# A Coarse Approach to the Freudenthal Compactification and Ends of Groups

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*To the loving memory of my father, Sami, to my beloved mother, Malikah, to my dear wife, Fatimah, and to my greatest pride and joy, my kids Hadi and Adam.*

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# Abstract

The main purpose of this work is to present a coarse counterpart to the Freudenthal compactification and its corona (the space of ends) that generalizes the Freudenthal compactification of a Freudenthal topological space $X$ (connected, locally connected, locally compact and $\sigma$-compact) and its corona; then applying it to groups as coarse space to obtain generalizations to many well-known results in the theory of ends of groups. To this end, we present two constructions:

1. The Coarse Freudenthal compactification of a proper metric space which is a coarse compactification that coincides with the Freudenthal compactification when the metric space is geodesic.

2. The Freudenthal large scale compactification of a Hausdorff topological coarse space $X$ that coincides with the Coarse Freudenthal compactification when $X$ is a proper metric space equipped with the coarse structure induced by its metric.

Coarse spaces of special interest are coarse groups equipped with coarse structures induced by a generating family, a class of groups that includes: finitely generated groups, countable, metrizable groups, and topological groups.

Considering the space of coarse ends to coarse groups, we present generalization, under the framework of coarse geometry, to classical results in the theory of ends of finitely generated groups and the theory of ends of locally compact, metrizable connected groups; such as the result that the number of ends is either is infinite of at most 2, and the result that, in such groups, groups of two ends are characterized as having a copy of $\mathbb{Z}$ of either finite index (in the case of finitely generated groups) or of compact index in the sense that coset space is compact (in the case of locally compact, metrizable connected groups).

We give the necessary and sufficient conditions for metrizability of the space of coarse ends of a coarse group that is coarsely geodesic; and prove that if the space of coarse ends is infinite and metrizable, then it is a Cantor space; which generalizes the fact that the space of ends of a finitely generated group that has infinitely many ends is a Cantor space.

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# Chapter 1

# Introduction

## 1.1 From Freudenthal compactification and ends of finitely generated groups to Freudenthal large scale compactification and coarse ends of coarse groups

Historically, as noted in (11) on p.287, ends are the oldest coarse topological notion. Internally, ends are described as follows:

**Definition 1.1.1.** A **Freudenthal end** is a decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of components of $X \setminus K_n$, where $(K_n)_{n\in\mathbb{N}}$ is an **exhausting sequence**, i.e. $K_n$ is compact, $K_n \subset int(K_{n+1})$ for each $n \in \mathbb{N}$, and $\bigcup\limits_{n\in\mathbb{N}} K_n = X$. The family of ends of $X$ is denoted by $ends(X)$.

Ends were used by Freudenthal in 1930 in his famous compactification (see (33)):

**Theorem 1.1.2.** *Suppose $X$ is a **Freudenthal space** ($\sigma$-compact, locally compact, connected and locally connected Hausdorff space). It has a compactification $\bar{X}$ such that $\bar{X} \setminus X$ is of dimension 0 and $\bar{X}$ dominates any compactification $\hat{X}$ of $X$ whose corona is of dimension 0.*

**Notation 1.1.3.** *Given a Freudenthal space $(X, \mathcal{T})$. If $U \in \mathcal{T}$, we define the subset:*
$U_{end} := \{(C_n)_{n\in\mathbb{N}} \in ends(X) : C_n \subset U$ *for some* $n \in \mathbb{N}\}$ *and* $\widetilde{U}_{end} := U_{end} \cup U$.

The family $\mathcal{T} \cup \{\widetilde{U}_{end} : U \in \mathcal{T}\}$ is a basis for a topology $\mathcal{T}_{end}$ on $X \cup ends(X)$. The topological space $\bar{X} := X \cup ends(X)$ is a compactification of $X$ called the **Freudenthal compactification**. The space of ends $ends(X) = \bar{X} \setminus X$ is of dimension 0, and $\bar{X}$ dominates any compactification $\hat{X}$ of $X$ whose corona is of dimension 0. Moreover, the number of ends of $X$ is the supremum of $m_n \geq 0$ where $m_n$ is the number of all mutually disjoint unbounded components of $X \setminus K_n$, for all $n \in \mathbb{N}$.

Initially, ends were useful as properties of topological groups:

**Theorem 1.1.4.** ***(Freudenthal)*** *A path connected topological group has at most two ends.*

**Theorem 1.1.5.** ***(Leo Zippin (44))*** *If a locally compact, metrizable, connected topological group $G$ is two-ended, then $G$ contains a closed subgroup $T$ isomorphic to the group of reals such that the coset-space $G/T$ is compact; moreover, the space $G$ is the topological product of the axis of reals by a compact connected set homeomorphic to the space $G/T$.*

**Theorem 1.1.6.** ***(Freudenthal/Hopf)*** *Let $G$ be a finitely generated group. Then:*

1) $|ends(G)|$ *is either infinite or at most* 2.

2) $|ends(G)| = 0$ *if and only if $G$ is finite.*

3) $|ends(G)| = 2$ *if and only if $G$ contains an infinite cyclic subgroup of finite index.*

4) *If $|ends(G)| = \infty$, then the space of ends $ends(G)$ is a Cantor space.*

Realizing that the space of ends of a finitely generated group is a Stone's space (Compact Hausdorff totally disconnected), and that its Boolean algebra of all clopen subsets is isomorphic to the quotient Boolean algebra of the Boolean algebra consists of all almost right invariant subsets by the ideal of finite subsets ($A \subset G$ is called almost right invariant if the symmetric difference $A \Delta A \cdot g$ is finite for all $g \in G$); E. Specker (40) defined the space of ends of an arbitrary group as the Stone's space of the quotient Boolean algebra of the Boolean algebra of all almost right invariant subsets by the ideal of finite subsets. Specker generalized the first three parts of 1.1.6:

**Theorem 1.1.7.** ***(Specker)*** *Let $G$ be an arbitrary group. Then:*

1) $|ends(G)|$ *is either infinite or at most* 2.

2) $|ends(G)| = 0$ *if and only if $G$ is finite.*

3) $|ends(G)| = 2$ *if and only if $G$ contains an infinite cyclic subgroup of finite index.*

Also, considering Specker's definition of space of ends, it is known that for infinite locally finite groups the space of ends is either infinite or of cardinality 1.

**Theorem 1.1.8.** ***(Dicks-Dunwoody, Holt)*** *Let $G$ be an infinite locally finite group. Then:*
1) $|ends(G)| = \infty$*, if $G$ is countable.*
2) $|ends(G)| = 1$*, if $G$ is uncountable.*

Part (1) is due to W. Dicks and M. J. Dunwoody (8); whereas the second part of the above theorem is due to Holt (23). Adopting Specker's approach, W. Dicks and M. J. Dunwoody (8), they proved the following result that is a generalization of the famous theorem, for finitely generated groups, of Stallings (42):

**Theorem 1.1.9.** ***(Dicks-Dunwoody)*** *A group $G$ has infinitely many ends if and only if one of the following conditions holds:*
1) *$G$ can be expressed as an amalgamated free product $A *_C B$, where $C$ is a proper finite subgroup of $A$ and $B$ such that $[B : C] \geq 3$, or;*
2) *$G$ can be expressed as an HNN extension $A*_C$, where $C$ is a proper finite subgroup of $A$.*

Finiteness in coarse groups equipped with the coarse structure induced by the generating set of finite subsets corresponds to boundedness in coarse groups in general.

When a group $G$ is equipped with the coarse structure induced by the generating set of all finite subsets, its almost right invariant subsets are exactly the coarsely clopen subsets of $G$, and its finite subsets are exactly its coarsely bounded subsets. The concept of coarsely clopen subsets is defined for any coarse space $X$. Furthermore, the family of all coarsely clopen subsets $\mathcal{CC}(X)$ of a given coarse space $X$ is a Boolean algebra. Consequently, it is natural to define the space of coarse ends $Ends(X)$ of a coarsely connected coarse space $X$ as the Stone's space of the quotient Boolean algebra of $\mathcal{CC}(X)$ by the ideal of all bounded subsets. Evidently, for a group $G$ equipped with the coarse structure induced by the generating set of all finite subsets, $Ends(G)$ coincide with Specker's space of ends; and therefore our construction of the space of coarse ends of a coarse space naturally generalizes Specker's space of ends of a group.

Beside being a Stone's space, the space of coarse ends $Ends(X)$ of a coarse space $X$ has the following properties:

1. It is a coarse invariant of coarse spaces.

2. $X \cup Ends(X)$ is a large scale compactification of $X$, called the Freudenthal large scale compactification, that coincides with the Coarse Freudenthal compactification of $X$, if $X$ is a proper metric space, and coincides with the Freudenthal compactification, when $X$ is a geodesic proper metric space.

3. If $X$ is a Freudenthal space, there exists a coarse structure on $X$ such that the space of coarse ends and the space of Freudenthal ends of $X$ are homeomorphic.

Many concepts involve finiteness can be generalized to the setting of boundedness. For instance, we can define boundedly generated groups, subgroups of bounded index and locally bounded groups as coarse counterparts of finitely generated groups, subgroups of finite index and locally finite groups, respectively. Consequently, many results involving finiteness hold, in general, in the realm of coarse groups.

Given a group $G$ and a symmetric subset $K \subset G$ generating $G$; $G$ can naturally be equipped with the norm $|.|_K : G \to [0, \infty)$ given by:

$$|x|_K = \begin{cases} 0 & \text{iff } x = 1_G \\ \inf\{n \in \mathbb{N} : x = x_1 \cdot \ldots \cdot x_n \text{ and } x_1, \ldots, x_n \in K \setminus \{1_G\}\} & \text{if } x \in G \setminus \{1_G\} \end{cases}$$

• The norm $|.|_K : G \to [0, \infty)$ on $G$ induces a left-invariant metric $d_K : G \times G \to [0, \infty)$ given by $d_K(x, y) := |x^{-1}y|$ and it is called the **Cayley** metric with respect to the subset $K$.

• The Cayley metric $d_K$ induces a graph $Cay(G, K) = (V, E)$ called the **Cayley graph of** $G$ **with respect to the subset** $K$ where $V = G$ and $\{x, y\} \in E$ if and only if $d_K(x, y) = 1$ if and only if $y = a \cdot x$ for some $a \in K \setminus \{1_G\}$.

• Passing to the geometric realization $X_K$, we can think of the Cayley graph as a geodesic metric space.

When $K$ is finite, the Cayley metric $d_K : G \times G \to [0, \infty)$ is a proper left-invariant metric that induces its coarse structure. Moreover, $G$ is coarsely equivalent to the geodesic metric space $X_K$. One can observe for any bounded subset $B \subset G$ the following hold:

$(P_1)$ $G \backslash B$ contains only finitely many unbounded (infinite) $K$-components (or 1-components) and the union of all bounded (finite) $K$-components of $G \setminus B$ is bounded. By a $K$-component of $G \setminus B$ we mean the largest subset $C$ of $G \setminus B$ such that for all $a, b \in C$, there exists a finite sequence $g_1, \ldots, g_n$ of elements of $G$ such that $a = g_1$ and $b = g_n$ with $g_i^{-1} \cdot g_{i+1} \in K$ for all $i < n$ (equivalently $d_K(g_i, g_{i+1}) = 1$ for all $i < n$).
$(P_2)$ $B$ can be covered by finitely many subsets of the form $g \cdot K$, where $g \in G$.

