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I am submitting herewith a thesis written by John Christopher Knight entitled "Non-Split Links: Links as sums of Tangles." I have examimed this thesis for form and content and reccommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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Non-Split Links: Links as Sums of Prime and Rational Tangles

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Mostly, I would like to thank my wife, Lisa. Without her love, support, and encouragement, I would certainly never have achieved this accomplishment and reached this moment.

ABSTRACT

This paper is concerned with the investigation of links that can be constructed as sums of prime and rational tangles. The investigation of these links includes extensions of work done by Manasco and analysis of the intersection pattern of surface and link projection. Manasco's methods are extended to a new situation to find a bound on the Euler characteristic of the surface in the complement of the link.

The work shows that the Euler characteristic of a surface in the complement of links constructed as semi-alternating sums of prime and rational tangles is not 2. Therefore, no component of the link may be bounded by a 2-sphere (not intersecting the link) or the link is said to be non-split. The non-split result holds for links that are the sum of two tangles: one prime and one rational. Also the non-split result applies to a daisy chain of tangles at least one of which is prime and the rest are at worst rational.

Contents

1 Introduction

The purpose of this paper is to present conditions to determine if a given link diagram represents a non-split link. This paper will only be concerned with classical knots or links in the Piecewise Linear Category. This paper is inspired by the results of an article by W. Menasco [M]. Many of the results and techniques of proof of Menasco will be carried on in this paper. In a sense this paper is a generalization of some of Menasco's results since Menasco only dealt with alternating links. A link L in R^3 is considered split if there exists a 2-sphere in the complement of the link $(R^3 - L)$ not bounding a ball. A link will be considered an embedding of $\bigcup_{i=1}^{k}\{S_i{}^1\}$ in R^3. While Menasco showed that any alternating connected link projection diagram represented a non-split link, we will not require that our link projections be alternating; however, we will consider link projection diagrams that are the sum of two alternating tangle projection diagrams.

Definition 1. A *projection diagram* or just *diagram* of a link is a projection $\pi(L)$ with gaps to indicate one arc passing over another. We will consider a projection diagram of a link in $R^2 + \infty = S^2$.

This paper is also closely related to the results of W. B. Raymond Lickorish [L]. Lickorish is involved with the production of prime knots by summing prime tangles; however, he did not consider the case of summing trivial tangles. We will also use a more general definition of a tangle; while he considered a tangle as two arcs spanning a 3-ball, we use a more general definition.

Note: Diagram 1.1(a) is an alternating projection while Diagram 1.1(b) is non-alternating.

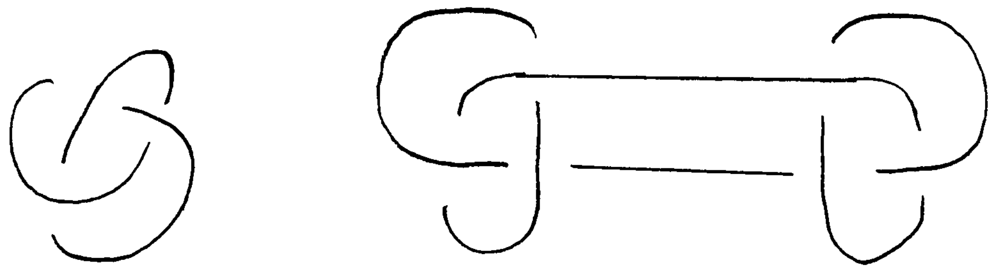


Diagram 1.1 (a) (b)

Definition 2. A *tangle* is a pair (B, t) where B is a 3-ball and t is a 1-dimensional properly embedded submanifold such that $\partial B \cap t = \partial t$ and t intersects ∂B in exactly four points.

Two tangles will be considered equivalent if there is a homeomorphism H between them so that $H : (\partial B, \partial t) \rightarrow (\partial B, \partial t_1)$ is the identity. Tangles also have projection diagrams; however, we will consider projection diagrams of tangles on the equatorial disk of B. Diagram 1.2(a) is an example of a tangle projection diagram. We will adopt the definition from Lickorish for a link as the sum of two or more tangles.

Definition 3. Given two tangles (B,t) and (B_1, t_1), we mean that a link is their *sum* $L = (B, t) \cup_f (B_1, t_1)$, where $(B, t) \cup_f (B_1, t_1)$ is defined by a homeomorphism $f : (\partial B, \partial t) \rightarrow (\partial B_1, \partial t_1)$.

We should also note that our sums will be less general than Lickorish's sums. In this paper sums will have the form as in Diagram 1.2(b) or the partial sum as in Diagram 1.2(c). Diagram 1.2(a) is an example of a link as a sum of alternating tangles.

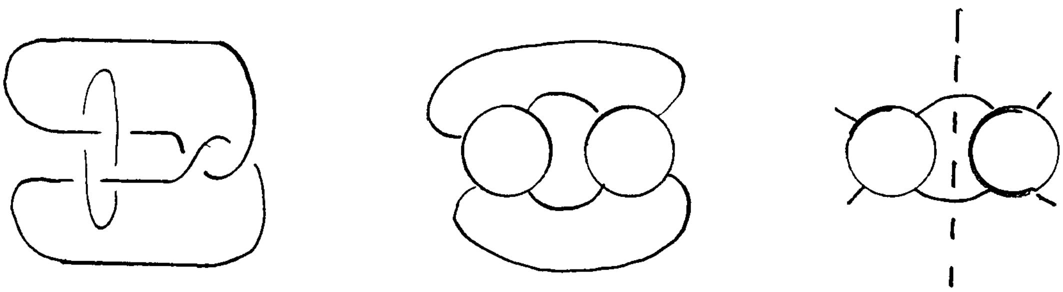


Diagram 1.2 (a) (b) (c)

A tangle (B, t) will be considered prime if any 2-sphere in B intersecting t exactly twice must bound a ball containing an unknotted arc of t. We will add the further restriction that it may have no crossing incident to the exterior of B. We will use the standard convention that the untangle is not considered prime, so any rational tangles will not be considered prime. Thus, we will show that if $L = (B, t) \cup_f (B_1, t_1)$ where (B, t) and (B_1, t_1) are alternating tangles at least one of which is prime and the other having more than one crossing, then L is non-split. We notice that $\pi(L)$ is not required to be alternating; in fact we will only be considering non-alternating $\pi(L)$. If the resulting sum is alternating, then the results of Menasco's paper would apply. The results will show, using certain methods of summing tangles, that more than two tangles may be summed and characterized as non-split.

Example 1.1

We notice that in Example 1.1 the second tangle has only one crossing and the link is split despite the primality of the first tangle. This example provides the motivation for the restriction that each additional tangle must have at least two crossings.

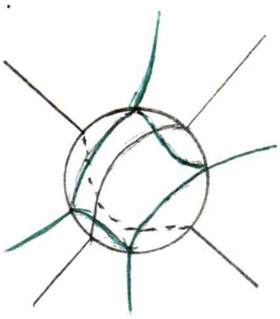
2 Historical Connections

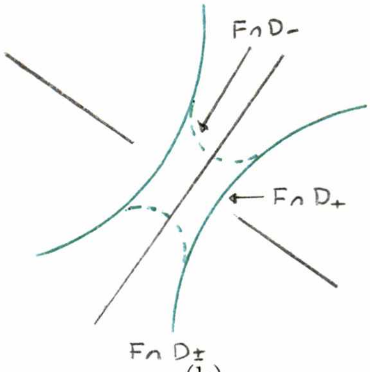
We will show that much of Menasco's work with alternating link projection diagrams applies to alternating tangle projection diagrams. We consider a surface F. Although in general F may not always be a 2-sphere, here we are concerned with 2-spheres, as our purpose is to provide sufficient conditions for a link to be non-split. Since Menasco's results are local, it is no surprise that the results easily carry over to the situation of tangles. We will be considering the intersection pattern of the surface and certain 2-spheres S_+ and S_- associated with the projection diagram defined as the projection of the tangle t onto the equatorial disk of B.

At each crossing of t on the equatorial disk of the tangle's projection, we will insert a small 2-sphere to replace a small disk in a neighborhood of the crossing, so that the over passing arc lies on the upper hemisphere of the 2-sphere while the under passing arc will lie on the lower hemisphere. This inserted 2-sphere will be referred to as a "bubble." We should also note that any surface meeting a bubble can be isotoped in the equatorial disk to meet the bubble in a saddle-shaped disk.[M]

Definition 4. *D_+ (respectively D_-)* is the equatorial disk in union with either the upper (respectively lower) hemisphere of each bubble (each disk in a neighborhood of each crossing of t has been replaced by a bubble).

