and Ω_∞. Actually the solution, outside the support of $\underline{f}$, can be written explicitly in terms of a boundary integral involving the velocity and the stresses at the boundary. Existence and uniqueness of the coupled problem is studied as well as finite element approximation which turns out to be a one parameter approximation.

CHAPTER 2

ON THE EXISTENCE, UNIQUENESS AND REGULARITY OF THE EXTERIOR STOKES PROBLEM IN R^3

2.1. Introduction

We consider the Stokes problem in an exterior domain Ω

$$-\Delta \underline{u}+\nabla p=f \text{ in } \Omega \tag{2.1}$$

$$\operatorname{div} \underline{u}=0 \text{ in } \Omega \tag{2.2}$$

$$\underline{u}\,|_{\Gamma}=0 \tag{2.3}$$

$$\lim_{R\to\infty} \underline{u}(x)=0 \tag{2.4}$$

Problem (2.1)–(2.3) is well posed in the case of a bounded domain, i.e., (2.1)–(2.3) have a unique solution depending continuously on the data. The situation is different in exterior domains, where an important question related to the infinite domain arises: for which values of ϵ does a unique solution $O(r^{\epsilon})$ for r large enough, exist ? If ϵ is small and negative a solution might not exist for a large class of "*physically reasonable*" data , while for ϵ large and positive the solution might be degenerate at infinity and non-uniqueness will result. To clarify the idea, we mention two standard examples. First, consider the Dirichlet problem in an exterior domain

$$\Delta\phi=f \text{ in } \Omega$$

$$\phi\,|_{\delta\Omega}=0$$

If ϕ is regular at infinity, i.e., ϕ $=O(1/r)$, then the solution is unique. Even under the weaker assumption $\phi=O(1/r^{1/2+\delta})$ one can get a similar result. Next, for the same problem, if we choose $\epsilon=0$, i.e., we are interested in bounded solutions, one can show examples where non-uniqueness results (Consider the exterior of a sphere of unit radius and let $\phi=\{1-1/r\}$; it is clear that ϕ is in the null space of the operator).

The first problem that we encounter is the choice of a function space in which to pose the problem. The class of solutions, we are interested in, is the class of *physically realizable* problems. This class is characterized by a finite Dirichlet integral $\mathcal{D}(\underline{u})$, defined by

$$\mathcal{D}(\underline{u})=\int_{\Omega} |\text{grad } \underline{u}|^2 \, dx$$

In [8] and [16] the problem has been treated for the case of $\underline{f}$ with compact support and the conclusion was that the velocity $\underline{u}$ necessarily goes to its asymptotic value at infinity at the rate of $O(1/r)$ while the pressure goes to zero at the rate $O(1/r^2)$. This, of course, shows that the solution does not have a finite kinetic energy, i.e, the solution is not in $L^2(\Omega)$, and consequently, eliminates the possibility of working with the usual Sobolev spaces used in the case of bounded domains. Even a stronger result has been presented for the non-linear problem, i.e., the Navier-Stokes equation; in [7] Finn proved that a class of solutions with a finite Dirichlet integral, that also belongs to $L^2(\Omega)$, consists only of the trivial solution ($\underline{u}$ identically zero).

One of the ways to handle that problem is by the use of mixed norms, i.e., derivatives of $\underline{u}$ belonging to $L^2(\Omega)$, while $\underline{u}$ belonging to $L^q(\Omega)$, where $q > 2$ is chosen such that finite improper integrals would result for this class of solutions.

(In three dimensions, q is chosen to be 6). But this, in addition to complicating the analysis, could deprive us of the advantage of working in a Hilbert space, which is later desirable for the purpose of approximating the solution of the problem by finite element methods. Another possibility is to work in a weighted function space with the weight chosen to have enough decay properties at infinity to include the class of solutions we are interested in.

Problem (2.1)-(2.4) has been treated previously using two different techniques: the first is the use of the theory of hydrodynamical potentials while the other is a variational technique [16].

In the first, the solution pair $(\underline{u},p)$ is represented as the sum of a volume potential and a surface potential. The results of this approach are of interest whenever coupling of boundary integral and finite element methods is desired in the approximation of the problem on an arbitrary fixed bounded sufficiently large domain.

In the other, only a velocity space $H(\Omega)$ was introduced and defined as follows:

Definition 2.1: Define $H(\Omega)$ to be the closure of

$$\{\ \underline{u},\ u_i \in C_0^{\infty}(\Omega)\quad i=1,2,3\ ,\ \operatorname{div}\underline{u}=0\ \}$$

with respect to $\left(\int_\Omega |\operatorname{grad}\underline{u}|^2\,dx\right)^{1/2}$. ∎

The variational formulation obtained was:

Seek $\underline{u} \in H(\Omega)$ such that:

$$a(\underline{u},\underline{v})=f(\underline{v}) \qquad \forall\ \underline{v} \in H(\Omega)$$

where,

$$a(\underline{u},\underline{v})=\int_\Omega \operatorname{grad}\underline{u}:\operatorname{grad}\underline{v}\ dx$$

$$f(\underline{v})=\int_{\Omega} \underline{f}.\underline{v}\ dx$$

In this treatment, in addition to the fact that the pressure space has not been specified for the problem, it is obvious that the formulation would severely restrict the choice of finite element approximations of the problem to divergence free finite element spaces.

In bounded domains two important aspects play a role in the existence proof, namely the Poincare inequality in H_0^1 and the Helmholtz decomposition of vector fields in R^n, $n \geq 3$. For a reference of how these are vital to the existence proof we refer the reader to [9] and [24]. Many counter examples can be given to show that the Poincare inequality does not hold in general unbounded domains, and in particular, for exterior domains. The type of domains for which such an inequality holds are at most slab domains ,i.e., domains that can be confined between two parallel planes. The reason such an inequality does not hold is mainly the asymptotic rate at the infinite boundary. To illustrate the idea we have the following: for a bounded and continuous function to be square integrable over an exterior domain in R^3, its decay rate at infinity should be at least $O(1/r^{3/2+\epsilon})$, where $\epsilon > 0$. A simple calculation shows that if the Dirichlet integral of a smooth function ϕ is finite, then $\nabla\phi$ is $O(1/r^{3/2+\epsilon})$, while the function ϕ itself will be $O(1/r^{1/2+\epsilon})$ and hence is not square integrable (finite energy). This is opposite to the case of bounded domains in which a distribution with finite Dirichlet integral will be square integrable up to an additive constant, i.e., where we have the inequality

$$\inf_{u_1 - u = a} \int_{\Omega} u^2\ dx \leq C \int_{\Omega} (\text{grad}\ u_1)^2\ dx$$

where a is a real number. Fortunately, we are not empty handed in the case of unbounded domains; the Hardy inequality plays the role of the Poincare inequality. This will be explained in the following sections.

The spaces we use here have been introduced by Hanouzet [12] and used by Nedelec and Planchard in their treatment of the Poisson equation in R^3 [20].

In what follows, we give a more precise formulation of both pressure and velociy spaces as well as regularity results. The variational form obtained will reduce to a Brezzi sadle point problem [4], which will enable us to apply many results of that abstract theory.

2.2. Preliminaries

In this section we introduce the notation and the spaces, and state various results concerning these spaces, which will be used in the sequel. Throughout this work C will denote a generic constant (positive) and ϵ will denote a small positive arbitrary number.

Definition 2.2: A set Ω_1 is <u>star shaped</u> with respect to a given point a in the set if for every $x \in \delta\Omega$ and $0 \le \lambda < 1$, the points $a+\lambda(x-a)$ belong to the set.■

Remark:A convex set is star shaped with respect to all its points.

Let Ω_1 be a bounded open set in R^3 which is star shaped with respect to the origin. Let Ω be the complement of its closure in R^3. We will denote by $x=(x_1,x_2,x_3)$ a generic point in R^3 and we let r denote the distance of the point to the origin

$$r=|x| = \left\{ \sum_{i=1}^{i=3} x_i^2 \right\}^{1/2}$$

We let $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ with λ_i a non-negative integer be a multi-index and let

$$|\lambda|=\lambda_1+\lambda_2+\lambda_3$$

We denote the distributional derivative $\partial/\partial x_i$ by D_i and for a multi-index λ, we let

$$D^{\lambda} = D_1^{\lambda_1} D_2^{\lambda_2} D_3^{\lambda_3} = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2} \partial x_3^{\lambda_3}}$$

We will use the following spaces introduced by Hanouzet [12]

Definition 2.3: For m a non-negative integer and $\alpha \in R$, we define the space

$$W^{m,\alpha}(\Omega)=\left\{ u \in \mathcal{D}'(\Omega) \mid (1+r^2)^{(\alpha-m+|\lambda|)/2} D^{\lambda}u \in L^2(\Omega)\ ,\ |\lambda| \leq m \right\}$$

where $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. ■

The following series of results are also due to Hanouzet [12].

1. $W^{m,\alpha}(\Omega)$ is a Hilbert space equipped with the inner product

$$(u,v)_{W^{m,\alpha}(\Omega)} = \sum_{|\lambda|\leq m} \left\{ \int_{\Omega} (1+r^2)^{\alpha-m+|\lambda|} D^{\lambda}u\, D^{\lambda}v\, dx \right\}$$

and norm

$$\| u \|_{W^{m,\alpha}(\Omega)} = [(u,u)_{W^{m,\alpha}(\Omega)}]^{1/2}$$

2. The folowing imbeddings are continuous:

$$W^{m,\alpha}(\Omega) \rightarrow W^{m-1,\alpha-1}(\Omega) \rightarrow \ldots\ldots \rightarrow W^{0,\alpha-m}(\Omega)\ ;$$

3. $L^2(\Omega)=W^{0,0}(\Omega)$;

4. The mapping $(1+r^2)^{\beta/2}u$ is an isomorphism that maps $W^{m,\alpha}(\Omega)$ onto $W^{m,\alpha-\beta}(\Omega)$;

5. The mapping $(1+r^2)^{-\beta/2}u$ is the inverse isomorphism ;

6. The mapping $(1+r^2)^{\beta/2}D^{\lambda}u$ is continuous from $W^{m,\alpha}(\Omega)$ into $W^{m-|\lambda|,\alpha-\beta}(\Omega)$.

7. There exist a linear operator $\gamma=(\gamma_0,\gamma_1,\ldots,\gamma_{m-1})$ with:

$$\gamma:W^{m,\alpha}(\Omega)\rightarrow\prod_{j=0}^{j=m-1}W^{m-j-1/2,\alpha}(\delta\Omega)$$

such that

$$\gamma u=(u(\delta\Omega),\partial u/\partial n(\delta\Omega),\ldots,\partial^{m-1}u/\partial n^{m-1}(\delta\Omega))$$

where $\delta\Omega$ denotes the boundary of Ω, $\partial/\partial n$ denotes the normal derivative and $W^{m-j-1/2,\alpha}(\delta\Omega)$ denote the trace spaces associated with the weighted Sobolev spaces defined above;

8. We also have that γ is onto; and

9. $\gamma^{-1}(0)=\mathring{W}^{m,\alpha}(\Omega)$ where $\mathring{W}^{m,\alpha}(\Omega)$ denotes the completion of $C_0^{\infty}(\Omega)$ in the $W^{m,\alpha}(\Omega)$norm.

10. The dual space of $\mathring{W}^{m,\alpha}(\Omega)$ is denoted by $W^{-m,-\alpha}(\Omega)$.

■

We will also need the following result.

Lemma 2.4: For an exterior domain,

$$W^{m-j-1/2,\alpha}(\delta\Omega)=H^{m-j-1/2}(\delta\Omega)$$

where $H^{m-j-1/2}(\delta\Omega)$ are the trace spaces asociated with the usual Sobolev spaces [1].

Proof: On $\delta\Omega$, the weight $(1+r^2)$ is bounded from above and below and thus the norms on $W^{m-j-1/2}(\delta\Omega)$ and $H^{m-j-1/2}(\delta\Omega)$ are equivalent. The result then follows.

■

Definition 2.5: We define $C_b^{\infty}(\Omega)$ to be the set of infinitely differentiable functions with bounded support in Ω.

Remarks:

1. The specific spaces which we will use most frequently are

$$W^{1,0}(\Omega)=\left\{u \in \mathcal{D}'(\Omega) \;\middle|\; (1+r^2)^{-1/2}u \in L^2(\Omega)\ ,\ Du \in L^2(\Omega)\right\}$$

and

$$W^{2,1}(\Omega)=\left\{u \in \mathcal{D}'(\Omega) \;\middle|\; (1+r^2)^{-1/2}u \in L^2(\Omega)\ ,\ Du \in L^2(\Omega),\ (1+r^2)^{1/2}D^2u \in L^2(\Omega)\right\}$$

2. If $u=O(1/r^{1/2+\epsilon})$ and grad $u=O(1/r^{3/2+\epsilon})$ as $r\to\infty$, then $u \in W^{1,0}(\Omega)$. Thus in this class of solutions we have functions whose decay rate at infinity is slower than $1/r$, the fundamental solution of the Laplacian in R^3.

3. The seminorm $|\ u\ |_1= (\int_\Omega \text{gradu.gradu}\ dx)^{1/2}$ is a norm equivalent to the norm defined on $\mathring{W}^{1,0}(\Omega)$. This follows from the Hardy inequality [13]:

$$\int_\Omega \frac{1}{1+|\ x\ |^2}|\ u\ |^2\,dx \le C\int_\Omega |\ \text{gradu}\ |^2\,dx$$

Lemma 2.6: $C_b^{\infty}(\Omega)$ is dense in $W^{1,0}(\Omega)$.

Proof: Let u_1 be in $W^{1,0}(\Omega)$, we have by Lemma {2.4} that $\gamma u_1 \in H^{1/2}(\delta\Omega)$, where γ is the trace operator introduced earlier. Now by the trace theorems for bounded domains, there exists at least one $u_2 \in H^1(\Omega_1)$ such that $\gamma u_1 = \gamma u_2$. Define u to be

$$
\begin{aligned}
u &= u_2 && \text{for } x \in \bar{\Omega}_1 \\
u &= u_1 && \text{for } x \in \Omega
\end{aligned}
$$

It is clear that $u \in W^{1,0}(R^3)$ and by the density of $C_0^{\infty}(R^3)$ in $W^{1,0}(R^3)$, there exist an approximating sequence denoted by $\{ \phi_k \}$ with $\phi_k \in C_0^{\infty}(R^3)$ for every k, i.e.,

$$\lim_{k\to\infty} \| \phi_k - u \|_{W^{1,0}(\Omega_R)} = 0$$

Now let Ξ_Ω denote the characteristic function of Ω and consider the sequence

$$\{ \Xi_\Omega \phi_k \} \text{ for } k=1,2,\ldots$$

This sequence is contained in $C_b^{\infty}(\Omega)$ and we have

$$\| \Xi_\Omega \phi_k - u_1 \|_{W^{1,0}(\Omega)} \leq \| \phi_k - u \|_{W^{1,0}(\Omega_R)}$$

Hence

$$\lim_{k\to\infty} \| \Xi_\Omega \phi_k - u_1 \|_{W^{1,0}(\Omega)} = 0$$

i.e., for $u_1 \in W^{1,0}(\Omega)$ there exists an approximating sequence in $C_b^{\infty}(\Omega)$ in the norm $\| \cdot \|_{W^{1,0}(\Omega)}$. This completes the proof.

■

2.3. Variational formulation of the problem

We now give a derivation of the Galerkin formulation of the problem (2.1)-(2.4) which we will use. We will employ the spaces $[\mathring{W}^{1,0}(\Omega)]^3$ for the velocity and $L^2(\Omega)$ for the pressure. Since $[C_0^{\infty}(\Omega)]^3$ is dense in $[\mathring{W}^{1,0}(\Omega)]^3$, we take the inner product of (2.1) with a vector valued function $\underline{v} \in [C_0^{\infty}(\Omega)]^3$ and then integrate both terms on the left hand side by parts to yield

$$\int_\Omega \mathrm{grad}\underline{u}:\mathrm{grad}\underline{v}\ dx - \int_\Omega p\,\mathrm{div}\underline{v}\ dx = \int_\Omega \underline{f}.\underline{v}dx$$

where the colon denotes the scalar product of the two tensors standing of each side of it. We also multiply (2.2) by a function $q \in L^2(\Omega)$ to yield

$$\int_\Omega q\,\mathrm{div}\underline{u}\ dx = 0$$

We define the bilinear forms

$$a(\underline{u},\underline{v}) = \int_\Omega \mathrm{grad}\underline{u}:\mathrm{grad}\underline{v}\ dx \tag{2.5}$$

$$b(p,v) = -\int_\Omega p\,\mathrm{div}\underline{v}\ dx \tag{2.6}$$

where $\underline{v} \in [C_0^{\infty}(\Omega)]^3$. The bilinear form a(.,.) defined above is densely defined and continous on $[\mathring{W}^{1,0}(\Omega)]^3 \times [\mathring{W}^{1,0}(\Omega)]^3$, and hence can be uniquely extended to a bilinear form, still denoted by a(.,.), for $\underline{v} \in [\mathring{W}^{1,0}(\Omega)]^3$. Then the weak formulation of (2.1)-(2.4) is given by:

seek $\underline{u} \in [\mathring{W}^{1,0}(\Omega)]^3$ and $p \in L^2(\Omega)$ such that:

$$a(\underline{u},\underline{v}) + b(p,\underline{v}) = \langle \underline{f},\underline{v} \rangle \qquad \forall\ \underline{v} \in [\mathring{W}^{1,0}(\Omega)]^3 \tag{2.7}$$

$$b(q,\underline{u}) = 0 \qquad \forall\ q \in L^2(\Omega) \tag{2.8}$$

where ⟨...⟩ denotes the $L^2(\Omega)$-inner product. The problem (2.7)-(2.8) is of the type considered in [4], where the following result is proved.

Theorem 2.7: Let V and S denote two real Hilbert spaces with inner products $(\ldots)_V$ and $(.,.)_S$ and norms $\| \cdot \|_V$ and $\| \cdot \|_S$, respectively. Let F(.) denote a bounded linear functional on V, and let a(.,.) and b(.,.) denote continuous bilinear forms on V×V and S×V respectively. Let the bilinear form a(.,.) be coercive on V , i.e.,

$$a(v,v) \geq \alpha_1 \| v \|_V^2 \quad \forall\, v \in V \tag{2.9}$$

for some constant $\alpha_1 > 0$, and let the bilinear form b(.,.) satisfy the stability condition

$$\sup_{\substack{v \in V \\ v \neq 0}} \frac{b(p,v)}{\| v \|_V} \geq \alpha_2 \| p \|_S \quad \forall\, p \in S \tag{2.10}$$

for some constant $\alpha_2 > 0$. Then there exists a unique pair $u \in V$ and $p \in S$ which satisfy

$$a(u,v) + b(p,v) = F(v) \qquad \forall\, v \in V \tag{2.11}$$

$$b(q,u) = 0 \qquad \forall\, q \in S \tag{2.12}$$

Moreover there exists a constant $C > 0$ such that:

$$\| u \|_V + \| p \|_S \leq C \| F \|_{V'}. \tag{2.13}$$

■

Clearly if we let $V=[\mathring{W}^{1,0}(\Omega)]^3$, $S=L^2(\Omega)$ and $F(.)=\langle f,.\rangle$, the problem (2.7)-(2.8) is of the type (2.11)-(2.12). Therefore the task at hand is the verfication of the hypotheses of Theorem {2.7}.

2.4. Existence, uniqueness and regularity

2.4.1. Existence and uniqueness of the solution

The greatest difficulty is encountered in verifying (2.10) , so we will begin with that condition. We will need the following preliminary results.

Lemma 2.8: Given $p \in L^2(\Omega)$, there exists a unique $\phi \in W^{1,0}(\Omega)$ such that:

$$-\mathrm{div}(1+r^2)^{1/2}\mathrm{grad}\phi \;=\; p \text{ in } \Omega \tag{2.14}$$

$$\frac{\partial\phi}{\partial n} = 0 \text{ on } \delta\Omega \tag{2.15}$$

Proof: Equation (2.14) may be rewritten in the form

$$-\Delta\phi - \mathrm{grad}[\ln(1+r^2)^{1/2}].\mathrm{grad}\phi = p/(1+r^2)^{1/2} \text{ in } \Omega$$

Multiplying this equation by a function $\psi \in C_b^{\infty}(\Omega)$, integrating over Ω , and then integrating the first term by parts yields

$$\int_\Omega \mathrm{grad}\phi.\mathrm{grad}\psi \; dx - \int_\Omega \psi\,\mathrm{grad}[\ln(1+r^2)^{1/2}].\mathrm{grad}\phi \; dx = \int_\Omega \psi p/(1+r^2)^{1/2} \; dx$$

where we have used (2.15) as well. Now, we define the bilinear form B(.,.)

$$B(\phi\,,\psi) = \int_\Omega \mathrm{grad}\phi.\mathrm{grad}\psi \; dx - \int_\Omega \psi\,\mathrm{grad}[\ln(1+r^2)^{1/2}].\mathrm{grad}\phi \; dx$$

and the linear functional G(.)

$$G(\psi) = \int_\Omega p\psi/(1+r^2)^{1/2} \; dx$$

It is clear that $G(\psi)$ is a bounded linear functional on $W^{1,0}(\Omega)$ whenever p is in $L^2(\Omega)$, since

$$|G(\psi)| \leq \left[\int_\Omega p^2 \, dx \right]^{1/2} \left[\int_\Omega \psi^2/(1+r^2) \, dx \right]^{1/2}$$

$$\leq \| p \|_{L^2(\Omega)} \| \psi \|_{W^{1,0}(\Omega)} \qquad \forall \psi \in W^{1,0}(\Omega)$$

With $\underline{e}_r$ denoting the unit vector in the radial direction, we also have

$$|B(\phi,\psi)| \leq \| \mathrm{grad}\phi \|_{L^2(\Omega)} \| \mathrm{grad}\psi \|_{L^2(\Omega)} + \left| \int_\Omega \frac{r\psi}{(1+r^2)} \underline{e}_r . \mathrm{grad}\phi \, dx \right|$$

$$\leq \| \mathrm{grad}\phi \|_{L^2(\Omega)} \| \mathrm{grad}\psi \|_{L^2(\Omega)}$$

$$+ \sup_{r \in \Omega} \frac{r}{(1+r^2)^{1/2}} \left\| \frac{\psi}{(1+r^2)^{1/2}} \right\|_{L^2(\Omega)} \| \mathrm{grad}\phi \|_{L^2(\Omega)}$$

$$\leq 2 \| \phi \|_{W^{1,0}(\Omega)} \| \psi \|_{W^{1,0}(\Omega)} \ \forall \phi, \psi \in W^{1,0}(\Omega)$$

so that the bilinear form B(.,.) is continuous on $W^{1,0}(\Omega) \times W^{1,0}(\Omega)$.

Noting that the bilinear form B(.,.) as well as the linear functional G(.) are densely defined and continuous and therefore can be extended to all of $W^{1,0}(\Omega)$, we arrive at the following weak formulation of (2.14)-(2.15). Given $p \in L^2(\Omega)$, we seek $\phi \in W^{1,0}(\Omega)$ such that:

$$B(\phi,\psi) = G(\psi) \qquad \forall \psi \in W^{1,0}(\Omega) \tag{2.16}$$

Now note that for $\phi \in C_b^\infty(\Omega)$

$$B(\phi,\phi) = \int_\Omega |\mathrm{grad}\phi|^2 \, dx - \int_\Omega \phi \, \mathrm{grad}[\ln(1+r^2)^{1/2}].\mathrm{grad}\phi \, dx$$

Integrating the second integral by parts then yields

$$\int_\Omega \phi \, \mathrm{grad}[\ln(1+r^2)^{1/2}].\mathrm{grad}\phi \, dx = \int_\Omega \mathrm{grad}[\ln(1+r^2)^{1/2}].\mathrm{grad}(\phi^2/2) \, dx$$

$$= -\int_\Omega \phi^2 \, \Delta \, [\ln(1+r^2)^{1/2}]/2 \, dx$$

$$+ \int_{\delta\Omega} \frac{r\phi^2}{2(1+r^2)} \underline{e}_r.\underline{n}\, dS$$

where $\underline{n}$ denotes the outer normal to the domain Ω. Since Ω_1 is star shaped, we have that $\underline{e}_r.\underline{n} \leq 0$, and since $\Delta[\ln(1+r^2)^{1/2}]$ is bounded below by 1 we get:

$$B(\phi, \phi) \geq \int_\Omega |\text{grad}\phi|^2\, dx + 1/2\int_\Omega \phi^2/(1+r^2)\, dx \geq 1/2\| \phi \|^2_{W^{1,0}(\Omega)} \qquad (2.17)$$

Inequality (2.17) holds for $\phi \in C_b^\infty(\Omega)$ and by the density of $C_b^\infty(\Omega)$ in $W^{1,0}(\Omega)$ and the continuity of the form B(.,.) holds for $\phi \in W^{1,0}(\Omega)$, showing that the bilinear form is coercive on $W^{1,0}(\Omega)$. Thus we have verified that the problem (2.16) satisfies the hypotheses of the Lax-Milgram theory, and thus possesses a unique solution in $W^{1,0}(\Omega)$, which, of course is weak solution of (2.14)-(2.15). ■

Remark: Note that the Neumann problem (2.14)-(2.15) has a unique solution in $W^{1,0}(\Omega)$. This should be contrasted with the case of a bounded domain, wherein the Neumann problem must be treated in $H^1(\Omega)/R$. We again observe that in unbounded domains.$W^{1,0}(\Omega)/R=W^{1,0}(\Omega)$, i.e., $W^{1,0}(\Omega)$ does not contain the constant function.

Lemma 2.9: [Regularity of the weak solution of problem (2.16)] Let $\phi \in W^{1,0}(\Omega)$ be the weak solution in $W^{1,0}(\Omega)$ of (2.14)-(2.15). If $\delta\Omega$ is sufficiently smooth, then ϕ is also in $W^{2,1}(\Omega)$.

