

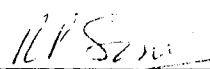
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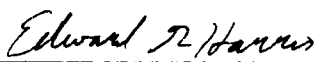
I am submitting herewith a dissertation written by Katsuya Yamada entitled "On Jacobi's Decomposition of the Motion of a Heavy Symmetrical Top into the Motions of Two Torque-Free Triaxial Tops." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

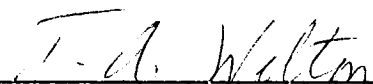


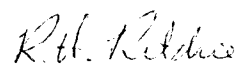
S. Y. Shieh, Major Professor

We have read this dissertation
and recommend its acceptance:









Accepted for the Council:



Vice Chancellor
Graduate Studies and Research

ON JACOBI'S DECOMPOSITION OF THE MOTION OF A HEAVY SYMMETRICAL TOP
INTO THE MOTIONS OF TWO TORQUE-FREE TRIAXIAL TOPS

A Dissertation
Presented for the
Doctor of Philosophy
Degree
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Katsuya Yamada

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ABSTRACT

According to Jacobi's decomposition theorem, the motion of a heavy symmetrical top can be expressed as a composite motion of two torque-free triaxial tops. It is natural to seek the connection between the dynamical constants of the three top motions (one heavy symmetrical top and two free triaxial tops). From the investigation, it is found that the formulas connecting these constants are simply projective transformations (fractional linear transformations). However, from the geometrical analysis on the decomposition theorem which will be stated below, it is found that a common constant term must be added to each of the above mentioned connecting formulas for one of the free triaxial tops.

The geometrical analysis is employed to obtain the clear physical contents of the theorem. It is found that this geometrical method gives not only the fundamental requirements for the theorem but also a clear physical picture showing how the composite motion, that is, the motions of two free triaxial tops, is related to the given motion of the heavy symmetrical top. Since an elaborate manipulation of the elliptic functions and the theta functions is involved in Jacobi's analysis, it is difficult to extract a clear picture of the theorem from Jacobi's analysis alone. Instantaneous positions of the three tops are specified by the three sets of Euler angles which can be computed using the connecting formulas mentioned above. However, the geometrical method gives some particular angular relations between the three sets of

Euler angles and these relations greatly simplify the computation of the Euler angles and clarify the relative positions of the three tops.

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CHAPTER I

INTRODUCTION

The dynamics of a rigid body has many applications such as molecular physics, nuclear physics, astrophysics, etc. However, because of the nonlinear nature of the rigid body motion, the number of soluble cases is very limited. At present (1982), there are only three soluble cases which can be solved under general initial conditions:

(1) Lagrange's case, (2) Euler's case, and (3) Kowalevski's case.

Lagrange's case is normally referred to as the "heavy symmetrical top" which is the motion of a symmetrical rigid body ($I_1 = I_2$) with one point fixed under the uniform gravitational field [1,2]. Its center of gravity lies along the axis of symmetry. Euler's case is the motion of an asymmetrical rigid body with one point fixed in space where there is no force field or it is placed in the uniform gravitational field but the center of gravity coincides with the fixed point giving no torque. Thus the Euler's case is referred to as the "torque-free triaxial top" [2,3]. Kowalevski's case is the motion of a symmetrical rigid body ($I_1 = I_2 = 2I_3$) with one point fixed in the uniform gravitational field but the center of gravity lies in the equatorial plane [4]. The decomposition theorem is concerned with the first two soluble cases, the heavy symmetrical top and the torque-free triaxial top.

The position of a rotating rigid body with one point fixed is completely determined by three independent variables, three Euler

angles for example. Therefore once the Euler angles as functions of time are found, the further computation of the orthogonal matrix whose nine elements are the direction cosines of the moving body axes with respect to the fixed space axes, may seem to be an unnecessarily tedious task without yielding any further new information. However, Jacobi found that the orthogonal matrix M , which describes the motion of a heavy symmetrical top, can be decomposed into two orthogonal matrices M_1 and M_2 , each of which represents the motion of a torque free triaxial top [5]. Jacobi expressed the matrix elements of all the three matrices explicitly in terms of the theta functions in establishing the relation

$$M = M_1 \tilde{M}_2 \quad (1.1)$$

where $\tilde{M}_2$ is the transpose of M_2 . The physical meaning of (1.1) is that the motion of a heavy symmetrical top can be decomposed into two motions of two torque-free triaxial tops. Let us consider the motion of a heavy symmetrical top M with the principal axes (x_2, y_2, z_2) and the motion of a free triaxial top M_1 with the principal axes (x_1, y_1, z_1) , both of which are described with respect to a frame of reference (x, y, z) fixed in space. Then the motion of the torque-free triaxial top observed from the fixed space is described by the orthogonal matrix M_1 and the same motion when observed from the frame of the heavy symmetrical top (i.e., the observer moving with the heavy symmetrical top) is described by the orthogonal matrix M_2 . The relative motion M_2 turns out to be also representing the motion of a torque-free triaxial top.

But not only the dynamical constants but also the kinematical constants (e.g., moments of inertia) of the two free triaxial tops underlying the motions M_1 and M_2 are different.

The precise description of the position of a rigid body at any instant requires three transcendental constants: the modulus of the elliptic functions, the time scaling factor and the additive constant in the argument of the elliptic functions. Klein and Sommerfeld call these constants transcendental in contrast to elementary physical constants which represent the energy and the angular momenta of the motion [5].

Among the three top motions M , M_1 , M_2 involved in Jacobi's decomposition theorem, the equality of the three moduli and that of the three time scaling factors are obvious because the elliptic functions with different moduli represent completely different functions and the different time scaling factors would make the concordant repetition of the periodic motions of the three tops impossible. Jacobi found a simple addition relation (sum and difference) for the third type of transcendental constants (the additive constants) $p = q_1 \pm q_2$.

There is a possibility that the three sets of physical constants of the three top motions may be connected by some simple relations due to the following two observations:

1. The squared modulus k^2 of the elliptic functions is the cross ratio of the four physical constants and the constancy of the cross ratio under a projective transformation (fractional linear transformation $x' = \frac{ax + b}{cx + d}$) gives a hint that these three sets of physical constants may be connected by the projective transformation which will

ensure the equality of the moduli of all the elliptic functions used in the description of the three top motions [7].

2. The formula for the addition theorem of the elliptic functions is fractional (the trigonometric functions are special elliptic functions with zero modulus and the denominator for the trigonometric addition theorem degenerates into unity). The addition relation $p = q_1 \pm q_2$ discovered by Jacobi for the third type of transcendental constants when combined with the fractional addition theorem of the elliptic functions also hints at the projective relation between the three sets of physical constants.

The above two observations indeed lead to the remarkable result: the three sets of physical constants in the decomposition theorem are connected to each other by simple projective transformations. These connection formulas in the form of projective transformation are essential to describe the relative motions of the three tops M , M_1 , and M_2 . Once the motion of a heavy symmetrical top is given (initial condition), the physical constants of its composite motion, that is, the motions of the two free triaxial tops, can be immediately determined by the connection formulas. Further, since all the constants appearing in the expressions of the three sets of Euler angles which are functions of time are these physical and transcendental constants, the three sets of Euler angles are completely determined.

The method of geometrical analysis of the decomposition theorem is employed in this research. The principal axes (x_1, y_1, z_1) of a free triaxial top and the principal axes (x_2, y_2, z_2) of a heavy symmetrical top are rotating with respect to the frame of reference (x, y, z) fixed

in space (the z axis is antiparallel to the gravity). The three origins of the three coordinate systems meet together at a common point which is fixed in time. The motion of the free triaxial top observed from the fixed inertial frame (x, y, z) is called the 1st free triaxial top and the same motion observed from the frame of the heavy symmetrical top, which is the motion of another free triaxial top, is called the 2nd free triaxial top. The principal axes of these two free triaxial top are the same, i.e., both are (x_1, y_1, z_1) . A sphere with an arbitrary radius is drawn centered at the common origin of the three coordinate systems. The three planes $(x-y)$, (x_1-y_1) and (x_2-y_2) will cut the surface of the sphere forming, in general, a spherical triangle on the surface. The three $(x-y)$ planes intersect each other and these three intersecting lines become three lines of nodes. These are the lines drawn from the center of the sphere to the three vertices of the spherical triangle.

Three sets of Euler's angles can be assigned with respect to this spherical triangle. Since all the Euler angles are functions of time, this spherical triangle moves along the $(x-y)$ plane of the inertial frame, which is an invariable plane, and its shape and size vary as the time goes on. As mentioned earlier, since a given heavy top motion (initial condition) will determine all the three sets of Euler angles of the three top motions when the connection formulas are used, both qualitative and quantitative observation of the three top motions are possible through the observation of the variation of the spherical triangle. One salient and important feature of this geometrical method is that the spherical triangle collapses periodically and at these

moments, all the three nutations stop momentarily and the three lines of nodes degenerate into a single line on the invariable plane (x-y). This indicates that the line of nodes of the heavy symmetrical top and that of the 1st free triaxial top have a race on the invariable plane, that is, when one is accelerated, the other is decelerated and vice versa. Whenever one catches up with the other, the spherical triangle collapses. This observation gives a conclusion that the motion of the heavy symmetrical top and that of the 1st free triaxial top are out of phase and the motions of the two free triaxial tops are in phase.

When the motion of the 1st free triaxial top is observed from the rotating frame of the heavy symmetrical top (the angular velocity $\vec{\omega}$ is not constant!), there is no guarantee that this relative motion represents the motion of another torque-free triaxial top because there is no knowing whether or not this relative motion satisfies the Euler's equations. The conditions which guarantee that the relative motion represents the motion of a free triaxial top will give the fundamental requirements for the decomposition theorem and these conditions are easily obtained from the observation of the motion of the spherical triangle.

CHAPTER II

SUMMARY OF THE PERTINENT FORMULAS DESCRIBING THE INSTANTANEOUS POSITIONS OF THE TOPS

1. A Heavy Symmetrical Top [1,2]

Denote the principal moments of inertia of a rigid body by (I_1, I_2, I_3) with respect to the principal axes (x_2, y_2, z_2) . Then for the motion of a symmetrical top $(I_1 = I_2)$ with one point fixed under the action of the uniform gravitational force, the total energy E of the top motion is given by [1]

$$E = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} \omega_{z_2}^2 + Mg\ell \cos \theta \quad (2.1)$$

where θ, ϕ, ψ represent the Euler angles specifying the orientation of the rigid body with respect to an inertial frame of reference (x, y, z) fixed in space and the z axis is antiparallel to the gravity. M is the total mass and ℓ is the length between the center of gravity and the fixed point. Eq. (2.1) can be rewritten as [1]

$$E' = E - \frac{I_3}{2} \omega_{z_2}^2 = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + Mg\ell \cos \theta. \quad (2.2)$$

Two constants v and w are defined by

$$v = \left(\frac{2E'}{I_1} \right)^{1/2} \quad \text{and} \quad w = \left(\frac{2Mg\ell}{I_1} \right)^{1/2} \quad (2.3)$$

where both v and w have the dimension of angular velocity. In terms of v and w , the three dimensionless physical constants of the motion of

the heavy symmetrical top will be [1]

$$\alpha' = \frac{V}{w}, \quad (2.4)$$

a = the angular momentum component along the axis of symmetry (z_2) divided by wI_1 , (2.5)

b = the angular momentum component along the space z axis (antiparallel to the gravity) divided by wI_1 . (2.6)

It is always understood that any angular velocity is divided by w to make it dimensionless.

The expression for the dimensionless angular velocity $\dot{\theta}$ of the nutation is given by [1]

$$\dot{\theta}^2 \sin^2 \theta = (1 - \cos^2 \theta)(\alpha' - \cos \theta) - (b - a \cos \theta)^2. \quad (2.7)$$

If we put $x = \cos \theta$, Eq. (2.7) becomes a cubic expression in x

$$\begin{aligned} \dot{x}^2 &= (1 - x^2)(\alpha' - x) - (b - ax)^2 = (x - x_1)(x - x_2)(x - x_3) \\ &= f(x) \end{aligned} \quad (2.8)$$

where x_1, x_2, x_3 are the roots of the cubic equation $f(x) = 0$. It can be shown that the three roots are all real and [1]

$$x_1 > 1 > x_2 > x_3 > -1 \quad (2.9)$$

and the physical range of $x (= \cos \theta)$ is

$$x_2 \leq x \leq x_3. \quad (2.10)$$

The cubic equation $f(x) = 0$ has an interesting structure in it; when a and b are interchanged in Eq. (2.8) and if α' is replaced by α which is given by

$$\alpha = a^2 - b^2 + \alpha' \quad (2.11)$$

then it can be shown that the three roots x_1, x_2, x_3 will still remain unchanged. That is, Eq. (2.8) can be rewritten as

$$f(x) = (1 - x^2)(\alpha - x) - (a - bx)^2 = (x - x_1)(x - x_2)(x - x_3) \quad (2.12)$$

with the same set of the three roots.