This definition parallels Menasco's S_+ (respectively S_-) where $\pi(L)$ is in S^2. Diagram 2.1(a) shows the saddle formed by the intersection of a surface and $D_\pm$ at a bubble. Diagram 2.1(b) shows the intersection of a transverse surface and a bubble in $D_{\pm}$.





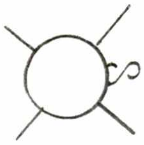


Diagram 2.1 (a) (b) (c)

Further we may assume that the surface F meets the D_+ and D_- transversely in a system of arcs. Secondly, we should assume that all links and tangles have been isotoped to eliminate any trivial crossings like Diagram 2.1(c).

Now we are ready to directly apply Menasco's results about alternating link diagrams to alternating tangle diagrams.

Proposition 1.

Let (B, t) be a tangle with an alternating connected projection diagram and F' be a surface in the complement of (B, t). Then we may isotope F' to F so that the following conditions hold:

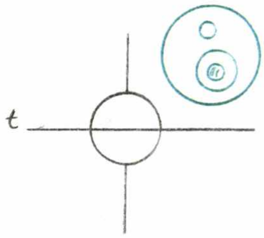
(i) There does not exist any simple closed curve in $D_{\pm} \cap F$.

(ii) Each arc of $D_{\pm} \cap F$ encounters the same bubble no more than once.

Proof.

(i) Suppose that there does exist a simple closed curve in $D_{\pm} \cap F'$ which does not intersect a bubble as in Diagram 2.2(a). Then by Menasco [M], there is a "dome" contained in the surface F' which can be isotoped through the equatorial disk so that it no longer intersects $D_{\pm}$. Likewise, any other simple closed curves not intersecting a bubble in $D_{\pm}$ can be isotoped away from $D_{\pm}$. The resulting surface would then be called F and contain only the arcs which intersect a bubble.

If there exists a simple closed curve C' that does intersect a bubble, then there must exist a innermost curve C that intersects a bubble as in Diagram 2.2(b). C must intersect another bubble so that the arc of t may escape the simple closed curve. Since the tangle is alternating, the next bubble encountered must be on the inside of C. Now an arc C'' must exist inside C to form a saddle. Since two arcs of $D_{\pm} \cap F$ cannot intersect, C'' must be a simple closed curve inside of C. C was assumed to be innermost so by contradiction the proposition holds.■



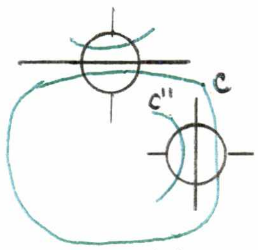


Diagram 2.2 (a) (b)

To visualize the first part of the proof of (i), we imagine a cork bobbing in the water. The intersection of the cork and the surface of the water would form a simple closed curve. Neither the cork nor the water will be significantly changed if the cork were weighted so that it sank beneath the surface of the water. However, the intersection of the cork and the surface of the water would now be empty.

(ii) Suppose an arc C does meet the upper hemisphere H of a bubble more than once. Let $D \subset D_+$ be a disk so that $D \cap H$ contains a rectangle R whose boundary is two arcs of C and two arcs of ∂H. Replace C if necessary with another component of $F' \cap D_+$ inside D (innermost condition). Now there are two possibilities: either as in Diagram 2.3(a) $R \cap t = \emptyset$ or as in Diagram 2.3(b) $R \cap t \neq \emptyset$. In either case, Menasco [M] shows that the surface F' may be isotoped to a surface F in order to eliminate saddles containing C. The methods used only involve the local area of the bubble and still hold for the tangle situation. The final result is a surface F such that (ii) holds.■

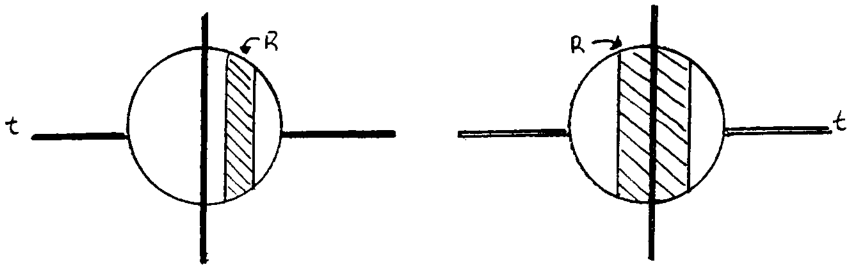


Diagram 2.3 (a) (b)

Definition 5. If F satisfies the two conditions in the statement of Proposition 1, then F will be considered to be in the *standard position.*

In order to classify the complexity of a given intersection pattern of a surface and a projection diagram, the complexity of a given intersection pattern will be given by the pair (σ, δ) ordered lexicographically where σ is the number of saddles and δ is the number of arcs in $D_{\pm}$. From this point on we will only concern ourselves with intersection patterns with minimal complexity; further we recall F is to be in the standard position for the duration of this paper.

3 Properties of Intersection Pattern

Let (B, t) be a connected alternating tangle and F be a 2-sphere in the complement. Further let n_1, n_2, n_3 and n_4 be the number of arcs in each quadrant of $D_{\pm} \cap F$ as indicated in Diagram 3.1(a).

Proposition 2.

If $n_i \neq 0$ for some i, then

(i) $n_i \neq 0 \ \forall i$

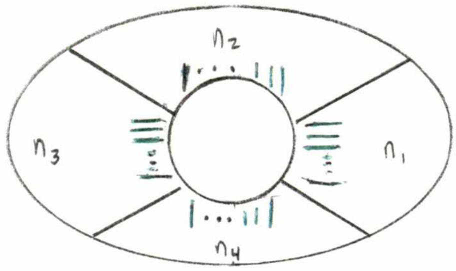
(ii) $n_1 = n_3$ and $n_2 = n_4$.

Lemma 1.

No arc of $D_{\pm} \cap F$ may enter and leave through the same quadrant.

Proof.

Assume without loss of generality that there exists an arc that enters and leaves through quadrant 1; then there must be an outermost such arc. Consider that outermost arc K. Suppose that this arc does not enter and leave through the same bubble, as in Diagram 3.1(b). The arc may be assumed to cross on the right hand side of the bubble, as in the bottom of Diagram 3.1(b). If this were not so, there would be another arc K' on the outside of the bubble; however, K was outermost. K' would have to enter and leave through quadrant 1 since it cannot intersect K and it cannot be a simple closed curve (Proposition 1(ii)). Since the tangle is alternating, the next bubble encountered must be met on the left or inner side. K must in fact meet another bubble or K would be entering and leaving through different quadrants. Then there exists a corresponding arc on one of the bubbles that is outside the outermost arc by the same argument for K', which is a contradiction. Therefore, the arc must not enter and leave through the same quadrant.■



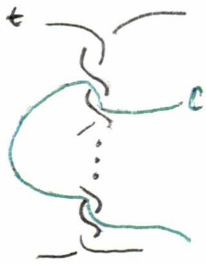
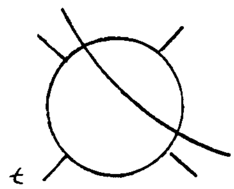


Diagram 3.1 (a) (b)

Proof of Proposition 2.

By the lemma we know that if $n_i \neq 0$ for some i then $n_j \neq 0$ for some $j \neq i$ since each arc must enter and leave through a different quadrant. Consider each crossing in the following way  becomes  . Certainly there may be more than one saddle at any particular crossing (the notation for this situation will be dealt with later and does not significantly impact the proof). Now suppose there exists some offending arc that does not enter and exit in opposite quadrants as in Diagram 3.2(a). Then there is a disk D formed by the outside of t and the offending arc. However, there is an odd number of arcs of t entering into D: two arcs from any crossing and one arc from the arc of t going to the boundary of B as in Diagram 3.2(b). Since it is impossible for an odd number of arcs to all join, we have a contradiction and the offending arc cannot exist. Thus, the number of arcs in opposite quadrants is equal or $n_1 = n_3$ and $n_2 = n_4$.■



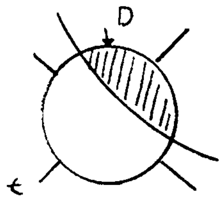


Diagram 3.2 (a) (b)

4 Rational Tangles

We will consider a rational tangle as any tangle that can be constructed by beginning with the untangle and creating a crossing by twisting any two incident arcs. By this construction we can see that any rational tangle is isotopic to the untangle. *Note*: This isotopy does not fix ∂B. However, by our notion of equivalence where the boundary is fixed by the homeomorphism, the tangles in Diagram 4.1 are pairwise inequivalent. Diagrams 7(a), 7(b), 7(c) and 7(d) are examples of some common rational tangles.