However, in the realm of coarse groups, for a symmetric bounded subset $K \subset G$ generating $G$, the Clayey metric $d_K : G \times G \to [0, \infty)$ does not necessarily induce its coarse structure. Moreover, even if $d_K : G \times G \to [0, \infty)$ induces its coarse structure, property $(P_1)$ and property $(P_2)$ need not to be satisfied.

It turns out that the sufficient condition for a coarse group to be boundedly generated is to satisfy the property $(P_1)$ for some symmetric bounded subset $K \subset G$. $(P_1)$ is also the necessary and sufficient condition for the space of coarse ends of a boundedly generated coarse group $G =< K >$ with a coarse structure induced by a Cayley metric $d_K$ to be metrizable. In the case when $Ends(G)$ is metrizable, its ends can be described in a similar manner as the Freudenthal ends.

We call coarse groups that satisfy property $P_2$) coarse groups of **bounded geometry**. This class of groups includes coarse groups with coarse structure induced by the generating set of finite subsets as well as locally compact groups with the coarse structure induced by the family of all relatively compact subsets. Being of coarse geometry is an important aspect to generalize the fact that a group $G$ is 2-ended if and only if it contains an infinite cyclic group of finite index.

For a coarse group to be boundedly generated, for its coarse structure to be induced by a metric or by a Cayley metric, to be of coarse geometry, and how these properties are related, is the core of **Chapter** 8.

The main results about the space of coarse ends of coarse groups are obtained in **Chapter** 9. We generalize 1.1.5, 1.1.6 and partially generalize 1.1.8; that is, we give a coarse version of the following classical results:

1. Locally finite groups are infinitely-ended.

2. The space of ends of a finitely generated $G$ is either infinite or it has at most 2 (discrete) elements.

3. If the space of ends of a finitely generated $G$ is infinite then it is a Cantor space.

4. A group $G$ is 2-ended if and only if it contains an infinite cyclic group of finite index. In case $G$ is a locally compact, metrizable, connected topological group and two-ended, then $G$ contains a closed subgroup $H$ such that the coset-space $G/H$ is compact.

The coarse counterparts of these results are:

1. Locally bounded coarse groups with metrizable coarse structures are infinitely-ended.

2. If the coarse group $G$ is a countable union $\{G_n\}_{n\in\mathbb{N}}$ of coarse subgroups each of which has a coarse structure, as a subspace of $G$, metrizable by a Cayley metric, and that every bounded subset is contained in one of these subgroups; then the number of coarse ends of $G$ is either infinite or at most 2.

3. If the coarse group $G$ is equipped with a coarse structure induced by a Cayley metric, and its space of coarse ends is infinite and topologically metrizable, then it is a Cantor space.

4. Let $G$ be a coarse group of bounded geometry that is $\sigma$-bounded. If $G$ has 2 coarse ends, then it contains an unbounded cyclic group of bounded index.

For completion, we also present a proof for 1.1.7, in the language of coarse geometry.

## 1.2 Outline

**Chapter** 2 is devoted to the study of compactifications of a topological space. More specifically, it gives the necessary and sufficient conditions for a topological space to admit a compactification; it discusses the domination relation between compactifications of a completely regular space $X$; and it presents two (equivalent) ways to construct a compactification of a completely regular space $X$. The first of which is by embedding it in a compact Hausdorff space; and the second way is by means of $C^*$-algebra employing the **Gelfand Theorem** that proves that every compactification of $X$ is the maximal ideal space of a unital $C^*$-subalgebra of $C_b(X, \mathbb{C})$ separating points from closed sets, and the maximal ideal space of a unital $C^*$-subalgebra of $C_b(X, \mathbb{C})$ separating points from closed sets determines a compactification of $X$.

The **Stone's Representation Theorems** is the topic of **Chapter** 3. It allows us to go back and fourth between Boolean algebras and Stone's spaces (Totally disconnected compact Hausdorff spaces). Stone's spaces are central to the theory of ends as both the space of Freudenthal ends and coarse ends are Stone's spaces. Additionally, every compactification of a discrete space $X$ can be realized as the Stone's space of some sub-Boolean algebra of the power set $\mathcal{P}(X)$ of $X$ that separates points; and every sub-Boolean algebra of the power set $\mathcal{P}(X)$ of $X$ that separates points determines a compactification of $X$.

**Chapter** 4 is the motivation of the dissertation, and it is concerned with classical theory of the Freudenthal compactification of a Freudenthal space ($\sigma$-compact, locally compact, connected and locally connected Hausdorff space). It serves as a survey on the basic facts about the Freudenthal compactification of a Freudenthal space in general and of a proper geodesic metric space in particular. The ends of finitely generated groups are discussed in that section.

We provide an explicit metric on the space of ends of a Freudenthal space and prove that the space of ends, when it is infinite, is a Cantor space, which generalizes the fact that the space of ends of a finitely generated group that has infinitely many ends is a Cantor space.

When the Freudenthal space is a proper geodesic metric space, we prove that its ends are in one-to-one correspondence with the components of its Higson corona. This observation enables to generalize the Freudenthal compactification to be the quotient space of its Higson compactification obtained by collapsing its the components of its Higson corona to a single point. This motivates the construction of the Coarse Freudenthal compactification, the spirit of **Chapter** 6.

**Chapter** 5 is devoted to introductory material on coarse geometry and it builds an appropriate framework that we use to construct and investigate the space of coarse ends. A proper coarse space is defined in (36) as a coarse space $X$ equipped with a Hausdorff paracompact topology that is compatible with its coarse structure in the sense that the diagonal is contained in an open entourage $E \subset X \times X$ and that every coarsely bounded subset is relatively compact. We call a coarse space $X$, that is equipped with a topology such that the diagonal is contained in an open entourage $E \subset X \times X$, a **topological coarse space**; and call a coarse space $X$, that is equipped with a topology such that every coarsely bounded subset is relatively compact, a **a coarsely locally compact space**. Then we investigate each of these space individually and prove that a Hausdorff topological coarse space that is coarsely locally compact is indeed paracompact and hence it is a proper coarse space.

Then, we investigate groups as coarse space separately in **Chapter** 8. More precisely, we study connectivity of coarse groups, metrizability of the coarse structure of coarse groups, and we answer the question to when their coarse structure is induced by a Cayley metric. The property that a coarse group possesses a bounded subset $K$ such that every bounded subset is covered by finitely many sets of the form $g \cdot K$ is of particular interest. We call such coarse groups, coarse groups of **bounded geometry**.

We obtain our main results in **Chapter** 6, **Chapter** 7 and **Chapter** 9. More specifically, in **Chapter** 6, we construct the Coarse Freudenthal compactification of a proper metric space and apply it to countable groups equipped with some proper left-invariant metric. A further generalization of the Freudenthal compactification is developed in **Chapter** 7. In **Chapter** 9, we apply our generalization to coarse groups to unify the theory of ends of finitely generated groups, the theory of ends of locally compact, metrizable connected groups as well as the theory of ends of arbitrary groups in the sense of Specker's definition under the framework of coarse geometry. As a result, we obtain generalized versions of several classical results in the theory of ends of groups.

# Chapter 2

# Compactifications

Compactifications of a topological space is a central topic in this thesis. To make thesis reasonably self-contained, we discuss the basic facts about compactifications that will be used in the subsequent chapters. In this chapter, we give the necessary and sufficient condition for the existence of a compactification, we study the relationship between compactifications, as well as we give two ways to construct a compactification of a topological space that admits a compactification. The basic reference for the first two sections of this chapter is (6).

## 2.1 Necessary and sufficient condition for the existence of a compactification

A topological space $X$ admits a compactification $Y$ if $Y$ is a compact Hausdorff space that contains a dense subset homeomorphic to $X$. Clearly, $X$ admits a compactification if and only if it is embedded in a Hausdorff compact space. Therefore, the existence of a compactification of $X$ is equivalent to the existence of an embedding from $X$ into some Hausdorff compact space. In this section, we show that a space $X$ admits a compactification if and only if it is completely regular.

**Definition 2.1.1.** Let $X$ be a topological space and $\mathcal{F} := \{f_\alpha : X \to X_\alpha : \alpha \in I\}$ be a family of functions where each $X_\alpha$ is a topological space:

• The family $\mathcal{F}$ is said to **separate points** if for each pair of distinct elements $x, y \in X$, there exists $f_\alpha \in \mathcal{F}$ such that $f_\alpha(x) \neq f_\alpha(y)$.

• The family $\mathcal{F}$ is said to **separate points from closed sets** if for each closed subset $F \subset X$ and an element $x \in X \setminus F$, there exists $f_\alpha \in \mathcal{F}$ such that $f_\alpha(x) \notin cl_{X_\alpha}(f_\alpha(F))$.

• The **evaluation map determined by $\mathcal{F}$** is the function $e_{\mathcal{F}} : X \to \prod_{\alpha \in I} X_\alpha$ that maps each $x$ to $(f_\alpha(x))_{\alpha \in I}$.

**Definition 2.1.2.** A topological space is **completely regular** if $C(X, \mathbb{R})$ separates points from closed sets, where $C(X, \mathbb{R})$ is the family of all continuous real-valued functions on $X$. The family of all continuous complex-valued functions on $X$ is denoted by $C(X, \mathbb{C})$.

**Proposition 2.1.3.** *A topological space $X$ is completely regular if and only if for every closed subset $F \subset X$ and an element $x \in X \setminus F$ outside $F$, there exists a continuous function $g : X \to [0, 1]$ such that $g(x) = 0$ and $g(F) = \{1\}$.*

*Proof.* Assume that $X$ is completely regular, $F \subset X$ is a closed subset and $x \in X \setminus F$. There exists a continuous real-valued function $f \in C(X, \mathbb{R})$ such that $f(x) \notin cl(f(F)) \subset \mathbb{R}$. Choose $\varepsilon > 0$ such that $(f(x) - \varepsilon, f(x) + \varepsilon) \cap cl(f(F)) = \emptyset$. The function $h : \mathbb{R} \to [0, 1]$ defined by:

$$h(y) = \min\{\frac{|y - f(x)|}{\varepsilon}, 1\}$$

is continuous. Let $g = h \circ f$, then $g : X \to [0, 1]$ is continuous satisfying $g(x) = 0$ and $g(F) = \{1\}$. The other direction is evident. $\square$

**Proposition 2.1.4.** *Let $X$ be topological space, the following are equivalent:*

*(a) $X$ is completely regular.*

*(b) The family $\mathcal{F} := \{f : X \to [0, 1] : f \in C(X, \mathbb{R})\}$ separates points from closed sets.*

*(c) $C_b(X, \mathbb{R})$ separates points from closed sets, where $C_b(X, \mathbb{R})$ is the family of all continuous bounded real-valued functions on $X$.*

*(d) $C(X, \mathbb{C})$ separates points from closed sets.*

*Proof.* $(a) \implies (b)$: Apply 2.1.3.