The period T of the heavy top motion is given by [5]

$$T = \frac{2K}{n} \quad (2.13)$$

where n , the time scaling factor, is defined by [5]

$$n = \frac{1}{2} (x_1 - x_3)^{1/2} . \quad (2.14)$$

K in (2.13) is the value of the complete elliptic integral of the first kind [8]

$$K = \int_0^{\pi/2} \frac{dy}{(1 - k^2 \sin^2 y)^{1/2}} \quad (2.15)$$

where k is the modulus [5]

$$k^2 = \frac{x_2 - x_3}{x_1 - x_3} . \quad (2.16)$$

In order to express the Euler angles ϕ and ψ as explicit functions of time, Jacobi introduced the following two constants q_1 and q_2 which are expressed in terms of the three roots x_1, x_2, x_3 through the elliptic function sn [5]

$$\text{sn}(iq_1) = i \left(\frac{x_1 - 1}{1 - x_2} \right)^{1/2}, \quad \text{sn}(iq_2) = i \left(\frac{x_1 - x_3}{1 + x_3} \right)^{1/2}. \quad (2.17)$$

Since $\text{cn}^2 = 1 - \text{sn}^2$ and $\text{dn}^2 = 1 - k^2 \text{sn}^2$, (2.17) gives

$$\text{cn}(iq_1) = \left(\frac{x_1 - x_2}{1 - x_2} \right)^{1/2}, \quad \text{cn}(iq_2) = \left(\frac{1 + x_1}{1 + x_3} \right)^{1/2} \quad (2.18)$$

$$\text{dn}(iq_1) = \left(\frac{(x_1 - x_2)(1 - x_3)}{(x_1 - x_3)(1 - x_2)} \right)^{1/2}, \quad \text{dn}(iq_2) = \left(\frac{1 + x_2}{1 + x_3} \right)^{1/2} \quad (2.19)$$

where (2.16) has been used for (2.19). In accordance with Klein and Sommerfeld, the constants n (the time scaling factor), k (the modulus), q_1 and q_2 introduced above are called "transcendental constants" [6]. While the constants $a, b, \alpha'(\alpha), x_1, x_2$ and x_3 , all of which appear in the cubic expressions (2.8) and (2.12) are called "physical constants."

In terms of these constants, the three Euler angles θ, ϕ, ψ at any instant can be expressed [5]

$$\cos \theta = x_3 + (x_2 - x_3) \text{sn}^2(u) \quad (2.20)$$

$$\phi = (\mu_2 \pm \mu_1)u + [\lambda_2(u) \pm \lambda_1(u)] \quad (2.21)$$

$$\psi = \left[\left(\frac{I_1 - I_3}{I_3} \right) \frac{a}{n} + (\mu_2 \mp \mu_1) \right] u + [\lambda_2(u) \mp \lambda_1(u)] \quad (2.22)$$

where

$$\mu_1 = \frac{d}{dq_1} \ln H_1(iq_1), \quad \mu_2 = \frac{d}{dq_2} \ln H(iq_2) \quad (2.23)$$

$$\lambda_1(u) = \frac{1}{2i} \ln \frac{\theta_1(u + iq_1)}{\theta_1(u - iq_1)}, \quad \lambda_2(u) = \frac{1}{2i} \ln \frac{\theta(u + iq_2)}{\theta(u - iq_2)} \quad (2.24)$$

$$u = nt \quad (t = \text{time}) . \quad (2.25)$$

H , H_1 , θ and θ_1 are the Jacobian theta functions. The constants q_1 and q_2 appear in the arguments of these theta functions as shown above. u is the variable proportional to the time as shown in (2.25) and because of this, the constant n is called "time scaling factor." μ_1 and μ_2 are constants but λ_1 and λ_2 expressed by (2.24) are functions of u and these are periodic functions. The signs appearing in Eqs. (2.21) and (2.22) are determined according to

$a < b$ for the upper sign,

$a > b$ for the lower sign.

In order to see the structures of Eqs. (2.21) and (2.22), it may be convenient to write them as

$$\phi = S_1 u + f_1(u) \quad (2.26)$$

$$\psi = S_2 u + f_2(u) \quad (2.27)$$

where

$$S_1 = \mu_2 \pm \mu_1 = \text{constant} \quad (2.28)$$

$$S_2 = \left(\frac{I_1 - I_3}{I_3} \right) \frac{a}{n} + (\mu_2 \mp \mu_1) = \text{constant} \quad (2.29)$$

$$f_1(u) = \lambda_2(u) \pm \lambda_1(u) \quad (2.30)$$

$$f_2(u) = \lambda_2(u) \mp \lambda_1(u) \quad (2.31)$$

From Eqs. (2.26) and (2.27), clearly the Euler angles ϕ and ψ consist of two terms: a secular term which is proportional to time and a periodic term. Because of the secular terms, the angles ϕ and ψ increase indefinitely and this corresponds to "rotation." While cosine of θ , which is expressed by (2.20), oscillates between x_2 and x_3 and never increases indefinitely in contrast to ϕ and ψ : this observation corresponds to Eq. (2.10) and this motion is so-called "nutation."

2. A Torque-Free Triaxial Top [2,3]

Denote the principal moments of inertia of a torque-free triaxial top by (A_1, B_1, C_1) with respect to the principal axes x_1, y_1, z_1 , which are rotating with respect to an inertial frame of reference (x, y, z) fixed in space where there is no external force field. The orientation of the top relative to the reference frame (x, y, z) is specified by Euler angles θ_1, ϕ_1, ψ_1 .

The four physical constants needed for the description of the motion of the free triaxial top are [5]

$$\frac{\ell_1}{A_1}, \quad \frac{\ell_1}{B_1}, \quad \frac{\ell_1}{C_1}, \quad \frac{h_1}{\ell_1}$$

where ℓ_1 is the magnitude of the total angular momentum and h_1 is twice the total energy (the kinetic energy). Each of these constants has the dimension of angular velocity. In the decomposition theorem, since the motion of a heavy symmetrical top is connected with the motions of two free triaxial tops, each of the above constants will be divided by w

which is defined by (2.3) to make it dimensionless as was done in the case of the heavy symmetrical top. Therefore, it is always understood that the four physical constants given above have already been divided by w and they are always dimensionless. It should be understood that the physical constants of the motion of a free triaxial top can be divided by w only in view of the decomposition theorem since w is the quantity defined in the motion of the heavy symmetrical top, not the free triaxial top. For the sake of convenience, the four physical constants are written as

$$s_0 = \frac{h_1}{\ell_1}, \quad s_1 = \frac{\ell_1}{C_1}, \quad s_2 = \frac{\ell_1}{A_1}, \quad s_3 = \frac{\ell_1}{B_1}. \quad (2.32)$$

The period T_1 of the motion is given by [5]

$$T_1 = \frac{2K_1}{n_1} \quad (2.33)$$

where n_1 , the time scaling factor, is defined by [5]

$$n_1 = [(s_0 - s_2)(s_1 - s_3)]^{1/2} \quad (2.34)$$

K_1 in (2.33) is the complete elliptic integral [8]

$$K_1 = \int_0^{\frac{\pi}{2}} \frac{dy}{(1 - k_1^2 \sin^2 y)^{1/2}}. \quad (2.35)$$

k_1 is the modulus given by [5]

$$k_1^2 = \frac{(s_3 - s_2)(s_1 - s_0)}{(s_0 - s_2)(s_1 - s_3)} \quad (2.36)$$

In order to express the Euler angle ϕ_1 (but not ψ_1) of the free triaxial top as an explicit function of time, Jacobi introduced a constant p_1 [5]

$$\operatorname{sn}(ip_1) = i \left(\frac{s_0 - s_2}{s_1 - s_0} \right)^{1/2}. \quad (2.37)$$

Since $\operatorname{cn}^2 = 1 - \operatorname{sn}^2$ and $\operatorname{dn}^2 = 1 - k_1^2 \operatorname{sn}^2$, (2.37) leads to

$$\operatorname{cn}(ip_1) = \left(\frac{s_1 - s_2}{s_1 - s_0} \right)^{1/2} \quad (2.38)$$

$$\operatorname{dn}(ip_1) = \left(\frac{s_1 - s_2}{s_1 - s_3} \right)^{1/2} \quad (2.39)$$

where (2.36) has been used for (2.39).

Since two torque-free triaxial tops are involved in the decomposition theorem, the four physical constants for the motion of the 2nd free triaxial top are defined as

$$s'_0 = \frac{h_2}{\ell_2}, \quad s'_1 = \frac{\ell_2}{C_2}, \quad s'_2 = \frac{\ell_2}{A_2}, \quad s'_3 = \frac{\ell_2}{B_2}. \quad (2.40)$$

Like the case of the first free triaxial top, the period T_2 , the time scaling factor n_2 , the complete integral K_2 , the modulus k_2 and the additive constant p_2 are given by [5]

$$T_2 = \frac{2K_2}{n_2} \quad (2.41)$$

$$n_2 = [(s'_0 - s'_2)(s'_1 - s'_3)]^{1/2} \quad (2.42)$$

$$k_2^2 = \frac{(s_3' - s_2')(s_1' - s_0')}{(s_0' - s_2')(s_1' - s_3')} \quad (2.43)$$

$$\operatorname{sn}(ip_2) = i \left(\frac{s_0' - s_2'}{s_1' - s_0'} \right)^{1/2} \quad (2.44)$$

$$\operatorname{cn}(ip_2) = \left(\frac{s_1' - s_2'}{s_1' - s_0'} \right)^{1/2} \quad (2.45)$$

$$\operatorname{dn}(ip_2) = \left(\frac{s_1' - s_2'}{s_1' - s_3'} \right)^{1/2} . \quad (2.46)$$

Again the constants $n_1, k_1, p_1, n_2, k_2, p_2$ are called "transcendental constants."

In terms of the physical constants and the transcendental constants, the Euler angles of the torque-free triaxial tops are expressed as functions of time:

The first free triaxial top [2,3,5]

$$\cos \theta_1 = P_1 \operatorname{dn}(u_1) \quad (2.47)$$

$$\phi_1 = M_1 u_1 \pm g_1(u_1) \quad (2.48)$$

$$\tan \psi_1 = Q_1 \frac{\operatorname{cn}(u_1)}{\operatorname{sn}(u_1)} \quad (2.49)$$

where

$$P_1 = \left(\frac{s_0 - s_2}{s_1 - s_2} \right)^{1/2} \quad (2.50)$$

$$M_1 = \frac{1}{s_1 - s_2} \left[s_2 \frac{d}{dp_1} \ln H(ip_1) - s_1 \frac{d}{dp_1} \ln \Theta(ip_1) \right] \quad (2.51)$$

$$g_1(u) = \frac{1}{2i} \ln \frac{\Theta(u_1 + ip_1)}{\Theta(u_1 - ip_1)} \quad (2.52)$$

$$Q_1 = \left(\frac{s_1 - s_3}{s_1 - s_2} \right)^{1/2} \quad (2.53)$$

$$u_1 = n_1 t \quad (2.54)$$

ϕ_1 in (2.48) consists of the secular term $M_1 u_1$, where M_1 is a constant given by (2.51), and the periodic term $g_1(u_1)$ which is given by (2.52). $\cos \theta_1$ in (2.47) varies purely periodically (nutaton). From (2.49), it can be shown that ψ_1 (the spin angle) actually also consists of a secular term and a periodic term which can be seen from the property of the "tangent."

The second free triaxial top

$$\cos \theta_2 = P_2 \operatorname{dn}(u_2) \quad (2.55)$$

$$\phi_2 = M_2 u_2 \pm g_2(u_2) \quad (2.56)$$

$$\tan \psi_2 = Q_2 \frac{\operatorname{cn}(u_2)}{\operatorname{sn}(u_2)} \quad (2.57)$$

where

$$P_2 = \left(\frac{s'_0 - s'_2}{s'_1 - s'_2} \right)^{1/2} \quad (2.58)$$

$$M_2 = \frac{1}{s'_1 - s'_2} \left[s'_2 \frac{d}{dp_2} \ln H(ip_2) - s'_1 \frac{d}{dp_2} \ln \Theta(ip_2) \right] \quad (2.59)$$

$$g_2(u_2) = \frac{1}{2i} \ln \frac{\Theta(u_2 + ip_2)}{\Theta(u_2 - ip_2)} \quad (2.60)$$

$$Q_2 = \left(\frac{s_1' - s_3'}{s_1' - s_2'} \right)^{1/2} \quad (2.61)$$

$$u_2 = n_2 t. \quad (2.62)$$

The nature of the structure of the 2nd free triaxial top is exactly the same as that of the 1st free triaxial top. However, at this stage, the above two free triaxial tops have no relation to each other, that is, they are two independent free triaxial tops. The expressions with the same forms as the above expressions can be obtained; even the physical constants are not divided by w . The expressions introduced in this chapter will be used for the following chapters.

CHAPTER III

DERIVATION OF THE PROJECTIVE TRANSFORMATIONS CONNECTING THE THREE SETS OF PHYSICAL CONSTANTS

In the preceding chapter, the expression of Euler angles as functions of time for each top motion have been presented in their explicit form. Further, it is clearly presented that each expression is expressed in terms of the physical constants and the transcendental constants corresponding to each top motion. Since according to the decomposition theorem, the given motion of a heavy symmetrical top can be decomposed into the motions of two free triaxial tops, there must be some definite relations between the three sets of transcendental and physical constants. When one finds these relations, then all the three sets of Euler angles describing the three top motions can be completely known and instantaneous relative positions of the three tops can be specified.