Proof: See the appendix for the proof. ■

Lemma 2.10: Given $\underline{a} \in H^{1/2}(\delta\Omega)$ with $\underline{a}.\underline{n}=0$, there exists $\underline{v}_2 \in [W^{1,0}(\Omega)]^3$ such that:

$$\text{div } \underline{v}_2 = 0 \text{ in } \Omega$$

$$\underline{v}_2=\underline{a} \text{ on } \delta\Omega$$

Moreover, there exists a constant $C > 0$ such that:

$$\| \underline{v}_2 \|_{W^{1,0}(\Omega)} \leq C \| \underline{a} \|_{H^{1/2}(\delta\Omega)}$$

with C independent of $\underline{a}$.

Proof: First, it is well known [24], that given $\underline{a} \in H^{1/2}(\delta\Omega)$, $\underline{a}.\underline{n}=0$, there exists $\underline{d} \in [H^2(\Omega_1)]^3$ such that curl $\underline{d}|_{\delta\Omega}=\underline{a}$ and

$$\| \underline{d} \|_{H^2(\Omega_1)} \leq C \| \underline{a} \|_{H^{1/2}(\delta\Omega)}$$

with C independent of $\underline{a}$. It is also well known [1],that there exists an extension operator E,

$$E : H^2(\Omega_1) \rightarrow H^2(R^3)$$

such that $E\underline{d} \in H^2(R^3)$ whenever $\underline{d} \in H^2(\Omega_1)$ and supp($E\underline{d}$) is contained in a ball of finite radius R, where R depends only on the geometry and the smoothness of the boundary but not on $\underline{d}$. We also have that the traces of $\underline{d}$ and $E\underline{d}$ are the same on the boundary, i.e.,

$$E\underline{d}|_{\delta\Omega}=\underline{d}|_{\delta\Omega}$$

and

$$\text{grad } E\underline{d}|_{\delta\Omega}=\text{grad } \underline{d}|_{\delta\Omega}$$

and that

$$\| E\underline{d} \|_{H^2(R^3)} \leq C \| \underline{d} \|_{H^2(\Omega_1)}$$

with C independent of $\underline{d}$. Now, let R denote the restriction operator from R^3 to Ω. Then

$$R\underline{v}=\underline{v}\Xi_{\Omega}$$

where Ξ_{Ω} is the characteristic function of Ω. Then the vector $\underline{\partial}=RE\underline{d}$ has the following properties:

1. supp($\underline{\partial}$) is compact in R^3 independent of $\underline{d}$;
2. curl $\underline{\partial}|_{\delta\Omega}$=curl$RE\underline{d}|_{\delta\Omega}$=curl $E\underline{d}|_{\delta\Omega}$=curl $\underline{d}|_{\delta\Omega}$= $\underline{a}$;
3. $\| \underline{\partial} \|_{H^2(\Omega)} \leq \| E\underline{d} \|_{H^2(R^3)} \leq C \| \underline{d} \|_{H^2(\Omega_1)} \leq C \| \underline{a} \|_{H^{1/2}(\delta\Omega)}$; and
4. $\| \text{curl}\underline{\partial} \|_{H^1(\Omega)} \leq \| \underline{\partial} \|_{H^2(\Omega)}$.

Now if $\underline{v}_2$=curl $\underline{\partial}$, we have the desired results, since for functions of compact support in R^3, the norms $\| \cdot \|_{H^1(\Omega)}$ and $\| \cdot \|_{W^{1,0}(\Omega)}$ are equivalent.

■

Lemma 2.11: Given $p \in L^2(\Omega)$, there exists at least one $\underline{v} \in [\mathring{W}^{1,0}(\Omega)]^3$ such that:

$$\text{div } \underline{v}=p \text{ in } \Omega$$

$$\underline{v}=0 \text{ on } \delta\Omega$$

Moreover

$$\| \underline{v} \|_{W^{1,0}(\Omega)} \leq C \| p \|_{L^2(\Omega)}$$

with $C > 0$ independent of p.

Proof: Let $\underline{v}_1=(1+r^2)^{1/2}\text{grad}\phi$ where ϕ satisfies (2.14)-(2.15). By Lemmas {2.8}, {2.9}, such a ϕ exists uniquely and $\phi \in W^{2,1}(\Omega)$. By observation (6) in Section {2.2}, $(1+r^2)^{1/2}\text{grad}\phi$ maps $W^{2,1}(\Omega)$ into $W^{1,0}(\Omega)$. Thus $\underline{v}_1 \in W^{1,0}(\Omega)$ satisfies

$$\text{div } \underline{v}_1 = p \text{ in } \Omega$$

$$\underline{v}_1 . \underline{n} = 0 \text{ on } \delta\Omega$$

and by Lemma {2.8}

$$\| \underline{v}_1 \|_{W^{1,0}(\Omega)} \leq C \| \phi \|_{W^{2,1}(\Omega)} \leq C \| p \|_{L^2(\Omega)}$$

Now let $\underline{a} = \underline{n} \times \underline{v}_1 |_{\delta\Omega}$ so that

$$\| \underline{a} \|_{H^{1/2}(\delta\Omega)} \leq C \| \underline{v}_1 \|_{W^{1,0}(\Omega)}$$

Then, we seek $\underline{v}_2 \in W^{1,0}(\Omega)$ such that:

$$\text{div } \underline{v}_2 = 0 \text{ in } \Omega$$

$$\underline{v}_2 |_{\delta\Omega} = \underline{a} = \underline{n} \times \underline{v}_1 \text{ on } \delta\Omega$$

The existence of such a $\underline{v}_2$ was established in Lemma {2.10}. Moreover,

$$\| \underline{v}_2 \|_{W^{1,0}(\Omega)} \leq C \| \underline{a} \|_{H^{1/2}(\delta\Omega)} \leq C \| \underline{v}_1 \|_{W^{1,0}(\Omega)} \leq C \| p \|_{L^2(\Omega)}$$

With $\underline{v} = \underline{v}_1 - \underline{v}_2$ the desired results are attained.

■

Lemma 2.12: The bilinear form b(.,.) defined by (2.6) satisfies the condition

$$\sup_{\substack{\underline{v} \in W^{1,0}(\Omega) \\ \underline{v} \neq 0}} \frac{b(p,\underline{v})}{\| \underline{v} \|_{W^{1,0}(\Omega)}} \geq C \| p \|_{L^2(\Omega)} \; \forall \; p \in L^2(\Omega)$$

Proof: By Lemma {2.11}, given $p \in L^2(\Omega)$, there exists at least one vector $\underline{w}$ $\in [\mathring{W}^{1,0}(\Omega)]^3$ such that div $\underline{w} = p$ and

$$\| \underline{w} \|_{W^{1,0}(\Omega)} \leq C \, \| p \|_{L^2(\Omega)}$$

Then,

$$\sup_{\substack{\underline{v} \in [\mathring{W}^{1,0}(\Omega)]^3 \\ \underline{v} \neq 0}} \frac{b(p,\underline{v})}{\| \underline{v} \|_{W^{1,0}(\Omega)}} \geq \frac{\int_\Omega p \operatorname{div} \underline{w} \, dx}{\| \underline{w} \|_{W^{1,0}(\Omega)}}$$

$$= \frac{\| p \|^2_{L^2(\Omega)}}{\| \underline{w} \|_{W^{1,0}(\Omega)}}$$

$$\geq C^{-1} \, \| p \|_{L^2(\Omega)}$$

which completes the proof. ∎

Lemma {2.12} shows that (2.10) holds for the problem (2.7)–(2.8). The remaining conditions of Theorem {2.7} are verified in the following result.

Lemma 2.13: Given $\underline{f} \in [W^{-1,0}(\Omega)]^3$, then $\langle f,.\rangle$ defines a continuous linear functional on $[\mathring{W}^{1,0}(\Omega)]^3$ and

$$|\langle \underline{f}, \underline{v} \rangle| \leq \| \underline{f} \|_{W^{-1,0}(\Omega)} \, \| \underline{v} \|_{W^{1,0}(\Omega)} \tag{2.18}$$

Furthermore, the bilinear form $a(.,.)$ defined by (2.9) is continuous and coercive on $[\mathring{W}^{1,0}(\Omega)]^3$.

Proof: Clearly $\langle f,.\rangle$ is a bounded linear funtional on $[\mathring{W}^{1,0}(\Omega)]^3$ and the estimate (2.18) holds. Now

$$|a(\underline{u},\underline{v})| = \left| \int_\Omega \operatorname{grad}\underline{u} : \operatorname{grad}\underline{v} \, dx \right| \leq \| \operatorname{grad}\underline{u} \|_{L^2(\Omega)} \, \| \operatorname{grad}\underline{v} \|_{L^2(\Omega)}$$

Also, the form $a(.,.)$ is coercive on $[\mathring{W}^{1,0}(\Omega)]^3 \times [\mathring{W}^{1,0}(\Omega)]^3$ since

$$a(\underline{u},\underline{u}) = |\nabla \underline{u}|^2_{L^2(\Omega)}$$

where, by the Hardy inequality, the right hand side is a norm equivalent to the norm defined on $[\mathring{W}^{1,0}(\Omega)]^3$. ∎

With the results of Lemmas {2.12} and {2.13} Theorem {2.7} immediately leads to the following result.

Theorem 2.14: There exists a unique solution pair $(\underline{u},p) \in [\mathring{W}^{1,0}(\Omega)]^3 \times L^2(\Omega)$ of the variational problem (2.7)–(2.8). Moreover

$$\| \underline{u} \|_{W^{1,0}(\Omega)} + \| p \|_{L^2(\Omega)} \leq C \| \underline{f} \|_{W^{-1,0}(\Omega)} \tag{2.19}$$

∎

2.4.2. Regularity results for the Stokes problem in R^3

We state without proof the following regularity theorem; the proof follows standard arguments similar to those used in the appendix. See, for example [12]

Theorem 2.15: The variational problem (2.7)–(2.8) has exaclty one solution

$$(\underline{u},p) \in [\mathring{W}^{1,0}(\Omega)]^3 \times L^2(\Omega)$$

Moreover, if $\underline{f} \in [W^{m-1,m}(\Omega)]^3$, and $\delta\Omega$ is sufficiently smooth (Ω is a C^m domain, see, e.g., [1]), then,

$$\underline{u} \in [W^{m+1,m}(\Omega)]^3 \bigcap [\mathring{W}^{1,0}(\Omega)]^3]$$

and

$$p \in W^{m,m}(\Omega)$$

Furthermore, there exists a constant $C > 0$ such that:

$$\| \underline{u} \|_{W^{m+1,m}(\Omega)} + \| p \|_{W^{m,m}(\Omega)} \leq C \| \underline{f} \|_{W^{m-1,m}(\Omega)}$$

∎

2.5. Inhomogeneous boundary conditions

In the case of inhomogeneous boundary conditions at the finite boundary $\delta\Omega$,i.e., if

$$\underline{u}|_{\delta\Omega}=\underline{a} \in H^{m+1/2}(\delta\Omega)$$

with

$$\int_{\delta\Omega} \underline{a.n} \, ds = 0$$

we can use a technique similar to the one used in the sequence of lemmas that led to the stability condition on b(.,.) and find a vector

$$\underline{u}_1 \in W^{m+1,m}(\Omega)$$

such that

$$\text{div } \underline{u}_1 = 0 \text{ in } \Omega$$

and

$$\underline{u}_1|_{\delta\Omega} = \underline{a}$$

with the following estimate holding

$$\| \underline{u}_1 \|_{W^{m+1,m}(\Omega)} \leq C \| \underline{a} \|_{H^{m+1/2}(\delta\Omega)}$$

Using $\underline{u}_1$ we can reduce the problem to the homogeneous case, and thus establish the existence and uniqueness of the solution of the inhomogeneous problem. Furthermore, we get the estimate

$$\| \underline{u} \|_{W^{m+1,m}(\Omega)} + \| p \|_{W^{m,m}(\Omega)} \leq C \| \underline{f} \|_{W^{m-1,m}(\Omega)} + \| \underline{a} \|_{H^{m+1/2}(\delta\Omega)}$$

Next, we consider the case where $\lim_{|x|\to\infty} \underline{u}(x) = \underline{U}_\infty$. Since constants are not in the spaces $W^{m+1,m}(\Omega)$, we use linearity to reduce the problem to the homogeneous case. In this case we get the estimate

$$\| \underline{u} - \underline{U}_\infty \|_{W^{m+1,m}(\Omega)} + \| p \|_{W^{m,m}(\Omega)} \leq C \| \underline{f} \|_{W^{m-1,m}(\Omega)}$$

Combining the two cases we get

$$\| \underline{u}(x) - \underline{U}_\infty \|_{W^{m+1,m}(\Omega)} + \| p \|_{W^{m,m}(\Omega)} \leq C \left\{ \| \underline{f} \|_{W^{m-1,m}(\Omega)} + \| \underline{a} - \underline{U}_\infty \|_{H^{m+1/2}(\delta\Omega)} \right\}.$$

CHAPTER 3

ON THE TRUNCATED PROBLEM ON A FINITE SUBDOMAIN

3.1. Introduction

Let $\Omega_R=\Omega\bigcap B(0;R)$, where $B(0;R)$ is the ball of radius R centered at the origin, with R to be chosen large enough. Now the boundary of the domain Ω_R is:

$$\delta\Omega \bigcup \{ \; x : | \; x \; |=R \; \}$$

By the truncated problem we mean:

$$-\Delta \underline{u}_R + \nabla p_R=f_R \qquad \text{in } \Omega_R \tag{3.1}$$

$$\operatorname{div}\underline{u}_R=0 \text{ in } \Omega_R \tag{3.2}$$

$$\underline{u}_R \,|\, _{\delta\Omega}=0 \tag{3.3}$$

$$B_k(\underline{u}_R)=0 \tag{3.4}$$

where B_k is a boundary operator imposed at the artificial boundary $\delta\Omega_R$. These artifical boundary conditions are of interest due to the lack of the information at the large boundary needed to establish existence and uniqueness of the solution of the truncated problem. Families of boundary operators at $\delta\Omega_R$ have been discussed, in general, for second order elliptic partial differential equations and, in particular, for the Helmholtz equation [11]. This sequence of boundary operators is defined by taking linear combinations of the different derivatives at the artificial boundary. This method is based essentially on a careful examination of the asymptotic expansion of the solution valid near infinity to generate boundary conditions that will enforce the asymptotic behavior of the actual solution on the solution of the truncated problem to a desired degree of accuracy. It is anticipated, as is the case for the Helmholtz equation, that the higher order of the boundary conditions (with the proper choice of the constants), would yield higher accuracy [11]. Unfortunately, the higher the order of the boundary operator B_k, the smoother the function spaces must be chosen to pose the truncated problem . This is of course undesirable for the sake of approximating the problem in a finite element space. Of the possible boundary operators, we consider two choices for B_k, k=0,1.

1. The zero velocity artificial boundary condition:

$$B_0(\underline{u}_R)=\underline{u}_R=0$$

2. The zero stress artificial boundary condition:

$$B_1(\underline{u}_R)=-p\underline{n}+\partial_n\underline{u}_R=0$$

where ∂_n denotes the derivative in the direction of the outer normal to the surface $\delta\Omega_R$.

In Chapter {2} we noted the following result.

Proposition 3.1: For R bounded we have:

$$\overset{\bullet}{W}{}^{1,0}(\Omega_R)=H^1(\Omega_R)$$

■

Remarks:

1. If the support of f is not compact in Ω, then, the truncated problem that

will be considered will correspond to an external force $\underline{f}_R$, with $\underline{f}_R$ being a good approximation of $\underline{f}$. In that case we will need the intermediate estimate:

$$\| \underline{u}-\underline{u}_1 \|_{W^{1,0}(\Omega)} + \| p-p_1 \|_{L^2(\Omega)} \leq C \| \underline{f}-\underline{f}_R \|_{W^{-1,0}(\Omega)}$$

This estimate follows directly from the well posedness of the original problem already discussed in Chapter {2}, where $\underline{u}_1$ and p_1 represent the solution pair of :

$$-\Delta\underline{u}_1 + \nabla p_1 = f_R \quad \text{in } \Omega \tag{3.5}$$

$$\operatorname{div}\underline{u}_1 = 0 \text{ in } \Omega \tag{3.6}$$

$$\underline{u}_1 |_{\delta\Omega} = 0 \tag{3.7}$$

$$\lim_{|x| \to \infty} \underline{u}_1(x) = 0 \tag{3.8}$$

and where $\underline{f}_R$ is chosen to be of compact support and to be an approximation to $\underline{f}$. Now using linearity we have that the difference $(\underline{u}-\underline{u}_1, p-p_1)$ satisfies:

$$-\Delta(\underline{u}-\underline{u}_1) + \nabla(p-p_1) = \underline{f}-\underline{f}_R \quad \text{in } \Omega$$

$$\operatorname{div}(\underline{u}-\underline{u}_1) = 0 \text{ in } \Omega$$

$$(\underline{u}-\underline{u}_1) |_{\delta\Omega} = 0$$

$$\lim_{|x| \to \infty} \left[\underline{u}(x)-\underline{u}_1(x) \right] = 0$$

Hence the required estimate follows from (2.19).

2. Note that in the case supp($\underline{f}$) is bounded in Ω we can choose R such that $\| \underline{f}-\underline{f}_R \|_{-1}=0$.

3. Due to the above reduction to the case with $\underline{f}_R$ of compact support, in the sequel we need only consider $\underline{f}$ having compact support.

4. For the purpose of estimating errors between the truncated problem and the unbounded problem, we need to be aware of the behavior, as R increases, of the constants involved in the different estimates. Therefore, although the spaces $[\mathring{W}^{1,0}(\Omega_R)]^3$ and $[\mathring{H}^1(\Omega_R)]^3$ are equivalent, we will pose the problem on $[\mathring{W}^{1,0}(\Omega_R)]^3$ rather than $[\mathring{H}^1(\Omega_R)]^3$ since the constant involved in the Hardy inequality on $[\mathring{W}^{1,0}(\Omega_R)]^3$ is independent of R whereas on $[\mathring{H}^1(\Omega_R)]^3$ the Poincare inequality holds with a constant dependent on R.

3.2. On the truncated problem with zero velocity artificial boundary condition

The spaces we use in this case are:

$$[\mathring{W}^{1,0}(\Omega_R)]^3=\left\{ \quad \underline{u} \in W^{1,0}(\Omega_R) \quad \middle| \quad \underline{u}|_{\delta\Omega\cup\delta\Omega_R}=0 \quad \right\}$$

$$L_0^2(\Omega_R)= \{ \ q \in L^2(\Omega_R) \ , \int_{\Omega_R} q \ dx= 0 \ \}$$

The variational fomulation of the problem is:

seek $\underline{u}_R,p_R \in [\mathring{W}^{1,0}(\Omega_R)]^3\times L_0^2(\Omega_R)$ such that:

$$a(\underline{u}_R,\underline{v}_R) + b(p_R,\underline{v}_R) = \langle \underline{f},\underline{v}_R\rangle \quad \forall\ \underline{v}_R \in [\mathring{W}^{1,0}(\Omega_R)]^3 \tag{3.9}$$

$$b(q_R,\underline{u}_R)=0 \quad \forall\ q_R \in L_0^2(\Omega_R) \tag{3.10}$$

where,

$$a(\underline{u}_R,\underline{v}_R)= \int_{\Omega_R} \operatorname{grad}\underline{u}_R:\operatorname{grad}\underline{v}_R \, dx$$

$$b(q_R,\underline{u}_R)= \int_{\Omega_R} q_R \operatorname{div}\underline{u}_R \, dx$$

Proposition 3.2: Given $\underline{a} \in W^{1/2,0}(\delta\Omega_R \bigcup \delta\Omega)$ such that:

$$\underline{a}.\underline{n}\,|_{\delta\Omega}=0$$

$$\underline{a}.\underline{n}\,|_{\delta\Omega_R}=0$$

Assume there exists $\underline{v}_2 \in [W^{1,0}(\Omega_R)]^3$ with:

$$\operatorname{div}\ \underline{v}_2=0 \text{ in } \Omega_R$$

$$\underline{v}_2\,|_{\delta\Omega\cup\delta\Omega_R}=\underline{a}$$

and

$$\|\ \underline{v}_2\ \|_{W^{1,0}(\Omega_R)} \leq C_1\ \|\ \underline{a}\ \|_{W^{1/2,0}(\delta\Omega\cup\delta\Omega_R)} \tag{3.11}$$

with C_1 independent of R. Then, given $p \in L_0^2(\Omega_R)$, there exists a $\underline{v} \in [\mathring{W}^{1,0}(\Omega_R)]^3$ and a constant $C > 0$, independent of R, such that:

$$\operatorname{div}\ \underline{v}=-p$$

and

$$\| \underline{v} \|_{W^{1,0}(\Omega_R)} \leq C \| p \|_{L^2(\Omega_R)}$$

Proof: See [9].

■

Remark: In spite of the assumption that C_1 is independent of R we will show that the convergence rate of $(\underline{u}_R, p_R)$ to $(\underline{u}, p)$ is only of order of $R^{-1/2}$, which is not satisfactory.

Let $\mathcal{E}$ denote the extension operator by zero outside the truncated domain:

$$\mathcal{E}u = u \qquad \text{for } x \in \Omega_R \tag{3.12}$$

$$\mathcal{E}u = 0 \qquad \text{for } x \notin \Omega_R$$

It is clear that $\mathcal{E}[\mathring{W}^{1,0}(\Omega_R)]^3$ is included in $[\mathring{W}^{1,0}(\Omega)]^3$, while $\mathcal{E}L^2_0(\Omega_R)$ is included in $L^2(\Omega)$. By a technique similar to the techniques used in Chapter {2}, we can conclude that a unique solution of the truncated problem exists. Furthermore the following estimate is true:

$$\| \underline{u}_R \|_{W^{1,0}(\Omega_R)} + \| p \|_{L^2(\Omega_R)} \leq C \| \underline{f}_R \|_{W^{-1,0}(\Omega_R)} \leq C \| \underline{f}_R \|_{L^2(\Omega_R)}$$

where C is a constant independent of R.

Remark: The variational form (3.9)–(3.10) for the first truncated problem is of the Brezzi type and therefore we have the following approximation result.

3.2.1. Abstract approximation results

Let X,M be two Hilbert spaces and consider the problem:

Seek $v,\lambda \in X{\times}M$ such that:

$$a(v,w)+b(w,\lambda)=\langle f,w\rangle \qquad \forall\ w \in X \tag{3.13}$$

$$b(\mu,v) \qquad =0 \qquad \forall\ \mu \in M \tag{3.14}$$

It is well known ([4]; see also Chapter {2}) that if a(.,.) is a continuous bilinear form i.e. ,

$$|\ a(v,w)\ | \le \|\ a\ \|\ \|\ v\ \|_X \|\ w\ \|_X$$

which is coercive on X×X, i.e.,

$$a(v,v) \ge \alpha_1 \|\ v\ \|_X^2 \qquad \forall\ v \in X \tag{3.15}$$

and b(.,.) is continuous on M×X, i.e.,

$$|\ b(\mu,v)\ | \le \|\ b\ \|\ \|\ \mu\ \|_M \|\ v\ \|_X$$

and satisfies the stability condition:

$$\sup_{\substack{v\in M\\ v\neq 0}} \frac{b(\psi,v)}{\|\ v\ \|_X} \ge \alpha_2 \|\ \lambda\ \|_M \quad \forall\ \lambda \in M \tag{3.16}$$

then there exists a unique solution pair $(v,\lambda) \in X{\times}M$ that continuously depends on the data, i.e,

$$\|\ v\ \|_X + \|\ \lambda\ \|_M \le C \|\ f\ \|_{X'}. \tag{3.17}$$

Now let X_R, M_R be two closed subspaces of X and M respectively. We pose the following problem:

seek $v_R, \lambda_R \in X_R \times M_R$ such that:

$$a(v_R,w_R)+b(w_R,\lambda_R)=\langle f,w_R\rangle \quad \forall\ w_R \in X_R \tag{3.18}$$

$$b(\mu_R,v_R)=0 \qquad \forall\ \mu_R \in M_R \tag{3.19}$$

Problem (3.18)–(3.19) is an approximation of problem (3.13)–(3.14).

Theorem 3.3: [4] Assume that the forms a(.,.), b(.,.) and satisfy the conditions (3.15)–(3.16). Assume in addition that:

$$\sup_{\substack{v_R \in X_R \\ v_R \neq 0}} \frac{b(\psi_R,v_R)}{\| v_R \|_X} \geq \alpha_2^* \| \lambda_R \|_M \quad \forall\ \lambda_R \in M_R \tag{3.20}$$

Then there exists a unique solution of the approximated problem (3.18)–(3.19). Moreover we have the following result:

$$\| v-v_R \|_X + \| \lambda-\lambda_R \|_M \tag{3.21}$$

$$\leq C \left\{ \inf_{w_R \in X_R} \| v-w_R \|_X + \inf_{\mu_R \in M_R} \| \lambda-\mu_R \|_M \right\}$$

where C is a constant depending on $\| \cdot a \|$, $\| b \|$, α_1 α_1^*, α_2 and α_2^* (We should note that it is essential to be aware of the dependence of the constants upon the truncation parameter R).