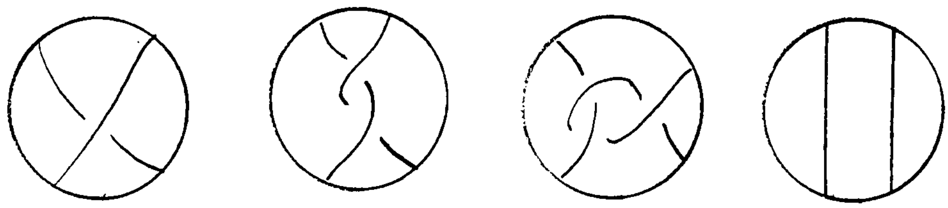


Diagram 4.1 (a) (b) (c) (d)

Proposition 3.

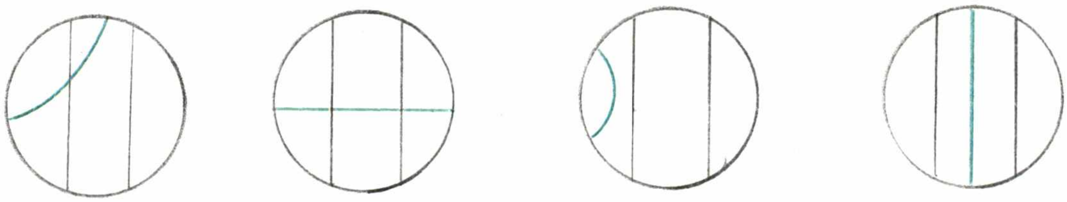
The only essential surface in the complement of a rational tangle is the union of disks.

Proof a.

Consider the doubling of two uncrossed arcs. A link of two unknotted, unlinked components is formed. The fundamental group of the complement is a free group of rank 2. Any fundamental group of an essential surface in the complement will have to be isomorphic to a subgroup of the free group, which is also free. However, the fundamental group of a closed surface is never free unless the group is trivial; thus, the only essential surface in the complement of the doubling must be a union of 2-spheres. This corresponds to the union of disks doubled in the complement of the untangle.■

Proof b.(alternate proof)

We will consider the intersection pattern of any surface in the complement of a rational tangle and then isotope the tangle back into the untangle form  . Now any arc of the intersection will fit one of the the following four cases.



Case 1 Case 2 Case 3 Case 4

Diagram 4.2

In cases 1-3, there is an outermost such arc which can be "pushed out" by means of isotoping the intersecting surface so that the arcs are removed from the intersection pattern; thus, all such arcs are non-essential. Clearly these patterns may exist in combination with the exception that case 2 and case 4 cannot exist simultaneously. It is a simple matter to deal with each case as it occurs in combination via an innermost removal of arcs. An arc intersecting t is simply a projection of a surface that contains an arc isotopic to a meridian curve. However, there is no means by which to eliminate arcs in case 4 which is isotopic to a disk in (B, t). Suppose that the disk contains some hole, then there will exist a compression disk in the surface. A compression disk cannot exist an essential surface; therefore, the proposition holds.■

Now that we know what we are dealing with in the complement of these rational tangles, it is helpful to determine the intersection pattern for any rational tangle. However, we must first clearly understand a geometric maneuver which crosses or uncrosses two arcs and how this maneuver affects the intersection pattern. Consider the intersection pattern of $F \cap D_{\pm}$ of a rational tangle (B, t). We wish to see the change in the intersection pattern as an incident crossing is uncrossed.

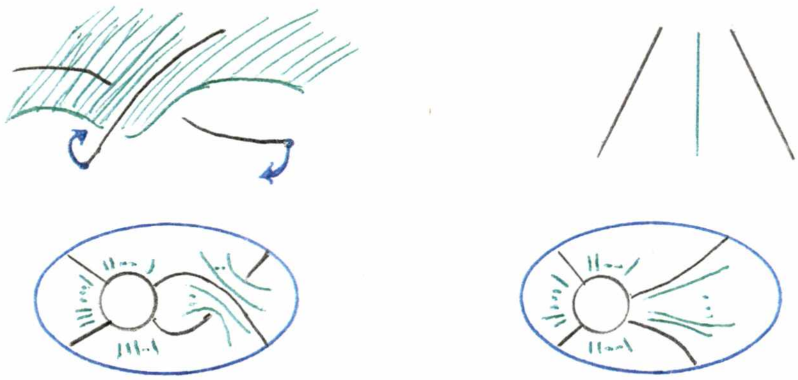


Diagram 4.3

The surface is untwisted as the crossing is eliminated by the isotopy, Diagram 4.3. Essentially this can be reduced to a local question of a disk in the complement of a 0-crossing rational tangle. We can think of the disk as a sheet separating the two arcs and how it would transversely intersect $D_{\pm}$. A saddle is formed as the two arcs are crossed as in the first part of Diagram 4.3. Then as the sheet is jerked taut or flat, the arcs are forced apart. The sheet still remains between the arcs as in the second part of Diagram 4.3, so the intersection appears drastically different.

Another way to consider this problem is to think of the 3-dimensional situation. Diagram 4.4(a) attempts to illustrate the complete geometric situation, where the ball and projection diagrams are given together. Diagram 4.4(b) is like Diagram 4.4(a) except that a nonessential arc is included to help visualize the disk separating the arcs. We should also note that this change is local and unique up to isotopy in the disk. Locally the maneuver involves a disk in the complement, which has a unique intersection pattern. The disk is changed to a saddle, which also has a unique intersection pattern formed by the crossing. This maneuver proves to be the key to the next several propositions. A diagram will be considered unique up to isotopy in the disk.

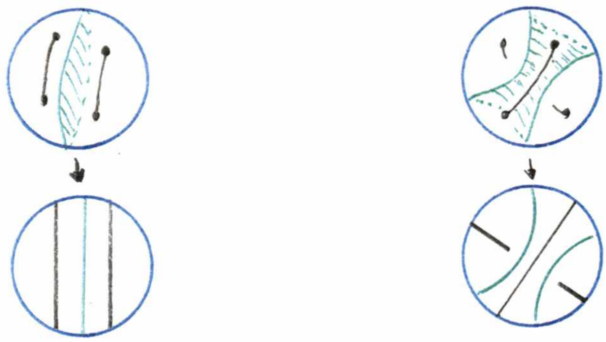


Diagram 4.4 (a) (b)

Proposition 4.

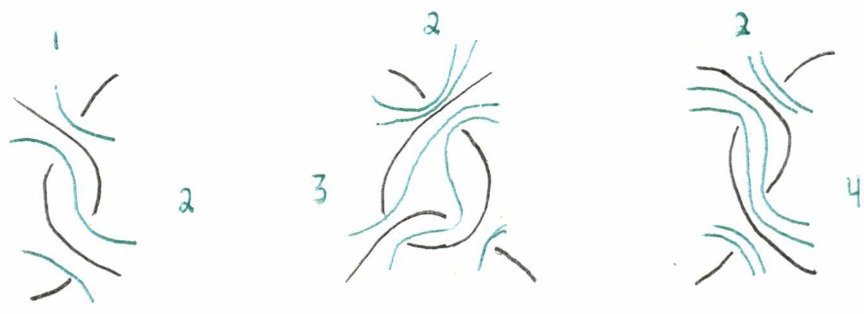
The intersection pattern of a surface F with a rational tangle (B, t) is unique up to isotopy in the disk $D_{\pm}$.

Proof.

We may assume that F has been isotoped to eliminate any trivial intersections with

$D\pm$. Because the union of disks is the only essential surface in the complement of the 0-crossing tangles, its intersection pattern is unique. By the inductive hypothesis, we may assume that the intersection pattern is unique for an n-crossing rational tangle. Consider an intersection pattern for a (n+1)-crossing rational tangle. We know that there is a crossing incident to the exterior of $D_{\pm}$. However, the incident crossing of the (n+1)-crossing tangle can be uncrossed using the previously mentioned geometric maneuver resulting in an n-crossing tangle with a unique intersection pattern. Thus, if there is more than one distinct intersection pattern for the (n+1)-crossing tangle, then the geometric maneuver implies there is more than one distinct intersection pattern for the n-crossing tangle which is a contradiction of the inductive hypothesis. Therefore, there is a unique intersection pattern for any rational tangle.■

Example 4.1(a), (b) and (c) provide some particular examples of rational tangles and their intersection patterns.



(a) (b) (c)

Example 4.1

At this time it is convenient to make several observations concerning rational tangles.