Because of the concordant periodic motions of the three tops, the periods of the three motions must be the same and further the nature of all the elliptic functions describing the three top motions must also be the same, that is, the three moduli must be equal. Then the fundamental requirements for the decomposition theorem are

$$k = k_1 = k_2 \tag{3.1}$$

$$n = n_1 = n_2 \tag{3.2}$$

which will also be obtained from a pure geometrical consideration which is discussed in the next chapter. The expressions of the three squared moduli

$$k^2 = \frac{x_2 - x_3}{x_1 - x_3}$$

$$k_1^2 = \frac{(s_3 - s_2)(s_1 - s_0)}{(s_0 - s_2)(s_1 - s_3)}$$

$$k_2^2 = \frac{(s_3' - s_2')(s_1' - s_0')}{(s_0' - s_2')(s_1' - s_3')}$$

indicate that they are cross ratios of (x_1, x_2, x_3) , (s_0, s_1, s_2, s_3) and (s_0', s_1', s_2', s_3') . There is a theorem which states that the cross ratio of four numbers is invariant under a projective transformation (fractional linear transformation) [7]. Therefore the following projective transformation between s and x may well be considered [7]:

$$s = \frac{Ax + B}{Cx + D} \quad (3.3)$$

where A, B, C, D are constants. The transformation (3.3) carries the four numbers (∞, x_1, x_2, x_3) into the four numbers (s_0, s_1, s_2, s_3) and automatically ensures the equality of the two squared moduli $k^2 = k_1^2$ (and $k = k_1$ since they are positive quantities). Similarly the equality $k^2 = k_2^2$ will be guaranteed by the following transformation:

$$s' = \frac{A'x + B'}{C'x + D'} \quad (3.4)$$

which carries the four numbers (∞, x_1, x_2, x_3) into the four numbers (s_0', s_1', s_2', s_3') .

The goal is to find the explicit forms of the projective transformations (3.3) and (3.4) since they represent the connection formulas between the physical constants of the three top motions. It is expected that all the constants in (3.3) and (3.4) are expressed in terms of the remaining physical constants $a, b, \alpha(\alpha')$ of the heavy symmetrical top. From (3.3) and (3.4), it can be shown that $x = x_0 = \infty$ is transformed into $s = s_0$ by (3.3) and the same point $x = x_0$ is transformed into $s' = s'_0$ by (3.4). Therefore (3.3) and (3.4) can be rewritten as

$$s - s_0 = \frac{BC - AD}{C(Cx + D)} \quad \text{with } s_0 = \frac{A}{C} \quad (3.5)$$

$$s' - s'_0 = \frac{B'C' - A'D'}{C'(C'x + D')} \quad \text{with } s'_0 = \frac{A'}{C'}. \quad (3.6)$$

The right hand side of (3.5) can also be expressed in terms of the transcendental constant p_1 which is defined by (2.37). This can be done by algebraic manipulation of (2.34), (2.36), (2.37) through (2.39).

The results are [10].

$$\left. \begin{aligned} s_1 - s_0 &= \pm n_1 \frac{i \operatorname{dn}(ip_1)}{\operatorname{sn}(ip_1) \operatorname{cn}(ip_1)} \\ s_2 - s_0 &= \pm n_1 \frac{\operatorname{sn}(ip_1) \operatorname{dn}(ip_1)}{i \operatorname{cn}(ip_1)} \\ s_3 - s_0 &= \pm n_1 \frac{k_1'^2 \operatorname{sn}(ip_1)}{i \operatorname{cn}(ip_1) \operatorname{dn}(ip_1)} \end{aligned} \right\} \quad (3.7)$$

where $k_1'^2 = 1 - k_1^2$. Similarly the right hand side of (3.6) can also be expressed in terms of the transcendental constant p_2 using Eqs. (2.42) through (2.46) and the expressions similar to (3.7) can be obtained.

Since the structural nature of the two free triaxial tops is the same, only the derivation of the transformation (3.3) will be considered, that is, the connection formulas between the physical constants of the heavy symmetrical top and of the 1st free triaxial top. To pursue the object, it is necessary to know the connections between the physical constants of the heavy symmetrical top alone.

From either (2.8) or (2.12), by putting $x = \pm 1$, the following two expressions will be obtained:

$$a - b = \pm [(x_1 - 1)(1 - x_2)(1 - x_3)]^{1/2} \quad (3.8)$$

$$a + b = [(x_1 + 1)(1 + x_2)(1 + x_3)]^{1/2} \quad (3.9)$$

Putting $x = \alpha'$, $x = x_1$ and $x = x_2$ separately in (2.8) gives

$$b - a\alpha' = \pm [(x_1 - \alpha')(\alpha' - x_2)(\alpha' - x_3)]^{1/2} \quad (3.10)$$

$$ax_1 - b = \pm [(x_1^2 - 1)(x_1 - \alpha')]^{1/2} \quad (3.11)$$

$$b - ax_2 = \pm [(1 - x_2^2)(\alpha' - x_2)]^{1/2} \quad (3.12)$$

Also putting $x = \alpha$, $x = x_1$ and $x = x_2$ separately in (2.12) gives

$$a - b\alpha = \pm [(x_1 - \alpha)(\alpha - x_2)(\alpha - x_3)]^{1/2} \quad (3.13)$$

$$bx_1 - a = \pm [(x_1^2 - 1)(x_1 - \alpha)]^{1/2} \quad (3.14)$$

$$a - bx_2 = \pm [(1 - x_2^2)(\alpha - x_2)]^{1/2} . \quad (3.15)$$

Since all the quantities in the above expressions are real and because of the inequality relations (2.9), the expressions (3.11), (3.12), (3.14) and (3.15) will give

$$\left. \begin{array}{l} x_1 > \alpha' > x_2 > x_3 \\ x_1 > \alpha > x_2 > x_3 \end{array} \right\} \quad (3.16)$$

It is already shown that there exist connections between the transcendental constants of the three top motions as shown by (3.1) and (3.2). There exist two more sets of connection formulas for the transcendental constants q_1, q_2, p_1, p_2 all of which were defined in Chapter II and these constants are called "additive constants" since they appear as additive constants in the arguments of the Jacobian theta functions as in (2.24), (2.52) and (2.60). The connection formulas between these constants were discovered by Jacobi and they are given by [5]

$$p_1 = q_1 \pm q_2 . \quad (3.17)$$

$$p_2 = q_1 \mp q_2 \quad (3.18)$$

The only difference in the above two expressions lies in the choice of sign. Although the proof will not be given here, from the observation of the following algebra, it can be easily shown that the expressions (3.17) and (3.18) can be exchanged if the physical constants a and b are interchanged and this depends on $a > b$ or $a < b$ which affect the signs of (3.17) and (3.18). The following algebra is based on:

$$p_1 = q_1 + q_2 . \quad (3.19)$$

The application of the addition theorem of the elliptic function sn to (3.19) gives [8]

$$\operatorname{sn}(ip_1) = \operatorname{sn}(iq_1 + iq_2)$$

$$= \frac{\operatorname{sn}(iq_1) \operatorname{cn}(iq_2) \operatorname{dn}(iq_2) + \operatorname{sn}(iq_2) \operatorname{cn}(iq_1) \operatorname{dn}(iq_1)}{1 - k^2 \operatorname{sn}^2(iq_1) \operatorname{sn}^2(iq_2)}. \quad (3.20)$$

Substituting (2.17) through (2.19) into (3.20), one obtains

$$\begin{aligned} \operatorname{sn}(ip_1) &= i \frac{[(x_1^2 - 1)(1 - x_2^2)]^{1/2} + (x_1 - x_2)(1 - x_3^2)^{1/2}}{1 - x_1x_2 + x_1x_3 - x_2x_3} \\ &= i \frac{[(x_1 - 1)(1 + x_2)(1 + x_3)]^{1/2} + [(x_1 + 1)(1 - x_2)(1 - x_3)]^{1/2}}{[(x_1 + 1)(1 - x_2)(1 + x_3)]^{1/2} - [(x_1 - 1)(1 + x_2)(1 - x_3)]^{1/2}} \\ &= i \frac{\left(\frac{x_1 - 1}{x_1 + 1}\right)^{1/2} (a + b) + \left(\frac{x_1 + 1}{x_1 - 1}\right)^{1/2} (a - b)}{\left(\frac{1 - x_2}{1 + x_2}\right)^{1/2} (a + b) - \left(\frac{1 + x_2}{1 - x_2}\right)^{1/2} (a - b)} \\ &= i \left(\frac{1 - x_2^2}{x_1^2 - 1}\right)^{1/2} \frac{bx_1 - a}{a - bx_2} \end{aligned} \quad (3.21)$$

in which (3.8) and (3.9) have been used. Substituting (3.14) and (3.15) into (3.21), one obtains

$$\operatorname{sn}(ip_1) = i \left(\frac{x_1 - \alpha}{\alpha - x_2}\right)^{1/2}. \quad (3.22)$$

Since $\operatorname{cn}^2 = 1 - \operatorname{sn}^2$, $\operatorname{dn}^2 = 1 - k^2 \operatorname{sn}^2$ and $k^2 = \frac{x_2 - x_3}{x_1 - x_3}$, (3.22) leads to

$$\operatorname{cn}(ip_1) = \left(\frac{x_1 - x_2}{\alpha - x_2}\right)^{1/2} \quad (3.23)$$

$$\operatorname{dn}(ip_1) = \left[\frac{(x_1 - x_2)(\alpha - x_3)}{(x_1 - x_3)(\alpha - x_2)} \right]^{1/2} \quad (3.24)$$

From (2.14) and (3.2), one has

$$n_1 = n = \frac{1}{2} (x_1 - x_3)^{1/2}. \quad (3.25)$$

From (2.16) and (3.1), one has

$$k_1'^2 = k'^2 = 1 - k^2 = \frac{x_1 - x_2}{x_1 - x_3}. \quad (3.26)$$

Substituting (3.22) through (3.26) into (3.7), one obtains

$$s_i - s_0 = \frac{a - bx_i}{2(\alpha - x_i)}, \quad i = 1, 2, 3. \quad (3.27)$$

where (3.13) has been used. Dropping the subscript i , (3.27) can be written

$$s - s_0 = \frac{a - bx}{2(\alpha - x)}. \quad (3.28)$$

Comparing (3.28) with (3.5), one can put

$$\frac{BC - AD}{C(Cx + D)} = \frac{a - bx}{2(\alpha - x)}, \quad s_0 = \frac{A}{C}.$$

Since this relation must hold for any x , one obtains

$$s_0 = \frac{A}{C} = \frac{b}{2}. \quad (3.29)$$

Thus (3.27) can be written

$$s_i = \frac{b}{2} + \frac{a - bx_i}{2(\alpha - x_i)} = \frac{a - bx_i}{2(\alpha - x_i)} \quad (3.30)$$

where the subscript index i runs 0, 1, 2, 3. $x_0 = \infty$ is transformed into $s_0 = \frac{b}{2}$ by (3.30). Dropping the subscript i , (3.30) can be written

$$s = \frac{a - bx}{2(\alpha - x)} \quad (3.31)$$

which has the same form as (3.3) and the desired projective transformation between s_i and x_i has been obtained. Eq. (3.30) is the connection formula between the physical constants of the heavy symmetrical top and those of the 1st free triaxial top.

To obtain the connection formulas between the physical constants of the heavy symmetrical top and those of the 2nd free triaxial top, that is, the connection between $(x_1, x_2, x_3, a, b, \alpha')$ and (s'_0, s'_1, s'_2, s'_3) , the addition theorem of the elliptic function sn will be applied to $p_2 = q_1 - q_2$ and the expressions (3.10) through (3.12) are used. The procedure is exactly the same as the one used for the derivation of (3.27) through (3.31). The results are

$$s'_i - s'_0 = \frac{b - a\alpha'}{2(\alpha' - x_i)}, \quad i = 1, 2, 3. \quad (3.32)$$

$$s'_0 = \frac{a}{2} \quad (3.33)$$

$$s'_i = \frac{b - ax_i}{2(\alpha' - x_i)}, \quad i = 0, 1, 2, 3. \quad (3.34)$$

$$s' = \frac{b - ax}{2(\alpha' - x)} \quad (3.35)$$

(3.35) has the same form as (3.4). $x_0 = \infty$ is carried into $s'_0 = \frac{a}{2}$ by (3.34). It is interesting to note that all the above expressions,

(3.32) through (3.35), can be obtained by merely interchanging a and b , and replacing α by α' in the expressions (3.27) through (3.31).

If the original symbols for the physical constants of the two free triaxial tops are used referring to (2.32) and (2.40), the transformation formulas (3.30) and (3.34) can be expressed explicitly:

$$\left. \begin{aligned} s_2 &= \frac{\ell_1}{A_1} = \frac{a - bx_2}{2(\alpha - x_2)} & s'_2 &= \frac{\ell_2}{A_2} = \frac{b - ax_2}{2(\alpha' - x_2)} \\ s_3 &= \frac{\ell_1}{B_1} = \frac{a - bx_3}{2(\alpha - x_3)} & s'_3 &= \frac{\ell_2}{B_2} = \frac{b - ax_3}{2(\alpha' - x_3)} \\ s_1 &= \frac{\ell_1}{C_1} = \frac{a - bx_1}{2(\alpha - x_1)} & s'_1 &= \frac{\ell_2}{C_2} = \frac{b - ax_1}{2(\alpha' - x_1)} \\ s_0 &= \frac{h_1}{\ell_1} = \frac{b}{2} & s'_0 &= \frac{h_2}{\ell_2} = \frac{a}{2} \end{aligned} \right\} \quad (3.36)$$

Thus it has been shown that the two sets of physical constants of the two free triaxial top motions are connected to the physical constants of the heavy top motion in the form of projective transformation.