■

3.2.2. Error estimates for the truncated problem

Now we apply this result to obtain an estimate on the error between the truncated problem and the original one. Problem (3.9)–(3.10) can be restated to be:

seek $\underline{u}_R, p_R \in E[\mathring{W}^{1,0}(\Omega_R)]^3 \times EL_0^2(\Omega_R)$ such that:

$$a_R(\underline{u}_R, \underline{v}) + b_R(p_R, \underline{v}) = \langle \underline{f}, \underline{v} \rangle \quad \forall\ \underline{v} \in E[\mathring{W}^{1,0}(\Omega_R)]^3 \tag{3.22}$$

$$b_R(q_R, \underline{u}_R) \qquad = 0 \qquad \forall\ q_R \in EL_0^2(\Omega_R) \tag{3.23}$$

where as before E is the extension operator defined in (3.12). Note that the extended spaces are closed subspaces of $[\mathring{W}^{1,0}(\Omega)]^3$ and $L^2(\Omega)$ respectively. Therefore we are ready to apply the Brezzi approximation result and get:

$$\| \underline{u}-\underline{u}_R \|_{W^{1,0}(\Omega_R)} + \| p-p_R \|_{L^2(\Omega_R)} \tag{3.24}$$

$$\leq C\left\{ \inf_{\underline{w}_R \in EW^{1,0}(\Omega_R)} \| \underline{u}-\underline{w}_R \|_{W^{1,0}(\Omega_R)} + \inf_{\mu_R \in EL_0^2(\Omega_R)} \| p-\mu_R \|_{L^2(\Omega_R)} \right\}$$

Now we need to estimate the right hand side of (3.24). To do this we need the following lemmas.

Lemma 3.4: There exists a map Π_R

$$\Pi_R : [\mathring{W}^{1,0}(\Omega)]^3 \to E[\mathring{W}^{1,0}(\Omega_R)]^3$$

such that $\forall\ \underline{w} \in [\mathring{W}^{1,0}(\Omega)]^3$

$$\lim_{R\to\infty} \| \Pi_R \underline{w}-\underline{w} \|_{W^{1,0}(\Omega_R)} = 0$$

In addition, if $\underline{w}=O(r^{-1})$ we have that:

$$\| \underline{w}-\Pi_R \underline{w} \|_{W^{1,0}(\Omega_R)} \leq \frac{C}{R^{1/2}}$$

where C is independent of R.

Proof: We define $\phi(x)$ to be an infinitely differentiable function with the following properties:

$$0 \leq \phi(x) \leq 1$$

$$\phi(x)=1 \quad \text{for } |x| \leq 1$$

$$\phi(x)=0 \quad \text{for } |x| \geq 2$$

We denote $\phi_R(x)=\phi(\frac{2x}{R})$.It is clear that the support of $\phi_R(x)$ is contained in B(0;R) which is the ball of radius R centered at the origin. Now define the map Π_R by:

$$\Pi_R \underline{w} = \phi_R(x)\underline{w} \quad \forall \; \underline{w} \in [\mathring{W}^{1,0}(\Omega)]^3$$

It is clear that $\Pi_R \underline{w} \in [\mathring{W}^{1,0}(\Omega_R)]^3$ since :

$$\Pi_R \underline{w}|_{\delta\Omega} = \phi_R(x)\underline{w}|_{\delta\Omega} = 0$$

$$\Pi_R \underline{w}|_{\delta\Omega_R} = \phi_R(x)\underline{w}|_{\delta\Omega_R} = 0$$

where the above equalities are to be understood in the sense of traces. Further,

$$\| \Pi_R \underline{w} \|^2_{W^{1,0}(\Omega_R)} = \int_{\Omega_R} \frac{1}{1+|x|^2} |\phi_R(x)\underline{w}|^2 \, dx + \int_{\Omega_R} |\nabla(\phi_R(x)\underline{w}|^2 \, dx$$
$$\leq \int_{\Omega_R} \frac{|\underline{w}|^2}{1+|x|^2} dx + \int_{\Omega_R} |\nabla \underline{w}|^2 \, dx + \int_{\Omega_R} |\underline{w}.\nabla\phi_R|^2 \, dx$$

But for ϕ_R we have that there exists a contant $C > 0$, independent of R, such that:

$$|D^m \phi_R(x)| \leq \frac{C}{R^{|m|}}$$

where m is a multiindex. Therefore

$$\int_{\Omega_R} |\underline{w}.\nabla\phi_R|^2\, dx \le \int_{\Omega_R} |\underline{w}|^2 |\nabla\phi_R|^2\, dx$$

$$\le \int_{\Omega_R} \frac{C^2}{R^2} |\underline{w}|^2\, dx$$

$$\le C^2 \int_{\Omega_R} \frac{1}{1+|x|^2} |\underline{w}|^2\, dx$$

$$\le C^2 \|\underline{w}\|^2_{W^{1,0}(\Omega_R)}$$

Therefore

$$\|\Pi_R\underline{w}\|_{W^{1,0}(\Omega_R)} \le C\|\underline{w}\|_{W^{1,0}(\Omega_R)} \le C\|\underline{w}\|_{W^{1,0}(\Omega)}$$

where C is a constant independent of R. Thus we have shown that $\Pi_R\underline{w} \in [\mathring{W}^{1,0}(\Omega_R)]^3$. Now,

$$\|\Pi_R\underline{w}-\underline{w}\|_{W^{1,0}(\Omega_R)}$$

$$=\int_{\Omega_R} \frac{1}{1+|x|^2} (\phi_R(x)-1)^2 |\underline{w}|^2\, dx + \int_{\Omega_R} |\nabla(\phi_R(x)-1)\underline{w})|^2\, dx$$

$$=I_1 + I_2$$

Let $\Omega^R_{R/2}$ be the ring subtended between the spheres centered at the origin with radii R/2 and R respectively, i.e.,

$$\Omega^R_{R/2} = \Omega \cap \left\{ B(0;R)\backslash\bar{B}(0;R/2) \right\}$$

Then

$$I_1 \le \int_{\Omega^R_{R/2}} \frac{1}{1+|x|^2} |\underline{w}|^2\, dx$$

$$I_2 \le \int_{\Omega_R} |\,(\phi_R - 1)\nabla \underline{w}\,|^2\,dx + \int_{\Omega_R} |\,\underline{w}.\nabla\phi_R\,|^2\,dx$$

$$\le \int_{\Omega^R_{R/2}} |\,\nabla\underline{w}\,|^2\,dx + C^2 \int_{\Omega^R_{R/2}} \frac{|\,\underline{w}\,|^2}{R^2}\,dx$$

$$\le \left\{ C_1 \int_{\Omega^R_{R/2}} |\,\underline{w}\,|^2\,dx + \int_{\Omega^R_{R/2}} \frac{|\,\underline{w}\,|^2}{1+|\,x\,|^2}\,dx \right\}$$

$$= C_1 \|\,\underline{w}\,\|^2_{1;\Omega^R_{R/2}}$$

where C_1 is independent of R. Now since $\underline{w} \in [\mathring{W}^{1,0}(\Omega)]^3$ we get that:

$$\lim_{R\to\infty} I_1 = 0$$

and

$$\lim_{R\to\infty} I_2 = 0$$

Finally for $\underline{w} = O(r^{-1})$, we have the estimate

$$\|\,\underline{w} - \Pi_R\underline{w}\,\|^2_{W^{1,0}(\Omega_R)} \le C_1 \left\{ \int_{\Omega^R_{R/2}} \frac{|\,w\,|^2}{1+|\,x\,|^2}\,dx + \int_{\Omega^R_{R/2}} |\,\nabla\underline{w}\,|^2\,dx \right\}$$

$$\le C_2 \left\{ \int_{\Omega^R_{R/2}} \frac{1}{|\,x\,|^2(1+|\,x\,|^2)}\,dx + \int_{\Omega^R_{R/2}} \frac{1}{|\,x\,|^4}\,dx \right\}$$

$$= C_2 \left\{ \int_{\Omega} \int_{R/2}^{R} \frac{1}{1+r^2}\,dr\,d\omega + \int_{\Omega} \int_{R/2}^{R} \frac{1}{r^2}\,dr\,d\omega \right\}$$

$$\le C_3 \left\{ \int_{R/2}^{\infty} \frac{1}{1+r^2}\,dr + \int_{R/2}^{\infty} \frac{1}{r^2}\,dr \right\}$$

$$= C \left\{ \tan^{-1}\left(\frac{2}{R}\right) + \frac{2}{R} \right\}$$

But $\tan^{-1}(\frac{2}{R}) \sim \frac{2}{R}$ for R sufficiently large. Therefore we have

$$\| \underline{u}-\Pi_R\underline{u} \|_1 \le \frac{C}{R^{1/2}}$$

where C is a constant independent of R, which completes the proof. ∎

Lemma 3.5: Assume that $p \in L^{2-\epsilon}(\Omega)\bigcap L^2(\Omega)$ for some $\epsilon > 0$, then there exists a map denoted by β_R

$$\beta_R : L^{2-\epsilon}(\Omega)\bigcap L^2(\Omega) \to \mathcal{E}L_0^2(\Omega_R)$$

such that

$$\lim_{R\to\infty} \| \beta_R p-p \|_{L^2(\Omega)} = 0 \ \forall \ p \in L^{2-\epsilon}(\Omega)\bigcap L^2(\Omega)$$

In addition, for $p=O(r^{-2})$ for r large, we can get the following estimate:

$$\| p-\beta_R p \| \le \frac{C}{R^{1/2}}$$

with C independent of R.

Proof: Consider the map β_R

$$\beta_R : L^{2-\epsilon}(\Omega)\bigcap L^2(\Omega) \to \mathcal{E}L_0^2(\Omega_R)$$

defined by

$$\beta_R p = \Xi_R\left\{ p - \frac{1}{\mu(\Omega_R)} \int_{\Omega_R} p \, dx \right\}$$

$$= \Xi_R\left\{ p - \frac{1}{\mu(\Omega_R)} \int_{\Omega_R} \Xi_R p \, dx \right\}$$

where Ξ_R is the characteristic function of the set Ω_R, and $\mu(.)$ is the measure in R^3.

Now we show that $\beta_R p \in L_0^2(\Omega_R)$; first we observe that:

$$\int_{\Omega_R} \beta_R \, p \, dx = 0$$

Also, since $\beta_R p=0$ outside Ω_R, it is sufficient to show that $\| \beta_R p \|_{L^2(\Omega_R)} < \infty$. In fact,

$$\| \beta_R p \|^2_{L^2(\Omega_R)} = \int_{\Omega_R} \{ p - \frac{1}{\mu(\Omega_R)} \int_{\Omega_R} p \, dx \}^2 dx$$

$$= \int_{\Omega_R} | p |^2 dx - \frac{1}{\mu(\Omega_R)} | \int_{\Omega_R} p \, dx |^2$$

$$\leq \| p \|^2_{L^2(\Omega)} + I_3$$

where

$$I_3 = \frac{1}{\mu(\Omega_R)} | \int_{\Omega_R} p \, dx |^2$$

$$\leq \frac{1}{\mu(\Omega_R)} \{ \int_{\Omega_R} p^2 \, dx \} . \{ \int_{\Omega_R} 1 \, dx \}$$

$$= \int_{\Omega_R} p^2 \, dx$$

Thus

$$\| \beta_R p \|_{L^2(\Omega_R)} \leq 2 \| p \|_{L^2(\Omega)}$$

Now we are ready to estimate the error between $\beta_R p$ and p in the norm $\| \, . \, \|_{L^2(\Omega_R)}$:

$$\| \beta_R p - p \|^2_{L^2(\Omega)} = \int_{\Omega_R} \{ p - \frac{1}{\mu(\Omega_R)} \int_{\Omega_R} p \, dx - p \}^2 dx$$

$$= \frac{1}{\mu(\Omega_R)} \{ \int_{\Omega_R} p \, dx \}^2$$

Now

$$\{ \int_{\Omega_R} p \, dx \}^2 \leq \{ \int_{\Omega_R} | p |^{2-\epsilon} dx \}^{\frac{2}{2-\epsilon}} \{ \int_{\Omega_R} 1 \, dx \}^{2-\frac{2}{2-\epsilon}}$$

$$= \| p \|^2_{L^{2-\epsilon}(\Omega_R)} \, (\mu(\Omega_R))^{2-\frac{2}{2-\epsilon}}$$

$$\leq \| p \|^2_{L^{2-\epsilon}(\Omega)} \, (\mu(\Omega_R))^{2-\frac{2}{2-\epsilon}}$$

Therefore,

$$\| \beta_R p - p \|^2_{L^2(\Omega)} = \frac{\| p \|^2_{L^{2-\epsilon}(\Omega)}}{\mu(\Omega_R)^{\epsilon/(2-\epsilon)}}$$

Consequently,

$$\lim_{R\to\infty} \| \beta_R p - p \|_{L^2(\Omega)} = 0$$

Now for $p=O(r^{-2})$, and $\epsilon > 0$ we have:

$$\| \beta_R p - p \|^2_{L^2(\Omega_R)} \leq C\left\{ \frac{1}{R} \right\}$$

or

$$\| \beta_R p - p \|_{L^2(\Omega_R)} \leq C \left\{ \frac{1}{R^{1/2}} \right\}$$

where C is a constant depending on p but not on R.

■

In the case where the support of $\underline{f}$ is bounded in R^3, we have that $\underline{u}(x)=O(R^{-1})$ and $p(x)=O(R^{-2})$ for $| x |=R$ large enough. With those decay rates we can calculate the asymptotic rate at which the errors $\| \underline{u}_R - \underline{u} \|_{W^{1,0}(\Omega_R)}$ and $\| p - \beta_R p \|_{L^2(\Omega_R)}$ go to 0. From (3.24) we have

$$\| p_R - p \|_{W^{1,0}(\Omega_R)} + \| \underline{u}_R - \underline{u} \|_{L^2(\Omega_R)} \tag{3.25}$$

$$\leq C \left\{ \| \underline{u} - \Pi_R \underline{u} \|_{W^{1,0}(\Omega_R)} + \| p - \beta_R p \|_{L^2(\Omega_R)} \right\}$$

Thus the following result follow from Lemma {3.4} and {3.5}

Theorem 3.6: Assume that supp($\underline{f}$) is compact, then there exists a constant C depending on $\underline{u}$,p such that:

$$\| \underline{u}-\underline{u}_R \|_{W^{1,0}(\Omega_R)} + \| p_R - p \|_{L^2(\Omega_R)} \leq C \frac{1}{R^{1/2}} \tag{3.26}$$

■

Remarks:

1. The asymptotic rate at which the solution of the truncated problem approaches the solution of the original one is not satisfactory (only $O(R^{-1/2})$). To achieve any improvement we need to consider other boundary conditions at the large boundary $\delta\Omega_R$. This will enable us to force the solution of the truncated problem to behave asymptotically in a larger neighborhood near the boundary $\delta\Omega_R$. One possible type to be consisdered under this treatment is the zero stress boundary condition (k=1), given by:

$$\left[-p\underline{n} + \underline{n}.\nabla\underline{u} \right] \Big|_{\delta\Omega_R} = 0$$

or in tensorial form

$$(-pn_i + u_{j,i}) \Big|_{\delta\Omega_R} = 0$$

2. Again we should emphasize the fact that the estimate (3.26) has been derived under the asumption that the constant C_1 in the estimate (3.11) in Proposition {3.2} was independent of R. Also the constant C in (3.11) in Proposition {3.2} grows with C_1. Therefore under the possible assumption that C_1 grows with R, the result could be worse, i.e.,

$$\| \underline{u}-\underline{u}_R \|_{W^{1,0}(\Omega_R)} + \| p_R - p \|_{L^2(\Omega_R)} \leq C \frac{1}{R^\delta} \qquad (3.27)$$

where $\delta < 1/2$ and even may be negative in which case no convergence might occur.

3.3. The truncated problem with zero stress at the artificial boundary

We only consider the case where $\underline{f}$ belongs to $L^2(\Omega_R)$ and the support of $\underline{f}$ is either compact in R^3 or has a sufficient decay rate at infinity. Similar to the previous case, we need only treat the case of support $\underline{f}$ compact in R^3, and choose R large enough so that the support of $\underline{f}$ is contained in Ω_R. In that case the truncated problem is

$$-\Delta\underline{u}_R + \nabla p_R = \underline{f} \quad \text{in } \Omega_R \qquad (3.28)$$

$$\mathrm{div}\,\underline{u}_R = 0 \text{ in } \Omega_R \qquad (3.29)$$

$$\underline{u}_R |_{\delta\Omega} = 0 \qquad (3.30)$$

$$-p_R\underline{n} + \frac{\partial}{\partial n}\underline{u}_R = 0 \text{ on } \delta\Omega_R \qquad (3.31)$$

The variational formulation of the problem is sought in the following space for the velocity

$$W^R(\Omega_R) = \left\{ \ \Xi_R \underline{u} \mid \underline{u} \in [\mathring{W}^{1,0}(\Omega)]^3 \ \right\}$$

where Ξ_R is the characteristic function of the domain Ω_R. For the pressure space we use $L^2(\Omega_R)$.

Proposition 3.7: Equipped with the norm $\| \cdot \|_{W^{1,0}(\Omega_R)}$, $W^R(\Omega_R)$ is a Hilbert space. ∎

Proposition 3.8:

$$W^R(\Omega_R) = H^R = \left\{ \ \underline{u} \in [H^1(\Omega_R)]^3 \ , \ \underline{u}|_{\delta\Omega} = 0 \ \right\}$$

∎

Remark: The proofs of the Propositions {3.7} and {3.8} use the equivalence of the weighted and the unweighted norms on the bounded domain Ω_R.

Proposition 3.9: The seminorm $|\quad|_{W^{1,0}(\Omega_R)}$ defined by:

$$|\ \underline{u}\ |_{W^{1,0}(\Omega_R)} = \left\{ \int_{\Omega_R} |\ \mathrm{grad}\underline{u}\ |^2\, dx \right\}^{1/2}$$

is a norm on $W^R(\Omega_R)$ equivalent to the usual norm.

Proof: We use the inequality [13], [16]

$$\int_{\Theta} \frac{\phi^2}{1+|\ x\ |^2}\, dx \le C \int_{\Theta} |\ \underline{v}\ |^2\, dx$$

where Θ is an unbounded domain, and where

$$\phi = \int_{L_i} \underline{v}.d\underline{l}$$

with L_i is a line segment $[x_0,x]$ for x_0, $x \in \Theta$. Now we can easily establish the inequality on the domain Ω_R for the set:

$$\underline{u} \in K = \left\{ \ \underline{u} = \Xi_R \mathrm{grad}\Psi,\ \Psi \in C_0^{\infty}(\Omega) \ \right\}$$

where now x_0 is any point outside the support of Ψ. Then the result can be extended to $W^R(\Omega_R)$ using the density of K in $W^R(\Omega_R)$.

∎

3.3.1. Variational formulation of the problem

The variational formulation of problem (3.28)-(3.31) can be obtained by multiplying (3.28) and (3.29) by functions in the testing velocity space $W^R(\Omega_R)$ and testing pressure space $L^2(\Omega_R)$ respectively. After integrating by parts we get the variational problem:

seek $\underline{u}_R, p_R \in W^R(\Omega_R)\times L^2(\Omega_R)$ such that:

$$a(\underline{u}_R,\underline{v}_R) + b(p_R,\underline{v}_R)=\langle \underline{f},\underline{v}_R\rangle \qquad \forall\ \underline{v}_R \in W^R(\Omega_R) \qquad (3.32)$$

$$b(q_R,\underline{u}_R) \qquad =0 \qquad \forall\ q_r \in L^2(\Omega_R) \qquad (3.33)$$

where

$$a(\underline{u}_R,\underline{v}_R)=\int_{\Omega_R} \text{grad}\ \underline{u}_R:\text{grad}\underline{v}_R\ dx$$

$$b(p_R,\underline{v}_R)=-\int_{\Omega_R} p_R\ \text{div}\underline{v}_R\ dx$$

This variational problem is of the Brezzi type and in what follows we verify the conditions needed to guarantee existence and uniqueness.

3.3.2. Verification of the conditions needed for the existence of the solution

3.3.2.1. Continuity and coercivity of b(.,.)

The bilinear form b(.,.) is defined on $L^2(\Omega_R)\times W^R(\Omega_R)$. To verify the continuity of b(.,.) we have the estimate

$$|\ b(p_R,\underline{v}_R)\ | = |\ -\int_{\Omega_R} p_R\ \text{div}\underline{v}_R\ dx\ |$$
$$\leq \|\ p_R\ \|_{L^2(\Omega_R)} \|\ \underline{v}_R\ \|_{W^{1,0}(\Omega_R)}$$

where we have used Schwartz inequality.

Now to show the stability condition on b(.,.) we need the following lemma.

Lemma 3.10: Given $p \in L^2(\Omega_R)$, there exists at least one vector $\underline{v}_R \in W^R(\Omega_R)$ such that:

$$\operatorname{div} \underline{v}_R = -p$$

$$\underline{v}_R \mid_{\delta\Omega} = 0$$

Further, there exsists a constant C independent of R such that

$$\| \underline{v} \|_{W^{1,0}(\Omega_R)} \leq C \| p \|_{L^2(\Omega_R)}$$

Proof: The proof uses the results of Chapter {2} which led to the sup condition for the unbounded problem. Given $p_R \in L^2(\Omega_R)$, we extend p_R outside Ω_R by zero. If we denote this function by ν, then:

$$\nu \in L^2(\Omega)$$

and

$$\| \nu \|_{L^2(\Omega)} = \| p_R \|_{L^2(\Omega_R)}$$

Now using Lemma {2.11}, we can find at least one vector $\underline{v} \in [\mathring{W}^{1,0}(\Omega)]^3$ such that:

$$\operatorname{div} \underline{v} = \nu$$

with the estimate

$$\| \underline{v} \|_{W^{1,0}(\Omega)} \leq C \| \nu \|_{L^2(\Omega)}$$

Now taking $\underline{v}_R$ to be the restriction of $\underline{v}$ to Ω_R, i.e.,

$$\underline{v}_R = \Xi_R \underline{v}$$

we get the required vector. Moreover,

$$\begin{aligned} \| \underline{v}_R \|_{W^{1,0}(\Omega_R)} &\le \| \underline{v} \|_{W^{1,0}(\Omega)} \\ &\le C \| \nu \|_{L^2(\Omega)} \\ &= C \| p_R \|_{L^2(\Omega_R)} \end{aligned}$$

■

Therefore we have the following result:

Lemma 3.11: For $p_R \in L^2(\Omega_R)$

$$\sup_{\substack{\underline{w} \in W^R(\Omega_R) \\ \underline{w} \neq 0}} \frac{b(p_R, \underline{w})}{\| \underline{w} \|_{W^{1,0}(\Omega_R)}} \ge \beta \| p_R \|_{L^2(\Omega_R)}$$

with β independent of R.

Proof: Let $\underline{v}_R$ be the outcome of the previous lemma; we have:

$$\sup_{\substack{\underline{w} \in W^R(\Omega_R) \\ \underline{w} \neq 0}} \frac{b(p_R, \underline{w})}{\| \underline{w} \|_{W^{1,0}(\Omega_R)}} \ge \frac{b(p_R, \underline{v}_R)}{\| \underline{v}_R \|_{W^{1,0}(\Omega_R)}} \ge \beta \| p_R \|_{L^2(\Omega_R)}$$

where $\beta = C^{-1}$.

■

3.3.2.2. Continuity and coercivity of a(.,.)

The continuity of a(.,.) is obvious since,

$$| \, a(\underline{u}_R,\underline{v}_R) \, | = | \int_{\Omega_R} \mathrm{grad}\underline{u}_R : \mathrm{grad}\underline{v}_R \, dx \, |$$

$$\leq \| \underline{u}_R \|_{W^{1,0}(\Omega_R)} \| \underline{v}_R \|_{W^{1,0}(\Omega_R)}$$

while the coercivity of a(.,.) follows from the fact that the seminorm $| \, . \, |_{W^{1,0}(\Omega_R)}$ is a norm on $W^R(\Omega_R)$ equivalent to the usual one.

Having verified the Brezzi conditions for the variational problem (3.32)–(3.33) we then have the following result.

Theorem 3.12: The variational problem

seek $\underline{u}_R, \ p_R \in W^R(\Omega_R) \times L^2(\Omega_R)$ such that:

$$a(\underline{u}_R,\underline{v}_R) + b(p_R,\underline{v}_R) = \langle \underline{f},\underline{v}_R \rangle \qquad \forall \ \underline{v}_R \in W^R(\Omega_R)$$

$$b(q_R,\underline{u}_R) = 0 \qquad \forall \ q_R \in L^2(\Omega_R)$$

has a unique solution. Further there exists a constant C >0 such that

$$\| \underline{u}_R \|_{W^{1,0}(\Omega_R)} + \| p_R \| \leq C \| \underline{f} \|_{L^2(\Omega_R)} .$$

■

3.4. Asymptotic error in zero stress boundary condition

We now consider the asymptotic error in the boundary condition (3.31).

Lemma 3.13: The fundamental solution $\underline{U}^k(x,y)$ of the Stokes equation satisfies:

$$n_i \frac{\partial}{\partial x_i} \underline{U}^k_j(x,y) \, \Big|_{x=R} = O(\frac{1}{R^2})$$

where the summation convention holds on the index i while k,j are fixed.

■

Remarks:

1. The proof of the previous lemma is a direct consequence of the behavior of $\underline{U}^k(x,y)$. Actually since $\underline{U}_j^k(x,y)=O(R^{-1})$, one can easily show that derivatives of $\underline{U}_j^k(x,y)$ would be $O(R^{-2})$.

2. The fundamental solution for the pressure will not cause any trouble in the estimates since for R large $P^k(x,y)$ is $O(R^{-2})$.

Lemma 3.14: For supp(f) compact, the solution $\underline{u}(x)$ of the problem on the unbounded domain Ω satisfies

$$n_l \frac{\partial}{\partial x_l} u_k(x) = O(\frac{1}{R^2})$$

where $1 \leq k \leq 3$ and the summation convention holding for the index l.

Proof: The result follows easily from the representation of the solution:

$$u_k(x) = \int_{\Omega_R} \underline{U}_j^k(x,y) f_i(y)\, dy + W_k(x,\underline{u}) + V_k(x, T_j(\underline{u}) n_j)$$

(where the expressions and the symbols involved are explained in the appendix) and the properties of the fundamental solutions.

■

Remark: Actually we can improve the result obtained in the previous lemma by getting a more precise estimate.