Remark 1. If (B, t) has more than one crossing, then $n \neq m$ in the intersection pattern as in Diagram 4.5 since at the formation of any additional crossing the new number of arcs in the non-incident quadrant becomes m + n $\neq$ n, m. We will not be considering the case of the single crossing; therefore, we may assume $n \neq m$.

Remark 2. Adding another disk in the complement simply adds a copy of the basic intersection pattern, Example 4.1(c).

Remark 3. If we have a rational tangle with an intersection pattern as in Diagram 4.5(a), then creating a crossing at the left or right will result in Diagram 4.5(b). This is useful because if $n < m$, then we know that there exists a crossing incident to the exterior in one of the quadrants containing the smallest number of arcs. In fact if a tangle contains a crossing incident to the exterior, then the smallest number of arcs in $F \cap D_{\pm}$ occurs in the corresponding quadrant.

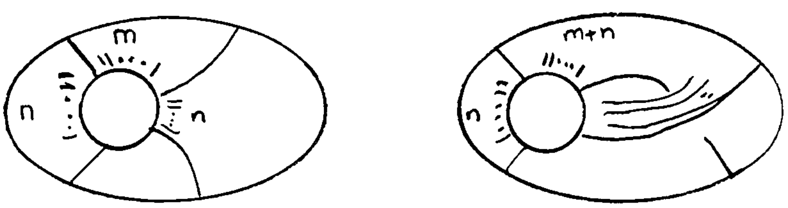


Diagram 4.5 (a) (b)

Remark 4. $n \times m$ intersection patterns can be constructed from a rational tangle. Intersection patterns are built up recursively, so some intersection patterns only occur as multiples of smaller patterns. For example, a 12×4 intersection pattern is just a 3×1 intersection pattern for a rational tangle with 3 extra disks in the complement. In general a $ps \times s$ intersection pattern is a $p \times 1$ intersection pattern with $s - 1$ disks added to the complement. If we restrict consideration to a single disk in the complement, then the intersection pattern can be show to be unique up to isotopy in the disk.

Remark 4 deserves some explanation; this explanation turns out to be quite elegant and is considered a proposition.

Proposition 5.

For any $n, m \, \epsilon N$ then there exists a rational tangle with an $m \times n$ intersection pattern.

The proof is based on the following process. Begin with a $m \times n$ intersection pattern with $n < m$ and reduce it to a $(m-n) \times n$ intersection pattern by untwisting an incident crossing. Consider $min\{m - n, n\}$ and $max\{m - n, n\}$; subtract the *minimum* from the *maximum* and repeat the process. Of course this process is really finding the greatest common divisor which exists for any two natural numbers. The result is an $\alpha \times \alpha$ intersection pattern where $\alpha = GCD\{n, m\}$. Again we subtract and are left with $\alpha \times 0$ intersection pattern, which is a 0-crossing rational tangle with α disks in the complement. The original $m \times n$ intersection pattern can then be recovered by creating crossings in the manner opposite to subtracting minimums from maximums.

Proof.

If n = m, then the single crossing with n disks satisfies the proposition. So without loss of generality assume $n < m$. Let c_1 be the minimum integer so that

$$g_1 = m - c_1 n \leq n.$$

Then let c_2 be the minimum integer so that

$$g_2 = n - c_2 g_1 \leq g_1$$

or in general c_i is the minimum integer such that

$$g_i = g_{i-2} - c_i g_{i-1} \leq g_{i-1}.$$

Let c_k be the c_i so that $g_k = g_{k-1}$. c_k must exist since n and m are finite and this process is just finding the greatest common divisor of m and n. Now we let $g_k = GCD\{m, n\}$. Begin with g_k disks in the complement of a 0-crossing tangle and then create one crossing. Next create c_k crossings in the East using the geometric maneuver and a $g_k \times (g_k + c_k g_k)$ intersection pattern will result. Since $g_k = g_{k-1}$, we have $g_k \times (g_k + c_k g_{k-1})$ by substitution. Solving for g_{k-2} in the previous equations we find the result:

$$g_k + c_k g_{k-1} = g_{k-2}.$$

Diagram 4.6 is a schematic illustration of this construction process. The final product of the c_k crossings is a $g_{k-1} \times g_{k-2}$ intersection pattern. Now we create c_{k-1} crossings in the South direction so that a $g_{k-3} \times g_{k-2}$ intersection pattern is produced. We continue this process alternating East and South direction until the desired $n \times m$ pattern is achieved. Since all new crossings created in this manner are incident to the exterior, the resulting tangle is rational.■

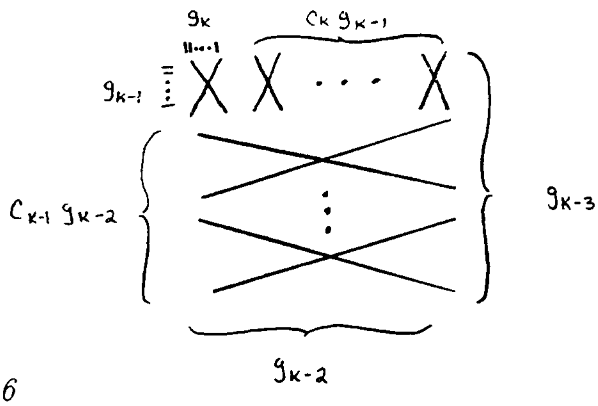
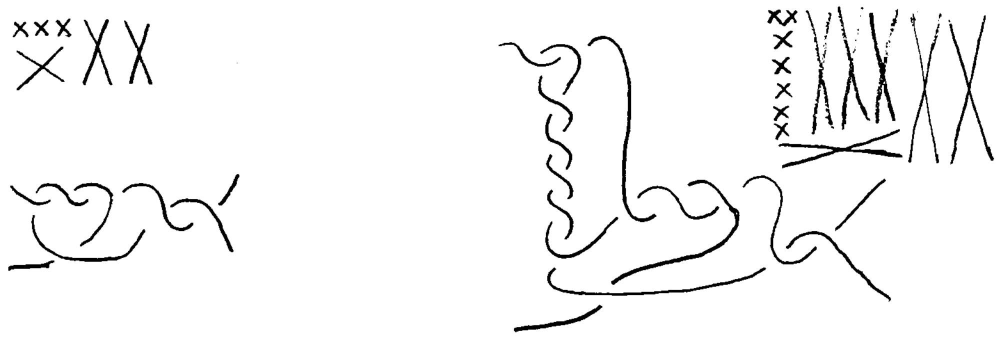


Diagram 4.6

It is convenient to develop a formula for the resulting n, m in any given intersection pattern of a rational tangle; our restriction of only allowing one disk in the complement will now be added, as well as knowledge of the number and direction of the formation of incident crossings. Incident crossings need only be made to the right or down since crossings incident to the right or left, moreover, top or bottom may be shifted to the opposite direction by utilizing a flyping maneuver. When flyping a tangle, the boundary of the ball containing the tangle remains fixed while the entire interior of the ball is isotoped so that the incident crossing on one side is uncrossed while simultaneously an incident crossing is created on the opposite side. Flyping is illustrated in Diagram 4.7.

Diagram 4.7

By convention, we will make right crossings first followed by downward crossings. This process will be continued in an alternating fashion. We will call a rational tangle formed in this fashion a *standard* rational tangle. Let $a_1, a_2, ...a_k$ signify a_1 right crossings followed by a_2 bottom crossings etc.... Example 4.2(a) is a 3, 1, 2 and Example 4.2(b) is a 2, 5, 3, 1, 2.



(a)

(b)

Example 4.2

Now we introduce the following notation for continued fractions, as borrowed from Conway [C]:

$$[a_1, a_2, ...a_k] = a_k + \frac{1}{[a_1, a_2, ...a_{k-1}]} \text{ and } [a_1] = a_1$$

Corollary.

If there is only one disk in the complement of a rational tangle, then the intersection pattern of $F \cap D_{\pm}$ is unique up to isotopy in the disk.

Proof.

For each $p \in N$ let a_{2p+1} be the number of consecutive right crossings and a_{2p} the number of consecutive bottom crossings; then $[a_1, a_2, ...a_k]$ is the record of twists for any given rational tangle. Consider that for the intersection pattern

$a_1 \times 1 \Leftrightarrow [a_1] = \frac{a_1}{1}$.

Now we can assume the inductive hypothesis on p

$a_n \times a_{n-1} \Leftrightarrow [a_1, a_2, ...a_{2p}] = a_{2p} + \frac{1}{[a_1, a_2, ...a_{2p-1}]} = \frac{a_n}{a_{n-1}}$.