Here is one important remark on the result: In the motion of a free triaxial top, if A, B, C represent the principal moments of inertia, h represents twice the kinetic energy and ℓ the magnitude of the total angular momentum, then it can be shown that for any possible motion of the free top, the parameter h/ℓ must lie between ℓ/A and ℓ/C for $A > B > C$ or $A < B < C$, otherwise no physical motion is possible [2, 3, 5]. The results shown in (3.36) meet this criterion: The expressions in (3.27) have the common numerator $(a - b\alpha)$ and also the expressions in

(3.32) have the common numerator $(b - a\alpha')$. Then the inequality relations in (3.16) guarantee the inequality relations $s_2 > s_3 > s_0 > s_1$ for $a - b\alpha > 0$ (or the opposite order of sequence if $a - b\alpha < 0$) and $s'_2 > s'_3 > s'_0 > s'_1$ for $b - a\alpha' > 0$ (or the opposite order of sequence if $b - a\alpha' < 0$). If (2.32) and (2.40) are used, the above argument can be written

$$\left. \begin{aligned} \frac{\ell_1}{A_1} &> \frac{\ell_1}{B_1} > \frac{h_1}{\ell_1} > \frac{\ell_1}{C_1} \text{ or the opposite sequence} \\ \frac{\ell_2}{A_2} &> \frac{\ell_2}{B_2} > \frac{h_2}{\ell_2} > \frac{\ell_2}{C_2} \text{ or the opposite sequence} \end{aligned} \right\} \quad (3.37)$$

This result (3.37) certainly meets the criterion.

CHAPTER IV

GEOMETRICAL ANALYSIS OF THE DECOMPOSITION THEOREM

Consider a set of three Cartesian coordinate systems expressed by the following three column matrices (vectors):

$$\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \vec{X}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \vec{X}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$\vec{X}$ represents an inertial frame of reference fixed in space, $\vec{X}_1$ represents the frame of a torque-free triaxial top (the principal axes) and $\vec{X}_2$ is the frame of a heavy symmetrical top (the principal axes). The z axis of the inertial frame is antiparallel to the gravity in view of the motion of the heavy symmetrical top. The coordinate origin is common to all the three coordinate systems and fixed in time.

Now, when the rotation of the frame $\vec{X}_1$ is observed from the inertial frame $\vec{X}$, the motion of $\vec{X}_1$ can be written as [1]

$$\vec{X} = M_1 \vec{X}_1 \tag{4.1}$$

where M_1 is a 3 x 3 orthogonal matrix whose nine elements are the direction cosines expressed in terms of the Euler angles θ_1 , ϕ_1 , ψ_1 which are expressed as functions of time as in (2.47), (2.48) and (2.49). When the rotation of the frame $\vec{X}_2$ is observed from the inertial frame $\vec{X}$, the motion of $\vec{X}_2$ is

$$\vec{X} = M \vec{X}_2 \quad (4.2)$$

where M stands for a 3×3 orthogonal matrix whose nine elements are the direction cosines expressed in terms of the Euler angles θ , ϕ , ψ which are expressed as functions of time as in (2.20), (2.21) and (2.22).

Now, if the rotation of the frame $\vec{X}_1$ (the torque-free triaxial top) is observed from the frame of the heavy symmetrical top $\vec{X}_2$ which itself is rotating, this relative motion can still be expressed in matrix form

$$\vec{X}_2 = M_2 \vec{X}_1 \quad (4.3)$$

where M_2 represents a 3×3 orthogonal matrix whose nine elements are the direction cosines of $\vec{X}_1$ with respect to the rotating coordinates $\vec{X}_2$ (the observer is fixed to the heavy symmetrical top) and they are expressed in terms of the 3rd set of Euler angles θ_2 , ϕ_2 and ψ_2 which specify the orientation of the coordinate system $\vec{X}_1$ relative to the frame of the heavy symmetrical top $\vec{X}_2$. Because of the orthogonality of the matrix M_2 , (4.3) can be inverted as

$$\vec{X}_1 = \tilde{M}_2 \vec{X}_2 \quad (4.4)$$

where $\tilde{M}_2$ represents the transposed form of M_2 . By combining (4.1), (4.2) and (4.4), one obtains

$$M = M_1 \tilde{M}_2. \quad (4.5)$$

Therefore, the matrix M which describes the motion of a heavy symmetrical top can be always expressed as a product of two matrices M_1 and $\tilde{M}_2$.

Although the matrix M_1 represents the motion of a torque-free triaxial top, its relative motion M_2 observed from the frame of the rotating heavy symmetrical top does not necessarily represent the motion of a torque-free triaxial top because there is no guarantee that this relative motion M_2 satisfies the Euler's equations (remember the angular velocity of the heavy symmetric top is not constant). Then the problem is under what conditions both matrices M_1 and M_2 represent the motions of two torque-free triaxial tops for the given motion of a heavy symmetrical top M .

Consider a sphere of an arbitrary radius whose center is chosen at the common origin of the three coordinate systems $\vec{X}$, $\vec{X}_1$ and $\vec{X}_2$. The three planes $(x-y)$, (x_1-y_1) and (x_2-y_2) will cut the surface of the sphere. The three cutting lines which are great circles on the surface of the sphere will form eight spherical triangles in general. One particular spherical triangle is chosen so that the inclinations of the two planes (x_1-y_1) and (x_2-y_2) with respect to the invariable plane $(x-y)$ correspond to the inclinations of the z_1 axis and the z_2 axes with respect to the fixed z axis respectively. Figure 4.1 shows such a spherical triangle. The three planes (x_1-y_1) , (x_2-y_2) and $(x-y)$ intersect each other and these intersecting lines form three lines of nodes which are the lines drawn from the center of the sphere to each vertex of the spherical triangle A, B, C , that is, $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ are the lines of nodes. The instantaneous positions of the three axes x, x_1, x_2 are also indicated in Figure 4.1. Then the three sets of Euler angles can be assigned with regard to the spherical triangle $\triangle ABC$. By definition, the angles ϕ and ϕ_1 are both measured from the x axis of the inertial

frame counterclockwisely to the lines of nodes $\overline{OC}$ and $\overline{OB}$ respectively and the angle ϕ_2 is measured from the x_2 axis of the heavy symmetrical top to the line of node $\overline{OA}$. Also by definitions, each of the angles ψ_1, ψ_2, ψ is measured from the corresponding line of nodes to the corresponding x axis, i.e., x_1 and x_2 axes. The angle θ is the inclination angle of the (x_2-y_2) plane of the heavy top relative to the invariable plane $(x-y)$, θ_1 is the inclination angle of the (x_1-y_1) plane of the free top relative to also the invariable plane $(x-y)$ and θ_2 is the inclination angle of the (x_1-y_1) plane relative to the moving plane (x_2-y_2) of the heavy top. Then from Figure 4.1, the following expressions can be obtained:

$$\left. \begin{aligned} \widehat{Ax_2} &= -\phi_2 \\ \widehat{Ax_1} &= -\psi_2 \end{aligned} \right\} \quad (4.6)$$

$$\left. \begin{aligned} \angle A &= \theta_2 \\ \angle B &= \pi - \theta_1 \\ \angle C &= \theta \end{aligned} \right\} \quad (4.7)$$

The symbol $\frown$ in (4.6) represents an angle, not an arc length. The angles of the spherical triangle $\triangle ABC$ are expressed in terms of the Euler's angles

$$\widehat{BC} = |\phi - \phi_1| = \text{the angle between the lines of nodes } \overline{OB} \text{ and } \overline{OC} \quad (4.8)$$

$$\widehat{AC} = |\psi + \phi_2| = \text{the angle between the lines of nodes } \overline{OC} \text{ and } \overline{OA} \quad (4.9)$$

$$\widehat{AB} = |\psi_1 - \psi_2| = \text{the angle between the lines of nodes } \overline{OA} \text{ and } \overline{OB}. \quad (4.10)$$

Note that the negative signs in (4.6) are due to the particular positions of the axes x_1 and x_2 as shown in Figure 4.1. However, it can be easily shown that no matter where these axes are, the descriptions of the angles expressed by (4.8) through (4.10) will not be affected as long as the angles measured counterclockwise are positive and the angles measured clockwise are negative.

Now consider the motion of a heavy symmetrical top with principal axes (x_2, y_2, z_2) whose orientations with respect to the inertial frame (x, y, z) are specified the Euler angles (θ, ϕ, ψ) and also consider the motion of a torque-free triaxial top with the principal axes (x_1, y_1, z_1) whose orientations with respect to the inertial frame (x, y, z) are specified by the Euler angles $(\theta_1, \phi_1, \psi_1)$. The origins of these three coordinate systems are common and fixed. At this stage it is assumed that these two top motions are independent from each other. Therefore, there is no knowing whether or not the given motion of the free triaxial top observed from the moving frame (x_2, y_2, z_2) of the heavy symmetrical top represents the motion of another free triaxial top because there is no guarantee that this relative motion satisfies the Euler's equations. Observing Figure 4.1, since the angles θ and θ_1 simply oscillate due to the nutations (but the oscillation is nonlinear), $\dot{\theta} = 0$ and $\dot{\theta}_1 = 0$ must occur periodically, where the "dot . over" represents the time derivative. Note that $\dot{\theta} = 0$ and $\dot{\theta}_1 = 0$ may not occur simultaneously. However, if the relative motion, which is the motion of the free triaxial top observed from the heavy symmetrical top, also represents the

motion of another free triaxial top, since θ_2 becomes the nutation angle of the motion of this "2nd free triaxial top," $\dot{\theta}_2 = 0$ must also occur periodically. But by observing Figure 4.1, it can be seen that $\dot{\theta}_2 = 0$, which occurs instantaneously, is possible only when both $\dot{\theta} = 0$ and $\dot{\theta}_1 = 0$ occur simultaneously. Therefore, if the relative motion represents the motion of another free triaxial top, then at least $\dot{\theta} = 0$, $\dot{\theta}_1 = 0$ and $\dot{\theta}_2 = 0$ must occur simultaneously and periodically. It can be shown that in each top motion, independent of the decomposition theorem, the angular velocity components in Euler angles representation are all expressed as functions of the nutation angle which is purely periodic. Therefore the simultaneity $\dot{\theta} = \dot{\theta}_1 = \dot{\theta}_2 = 0$ implies that there must exist a period which is common to all the three top motions. Further, the simultaneity condition $\dot{\theta} = \dot{\theta}_1 = \dot{\theta}_2 = 0$ implies that all the three top motions are either in phase or out of phase. Since the transcendental constants determine the characteristics of the periodicity of the top motions, the above argument indicates that the three sets of the transcendental constants which were introduced in Chapter II are connected by

$$k = k_1 = k_2 \text{ (equality of the moduli)} \quad (4.11)$$

$$n = n_1 = n_2 \text{ (equality of the time scaling factors)} \quad (4.12)$$

$$K = K_1 = K_2 \text{ (equality of the complete integrals)} . \quad (4.13)$$

Eq. (4.11) guarantees that all the periodic functions, which are either the elliptic functions or the theta functions describing the top motions, have a common character since the characters of these functions are completely determined by the moduli.

Eq. (4.12) guarantees that the three top motions are either in phase or out of phase. Eq. (4.13) is the immediate result of (4.11). Also, (4.12) and (4.13) give the condition that the three periods are the same, which can be seen from (2.13), (2.33) and (2.41). The common period is

$$T = \frac{2K}{n} \quad (4.14)$$

but the period in the unit of u is $2K$. Further, imposing the condition (4.12) on (2.25), (2.54) and (2.62) gives

$$u = u_1 = u_2 . \quad (4.15)$$

Due to the condition (4.15), the expressions of the three sets of Euler angles (2.20), (2.26), (2.27), (2.47) through (2.49) and (2.55) through (2.57) are written:

A. The heavy symmetrical top $(x_2, y_2, z_2, \theta, \phi, \psi)$

$$\cos\theta = x_3 + (x_2 - x_3) \operatorname{sn}^2(u) \quad (4.16)$$

$$\phi = S_1 u + f_1(u) \quad (4.17)$$

$$\psi = S_2 u + f_2(u) . \quad (4.18)$$

B. The first free triaxial top $(x_1, y_1, z_1, \theta_1, \phi_1, \psi_1)$

$$\cos\theta_1 = P_1 \operatorname{dn}(u) \quad (4.19)$$

$$\phi_1 = M_1 u \pm g_1(u) \quad (4.20)$$

$$\tan\psi_1 = Q_1 \frac{\operatorname{cn}(u)}{\operatorname{sn}(u)} . \quad (4.21)$$

C. The 2nd free triaxial top $(x_1, y_1, z_1, \theta_2, \phi_2, \psi_2)$

Note that the principal axes of the 2nd free top are the same as those of the 1st free top, i.e., both are (x_1, y_1, z_1) .

$$\cos \theta_2 = P_2 \operatorname{dn}(u) \quad (4.22)$$

$$\phi_2 = M_2 u \pm g_2(u) \quad (4.23)$$

$$\tan \psi_2 = Q_2 \frac{\operatorname{cn}(u)}{\operatorname{sn}(u)} \quad (4.24)$$

Now, in the motion of a given heavy symmetrical top, since all of the physical and transcendental constants are determined by an initial condition, all the Euler angles expressed by (4.16), (4.17), (4.18) can be completely specified.