Lemma 3.15: There exists a constant $C > 0$ such that for $|x|=R$ large enough we have

$$\left| n_l \frac{\partial}{\partial x_l} u_k(x) - p n_k \right|_{|x|=R} \leq \frac{C\| \underline{f} \|_{L^2(\Omega_R)}}{R^2}$$

Proof: We use the integral representation of the solution and the Schwartz inequality to obtain that:

$$\left| n_l \frac{\partial}{\partial x_l} u_k(x) - p n_k \right|_{|x|=R}$$

$$\leq \| \underline{f} \|_{L^2(\Omega_R)} \left\{ \int_{\mathrm{supp}\underline{f}} \left| n_l \frac{\partial}{\partial x_l} \underline{U}_j^k(x,y) + P^j(x,y) n_k \right|^2 dy \right\}^{1/2}$$

$$+ \| \underline{u} \|_{H^{1/2}(\delta\Omega)} \left\{ \int_{\delta\Omega} \left| \left\{ n_l \frac{\partial}{\partial x_l} \right\} T^*_{ij}(\underline{U}^k(x,y))_y n_j + \frac{\partial}{\partial x_j} P^i(x,y) n_k \right|^2 dS_y \right\}^{1/2}$$

$$+ \| T_{ij}(\underline{u}) \|_{H^{1/2}(\delta\Omega)} \left\{ \int_{\delta\Omega} \left| \left\{ n_l \frac{\partial}{\partial x_l} \right\} \underline{U}_j^k(x,y) + P^i(x,y) n_k \right|^2 dS_y \right\}^{1/2}$$

where, as usual, the summation convention holds for the indices i, l, j. Now using the estimates

$$\| \underline{u} \|_{W^{2,1}(\Omega)} + \| p \|_{W^{1,1}(\Omega)} \leq C \| \underline{f} \|_{L^2(\Omega)}$$

along with the trace inequalities

$$\| \underline{u} \|_{H^{1/2}(\delta\Omega)} \leq C \| \underline{u} \|_{W^{2,1}(\Omega)}$$

and

$$\| T_{ij}(\underline{u}) \|_{H^{1/2}(\delta\Omega)} \leq C \left\{ \| \underline{u} \|_{W^{2,1}(\Omega)} + \| p \|_{W^{1,1}(\Omega)} \right\}$$

we obtain the result.

∎

3.5. Error estimates for the truncated problem with zero stress at the artificial boundary

Now we study the error between the truncated domain and the infinite domain problems. Let $\underline{e}_R$, μ_R be the difference between the two solutions, i.e,

$$\underline{e}_R=\underline{u}-\underline{u}_R$$

$$\mu_R=p-p_R$$

Then $\underline{e}_R$, μ_R satisfy:

$$-\Delta\underline{e}_R+\nabla\mu_R=0$$

$$\mathrm{div}\,\underline{e}_R=0$$

$$\underline{e}_R|_{\delta\Omega_R}=0$$

$$\frac{\partial}{\partial n}\underline{e}_R-\mu_R\underline{n}_R=\frac{\partial}{\partial n}\underline{u}-p\underline{n}_R=g\in W^{-1/2,0}(\delta\Omega_R)$$

Now this problem of finding $\underline{e}_R$, μ_R can be formulated variationally as follows:

seek $\underline{e}_R$, $\mu_R \in W^R(\Omega_R)\times L^2(\Omega_R)$ such that:

$$a(\underline{e}_R,\underline{v}_R)+b(\mu_R,\underline{v}_R)=\langle g,\underline{v}_R\rangle \qquad \forall\ \underline{v}_R\in W^R(\Omega_R) \tag{3.34}$$

$$b(q_R,\underline{e}_R)=0 \qquad \forall\ q_R\in L^2(\Omega_R) \tag{3.35}$$

Problem (3.34)-(3.35) has a unique solution pair $(\underline{e}_R,\mu_R)$ in $W^R(\Omega_R)\times L^2(\Omega_R)$, and there exists a constant $C>0$ such that:

$$\| \underline{e}_R \|_{W^{1,0}(\Omega_R)} + \| \mu_R \|_{L^2(\Omega_R)} \leq C \| g \|_{W^{-1/2,0}(\delta\Omega_R)}$$

i.e.,

$$\| \underline{u}-\underline{u}_R \|_{W^{1,0}(\Omega_R)} + \| p-p_R \|_{L^2(\Omega_R)} \leq C \| g \|_{W^{-1/2,0}(\delta\Omega_R)}$$

Now for $\underline{f}$ in $L^2(\Omega_R)$ we obtain that g belongs to $W^{1/2,0}(\delta\Omega_R)$ and consequently belongs to $L^2(\delta\Omega_R)$. Therefore we have to estimate:

$$\| g \|_{W^{-1/2,0}(\delta\Omega_R)} = \sup_{\substack{\underline{v}\in W^{1/2,0}(\delta\Omega_R) \\ \underline{v}\neq 0}} \frac{\langle g,\underline{v}\rangle_{\delta\Omega_R}}{\| \underline{v} \|_{W^{1/2,0}(\delta\Omega_R)}}$$

$$= \sup_{\substack{\underline{v}\in H^{1/2}(\delta\Omega_R) \\ \underline{v}\neq 0}} \frac{\langle g,\underline{v}\rangle_{\delta\Omega_R}}{\| \underline{v} \|_{H^{1/2}(\delta\Omega_R)}}$$

where the equality holds due to the eqivalence of the norms on the bounded domain $\delta\Omega_R$, and where

$$\langle g,\underline{v}\rangle_{\delta\Omega_R} = \int_{\delta\Omega_R} g_i \, v_i \, dS$$

But

$$| \langle g,\underline{v}\rangle_{\delta\Omega_R} | \leq \| g \|_{L^2(\delta\Omega_R)} \| \underline{v} \|_{L^2(\delta\Omega_R)}$$

and

$$\| \underline{v} \|_{L^2(\delta\Omega_R)} \leq \| \underline{v} \|_{H^{1/2}(\delta\Omega_R)}$$

Also for $\underline{f}$ in $L^2(\Omega)$ we have the inequality:

$$| g |_{\delta\Omega_R} \leq \frac{C \| \underline{f} \|_{L^2(\Omega_R)}}{R^2}$$

Thus we get

$$\| g \|_{W^{-1/2,0}(\delta\Omega_R)} \leq \| g \|_{L^2(\delta\Omega_R)}$$

$$=\left\{ \int_{\delta\Omega_R} | g |^2 \, dS \right\}^{1/2}$$

$$\leq C \| f \|_{L^2(\Omega_R)} \left\{ \int_{\delta\Omega_R} \frac{1}{R^4} R^2 \, d\omega \right\}^{1/2}$$

$$= \frac{C}{R} \| f \|_{L^2(\Omega_R)}$$

These results can be summarized in the following theorem.

Theorem 3.16: There exists a constant $C > 0$, independent of R, such that:

$$\| \underline{u} - \underline{u}_R \|_{W^{1,0}(\Omega_R)} + \| p - p_R \|_{L^2(\Omega_R)} \leq \frac{C \| \underline{f} \|_{L^2(\Omega_R)}}{R} .$$

■

CHAPTER 4

ON THE DISCRETE APPROXIMATION OF THE TRUNCATED PROBLEM

4.1. Introduction

Without loss of generality assume that Ω_R is a polygonal domain in R^3. Let $\{ \tau_h \}$ be a family of regular triangulization of Ω_R such that:

$$\bar{\Omega}_R = \bigcup_{K \in \tau_h} K$$

where K denotes a simplex in R^3. We set h(K) to be the length of the largest edge of K and let

$$h = \max_{K \in \tau_h} h(K)$$

and

$$h_{min} = \min_{K \in \tau_h} h(K)$$

and assume that

$$\frac{h}{h_{min}} \leq \beta$$

for some constant β. In addition, if ρ_K is the diameter of the largest sphere inscribed in K we set

$$\sigma_K = \frac{h_K}{\rho_K}$$

The family of triangulization is regular if there exists $\sigma > 0$ such that:

$$\sigma_K \leq \sigma \ \forall\ K \in \tau_h \quad .$$

By $P_m(K)$, we will denote the set of all polynomials of degree less than or equal m defined on the simplex K.

Remark: Again, although we have the equivalence of the usual norms and the weighted norms on the bounded domain Ω_R, we will keep on using the weighted norms since our final goal is to derive estimates where the constants depend neither on the truncation parameter R nor the discretization parameter h.

Definition 4.1: The discrete divergence operator denoted by div_h is defined by:

$$(q_h, \mathrm{div}_h \underline{v}_h) = \sum_{K \in \tau_h} \int_K q_h \mathrm{div} \underline{v}_h \, dx$$

■

4.2. The discrete problem for zero velocity artificial boundary condition

Let $\mathring{S}_h$ and Q_h be finite dimensional subspaces of $[\mathring{W}^{1,0}(\Omega_R)]^3$ and $L_0^2(\Omega_R)$ respectively. We have the following problems:

1. The original problem on the unbounded domain given by:

seek $(\underline{u},p) \in [\mathring{W}^{1,0}(\Omega)]^3 \times L^2(\Omega)$ such that:

$$a(\underline{u},\underline{v})+b(p,\underline{v})=\langle \underline{f},\underline{v}\rangle \qquad \forall\ \underline{v} \in [\mathring{W}^{1,0}(\Omega)]^3 \tag{4.1}$$

$$b(q,\underline{u}) \qquad =0 \qquad \forall\ q \in L^2(\Omega) \tag{4.2}$$

2. The truncated problem on the bounded domain Ω_R given by:

seek $(\underline{u}_R, p) \in [\mathring{W}^{1,0}(\Omega_R)]^3 \times L_0^2(\Omega_R)$ such that:

$$a(\underline{u}_R, \underline{v}_R) + b(p, \underline{v}_R) = \langle \underline{f}, \underline{v}_R \rangle \qquad \forall\ \underline{v}_R \in [\mathring{W}^{1,0}(\Omega_R)]^3 \tag{4.3}$$

$$b(q_R, \underline{u}_R) \qquad = 0 \qquad \forall\ q \in L_0^2(\Omega_R) \tag{4.4}$$

3. And finally the discrete problem given by:

seek $(\underline{u}_{Rh}, p_{Rh}) \in \mathring{S}_h \times Q_h$ such that:

$$a(\underline{u}_{Rh}, \underline{v}_{Rh}) + b(p_{Rh}, \underline{v}_{Rh}) = \langle \underline{f}, \underline{v}_{Rh} \rangle \qquad \forall\ \underline{v}_{Rh} \in \mathring{S}_h \tag{4.5}$$

$$b(q_{Rh}, \underline{u}_{Rh}) \qquad = 0 \qquad \forall\ q_{Rh} \in Q_h \tag{4.6}$$

We have established the well-posedness of (4.1)-(4.2) in Chapter {2}, as well as of (4.3)-(4.4) in Chapter {3}. Also, we have obtained estimate on the difference $\underline{u}-\underline{u}_R$ as well as $p-p_R$. The estimate was:

$$\| \underline{u}-\underline{u}_R \|_{W^{1,0}(\Omega_R)} + \| p-p_R \|_{L^2(\Omega_R)} \le C \| f \|_{L^2(\Omega_R)} R^{-1/2}$$

Remark: We remind the reader that in Chapter {3}, we have made an assumption that led to the result that C is independent of R.

The conditions under which the well posedness of the discrete problem is ensured, as well as approximation results are found in the following theorem.

Theorem 4.2: Assume:

$$\sup_{\substack{\underline{v}_{Rh} \in \mathring{S}_h \\ \underline{v}_{Rh} \neq 0}} \frac{b(q_{Rh}, \underline{v}_{Rh})}{\| \underline{v}_{Rh} \|_{W^{1,0}(\Omega_R)}} \ge \beta^* \| q_{Rh} \|_{L^2(\Omega_R)} \quad \forall\ q_{Rh} \in Q_h$$

with β^* independent of h. Then problem (4.5)–(4.6) has a unique solution pair $(\underline{u}_{Rh},p_{Rh})$ in $\overset{\circ}{S}_h\times Q_h$ with the following estimate holding:

$$\| \underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_{Rh} \|_{L^2(\Omega_R)} \le C \| \underline{f} \|_{L^2(\Omega_R)}$$

Moreover, there exists a constant $C_1 > 0$ such that

$$\| \underline{u}_R-\underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_R-p_{Rh} \|_{L^2(\Omega_R)}$$
$$\le C_1 \left\{ \inf_{\underline{w}_{Rh}\in\overset{\circ}{S}_h} \| \underline{u}_R-\underline{w}_{Rh} \|_{W^{1,0}(\Omega_R)} + \inf_{q_{Rh}\in Q_h} \| p_R-q_{Rh} \|_{L^2(\Omega_R)} \right\}$$

■

4.2.1. Finite element spaces and well posedness of the discrete problem

Definition 4.3: Let ${}_U|\quad|$ and ${}_W|\quad|$ denote the unweighted and weighted seminorms respectively where:

$${}_U| \phi |_m^2 = \int_{\Omega_R} | D^\mu \phi |^2 \, dx$$

$${}_W| \phi |_m^2 = \int_{\Omega_R} (1+| x |^2)^{\mu-1} | D^\mu \phi |^2 \, dx$$

with μ a multiindex and $| \mu |=m$.

■

The stability condition in Theorem {4.2} has been verified for different finite element spaces for both velocity and pressure for bounded domains using the unweighted norms. A crucial observation in the following lemma will enable us to extend the different results obtained for the bounded domains in the unweighted norms to the case of the truncated problem with the weighted norms.

Lemma 4.4: There exists a map denoted by Π_h

$$\Pi_h: W^{m,m-1}(\Omega_R) \to \overset{\circ}{S}_h$$

or

$$\Pi_h: H^m(\Omega_R) \to \overset{\circ}{S}_h$$

such that:

$$_W| \ v-\Pi_h v \ |_\nu \leq C \ h^{\mu-\nu} \ _W| \ v \ |_\mu$$

for $\nu=0$, 1 and $\mu \geq \nu+1$.

Proof: Such a map exists in the unweighted norm. Actually we have

$$_U| \ v-\Pi_h v \ |_\nu \leq C \ h^{\mu-\nu} \ _U| \ v \ |_\mu$$

Now due to the fact that

$$_W| \quad |_\nu \leq \ _U| \quad |_\nu \ \text{ for } \nu=0,1$$

and

$$_U| \quad |_\tau \leq \ _W| \quad |_\tau \ \text{ for } \tau \geq 1$$

we easily obtain the result. ■

Remark:

1. Due to the equivalence of the weighted and the unweighted norms on the bounded domain Ω_R, the projection operators from the weighted and the unweighted spaces onto $\overset{\circ}{S}_h$ respectively, coincide.

2. Another type of result that can be obtained is

$$_W| \ v-\Pi_h v \ |_\tau \leq C \ h^{\mu-\tau} \ _U| \ v \ |_\mu$$

One can therefore establish results similar to the case of bounded domains once one chooses particular finite element spaces for both the velocity and the pressure. In particular, we mention the work of Temam [24], Crouzeix and Raviart [6] Girault and Raviart [9] etc.... The following is a summary of some of the possible choices of spaces and the corresponding approximation results.

1. Conforming elements.

a. If the pair $(\underline{u}_R, p_R) \in \left\{ W^{2,1}(\Omega_R) \cap [\mathring{W}^{1,0}(\Omega_R)]^3 \right\} \times \left\{ L_0^2(\Omega_R) \cap W^{1,1}(\Omega_R) \right\}$ one can choose:

$$\mathring{S}_h = \left\{ \underline{v} \mid v_i \in C^0(\Omega_R) \cap W^{1,0}(\Omega_R) \ , \ v_i|_K \in P_2(K) \right\}$$

where i=1,2,3, and

$$Q_h = \left\{ q \in L_0^2(\Omega_R) \mid \quad q|_K \in P_0(K) \right\}$$

The result obtained in this case

$$\| \underline{u}_R - \underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_R - p_{Rh} \|_{L^2(\Omega_R)} \leq C\, h \left\{ \| \underline{u}_R \|_{W^{2,1}(\Omega_R)} + \| p_R \|_{W^{1,1}(\Omega_R)} \right\}$$

b. If the pair $(\underline{u}_R, p_R) \in \left\{ W^{3,2}(\Omega_R) \cap [\mathring{W}^{1,0}(\Omega_R)]^3 \right\} \times \left\{ L_0^2(\Omega_R) \cap W^{2,2}(\Omega_R) \right\}$ one can also choose:

$$\mathring{S}_h = \left\{ \underline{v} \mid v_i \in C^0(\Omega_R) \cap W^{1,0}(\Omega_R) \ , \ v_i|_K \in P_2(K) + \text{"bubble extension"} \right\}$$

where for "bubble extension" we refer the reader to Crouzeix and Raviart [6], and

$$Q_h = \left\{ q \in L_0^2(\Omega_R) \mid \quad q|_K \in P_1(K) \right\}$$

and the result obtained is:

$$\| \underline{u}_R-\underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_R-p_{Rh} \|_{L^2(\Omega_R)} \leq C\ h^2 \left\{ \| \underline{u}_R \|_{W^{3,2}(\Omega_R)} + \| p_R \|_{W^{2,2}(\Omega_R)} \right\}$$

c. Other possible choices of spaces are available in [3].

2. Non-conforming case (Crouzeix and Raviart [6]). In this case the inclusion property for the approximating velocity space does not hold. However, the choices

$$\mathring{S}_h = \left\{ \underline{v} \mid v_i|_K \in P^1(K) \right\}$$

$$Q_h = \left\{ q \in L^2(\Omega_R) \mid q|_K \in P_0(K) \right\}$$

yield the error estimate:

$$\| \underline{u}_R-\underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_R-p_{Rh} \|_{L^2(\Omega_R)} \leq C\ h \left\{ \| \underline{u}_R \|_{W^{2,1}(\Omega_R)} + \| p_R \|_{W^{1,1}(\Omega_R)} \right\}$$

Now we turn to the case of the other boundary condition.

4.3. The discrete problem for zero stresses boundary condition

Let W^h and Q_h denote finite dimensional subpsaces of $W^R(\Omega_R)$ and $L^2(\Omega_R)$ respectively. Again we have the following problems:

1. The original problem on the unbounded domain given by:

seek $(\underline{u},p) \in [\mathring{W}^{1,0}(\Omega)]^3 \times L^2(\Omega)$ such that:

$$a(\underline{u},\underline{v})+b(p,\underline{v})=\langle \underline{f},\underline{v}\rangle \qquad \forall\ \underline{v} \in [\mathring{W}^{1,0}(\Omega)]^3 \qquad (4.7)$$

$$b(q,\underline{u}) \qquad =0 \qquad \forall\ q \in L^2(\Omega) \qquad (4.8)$$

2. The truncated problem on the bounded domain Ω_R given by:

seek $(\underline{u}_R, p_R) \in W^R(\Omega_R) \times L^2(\Omega_R)$ such that:

$$a(\underline{u}_R, \underline{v}_R) + b(p_R, \underline{v}_R) = \langle \underline{f}, \underline{v}_R \rangle \qquad \forall\ \underline{v}_R \in W^R(\Omega_R) \tag{4.9}$$

$$b(q_R, \underline{u}_R) \qquad = 0 \qquad \forall\ q_R \in L^2(\Omega_R) \tag{4.10}$$

3. And finally the discrete problem given by:

seek $(\underline{u}_{Rh}, p_{Rh}) \in W^h \times Q_h$ such that:

$$a(\underline{u}_{Rh}, \underline{v}_{Rh}) + b(p_{Rh}, \underline{v}_{Rh}) = \langle \underline{f}, \underline{v}_{Rh} \rangle \qquad \forall\ \underline{v}_{Rh} \in W^h \tag{4.11}$$

$$b(q_{Rh}, \underline{u}_{Rh}) \qquad = 0 \qquad \forall\ q_{Rh} \in Q_h \tag{4.12}$$

where the expression for a(.,.) , b(.,.) have already been defined in Chapter {3}.

Theorem 4.5: Assume,

$$\sup_{\substack{\underline{v}_h \in W^h \\ \underline{v}_h \neq 0}} \frac{b(q_h, \underline{v}_h)}{\| \underline{v}_h \|_{W^{1,0}(\Omega_R)}} \geq \alpha_1 \| q_h \|_{L^2(\Omega_R)} \qquad \forall\ q_h \in Q_h \tag{4.13}$$

Then problem (4.11)-(4.12) has a unique solution pair $(\underline{u}_{Rh}, p_{Rh}) \in W^h \times Q_h$. Moreover, there exists constants C, $C_1 > 0$ such that:

$$\| \underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_{Rh} \|_{L^2(\Omega_R)} \leq C \| \underline{f} \|_{L^2(\Omega_R)}$$

and

$$\| \underline{u}_R - \underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_R - p_{Rh} \|_{L^2(\Omega_R)}$$

$$\leq C_1 \left\{ \inf_{\underline{v}_{Rh} \in W^h} \| \underline{u}_R - \underline{v}_{Rh} \|_{W^{1,0}(\Omega_R)} + \inf_{q_{Rh} \in Q_h} \| p_R - q_{Rh} \|_{L^2(\Omega_R)} \right\}$$

Proof: The proof is the Brezzi approximation result.

■

Remark: If we denote by Z^h the subspace of W^h where:

$$Z^h = \left\{ \underline{v}_{Rh} : b(q_h, \underline{v}_{Rh}) = 0 \ \forall \ q_h \in Q_h \right\}$$

then, the strong coercivity of a(.,.) is necessary and sufficient for the well-posedness of the problem:

seek $\underline{u}_{Rh} \in Z^h$ such that:

$$a(\underline{u}_{Rh}, \underline{v}_{Rh}) = \langle \underline{f}, \underline{v}_{Rh} \rangle \qquad \forall \ \underline{v}_{Rh} \in Z^h \tag{4.14}$$

Now we discuss the hypotheses sufficient to ensure the condition (4.13) in the previous theorem.

<u>Hypothesis (H1).</u>

There exists a mapping denoted by $r_h \in L([W^{2,1}(\Omega_R)]^3 \cap W^R(\Omega_R); Z^h)$ and a positive integer ν such that:

$$b(q_{Rh}, \text{div } r_h\underline{v} - \underline{v}) = 0 \ \forall \ q_h \in Q_h$$

with

$$\| r_h\underline{v} - \underline{v} \|_{W^{1,0}(\Omega_R)} \leq C \, h^m \, \| \underline{v} \|_{W^{m+1,m}(\Omega_R)} \quad \forall \ \underline{v} \in W^{m+1,m}(\Omega_R)$$

and where $1 \leq m \leq \nu$.

Hypothesis (H2).

The orthogonal projection denoted by $\rho_h \in L(W^{m,m}(\Omega_R)\cap L^2(\Omega_R);Q_h)$ satisfies:

$$\| \rho_h q-q \|_{L^2(\Omega_R)} \leq C\ h^m \| q \|_{W^{m,m}(\Omega_R)}\ \forall\ q \in W^{m,m}(\Omega_R)\cap L^2(\Omega_R)$$

Theorem 4.6: Under the assumptions (H1)-(H2) problem (4.14) and consequently (4.11)-(4.12) has exactly one solution $\underline{u}_{Rh} \in Z^h$ and

$$\lim_{h\to 0} \| \underline{u}_{Rh}-\underline{u}_R \|_{W^{1,0}(\Omega_R)}=0 \tag{4.15}$$

Moreover, if the pair $(\underline{u}_R,p_R) \in \left\{ W^{s+1,s}(\Omega_R)\cap W^R(\Omega_R) \right\}\times\left\{ W^{s,s}(\Omega_R) \right\}$ we have the bound:

$$\| \underline{u}_R-\underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} \leq C\ h^m\left\{ \| \underline{u}_R \|_{W^{m+1,m}(\Omega_R)} + \| p_R \|_{W^{m,m}(\Omega_R)} \right\} \tag{4.16}$$

for $1 \leq m \leq s$.

Proof: First we show the existence of the discrete problem for $\underline{u}_{Rh}$. According to hypotesis (H1) we can deduce that:

$$Z^h\neq\left\{ \underline{0} \right\}$$

Now using the strong coercivity of a(.,.) , we obtain the well posedness of problem (4.14). Now we are in a position to verify (4.15). Actually from the approximation properties the Brezzi saddle point problem we have the estimate

$$\| \underline{u}_R-\underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} \leq$$

$$C_1 \left\{ \inf_{\underline{v}_{Rh}\in W^h} \| \underline{u}_R-\underline{v}_{Rh} \|_{W^{1,0}(\Omega_R)} + \inf_{q_{Rh}\in Q_h} \| p_R-q_{Rh} \|_{L^2(\Omega_R)} \right\}$$

Now assume that $\underline{u}_R \in W^{2,1}(\Omega_R) \bigcap W^R(\Omega_R)$ and that $p_R \in W^{1,1}(\Omega_R)$. Using hypotheses (H1) as well as (H2) for m=1 we get the estimate (4.16) for m=1. Thus for such pair $(\underline{u}_R, p_R)$ the result (4.15) holds. And by the density of $W^{2,1}(\Omega_R) \bigcap W^R(\Omega_R)$ in $W^R(\Omega_R)$ and the density of $W^{1,1}(\Omega_R)$ in $L^2(\Omega_R)$ the result (4.15) holds for $(\underline{u}_R, p_R) \in W^R(\Omega_R) \times L^2(\Omega_R)$. Finally a similar argument can lead to estimate (4.16) for:

$$(\underline{u}_R, p_R) \in \left\{ W^{s+1,s}(\Omega_R) \bigcap W^R(\Omega_R) \right\} \times \left\{ W^{s,s}(\Omega_R) \right\}$$

This completes the proof. ∎

<u>Hypothesis (H3).</u>

For every $q_h \in Q_h$, there exists a $\underline{w}_{Rh} \in W^h$ such that:

$$(q_h - \mathrm{div}_h \underline{w}_{Rh}, s_h) = 0 \quad \forall\ s_h \in Q_h$$

with

$$\| \underline{w}_{Rh} \|_{W^{1,0}(\Omega_R)} \leq C \| q_h \|_{L^2(\Omega_R)}$$

Theorem 4.7: Under the hypotheses (H1), (H2) and (H3) problem (4.11)-(4.12) has exactly one pair $(\underline{u}_{Rh}, p_{Rh}) \in Z^h \times Q_h$ and

$$\lim_{h \to 0} \left\{ \| \underline{u}_{Rh} - \underline{u}_R \|_{W^{1,0}(\Omega_R)} + \| p_R - p_{Rh} \|_{L^2(\Omega_R)} \right\} = 0 \tag{4.17}$$

Moreover if the pair $(\underline{u}_R, p_R) \in \left\{ W^{s+1,s}(\Omega_R) \bigcap W^R(\Omega_R) \right\} \times \left\{ W^{s,s}(\Omega_R) \right\}$ we have the bound:

$$\| \underline{u}_R - \underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_R - p_{Rh} \|_{L^2(\Omega_R)} \tag{4.18}$$

$$\leq C\, h^{m}\left\{ \| \underline{u}_R \|_{W^{m+1,m}(\Omega_R)} + \| p_R \|_{W^{m,m}(\Omega_R)} \right\}$$

for $1 \leq m \leq s$.