$(b) \implies (c)$: Straightforward.

$(c) \implies (d)$: Let $F \subset X$ be a closed subset and $x \in X \setminus F$. There exists a bounded continuous real-valued function $f \in C_b(X, \mathbb{R})$ such that $f(x) \notin cl_{\mathbb{R}}(f(F)) \subset \mathbb{R}$. Since $\mathbb{R}$ is a closed subspace of $\mathbb{C}$, then $cl_{\mathbb{R}}(f(F)) = cl_{\mathbb{C}}(f(F))$.

$(d) \implies (a)$: Let $F \subset X$ be a closed subset and $x \in X \setminus F$. There exists a continuous complex-valued function $f \in C(X, \mathbb{C})$ such that $f(x) \notin cl_{\mathbb{C}}(f(F))$. Choose $\varepsilon > 0$ such that $B(f(x), \varepsilon) \cap cl_{\mathbb{C}}(f(F)) = \emptyset$. The function $h : \mathbb{C} \to [0, 1]$ defined by:

$$h(y) = \min\{\frac{\|y - f(x)\|}{\varepsilon}, 1\}$$

is continuous. Let $g = h \circ f$, then $g : X \to [0, 1]$ is continuous satisfying $g(x) = 0$ and $g(F) = \{1\}$. Hence, $\mathcal{F} := \{f : X \to [0, 1] : f \in C(X, \mathbb{R})\}$ separates points from closed sets, and a fortiori, $C(X, \mathbb{R})$ separates points from closed sets. $\square$

By Urysohn's Lemma, every normal space is completely regular . In particular, Hausdorff compact spaces are completely regular. To show that if $X$ admits a compactification, then it is completely regular it suffices to show that a subspace of a completely regular space is completely regular.

**Proposition 2.1.5.** *Let $A$ be a subset of a completely regular space $X$. Then, $A$, as a subspace of $X$, is completely regular.*

*Proof.* Let $F_A$ be a closed subset of $A$ and $x \in A \setminus F_A$. Choose a closed subset $F$ such that $F_A = F \cap A$. Notice, $x \notin F$ as $x \in A \setminus F_A$ and therefore there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$. The function $g =: f|_A$ is bounded continuous function on $A$ with $g(x) = 0$ and $g(F_A) = \{1\}$. Hence, $A$ is completely regular. $\square$

The necessary and sufficient condition for a space $X$ to admit a compactification is to be completely regular.

**Lemma 2.1.6.** *Let $X$ be a topological space and $\mathcal{F} := \{f_\alpha : X \to X_\alpha : \alpha \in I\}$ be a family of functions where each $X_\alpha$ is a topological space:*

*(a) The evaluation map $e_{\mathcal{F}} : X \to \prod_{\alpha \in I} X_\alpha$ is continuous if and only if each $f_\alpha \in \mathcal{F}$ is continuous.*

*(b) If $\mathcal{F}$ separates points if and only if $e_{\mathcal{F}} : X \to \prod_{\alpha \in I} X_\alpha$ is injective.*

*(c) If $\mathcal{F}$ separates points from closed sets, then $e_{\mathcal{F}} : X \to \prod_{\alpha\in I} X_\alpha$ is open.*

*Proof.* $(a)$ Assume that $e_{\mathcal{F}} : X \to \prod_{\alpha\in I} X_\alpha$ is continuous. Since the projection function $\pi_\beta : \prod_{\alpha\in I} X_\alpha \to X_\beta$ is continuous for each $\beta \in I$, so that $f_\beta = \pi_\beta \circ e_{\mathcal{F}}$ is continuous. Conversely, assume that each $f_\alpha : X \to X_\alpha$ is continuous. Let, $U$ be a subbasis element of $\prod_{\alpha\in I} X_\alpha$, then $U = \pi_\alpha^{-1}(U_\alpha)$ for some open subset $U_\alpha$ of $X_\alpha$. Now, $e_{\mathcal{F}}^{-1}(U) = e_{\mathcal{F}}^{-1}(\pi_\alpha^{-1}(U_\alpha)) = (\pi_\beta \circ e_{\mathcal{F}})^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha)$ which is open in $X$.

$(b)$ Evident.

$(c)$ Let $U$ be an open subset of $X$ and $a \in e_{\mathcal{F}}(U)$. We aim to find an open subset $O$ of $e_{\mathcal{F}}(X)$ such that $a \in O \subset e_{\mathcal{F}}(U)$. Let $a = e_{\mathcal{F}}(x)$ for some $x \in U$; and choose $f_\alpha \in \mathcal{F}$ such that $f(x) \notin cl_{X_\alpha}(f_\alpha(X \setminus U))$. Consider the open subset $V = \pi_\alpha^{-1}(X_\alpha \setminus cl_{X_\alpha}(f_\alpha(X \setminus U)))$ of $\prod_{\alpha\in I} X_\alpha$. Then, $O := V \cap e_{\mathcal{F}}(X)$ is an open subset of $e_{\mathcal{F}}(X)$ containing $a \in e_{\mathcal{F}}(U)$. We claim that $O$ is contained in $e_{\mathcal{F}}(U)$. Seeking contradiction, assume that there exists $y \in X \setminus U$ such that $e_{\mathcal{F}}(y) \in V$. On one hand, as $y \in X \setminus U$, we have $f_\alpha(y) \in f_\alpha(X \setminus U)$. On the other hand, as $e_{\mathcal{F}}(y) \in V = \pi_\alpha^{-1}(X_\alpha \setminus cl_{X_\alpha}(f_\alpha(X \setminus U)))$, we have $f_\alpha(y) = \pi_\alpha \circ e_{\mathcal{F}}(y) \in X_\alpha \setminus (cl_{X_\alpha}(f_\alpha(X \setminus U)) \subset X_\alpha \setminus (f_\alpha(X \setminus U))$, a contradiction. Therefore, $a \in O \subset e_{\mathcal{F}}(U)$, as desired. □

**Corollary 2.1.7.** *A topological space $X$ is completely regular if and only if $X$ admits a compactification.*

*Proof.* Assume that $X$ is completely regular. Notice that $\mathcal{F} := \{f : X \to [0,1] : f \in C(X,\mathbb{R})\}$ separates points from closed sets, and since $X$ is Hausdorff, it also separates points. Therefore, $e_{\mathcal{F}} : X \to \prod_{f\in\mathcal{F}} [0,1]$ is an embedding by above lemma. Since $\prod_{f\in\mathcal{F}} [0,1]$ is compact, $X$ admits a compactification.

Conversely, if $X$ admits a compactification, then it is a subspace of a compact space. The result follows from the fact that a compact Hausdorff space is completely regular and a subspace of completely regular space is completely regular. □

From now on, unless otherwise stated, all topological spaces are assumed to be completely regular.

## 2.2 Relationship between compactifications

**Definition 2.2.1.** Let $\alpha_1 X$ and $\alpha_2 X$ be compactifications of $X$:

• We say that $\alpha_1 X$ **dominates** $\alpha_2 X$ and we write $\alpha_1 X \geqslant \alpha_2 X$ if there exists a continuous function $f : \alpha_1 X \to \alpha_2 X$ such that $f \circ \alpha_1 = \alpha_2$. A such function $f : \alpha_1 X \to \alpha_2 X$ is called a **domination** and $\alpha_1 X$ is called **dominated by** $\alpha_1 X$.

• We say that $\alpha_1 X$ and $\alpha_2 X$ are **equivalent** and we write $\alpha_1 X \simeq \alpha_2 X$ if there exists a homeomorphism $f : \alpha_1 X \to \alpha_2 X$ such that $f \circ \alpha_1 = \alpha_2$.

**Lemma 2.2.2.** *Given a topological space $X$, let $\mathcal{C}_X$ be the family of all compactifications of $X$:*

*(a) The relation $\simeq$ on $\mathcal{C}_X$ is an equivalence relation.*

*(b) Let $\alpha_0 X, \alpha X, \beta_0 X, \beta X \in \mathcal{C}_X$ such that $\alpha_0 X \simeq \alpha X$ and $\beta_0 X \simeq \beta X$. If $\alpha_0 X \geqslant \beta_0 X$, then $\alpha X \geqslant \beta X$.*

*Proof.* $(a)$ If $\alpha_1 X \simeq \alpha_2 X$ and $\alpha_2 X \simeq \alpha_3 X$ then $\alpha_1 X \simeq \alpha_3 X$. Indeed, let $f : \alpha_1 X \to \alpha_2 X$ and $g : \alpha_2 X \to \alpha_3 X$ be homeomorphisms such that $f \circ \alpha_1 = \alpha_2$ and $g \circ \alpha_2 = \alpha_3$. Them, $g \circ f : \alpha_1 X \to \alpha_3 X$ is a homeomorphism with $(g \circ f) \circ \alpha_1 = \alpha_3$.

$(b)$ let $f_0 : \alpha_0 X \to \beta_0 X$ be continuous such that $f_0 \circ \alpha_0 = \beta_0$; and $\varphi : \alpha X \to \alpha_0 X$ and $\tau : \beta_0 X \to \beta X$ be homeomorphisms such that $\varphi \circ \alpha = \alpha_0$ and $\tau \circ \beta_0 = \beta$. The continuous function $f := \tau \circ f_0 \circ \varphi$ satisfies $f \circ \alpha = \beta$. □

**Notation 2.2.3.** *Let $X$ be a topological space, the set of all equivalence classes of $\mathcal{C}_X$ is denoted by $K(X)$.*

We aim to show that the set $K(X)$ is partially ordered by the domination relation $\geqslant$. Moreover, $(K(X), \geqslant)$ admits a supremum and it has an infimum if and only if $X$ is locally compact. Clearly $\geqslant$ is reflexive, and by 2.2.2, it is transitive. In order to see that $\geqslant$ on $K(X)$, we first observe that every equivalence class in $K(X)$ contains a compactification of $X$ in which $X$ is embedded by the inclusion.