While in the motions of two free triaxial tops, from the observation of the expressions (2.32) through (2.62), it is seen that all of the constants appearing in the expressions of the two sets of Euler angles are expressed in terms of the physical constants only, that is, (s_0, s_1, s_2, s_3) and (s'_0, s'_1, s'_2, s'_3) since all the transcendental constants are expressed in terms of the physical constants as seen from (2.34), (2.36), (2.37), (2.42), (2.43), (2.44) and thus the Euler angles of the 1st free top motion given by (4.19), (4.20) and (4.21) are completely expressed in terms of (s_0, s_1, s_2, s_3) . While the Euler angles for the 2nd free top motion given by (4.22), (4.23) and (4.24) are completely expressed in terms of (s'_0, s'_1, s'_2, s'_3) . However, due to the expressions in (3.36), the two sets of physical constants of the two free triaxial tops are completely specified in terms

of the physical constants of the heavy symmetrical top. All the expressions in (3.36) were derived from the conditions on the relations between the transcendental constants of the three top motions which are given by (3.1), (3.2), (3.17) and (3.18). The above argument gives a conclusion that once an initial condition for the motion of a given heavy symmetrical top is given, all the three sets of Euler angles of the three top motions can be completely specified.

However, the above conclusion still does not give a clear physical picture of the three top motions, that is, how each top moves relative to the other tops. It is already shown that the nutations of the three top motions stop instantaneously and simultaneously. The nutation velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ of the two free triaxial tops are given by [3]

$$\dot{\theta}_1 = (s_2 - s_3) \sin\theta_1 \sin\psi_1 \cos\psi_1 \quad (4.25)$$

$$\dot{\theta}_2 = (s_2' - s_3') \sin\theta_2 \sin\psi_2 \cos\psi_2 . \quad (4.26)$$

The case $\theta_1 = N\pi$ or $\theta_2 = N\pi$, where $N = 0, \pm 1, \pm 2, \dots$, should be excluded since such a case will not give any periodic motion, i.e., no nutation. The same situation occurs to the motion of a heavy symmetrical top (the heavy top motion with $\theta = 0$ is called a sleeping top). Therefore, it is always understood that $\sin\theta \neq 0$, $\sin\theta_1 \neq 0$ and $\sin\theta_2 \neq 0$. Then, from (4.25) and (4.26), it can be seen that there are only two cases which give $\dot{\theta}_1 = 0$ and $\dot{\theta}_2 = 0$:

$$\psi_1 = m\pi, \quad \psi_2 = m\pi \quad (4.27)$$

or

$$\psi_1 = \frac{(2m+1)}{2} \pi, \quad \psi_2 = \frac{(2m+1)}{2} \pi \quad (4.28)$$

where $m = 0, \pm 1, \pm 2, \pm 3, \dots$. In (4.27), even $\psi_1 = m\pi$ and $\psi_2 = m\pi$ do not occur simultaneously, $\dot{\theta}_1 = 0$ occurs and the same situation occurs to (4.28). However, the simultaneity condition $\dot{\theta} = \dot{\theta}_1 = \dot{\theta}_2 = 0$ requires

$$\left. \begin{aligned} \psi_1 &= \psi_2 = m\pi && (\text{simultaneous}) \\ \psi_1 &= \psi_2 = \frac{2m+1}{2} \pi && (\text{simultaneous}) \end{aligned} \right\} \quad (4.29)$$

which indicate that the variations of ψ_1 and ψ_2 must be in phase. This can be seen from (4.21) and (4.24):

$$\tan \psi_1 = Q_1 \frac{\text{cn}(u)}{\text{sn}(u)} \quad (4.30)$$

$$\tan \psi_2 = Q_2 \frac{\text{cn}(u)}{\text{sn}(u)} \quad (4.31)$$

which obviously shows that ψ_1 and ψ_2 are in phase. The values of the elliptic functions $\text{cn}(u)$ and $\text{sn}(u)$ at $u = mK$ with $m = 0, \pm 1, \pm 2, \pm 3, \dots$, are [8]

$$\text{cn}(2mK) = \pm 1 \quad \text{sn}(2mK) = 0$$

$$\text{cn}[(2m+1)K] = 0 \quad \text{sn}[(2m+1)K] = \pm 1.$$

[Remember $K = K_1 = K_2$ due to (4.13).]

Then from (4.30) and (4.31), one obtains

$$\psi_1 = \psi_2 \begin{cases} = \frac{2m+1}{2} \pi & \text{at } u = 2mK \\ = m\pi & \text{at } u = (2m+1)K \end{cases} \quad (4.32)$$

which is consistent with (4.29). Thus ψ_1 becomes equal to ψ_2 at $u = mK$ but $\psi_1 \neq \psi_2$ at any other time since $Q_1 \neq Q_2$. When (4.32) is combined with (4.25) and (4.26), one obtains

$$\dot{\theta}_1 = \dot{\theta}_2 = 0 \text{ at } u = mK, m = 0, \pm 1, \pm 2, \dots$$

The nutation angle of the heavy symmetrical top is, according to (4.16), given by

$$\cos \theta = x_3 + (x_2 - x_3) \operatorname{sn}^2(u).$$

Knowing $\sin \theta \neq 0$ and $u = nt$, when this expression is differentiated with respect to time, one obtains [8]

$$\dot{\theta} \propto \operatorname{sn}(u) \operatorname{cn}(u) \operatorname{dn}(u)$$

which gives $\dot{\theta} = 0$ at $u = mK$. Thus the simultaneity condition is

$$\dot{\theta} = \dot{\theta}_1 = \dot{\theta}_2 = 0 \text{ at } u = mK. \quad (4.33)$$

This should be so since the whole argument started with the condition $\dot{\theta} = \dot{\theta}_1 = \dot{\theta}_2 = 0$.

Now, with regard to the spherical triangle $\triangle ABC$ of Figure 4.1 (p. 31), there are following two laws:

Law of sines [9]

$$\frac{\sin \widehat{BC}}{\sin \theta_2} = \frac{\sin \widehat{AC}}{\sin \theta_1} = \frac{\sin \widehat{AB}}{\sin \theta} \quad (4.34)$$

where (4.7) has been used.

Law of cosines [9]

$$\cos \theta = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \widehat{AB} \quad (4.35)$$

where (4.7) has been used. Now because of (4.32), (4.10) gives

$$\widehat{AB} = |\psi_1 - \psi_2| = 0 \text{ at } u = mK .$$

Then since the case $\sin \theta_2 = \sin \theta_1 = \sin \theta = 0$ is excluded due to the reason stated earlier, the law of sine (4.34) gives

$$\widehat{AB} = \widehat{AC} = \widehat{BC} = 0 \text{ at } u = mK$$

which says that the spherical triangle $\triangle ABC$ of Figure 4.1 (p. 31) collapses at $u = mK$. Therefore, from (4.8), (4.9) and (4.10), one can conclude

$$\left. \begin{array}{l} \phi = \phi_1 \\ \psi = -\phi_2 \\ \psi_1 = \psi_2 \end{array} \right\} \text{ at } u = mK . \quad (4.36)$$

When $\widehat{AB} = 0$, the law of cosines (4.35) becomes

$$\cos \theta = \cos(\theta_1 - \theta_2) .$$

Therefore

$$\theta = \pm(\theta_1 - \theta_2) \text{ at } u = mK \quad (4.37)$$

which is the relation between the three nutation angles at $u = mK$. Now, if $u = mK$ is substituted into the expressions of the angles ϕ , ψ , ϕ_1 and ϕ_2 which are given by (4.17), (4.18), (4.20) and (4.23), respectively, it can be shown that all the periodic parts of these expressions vanish [5]. That is,

$$f_1(mK) = f_2(mK) = g_1(mK) = g_2(mK) = 0.$$

Then at these moments $u = mK$, the first two expressions of (4.36) give

$$\left. \begin{aligned} S_1 &= M_1 \\ S_2 &= -M_2 \end{aligned} \right\}, \quad (4.38)$$

that is, the constant multiplier M_1 in the secular term of ϕ_1 of the 1st free triaxial top is equal to the multiplier S_1 of the secular term of ϕ of the heavy symmetrical top, and the multiplier M_2 with a negative sign in the secular term of ϕ_2 of the 2nd free triaxial top is equal to the multiplier S_2 of the secular term of ψ of the heavy symmetrical top. These relations are satisfied at any instant. Therefore, the expressions for M_1 and M_2 given by (2.51) and (2.59), respectively, will not be needed for their evaluations. These constant multipliers M_1 , M_2 , S_1 and S_2 simply represent the angular velocities $\dot{\phi}_1$, $\dot{\phi}_2$, $\dot{\phi}$ and $\dot{\psi}$, respectively, of the secular terms at $u = mK$. Therefore, using (4.17), (4.18), (4.20) and (4.23) along with (4.38), these angular velocities can be written, knowing $u = u_1 = u_2 = nt$,

$$\frac{1}{n} \dot{\phi} = S_1 + \frac{d}{du} f_1(u) , \quad \frac{1}{n} \dot{\phi}_1 = S_1 \pm \frac{d}{du} g_1(u) \quad (4.39)$$

$$\frac{1}{n} \dot{\psi} = S_2 + \frac{d}{du} f_2(u) , \quad \frac{1}{n} \dot{\phi}_2 = -S_2 \mp \frac{d}{du} g_2(u)$$

which indicates that the precessional motion of the first free triaxial top ($\dot{\phi}_1$) is connected to the precessional motion of the heavy symmetrical top ($\dot{\phi}$). On the other hand, however, the precessional motion of the 2nd free triaxial top ($\dot{\phi}_2$) is connected with the spin motion of the heavy symmetrical top ($\dot{\psi}$). Figure 4.1 (p. 31) gives some implication of the above argument.

Now since the three nutation angles θ , θ_1 , θ_2 oscillate, they assume their maximum and minimum values alternately. The extremum of each of these angles, which occurs at $u = mK$ as shown in (4.33), needs to be evaluated. The values of the elliptic functions $\text{sn}^2(u)$ and $\text{dn}(u)$ at $u = mK$, which appear in (4.16), (4.19) and (4.22), are given by [8]

$$\text{sn}^2(2mK) = 0 \quad \text{sn}^2[(2m+1)K] = 1$$

$$\text{dn}(2mK) = 1 \quad \text{dn}[(2m+1)K] = k'$$

where $k' = (1 - k^2)^{1/2}$ and $0 < k' < 1$. From (4.16),

$$\left. \begin{aligned} \theta &= \cos^{-1} x_3 = \text{the maximum of } \theta \\ &\equiv \gamma_1 \quad \text{at } u = 2mK \\ \theta &= \cos^{-1} x_2 = \text{the minimum of } \theta \\ &\equiv \gamma_2 \quad \text{at } u = (2m+1)K \end{aligned} \right\} \quad (4.40)$$

where (2.10) has been used. From (4.19),

$$\left. \begin{aligned}
 \theta_1 &= \cos^{-1} P_1 = \text{the minimum of } \theta_1 \\
 &\equiv \alpha_1 \quad \text{at } u = 2 \text{ mK} \\
 \theta_1 &= \cos^{-1}(k \cdot P_1) = \text{the maximum of } \theta_1 \\
 &\equiv \beta_1 \quad \text{at } u = (2m + 1)K
 \end{aligned} \right\} \quad (4.41)$$

From (4.22),

$$\left. \begin{aligned}
 \theta_2 &= \cos^{-1} P_2 = \text{the minimum of } \theta_2 \\
 &\equiv \alpha_2 \quad \text{at } u = 2 \text{ mK} \\
 \theta_2 &= \cos^{-1}(k \cdot P_2) = \text{the maximum of } \theta_2 \\
 &\equiv \beta_2 \quad \text{at } u = (2m + 1)K .
 \end{aligned} \right\} \quad (4.42)$$

Then the expression (4.37) can be written

$$\left. \begin{aligned}
 \gamma_1 &= \pm(\alpha_1 - \alpha_2) \quad \text{at } u = 2 \text{ mK} \\
 \gamma_2 &= \pm(\beta_1 - \beta_2) \quad \text{at } u = (2m + 1)K
 \end{aligned} \right\} \quad (4.43)$$

where

$$\left. \begin{aligned}
 \gamma_1 &> \gamma_2 \\
 \alpha_1 &< \beta_1 \\
 \alpha_2 &< \beta_2
 \end{aligned} \right\} \quad (4.44)$$

Geometrically the situation (4.43) occurs when the three axes z , z_1 and z_2 lie in a plane which is of course perpendicular to the invariable plane (x - y). Further, because of the relation (4.32) and the definitions of the Euler's angles ψ_1 and ψ_2 , it can be shown that $\gamma_1 = \pm(\alpha_1 - \alpha_2)$ occurs when the four axes z , z_1 , z_2 and x_1 lie in a plane and

$\gamma_2 = \pm(\beta_1 - \beta_2)$ occurs when the four axes z , z_1 , z_2 and y_1 lie in another plane. These planes are perpendicular to the invariable plane. These situations are shown in Figure 4.2. Since the spherical triangle collapses at these moments $u = 2 mK$ and $u = (2m + 1)K$, the three lines of nodes combine together in parallel becoming a single line of nodes coming out of the origin and it is perpendicular to the planes where the axes z , z_1 , z_2 and x_1 or y_1 lie.