Proof: It is clear that (H3) implies the discrete stability condition on $b(.,.)$, since for every $q_h \in Q_h$, we have

$$\sup_{\substack{\underline{v}_{Rh} \in W^h \\ \underline{v}_{Rh} \neq 0}} \frac{b(q_h, \underline{v}_{Rh})}{\| \underline{v}_{Rh} \|_{W^{1,0}(\Omega_R)}} \geq \frac{(q_h, q_h)}{\| \underline{v}_{Rh} \|_{W^{1,0}(\Omega_R)}} \geq \beta \| q_h \|_{L^2(\Omega_R)}$$

hence by the Brezzi approximation result we get the estimate

$$\| p_R - p_{Rh} \|_{L^2(\Omega_R)} \leq C \left\{ \| \underline{u}_R - \underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \inf_{q_h \in Q_h} \| p_R - q_h \|_{L^2(\Omega_R)} \right\}$$

Hence using (H1) and (H2) we obtain the results (4.17) and (4.18).

■

4.4. Some examples of finite element spaces

Let

$$W^h = \left\{ \underline{u}_h \in C^0(\Omega_R) \cap W^R(\Omega_R) : \underline{u}_h|_K \in P_2(K) \right\}$$

Also let

$$Q_h = \left\{ p \in L^2_0(\Omega_R) : p|_K \in P_0(K) \right\}$$

This sequence of lemmas is to verify the conditions in Theorem {4.5}.

Lemma 4.8: [9]For each function $\underline{v} \in C^0(\overline{K})$, there exists a unique function denoted by:

$$\Pi_K \underline{v} \in P_2(K)$$

defined by

$$\Pi_K \underline{v}(a_i) = \underline{v}(a_i)$$

$$\int_{\delta K_i} \left\{ \Pi_K \underline{v} - \underline{v} \right\} dS = 0$$

where $1 \le i \le 4$ and δK_i is the ith face of the simplex K, and for $\underline{v} \in W^{m+1,m}(\Omega_R)$

$$_W| \ \underline{v} - \Pi_K \underline{v} \ |_j \le C \, h_K^i \rho_K^{-j} \ _W| \ \underline{v} \ |_i$$

where j=0, 1 and $j \le i \le m+1$.

■

Lemma 4.9: For the spaces defined above, there exists a map denoted by r_h

$$r_h: W^{2,1}(\Omega_R) \cap W^R(\Omega_R) \to W^h$$

with r_h satisfying the hypothesis (H1).

Proof: We construct the mapping in the usual manner.(see [9])

<u>Step 1.</u>Given $q_h \in Q_h$, there exists a unique $\underline{v}_1$ in the orthogonal complement of Z such that:

$$\mathrm{div}\underline{v}_1 = q_h$$

and

$$\| \ \underline{v}_1 \ \|_{W^{1,0}(\Omega_R)} \le C \ \| \ q_h \ \|_{L^2(\Omega_R)}$$

Step 2. Since $\underline{v}_1$ need not to be continuous, we obtain the projection $\underline{v}_{1h}$ of $\underline{v}_1$ on W^h by choosing $\underline{v}_{1h} \in W^h$ such that:

$$a(\underline{v}_{1h},\underline{v})=a(\underline{v}_1,\underline{v}) \quad \forall\ \underline{v} \in W^h \tag{4.19}$$

From the properties of the projection we have

$$\| \underline{v}_{1h} \|_{W^{1,0}(\Omega_R)} \leq C \| \underline{v}_1 \|_{W^{1,0}(\Omega_R)}$$

Step 3. Now on each simplex K we define the function $r_h\underline{v}$

$$r_h\underline{v}(a_i)=\underline{v}_{1h}(a_i) \quad \text{for } a_i \text{ a vertex}$$

$$\int_{\delta K_i} r_h\underline{v}-\underline{v}\ dS=0 \tag{4.20}$$

for every δK_i the ith face belonging to the simplex K. Thus $r_h\underline{v}$ is the required map.

■

Lemma 4.10: [9]The map r_h defined in the previous lemma satisfies:

$$\| r_h\underline{v} \|_{W^{1,0}(\Omega_R)} \leq C \| q_h \|_{L^2(\Omega_R)} \tag{4.21}$$

Moreover we have the property in (H3), i.e.,

$$(s_h,\mathrm{div}_h r_h\underline{v}_1-q_h)=0 \quad \forall\ s_h \in Q_h \tag{4.22}$$

Proof: The proof of (4.21) can be found in [9], and by using the fact that $s_h|_K \in P_0(K)$ and applying the divergence theorem we get:

$$(s_h,\mathrm{div}_h r_h\underline{v}_{1h}-q_h)= \sum_{K\in\tau} \int_K s_h \mathrm{div} r_h\underline{v}_{1h}\ dx$$

$$= \sum_{K\in\tau} s_h|_K \int_K \operatorname{div} r_h \underline{v}_{h-1h}\, dx$$

$$= \sum_{K\in\tau} s_h|_K \int_K \operatorname{div} \underline{v}_1\, dx$$

$$= \sum_{K\in\tau} s_h|_K \int_K q_h\, dx$$

$=(s_h, q_h)$ which compltes the proof.

■

Lemma 4.11: There exists a constant $C > 0$, independent of R and h such that:

$$\| \underline{u}_R - \underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_R - p_{Rh} \|_{L^2(\Omega_R)} \leq C\, h \left\{ \| \underline{u}_R \|_{W^{2,1}(\Omega_R)} + \| p \|_{W^{1,1}(\Omega_R)} \right\}$$

■

Remarks:

1. Modifying the pressure space to be:

$$Q_h = \left\{ q : q \in C^0(\Omega_R \cap L_0^2(\Omega_R)\ , q|_K \in P^1(K) \right\}$$

and the velocity space to be

$$W^h = \left\{ \underline{v}_{Rh} \in C^0(\Omega_R) \cap W^R(\Omega_R)\ \underline{u}_{Rh}|_K \in P_2(K) + \text{"bubble extension"} \right\}$$

we can improve the approximation result and get quadratic convergence in the $W^{1,0}(\Omega_R)$ norm.

2. Using the extension of Aubin-Nitsche lemma for the Brezzi saddle point problem, one can obtain estimates for the difference $(\underline{u}_R-\underline{u}_{Rh})$ in the norm $\| \quad \|_{W^{0,-1}(\Omega_R)}$ given by:

$$\| \phi \|_{W^{0,-1}(\Omega_R)} = \left\{ \int_{\Omega_R} \frac{1}{1 + | x |^2} | \phi |^2 \, dx \right\}^{1/2}$$

In this case we use the fact that the dual problem:

seek $(\underline{\nu}_g, \pi_g) \in W^R(\Omega_R) \times L^2(\Omega_R)$ such that

$$a(\underline{\omega}, \underline{\nu}_g) + b(\pi_g, \underline{\omega}) = \langle g, \underline{\omega} \rangle \qquad \forall\ \underline{\omega} \in W^R(\Omega_R)$$

$$b(q, \underline{\nu}_g) \qquad = 0 \qquad \forall\ q \in L^2(\Omega_R)$$

is well posed. This can be done using the same techniques already used for the original problem. Therefore we have a regular dual problem. Now using the approximation properties in (H1) we can show that:

$$\| \underline{u}_R-\underline{u}_{Rh} \|_{W^{0,-1}(\Omega_R)} \leq C\, h \left\{ \| \underline{u}_R-\underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_R-p_{Rh} \|_{L_0^2(\Omega_R)} \right\}$$

which finally gives

$$\| \underline{u}_R-\underline{u}_{Rh} \|_{W^{0,-1}(\Omega_R)} \leq C\, h^{d+1} \left\{ \| \underline{u}_R \|_{W^{2,1}(\Omega_R)} + \| p_R \|_{W^{1,1}(\Omega_R)} \right\}$$

where d=1,2 depends on the degree of approximation in the $W^{1,0}(\Omega_R)$ norm.

3. If we let $| \cdot |_{H^m(\Omega_R)}$ be the unweighted mth seminorm defined by:

$$| \phi |^2_{H^m(\Omega_R)} = \sum_{|\mu|=m} \int_{\Omega_R} | D^{\mu}\phi |^2 \, dx$$

where μ is a multi-index, we can obtain the alternate estimates:

$$\| \underline{u}_R - \underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p_R - p_{Rh} \|_{L^2(\Omega_R)} \leq C\, h^d \left\{ \mid \underline{u}_R \mid_{H^2(\Omega_R)} + \mid p \mid_{H^1(\Omega_R)} \right\}$$

$$\| \underline{u}_R - \underline{u}_{Rh} \|_{W^{0,-1}(\Omega_R)} \leq C\, h^{d+1} \left\{ \mid \underline{u}_R \mid_{H^2(\Omega_R)} + \mid p_R \mid_{H^1(\Omega_R)} \right\}$$

These estimates will be useful when we consider mesh grading techniques.

4.5. A two parameter approximation

Finally we combine the truncation error as well as the discretization error to obtain estimates on the difference $(\underline{u}-\underline{u}_{Rh})$ as well as the difference $(p-p_{Rh})$

Theorem 4.12: There exists a constant $C' > 0$ such that:

$$\| \underline{u} - \underline{u}_{Rh} \|_{W^{1,0}(\Omega_R)} + \| p - p_{Rh} \|_{L^2(\Omega_R)} \leq C \left\{ h^d + R^{-\delta(k)} \right\}$$

where d and $\delta(k)$ depend on the choice of the discrete spaces and of the artificial boundary condition, respectively.

■

4.6. A mesh grading technique

Mesh grading techniques are used to balance the errors arising from different sources in the approximation process. These procedures take into account:

1. The decreasing effect of the seminorms in the right hand side of the estimate as the distance of the simplex from the origin increases.

2. The mesh size in the near field (usually near field denotes the domain in which the function does not assume its decay rate)

3. The truncation error.

We will explain this balancing procedure in the following example, where we will attempt to balance a discretization error $O(h^{\delta})$ with a truncation error $O(R^{-\mu})$. Let Ω_0 be chosen such that:

$$\mathrm{supp}(\underline{f}) \subset \Omega_0$$

and, to simplify the discusion, we will assume that:

$$\Omega_0 = \Omega \cap B(0;1)$$

where, as defined in Chapter {2}, B(0;1) denotes the unit ball centered at the origin. Now let

$$\Omega_j = \left\{ x : 2^{j-1} \leq | x | \leq 2^j \right\}, j=1,2,\ldots,J_R$$

where J_R is to be determined such that

$$\bar{\Omega}_R = \bigcup_{0 \leq j \leq J_R} \Omega_j$$

We take

$$R^{\mu} = h^{-\delta}$$

so that we can balance the truncation error with the discretization error, where $h \in (0,1)$ is a parameter to be explained shortly. Let W^h_j and P^h_j denote finite element subspaces for the velocity and the pressure respectively that are associated with the ring Ω_j, i.e., there exists a $X_j \in W^h_j$ and a $\rho_j \in P^h_j$ such that:

$$\int_{\Omega_j} | \operatorname{grad}(\underline{u}_R - X_j) |^2 \, dx \leq C \, h_j^{\delta} \, {}_w| \underline{u}_R |^2_{\delta+1;\Omega_j}$$

and,

$$\int_{\Omega_j} |\, p_R - \rho_j \,|^2 \, dx \le C \, h_j^{\delta} \, |\, p_R \,|^2_{w\delta.\delta(\Omega_j)}$$

where ${}_w|\quad|$ denotes the weighted semi-norm associated with the velocity space, and

$$|\, \phi \,|^2_{w\delta.\delta(\Omega_j)} = \sum_{|\,\tau\,|=\delta} \int_{\Omega_j} (1+|\, x \,|^2)^{\delta} \; |\, D^{\tau}\phi \,|^2 \, dx$$

denotes the weighted seminorm of order δ associated with the pressure space. In addition, assume that outside the unit ball that $\underline{u}_R, p_R$ assume their decay properties, i.e., for $r \ge 1$ we have,

$$|\, \underline{u}_R \,| \le C \, r^{-1}$$

and

$$|\, p_R \,| \le C \, r^{-2}$$

where C is a positive constant dependent on both $\underline{u}_R$ and p_R but not on r.

Now on Ω_j we have for τ a multi-index:

$$\begin{aligned} {}_w|\, \underline{u}_R \,|^2_{\delta+1;\Omega_j} &= \sum_{|\,\tau\,|=\delta+1} \int_{\Omega_j} (1+|\, x \,|^2)^{\delta} \; |\, D^{\tau}\underline{u}_R \,|^2 \, dx \\ &\le C \int_{2^{j-1}}^{2^j} \frac{(1+r^2)^{\delta}}{r^{2\delta+4}} \, r^2 \, dr \\ &\le C \int_{2^{j-1}}^{\infty} \frac{(1+r^2)^{\delta}}{r^{2\delta+4}} \, r^2 \, dr \\ &\le C \, 2^{-(j-1)} \end{aligned}$$

Similarly, for the pressure we obtain that

$$|\, p_R \,|^2_{w\delta.\delta} = \sum_{|\,\tau\,|=\delta} \int_{\Omega_j} (1+|\, x \,|^2)^{\delta} \; |\, D^{\tau}p_R \,|^2 \, dx$$

$$\leq C\ 2^{-(j-1)}$$

Now, we can choose h_j such that

$$\int_{\Omega_j} |\ \mathrm{grad}(\underline{u}_R - X_j)\ |^2\, dx \leq C\ h_j^{2\delta} 2^{-(j-1)} \leq C\ h^{2\delta}$$

and,

$$\int_{\Omega_j} |\ p_R - \rho_j\ |^2\, dx \leq C\ h_j^{2\delta} 2^{-(j-1)} \leq C\ h^{2\delta}$$

where h is the mesh size in Ω_0, in the following manner:

$$h_0 = h,\ h_1 = 2^k h, \ldots,\ h_j = 2^{kj} h, \ldots,\ h_{J_R} = 2^{kJ_R} h$$

with

$$0 < k < 1/(2\delta)$$

where the case k=0 corresponds to no grading. Now, to calculate J_R, we use the fact that the outer boundary of the last ring should occupy the position R; hence:

$$J_R = [\ J_R^*\ ] + 1$$

where $[\quad]$ denotes the integer part of the real number. Now for our illustration

$$J_R^* = \log_2(R) = -\frac{\delta}{\mu} \log_2 h$$

so that we would balance the discretization error with the truncation error. Now we calculate the error estimate in terms of the parameter h

$$\|\ \underline{u}_R - \underline{u}_{Rh}\ \|^2_{W^{1,0}(\Omega_R)} \leq C \sum_{j=0}^{J_R} h_j^{2\delta} 2^{-(j-1)}$$

$$\leq C\ h^{2\delta} \sum_{j=0}^{j=J_R} 2^{(2\delta)^{-1} j} . 2^{-(2\delta)^{-1}}$$

$$\leq C\, h^{2\delta} \sum_{j=0}^{j=\infty} 2^{(2\delta)^{-1} j} . 2^{-(2\delta)^{-1}}$$

$$\leq C\, h^{2\delta}$$

with a constant C independent of R.

Thus, letting h to be the discretiztion parameter, we can find J_R to be the number of rings, and h_j to be the size of the uniform mesh to be used inside each ring Ω_j.

Fortunately enough, this mesh grading can be imporved, i.e., we can increase the mesh size, due to the fact that we have been using the weighted seminorms in the estimates of the approximation properties of the finite element spaces. Actually, one can use the unweighted norms and obtain a better estiamte for the constant k and consequently a larger mesh grading with

$$0 < k < 1 + (2\delta)^{-1}$$

This is due to the fact that for $\underline{u}_R \leq Cr^{-1}$ and $p_R \leq Cr^{-2}$

$$_U| \ \underline{u}_R \ |^2_{\delta+1} \leq C\ 2^{-(2\delta+1)(j-1)}$$

and

$$_U| \ p_R \ |^2_{\delta} \leq C\ 2^{-(2\delta+1)(j-1)}$$

Therefore, one can improve the estimates on the size of the mesh to obtain a larger grading and an error $O(h^{\delta})$ with h, as defined above, to be the maximum of the mesh sizes in Ω_0.

CHAPTER 5

ON THE COUPLING OF BOUNDARY INTEGRAL AND FINITE ELEMENT METHODS

5.1. Introduction

This method essentially follows the work of J.Nedelec [19] and A.Sequira [22] for their treatment of the Laplace equation and the exterior Stokes problem, respectively, in two dimensions. We introduce the smooth boundary $\delta\Omega_R$ separating Ω_R from Ω_∞ (Ω_∞ is the unbounded part of the fluid exterior to $\delta\Omega_R$ and R is chosen large enough to have the support of $\underline{f}$ included in Ω_R, i.e, R need not be too large). The problem we solve can now be rewritten as

$$-\Delta\underline{u}_R + \nabla p_R = f \qquad \text{in } \Omega_R \tag{5.1}$$

$$\operatorname{div}\underline{u}_R = 0 \text{ in } \Omega_R \tag{5.2}$$

$$-\Delta\underline{u}_\infty + \nabla p_\infty = 0 \qquad \text{in } \Omega_\infty \tag{5.3}$$

$$\operatorname{div}\underline{u}_\infty = 0 \text{ in } \Omega_\infty \tag{5.4}$$

$$\underline{u}_R \,|\, \delta\Omega = 0 \tag{5.5}$$

$$\underline{u}_{(int)} \,|\, \delta\Omega_R = \underline{u}_{(ext)} \,|\, \delta\Omega_R \tag{5.6}$$

$$\underline{\lambda}_{(int)} \,|\, \delta\Omega_R = \underline{\lambda}_{(ext)} \,|\, \delta\Omega_R \tag{5.7}$$

where $(\ )_{(int)}$ and $(\ )_{(ext)}$ represent the value at the surface $\delta\Omega_R$ evaluated at the inner and outer sides, respectively, and $\underline{\lambda}$ represents the stress exerted on the surface $\delta\Omega_R$ and is given by

$$\lambda_i = T_{ij}(\underline{u})n_j$$

where the summation convention holds for the index j.

Physically, conditions (5.6)-(5.7) are needed so that we can dynamically patch the fluid portions in Ω_R and Ω_∞. While condition (5.7) implies that the stresses exerted by the fluid in Ω_R on the fluid in Ω_∞ and vice versa are equal, condition (5.6) implies that the particles adjacent to the inner and outer sides of the surface $\delta\Omega_R$ move with the same velocity. The treatment of the problem will be reduced to a search in Ω_R for velocities and stresses such that the coupling of the fluid in Ω_R and Ω_∞ is feasible. Mathematically, we will show that the problem on Ω_R is uniquely solvable if we enforce the actual stress and the actual velocity at the boundary. A variational enforcement of the constraints will be considered and analyzed in detail. Once the stresses along with the velocity at the boundary $\delta\Omega_R$ are determined, we can uniquely determine the solution pair ($\underline{u}$,p) at any point x in Ω_∞ by using an intergral representation of the solution. Actually, Ω_R will be taken to be the region in which we are interested in computing the solution pair ($\underline{u}$,p) so there would be no need for extra computation using numerical quadratures.

5.2. Variational formulation of the problem

In Ω_∞ the solution $\underline{u}_{\Omega_\infty}$, p_{Ω_∞} satisfies (5.3)- (5.4). Using Green's formulas for the Stokes problem and the fudamental solution pair $(\underline{U}^k(x,y),P^k(x,y))$ (see Appendix B) we can, through the customary method, obtain an integral representation of the solution for x in Ω_∞ as well as for x on $\delta\Omega_R$. For $x \in \Omega_\infty$, the solution is given by:

$$u_k(x)=-\int_{\delta\Omega_R} T^*_{ij}(\underline{U}^k,P^k)u_i(y)n_j(y)\ dS_y + \int_{\delta\Omega_R} \lambda_i(y)U_i^k(x-y)\ dS_y \tag{5.8}$$

and for $x \in \delta\Omega_R$, the solution is given by:

$$\frac{1}{2}\, u_k(x) = -\int_{\delta\Omega_R} T^*_{ij}(\underline{U}^k, P^k)\, u_i(y)\, n_j(y)\, dS_y + \int_{\delta\Omega_R} \lambda_i(y)\, U_i^k(x-y)\, dS_y. \qquad (5.9)$$

From (5.8), (5.9) it is clear that the stresses and velocities at $\delta\Omega_R$ uniquely determine $\underline{u}$,p in Ω_∞.

Remark: The factor of $\frac{1}{2}$ appearing on the left hand side of equation (5.9) is due to the fact that the deleted neighborhood near the boundary will have a solid angle of 2π and not 4π as the case of $x \in \Omega_\infty$. Noting that

$$\begin{aligned} K_{ik}(x,y) &= T^*_{ij}(\underline{U}^k, P^k)\, n_j(y) \\ &= -\frac{3}{4\pi}\, \frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\ x-y\ |^5}\, n_j(y) \end{aligned}$$

we can write

$$a(x)\, u_k(x) = -\int_{\delta\Omega_R} K_{ik}(x,y)\, u_i(y)\, dS_y + \int_{\delta\Omega_R} \lambda_i(y)\ U_i^k(x,y)\, dS_y$$

where

$$\begin{aligned} a(x) &= 1, \text{ for } x \in \Omega_\infty \\ &= \frac{1}{2} \text{ for } x \in \delta\Omega_R\ . \end{aligned}$$

To derive the variational formulation of the coupled problem we use the Green's formula in Ω_R given by:

$$\begin{aligned} \frac{1}{2}\int_{\Omega_R} (v_{i,j}+v_{j,i})(u_{i,j}+u_{j,i})\, dx &= \int_{\Omega_R} (-u_{i,jj}+p_{,i})\ v_i + p\ v_{i,i}\ dx \\ &\quad - \int_{\delta\Omega \cup \delta\Omega_R} T_{ij}(\underline{u},p)\, n_j(y)\, v_i(y)\, dS_y\ . \end{aligned}$$

The spaces we use are W^R for the velocity, defined by

$$W^R=\left\{ \ \underline{u} \in H^1(\Omega_R) \ \mid \ \underline{u}|_{\delta\Omega}=0 \ \right\};$$

for the pressure we use $L_0^2(\Omega_R)$ defined by

$$L_0^2(\Omega_R)=\left\{ \ q \in L^2(\Omega_R) \ \mid \ \int_{\Omega_R} q \ dx = 0 \ \right\};$$

and for the stresses the space T defined by

$$T=\left\{ \ \underline{\nu} \in H^{-1/2}(\delta\Omega_R) \quad \langle \underline{\nu},\underline{1}\rangle_{\delta\Omega_R}=0 \ \right\}.$$

We set

$$a(\underline{u},\underline{v})=\frac{1}{2}\int_{\Omega_R} (u_{i,j}+u_{j,i})(v_{i,j}+v_{j,i}) \ dx$$

$$(p,\mathrm{div}\underline{v})=\int_{\Omega_R} p\,\mathrm{div}\underline{v} \ dx$$

$$(\underline{f},\underline{v})=(f_i,v_i)=\int_{\Omega_R} f_i v_i \ dx$$

and for $\underline{\lambda}$ in $H^{-1/2}(\delta\Omega_R)$

$$\langle\underline{\lambda},\underline{v}\rangle_{\delta\Omega_R}=\int_{\delta\Omega_R} \lambda_i(x)v_i(x)dS_x$$

with the summation convention taken into account whenever needed, and where $\langle .,. \rangle_{\delta\Omega_R}$ denotes the duality pairing between $H^{-1/2}(\delta\Omega_R)$ and $H^{1/2}(\delta\Omega_R)$. Using the above notation, we are now ready to rewrite Green's formula. In fact, we get:

$$a(\underline{u},\underline{v})-(p,\mathrm{div}\underline{v})+\langle\underline{\lambda},\underline{v}\rangle_{\delta\Omega_R}=(\underline{f},\underline{v}) \ .$$

Now to enforce the constraints on $\underline{\lambda}$ and $\underline{u}$ at the boundary $\delta\Omega_R$ we multiply equation (5.9) by $\underline{\mu}\in T$ and integrate over the boundary $\delta\Omega_R$ to yield:

$$2b(\underline{\lambda},\underline{\mu})-\langle\underline{u},\underline{\mu}\rangle_{\delta\Omega_R}+2\langle G\underline{u},\underline{\mu}\rangle_{\delta\Omega_R}=0$$

where

$$b(\underline{\lambda},\underline{\mu})=\int_{\delta\Omega_R}\int_{\delta\Omega_R}\lambda_i(x)\ U_i^k(x-y)\mu_k(y)\ dS_x\ dS_y$$

and,

$$G_k\underline{u}(x)=\int_{\delta\Omega_R}T_{ij}(\underline{U}^k(x-y),P^k(x-y))u_i(y)n_j(y)\ dS_y$$

and

$$\langle G\underline{u},\underline{\mu}\rangle_{\delta\Omega_R}=\langle G_k\underline{u},\mu_k\rangle_{\delta\Omega_R}$$

Combining the conditions of coupling as well as the incompressibility condition we get the variational formulation of problem (5.1)-(5.7):

seek $(\underline{u},\underline{\lambda},p)\in W^R\times T\times L_0^2(\Omega_R)$ such that:

$$a(\underline{u},\underline{v})-(p,\mathrm{div}\underline{v})+\langle\underline{\lambda},\underline{v}\rangle_{\delta\Omega_R}=((\underline{f},\underline{v}))\quad\forall\ \underline{v}\in W^R \tag{5.10}$$

$$2b(\underline{\lambda},\underline{\mu})-\langle\underline{\mu},\underline{u}\rangle_{\delta\Omega_R}+2\langle G\underline{u},\underline{\mu}\rangle_{\delta\Omega_R}=0\quad\forall\ \underline{\mu}\in T \tag{5.11}$$

$$(q,\mathrm{div}\underline{u})=0\quad\forall\ q\in L_0^2(\Omega_R)\ . \tag{5.12}$$

In the following we discuss the solvability of (5.10)-(5.12) in terms of conditions on a(.,.), b(.,.) as well as the operator G.