**Lemma 2.2.4.** *Let $(Y, \mathcal{T}_Y)$ be a topological space and $X$ be a set. If $f : X \to Y$ is bijective, then there exists a topology $\mathcal{T}_{X_\alpha}$ on $X$ such that $f : X \to Y$ is a homeomorphism.*

*Proof.* Take $\mathcal{T}_X = \{U \subset X : f(U) \in \mathcal{T}_Y\}$. $\square$

**Lemma 2.2.5.** *Let $Z$ be a compact space and $f : X \to Z$ be an embedding. There exists a unique (up to equivalent) compactification $Y$ of $X$ that contains $X$ as a subspace and there exists an embedding $\widetilde{f} : Y \to Z$ such that $\widetilde{f}|_X = f$.*

*Proof.* Notice that $cl_Z(f(X))$ is a compactification of $X$. Without loss of generality assume that $A := cl_Z(f(X)) \setminus f(X)$ is disjoint from $X$, and let $Y := X \cup A$. Define the map $\widetilde{f} : Y \to Z$ by:

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X, \\ x & \text{if } x \in A. \end{cases}$$

Clearly $\widetilde{f}$ is bijective and hence $\mathcal{T}_Y = \{U \subset Y : \widetilde{f}(U) \in \mathcal{T}_Z\}$ is a topology on $Y$. Moreover, every open subset of $X$ is contained in $\mathcal{T}_Y$. Notice that $\widetilde{f} : Y \to Z$ is an embedding and $\widetilde{f}|_X = f$. Now, assume that $Y'$ is a compactification of $X$ that contains $X$ as a subspace such that there exists an embedding $g : Y' \to Z$ with $g|_X = f$. Notice that $\widetilde{f}(Y) = \widetilde{f}(cl_Y(X)) = cl_Z(f(X))$ and similarly one has $g(Y') = g(cl_{Y'}(X)) = cl_Z(f(X))$. Therefore, $\widetilde{f}^{-1} \circ g : Y' \to Y$ is a homeomorphism with $(\widetilde{f}^{-1} \circ g)|_X = id_X$. $\square$

**Corollary 2.2.6.** *Let $\alpha X$ be a compactification of $X$; then there exists a unique (up to equivalent) compactification $Y_\alpha$ of $X$ that is equivalent to $\alpha X$ and contains $X$ as a subspace.*

*Proof.* Apply 2.2.5. $\square$

**Lemma 2.2.7.** *Let $A$ be a subspace of $X$ and $f : A \to Y$ be a continuous map. $f : A \to Y$ admits at most one continuous extension $\widetilde{f} : cl_X(A) \to Y$.*

*Proof.* Let $\widetilde{f} : cl_X(A) \to Y$ and $\widetilde{f}_0 : cl_X(A) \to Y$ be continuous extensions of $f : A \to Y$. Seeking contradiction, assume that there exists $x \in cl_X(A) \setminus A$ such that $\widetilde{f}(x) \neq \widetilde{f}_0(x)$. Let $U$ and $V$ be disjoint open subsets of $Y$ containing $\widetilde{f}(x)$ and $\widetilde{f}_0(x)$, respectively. Choose an open neighborhood $O$ of $x$ such that $\widetilde{f}(O) \subset U$ and $\widetilde{f}_0(O) \subset V$. Now, since $O \cap A \neq \emptyset$, choose $a \in O \cap A$. Then, $f(a) \in U \cap V$, a contradiction. $\square$

An application of the above lemma is that if $g : cl_X(A) \to Y$ is continuous such that $g|_A = id_A$, then $g = id_{cl_X(A)}$.

**Proposition 2.2.8.** *Let $Y$ and $Z$ be compactifications of $X$ containing $X$. $Y$ dominates $Z$ if and only if there exists a unique quotient map $f : Y \to Z$ such that $f|_X = id_X$.*

*Proof.* Let $f : Y \to Z$ be a domination. Since $X \subset f(Y)$, $f(Y)$ is dense in $Z$. Let $z \in Z$, and $(z_\alpha)_{\alpha \in I} \subset f(Y)$ be a net converging to $z$. For each $\alpha \in I$, let $y_\alpha \in Y$ such that $f(y_\alpha) = z_\alpha$. Since $Y$ is compact, the net $(y_\alpha)_{\alpha \in I}$ has a convergent subnet $(y_\beta)_{\beta \in J}$. Assume that $(y_\beta)_{\beta \in J}$ converges to $y \in Y$. By the continuity of $f : Y \to Z$, the subnet $(z_\beta)_{\beta \in J}$ of $(z_\alpha)_{\alpha \in I}$ where $z_\beta = f(y_\beta)$, converges to $f(y)$. $(z_\alpha)_{\alpha \in I}$ converges to a unique point $z$ and every subnet of $(z_\alpha)_{\alpha \in I}$ converges to the same point $z$, hence $f(y) = z$. Therefore, $f : Y \to Z$ is onto which implies that $f : Y \to Z$ is a quotient map as it is closed. If $g : Y \to Z$ is a continuous map such that $g|_X = id_X$, then as $X$ is dense in $Y$, $f = g$. □

**Proposition 2.2.9.** *Let $\alpha_1 X$ and $\alpha_2 X$ be compactifications of $X$. The compactifications $\alpha_1 X$ and $\alpha_2 X$ are equivalent if and only if $\alpha_1 X \geqslant \alpha_2 X$ and $\alpha_2 X \geqslant \alpha_1 X$.*

*Proof.* Assume that $\alpha_1 X \simeq \alpha_2 X$ and let $f : \alpha_1 X \to \alpha_2 X$ be a homeomorphism such that $f \circ \alpha_1 = \alpha_2$. Clearly, $\alpha_1 X \geqslant \alpha_2 X$. Since $f^{-1} \circ \alpha_2 = \alpha_1$, one has $\alpha_2 X \geqslant \alpha_1 X$.

Conversely, assume that $\alpha_1 X \geqslant \alpha_2 X$ and $\alpha_2 X \geqslant \alpha_1 X$. Let $Y_1$ and $Y_2$ be compactifications of $X$ containing $X$ such that $\alpha_1 X \simeq Y_1$ and $\alpha_2 X \simeq Y_2$. Since $\alpha_1 X \geqslant \alpha_2 X$ and $\alpha_2 X \geqslant \alpha_1 X$, then $Y_1 \geqslant Y_2$ and $Y_2 \geqslant Y_1$. Let $f_1 : Y_1 \to Y_2$ and $f_2 : Y_2 \to Y_1$ be continuous such that $f_1|_X = f_2|_X = id_X$. Now, as $(f_1 \circ f_2)|_X = id_X$ and $(f_2 \circ f_1)|_X = id_X$, then by 2.2.7, $f_1 \circ f_2 = id_{Y_2}$ and $f_2 \circ f_1 = id_{Y_1}$. Therefore, $f_1 : Y_1 \to Y_2$ is a bijective continuous map between compact Hausdorff spaces and hence it is a homeomorphism. Since $\simeq$ is transitive and $Y_1 \simeq Y_2$, thus $\alpha_1 X \simeq \alpha_2 X$ as wanted. □

**Corollary 2.2.10.** *Let $X$ be a topological space, the domination relation $\geqslant$ on $K(X)$ is a partial order relation on $K(X)$.*

*Proof.* Apply 2.2.9. □

Let $C_b(X, \mathbb{C})$ be the family of all bounded complex-valued functions on $X$, and for each $f \in C_b(X, \mathbb{C})$, let $B_f$ be a closed ball in $\mathbb{C}$ containing $f(X)$. Notice that if $\mathcal{F} \subset C_b(X, \mathbb{C})$ separates points from closed sets, then $e_{\mathcal{F}} : X \to \prod_{f \in \mathcal{F}} B_f$ is an embedding. Therefore,

$e_{\mathcal{F}}X = cl(e_{\mathcal{F}}(X))$ is a compactification of $X$, where the closure is taken in the compact space $\prod_{f\in\mathcal{F}} B_f$. In particular, a subset $\mathcal{F}$ of $C_b(X,\mathbb{C})$ that separates points from closed sets determines a compactification $e_{\mathcal{F}}X$ of $X$. Conversely, every compactification $\alpha X$ of $X$ is determined, up to equivalent, by a subset $\mathcal{F}$ of $C_b(X,\mathbb{C})$ that separates points from closed sets.

**Theorem 2.2.11.** *Let $\alpha X$ be a compactification of $X$. There exists a subset $\mathcal{F}_\alpha$ of $C_b(X,\mathbb{C})$ that separates points from closed sets such that $e_{\mathcal{F}_\alpha}X \simeq \alpha X$.*

*Proof.* Let $Y$ be a compactification of $X$ such that $X \subset Y$ and $Y \simeq \alpha X$. Set $\mathcal{F}_\alpha := \{f|_X : f \in C(Y,\mathbb{C})\}$. Clearly $\mathcal{F}_\alpha$ is a subset of $C_b(X,\mathbb{C})$ that separates points from closed sets as in the proof of 2.1.5. Hence, $e_{\mathcal{F}_\alpha}X$ is a compactification of $X$. We claim that $e_{\mathcal{F}_\alpha}X \simeq \alpha X$. To that end, it suffices to show that $e_{\mathcal{F}_\alpha}X \simeq Y$. Notice that $\mathcal{F} := C(Y,\mathbb{C})$ separates points from closed sets of $Y$ and therefore $e_{\mathcal{F}} : Y \to \prod_{f\in\mathcal{F}} B_f$ is an embedding. Clearly $e_{\mathcal{F}} : Y \to \prod_{f\in\mathcal{F}} B_f$ is a continuous extension of $e_{\mathcal{F}_\alpha} : X \to \prod_{f\in\mathcal{F}} B_f$ and $Y \simeq e_{\mathcal{F}}(Y) = e_{\mathcal{F}}(cl_Y(X)) = \overline{e_{\mathcal{F}}(X)} = e_{\mathcal{F}_\alpha}X$. Thus, $e_{\mathcal{F}} : Y \to e_{\mathcal{F}_\alpha}X$ is a homeomorphism with $e_{\mathcal{F}} \circ id_X = e_{\mathcal{F}_\alpha}$. $\square$

So far, we have seen that if $Y_\alpha$ is a compactification of $X$ containing $X$, then the subset $C_\alpha := \{f|_X : f \in C(Y_\alpha)\}$ of $C_b(X,\mathbb{C})$ determines $Y_\alpha$. Moreover, every compactification $\alpha X$ of $X$ is determined by the subset $C_\alpha := \{f|_X : f \in C(Y_\alpha,\mathbb{C})\}$ of $C_b(X,\mathbb{C})$, where $Y_\alpha$ is a compactification of $X$ containing $X$ and $\alpha X \simeq Y_\alpha$. We conclude this section by showing that $\alpha_1 X$ is dominated by $\alpha_2 X$ if and only $C_{\alpha_1}$ is contained in $C_{\alpha_2}$.