Notice by Remark 3 that $a_{n+1} = (a_n)(a_{2p+1}) + a_{n-1}$ since a_{2p+1} corresponds to right hand crossings or more importantly we have

$a_{2p+1} = \frac{a_{n+1} - a_{n-1}}{a_n}$

Therefore,

$a_n \times a_{n+1} \Leftrightarrow [a_1, a_2, ...a_{2p+1}] = a_{2p+1} + \frac{1}{[a_1, a_2, ...a_{2p}]} = a_{2p+1} + \frac{a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n}$.

Since continued fractions have unique representations, the construction of right and bottom crossings produces a unique fraction with a unique rational tangle construction [C]. If there were another intersection pattern, it would have to have a continued fraction representation equivalent to the first; thus, the intersection patterns are equivalent up to isotopy.■

5 Lattice Diagrams

We can develop a method of calculating the number of saddles in an arbitrary $n \times m$ intersection pattern if we know the pattern is for a rational tangle and we know the number of disks in the complement. In fact it can be shown that $\delta_+ - \sigma = k$ (repectively $\delta_- - \sigma = k$) where δ_+ (respectively δ_-) is the number of arcs in D_+ (respectively D_-), σ is the number of saddles and k is the number of disks in the complement. Again we consider the geometric maneuver with k disks, as in Diagram 5.1. We find what happens to $\delta_\pm$ and σ as a new crossing is created or removed. The difference $\delta_\pm - \sigma$ remains fixed since both $\delta_\pm$ and σ increase by k as a new crossing is created and both decrease by k if a crossing is eliminated. Thus, we can untwist all the crossings of a rational tangle back to the untangle and $\delta_\pm - \sigma$ will remain fixed. In the untangle case, $\sigma = 0$ and $\delta_\pm$ is the number of disks in the complement i.e. $\delta_\pm = k$. Since this reduction is possible for any rational tangle, we have the result $\delta_\pm - \sigma = k$ or $\sigma = \delta_\pm - k$; however, this exact number for σ is not incredibly helpful since it depends on the number of disks in the complement which is not often readily known. We can develop a better estimate on σ with the concepts of lattice diagrams.

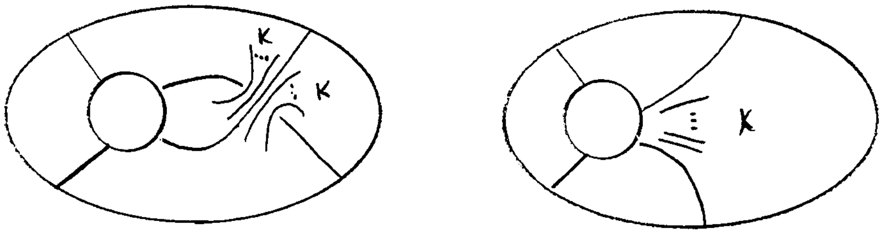
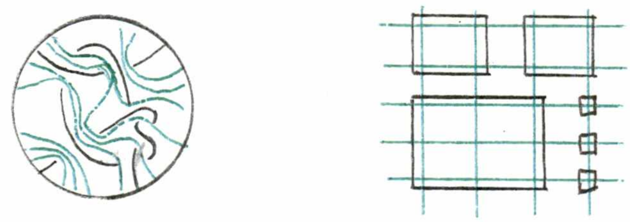


Diagram 5.1

In the process of looking for a better estimate for σ , we develop the following ideas. Recall the use of  to represent a crossing in a tangle projection diagram. This symbolism induces a lattice diagram that represents the reduced intersection pattern of a surface F and an alternating tangle diagram. Each box in the lattice diagram (also referred to as a *crossing square*) represents a single crossing. We also find each box must be square; thus, an $n \times n$ box represents an n-crossing square with n saddles. Example 5.1 gives some examples of $F \cap (B, t)$ and the corresponding lattice diagrams.

Example 5.1(a)



Example 5.1(b)

Now we would like to introduce the means by which a tangle diagram can be generated from a lattice diagram. *Note*: When joining crossing squares, the arcs may not cross any lattice arc, crossing square, or arc joining other crossing squares.

Proposition 6.

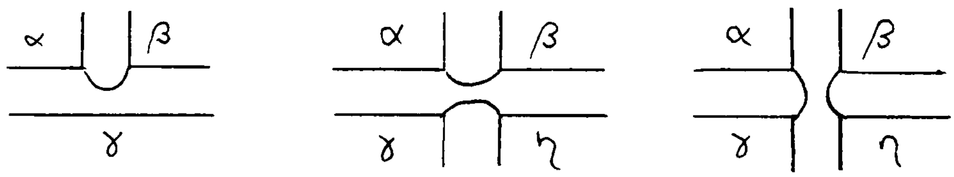
Given any lattice diagram and crossing square, there exists a corresponding tangle projection diagram.

Proof.

We need only to consider the ways that the crossing squares are arranged so that they may be connected to form a tangle projection. Let $\alpha, \beta, \gamma, \eta$ be crossing squares; then the squares can only occur in the two general cases.

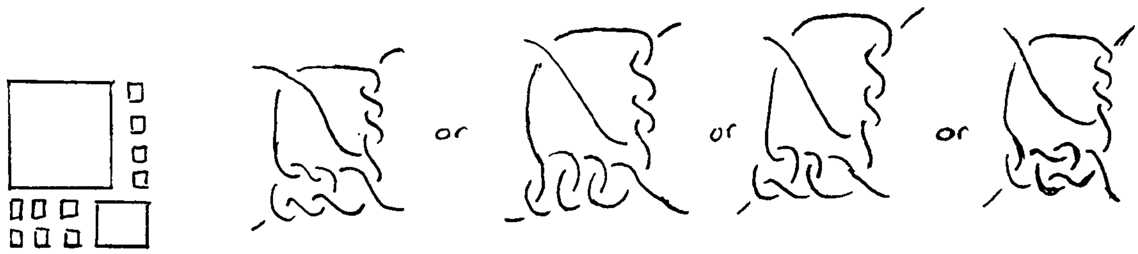
In case 1, α, β can be joined by an arc. Case 2 presents two possibilities: either join α to β and γ to η or join α to γ and β to η with arcs. In both cases all crossing squares can be joined with arcs that connect the crossings. Thus, if these connection arcs are constructed at each meeting of the crossing squares, then each lattice will represent at least one tangle.■

Each time case 2 occurs, two tangles may result as in Example 5.2(a) although two tangles might not result as in Example 5.2(b).

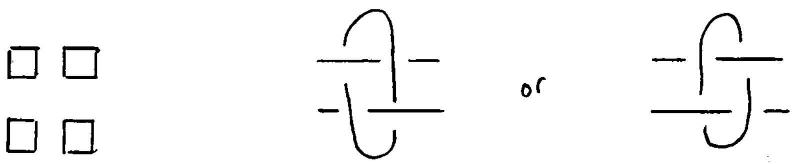


Case 1 Case 2(a) Case 2(b)

In Example 5.2(a) we see how a tangle can be constructed from a lattice diagram. Although it is a simple tangle, Example 5.2(b) shows that the same tangle may result regardless of how the connections are produced. Notice that case 2 occurs twice, so 4 possible tangles result from the construction.



Example 5.2(a)



Example 5.2(b)

Using lattice diagrams and crossing squares gives a precise method of counting saddles in the tangle projection diagram since an $n \times n$ square contains n saddles. Thus, if $\{a_i\}_{i=1}^n$ is a set of all the crossing squares and $\|a_i\|$ is the side length of a_i, then $\sigma = \sum_{i=1}^n \|a_i\|$ where σ is the number of saddles in the associated tangle projection diagram. Therefore, we have $\sigma \leq nm$ where $\sigma = nm$ only if each crossing square contains only a single saddle: that is, all crossing squares are 1×1. The upper bound is only achieved in the event that either $m = 1$ or $n = 1$. If the upper bound is achieved, then $\delta_{\pm} - \sigma = k$ or $(m+n) - nm = k > 0$ which can only happen if neither $m \neq 1$ nor $n \neq 1$. In the case of the rational tangle, we can produce an accurate lower bound on σ. Recall that a rational tangle has one crossing incident to the exterior, so in any lattice diagram there must exist an n-crossing square if the tangle has a $n \times m$

lattice diagram where $n < m$.

Proposition 7.

If $F \cap D_{\pm}$, then $2n \leq \sigma \leq nm$ where σ is the number of saddles in $F \cap D_{\pm}$ and F is a 2-sphere.

Proof.