5.3. T-ellipticity of b(.,.)

Now we show that the symmetric bilinear form b(.,.) defined on T×T is T-elliptic, i.e., there eixsts a constant α such that:

$$b(\underline{\lambda},\underline{\lambda})=\int_{\delta\Omega_R}\int_{\delta\Omega_R}\lambda_i(x)U_i^k(x,y)\lambda(y)\ dS_x\ dS_y\ \geq\ \alpha\ \|\ \underline{\lambda}\ \|_{H^{-1/2}(\delta\Omega_R)}\ .$$

To do this we need the following sequence of lemmas.

Lemma 5.1: Given $\underline{u}_0 \in H^{1/2}(\delta\Omega_R)$ with,

$$\int_{\delta\Omega_R} \underline{u}_0 \cdot \underline{n} \, dS=0$$

there exists a unique pair $(\underline{v},\pi) \in W^{1,0}(R^3) \times L^2(R^3)$ such that:

$$-\Delta\underline{v}+\nabla\pi=0 \text{ in } R^3, \tag{5.13}$$

$$\text{div}\underline{v}=0 \text{ in } R^3, \tag{5.14}$$

$$\underline{v}\,|_{\delta\Omega_R}=\underline{u}_0 \,. \tag{5.15}$$

Proof: Given $\underline{u}_0 \in H^{1/2}(\delta\Omega_R)$ we can find a vector $\underline{\phi}$ in $W^{1,0}(R^3)$ such that:

$$\underline{\phi}\,|_{\delta\Omega_R}=\underline{u}_0 \,.$$

Now using the linearity of the problem (5.13)-(5.15) we get:

$$-\Delta(\underline{v}-\underline{\phi}) + \nabla\pi=\Delta\underline{\phi} \text{ in } R^3, \tag{5.16}$$

$$\text{div}(\underline{v}-\underline{\phi}) = -\text{div}\underline{\phi} \text{ in } R^3, \tag{5.17}$$

$$(\underline{v}-\underline{\phi})\,|_{\delta\Omega_R}=0 \,. \tag{5.18}$$

Let Ω_i, Ω_e denote the domains interior and exterior to the boundary $\delta\Omega_R$ respectively. In Ω_i the problem (5.16)-(5.18) can be posed variationally as follows:

Seek $\underline{v}_{(i)}, p_{(i)} \in [H_0^1(\Omega_i)]^3 \times L_0^2(\Omega_i)$ such that:

$$\int_{\Omega_i} \mathrm{grad}(\underline{\upsilon}_{(i)}-\underline{\phi}):\mathrm{grad}\,\underline{v}\; dx-\int_{\Omega_i} p_{(i)}\mathrm{div}\,\underline{v}\; dx=0 \qquad \forall\ \underline{v}\in H_0^1(\Omega_i) \tag{5.19}$$

$$\int_{\Omega_i} p\,\mathrm{div}(\underline{\upsilon}_{(i)}-\underline{\phi})\; dx = -\int_{\Omega_i} p\,\mathrm{div}\,\underline{\phi}\; dx \qquad \forall\ p\in L_0^2(\Omega_i) \tag{5.20}$$

while in the exterior domain Ω_c we have:

$$\int_{\Omega_c} \mathrm{grad}(\underline{\upsilon}_{(c)}-\underline{\phi}):\mathrm{grad}\,\underline{v}\; dx-\int_{\Omega_c} p_{(c)}\mathrm{div}\,\underline{v}\; dx \qquad \forall\ \underline{v}\in \mathring{W}^{1,0}(\Omega_i) \tag{5.21}$$

$$\int_{\Omega_c} p\,\mathrm{div}(\underline{\upsilon}_{(c)}-\underline{\phi})\; dx = -\int_{\Omega_c} p\,\mathrm{div}\,\underline{\phi}\; dx \qquad \forall\ p\in L^2(\Omega_c)\ . \tag{5.22}$$

Now the problem (5.19)-(5.20) as well as the problem (5.21)-(5.22) each admits a unique solution. This can be easily verified by checking the conditions needed for a Brezzi type variational problem: for the bounded domain, i.e., Ω_i case, we refer the reader to [9] while for the unbounded domain Ω_c we refer the reader to Chapter {2}. We should note that the solution pairs $(\underline{\upsilon}_i,p_i)$ and $(\underline{\upsilon}_c,p_c)$ are independent of the choice of the particular $\underline{\phi}$ whose trace on $\delta\Omega_R$ is $\underline{u}_0$. To see this, let $\underline{\upsilon}_{\phi}$, $\underline{\upsilon}_{\psi}$ be two different solutions corresponding to $\underline{\phi}$ and $\underline{\psi}$ respectively, where

$$\underline{\phi}\,|\,_{\delta\Omega_R}=\underline{\psi}\,|\,_{\delta\Omega_R}=\underline{u}_0$$

then if we denote the diference in Ω_i between $\underline{\upsilon}_{\phi}$ and $\underline{\upsilon}_{\psi}$ by $\underline{\omega}_i$, and also if we denote the difference in the pressure by π_i then we can easily see that $\underline{\omega}_i$ satisfies:

$$\int_{\Omega_i} \mathrm{grad}\,\underline{\omega}_{(i)}:\mathrm{grad}\,\underline{v}\; dx-\int_{\Omega_i} \pi_{(i)}\mathrm{div}\,\underline{v}\; dx=0 \qquad \forall\ \underline{v}\in H_0^1(\Omega_i)$$

$$\int_{\Omega_i} p\,\mathrm{div}\,\underline{\omega}_{(i)}\; dx=0 \qquad \forall\ p\in L_0^2(\Omega_i)\ .$$

Taking $\underline{v}=\underline{\omega}_i$ and $q=\pi_i$ we get that:

$$\int_{\Omega_i} |\ \mathrm{grad}\,\underline{\omega}_i\ |^2\; dx=0$$

so that, by the Poincare inequality,

$$\| \underline{\omega}_i \|_{L^2(\Omega_i)} \leq C \| \nabla \underline{\omega}_i \|_{L^2(\Omega_i)} = 0$$

which implies that $\underline{\omega}_i=0$. Similarly, for the exterior problem we repeat the same argument to conclude that:

$$\int_{\Omega_c} | \operatorname{grad} \underline{\omega}_c |^2 \, dx = 0$$

where now by the Hardy inequality we obtain that:

$$\int_{\Omega_c} \frac{\underline{\omega}_c}{1+| x |^2} \, dx \leq \int_{\Omega_c} | \operatorname{grad} \underline{\omega}_c |^2 \, dx = 0$$

hence $\underline{\omega}_e=0$. This completes the proof.

■

Now the uniqueness of the solution of problem (5.13)-(5.15) independent of the choice of $\underline{\phi}$ allows us to state the following lemma.

Lemma 5.2: Problem (5.13)-(5.15) admits a unique solution pair $(\underline{v},\pi)$ if the function $\underline{u}_0 \in H^{1/2}(\delta\Omega_R)$. Further the pair $(\underline{v},\pi)$ is in the subspace K of $W^{1,0}(R^3) \times L^2(R^3)$ defined by:

$$K = \Big\{ (\underline{v},p) \in W^{1,0}(R^3) \times L^2(R^3) \mid (-\Delta\underline{v}+\nabla\pi) \in W^{1,0}(R^3)',$$

$$\operatorname{supp}(-\Delta\underline{v}+\nabla\pi) \text{ is included in } \delta\Omega_R, \ \operatorname{div} \underline{v}=0 \text{ in } R^3 \Big\}$$

Moreover, problem (5.13)-(5.15) defines an isomorphism form $H^{1/2}(\delta\Omega_R)$ onto K.

■

Remark: Problem (5.13)-(5.15) allows us, following Lions and Magenes [18], to define the stress vectors at the inner and outer sides of the boundary $\delta\Omega_R$ corresponding to a pair $(\underline{u},p)$ as follows:

$$\langle T_{ij}(\underline{u},p)n_j|_{\delta\Omega_R}, \underline{v}|_{\delta\Omega_R}\rangle \tag{5.23}$$

$$= \int_{\Omega_i} \text{grad } \underline{u}\text{:grad}\underline{v}\ dx - \int_{\Omega_i} p\ \text{div}\underline{v}\ dx\ \forall\ \underline{v} \in H^1(\Omega_i)$$

and,

$$-\langle T_{ij}(\underline{u},p)n_j|_{\delta\Omega_R}, \underline{v}|_{\delta\Omega_R}\rangle \tag{5.24}$$

$$= \int_{\Omega_e} \text{grad } \underline{u}\text{:grad}\underline{v}\ dx - \int_{\Omega_e} p\ \text{div}\underline{v}\ dx\ \forall\ \underline{v} \in W^{1,0}(\Omega_e)$$

where n_j denotes the outer directional derivative of the normal to the surface $\delta\Omega_R$.

Now consider the problem of determining the flow of a fluid due to a single layer of density Ψ at the boundary $\delta\Omega_R$. The potentials of a single layer of density Ψ denoted by $\underline{V}(x,\Psi)$ and $Q(x,\Psi)$ and are given by:

$$V_j(x,\Psi)= \int_{\delta\Omega_R} \underline{U}_j^k(x,y)\Psi_k(y)\ dS_y \tag{5.25}$$

$$Q(x,\Psi)= \int_{\delta\Omega_R} P^k(x,y)\Psi_k(y)\ dS_y \tag{5.26}$$

where $\underline{U}^k(x,y)$ and $P^k(x,y)$ are the fundamental solutions of the Stokes system. For a discussion of those potentials we refer the reader to the Appendix B.

It is known [15] that $V(x,\Psi)$ is continuous across the boundary $\delta\Omega_R$ provided that Ψ is not " badly behaved ". However, the corresponding pressure $Q(x,\Psi)$ is not continuous. Further these functions satisfy the homogeneous Stokes system in $\Omega_i \cup \Omega_e$ since the kernels of the integral expressions do. To see this

$$-\Delta\underline{V}(x,\psi)+\nabla Q(x,\Psi)= \int_{\delta\Omega_R} \left\{ -\Delta_x\underline{U}^k(x,y)+\nabla_x P^k(x,y) \right\}\Psi_k(y)\ dS_y$$

$$=\int_{\delta\Omega_R} \left\{ -\delta(x-y)\underline{e}^k \right\}\Psi_k(y)\, dS_y .$$

This implies that $-\Delta\underline{V}(x,\Psi)+\nabla Q(x,\Psi)=0$ for $x \notin \delta\Omega_R$ and therefore we obtain that:

$$\mathrm{supp}(\ -\Delta\underline{V}(x,\Psi)+\nabla Q(x,\Psi)\ ) \subset \delta\Omega_R .$$

Similarly we can deduce that div $\underline{V}(x,\Psi)=0$ and conclude that the pair $(\underline{V},Q) \in K$.

Now by adding (5.23) and (5.24) we get the result that for any pair $(\underline{u},p)$:

$$\langle T_{ij}(\underline{u},p)_{(i)}n_j|_{\delta\Omega_R} - T_{ij}(\underline{u},p)_{(e)}n_j|_{\delta\Omega_R}, \underline{v}|_{\delta\Omega_R}\rangle$$

$$=\int_{R^3} \mathrm{grad}\underline{u}:\mathrm{grad}\underline{v}\, dx - \int_{R^3} p\ \mathrm{div}\underline{v}\, dx \quad \forall\ \underline{v} \in W^{1,0}(R^3)$$

subject to

$$\int_{R^3} q\,\mathrm{div}\underline{u}\, dx = 0 \quad \forall\ q \in L^2(R^3) .$$

But due to the fact that

$$T_{ij}(\underline{V},Q)_{(i)}n_j - T_{ij}(\underline{V},Q)_{(e)}n_j = \Psi_i$$

(see [16] for more details) we can rewrite the problem for $\underline{V},Q$ in the form

seek $\underline{V},Q \in K$ such that:

$$\langle \Psi,\underline{v}|_{\delta\Omega_R}\rangle = \int_{R^3} \mathrm{grad}\underline{V}:\mathrm{grad}\underline{v}\,dx - \int_{R^3} Q\ \mathrm{div}\underline{v}\, dx \qquad \forall\ \underline{v} \in W^{1,0}(R^3) \qquad (5.27)$$

$$\int_{R^3} q\ \mathrm{div}\underline{V}\, dx = 0 \qquad \forall\ q \in L^2(R^3) \qquad (5.28)$$

Remark: Due to Lemma {5.1} the follwing remark, we can easily see that if $\underline{u}_0$ is given in the space $H^{1/2}(\delta\Omega_R)$, then the solution is in $W^{1,0}(R^3)\times L^2(R^3)$ and consequently the stresses at the inner and outer sides of the boundary $\delta\Omega_R$ are in $H^{-1/2}(\delta\Omega_R)$, which shows that the single layer Ψ is also in $H^{-1/2}(\delta\Omega_R)$. Conversely, if Ψ is in $H^{-1/2}(\delta\Omega_R)$, then the variational problem (5.27)-(5.28) admits a unique solution in $W^{1,0}(R^3)\times L^2(R^3)$. Moreover, the solution is in the set K previously defined.

Lemma 5.3: If $\Psi \in C^{\infty}(\delta\Omega_R)$, the unique solution of (5.27)-(5.28) is given by (5.25)-(5.26).

Proof: We can easily verify that the system (5.27)-(5.28) is satisfied and that $\underline{V}$,Q $\in$ K.

■

Lemma 5.4: The map form $C^{\infty}(\delta\Omega_R)$ into K defined by the expressions (5.25)-(5.26) is continuous from $H^{-1/2}(\delta\Omega_R)$ into K (and into $H^{1/2}(\delta\Omega_R)$) and therefore can be uniquely extended to an isomorphism form $H^{-1/2}(\delta\Omega_R)$ onto K (also onto $H^{1/2}(\delta\Omega_R)$).

Proof: This follows form Lemma {5.1}, the following remark and Lemma {5.3}.

■

Theorem 5.5: The isomorphism in Lemma {5.4} corresponds to the variational problem:

seek $\Psi \in H^{-1/2}(\delta\Omega_R)$ such that:

$$b(\Psi,\Psi^*)=\langle \underline{u}_0,\Psi^*\rangle_{\delta\Omega_R} \quad \forall \; \underline{\Psi}^* \in H^{-1/2}(\delta\Omega_R)$$

where,

$$b(\Psi,\Psi^*)=\int_{\delta\Omega_R}\int_{\delta\Omega_R} \Psi_j(x)\underline{U}_j^k(x,y)\Psi^*_k(y)\; dS_x\; dS_y$$

and

$$\langle \underline{u}_0, \Psi^* \rangle_{\delta\Omega_R} = \int_{\delta\Omega_R} u_{k0}(y)\ \Psi^*_k(y)\ dS_y$$

where the left hand side is defined for Ψ, Ψ^* in $C^{\infty}(\delta\Omega_R)$, and is continuous on $H^{-1/2}(\delta\Omega_R)\times H^{-1/2}(\delta\Omega_R)$, and therefore, can be uniquely extended to a continuous bilinear form on

$$H^{-1/2}(\delta\Omega_R)\times H^{-1/2}(\delta\Omega_R)\ .$$

Moreover the expression

$$\| \Psi \| = \left\{ \int_{\delta\Omega_R} \int_{\delta\Omega_R} \Psi_k(x)\underline{U}^k_j(x,y)\Psi_j(y)\ dS_x\ dS_y \right\}^{1/2} \tag{5.29}$$

is a norm on $H^{-1/2}(\delta\Omega_R)$ equivalent to the usual norm.

Proof: The first part of the theorem has already been shown in the last remark and we have:

$$\begin{aligned}
| \ b(\Psi,\Psi') \ | &= \left| \int_{\delta\Omega_R} \int_{\delta\Omega_R} \Psi_k(x)\underline{U}^k_j(x,y)\Psi'_i(y)\ dS_x\ dS_y \right| \\
&= | \ \langle \Psi_i, V_i \rangle_{\delta\Omega_R} \ | \\
&\le \| \Psi_i \|_{H^{-1/2}(\delta\Omega_R)} \| V_i \|_{H^{1/2}(\delta\Omega_R)} \\
&\le C \| \Psi_i \|_{H^{-1/2}(\delta\Omega_R)} \| \Psi' \|_{H^{-1/2}(\delta\Omega_R)}
\end{aligned}$$

where the last inequality follows due to the isomorphism from $H^{-1/2}(\delta\Omega_R)$ onto $H^{1/2}(\delta\Omega_R)$ (see Lemma {5.4}). We should remind the reader that the tensorial summation convention is still used. Therefore it remains only to show the result (5.29). Let Ψ be in $H^{-1/2}(\delta\Omega_R)$; by resolving the variational problem (5.23)-(5.24) we can find a unique pair $(\underline{V},Q)$ in K and consequently we get that the trace of $\underline{V}$ is in the space $H^{1/2}(\delta\Omega_R)$. If Ψ is smooth enough, the solution pair $(\underline{V},Q)$ can be actually represented by the integral equations (5.27)-(5.28). Finally,

$$\langle\Psi,\underline{v}\rangle_{\delta\Omega_R} = \int_{R^3} \mathrm{grad}\underline{V}:\mathrm{grad}\underline{v}\ dx - \int_{R^3} Q\ \mathrm{div}\underline{v}\ dx \quad \forall\ \underline{v} \in W^{1,0}(R^3)$$

$$\int_{R^3} q\,\mathrm{div}\underline{V}\ dx = 0 \qquad \forall\ q \in L^2(R^3)\ .$$

But by repalacing $\underline{v}$ by $\underline{V}$ in the last equation and making use of the incompressibility condition we get:

$$\langle\Psi,\underline{V}|_{\delta\Omega_R}\rangle_{\delta\Omega_R} = \int_{R^3} |\ \mathrm{grad}\underline{V}\ |^2\ dx$$

$$= \int_{\delta\Omega_R}\int_{\delta\Omega_R} \Psi_k(x)\underline{U}_j^k(x,y)\Psi_j(y)\ dS_x\ dS_y\ .$$

Now by using the isomorphism in Lemma {5.4} we get

$$\int_{R^3} |\ \mathrm{grad}\underline{V}\ |^2\ dx \geq \alpha \|\ \Psi\ \|^2_{H^{-1/2}(\delta\Omega_R)}$$

(actually we have from problem (5.13)-(5.15) that $\|\ \Psi\ \|_{H^{-1/2}(\delta\Omega_R)} \leq C \|\ \underline{V}\ \|_{W^{1,0}(R^3)}$).

Now by using the fact that:

$$\langle\Psi,\underline{V}\rangle_{\delta\Omega_R} \leq \|\ \Psi\ \|_{H^{-1/2}(\delta\Omega_R)} \|\ \underline{V}\ \|_{H^{1/2}(\delta\Omega_R)}$$

$$\leq C \|\ \Psi\ \|^2_{H^{-1/2}(\delta\Omega_R)}$$

where the last inequality is obtained from (5.27)-(5.28). Thus we have established the equivalence of the norms. This completes the proof.

∎

5.4. Compactness of the operator G_k

Actually, we will show the comapactness of the operator:

$$G : H^{1/2}(\delta\Omega_R) \to H^{1/2}(\delta\Omega_R)$$

and this will enable us to establish the well posedness of the problem.

Lemma 5.6: The map E given by the integral

$$(E\underline{\phi})_i(x)=\omega_i(x)=\int_{\delta\Omega_R} K_{ij}(x,y)\phi_j(y)\ dS_y \tag{5.30}$$

is well defined for $\underline{\phi} \in C^{\infty}(\delta\Omega_R)$ and is continuous from $H^{1/2}(\delta\Omega_R)$ into $H^{3/2}(\delta\Omega_R)$, and therefore, can be uniquely extended to a continuous map defined on the whole space $H^{1/2}(\delta\Omega_R)$.

Proof: To show continuity of the map E from $H^{1/2}(\delta\Omega_R)$ into $H^{3/2}(\delta\Omega_R)$ we need the following series of estimates. First we show that

$$\| E\underline{\phi} \|_{L^2(\delta\Omega_R)} \le C \| \underline{\phi} \|_{L^2(\delta\Omega_R)} .$$

Since,

$$w_i(x)=\int_{\delta\Omega_R} K_{ij}(x,y)\ \phi_j(y)\ dS_y$$

we have that

$$| w_i(x) |^2 \le \left\{ \int_{\delta\Omega_R} | K_{ij}(x,y) |^{1/2} | K_{ij}(x,y) |^{1/2} | \phi_j(y) | dS_y \right\}^2$$

$$\le \left\{ \int_{\delta\Omega_R} | K_{ij}(x,y) | dS_y \right\} \left\{ \int_{\delta\Omega_R} | K_{ij}(x,y) || \phi_j(y) | dS_y \right\} .$$

But for $x \in \delta\Omega_R$ with $\delta\Omega_R$ smooth enough

$$\int_{\delta\Omega_R} |\ K_{ij}(x,y)\ |\ dS_y \leq C_1$$

Now by integarting both sides with respect to x we get:

$$\|\ w_i\ \|^2_{L^2(\delta\Omega_R)} = \int_{\delta\Omega_R} |\ w_i(x)\ |^2\ dS_x$$

$$\leq C_1 \int_{\delta\Omega_R} \left\{ \int_{\delta\Omega_R} |\ K_{ij}(x,y)\ |\ dS_x \right\} |\ \phi_j(y)\ |^2\ dS_y$$

$$= C_1^{\ 2}\ \|\ \underline{\phi}\ \|^2_{L^2(\delta\Omega_R)}$$

Next we show that

$$\|\ w_i\ \|_{H^{1/2}(\delta\Omega_R)} \leq C\ \|\ \underline{\phi}\ \|_{L^2(\delta\Omega_R)}$$

Let x,x' be two points on $\delta\Omega_R$. We have:

$$|\ w_i(x)-w_i(x')\ |^2 \leq \left\{ \int_{\delta\Omega_R} |\ K_{ij}(x,y)-K_{ij}(x',y)\ |\ |\ \phi_j(y)\ |\ dS_y \right\}^2$$

$$= \left\{ \int_{\delta\Omega_R} |\ K_{ij}(x,y)-K_{ij}(x',y)\ |^{1/2-\epsilon+1/2+\epsilon}\ |\ \phi_j(y)\ |\ dS_y \right\}^2$$

$$\leq \left\{ \int_{\delta\Omega_R} |\ K_{ij}(x,y)-K_{ij}(x',y)\ |^{1+2\epsilon}\ dS_y \right\}$$

$$\left\{ \int_{\delta\Omega_R} |\ k_{ij}(x,y)-K_{ij}(x',y)\ |^{1-2\epsilon} |\ \phi_j(y)\ |^2\ dS_y \right\}$$

But according to [15]

$$\int_{\delta\Omega_R} |\ K_{ij}(x,y)-K_{ij}(x',y)\cdot\ |^{1+2\epsilon}\ dS_y \leq C\ |\ x-x'\ |^{1-2\epsilon}$$

therefore we get

$$|\ w_i(x)-w_i(x')\ |^2 \leq C\ |\ x-x'\ |^{1-2\epsilon} \int_{\delta\Omega_R} |\ K_{ij}(x,y)-K_{ij}(x',y)\ |^{1-2\epsilon} |\ \phi_j(y)\ |^2\ dS_y$$

and

$$\int_{\delta\Omega_R} \int_{\delta\Omega_R} \frac{|\ w_i(x)-w_i(x')\ |^2}{|\ x-x'\ |^3}\ dS_x dS_{x'}$$

$$\leq C \int_{\delta\Omega_R} \int_{\delta\Omega_R} \frac{1}{|\ x-x'\ |^{2+2\epsilon}} \int_{\delta\Omega_R} |\ K_{ij}(x,y)-K_{ij}(x',y)\ |^{1-2\epsilon} |\ \phi_j(y)\ |^2\, dS_y\, dS_x\, dS_{x'}$$

$$= \int_{\delta\Omega_R} |\ \phi_j(y)\ |^2 \left\{ \int_{\delta\Omega_R} \int_{\delta\Omega_R} \frac{|\ K_{ij}(x,y)-K_{ij}(x',y)\ |^{1-2\epsilon}}{|\ x-x'\ |^{2+2\epsilon}}\, dS_x\, dS_{x'} \right\} dS_y\ .$$

But now since

$$\int_{\delta\Omega_R} \int_{\delta\Omega_R} \frac{|\ K_{ij}(x,y)-K_{ij}(x',y)\ |^{1-2\epsilon}}{|\ x-x'\ |^{2+2\epsilon}}\, dS_x\, dS_{x'} \leq C_3$$

we finally get

$$\|\ w\ \|_{H^{1/2}(\delta\Omega_R)} \leq C\ \|\ \phi\ \|_{L^2(\delta\Omega_R)} \leq C\ \|\ \phi\ \|_{H^{1/2}(\delta\Omega_R)}\ .$$

It remains to estimate the quantity

$$\int_{\delta\Omega_R} \int_{\delta\Omega_R} \frac{|\ w_i(x)_{;\alpha}-w_i(x')_{;\alpha}\ |^2}{|\ x-x'\ |^3}\, dS_x dS_{x'}.$$

where $(\ .\ )_{;\alpha}$ denotes the directional derivative along the surface $\delta\Omega_R$. Equation (5.30) can be rewitten in the form

$$w_i(x)-1/2\phi_i(x)=\int_{\delta\Omega_R} K_{ij}(x,y) \left\{ \phi_j(y)-\phi_j(x) \right\} dS_y$$

due to the fact that

$$\int_{\delta\Omega_R} K_{ij}(x,y)\ dS_y=1/2\ \delta^i_j \qquad \text{for } x\in\ \delta\Omega_R\ .$$

Now differentiating in the direction of the tangent to the surface $\delta\Omega_R$ and using the above notation for the derivative we get:

$$\left\{ w_i(x)-\phi_i(x) \right\}_{;\alpha}=\int_{\delta\Omega_R} K_{ij}(x,y)_{;\alpha} \left\{ \phi_j(y)-\phi_j(x) \right\} dS_y$$
$$+\int_{\delta\Omega_R} K_{ij}(x,y) \left\{ \phi_j(x) \right\}_{;\alpha}\ dS_y$$

where we can show that the second integral is equal to

$$\int_{\delta\Omega_R} K_{ij}(x,y)\left\{ \phi_j(x) \right\}_{;\alpha} dS_y = \left\{ \phi_j(x) \right\}_{;\alpha} \int_{\delta\Omega_R} K_{ij}(x,y)\, dS_y$$
$$= \frac{1}{2} \left\{ \phi_i(x) \right\}_{;\alpha} .$$

Therefore we get

$$w_i(x)_{;\alpha} = \int_{\delta\Omega_R} K_{ij}(x,y)_{;\alpha} \left\{ \phi_j(y) - \phi_j(x) \right\} dS_y .$$

Denoting by $J_i(x,x')$ the quantity

$$\left\{ w_i(x) - w_i(x') \right\}_{;\alpha}$$

we obtain

$$J_i(x,x') = J_i(x) - J_i(x')$$
$$= \int_{\delta\Omega_R} K_{ij}(x,y)_{;\alpha} \left\{ \phi_j(y) - \phi_j(x) \right\} dS_y$$
$$- \int_{\delta\Omega_R} K_{ij}(x',y)_{;\alpha} \left\{ \phi_j(y) - \phi_j(x') \right\} dS_y$$

Now refering to [15] the estimate

$$\int_{\delta\Omega_R} \int_{\delta\Omega_R} \frac{\left| J_i(x) - J_i(x') \right|^2}{\left| x - x' \right|^3} dS_x\, dS_{x'} \le C \left\| \underline{\phi} \right\|^2_{H^{1/2}(\delta\Omega_R)}$$

holds, so that by combining the three estimates we get:

$$\left\| \underline{w} \right\|_{H^{3/2}(\delta\Omega_R)} \le C \left\| \underline{\phi} \right\|_{H^{1/2}(\delta\Omega_R)}$$

which shows the continuity of the map E.