**Proposition 2.2.12.** *Let $X$ be a topological space:*
*(a) If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset C_b(X,\mathbb{C})$ and $\mathcal{F}_1$ separates points from closed sets in $X$; then $e_{\mathcal{F}_2}X$ dominates $e_{\mathcal{F}_1}X$.*
*(b) $\alpha_1 X$ is dominated by $\alpha_2 X$ if and only $C_{\alpha_1}$ is contained in $C_{\alpha_2}$*

*Proof.* (*a*) Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset C_b(X,\mathbb{C})$. As in the proof of 2.1.6, the projection map $\pi_{\mathcal{F}_1} : \prod_{f\in\mathcal{F}_2} B_f \to \prod_{g\in\mathcal{F}_1} B_g$ is continuous. Moreover, $\pi_{\mathcal{F}_1}(e_{\mathcal{F}_2}X) = e_{\mathcal{F}_1}X$ and hence $\pi_{\mathcal{F}_1|_{e_{\mathcal{F}_2}X}} : e_{\mathcal{F}_2}X \to e_{\mathcal{F}_1}X$ is continuous with $\pi_{\mathcal{F}_1|_{e_{\mathcal{F}_2}X}} \circ e_{\mathcal{F}_2} = e_{\mathcal{F}_1}$. Therefore, $e_{\mathcal{F}_2}X$ dominates $e_{\mathcal{F}_1}X$.
(*b*) If $C_{\alpha_1} \subset C_{\alpha_2}$, then by (*a*), $\alpha_2 X$ dominates $\alpha_1 X$. On the other hand, if $\alpha_2 X$ dominates $\alpha_1 X$, then $Y_{\alpha_2}$ dominates $Y_{\alpha_1}$. Let $\varphi : Y_{\alpha_2} \to Y_{\alpha_1}$ is a domination with $\varphi|_X = id_X$. If $f \in C_{\alpha_1}$,

then $f : X \to B_f$ extends continuously to $g : Y_{\alpha_1} \to B_f$. Notice that $g \circ \varphi : Y_{\alpha_2} \to B_f$ is continuous with $(g \circ \varphi)|_X = g|_X = f$ and hence $f \in C_{\alpha_2}$. □

**Corollary 2.2.13.** *The family $K(X)$ admits a supremum.*

*Proof.* Let $\mathcal{F} := C_b(X, \mathbb{C})$, then $e_{\mathcal{F}}X$ is the wanted compactification. □

**Definition 2.2.14.** Given a topological space $X$, the **Stone-Čech compactification** is the compactification $e_{\mathcal{F}}X$ where $\mathcal{F} := C_b(X, \mathbb{C})$. The Stone-Čech compactification of $X$ is denoted by $\beta X$.

Let $\beta X$ be the Stone-Čech compactification; by 2.2.11, $C_\beta = C_b(X, \mathbb{C})$ and therefore every function of $C_b(X, \mathbb{C})$ continuously and uniquely extends to $\beta X$. The Stone-Čech compactification has the following universal property:

**Proposition 2.2.15.** *A compactification $\alpha X$ is the Stone-Čech compactification of $X$ if and only if for any compact space $Y$, every continuous map $f : X \to Y$ extends continuously $\widetilde{f} : \alpha X \to Y$.*

*Proof.* Without loss of generality, we may assume that the compactifications $\alpha X$ and $\beta X$ to be containing $X$ as a subset. Let $\alpha X$ be the Stone-Čech compactification of $X$, $Y$ be compact and $f : X \to Y$ be continuous. Since $Y$ is completely regular, as in the proof of 2.1.7, it is embedded in $[0, 1]^I$ for some set $I$. Without loss of generality, assume that $Y \subset [0, 1]^I$. For each $i \in I$, put $f_i := \pi_i \circ f$. As each $f_i : X \to [0, 1]$ is continuous bounded, it admits a continuous extension $\widetilde{f_i} : \alpha X \to [0, 1]$. Consider the continuous function $\widetilde{f} : \alpha X \to [0, 1]^I$ that maps each $x \in \alpha X$ to $(f_i(x))_{i \in I}$. Notice that $\widetilde{f}(\alpha X) = \widetilde{f}(cl_{\alpha X}(X)) \subset cl_Y(f(X)) \subset Y$ and hence $\widetilde{f} : \alpha X \to Y$ is a continuous extension to $f : X \to Y$.

Conversely, the inclusion $\imath : X \to \beta X$ admits a continuous extension $\widetilde{\imath} : \alpha X \to \beta X$ which is evidently a domination and hence $\alpha X \simeq \beta X$. □

**Definition 2.2.16.** A topological space $X$ is **locally compact** if there exists a compact neighborhood $K_x$ for every $x \in X$; i.e., for every $x \in X$, there exist a compact subset $K_x$ and an open subset $U_x$ such that $x \in U_x \subset K_x$. Equivalently, for all $x \in X$ and an open neighborhood $U_x$ of $x$, there exists a relatively compact open neighborhood $V_x$ of $x$ such that

$x \in V_x \subset cl(V_x) \subset U_x$. The family of all open relatively compact subsets of a locally compact space $X$ consists a bases for $X$.

**Definition 2.2.17.** A compactification $Y$ of $X$ is called a **finite-point compactification** if its corona $\partial_Y X := Y \backslash X$ is finite. When $|\partial_Y X| = 1$, $Y$ is called (the) **one-point compactification**.

**Lemma 2.2.18.** *Let $(X, \mathcal{T})$ be a non-compact locally compact space and $\infty$ be an arbitrary point not in $X$. Set $X_\infty := X \cup \{\infty\}$, then:*
*(a) The family $\mathcal{T}_\infty := \mathcal{T} \cup \{U \subset X_\infty : \infty \in U$ and $X_\infty \setminus U$ is compact$\}$ is a topology on $X_\infty$.*
*(b) $(X_\infty, \mathcal{T}_\infty)$ is a compactification of $X$.*

*Proof.* $(a)$ Put $\mathcal{B}_\infty := \{U \subset X_\infty : \infty \in U$ and $X_\infty \setminus U$ is compact$\}$. Notice that $U \in \mathcal{B}_\infty$ if and only if there exists a compact subset $K_U$ of $X$ such that $U = X_\infty \setminus K_U$. Now:
(1) $\emptyset, X_\infty \in \mathcal{T}_\infty$ as $\emptyset \in \mathcal{T}$ and $X_\infty \in \mathcal{B}_\infty$.
(2) Let $U, V \in \mathcal{T}_\infty$:
• if $U, V \in \mathcal{B}_\infty$, then there exist $K_U$ and $K_V$ compact subsets of $X$ such that $U = X_\infty \setminus K_U$ and $V = X_\infty \setminus K_V$. Notice $U \cap V = X_\infty \setminus (K_U \cup K_V)$ and hence $U \cap V \in \mathcal{B}_\infty \subset \mathcal{T}_\infty$.
• if $U \in \mathcal{T}$ and $V \in \mathcal{B}_\infty$, then $U \cap V = U \cap (X_\infty \setminus K_V) = U \cap (X \setminus K_V)$ for some compact subset $K_V$ of $X$ and hence $U \cap V \in \mathcal{T} \subset \mathcal{T}_\infty$.
(3) If $U \in \mathcal{T}$ and $V \in \mathcal{B}_\infty$, then $U \cup V = U \cup (X_\infty \setminus K_V) = X_\infty \setminus (K_V \setminus U)$ for some compact subset $K_V$ of $X$ and hence $U \cup V \in \mathcal{B}_\infty \subset \mathcal{T}_\infty$. Now, let $\{U_\alpha : \alpha \in I\} \subset \mathcal{T}_\infty$ and let $I_1, I_2 \subset I$ such that $I = I_1 \cup I_2$ with $\{U_\alpha : \alpha \in I_1\} \subset \mathcal{T}$ and $\{U_\beta : \beta \in I_2\} \subset \mathcal{B}_\infty$. Notice that for all $\beta \in I_2$, there exists a compact subset $K_\beta$ of $X$ such that $U_\beta = X_\infty \setminus K_\beta$. Also, $\bigcup\limits_{\alpha \in I} U_\alpha = \bigcup\limits_{\alpha \in I_1} U_\alpha \cup \bigcup\limits_{\beta \in I_2} U_\beta = U \cup V$ where $U := \bigcup\limits_{\alpha \in I_1} U_\alpha$ and $V := X_\infty \setminus (\bigcap\limits_{\beta \in I_2} K_\beta)$. Since $U \in \mathcal{T}$ and $V \in \mathcal{B}_\infty$, so $U \cup V \in \mathcal{B}_\infty \subset \mathcal{T}_\infty$ as observed.
$(b)$ Clearly $X$ is an open dense subspace of $X_\infty$. So, we need to verify that $X_\infty$ is Hausdorff compact space.
• $X_\infty$ is Hausdorff. Indeed, let $x, y \in X_\infty$ be two distinct points. If $x \in X$ and $y = \infty$, we can find a relatively compact open neighborhood $U$ of $x$ in $X$. Notice that $V := X_\infty \setminus cl_X U$ is an open neighborhood of $y$ and $U \cap V = \emptyset$.
• $X_\infty$ is compact. Indeed, let $\{U_\alpha : \alpha \in I\}$ be an open cover of $X_\infty$. Choose an open neighborhood $U_\alpha$ of $\infty$ for some $\alpha \in I$. There exists a compact subset $K_\alpha$ of $X$ such that

$U_\alpha = X_\infty \setminus K_\alpha$. Now, $K_\alpha$ can be covered by finitely many elements of $\{U_\alpha : \alpha \in I\}$ and hence $X_\infty = U_\alpha \cup K_\alpha$ is covered by finitely many elements of $\{U_\alpha : \alpha \in I\}$. $\square$

If $Y$ is a compactification of $X$ such that $Y \setminus X$ is finite, then $X$ is an open subspace of the locally compact space $Y$. This fact along with the above proposition give us the following characterization for locally compact spaces.

**Corollary 2.2.19.** *A topological space $X$ is locally compact if and only if it admits a one-point compactification.*

All one-point compactifications are equivalent and therefore the one-point compactification is unique up to equivalent. Moreover, it is dominated by every compactification in $K(X)$ and hence it is the infimum of $K(X)$.

**Proposition 2.2.20.** *Let $(X, \mathcal{T})$ be a non-compact locally compact space and let $X_\infty$ be a one-point compactification of $X$. Then:*
*(a) If $X_{\infty_0}$ is a one-point compactifications of $X$, $X_{\infty_0} \simeq X_\infty$.*
*(a) If $\alpha X$ is a compactification of $X$, then $\alpha X \geqslant X_\infty$.*

*Proof.* $(a)$ Define the function $f : X_\infty \to X_{\infty_0}$ by:

$$f(x) = \begin{cases} x & \text{if } x \in X, \\ \infty_0 & \text{if } x = \infty. \end{cases}$$

Clearly, $f : X_\infty \to X_{\infty_0}$ is a bijective function between compact Hausdorff spaces and $f|_X = id_X$. Moreover, let $U$ be an open subset of $X_{\infty_0}$. If $U \subset X$, then $f^{-1}(U) = U$ is an open subset of $X_\infty$. Assume that $U \cap X_{\infty_0} \setminus X \neq \emptyset$. Then, $U = X_{\infty_0} \setminus K_U$ for some compact subset $K_U$ of $X$. Hence, $f^{-1}(U) = f^{-1}(X_{\infty_0} \setminus K_U) = X_\infty \setminus K_U$ which is an open subset of $X_\infty$. Thus, $f : X_\infty \to X_{\infty_0}$ is a homeomorphism.
$(b)$ Define the function $f : \alpha X \to X_\infty$ by:

$$f(x) = \begin{cases} x & \text{if } x \in X, \\ \infty & \text{if } x \in \alpha X \setminus X. \end{cases}$$

For the continuity of $f : \alpha X \to X_\infty$, choose a neighborhood $U$ of $\infty$ in $X_\infty$. Then, $U = X_\infty \setminus K_U$ for some compact subset $K_U$ of $X$. Hence, $f^{-1}(U) = f^{-1}(X_\infty \setminus K_U) = \alpha X \setminus K_U$ which is an open subset of $\alpha X$. Thus, $f : \alpha X \to X_\infty$ is a domination. □

**Corollary 2.2.21.** *If $X$ is locally compact, then $K(X)$ is a complete lattice.*

*Proof.* The one-point compactification is the infimum of $K(X)$ and the Stone-Čech compactification is the supremum of $K(X)$. □

The converse is also true; that is if $K(X)$ admits an infimum, then $X$ is locally compact.