Without loss of generality, we may assume that Diagram 5.2 represents the $n \times m$ lattice diagram. We know that there exist crossing squares $a_i (1 \leq i \leq q)$ where each a_i is a crossing square so that $m_1 \cap a_i \neq \emptyset$ and $\sum_{i=1}^{j} ||a_i|| = n$ since each crossing must be in some box. Further since $n < m$ and all boxes are square, we know there exists $b_i (1 \leq i \leq p)$ where each b_i is a crossing so that $m_m \cap b_i \neq \emptyset$ and $\sum_{i=1}^{q} ||b_i|| = n$. Also $a_i \neq b_j$ since $||a_i||, ||b_j|| \leq n < m \; \forall i \leq q, j \leq p$. We know the a_i's and the b_j's are disjoint and no square is larger than $n \times n$. Thus, $\sigma \geq 2n$ or $2n \leq \sigma \leq nm$.■

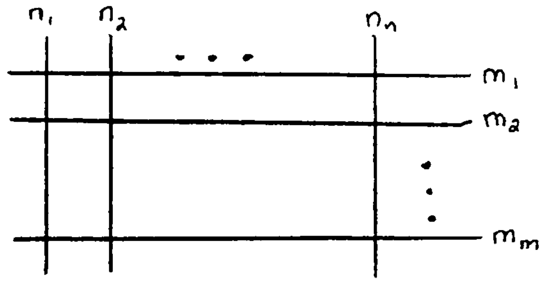


Diagram 5.2

Example 4.1(a) shows a rational tangle with exactly 2n saddles.

Corollary to Proposition 7.

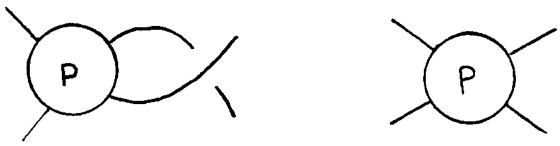
If (B, t) is a rational tangle and $F \cap D_{\pm}$ has an $n \times m$ lattice diagram, then $\sigma \geq m \geq n$.

Proof.

We need to sum the lengths of the crossing squares that intersect n_1 as in Diagram 5.2.■

6 Analysis of Prime Tangles

As with rational tangles, prime tangles also have projection diagrams and intersection patterns with some surface F that yield lattice diagrams, that is Propositions 1 and 2 apply equally to prime tangles. In fact, Example 5.2 (a)and Example 5.2(b) are both prime. However, we may assume that our prime tangles have been isotoped to eliminate any crossings incident to the exterior as well as to minimize the number of crossings. Example 6.1 shows the elimination of an incident crossing. Any incident crossing will simply be absorbed by the rational tangle by the flyping of the prime tangle.



Example 6.1

What does this mean for any corresponding $n \times m$ lattice diagram? Since any incident crossing can be eliminated then all crossing squares must have side length less than n where $n < m$. Now we can find a better lower bound on σ when the tangle is prime.

Proposition 8.

If (B, t) is a prime tangle, $F \cap B \neq \emptyset$ and F is in the standard position then $2m \leq \sigma \leq nm$ $(n < m)$ where $F \cap D_{\pm}$ has an associated $n \times m$ lattice diagram.

Proof.

The proof is similar to that of Proposition 7, so we will refer to Diagram 5.2. Let $\{a_i\}_{i=1}^{q}$ be the set of crossing squares such that $n_1 \cap a_i \neq \emptyset$; then $\sum_{i=1}^{q} ||a_i|| = n$. Thus, $\sigma \geq m$. Moreover, let $\{b_i\}_{i=1}^{p}$ be the set of crossing squares such that $n_n \cap b_i \neq \emptyset$; then, $\sum_{i=1}^{p} ||b_i|| = m$. In addition, $a_i \neq b_j$ since $||a_i||, ||b_j|| < n-1 \; \forall \; i, j$ (which is the result of no incident crossings); thus, $2m \leq \sigma \leq nm$.■

This concludes the analysis for general prime tangles, but we can say more for solid tangles. We can make the following definition.

Definition 6. A *solid* tangle (B, t) is a prime tangle such that any properly embedded disk D intersecting t twice separates t into at least one unknotted arc.

Proposition 9.

If (B, t) is a solid tangle, $F \cap B \neq \emptyset$ and F is in the standard position, then $2m + 1 \leq \sigma \leq nm$ $(n < m)$ where $F \cap D_{\pm}$ has an associated $n \times m$ lattice diagram. **Proof.**

Now we assume that the set of all crossing squares for a solid tangle (B, t) is $\{a_i\} \cap \{b_j\}$ (for a_i and b_i as discussed in the proof of Proposition 7) which implies $\sigma = 2m$. Now without loss of generality we may assume that $||a_1|| + ||b_1|| = n$. We will consider the following cases: (i) $||a_1|| = ||b_1||$ or (ii) $||a_1|| \neq ||b_1||$.

Case (i). If $||a_1|| = ||b_1||$, then $\exists \; a_2, b_2$ since $m \geq n$ and $||a_1|| = ||b_1|| \neq n$. If $||a_1|| \neq ||a_2||$, then the following graphical situation occurs as in Diagram 6.1(a) or Diagram 6.1(b). In either case a vertical line can be drawn representing a 2-sphere intersecting t twice and bounding a knotted arc which cannot occur since (B, t) is solid. Thus, $||a_n|| = ||a_{n+1}||$ and $||b_n|| = ||b_{n+1}||$ $\forall n$ which implies Diagram 6.1(c). If at any time in forming the link Diagram 6.1(d) occurs, then a 2-sphere can be constructed intersecting t twice and bounding a knotted arc. Thus, Case (i) cannot occur since our tangle is solid.

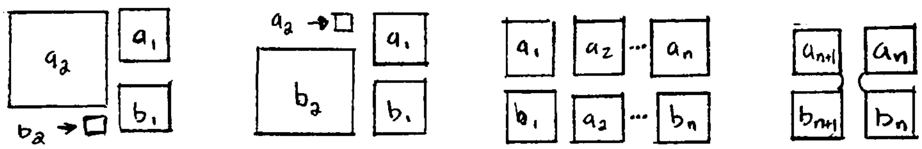


Diagram 6.1 (a) (b) (c) (d)

Case (ii). Without loss of generality assume that $||a_1|| < ||b_1||$, which implies $||a_1|| = ||a_2||$. If $lcm(||a_1||, ||b_1||) = ||b_1||$ as in Diagram 6.2(a), then $\sum_{i=1}^{n} ||a_i|| = ||b_1||$. This is essentially the same problem as Case (i) with the rectangle formed by $\{a_i\}_{i=1}^{n}$ playing the role of a_1, which we know may not occur for a solid tangle (B, t). If $lcm(||a_1||, ||b_1||) = \phi \neq ||b_1||$ as in Diagram 6.2(b), then $||b_1|| = ||b_2||$. Thus, $||a_1|| = ||a_n|| \forall n \leq N$ such that $\sum_{i=1}^{n} ||a_i|| = \phi$ and $||b_1|| = ||b_n|| \forall n \leq n_0$ such that $\sum_{i=1}^{n_0} ||b_i|| = \phi$.

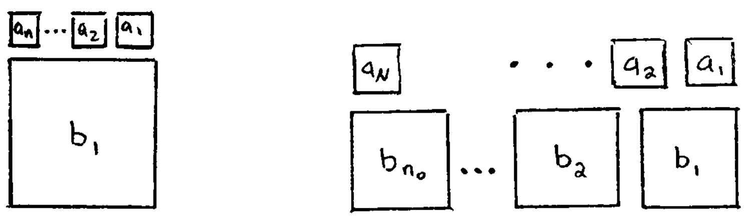
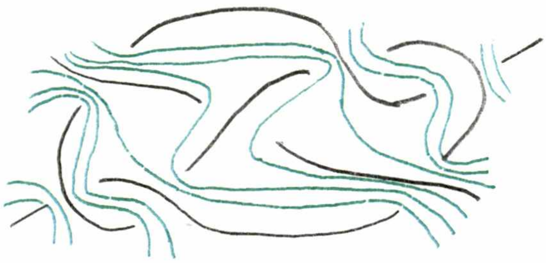


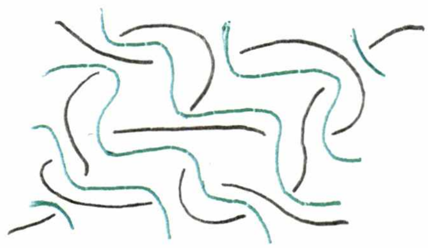
Diagram 6.2 (a) (b)

If $\phi = m$, then the link is not solid since a horizontal line between a_i and b_i will intersect t twice and bound a knotted arc. If $\phi \leq m$, then we have case (1) with rectangles $\{a_i\}_{i=1}^{N}$ and $\{b_j\}_{j=1}^{n_0}$ playing the roles of a_1, b_1 respectively. Thus, the set $\{a_i\} \cup \{b_j\}$ must not be all of the crossing squares; therefore, for a solid tangle $2m+1 \leq \sigma \leq nm$.■

Example 6.2(a) shows a solid tangle with a minimal number of $2m+1$ saddles and 6(b) shows a solid tangle with a maximal number of nm saddles.