■

Lemma 5.7: The map G_k defined by

$$G_k\underline{u}=\int_{\delta\Omega_R} T_{ij}(\underline{U}^k,P^k)(x-y)n_j(y)u_i(y)\ dS_y$$

is a densely defined map which is continuous from $H^{1/2}(\delta\Omega_R)$ into $H^{3/2}(\delta\Omega_R)$, and therefore, can be uniquely extended to a map still denoted by G_k on the whole of $H^{1/2}(\delta\Omega_R)$.

Proof: Apply Lemma{5.6} for G_k to get the required result.

■

Lemma 5.8: The operator G_k, viewed as an operator form $H^{1/2}(\delta\Omega_R)$ into $H^{1/2}(\delta\Omega_R)$, is a compact operator.

Proof: We make use of Lemma {5.7} as well as the fact that the imbedding of $H^{3/2}(\delta\Omega_R)$ into $H^{1/2}(\delta\Omega_R)$ is compact due to the boundedness of the boundary $\delta\Omega_R$. In fact we have:

$$W^R \rightarrow (H^1(\Omega_R))^3 \rightarrow (H^{1/2}(\delta\Omega_R))^3 \rightarrow (H^{3/2}(\delta\Omega_R)^3 \rightarrow (H^{1/2}(\delta\Omega_R))^3 \rightarrow T'$$

■

5.5. Coercivity of the form a(.,.)

Lemma 5.9: The bilinear form defined by

$$a(\underline{u},\underline{v})=\int_{\Omega_R} \frac{(u_{,ij}+u_{j,i})(v_{i,j}+v_{j,i})}{2}\ dx$$

is continuous on $W^R \times W^R$. Moreover, the form is W^R-elliptic.

Proof: Continuity of the form is obvious while the W^R ellipticity follows from a variation of Korn's inequality as follows. There exists a constant $C > 0$ that depends on Ω_R such that:

$$\| \underline{u} \|_1 \leq C \left\{ \int_{\Omega_R} | u_{i,j}+u_{j,i} |^2 \right\}^{1/2}$$

where the proof of the last inequality is carried through contradiction and compactness argument (See [5] for example).

∎

We are now in a position to study the existence of a solution of the variational formulation (5.10)–(5.12). We will show that (5.10)–(5.12) is a compact perturbation of a well posed problem. Therefore the associated operator is a Fredholm operator where the injectivity of the operator implies its surjectivity, i.e., uniqueness implies existence, yielding the well-posedness of the variational formulation.

5.6. Abstract study of the problem

In order to analyze, and later approximate, our problem we need the following theorem concerning the generalized saddle point problem. [20]

Theorem 5.10: Let V_1, V_2, S_1, S_2 be Hilbert spaces and $(.,.)_{V_i}$, $(.,.)_{S_i}$, $\| \ \|_{V_i}$ and $\| \ \|_{S_i}$ be the inner products and norms associated with the spaces respectively. Define

$$Z_1 = \left\{ v \in V_1 \mid B(q,v)=0 \ \forall \ q \in S_2 \right\}$$

$$Z_2 = \left\{ v \in V_2 \mid D(q,v)=0 \ \forall \ q \in S_1 \right\}$$

Assume that the bilinear forms $A(.,.)$, $B(.,.)$ and $D(.,.)$ are continuous on $V_1 \times V_2$, $S_1 \times V_2$ and $S_2 \times V_1$ respectively. In addition, assume

$$\sup_{\substack{u \in V_2 \\ u \neq 0}} \frac{B(p,u)}{\| u \|_{V_2}} \geq \beta_2 \| p \|_{S_1} \ \forall \ p \in S_1$$

$$\sup_{\substack{u \in V_1 \\ u \neq 0}} \frac{D(p,u)}{\| u \|_{V_1}} \geq \beta_1 \| p \|_{S_2} \ \forall \ p \in S_2$$

$$\sup_{\substack{v \in Z_2 \\ v \neq 0}} \frac{A(u,v)}{\| v \|_{V_2}} \geq \alpha \| u \|_{V_1} \ \forall \ u \in Z_1$$

$$\sup_{\substack{u \in Z_1 \\ u \neq 0}} \frac{A(u,v)}{\| u \|_{V_1}} > 0 \ \forall \ v \in Z_2$$

Then, the saddle point problem

seek $u,p \in V_1 \times S_1$ such that:

$$A(u,v) + B(p,v) = \langle f,v \rangle \qquad \forall \ v \in V_2$$

$$D(q,u) = 0 \qquad \forall \ q \in S_2$$

has a unique solution. Further, there exists a constant $C > 0$ such that:

$$\| u \|_{V_1} + \| p \|_{S_1} \leq C \| f \|_{V_2'}$$

where V_2' is the dual space of the Hilbert space V_2. ■

We define the Hilbert spaces $H = W^R \times T$, $H_0 = W_0 \times T$ where,

$$W_0 = \left\{ \underline{v} \in W^R \mid (p, \mathrm{div}\underline{v}) = 0 \ \forall \ p \in L_0^2(\Omega_R) \right\}$$

We denote by $(.,.)_H$ and $\| \ . \ \|_H$ to be the inner product and the norm associated with the Hilbert spaces defined above. We also introduce the forms A(.,.), B(.,.) and G(.,.) on $H \times H$. Also, we introduce the form D(.,.) continuous on $H \times L_0^2(\Omega_R)$ and, with the same notation, the associated operators A, B, $\mathcal{G} \in \mathcal{L}(\mathcal{H},\mathcal{H}')$ and D $\in$ $\mathcal{L}(\mathcal{H},L_0^2(\Omega_R)')$, where the notation ' denotes the dual space. These operators are defined by

$$B(\hat{u},\hat{v})=[B\hat{u},\hat{v}]=a(\underline{u},\underline{v})+\langle\underline{v},\underline{\lambda}\rangle_{\delta\Omega_R}-\langle\underline{u},\underline{\mu}\rangle_{\delta\Omega_R}+2b(\underline{\lambda},\underline{\mu})$$

$$\mathcal{G}(\hat{u},\hat{v})=[\mathcal{G}\hat{u},\hat{v}]=2\langle\mathcal{G}\underline{u},\underline{\mu}\rangle_{\delta\Omega_R}$$

$$A=B+\mathcal{G}$$

$$D(\hat{v},q)=(D\hat{v},q)=-(q,\mathrm{div}\underline{v})$$

for arbitrary elements $\hat{u}=(\underline{u},\underline{\lambda})$, $\hat{v}=(\underline{v},\underline{\mu}) \in \mathcal{H}$ and $q \in L_0^2(\Omega_R)$. Here $[.,.]$ and $(.,.)$ denote the duality between $\mathcal{H}$ and its dual $\mathcal{H}'$ and $L_0^2(\Omega_R)$ and its dual $L_0^2(\Omega_R)'$, respectively. Our problem can now be rewritten in a Brezzi-Nicolaides type setting:

seek $(\hat{u},p)=((\underline{u},\underline{\lambda}),p) \in \mathcal{H}\times L_0^2(\Omega_R)$ such that:

$$A(\hat{u},\hat{v}) + D(\hat{v},p)= [f,\hat{v}] \qquad \forall\ \hat{v} \in \mathcal{H} \tag{5.31}$$

$$D(\hat{u},q)= 0 \qquad \forall\ q \in L_0^2(\Omega_R) \tag{5.32}$$

The problem can be reformulated to be:

seek $\hat{u}=(\underline{u},\underline{\lambda}) \in \mathcal{H}_0$ such that:

$$A(\hat{u},\hat{v}) = [f,\hat{v}] \qquad \forall\ \hat{v} \in \mathcal{H}_0 \tag{5.33}$$

where the associted unperturbed problems can be formulated to be

seek $(\hat{w},p)=((\underline{w},\underline{\lambda}),p) \in \mathcal{H}\times L_0^2(\Omega_R)$ such that:

$$B(\hat{w},\hat{v}) + D(\hat{v},p) = [f,\hat{v}] \qquad \forall\ \hat{v} \in H \tag{5.34}$$

$$D(\hat{w},q) = 0 \qquad \forall\ q \in L_0^2(\Omega_R) \tag{5.35}$$

and,

seek $\hat{w}=(\underline{w},\underline{\lambda}) \in H_0$ such that:

$$B(\hat{u},\hat{v}) = [f,\hat{v}] \qquad \forall\ \hat{v} \in H_0 \tag{5.36}$$

Equivalently in operator form one can write

$$(\hat{u},p) \in H \times L_0^2(\Omega_R) \quad : \quad A\hat{u} + D^*p = f \quad \text{in } H'$$
$$D\hat{u} = 0 \quad \text{in } L_0^2(\Omega_R)'$$

$$(\hat{u},p) \in H \times L_0^2(\Omega_R) \quad : \quad A\hat{u} + D^*p = f \quad \text{in } H'$$
$$D\hat{u} = 0 \quad \text{in } L_0^2(\Omega_R)'$$

$$\hat{u} \in H_0 \quad : \quad A\hat{u} = f \quad \text{in } H_0'$$

$$\hat{w} \in H_0 \quad : \quad B\hat{w} = f \quad \text{in } H_0'$$

Remark: The notation * stands for the adjoint of the operator and the notation ' stands for the dual space.

5.7. Verification of the conditions needed for the existence of the solution

5.7.1. Well posedness of the unperturbed problem

To be able to verify the Brezzi-Nicolaides conditions, we need first to analyze problems (5.34)-(5.35) as well as (5.36). In fact we have:

1.B(.,.) is H elliptic since,

$$
\begin{aligned}
B(\hat{v},\hat{v}) \quad &= a(\underline{v},\underline{v}) + 2b(\underline{\lambda},\underline{\lambda}) \\
&\geq \| \underline{v} \|^2_{H^1(\Omega_R)} + 2a_2 \| \underline{\lambda} \|^2_{H^{-1/2}(\delta\Omega_R)} \\
&= \gamma \| \hat{v} \|^2_{H}
\end{aligned}
$$

2. The weak coercivity of D(.,.) on $H \times L^2_0(\Omega_R)$, i.e.,

$$
\sup_{\substack{v \in H \\ v \neq 0}} \frac{D(v,q)}{\| v \|_H} \geq \beta \| q \|_{L^2(\Omega_R)}
$$

This can be easily verified [9] by the fact that

$$
\sup_{\substack{v \in H \\ v \neq 0}} \frac{D(v,q)}{\| v \|_H} \geq \sup_{\substack{\underline{w} \in H^1_0(\Omega_R) \\ \underline{w} \neq 0}} \frac{D(\underline{w},q)}{\| \underline{w} \|_H} \geq \beta \| q \|_{L^2(\Omega_R)}
$$

Hence there exists one and only one solution of the problem (5.34)-(5.35) that continuously depends on the data f, i.e.

$$
\| \hat{v} \|_H + \| q \|_{L^2(\Omega_R)} \leq C \| f \|_{W^R},
$$

where C is a constant.

5.7.2. Well posedness of the perturbed problem

Now it suffices to prove the injectivity of the operator to conclude that problem (5.31)-(5.32) is well posed as well as showing the stability conditions on A(.,.).

Let Π denote the projection operator from H' into H'_0, then we have:

$$A : H \to H'$$

as well as,

$$\Pi A : H_0 \to H'_0$$

where H'_0 denotes the dual space of H_0.

Lemma 5.11: The operator $A \in L(H,H')$ is injective.

Proof: Let $\hat{w}=(\underline{w},\underline{\tau}) \in H$ such that $A\hat{w}=0$,i.e.,

$$a(\underline{w},\underline{v})+\langle\underline{v},\underline{\tau}\rangle_{\delta\Omega_R}=0 \qquad \forall\ \underline{v} \in H \tag{5.37}$$

$$2b(\underline{\tau},\underline{\mu})-\langle\underline{w},\underline{\mu}\rangle_{\delta\Omega_R}+2\langle G\underline{w},\underline{\mu}\rangle_{\delta\Omega_R}=0 \qquad \forall\ \underline{\mu} \in T\ . \tag{5.38}$$

Choosing the test functions in H to be $(\underline{w},\underline{\tau})$ and $(\underline{w},-\underline{\tau})$ respectively and by adding we get

$$a(\underline{w},\underline{w})+\langle\underline{w},\underline{\tau}\rangle_{\delta\Omega_R}=0 \tag{5.39}$$

$$2b(\underline{\tau},\underline{\tau})-\langle\underline{w},\underline{\tau}\rangle_{\delta\Omega_R}+\ 2\langle G\underline{w},\underline{\tau}\rangle_{\delta\Omega_R}=0 \tag{5.40}$$

Also we have

$$a(\underline{w},\underline{w}) - \langle \underline{w},\underline{\tau} \rangle_{\delta\Omega_R} = 0 \tag{5.41}$$

$$2b(\underline{\tau},\underline{\tau}) + \langle \underline{w},\underline{\tau} \rangle_{\delta\Omega_R} - 2\langle G\underline{w},\underline{\tau} \rangle_{\delta\Omega_R} = 0 \;. \tag{5.42}$$

Adding (5.39) to (5.41) we get:

$$2a(\underline{w},\underline{w})=0$$

where by the coecivity of a(.,.) we get $\underline{w}=0$. Similarly, by adding (5.40) to (5.42) we get that

$$4b(\underline{\tau},\underline{\tau})=0$$

which implies that $\underline{\tau}=0$ due to the T-ellipticity of b(.,.).Thus we get $\hat{w}=0$.

■

Corollary 5.12: The operator $\Pi A \in L(H_0,H_0')$ is injective.

■

Remark: It should be noted that whenever problem (5.33) possesses a unique solution $\hat{w}$ in H_0, problem (5.31)–(5.32) has a unique solution pair $(\hat{w},p) \in H \times L^2_0(\Omega_R)$.

We are now in a position to show the stability condtions on A(.,.).

Proposition 5.13: For problem (5.31)–(5.32) we have that

$$Z_1 = Z_2 = H_0$$

where for i = 1, 2 we have:

$$Z_i = \left\{ \hat{w} = (\underline{u},\underline{\lambda}) \in H = W^R(\Omega_R) \times T : (p,\mathrm{div}\underline{u})=0 \;\forall\; p \in L^2_0(\Omega_R) \right\}$$

■

Lemma 5.14: Given $\hat{w} \in H_0$, there exists a unique $\hat{\psi}$ such that:

$$A(\hat{\psi},\hat{\phi}) = B(\hat{w},\hat{\phi}) \quad \forall \ \hat{\phi} \in H_0 .$$

Moreover, there exists a constant $C > 0$ such that:

$$\| \hat{\psi} \|_H \leq C \| \hat{w} \|_H .$$

Proof: The proof of the lemma follows from the injectivity of the operator ΠA.

■

Corollary 5.15: Given $\hat{w} \in H_0$, there exists a unique $\hat{\psi}$ such that:

$$A(\hat{\phi},\hat{\psi})=B(\hat{\phi},\hat{w}) \quad \forall \ \hat{\phi} \in H_0$$

Moreover, there exists a constant $C > 0$ such that:

$$\| \hat{\psi} \|_H \leq C \| \hat{w} \|_H$$

■

Remark: While the form B(.,.) is symmetric and strongly coercive, the form A(.,.) is not symmetric but weakly coercive.

Lemma 5.16:

$$\sup_{\substack{\hat{\psi}\in H_0 \\ \| \hat{\psi} \|_H=1}} A(\hat{\phi},\hat{\psi}) \geq \beta_1 \| \hat{\phi} \|_H \ \forall \ \hat{\phi} \in H_0$$

$$\sup_{\substack{\hat{\phi}\in H_0 \\ \| \hat{\phi} \|_H=1}} A(\hat{\phi},\hat{\psi}) \geq \beta_2 \| \hat{\psi} \|_H \ \forall \ \hat{\psi} \in H_0 .$$

■

Remark: The proof of the lemma is an easy consequence of Lemma {5.14} and Corollary {5.15}.

5.8. Existence, Uniqueness and regularity

We summarize all the above in the following theorem, where we also state the regularity result for the problem (5.31)–(5.32).

Theorem 5.17: The variational problem (5.31)–(5.32) has exactly one solution $(\underline{u},\underline{\lambda},p) \in W^R \times T \times L_0^2(\Omega_R)$ where $(\underline{u},\underline{\lambda})$ is the unique solution of (5.33). Moreover, if $\underline{f} \in H^{m-1}(\Omega_R)$, then $\underline{u} \in H^{m+1}(\Omega_R)$, $\underline{\lambda} \in H^{m-1/2}(\delta\Omega_R)$ and $p \in H^m(\Omega_R) \bigcap L_0^2(\Omega_R)$, and there exists a constant $C > 0$ such that:

$$\| \underline{u} \|_{H^{m+1}(\Omega_R)} + \| p \|_{H^m(\Omega_R)} + \| \underline{\lambda} \|_{H^{m-1/2}(\delta\Omega_R)} \leq C \| \underline{f} \|_{H^{m-1}(\Omega_R)}$$

■

5.9. Finite element approximation

Let W_h, T_h and Q_h be finite dimensional subspaces of the spaces W, T and Q respectively. Let

$$H_h = W_h \times T_h$$

The variational problem on $W_h \times T_h \times Q_h$ can be rewritten to be:

Seek $(\hat{\underline{u}}_h, p_h) \in H_h \times Q_h$ such that:

$$A(\hat{\underline{u}}_h, \hat{\underline{v}}_h) + D(p_h, \hat{\underline{v}}_h) = \langle \underline{f}, \hat{\underline{v}}_h \rangle \qquad \forall\ \hat{\underline{v}}_h \in H_h \tag{5.43}$$

$$D(q_h, \hat{\underline{u}}_h) = 0 \qquad \forall\ q_h \in Q_h \tag{5.44}$$

Now define

$$Z_h = \left\{ \hat{\underline{v}}_h \in H_h : D(q_h,\hat{\underline{v}}_h)=0 \ \forall \in Q_h \right\}.$$

The associated problem is given by

seek $\hat{\underline{u}}_h \in Z_h$ such that:

$$A(\hat{\underline{u}}_h,\hat{\underline{v}}_h) = \langle \underline{f},\hat{\underline{v}}_h \rangle \qquad \forall\ \hat{\underline{v}}_h \in Z_h \tag{5.45}$$

Theorem 5.18: Assume

$$\sup_{\substack{\hat{\underline{v}}_h \in Z_h \\ \| \hat{\underline{v}}_h \|_H = 1}} A(\hat{\underline{u}}_h,\hat{\underline{v}}_h) \geq \alpha_1 \| \hat{\underline{u}}_h \|_H \qquad \forall\ \hat{\underline{u}}_h \in Z_h \tag{5.46}$$

$$\sup_{\substack{\hat{\underline{u}}_h \in Z_h \\ \| \hat{\underline{u}}_h \|_H = 1}} A(\hat{\underline{u}}_h,\hat{\underline{v}}_h) \geq \alpha_2 \| \hat{\underline{v}}_h \|_H \qquad \forall\ \hat{\underline{v}}_h \in Z_h \tag{5.47}$$

$$\sup_{\substack{\underline{v}_h \in W_h \\ \| \underline{v}_h \|_H = 1}} D(q_h,\underline{v}_h) \geq \beta \| q_h \|_{L^2(\Omega_R)} \qquad \forall\ q_h \in Q_h \tag{5.48}$$

Then, problem (5.43)–(5.44) has one and only one solution pair $(\hat{\underline{u}}_h,p_h) \in H_h \times Q_h$, where $\hat{\underline{u}}_h \in Z_h$ of problem (5.45).

Moreover, there exists a constant $C > 0$ such that:

$$\| \hat{u}-\hat{\underline{u}}_h \|_H + \| p-p_h \|_{L^2(\Omega_R)} \tag{5.49}$$

$$\leq C \left\{ \inf_{\hat{\underline{v}}_h \in H_h} \| \hat{u}-\hat{\underline{u}}_h \|_H + \inf_{q_h \in Q_h} \| p-q_h \|_{L^2(\Omega_R)} \right\}$$

Proof: Problem (5.45) has a unique solution by using conditions (5.46)–(5.47). These are Babuska type conditions needed for existence and uniqueness of (5.45).

Also the uniqueness of the pressure follows from the stability condition (5.48). Finally, using the Brezzi-Nicolaides approximation result, we get the estimate (5.49).

∎

Also, we can apply the extension of the Aubin Nitche duality argument to the Brezzi-Nicolaides saddle point problem, since $H^1(\Omega_R) \subset_d L^2(\Omega_R)$ with the imbedding being dense. Let $(.,.)_{L^2(\Omega_R)}$ and $\| \quad \|_{L^2(\Omega_R)}$ denote the inner product and the norm respectively.

Theorem 5.19: Assume conditions (5.46)-(5.48) satisfied. Given the problem:

seek $(\hat{\phi}_g, \pi_g) \in H \times L_0^2(\Omega_R)$ such that:

$$A(\hat{\psi},\hat{\phi}_g) + D(\pi_g,\hat{\psi}) = [g,\hat{\psi}] \qquad \forall\ \hat{\psi} \in H \tag{5.50}$$

$$D(\hat{\phi}_g,q) = 0 \qquad \forall\ q \in L_0^2(\Omega_R) \tag{5.51}$$

Then there exists a constant $C > 0$ such that:

$$\| \hat{u}-\underline{\hat{u}}_h \|_{L^2(\Omega_R)} \leq C \Big[\| \hat{u}-\underline{\hat{u}}_h \|_H + \| p-p_h \|_{L^2(\Omega_R)} \Big] \times$$

$$\sup_{\substack{g \in L^2(\Omega_R) \\ g \neq 0}} \Big[\frac{1}{\| g \|_{L^2(\Omega_R)}} \Big\{ \inf_{\psi_h \in Q_h} \| \pi_g - \psi_h \|_{L^2(\Omega_R)} + \inf_{\hat{\underline{z}}_h \in H} \| \hat{\phi}_g - \underline{\hat{u}}_h \|_H \Big\} \Big]$$

∎

Remark: G is a compact operator from $H^{1/2}(\delta\Omega_R)$ into $H^{1/2}(\delta\Omega_R)$. Therefore G can be approximated by a sequence of operators $\{ G_n \}$ with G_n, of finite rank, associated with the finite dimensional spaces H_h [24]. Now for n large enough (or equivalently h small enough), we can get

$$\| G - G_n \| \leq \epsilon$$

where $\| \, . \, \|$ is the operator norm. Now since the constant in the stability condition on A(.,.) is a lower bound on the eigen values of the operator A which is a compact perturbation of the strongly coercive operator B, we can find a G_n with n large enough so that the operator A restricted to the finite dimensional subspace would remain invertible. This is verified in the following lemma.

Lemma 5.20: Assume

$$B(\underline{\hat{w}}_h,\underline{\hat{w}}_h) \geq \alpha \parallel \underline{\hat{w}}_h \parallel^2_H \qquad\qquad \forall\ \underline{\hat{w}}_h \in H_h$$

with α independent of h, and

$$\inf_{\underline{\hat{v}}_h \in Z_h} \parallel (\Pi B)^{-1} (\Pi G)\ \underline{\hat{w}}_h - \underline{\hat{v}}_h \parallel_H \leq c_1(h) \parallel (\Pi B)^{-1} (\Pi G)\ \underline{\hat{w}}_h \parallel_U \qquad (5.52)$$

where U is the subspace of H in which the operator $(\Pi B)^{-1} (\Pi G)$ is bounded, and $c_1(h) \rightarrow 0$ as $h \rightarrow 0$. Then, for h small enough, condition (5.46) holds.