**Proposition 2.2.22.** *Let $X$ be a topological space. If $K(X)$ admits an infimum, then $X$ is locally compact.*

*Proof.* Let $\alpha X$ be the infimum of $K(X)$. Without loss of generality, assume that $X \subset \alpha X$. Seeking contradiction, assume that $X$ is not locally compact. If $X$ is an open subset of $\alpha X$, then it is locally compact, a contradiction. Therefore, $\alpha X \setminus X$ must be infinite. Choose distinct points $a, b \in \alpha X \setminus X$ and set $\mathcal{F}_\alpha := \{f|_X \in C_\alpha : f(a) = f(b)\}$. We claim that $\mathcal{F}_\alpha$ separates points from closed sets in $X$ and that $\mathcal{F}_\alpha$ is properly contained in $C_\alpha$. Indeed, if $F \subset X$ is a closed subset and $x \notin F$, then there exists a closed subset $F' \subset \alpha X$ such that $F = F' \cap X$. Notice that $F' \cup \{a, b\}$ is closed and $F = F' \cup \{a, b\} \cap X$, so we may assume that $a, b \in F'$. Choose a continuous real-valued function $f : \alpha X \to [0, 1]$ such that $f(x) = 0$ and $f(F') = \{1\}$. Since $f(a) = f(b) = 1$, $f|_X \in \mathcal{F}_\alpha$ and hence $\mathcal{F}_\alpha$ separates points from closed sets in $X$. Moreover, since $C(\alpha X, \mathbb{R})$ separates points from closed sets in $\alpha X$, it is, a fortiori, separates points and hence there exists a continuous function $g \in C(\alpha X, \mathbb{R})$ such that $g(a) \neq g(b)$. Thus, $g|_X \in C_\alpha \setminus \mathcal{F}_\alpha$ as wanted. By 2.2.12, the compactification $e_{\mathcal{F}_\alpha} X$ is dominated by $\alpha X$ and it does not dominate $\alpha X$, contradicting the fact that $\alpha X$ is the infimum of $K(X)$. Thus, $X$ is locally compact. □

**Corollary 2.2.23.** *A topological space $X$ is locally compact if and only if $K(X)$ is a complete lattice.*

Our next goal is to show that the one-point compactification, if exists, is induced by the family $C_0(X, \mathbb{C})$ of all continuous complex-valued functions that vanish at infinity; and that $C_0(X, \mathbb{C})$ separates points from closed sets in $X$ if and only if $X$ is locally compact.

**Definition 2.2.24.** Let $X$ be a topological space. A complex-valued function $f : X \to \mathbb{C}$ is said to **vanish at infinity** if for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon$ of $X$ such that $\|f(x)\| < \varepsilon$ for all $x \in X \setminus K_\varepsilon$. The family of all continuous functions that vanish at infinity is dented by $C_0(X, \mathbb{C})$.

**Lemma 2.2.25.** *Let $X$ be a topological space and $\alpha X$ be an arbitrary compactification of $X$; then $C_0(X, \mathbb{C}) \subset C_\alpha$.*

*Proof.* Let $f \in C_0(X, \mathbb{C})$ and define the function $g : \alpha X \to \mathbb{C}$ by:

$$g(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x \in \alpha X \setminus X. \end{cases}$$

We claim that $g : \alpha X \to \mathbb{C}$ is continuous. To this end, it suffices to show that $g|_{\alpha X \setminus X}$ is continuous. Let $a \in \alpha X \setminus X$ and let $\varepsilon > 0$. Choose a compact subset $K_\varepsilon$ of $X$ such that $\|f(x)\| < \varepsilon$ for all $x \in X \setminus K_\varepsilon$. Notice that $U := \alpha X \setminus K_\varepsilon$ is an open neighborhood of $a$ in $\alpha X$ and for all $x \in U$, $\|g(a) - g(x)\| = \|g(x)\| < \varepsilon$. Hence $g : \alpha X \to \mathbb{C}$ is a continuous extension of $f : X \to \mathbb{C}$ and hence $f \in C_\alpha$ as desired. $\square$

**Theorem 2.2.26.** *Let $X$ be a locally compact space. The family $C_0(X, \mathbb{C})$ separates points from closed sets in $X$ and the compactification $e_{C_0(X,\mathbb{C})}X$ is the one-point compactification.*

*Proof.* Let $F \subset X$ be a closed subset and $x \in X$ such that $x \notin F$. Since $X \setminus F$ is an open neighborhood of $x$ and $X$ is locally compact, there exists a relatively compact open neighborhood $U$ of $x$ such that $x \in U \subset cl(U) \subset X \setminus F$. Notice that $F \subset X \setminus cl(U) \subset X \setminus U$. Since $X$ is completely regular, and $X \setminus U$ is closed not containing $x$, there exists a continuous function $g : X \to [0, 1]$ such that $g(x) = 0$ and $g(X \setminus U) = \{1\}$. Let $\varphi : [0, 1] \to [0, 1]$ be a homeomorphism such that $\varphi(0) = 1$ and $\varphi(1) = 0$. Let $f := \varphi \circ g$, and notice that $f : X \to [0, 1]$ is continuous with $f(X \setminus cl(U)) = \{0\}$, so $f \in C_0(X, \mathbb{C})$. Moreover, as $f(x) = 1 \notin cl_{\mathbb{C}} f(F) = \{0\}$, $C_0(X, \mathbb{C})$ separates points from closed sets in $X$. By 2.2.12 and 2.2.25, the compactification $e_{C_0(X,\mathbb{C})}X$ is the infimum of $K(X)$. Therefore, $e_{C_0(X,\mathbb{C})}X$ is the one-point compactification. $\square$

**Corollary 2.2.27.** *The topological space $X$ is locally compact if and only if $C_0(X, \mathbb{C})$ separates points from closed sets in $X$.*

*Proof.* If $X$ is locally compact then $C_0(X, \mathbb{C})$ separates points from closed sets in $X$ by the preceding theorem. On the other hand, if $C_0(X, \mathbb{C})$ separates points from closed sets in $X$, then by 2.2.12 and 2.2.25, the compactification $e_{C_0(X,\mathbb{C})}X$ is the infimum of $K(X)$. By 2.2.22, $X$ must be locally compact. $\square$

We conclude this section by the fact that locally compact spaces are exactly the completely regular spaces having the property that the corona of any of their compactifications is compact.

**Lemma 2.2.28.** *Let $X$ be a dense subspace of $Y$, $F$ be a closed subset of $Y$ and $U$ is an open subset of $Y$. If $U \cap X \subset F$, then $U \subset F$.*

*Proof.* Notice that $U \cap X \neq \emptyset$ and that $(U \cap X) \cap Y \setminus F) = \emptyset$. Seeking contradiction assume that $U \setminus F \neq \emptyset$. Then $U \cap (Y \setminus F)$ is a non-empty open subset of $Y$ and hence $(X \cap U) \cap Y \setminus F) \neq \emptyset$, a contradiction. $\square$

**Proposition 2.2.29.** *Let $X$ be a locally compact dense subspace of $Y$. Then, $X$ is an open subspace. In particular, if $Y$ is a compactification of $X$, then $X$ is an open subspace of $Y$.*

*Proof.* Let $x \in X$ and choose a relatively compact open neighborhood $U$ of $x$ in $X$. Since $cl_X(U)$ is compact in $X$, it is compact in the Hausdorff space $Y$ and hence it is a closed subset of $Y$. Notice that there exists an open subset $V \subset Y$ such that $U = V \cap X \subset cl_X(U)$. Hence, by above lemma, $a \in V \subset cl_X(U) \subset X$ and thus $X$ is an open in $Y$ as desired. $\square$

**Corollary 2.2.30.** *Let $X$ be a completely regular space and $\alpha X$ be a compactification of $X$. $X$ is locally compact if and only if $\alpha(X)$ is an open subspace of $\alpha X$.*

*Proof.* Apply 2.2.29. $\square$

**Theorem 2.2.31.** *Let $X$ be a completely regular space and $\alpha X$ be a compactification of $X$. The following are equivalent:*

*(a) $X$ is locally compact.*

*(b) $X$ admits a one-point compactification.*

*(c) $K(X)$ admits an infimum.*

*(d) $C_0(X, \mathbb{C})$ separates points from closed sets in $X$.*

*(e) $X$ is an open subspace of $\alpha X$.*

*(e) The corona $\partial_\alpha X := \alpha X \setminus X$ is compact.*

*Proof.* Apply 2.2.19, 2.2.23, 2.2.27 and 2.2.30. □

## 2.3 Compactifications as maximal ideal spaces

For a locally compact space $X$, we utilize the Gelfand Theorem to find a duality between the category of its compactifications and the category of the unital $C^*$-subalgebras of $C_b(X, \mathbb{C})$, that separate points from closed sets. The proof of Theorems 2.3.14, 2.3.16 and 2.3.17 are omitted as they involve the definition of terms which are not central to our purpose. Reader who is interested in the proofs can find them in (29).

**Definition 2.3.1.** A ring $(A, +, \cdot)$ is called a **$\mathbb{C}$-algebra** if $\mathbb{C}$ acts on the abelian group $(A, +)$ and the action of $\mathbb{C}$ satisfies:

1. $\alpha(x + y) = \alpha x + \alpha y$,
2. $(\alpha + \beta)x = \alpha x + \beta x$,
3. $\alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y)$, for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in A$.

If the ring $A$ is commutative, $(A, +, \cdot)$ is called a **commutative $\mathbb{C}$-algebra**, and if the ring $A$ has a multiplicative identity $1_A$, $(A, +, \cdot)$ is called a **unital $\mathbb{C}$-algebra**.

**Definition 2.3.2.** A $\mathbb{C}$-algebra $A$ equipped with a norm $\|.\| : A \to [0, \infty)$ satisfying $\|x \cdot y\| \leq \|x\| \, \|y\|$, for all $x, y \in A$, is called a **normed algebra**. The normed algebra $A$ is called a **Banach algebra over** $\mathbb{C}$ if $A$ is a complete metric space with respect to the metric induced by its norm $\|.\| : A \to [0, \infty)$.

**Definition 2.3.3.** A Banach algebra $A$ over $\mathbb{C}$ is called a $C^*$-algebra if it is equipped with a map $\lambda_A : A \to A$ satisfying:

1. $\lambda_A \circ \lambda_A = id_A$,
2. $\lambda_A(x + y) = \lambda_A(x)\lambda_A(y)$,

3. $\lambda_A(x \cdot y) = \lambda_A(y)\lambda_A(x)$,

4. $\lambda_A(\alpha x) = \overline{\alpha}\lambda_A(x)$,

5. $\|\lambda_A(x) \cdot x\| = \|x\|^2$, for all $x, y \in A$ and $\alpha \in \mathbb{C}$.