(a)



(b)

Example 6.2

7 Concluding Results

It is convenient at this time to introduce a definition of the types of links to which the results will apply.

Definition 7. A link diagram L is said to be *semi-alternating* if it admits to a diagram D such that

i) D is non-alternating

ii) D is the sum (as in Diagram 7.1) of alternating diagrams D_1 and D_2 where D_1 is prime (i.e. has no incident crossings) and D_2 is a standard rational tangle diagram.

Once again we lean on Menasco's result; the Euler characteristic of a surface in the complement of a link is given by $\chi = \delta - \sigma$ where δ is the number of domes that is the number of distinct arcs in $F \cap S_\pm$ - and σ is the number of saddles in $F \cap S_\pm$. We can show that the Euler characteristic is not 2 for any surface in the complement of $L = (B,t) \cup_f (B_1, t_1)$ where (B,t) is prime and (B_1,t_1) is alternating connected and has more than one crossing (i.e. L is semi-alternating).

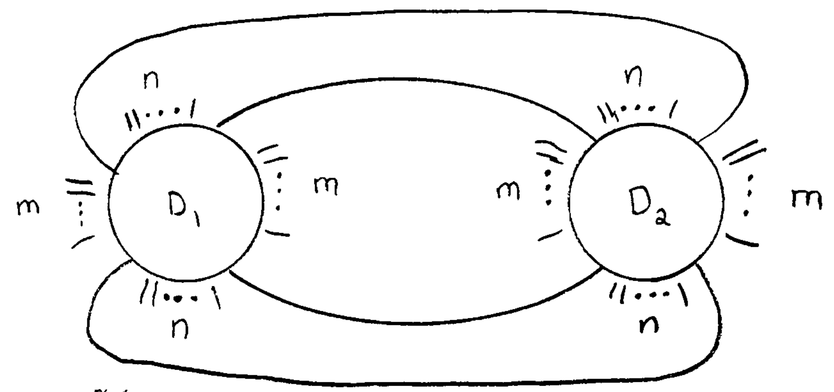


Diagram 7.1

Theorem 1.

If L is semi-alternating, then L is non-split.

Proof.

Let F be a 2-sphere. In the complement F needs to be standard as in [M], that is we may assume L is split. Consider $F \cap \pi(L)$. We will consider the reduced intersection pattern of $F \cap S_{\pm}$, where F is in standard position as in Diagram 7.1.

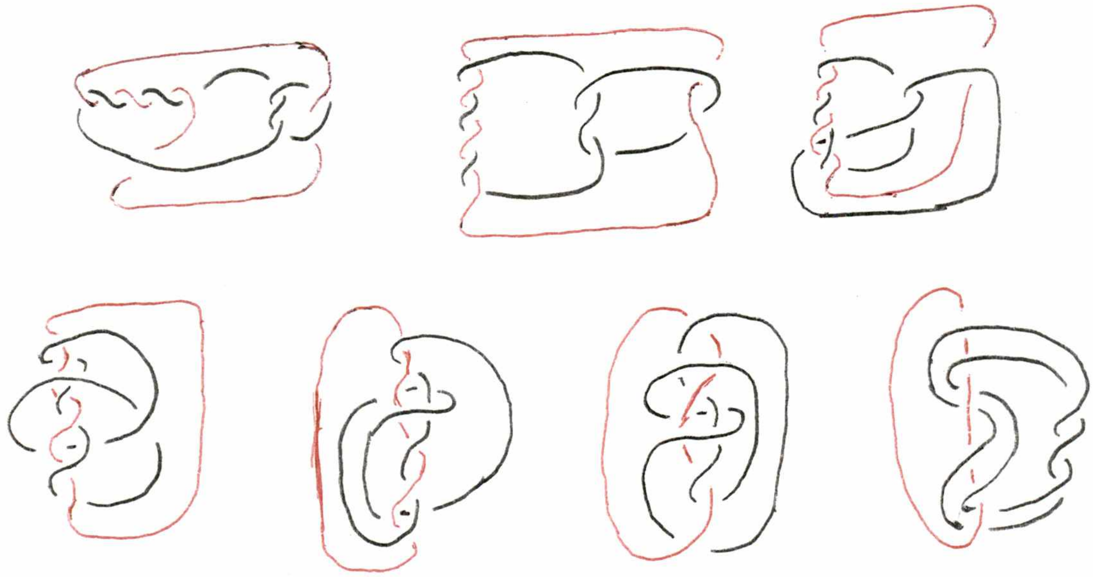
By Menasco we need only to find the Euler characteristic of the surface by counting arcs and saddles of the reduced intersection pattern. We may assume that $n \leq m$. If $m \leq n$, then the saddle contribution from D_1 is 2n and the saddle contribution from D_2 is at least 2m . The following inequality will still hold. Now we count the total number of saddles by looking inside each tangle. We know that $\sigma_{D_1} \geq 2m$ since D_1 is prime. Moreover we know $\sigma_{D_2} \geq 2n$. Thus we have:

$$\chi_F = (\delta_{D_1} + \delta_{D_2}) - (\sigma_{D_1} + \sigma_{D_2}) \leq 2(m+n) - 2m - 2n = 0$$

Therefore, F cannot be a 2 sphere; and it follows that L is non-split.■

Finally, Example 7.1(a) shows two tangles that have intersection patterns that simply will not match. The only rational tangle diagram that will match this prime tangle diagram is the single crossing with two disks in the complement as in Example 1.1, which is untangled. Example 7.1(b) shows how the link can be isotoped to see the torus in the complement. In Example 7.1(b) we have a 6×5 intersection pattern, so m = 6 and n = 5; thus, $\sigma_1 = 10$ and $\sigma_2 = 12$. $\chi = 2(6+5) - 22 = 0$, so we expect to find a torus in the complement.

Example 7.1(a)



Example 7.1(b)

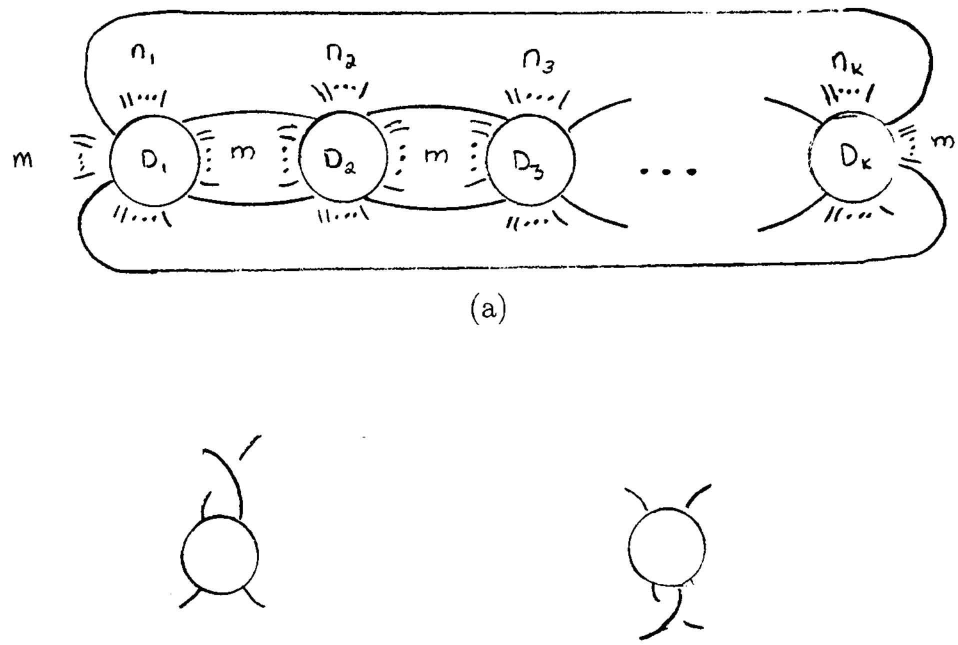
Now let's introduce another way to sum tangle diagrams by the following definition.

Definition 8. A Link L has a *Daisy Chain* diagram D if it admits to a partial summing of standard alternating tangle diagrams $\sum_{i=1}^{p} D_i$ in a semi-alternating fashion where the D_1 and D_p are joined to close the sum such that

i) D_i is prime for some i

ii) D_i has at least 2 crossings $\forall i$.