Now a summary will be given here: Figure 4.3 shows the pattern of the varying spherical triangle $\triangle ABC$ in which diagrams (a) and (c) correspond to the moments $u = 2 mK$ and $(2m + 1)K$, respectively, when the spherical triangle collapses. At these moments, the nutation angles satisfy $\gamma_1 = \alpha_1 - \alpha_2$ in (a) and $\gamma_2 = \beta_1 - \beta_2$ in (c). It is assumed that the precession angular velocity $\dot{\phi}$ of the heavy symmetrical top does not change its sign. In between (a) and (c), that is, $2 mK < u < (2m + 1)K$, a spherical triangle is formed as shown in (b). Referring to Figure 4.1 (p. 31) concurrently with Figure 4.3, in order that the above situation is possible, the line of nodes $\overline{OB}$ must be moving faster than the line of nodes $\overline{OC}$ on the invariable plane (x - y) at the moment $u = 2 mK$ corresponding to diagram (a) of Figure 4.3. That is, the relation between the precession speed of the heavy top and that of the first free top must be $\dot{\phi} < \dot{\phi}_1$ at $u = 2 mK$ (but $\phi = \phi_1$). Then the three lines of nodes will be separated to form the spherical triangle as in (b). While at $u = (2m + 1)K$ corresponding to diagram (c) of Figure 4.3, the spherical triangle again collapses. But this is possible only when the line of nodes $\overline{OB}$ is decelerated and the line of nodes $\overline{OC}$ is accelerated in the interval $2 mK < u < (2m + 1)K$ so that the line $\overline{OC}$ can catch up with $\overline{OB}$.

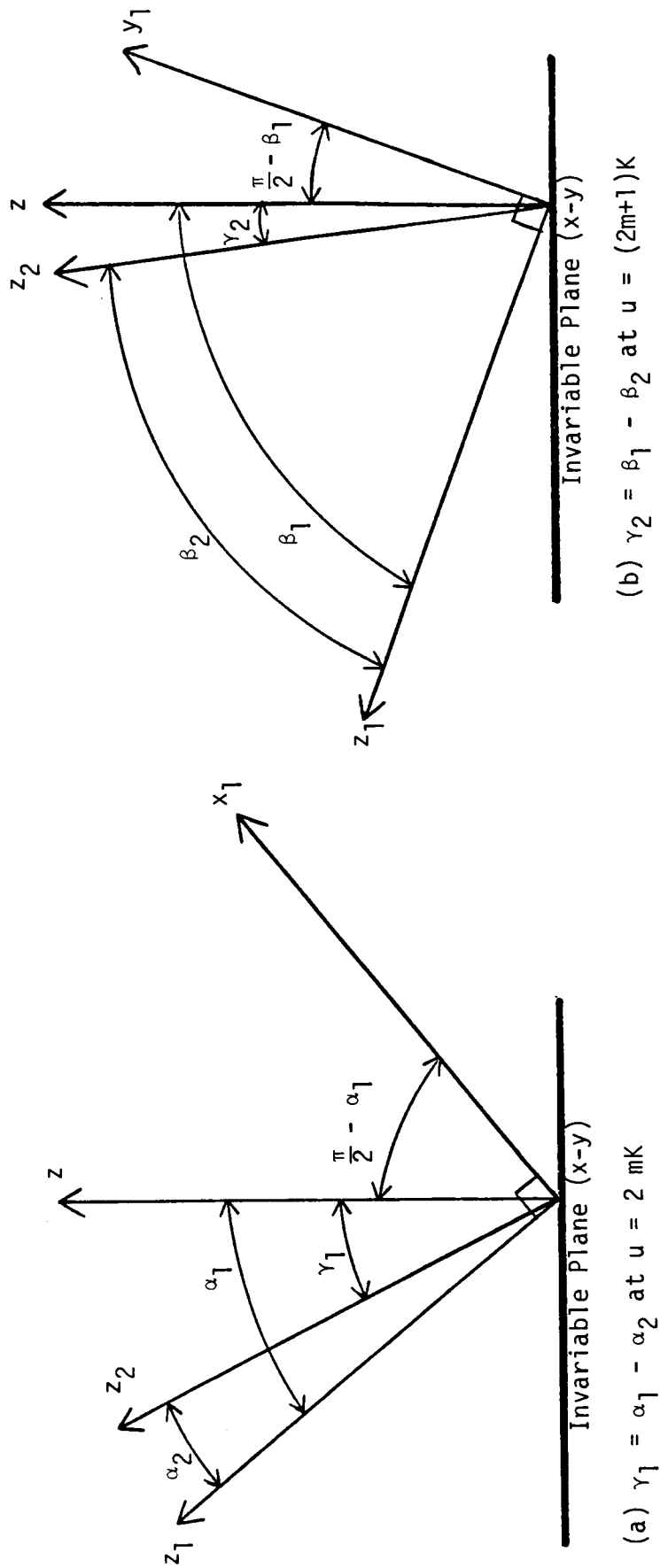
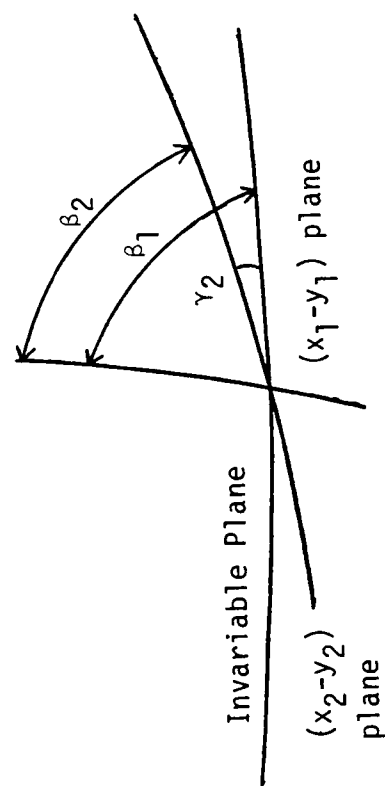
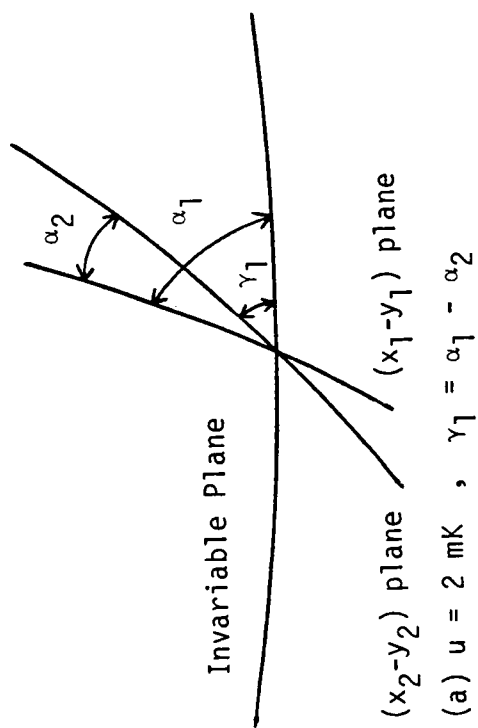
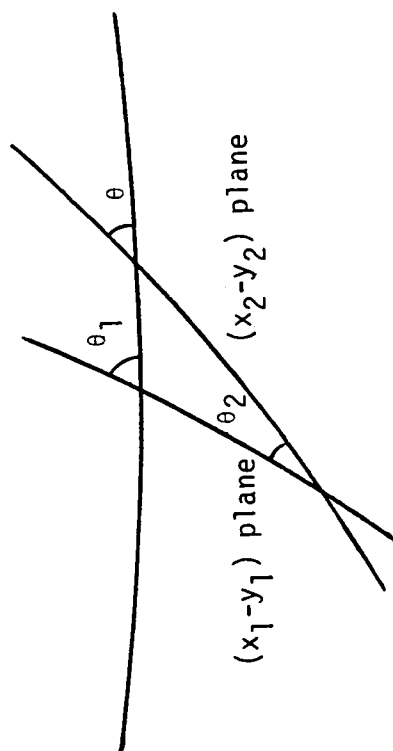
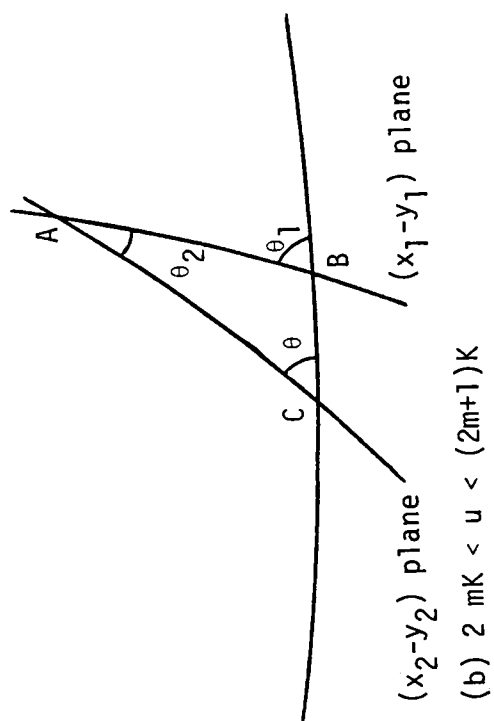


Figure 4.2. Four axes lie in the same planes at $u = mK$. At these moments, the spherical triangle collapses and $\delta = \delta_1 = \delta_2 = 0$. The single line of nodes is perpendicular to these planes.



(c) $u = (2m+1)K$, $\gamma_2 = \beta_1 - \beta_2$



(d) $u > (2m+1)K$

Figure 4.3. The variation of the spherical triangle.

The periodic parts of (4.17) and (4.20) are responsible for the acceleration and the deceleration. At $u = (2m + 1)K$ which corresponds to the diagram (c) of Figure 4.3, the relation between the precession angular speeds of the two tops must be $\dot{\phi} > \dot{\phi}_1$ (but again $\dot{\phi} = \dot{\phi}_1$) which is opposite to the relation at $u = 2mK$ (a). Then it is easy to see that after (c), i.e., $u > (2m + 1)K$, an inverted spherical triangle will be formed as shown in (d). After (d), the pattern regains the original situation (a). That is, the whole pattern is repeated with the period $2K$ in the unit of $u(=nt)$. Of course the position of the spherical triangle on the invariable plane is always changing due to the precession. Note that between $u = 2mK$ and $u = (2m + 1)K$, θ decreases from its maximum value γ_1 to its minimum value γ_2 and both θ_1 and θ_2 increase from their minimum values α_1 and α_2 to the maximum values β_1 and β_2 . For $u > (2m + 1)K$, the exactly opposite situation exists. Therefore θ_1 and θ_2 are in phase and they are out of phase with respect to θ . This can be seen also from (4.16), (4.19) and (4.22) since the elliptic functions $\text{sn}^2(u)$ and $\text{dn}(u)$ are out of phase. Further, as discussed earlier, when the line of nodes $\overline{OB}$ is accelerated, the line of nodes $\overline{OC}$ is decelerated and therefore ϕ and ϕ_1 must be out of phase. Thus it can be concluded that the motion of the heavy top and that of each free top are out of phase and the motions of the two free tops are in phase.

CHAPTER V

ANGULAR VELOCITY COMPONENTS IN ASSOCIATION WITH THE DECOMPOSITION THEOREM

1. A heavy symmetrical top [1]

With $x = \cos\theta$, the angular velocities are given by

$$\dot{x}^2 = (1 - x^2)(\alpha' - x) - (b - ax)^2 \quad (5.1)$$

$$\dot{\phi} = \frac{b - ax}{1 - x^2} \quad (5.2)$$

$$\dot{\psi} = \left(\frac{I_1 - I_3}{I_3} \right) a + \frac{a - bx}{1 - x^2} . \quad (5.3)$$

All the parameters appearing in the above expressions are already introduced in Chapter II.