A $C^*$-algebra $A$ is called a **commutative $C^*$-algebra** if $A$ as a ring is commutative; $A$ is called a **unital $C^*$-algebra** if $A$ as a ring has an identity $1_A$.

**Definition 2.3.4.** A map between $\varphi : A \to B$ between $C^*$-algebras is called ***-homomorphism** if:

1. $\varphi(\alpha x + y) = \alpha\varphi(x) + \varphi(y)$, that is $\varphi : A \to B$ is a linear transformation.

2. $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$, that is $\varphi : A \to B$ is a ring homomorphism.

3. $\varphi(\lambda_A(x)) = \lambda_B\varphi(x)$, for all $x, y \in A$ and $\alpha \in \mathbb{C}$.

A $*$-homomorphism $\varphi : A \to B$ is called a **$*$-isomorphism** if it is bijective.

**Definition 2.3.5.** A subset $B \subset A$ of a $C^*$-algebra $A$ is called **$C^*$-subalgebra** if $B$ with the operation inherited from $A$ itself a $C^*$-algebra. Equivalently,

1. $\alpha x + y \in B$,

2. $x \cdot y \in B$, for all $x, y \in B$ and $\alpha \in \mathbb{C}$; and,

3. $\lambda_A(B) \subset B$.

**Notation 2.3.6.** *The family of all bounded functions from the topological space $X$ to the metric space $Y$ is denoted by $B(X,Y)$. The family of all bounded complex-valued functions that vanish at infinity is denoted by $B_0(X,\mathbb{C})$.*

**Proposition 2.3.7.** *Equip the family $B(X,\mathbb{C})$ with the supremum norm $\|.\|_\infty : B(X,\mathbb{C}) \to [0,\infty)$ given by $\|f\|_\infty = \sup\limits_{x\in X} \|f(x)\|$. Then:*

*(a) The normed algebra $(B(X,\mathbb{C}), \|.\|_\infty)$ with the complex conjugation function $\lambda_{B(X,\mathbb{C})} : B(X,\mathbb{C}) \to B(X,\mathbb{C})$ is a commutative unital $C^*$-algebra.*

*(b) $B_0(X,\mathbb{C})$ is a closed ideal of $B(X,\mathbb{C})$, and therefore a $C^*$-subalgebra.*

*(c) $C_b(X,\mathbb{C})$ is a $C^*$-subalgebra of $B(X,\mathbb{C})$.*

*(d) $C_0(X,\mathbb{C})$ is a closed ideal of $C_b(X,\mathbb{C})$, and therefore a $C^*$-subalgebra.*

*(e) Let $Y$ be a compactification of $X$ containing $X$; then the family:*

*$C_Y := \{f \in C_b(X,\mathbb{C}) : f$ extends continuously to $Y\}$ is a unital $C^*$-subalgebra of $C_b(X,\mathbb{C})$. Furthermore, as a $C^*$-algebra, $C_Y$ is $*$-isomorphic to $C(Y,\mathbb{C})$.*

*Proof.* Indeed for any metric space $(Y, d)$, then the map $d_\infty : B(X,Y) \times B(X,Y) \to [0,\infty)$ given by $d_\infty(f,g) = \sup_{x\in X}\{d(f(x), g(x))\}$ is a metric on $B(X,Y)$ called the **uniform metric**. If $(Y,d)$ is a complete metric space, $(B(X,Y), d_\infty)$ is also complete. The completeness of $(B(X,Y), d_\infty)$ follows from the fact that a convergent sequence in $(B(X,Y), d_\infty)$ converges uniformly, and that the limit of a uniformly convergent sequence of bounded functions is itself a bounded function. The space $(C_b(X,Y), d_\infty)$ is a closed subspace of $(B(X,Y), d_\infty)$ as the limit of a uniformly convergent sequence of continuous functions is continuous. Therefore $(C_b(X,Y), d_\infty)$ is itself complete. By taking $Y = \mathbb{C}$, $(a)$ and $(c)$ are proved.

Notice that the constant function $c_1 : X \to \mathbb{C}$ mapping each $x \in X$ to $1 \in \mathbb{C}$ is the identity of the ring $B(X,\mathbb{C})$ and that $c_1 \in C_b(X,\mathbb{C})$. To see that $B_0(X,\mathbb{C})$ is a closed subset of $B(X,\mathbb{C})$, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $B_0(X,\mathbb{C})$ converging to $f \in B(X,\mathbb{C})$. We claim that $f \in B_0(X,\mathbb{C})$. Indeed, for $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\|f_n - f\|_\infty < \varepsilon/2$ for all $n \geq N$. Let $K \subset X$ be a compact subset such that $\|f_N(x)\| < \varepsilon/2$ for all $x \in X \setminus K$. Then, $\|f(x)\| \leq \|f_N(x) - f(x)\| + \|f_N(x)\| \leq \|f_n - f\|_\infty + \|f_N(x)\| < \varepsilon$, for all $x \in X \setminus K$; hence, $f \in B_0(X,\mathbb{C})$. Moreover, assume $f \in B(X,\mathbb{C})$ and $g \in B_0(X,\mathbb{C})$, and choose $r > 0$ such that $\|f\|_\infty < r$; and for $\varepsilon > 0$, choose a compact subset such that $\|f(x)\| < \varepsilon/r$ for all $x \in X \setminus K$. Then, $\|(f \cdot g)(x)\| < \varepsilon$ for all $x \in X \setminus K$, which proves that $B_0(X,\mathbb{C})$ is a closed ideal of $B(X,\mathbb{C})$. Similarly, $C_0(X,\mathbb{C})$ is a closed ideal of $C_b(X,\mathbb{C})$.

Finally, define $\varphi : C(Y,\mathbb{C}) \to C_b(X,\mathbb{C})$ by $\varphi(f) = f|_X$. It is straightforward to see that $\varphi : C(Y,\mathbb{C}) \to C_b(X,\mathbb{C})$ is a $*$-homomorphism. Utilizing the fact that the image of a $C^*$-algebra under a $*$-homomorphism is a $C^*$-algebra; we obtain that $\varphi(C(Y,\mathbb{C}))$ is a $C^*$-subalgebra of $C_b(X,\mathbb{C})$. Notice that $\varphi(C(Y,\mathbb{C})) = C_Y$. Moreover, $\varphi : C(Y,\mathbb{C}) \to C_b(X,\mathbb{C})$ is injective by 2.2.7. By the First Isomorphism Theorem of $C^*$-algebras, we conclude that $C_Y$ is $*$-isomorphic to $C(Y,\mathbb{C})$. □

**Observation 2.3.8.** *If $X$ is a non-compact topological space, and $f : X \to \mathbb{C}$ is a non-zero constant function; then $f \notin C_0(X,\mathbb{C})$. Moreover, as $C_0(X,\mathbb{C})$ is an ideal of $C_b(X,\mathbb{C})$, $c_1 \in C_0(X,\mathbb{C})$ if and only if $C_0(X,\mathbb{C}) = C_b(X,\mathbb{C})$. Therefore, $c_1 \in C_0(X,\mathbb{C})$ if and only if $X$ is compact.*

**Definition 2.3.9.** Given a metric space $(X, d)$. The function $f : X \to \mathbb{C}$ is called **slowly oscillating** if for any $r, \varepsilon > 0$ there is a bounded subset $K$ of $X$ such that if $x, y \in X \setminus K$ and $d(x, y) < r$ then $\|f(x) - f(y)\| < \varepsilon$. Equivalently, $f : X \to \mathbb{C}$ is slowly oscillating if for any sequence $(x_n) \subset X$ converging to infinity and $(y_n) \subset X$ a sequence such that $d(x_n, y_n) < r$ for some $r > 0$, $\|f(x_n) - f(y_n)\| \to 0$. The family of all bounded slowly oscillating functions is denoted by:

$$B_h(X, \mathbb{C}) := \{f \in B(X, \mathbb{C}) : f \text{ is slowly oscillating}\},$$

and the family of all bounded continuous slowly oscillating functions is denoted by:

$$C_h(X, \mathbb{C}) := \{f \in C_b(X, \mathbb{C}) : f \text{ is slowly oscillating}\}.$$

**Proposition 2.3.10.** *Let $(X, d)$ be a metric space. Then:*
*(a) The subset $B_h(X, \mathbb{C})$ of $B(X, \mathbb{C})$ is a unital $C^*$-subalgebra of $B(X, \mathbb{C})$ containing $B_0(X, \mathbb{C})$.*
*(b) The subset $C_h(X, \mathbb{C})$ of $C_b(X, \mathbb{C})$ is a unital $C^*$-subalgebra of $C_b(X, \mathbb{C})$ containing $C_0(X, \mathbb{C})$.*

*Proof.* $(a)$ It is straightforward to see that $B_h(X, \mathbb{C})$ is a unital normed subalgebra of $B(X, \mathbb{C})$. We just need to prove that $B_h(X, \mathbb{C})$ is a closed subset $B(X, \mathbb{C})$ to conclude that $B_h(X, \mathbb{C})$ is a unital $C^*$-subalgebra of $B(X, \mathbb{C})$. This follows directly from the fact that convergent sequence in $B(X, \mathbb{C})$ with respect to the supremum norm converges uniformly. Indeed, if $(f_n)_{n\in\mathbb{N}} \subset B_h(X, \mathbb{C})$ such that $f_n \to f \in B(X, \mathbb{C})$, then we claim that $f \in B_h(X, \mathbb{C})$. To this end, let $r, \varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $\|f - f_N\|_\infty < \varepsilon/3$. Since $f_N \in B_h(X, \mathbb{C})$, we can find a bounded subset $K \subset X$ such that for all $x, y \in X \setminus K$ such that $d(x, y) < r$ then $\|f_N(x) - f_N(y)\| < \varepsilon/3$. Notice that $\|f(x) - f(y)\| \le \|f(x) - f_N(x)\| + \|f_N(x) - f_N(y)\| + \|f_N(y) - f(y)\| < \varepsilon$. Hence, $B_h(X, \mathbb{C})$ is a unital $C^*$-subalgebra of $B(X, \mathbb{C})$. Assume that $f \in B_0(X, \mathbb{C})$, and $\varepsilon > 0$. Choose a compact subset $K \subset X$ such that $\|f(x)\| < \varepsilon/2$ for all $x \in X \setminus K$. Then, $\|f(x) - f(y)\| \le \|f(x)\| + \|f(y)\| < \varepsilon$, for all $x, y \in X \setminus K$. Since compact subsets are bounded in $X$, so $B_0(X,\mathbb{C}) \subset C_h(X,\mathbb{C})$.