A daisy chain is illustrated in Diagram 7.2(a).



(a)

Diagram 7.2 (b) (c)

We can tighten our hypothesis a bit more and produce a result concerning the sum of more than two tangles. We let L be a link that admits a daisy chain diagram. Without loss of generality, we may assume that all incident crossings of the prime tangle diagrams must occur in the vertical position as in Diagram 7.2(b), (c). If they do not, then they can be flyped so that the incident crossings are absorbed by the rational tangles or possibly cancel out crossings so that fewer crossings occur.

Lemma.

If L is a link that admits a daisy chain diagram where D_1 is the prime tangle and all are oriented as in Diagram 7.2(b) and (c), then L is non-split.

Proof.

Consider the intersection pattern of the projection of F, a surface in the complement of the link in standard position. Since all incident crossings occur as in Diagram 7.2 (b) and (c), then we know that $m \geq n_i \quad \forall i \geq 2$. As in Theorem 1, we need only to calculate the Euler characteristic χ. Since each arc in S_+(respectively S_-) must join another arc in S_+(respectively S_-), we know that total number of domes δ is bounded by the following inequality:

$$\delta \leq 2m + 2\sum_{i=1}^{k} \frac{n_i}{2}.$$

If we recall that all the n_1 arcs are counted elsewhere because they cannot reenter the same tangle we have $n_1 \leq \sum_{i=2}^{k} n_i$. Therefore, the resulting inequality follows for δ:

$$\delta \leq 2(m + \sum_{i=1}^{k} \frac{n_i}{2})$$

$$= 2(m + \frac{1}{2}n_1 + \sum_{i=2}^{k} \frac{n_i}{2})$$

$$\leq 2(m + \sum_{i=2}^{k} \frac{n_i}{2}) + \sum_{i=2}^{k} \frac{n_i}{2})$$

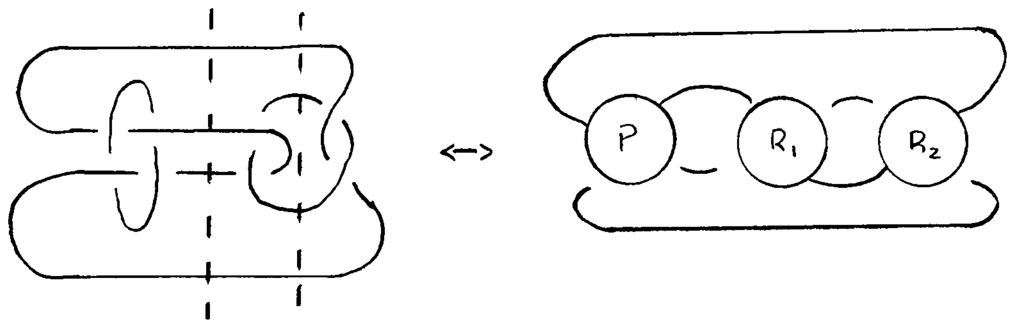
$$= 2(m + \sum_{i=2}^{k} n_i.$$

However, we know that $\sigma_{D_1} \geq max\{2m, 2n_1\}$ and $\sigma_{D_i} \geq 2n_i \ \forall i \geq 2$. Thus, if F is any surface in the complement of the link L, then

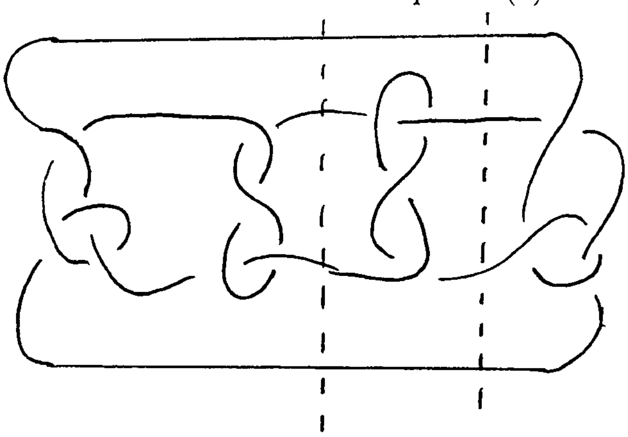
$$\chi_F = \delta - \sum_{i=1}^{n} \sigma_{D_i} \leq 2m + 2(\sum_{i=2}^{k} n_i) - max\{2m, 2n_1\} - \sum_{i=2}^{k} 2n_i \leq 0.$$

Thus, the surface cannot be a 2 sphere or L is not split.■

Example 7.2(a) provides a link that is a daisy chain of tangles and has Euler characteristic less than oe equal to 0 while Example 7.2 (b) has Euler characteristic less than oe equal to -6. These examples motivate us to take a closer look at tangles not of the type in Diagram 7.2(b),(c). Now with the use of the lemma we can prove a larger result.



Example 7.2(a)



Example 7.2(b)

Theorem 2.

If L is a link that has a daisy chain diagram so that at least one of the tangles is prime and all of the tangles are alternating with at least two crossings, then L is non-split (note that the orientation of the incident crossing is no longer a hypothesis).

Proof.

Consider the intersection pattern of the projection of F, a surface in the complement of the link in standard position. First we need to consider the case where incident crossings occur in a horizontal position in the tangles. Now we will focus our attention on the incident crossing and flyp that crossing one tangle at a time until all horizontal incident crossings are to the far right and all in a row of r consecutive crossings. Again we can use Diagram 7.2(a) as frame of reference with D_k as the rational tangle formed by the r horizontal crossings. The number of arcs produced in S_+ is $\frac{rm}{2}$ since each arc must meet with another arc in S_+. The contribution $\frac{rm}{2}$ is the same from S_-. Because the number of saddles in D_k is easily calculable as rm (Diagram 7.3), D_k provides a net contribution of 0 to the Euler characteristic. Thus, for counting purposes, the numerical value we obtain in the lemma for the Euler characteristic will still be valid. We have a similar final inquality as in the lemma.

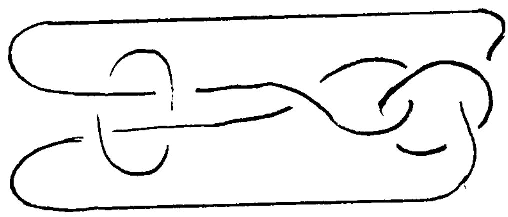
$$\chi_F = \delta - (\sum_{i=1}^{k-1} \sigma_{D_i} + \sigma_{D_i})$$

$$\leq 2m + 2(\sum_{i=2}^{k-1} n_i) + rm - max\{2m, 2n_1\} - \sum_{i=2}^{k-1} 2n_i - rm \leq 0$$

Thus, any Euler characteristic of any surface must be less than or equal to 0 and, therefore, not a 2-sphere.■

Diagram 7.3

Example 7.3 illustrates a tangle as discussed in Theorem 2 and the resulting Euler characteristic is less than oe equal to 0.



Example 7.3

Though the purpose of this paper has reached a conclusion, some speculation and avenues of interest remain. The application and analysis of solid tangles and their extreme denseness could be extended beyond the work presented in this paper. Also from my work and the many examples that I have produced for my own benefit, it appears that these techniques might be adapted to show that under these same hypotheses presented in Theorem 1 and Theorem 2 a far stronger result remains concerning the primality of the sums of tangles. Without exception, for all of my examples, the resulting links formed as the sums of rational and prime tangles were prime. It would seem that under the same hypotheses presented here that these links as sums of tangles are not only non-split but also in fact prime. Nevertheless, this speculation remains to be proven.

References

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Vita

J. Chris Knight was born in Macon, Georgia on May 28, 1972. In 1980 he moved to Knoxville, Tennessee where he finished his education in the Knox County Public School System. After he graduated valedictorian of his class from Doyle High School in May of 1990, in August of 1990 he enrolled in Carson-Newman College to pursue a Mathematics major with certification in secondary education. While attending Carson-Newman, Chris married his high school sweetheart Lisa on December 19, 1992. Upon completion of his B.A. in May of 1994 and graduating Summa Kum Laud, he was awarded a teaching assistanceship at the University of Tennessee Knoxville to pursue a M.S. in mathematics. At the completion of the course work in 1996, he accepted a position teaching mathematics and coaching soccer at Morristown High School West in Morristown, Tennessee. In July of 1998, Chris completed his thesis and was awarded the Master of Science degree in mathematics.

At present he continues to teach and coach in the Hamblen County Public School System in Morristown, Tennessee. He also continues to adjunct instruct for the University of Tennessee Knoxville and Walters' State Community College.