2. A torque-free triaxial top [2,3]

From the result of the dynamics of a free triaxial top, the three angular velocity components are given by [3]

$$\dot{\theta}_1 = (s_2 - s_3) \sin\theta_1 \sin\psi_1 \cos\psi_1 \quad (5.4)$$

$$\dot{\phi}_1 = s_2 \sin^2\psi_1 + s_3 \cos^2\psi_1 \quad (5.5)$$

$$\dot{\psi}_1 = \frac{\cos\theta_1}{\sin^2\theta_1} (s_1 - s_0) \quad (5.6)$$

3. The second free triaxial top

$$\dot{\theta}_2 = (s_2' - s_3') \sin \theta_1 \sin \psi_1 \cos \psi_1 \quad (5.7)$$

$$\dot{\phi}_2 = s_2' \sin^2 \psi_2 + s_3' \cos^2 \psi_2 \quad (5.8)$$

$$\dot{\psi}_2 = \frac{\cos \theta_2}{\sin^2 \theta_2} (s_1' - s_0') \quad (5.9)$$

4. The evaluation of the angular velocities at particular moments

$u = 2mK$ and $u = (2m + 1)K$, where $m = 0, \pm 1, \pm 2, \pm 3, \dots$

From (4.40)

$$\cos \theta = \cos \gamma_1 = x_3 \quad \text{at } u = 2mK$$

$$\cos \theta = \cos \gamma_2 = x_2 \quad \text{at } u = (2m + 1)K$$

From (4.41)

$$\cos \theta = \cos \alpha_1 = p_1 \quad \text{at } u = 2mK$$

$$\cos \theta_1 = \cos \beta_1 = k' p_1 \quad \text{at } u = (2m + 1)K$$

From (4.42)

$$\cos \theta_2 = \cos \alpha_2 = p_2 \quad \text{at } u = 2mK$$

$$\cos \theta_2 = \cos \beta_2 = k' p_2 \quad \text{at } u = (2m + 1)K$$

From (4.32)

$$\psi_1 = \psi_2 = \frac{2m + 1}{2} \pi \quad \text{at } u = 2mK$$

$$\psi_1 = \psi_2 = m\pi \quad \text{at } u = (2m + 1)K$$

P_1 is given by (2.50), P_2 is given by (2.58) and $k' = \sqrt{1 - k^2} = \sqrt{1 - k_1^2} = \sqrt{1 - k_2^2}$ with k_1 and k_2 are given by (2.36) and (2.43), respectively.

Then the angular velocities at these particular moments can be specified in terms of the physical constants.

A. $u = 2 \text{ mK}$

A heavy symmetrical top ($x = x_3$)

$$\dot{\theta} = 0 \quad (5.10)$$

$$\dot{\phi} = \frac{b - ax_3}{1 - x_3^2} \quad (5.11)$$

$$\dot{\psi} = \left(\frac{I_1 - I_3}{I_3} \right) a + \frac{a - bx_3}{1 - x_3^2} \quad (5.12)$$

The first free triaxial top

$$\dot{\theta}_1 = 0$$

$$\dot{\phi}_1 = s_2 = \frac{\ell_1}{A_1} \quad (5.13)$$

$$\dot{\psi}_1 = \frac{P_1}{1 - P_1^2} (s_1 - s_0) = [(s_1 - s_2)(s_0 - s_2)]^{1/2}$$

The second free triaxial top

$$\dot{\theta}_2 = 0$$

$$\dot{\phi}_2 = s_2' = \frac{\ell_2}{A_2} \quad (5.14)$$

$$\dot{\psi}_2 = \frac{P_2}{1 - P_2^2} (s_1' - s_0') = [(s_1' - s_2')(s_0' - s_1')]^{1/2}$$

B. $u = (2m + 1)K$

A heavy symmetrical top ($x = x_2$)

$$\dot{\theta} = 0 \quad (5.15)$$

$$\dot{\phi} = \frac{b - ax_2}{1 - x_2^2} \quad (5.16)$$

$$\dot{\psi} = \left(\frac{I_1 - I_3}{I_3} \right) a + \frac{a - bx_2}{1 - x_2^2} \quad (5.17)$$

The first free triaxial top

$$\dot{\theta}_1 = 0$$

$$\dot{\phi}_1 = s_3 = \frac{\ell_1}{B_1} \quad (5.18)$$

$$\dot{\psi}_1 = \frac{k'P_1}{1 - (k'P_1)^2} (s_1 - s_0) = [(s_1 - s_3)(s_0 - s_3)]^{1/2}$$

The second free triaxial top

$$\dot{\theta}_2 = 0$$

$$\dot{\phi}_2 = s'_3 = \frac{\ell_2}{B_2} \quad (5.19)$$

$$\dot{\psi}_2 = \frac{k'P_2}{1 - (k'P_2)^2} (s'_1 - s'_0) = [(s'_1 - s'_3)(s'_0 - s'_3)]^{1/2}$$

5. Using (3.36), all the angular velocities at $u = mK$ of the two free triaxial tops can be expressed in terms of the physical constants of the heavy symmetrical top.

A. $u = 2 \text{ mK}$, $(x = x_3)$

The first free triaxial top

$$\dot{\theta}_1 = 0 \quad (5.20)$$

$$\dot{\phi}_1 = \frac{1 - x_2^2}{2(a - bx_2)} \quad (5.21)$$

$$\dot{\psi}_1 = \left[\frac{(x_1 - x_2)(1 - x_2^2)}{4(1 - x_3^2)} \right]^{1/2} \frac{a - bx_3}{a - bx_2} \quad (5.22)$$

The second free triaxial top

$$\dot{\theta}_2 = 0 \quad (5.23)$$

$$\dot{\phi}_2 = - \left(\frac{I_1 - I_3}{I_3} \right) a - \frac{1 - x_2^2}{2(b - ax_2)} \quad (5.24)$$

$$\dot{\psi}_2 = \left[\frac{(x_1 - x_2)(1 - x_2^2)}{4(1 - x_3^2)} \right]^{1/2} \frac{b - ax_3}{b - ax_2} \quad (5.25)$$

B. $u = (2m + 1)K$, $(x = x_2)$

The first free triaxial top

$$\dot{\theta}_1 = 0 \quad (5.26)$$

$$\dot{\phi}_1 = \frac{1 - x_3^2}{2(a - bx_3)} \quad (5.27)$$

$$\dot{\psi}_1 = \left[\frac{(x_1 - x_3)(1 - x_3^2)}{4(1 - x_2^2)} \right]^{1/2} \frac{a - bx_2}{a - bx_3} \quad (5.28)$$

The second free triaxial top

$$\dot{\theta}_1 = 0$$

$$\dot{\phi}_2 = - \left(\frac{I_1 - I_3}{I_3} \right) a - \frac{1 - x_3^2}{2(b - ax_3)} \quad (5.29)$$

$$\dot{\psi}_2 = \left[\frac{(x_1 - x_3)(1 - x_3^2)}{4(1 - x_2^2)} \right]^{1/2} \frac{b - ax_2}{b - ax_3} \quad (5.30)$$

CHAPTER VI

A NUMERICAL EXAMPLE

Since the decomposition theorem states that a given heavy top motion can be expressed as a composite motion of two free triaxial tops, an initial condition of the heavy top determines all the dynamical and kinematical constants of the three top motions. There will be no loss of generality if the initial time is chosen when the nutation of the heavy top stops, that is, $\dot{\theta} = 0$. Then the following initial values are needed: the inclination angle θ_0 of the axis of symmetry of the heavy top with respect to the vertical axis which is antiparallel to the gravity and its cosine value is denoted by $x_2 = \cos\theta_0$, the initial precessional angular velocity $\dot{\phi}_0$ and the initial spin angular velocity $\dot{\psi}_0$. Then the dimensionless physical constants of the heavy top motion in terms of the above initial values are expressed as

$$\left. \begin{aligned} a &= \frac{I_3}{I_1} (\dot{\psi}_0 + \dot{\phi}_0 x_2) \\ b &= \dot{\phi}_0 (1 - x_2^2) + a x_2 \\ \alpha' &= \dot{\phi}_0^2 (1 - x_2^2) + x_2 \end{aligned} \right\} \quad (6.1)$$

which can be easily obtained from the dynamical expressions of the heavy symmetrical top [1]. Further, from the cubic expression (2.8) of the heavy top motion, the relations between the three physical constants a , b , α' and the three roots x_1 , x_2 , x_3 are given by

$$\left. \begin{aligned} a^2 + \alpha' &= x_1 + x_2 + x_3 \\ 2ab - 1 &= x_1x_2 + x_2x_3 + x_3x_1 \\ b^2 - \alpha' &= x_1x_2x_3 \end{aligned} \right\} \quad (6.2)$$

which can be obtained by simply expanding (2.8).

Now the following numerical values will be given:

$$\frac{I_3}{I_1} = \frac{2}{3}, \quad x_2 = \cos 60 = 0.5, \quad \dot{\phi}_0 = 2, \quad \dot{\psi}_0 = 1 \quad (6.3)$$

all of which are dimensionless since the angular velocities have been divided by w which is given by (2.3). Then (6.1) gives

$$a = \frac{4}{3}, \quad b = \frac{13}{6}, \quad \alpha' = \frac{7}{2}. \quad (6.4)$$

Combining (6.4) with (6.2), one obtains

$$\begin{aligned} x_1 &= 4.2103995 \\ x_2 &= 0.5673782 \\ x_3 &= 0.5 \end{aligned} \quad (6.5)$$

in which the value of x_2 and that of x_3 have been interchanged to give the following sequential order in the three roots as shown by (2.9).

$$x_1 > 1 > x_2 > x_3 > -1$$

The transcendental constants associated with the elliptic functions are:

$$k \text{ (modulus)} = 0.1347564$$

$$n \text{ (time scaling factor)} = 0.9631199$$

$$K \text{ (complete integral)} = 1.5780013$$

$$q_1 \text{ (additive constant)} = 1.7007607$$

$$q_2 \text{ (additive constant)} = 1.2268984$$

$$p_1 = q_1 + q_2 = 2.9276591$$

$$p_2 = q_1 - q_2 = 0.4738623$$

since $a < b$. The evaluation of q_1 and q_2 is based on the method given by E. T. Whittaker [3]. Jacobi gives explicit formulas for the evaluation of (2.23), (2.24), (2.52) and (2.60) [10].

With these preparations along with the connection formulas between the physical constants of the three top motions, which are given in (3.36), all the three sets of Euler angles can be completely evaluated. Table 6.1 shows the results. Table 6.2 shows the values of the angular velocity components of the three top motions at $u = mK$, which were evaluated using (5.10) through (5.12), (5.15) through (5.17) and (5.20) through (5.30). Table 6.1 clearly shows that the angles θ , θ_1 and θ_2 oscillate between the maxima and the minima due to the nutation. Also it shows that θ_1 and θ_2 are in phase and they are out of phase with respect to θ . Because of the rotation motions (the secular terms), the Euler angles ϕ , ϕ_1 , ϕ_2 , ψ , ψ_1 , ψ_2 are always increasing or decreasing. The angular relations in (4.36) are satisfied at $u = 0, K, 2K$. Also the relations in (4.43) are satisfied at these moments. As for the angular velocities, Table 6.2 shows that $\dot{\phi} < \dot{\phi}_1$ at $u = 0$, $\dot{\phi} > \dot{\phi}_1$ at $u = K$ and again $\dot{\phi} < \dot{\phi}_1$ at $u = 2K$. This shows that when ϕ_1 is accelerated, ϕ is decelerated and vice versa. Therefore ϕ and ϕ_1 are out of phase.

TABLE 6.1
VARIATION OF THE THREE SETS OF EULER'S ANGLES DURING ONE PERIOD

u	0	$\frac{K}{3}$	$\frac{2}{3} K$	K	$\frac{4}{3} K$	$\frac{5}{3} K$	2K
θ	60.00	58.87	56.59	55.43	56.59	58.87	60.00
ϕ	0.00	62.75	126.49	191.24	256.04	319.72	382.47
ψ	0.00	30.83	59.37	85.62	111.81	140.40	171.20
θ_1	3.79	5.42	7.70	8.62	7.70	5.42	3.79
ϕ_1	0.00	86.46	143.25	191.24	239.20	296.01	382.47
ψ_1	90.00	37.28	14.24	0.00	-14.24	-37.28	-90.00
θ_2	63.79	63.86	63.99	64.05	63.99	63.86	63.79
ϕ_2	0.00	-28.42	-56.96	-85.62	-114.28	-142.82	-171.20
ψ_2	90.00	59.83	29.83	0.00	-29.83	-59.83	-90.00

Note: All angles are expressed in degree.

TABLE 6.2
 DIMENSIONLESS ANGULAR VELOCITIES AT $u = 0, K$ and $2K$
 WHEN THE THREE LINES OF NODES COINCIDE

	0	K	2K
$\dot{\theta}$	0.0000000	0.0000000	0.0000000
$\dot{\phi}$	2.0000000	2.0796340	2.0000000
$\dot{\psi}$	1.0000000	0.8200609	1.0000000
$\dot{\theta}_1$	0.0000000	0.0000000	0.0000000
$\dot{\phi}_1$	3.2595729	1.5000000	3.2595729
$\dot{\psi}_1$	2.1810216	0.4214259	2.1810216
$\dot{\theta}_2$	0.0000000	0.0000000	0.0000000
$\dot{\phi}_2$	-0.9070935	-0.9166666	-0.9070935
$\dot{\psi}_2$	0.9652360	0.9522429	0.9652360

Although none of the Euler angles of the three tops is independent of the others, the variation of the three nutation angles θ , θ_1 , θ_2 and the relative variation of ϕ and ϕ_1 will essentially determine the variation of the shape of the spherical triangle. This can be seen from Figure 4.1 (p. 31) and Figure 4.3 (p. 46). Therefore, the above argument based on Table 6.1 and Table 6.2 are completely consistent with the general argument given in Chapter IV.

CHAPTER VII

DISCUSSION AND CONCLUSION

The dimensionless analytical method has been used. That is, any angular velocity or any quantity which has the dimension of angular velocity (reciprocal of time) is divided by the quantity w which is defined by

$$w = \left(\frac{2 Mg \ell}{I_1} \right)^{1/2}$$

where M is the total mass of the heavy symmetrical top, g is the gravitational field intensity having the dimension of acceleration, ℓ is the length between the fixed point and the center of mass and I_1 is the moment of inertia. Therefore w has the dimension of angular velocity. The quantity w can be immediately determined once a symmetrical top is chosen since it is independent of any initial condition. Then it is clear that any angular velocity is the relative velocity relative to w . This method greatly simplifies the analysis, especially the numerical analysis.