Proof: Given $\underline{\hat{w}}_h \in Z_h$ we take:

$$\underline{\hat{\psi}}_h = s_h\ (\Pi A)^{*^{-1}}\ (\Pi B)^{*}\underline{\hat{w}}_h\ .$$

where s_h denotes the orthogonal projection from H into H_{0h} defined by:

$$B(\underline{\hat{v}}_h,s_h\hat{u})=B(s_h\underline{\hat{v}}_h,\hat{u}) \quad \forall\ \hat{u} \in H$$

$$B(\underline{\hat{v}}_h,s_h\underline{\hat{u}}_h)=B(\underline{\hat{v}}_h,\underline{\hat{u}}_h) \quad \forall\underline{\hat{u}}_h \in H_{0h}$$

Then

$$A(\underline{\hat{w}}_h,\underline{\hat{\psi}}_h)=B(\underline{\hat{w}}_h,\underline{\hat{\psi}}_h) + G(\underline{\hat{w}}_h,\underline{\hat{\psi}}_h)\ .$$

Now

$$B(\underline{\hat{w}}_h,\underline{\hat{\psi}}_h)=B(\underline{\hat{w}}_h,s_h\ (\Pi A)^{*^{-1}}\ (\Pi B)^{*}\underline{\hat{w}}_h)$$

$$=B(\underline{\hat{w}}_h,\ (\Pi A)^{*^{-1}}\ (\Pi B)^{*}\underline{\hat{w}}_h)$$

$$=[\ (\Pi B)\ \underline{\hat{w}}_h,\ (\Pi A)^{*^{-1}}\ (\Pi B)^{*}\underline{\hat{w}}_h\ ]$$

$$=[\ (\Pi B)\ (\Pi A)^{-1}\ (\Pi B)\ \underline{\hat{w}}_h,\underline{\hat{w}}_h\ ]\ .$$

But using the identity

$$(\Pi A)^{-1} = (\Pi B)^{-1} - (\Pi A)^{-1}\ (\Pi \mathcal{G})\ \ (\Pi B)^{-1}$$

we obtain that

$$B(\underline{\hat{w}}_h,\underline{\hat{\psi}}_h) = [\ (\Pi B) \left\{\ I - (\Pi A)^{-1}\ (\Pi \mathcal{G})\ \right\} \underline{\hat{w}}_h,\underline{\hat{w}}_h\ ]$$

$$= B(\underline{\hat{w}}_h,\underline{\hat{w}}_h) - B(\ (\Pi B)^{-1}\ (\Pi \mathcal{G})\ \underline{\hat{w}}_h,\ (\Pi A)^{*^{-1}}\ (\Pi B)^{*}\underline{\hat{w}}_h)\ .$$

Also

$$\mathcal{G}(\underline{\hat{w}}_h,\underline{\hat{\psi}}_h) = (\mathcal{G}\underline{\hat{w}}_h, s_h\ (\Pi A)^{*^{-1}}\ (\Pi B)^{*}\underline{\hat{\psi}}_h)$$

$$=[\ (\Pi B)\ (\Pi B)^{-1}\ (\Pi \mathcal{G})\ \underline{\hat{w}}_h, s_h\ (\Pi A)^{*^{-1}}\ (\Pi B)^{*}\underline{\hat{\psi}}_h\ ]$$

$$= B(s_h\ (\Pi B)^{-1}\ (\Pi \mathcal{G})\ \underline{\hat{w}}_h,\ (\Pi A)^{*^{-1}}\ (\Pi B)^{*}\underline{\hat{\psi}}_h)\ .$$

Therefore

$$A(\underline{\hat{w}}_h,\underline{\hat{\psi}}_h) = B(\underline{\hat{w}}_h,\underline{\hat{w}}_h) + B((I-s_h)\ (\Pi B)^{-1}\ (\Pi \mathcal{G})\ \underline{\hat{w}}_h,\ (\Pi A)^{*^{-1}}\ (\Pi B)^{*}\underline{\hat{\psi}}_h)\ .$$

But according to (5.52)

$$\lim_{h\to 0} \left\|\ (I-s_h)\ (\Pi B)^{-1}\ (\Pi \mathcal{G})\ \right\| = 0\ .$$

Hence for h small enough we can get that:

$$A(\underline{\hat{w}}_h,\underline{\hat{\psi}}_h)\ \geq\ \alpha^{*}\left\|\ \underline{\hat{w}}_h\ \right\|_{H}^{2}\ .$$

Noting that:

$$\| \hat{\underline{\psi}}_h \|_H \leq C \| \hat{\underline{w}}_h \|_H$$

with C independent of h, we easily obtain the result.

∎

We now discuss the hypotheses underwhich we can establish the conditions in Theorem {5.18}.

Hypothesis (H1).

There exists a mapping denoted by $r_h \in L([H^2(\Omega_R)]^3 \cap W^R(\Omega_R);Z^h)$ and a positive integer l such that:

$$b(q_{Rh}, \text{div } r_h\underline{v}-\underline{v})=0 \quad \forall\ q_h \in Q_h$$

with

$$\| r_h\underline{v}-\underline{v} \|_{H^1(\Omega_R)} \leq C\ h^m \| \underline{v} \|_{H^{m+1}(\Omega_R)} \quad \forall\ \underline{v} \in H^{m+1}(\Omega_R)\ .$$

Hypothesis (H2).

The orthogonal projection denoted by $\mu_h \in L(H^m(\delta\Omega_R) \cap T;T_h)$ satisfies:

$$\| \mu_h\underline{\lambda}-\underline{\lambda} \|_{H^{-1/2}(\delta\Omega_R)} \leq C\ h^m \| \underline{\lambda} \|_{H^m(\delta\Omega_R)} \quad \forall\ \underline{\lambda} \in H^m(\delta\Omega_R) \cap T\ .$$

Hypothesis (H3).

The orthogonal projection denoted by $\rho_h \in L(H^m(\Omega_R) \cap L^2_0(\Omega_R);Q_h)$ satisfies:

$$\| \rho_h q - q \|_{L^2(\Omega_R)} \leq C\, h^m \| q \|_{H^m(\Omega_R)} \quad \forall\, q \in H^m(\Omega_R) \cap L^2_0(\Omega_R)\ .$$

Hypothesis (H4).

For every $q_h \in Q_h$, there exists a $\underline{w}_{Rh} \in Z_h$ such that:

$$(q_h - \mathrm{div}_h \underline{w}_{Rh}, s_h) = 0 \quad \forall\, s_h \in Q_h$$

with

$$\| \underline{w}_{Rh} \|_{H^1(\Omega_R)} \leq C \| q_h \|_{L^2(\Omega_R)}\ .$$

Similar to Chapter {4} we can state the following theorem.

Theorem 5.21: Under hypotheses (H1)-(H4) we can show that problem (5.43)-(5.44) has a unique solution pair $(\hat{\underline{u}}_h, p_h) \in \mathcal{H}_h \times Q_h$ with the following estimate holding:

$$\| \hat{\underline{u}} - \hat{\underline{u}}_h \|_{\mathcal{H}} + \| p - p_h \|_{L^2(\Omega_R)} \leq C\, h^m \left\{ \| \hat{\underline{u}} \|_{\mathcal{H}} + \| p \|_{H^m(\Omega_R)} \right\}$$

or

$$\| \underline{u} - \underline{u}_h \|_{H^1(\Omega_R)} + \| \underline{\lambda} - \underline{\lambda}_h \|_{H^{-1/2}(\delta\Omega_R)} + \| p - p_h \|_{L^2(\Omega_R)} \tag{5.53}$$

$$\leq C\, h^m \left\{ \| \underline{u} \|_{H^{m+1}(\Omega_R)} + \| \underline{\lambda} \|_{H^{m-1/2}(\Omega_R)} + \| p \|_{H^m(\Omega_R)} \right\}.$$

■

5.10. A sample choice of finite element spaces

Assume $(\underline{u},\underline{\lambda},p) \in \{ H^2(\Omega_R) \cap \Omega_R \} \times \{ H^{1/2}(\delta\Omega_R) \cap T \} \times \{ L_0^2(\Omega_R) \cap H^1(\Omega_R) \}$,

then we can choose the following finite element spaces:

$$Q_h = \{ q_h \in L^2(\Omega_R) : q_h|_K \in P_0(K) \}$$

$$T_h = \{ \underline{\lambda}_h \in C^0(\delta\Omega_R) \cap H^1(\delta\Omega_R) : \underline{\lambda}_h|_K \in P_1(K) \}$$

$$W_h = \{ \underline{v}_h \in C^0(\Omega_R) \cap \Omega_R : \underline{v}_h|_K \in P_2(K) \} .$$

In that case we obtain the estimate (5.53) for m=1.

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APPENDICES

APPENDIX A

ON THE REGULARITY OF THE SOLUTION OF 2.14 AND 2.15

The regularity of problem (2.14)-(2.15) was vital to the existence proof of the stability condition (2.10). Specifically we needed, in Lemma {2.9}, to show that if $\phi \in W^{1,0}(\Omega)$ and the right hand side $p/(1+r^2)^{1/2}$ belongs to $W^{0,1}(\Omega)$, then $\phi \in W^{2,1}(\Omega)$. Note that to show that $\phi \in W^{2,1}(\Omega)$, we need to show that $D\phi \in L^2(\Omega)$ and $(1+r^2)^{1/2}D^2\phi$ is also in $L^2(\Omega)$.

We define $\Omega_R=\Omega \bigcap B(0;R)$, where $B(0;R)$ is the open ball of radius R centered at the origin. Let R be chosen such that Ω_R is not empty. We also define Ω^{2R}_R to be the ring shaped region in Ω such that $R < |x| < 2R$ for $x \in \Omega^{2R}_R$.

Definition A.1: Let $0 \le X(x) \le 1$ be an infinitely differentiable function such that $X(x)=1$ for $x \in B(0;1)$ and $\mathrm{supp}(X(x))$ is contained in $B(0,2)$. We define $X_1(x)$ to be

$$X_1(x)=X(x/R) .$$

We also define $X_2(x)$ to be

$$X_2(x)= 1-X_1(x) \quad \text{for } x \in \Omega .$$

We denote by Q_i the support of X_i in R^3. We introduce the notation $\tilde{X}_i\Psi$ to denote the extension of $X_i\Psi$ by zero outside the support for any function Ψ. ∎

From these definitions, we have that $X_2(x)$ has its support Q_2 outside Ω_R. Furthermore, we may deduce that there exists a positve constant C such that

$$|D^{\tau}X_i| \le C/R^{|\tau|}$$

where τ is a multi-index.

Let A be the operator associated with the bilinear form B(.,.) defined in Lemma {2.6}, i.e.,

$$(A\phi, \psi)_{L^2(\Omega)} = B(\phi, \psi) \qquad \forall\ \psi \in W^{1,0}(\Omega)$$

We also introduce the notation $[A,X_i]$ to denote the commutator of the two operators A and X_i for i=1,2. This commutator is formally given by:

$$\begin{aligned}[A,X_i]\phi &= A(X_i\phi) - X_iA(\phi)\\ &= (1+r^2)^{-1/2}\ \mathrm{div}\ (1+r^2)^{1/2}\ \mathrm{grad}X_i\phi\ - X_i(1+r^2)^{-1/2}\mathrm{div}(1+r^2)^{1/2}\mathrm{grad}\phi\end{aligned}$$

Lemma A.2: [12](Regularity result for R^n) Let $\xi \in W^{k,k+\alpha}(R^3)$. If $A(\xi) \in W^{k-1,k-1-\alpha}(R^3)$, then $\xi \in W^{k+1,k+1+\alpha}(R^3)$.

Proof: See [12] for the proof. ∎

Lemma A.3(Regularity results for bounded domains) Let $\rho \in H^k(Q)$. If $A(\rho) \in H^{k-1}(Q)$, with Q bounded and δQ sufficiently smooth, then $\rho \in H^{k+1}(Q)$

Proof: See any standard reference on partial differential equations (e.g. [18]). ∎

Lemma A.4: For $\phi \in W^{1,0}(\Omega)$, we have the following a priori estimate:

$$[A,X_i]\phi \in W^{0,1}(Q_i) \text{ for } i=1,2 .$$

Proof: For the problem ((2.14)-(2.15)) the operator $[A,X_i]$ can be formally written as:

$$[A,X_i]\phi = \mathrm{grad}\phi.\mathrm{grad}X_i + \phi\Delta X_i + \phi\,\mathrm{grad}X_i.\mathrm{grad}[\ln(1+r^2)^{1/2}] \ .$$

Thus we need to derive an estimate in the norm $\| \ \|_{W^{0,1}}$, for each term of the right hand side. For i=1,2

$$\| \mathrm{grad}\phi.\mathrm{grad}X_i \|^2_{W^{0,1}(Q_i)} \le \int_{Q_i} (1+r^2) \ |\mathrm{grad}\phi|^2 |\mathrm{grad}X_i|^2 \ dx$$

$$\le C \int_{\Omega_R^{2R}} (1+r^2)/R^2 \ | \ \mathrm{grad}\phi \ |^2 \ dx$$

$$\le C \int_{\Omega_R^{2R}} |\mathrm{grad}\phi|^2 \ dx$$

$$\le C \| \ \phi \ \|^2_{W^{1,0}(\Omega)}$$

Also

$$\| \ \phi\Delta X_i \ \|^2_{W^{0,1}(Q_i)} = \int_{Q_i} (1+r^2) \ \phi^2 |\Delta X_i|^2 \ dx$$

$$\le C \int_{\Omega_R^{2R}} (1+r^2)/R^4 \ \phi^2 \ dx$$

$$\le C \int_{\Omega_R^{2R}} 1/(1+r^2) \ \phi^2 \ dx$$

$$\le C \| \ \phi \ \|^2_{W^{1,0}(\Omega)} \ .$$

Similarly we get the estimate

$$\| \ \phi\,\mathrm{grad}X_i.\mathrm{grad}[\ln(1+r^2)^{1/2}] \ \|^2_{W^{0,1}(Q_i)} \le \int_{Q_i} (1+r^2) \ \phi^2 \ |\mathrm{grad}[\ln(1+r^2)^{1/2}]|^2 \ dx$$

$$\le C \int_{\Omega_R^{2R}} (r/R)^2/(1+r^2) \ \phi^2 \ dx$$

$$\le C \| \ \phi \ \|^2_{W^{1,0}(\Omega)} \qquad \blacksquare$$

Remarks:

1. For the case i=1 these estimates hold also in the usual Sobolev spaces due to the equivalence of the norms (Q_1 is bounded)

2. Note also that the estimates obtained are independent of R.

We are now ready to prove Lemma {2.9}.

Lemma 2.9: Let $\phi \in W^{1,0}(\Omega)$ be the weak solution in $W^{1,0}(\Omega)$ of (2.14)-(2.15) for $p/(1+r^2)^{1/2} \in W^{0,1}(\Omega)$. If $\delta\Omega$ is sufficiently smooth, then ϕ is also in $W^{2,1}(\Omega)$.

Proof: Since Q_1 is bounded in R^3, we have from Lemma {A.3} that

$$[A,X_1]\phi \in W^{0,1}(Q_1)$$

Since by hypothesis, $A(\phi) \in W^{0,1}(\Omega)$, we have that

$$A(X_1\phi)=X_1A(\phi) +[A,X_1]\phi \in W^{0,1}(Q_1)=H^0(Q_1)$$

Hence by Lemma {A.3} we have that $X_1\phi \in H^2(\Omega_{2R})$ since Q_1 is contained in Ω_{2R}.

Similarly, using Lemmas {A.2} and {A.4}, we conclude that

$$\tilde{X}_2\phi \in W^{2,1}(\Omega)$$

and therefore, since $\tilde{X}_2\phi$ is an extension, that

$$X_2\phi \in W^{2,1}(\Omega) .$$

Then, since $\phi=X_1\phi+X_2\phi\ \forall\ x \in \Omega$, we conclude that

$$\phi \in W^{2,1}(\Omega) .$$

■

APPENDIX B

INVESTIGATION OF THE SOLUTION OF THE STOKES EQUATIONS IN CASE OF SUPPORT OF F COMPACT IN R^3

The solution of the Stokes problem has been investigated using the theory of hydrodynamic potentials. The following is a summary of the results obtained using that treatment. These results are dicussed in details in [16]. We will use tensorial notation.

1. The singular solution of the Stokes problem or, more precisely, the fundamental tensor is made up of solutions corresponding to an impulse directed along the various k directions (k=1,2,3) and satisfies:

$$-\Delta \underline{U}^k(x,y)+\nabla P^k(x,y)=-\delta(x-y)e^k$$

$$\mathrm{div}\underline{U}^k(x,y)=0$$

   where e^k is a unit vector in the direction of the k-axis and the derivatives are taken with respect to x. We also note that $\underline{U}^k(x,y)$ and $P^k(x,y)$ should approach zero as $|x|\rightarrow\infty$.

2. The fundamental solutions $\underline{U}^k(x,y),P^k(x,y)$ are given by:

$$\underline{U}^k_j(x,y)=-\frac{1}{8\pi}\left[\frac{\delta^k_j}{|x-y|}+\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^3}\right]$$

$$P^k(x,y)=-\frac{1}{4\pi}\,\frac{(x_k-y_k)}{|x-y|^3}$$

where δ^k_j is the Kronecker delta given by:

$$\delta^k_j = 1 \text{ for } j=k\ .$$
$$= 0 \text{ for } j\neq k$$

3. The Green's formulas corresponding to flow pairs $(\underline{u},p)$ and $(\underline{v},q)$ are given by:

$$\int_\Omega (u_{i,jj}-p_{,i})\ v_i\ dx=-\frac{1}{2}\int_\Omega (u_{i,k}+u_{k,i})(v_{i,k}+v_{k,i})\ dx + \int_Q T_{ij}(\underline{u})v_in_j\ dS$$

where

$(\ )_{i,j}$ denotes the derivative of the ith component of a vector with respect to the jth coordinate,

$(\ )_{,j}$ denotes the derivative of a scalar with respect to the jth coordinate, and,

$$T_{ij}(\underline{u})=-\delta^j_i p + (u_{i,j}+u_{j,i})$$

is the stress tensor corresponding to the flow pair $(\underline{u},p)$.

Another formula is given by

$$\int_\Omega (v_{i,jj}-q_i)u_i-(u_{i,jj}+p_i)v_i\ dx=\int_Q T_{ij}(\underline{v})u_in_j-T^*_{ij}(\underline{u})v_in_j\ dS \qquad \text{(B.1)}$$

where,

$$T^*_{ij}(\underline{u})=\delta^j_i p+(u_{i,j}+u_{j,i})$$

4. Using the customary method, equation (B.1) and the fundamental solution yield an integral representation of the solution of the Stokes system:

$$u_k(x)=\int_\Omega \underline{U}^k_j(x,y)\ f_j(y)\ dy+\int_Q T^*_{ij}(\underline{U}^k(x,y))_y u_i n_j\ dS$$

$$-\int_Q \underline{U}^k_j(x,y)T_{ji}(\underline{u})n_i dS$$

and,

$$p(x)=\int_\Omega P^k(x,y)f_k(y)\ dy-\int_Q P^k(x,y)T_{kj}(\underline{u})n_j\ dS$$

$$-2\int_Q P^k(x,y)_{,j}u_k n_j\ dS$$

5. *The volume potential.* We define $\rho(x)$ and $\mu(x)$ to be the potentials of a volume distribution of density $\underline{f}$, and these are given by:

$$\rho_j(x)=\int_\Omega \underline{U}^k_j(x,y)f_k(y)\ dy$$

$$\mu(x)=\int_\Omega P^k(x,y)f_k(y)dy$$

6. *The potential of a single layer.* We define $V(x,\underline{\psi})$ and $Q(x,\underline{\psi})$ to be the potentials of a single layer of density $\underline{\psi}$, and these are given by:

$$V_j(x,\underline{\psi})=-\int_Q \underline{U}_j^k(x,y)\psi_k(y)\ dS_y$$

$$Q(x,\underline{\psi})=-\int_Q P^k(x,y)\psi_k(y)\ dS_y$$

7. *The potential of a double layer.* We define $\underline{W}(x,\underline{\phi})$ and $\Pi(x,\underline{\phi})$ to be the potentials corresponding to a double layer of density $\underline{\phi}$ given by:

$$W_k(x,\underline{\phi})=\int_Q T_{ij}(\underline{U}^k(x,y))_y\phi_i(y)n_j(y)\ dS_y$$

$$\Pi(x,\underline{\phi})=-2\frac{\partial}{\partial x_j}\int_Q P^k(x,y)n_j(y)\phi_k(y)\ dS_y$$

8. The potential concept associted with the above definitions is a direct analogue to the case of electrostatic potentials in field theory.

9. If we substitute with the explicit expressions for $\underline{U}^k(x,y)$, $P^k(x,y)$ and $T^*_{ij}(\underline{U}^k(x,y))$ we get for the potentials of the single and double layers respectively:

$$V_j(x,\underline{\psi})=\frac{1}{8\pi}\int_Q\left[\frac{\delta^k_j}{|x-y|}+\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^3}\right]\psi_k(y)\ dS_y$$

$$Q(x,\underline{\psi})=\frac{1}{4\pi}\int_Q\frac{(x_k-y_k)}{|x-y|^3}\psi_k\ dS_y$$

and,

$$W_k(x,\underline{\phi})=-\frac{3}{4\pi}\int_Q \frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^5}\,\phi_i(y)n_j\,dS_y$$

$$\Pi(x,\underline{\phi})=\frac{1}{2\pi}\frac{\partial}{\partial x_j}\int_Q \frac{(x_k-y_k)}{|x-y|^3}\,n_j(y)\;\phi_k(y)\;dS_y$$

Or using a shorter notation we write:

$$K_{ki}(x,y)=\frac{3}{4\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^5}\,n_j(y)$$

$$K_k(x,y)=\frac{1}{2\pi}\frac{\partial}{\partial x_j}\frac{(x_k-y_k)}{|x-y|^3}\,n_j(y)$$

$$W_k(x,\underline{\phi})=\int_Q K_{ki}(x,y)\phi_i(y)dS_y$$

$$\Pi(x,\underline{\phi})=\int_Q K_j(x,y)\;\phi_j(y)\;dS_y$$

The kernel $K_{kj}(x,y)$ satisfies the following properties

a. $\quad | K_{ij}(x,y) | \leq \dfrac{C}{|x-y|}$ for $x,y \in Q$.

b. $\quad \int_Q | K_{ij}(x,y) | \, dS_y \leq C$ for Q smooth enough and $x \in S$.

c. $\quad | K_{ij}(x,y)-K_{ij}(x',y) | \leq \dfrac{C\,| x-x' |}{R^2}$

d. $\quad \left|\dfrac{\partial}{\partial x_a}K_{ij}(x,y)\right| \leq \dfrac{C}{|x-y|^2}$

e. $\quad \left|\dfrac{\partial}{\partial x_a}K_{ij}(x,y)-\dfrac{\partial}{\partial x_a}K_{ij}(x',y)\right| \leq \dfrac{C\,| x-x' |}{R^3}$

where $\dfrac{\partial}{\partial x_a}$ is the derivative along the surface Q and $R=\min\{|x-y| , |x'-y|\}$.

10. The potential pairs of the volume potential, the single layer potential and the double layer potential satisfy the Stokes system.

11. While the potential $\underline{V}(x,\underline{\psi})$ of the single layer is continuous across the surface S, the potential $\underline{W}(x,\underline{\phi})$ is not, and we have:

$$W_j(x)_{(i)}=\frac{1}{2}\ \phi_j(x) + \int_Q K_{ji}(x,y)\phi_i(y)dS_y$$

$$W_j(x)_{(e)}=-\frac{1}{2}\ \phi_j(x) + \int_Q K_{ji}(x,y)\phi_i(y)dS_y$$

where the indices (i),(e) denote the interior and exterior sides of the closed surface Q respectively.

12. Also, the stress tensor due to the single layer $\underline{\psi}(x)$ at the surface Q is discontinuous and in fact we have:

$$T_{kj}(\underline{V}(x,\underline{\psi}))\ n_j(x)_{(i)}=\frac{1}{2}\psi_k(x) + \int_Q K_{ki}(x,y)\phi_i(y)dS_y$$

$$T_{kj}(\underline{V}(x,\underline{\psi}))n_j(x)_{(e)}=-\frac{1}{2}\phi_k(x)+\int_Q K_{ki}(x,y)\phi_i(y)dS_y$$

13. Other important properties of the kernel of the potential of the double layer are

a. $$\int_Q |K_{ij}(x,y)-K_{ij}(x'-y)|^{1+2\epsilon}dS_y \le C|x-x'|^{1-2\epsilon}$$

b. $$\int_Q\int_Q \frac{|K_{ij}(x,y)-K_{ij}(x',y)|^{1-2\epsilon}}{|x-x'|^{2+2\epsilon}}\ dS_x\ dS_y \le C$$

c. $$\int_Q \left| \frac{\partial}{\partial x_\alpha} K_{ij}(x,y) \right|^2 |x-y|^3 \, dS_y \leq C$$

14. In the case of supp($\underline{f}$) compact, the solutions $\underline{u}$ and p are $O(|x|^{-1})$ and $O(|x|^{-2})$ respectively, due to the decay rate of the kernels in their integral representation.

VITA

Georges Halim Guirguis was born in Alexandria, Egypt on July 9, 1951. He attended High school in that city and was graduated from St. Marc High School in June 1968. In 1973 he received the Bachelor of Science degree in Electrical Engineering from Alexandria University. In 1976 he received the Bachelor of Science degree in Mathematics from Ain Shams University in Cairo. In 1979 he entered the Graduate School of the University of Tennessee, Knoxville. He received the Doctor of Philosophy degree in Mathematics in August 1983. The author will be employed with the department of Mathematics, North Carolina State University.