As indicated in Chapter II, there is a symmetric property in the motion of a heavy symmetrical top. That is, the cubic equations (2.8) and (2.12) along with (2.11) have this symmetric property under the interchange of the physical constants a and b . The three roots x_1, x_2, x_3 satisfy the two different cubic equations (2.8) and (2.12). In other words, there exist two different motions for a given set of three roots.

To see how this symmetric property appears in the decomposition theorem, Eqs. (3.30) and (3.34) may be rewritten as

$$s_i = \frac{1 - x_i^2}{2(a - bx_i)} \quad (7.1)$$

$$s'_i = \frac{1 - x_i^2}{2(b - ax_i)} \quad (7.2)$$

Eq. (2.12) has been used to derive (7.1) and Eq. (2.8) has been used to derive (7.2). Then all the expressions in (3.36) can be rewritten as:

<u>The first free top</u>	<u>The second free top</u>	
$\frac{\ell_1}{A_1} = \frac{1 - x_2^2}{2(a - bx_2)}$	$\frac{\ell_2}{A_2} = \frac{1 - x_2^2}{2(b - ax_2)}$	$\left. \vphantom{\begin{matrix} \frac{\ell_1}{A_1} \\ \frac{\ell_1}{B_1} \\ \frac{\ell_1}{C_1} \\ \frac{h_1}{\ell_1} \end{matrix}} \right\} \quad (7.3)$
$\frac{\ell_1}{B_1} = \frac{1 - x_3^2}{2(a - bx_3)}$	$\frac{\ell_2}{B_2} = \frac{1 - x_3^2}{2(b - ax_3)}$	
$\frac{\ell_1}{C_1} = \frac{1 - x_1^2}{2(a - bx_1)}$	$\frac{\ell_2}{C_2} = \frac{1 - x_1^2}{2(b - ax_1)}$	
$\frac{h_1}{\ell_1} = \frac{b}{2}$	$\frac{h_2}{\ell_2} = \frac{a}{2}$	

which clearly shows that the two sets of expressions are completely symmetric or exchangeable under the interchange of the constants a and b .

Now, in the expressions (2.23) and (2.24), since all the transcendental constants n , k , q_1 , q_2 are expressed in terms of the

three roots only (i.e., x_1, x_2, x_3), the quantities $\mu_1, \mu_2, \lambda_1, \lambda_2$ will not be affected by the action of interchanging the constants a and b . However, as indicated in Chapter II, the signs appearing in (2.21) and (2.22) are determined such that the upper sign is chosen for $a < b$ and the lower sign is chosen for $a > b$. This means that when a and b are interchanged, all the signs in (2.21) and (2.22) are also changed. Now, for the sake of convenience, the quantity D is defined

$$D = \left(\frac{I_1 - I_3}{I_3} \right) a \quad (7.4)$$

Now, suppose $D = 0$, which is the case of a spherical heavy top since $I_1 = I_3$, then the signs in the expressions (2.21) and (2.22) are interchanged for the interchange of the constants a and b so that (2.21) and (2.22) are exchanged. That is, ϕ and ψ are exchangeable under the interchange of a and b for $D = 0$. This can also be seen from (5.11) and (5.12). That is, if $D = 0$, which is the first term of (5.12), it is clear that the two angular velocities $\dot{\phi}$ and $\dot{\psi}$ become symmetric under the interchange of a and b . The expressions of the precession angular velocities $\dot{\phi}_1$ and $\dot{\phi}_2$ of the two free triaxial tops are given by (4.39). But using the definitions of S_1 and S_2 which are given by (2.28) and (2.29), respectively, and the definition of D given by (7.4), the two angular velocities can be written

$$\dot{\phi}_1 = n[(\mu_2 \pm \mu_1) \pm \frac{d}{du} g_1(u)] \quad (7.5)$$

$$\dot{\phi}_2 = -D - n[(\mu_2 \mp \mu_1) \pm \frac{d}{du} g_2(u)] \quad (7.6)$$

For the reason stated earlier, the $(\pm)$ signs in $(u_2 \pm u_1)$ of (7.5) and those in $(u_2 \mp u_1)$ of (7.6) are interchanged when the constants a and b are interchanged. The periodic functions $g_1(u)$ of (7.5) and $g_2(u)$ of (7.6) are defined by (2.52) and (2.60), respectively. These functions are expressed in terms of the transcendental constants p_1 and p_2 , that is, $g_1(u)$ in terms of p_1 and $g_2(u)$ in terms of p_2 . Further, p_1 is defined in terms of the four physical constants (s_0, s_1, s_2, s_3) as shown by (2.37) and p_2 is defined in terms of the physical constants (s'_0, s'_1, s'_2, s'_3) as shown by (2.44). But according to (7.3), these two sets of four physical constants are completely exchangeable under the interchange of the constants a and b . Consequently, the two periodic functions $g_1(u)$ and $g_2(u)$ are symmetric or exchangeable under the interchange of a and b . It is obvious that their derivative forms $\frac{d}{du} g_1(u)$ and $\frac{d}{du} g_2(u)$ are also symmetric under the interchange of a and b . Then, it can be concluded from (7.5) and (7.6) that if $D = 0$, $\dot{\phi}_1$ and $(-\dot{\phi}_2)$ are exchangeable under the interchange of a and b . However, in general, since $D \neq 0$, i.e., $I_1 \neq I_3$, $\dot{\phi}_1$ and $(-\dot{\phi}_2)$ are not exchangeable under the interchange of a and b .

Now, from (5.13), (5.14), (5.18) and (5.19) along with (7.3), the angular velocities $\dot{\phi}_1$ and $\dot{\phi}_2$ can be expressed in terms of the heavy top constants a, b, x_1, x_2 and x_3 :

At $u = 2 mK, m = 0, \pm 1, \pm 2, \dots$

$$\dot{\phi}_1 = s_2 = \frac{\ell_1}{A_1} = \frac{1 - x_2^2}{2(a - bx_2)} \quad (7.7)$$

$$\dot{\phi}_2 = s'_2 = \frac{\ell_2}{A_2} = \frac{1 - x_2^2}{2(b - ax_2)} \quad (7.8)$$

At $u = (2m + 1)K$

$$\dot{\phi}_1 = s_3 = \frac{\ell_1}{B_1} = \frac{1 - x_3^2}{2(a - bx_3)} \quad (7.9)$$

$$\dot{\phi}_2 = s'_3 = \frac{\ell_2}{B_2} = \frac{1 - x_3^2}{2(b - ax_3)} \quad (7.10)$$

Obviously, the above results [(7.7) through (7.10)] are not consistent with (7.5) and (7.6) regarding the exchangeability between $\dot{\phi}_1$ and $\dot{\phi}_2$ —even $D = 0$ in (7.6). With respect to the interchange of the constants a and b , $\dot{\phi}_1$ and $\dot{\phi}_2$ are exchangeable in the above results. While in (7.5) and (7.6), it has been shown that, if $D = 0$, $\dot{\phi}_1$ and $(-\dot{\phi}_2)$ are exchangeable. To accomplish this consistency, the negative sign should be put on the right hand side of (7.8) and (7.10). Further, $\dot{\phi}_2$ of (7.6) has the extra term $(-D)$, which is a constant given by (7.4). Since $D \neq 0$ in general, $\dot{\phi}_2$ of (7.8) and (7.10) should also have this term $(-D)$. Therefore, (7.8) and (7.10) must be written

$$\left. \begin{aligned} \dot{\phi}_2 = s'_2 = \frac{\ell_2}{A_2} &= -D - \frac{1 - x_2^2}{2(b - ax_2)} & \text{at } u = 2mK \\ \dot{\phi}_2 = s'_3 = \frac{\ell_2}{B_2} &= -D - \frac{1 - x_3^2}{2(b - ax_3)} & \text{at } u = (2m + 1)K \end{aligned} \right\} \quad (7.11)$$

where D is defined by (7.4). Then (7.6) and (7.11) are completely consistent regarding the exchangeability of $\dot{\phi}_1$ and $\dot{\phi}_2$ with respect to the interchange of a and b . That is, at $u = mK$, (7.5) becomes either (7.7) or (7.9) and (7.6) becomes (7.11).

It should be noted that in (7.5) and (7.6), the $(\pm)$ signs in front of $\frac{d}{du} g_1(u)$ and $\frac{d}{du} g_2(u)$ if observed from the inside of each

bracket $[\cdot]$, both are either plus or negative. Otherwise the exchangeability of $\dot{\phi}_1$ and $\dot{\phi}_2$ will be impossible for $D = 0$. Also note that the property of the exchangeability itself, in association with the two free top motions, originates from the results in (3.36) which led to (7.3) and the results in (4.38) which led to (7.5) and (7.6).

Now, the forms of the two physical constants s_2^2 and s_3^2 of the 2nd free top have been modified from the forms in (7.3) to the forms in (7.11). But it is natural to treat each of the four constants equally, that is, none is preferred to the others in the four constants. Therefore, the four physical constants $s_0^2, s_1^2, s_2^2, s_3^2$ of the 2nd free top should be written just like (7.11):

$$\left. \begin{aligned} \frac{\lambda_2}{A_2} &= -D - \frac{1 - x_2^2}{2(b - ax_2)} \\ \frac{\lambda_2}{B_2} &= -D - \frac{1 - x_3^2}{2(b - ax_3)} \\ \frac{\lambda_2}{C_2} &= -D - \frac{1 - x_1^2}{2(b - ax_1)} \\ \frac{h_2}{\lambda_2} &= -D - \frac{a}{2} \end{aligned} \right\} . \quad (7.12)$$

Eq. (7.12) can be justified as follows: By (2.37) and (2.44), the transcendental constants p_1 and p_2 are expressed in terms of the corresponding four physical constants of each free top motion. But these constants appear only in the form of difference of two constants $s_i - s_j$ or $s_i^2 - s_j^2$ with $i \neq j$. Explicitly:

$$\frac{s_0 - s_2}{s_1 - s_0} \text{ for the 1st free top motion} \quad (7.13)$$

$$\frac{s'_0 - s'_2}{s'_1 - s'_0} \text{ for the 2nd free top motion .} \quad (7.14)$$

Then it is obvious that the effects of the extra term D and the negative signs in (7.12) will not show in the expression of p_2 . Because of this, it can be easily seen that (7.13) and (7.14) are exchangeable under the interchange of a and b ; even (7.12) is used. Therefore, the two transcendental constants p_1 and p_2 are symmetric with respect to the interchange of a and b . Further, from the definitions of the other transcendental constants, the time scaling factor n_2 and the modulus k_2 which are given by (2.42) and (2.43), it is clear that the effects of D and the negative sign in (7.12) will not show in the result. Consequently, the periodic functions $g_1(u)$ and $g_2(u)$ are also exchangeable. That is, as far as the periodic parts alone are concerned, whether the physical constants of the 2nd free top motion in (7.3) are used or those in (7.12) are used, the exchangeability of the two free top motions remains the same. If only s'_2 and s'_3 are expressed as in (7.11), clearly the above argument will not hold.

In deriving (3.36), (3.32) was used and it has the form of difference of two constants $s'_1 - s'_0$. This is the reason why the extra term D did not come out in (3.36).

Instead of using the explicit solutions of the top motions involved in the decomposition theorem, Darboux, using only the dynamical aspect of the rigid body motions, gave some extensive analysis on the decomposition theorem [11]. Although Darboux showed an algebraic

proof of the existence of the decomposition theorem, it can be shown that the results in (7.12) can also be derived from Darboux's results.

All the formulas used for the description of the motion of any free triaxial top are based on the condition that the positive z axis (i.e., z_1 or z_2 in the decomposition theorem) is chosen in the direction of the total angular momentum [2,3]. If this condition is combined with the inequality relations in (3.16), it can be shown that the four constants in (7.12) are either all positive or all negative. In the case that all are negative, the direction of the total angular momentum $\vec{\lambda}_2$ of the 2nd free top must be in the negative z_2 direction (z_2 is the axis of symmetry of the heavy symmetrical top). The numerical example presented in Chapter VI belongs to the above case. If the angular velocities $\dot{\phi}$ and $\dot{\psi}$ of the heavy symmetrical top never change their sign, the four constants in (7.12) are always all negative. However, if either $\dot{\phi}$ or $\dot{\psi}$ (or both) changes its sign, for example, a cusp motion and a motion with a retrogression, there is no guarantee that the four constants in (7.12) represent physical quantities.

The total energy and the total angular momentum of each free top motion are related with the angular momentum component a or b of the heavy top motion. This can be seen from (7.3) and (7.12). That is,

$$\frac{h_1}{\lambda_1} = \frac{b}{2} \quad (7.15)$$

$$\frac{h_2}{\lambda_2} = -D - \frac{a}{2} \quad (7.16)$$

Again, as long as $\dot{\phi}$ and $\dot{\psi}$ of the heavy top motion do not change their sign during the motion, since the direction of the angular momentum $\vec{l}_2$ is in the negative z_2 axis, (7.16) represents a physical quantity.

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