$(b)$ Similar to $(a)$. $\square$

**Lemma 2.3.11.** *Given a non-empty set $X$ and a family of functions $\mathcal{F} := \{f_\alpha : X \to X_\alpha : \alpha \in I\}$, where each $X_\alpha$ is a topological space. The family $\mathcal{S} := \{f_\alpha^{-1}(U_\alpha) : U_\alpha \subset X_\alpha$ is open in $X_\alpha, \alpha \in I\}$ is a subbasis for a topology on $X$ called the* ***weak topology*** *on $X$ generated by $\mathcal{F}$.*

*Proof.* Straightforward. $\square$

**Definition 2.3.12.** A linear map $L : A \to B$ between normed $\mathbb{C}$-vector spaces $(A, \|.\|_A)$ and $(B, \|.\|_B)$ is called **bounded** if there exists $M > 0$ such that $\|L(x)\|_B \leq M\|x\|_A$, for all $x \in A$. When $B = \mathbb{C}$, $L : A \to B$ is called **linear functional**. The **dual** of $A$ is the family $A^* := \{f : A \to \mathbb{C} : f$ is bounded linear functional$\}$. The dual space $A^*$ is equipped with the supremum norm $\|f\|_\infty := \sup\limits_{x\in A}\{\|f(x)\| : \|a\|_A \leq 1\}$. For each $x \in A$, the **evaluation map** $ev_x : A^* \to \mathbb{C}$ that maps each $f \in A^*$ to $f(x)$. The **weak-$*$ topology** $\mathcal{T}_{\mathcal{W}^*}$ on $A^*$ is the weak topology generated by the family $\{ev_x : A^* \to \mathbb{C} : x \in A\}$.

**Proposition 2.3.13.** *Let $A$ be a normed $\mathbb{C}$-vector space. The weak-$*$ topology on $A^*$ is Hausdorff.*

*Proof.* Let $f, g \in A^*$ be distinct elements, and choose $x \in A$ such that $f(x) \neq g(x)$. Let $U$ and $V$ be disjoint neighborhoods in $\mathbb{C}$ of $f(x)$ and $g(x)$, respectively. As $ev_x : A^* \to \mathbb{C}$ is continuous, $ev_x^{-1}(U)$ and $ev_x^{-1}(V)$ are disjoint neighborhoods in $\mathcal{T}_{\mathcal{W}^*}$ of $f$ and $g$, respectively. $\square$

**Lemma 2.3.14.** *(**Alaoglu Theorem**) Let $A$ be a normed $\mathbb{C}$-vector space. The closed unit ball $B_1 := \{f \in A^* : \|f\|_\infty \leq 1\}$ of the normed space $(A^*, \|.\|_\infty)$ is compact in the weak-$*$ topology $\mathcal{T}_{\mathcal{W}^*}$ of $A^*$.*

*Proof.* See (29). $\square$

**Definition 2.3.15.** Let $A$ be a commutative unital Banch algebra; the family:
$\Omega(A) := \{\sigma : A \to \mathbb{C} : \sigma$ is linear functional and a ring homomorphism$\}$ is called the **maximal ideal space** of $A$.

The maximal ideal space $\Omega(A)$ is not only a subset of $A^*$, but it is a compact subset when $A^*$ is equipped with the weak-$*$ topology $\mathcal{T}_{\mathcal{W}^*}$.

**Theorem 2.3.16.** *Let $A$ be a commutative unital Banch algebra; then $\|\sigma\|_\infty = 1$, for all $\sigma \in \Omega(A)$.*

*Proof.* See (29). $\square$

Consequently, if $A$ is a commutative unital Banch algebra; then $\Omega(A)$ is a closed subset of closed unit ball $B_1 := \{f \in A^* : \|f\|_\infty \leq 1\}$ of the normed space $(A^*, \|.\|_\infty)$, and by Alaoglu Theorem, it is a compact subset of the Hausdorff space $(A^*, \mathcal{T}_{\mathcal{W}^*})$.

The evaluation map $ev_x : (A^*, \mathcal{T}_{\mathcal{W}^*}) \to \mathbb{C}$ is continuous for each $x \in A$. Therefore, $ev_x|_{\Omega(A)} \in C(\Omega(A), \mathbb{C})$, for all $x \in A$. Therefore, we can define a map $\Gamma : A \to C(\Omega(A), \mathbb{C})$ mapping each $x \in A$ to $ev_x|_{\Omega(A)}$.

**Theorem 2.3.17.** *(**Gelfand Theorem**) Let $A$ be a commutative unital $C^*$-algebra. The **Gelfand Transformation** $\Gamma : A \to C(\Omega(A), \mathbb{C})$ given by $\Gamma(x) = ev_x$ is an isometric $*$-isomorphism.*

*Proof.* See (29). $\square$

We utilize the Gelfand Theorem to prove that every unital $C^*$-subalgebra of $C_b(X, \mathbb{C})$, that separates points from closed sets, determines a compactification of $X$.

**Theorem 2.3.18.** *Let $A$ be a unital $C^*$-subalgebra $C_b(X, \mathbb{C})$ that separates points from closed sets. The maximal ideal space $\Omega(A)$ is a compactification of $X$.*

*Proof.* Notice that for each $x \in X$, the evaluation map $e_x : A \to \mathbb{C}$ that sends each $f \in A$ to $f(x)$ is a linear transformation and a ring homomorphism. Thus, $e_x \in \Omega(A)$, for all $x \in X$. Define $\psi : X \to \Omega(A)$ by $\psi(x) = e_x$.

**Claim 1:** $\psi : X \to \Omega(A)$ is continuous and injective.

**Proof of Claim 1:** Notice that a subbasis element $\Omega(A)$ is of the form $U_f := \{\sigma \in \Omega(A) : \sigma(f) \in U\}$ for some $f \in A$ and $U \subset \mathbb{C}$ an open subset of $\mathbb{C}$. Let $(x_\alpha)_{\alpha \in I}$ be a net in $X$ converging to $x \in X$. Then, for each $f \in A$, $f(x_\alpha) \to f(x)$ and therefore $e_{x_\alpha} \to e_x$ proving that $\psi(x_\alpha) \to \psi(x)$. The injectivity of $\psi : X \to \Omega(A)$ follows from the fact that $A$ separates points from closed sets. Indeed, if $\psi(x) = \psi(y)$, then $f(x) = f(y)$ for all $f \in A$, and as $A$ separates points from closed sets, it separates points which implies that $x = y$.

**Claim 2:** $\psi : X \to \Omega(A)$ is an open map.

**Proof of Claim 2:** Let $O \subset X$ be an open subset. We aim to show that $\psi(O) = \{e_x : x \in O\}$ is an open subset $\psi(X) = \{e_x : x \in X\}$. Indeed, let $\sigma \in \psi(O)$. Then, $\sigma = e_x$ for some $x \in O$. Since, $x \notin F := X \setminus O$, and $A$ separates points from closed sets, there exists $f \in A$ such that $f(x) \notin cl_{\mathbb{C}}(f(F))$. The subset $U := \mathbb{C} \setminus cl_{\mathbb{C}}(f(F))$ is an open subset of $\mathbb{C}$ containing $f(x)$. Therefore, the subset $U_f \cap \psi(X) = \{\sigma \in \Omega(A) : \sigma(f) \in U\} \cap \psi(X)$ is an open subset of $\psi(X)$ containing $\sigma = e_x$. Moreover, if $e_y \in U_f \cap \psi(X)$, then $f(y) \in U = \mathbb{C} \setminus cl_{\mathbb{C}}(f(F))$ and hence $f(y) \notin f(F)$. In particular, $y \notin F$ and thus $y \in O$, which implies that $e_x \in U_f \cap \psi(X) \subset \psi(O)$. Therefore, $\psi(O)$ is an open subset of $\psi(X)$.

**Claim 3:** $\psi(X)$ is dense in $\Omega(A)$.

**Proof of Claim 3:** As $X$ is homeomorphic to $\psi(X)$, and by Gelfand Theorem $A$ and $C(\Omega(A), \mathbb{C})$ are $*$-isomorphic, we may assume without loss of generality that $X = \psi(X)$ and $A = C(\Omega(A), \mathbb{C})$. Seeking contradiction, assume that there exists a non-empty open subset $U$ of $\Omega(A)$ such that $U \cap X = \emptyset$. Hence $X \subset \Omega(A) \setminus U$ and $\Omega(A) \setminus U$ is a closed subset of the Hausdorff compact space $\Omega(A)$. By Urysohn's lemma, there exists non-constant continuous function $f \in C(\Omega(A), \mathbb{C}) = A$ such that $f(\Omega(A) \setminus U) = \{0\}$. Hence, $f(X) = \{0\}$, a contradiction. Therefore, $\psi(X)$ is dense in $\Omega(A)$. ☐

**Corollary 2.3.19.** *Let $(X, d)$ be a proper metric space, then $\Omega(C_h(X, \mathbb{C}))$ is a compactification of $X$.*

*Proof.* By 2.3.10, $C_h(X, \mathbb{C})$ is a unital $C^*$-subalgebra of $C_b(X, \mathbb{C})$ containing $C_0(X, \mathbb{C})$. Since $X$ is locally compact, then by 2.2.26, $C_0(X, \mathbb{C})$ separates points from closed sets. Consequently, $C_h(X, \mathbb{C})$ is a unital $C^*$-subalgebra of $C_b(X, \mathbb{C})$ that separates points from closed sets. Apply 2.3.18. ☐

**Definition 2.3.20.** Let $(X, d)$ be a proper metric space, and $C_h(X, \mathbb{C})$ be the $C^*$-subalgebra of all continuous bounded complex-valued slowly oscillating functions. The **Higson compactification** of $X$ is the maximal ideal space $\Omega(C_h(X, \mathbb{C}))$ of $C_h(X, \mathbb{C})$. The Higson compactification is denoted by $hX$ and its corona by $\partial_h X$. Its corona $\partial_h X$ is referred to as the **Higson corona**.

**Theorem 2.3.21.** *Let $A$, $B$ and $C$ be unital commutative $C^*$-algebras; and let $X$ and $Y$ be topological spaces. Then:*

*(a) Every $*$-homomorphism $\varphi : A \to B$ induces a continuous map $\widetilde{\varphi} : \Omega(B) \to \Omega(A)$ between their maximal ideals spaces equipped with the weak-$*$ topology defined by $\widetilde{\varphi}(g) = g \circ \varphi$. If $\varphi : A \to B$ is a $*$-isomorphism, then $\widetilde{\varphi} : \Omega(B) \to \Omega(A)$ is a homeomorphism.*

*(b) Every continuous function $f : X \to Y$ induces a $*$-homomorphism $\widehat{f} : C(Y, \mathbb{C}) \to C(X, \mathbb{C})$. If $f : X \to Y$ is a homeomorphism, then $\widehat{f} : C(Y, \mathbb{C}) \to C(X, \mathbb{C})$ is a $*$-isomorphism.*

*(c) If $X$ and $Y$ are compact, and if $A = C(X, \mathbb{C})$ and $B = C(Y, \mathbb{C})$, then $X$ and $Y$ are homeomorphic if and only if $A$ and $B$ are $*$-isomorphic.*

*Proof.* $(a)$ Let $\varphi : A \to B$ be a $*$-homomorphism. Define $\widetilde{\varphi} : \Omega(B) \to \Omega(A)$ by $\widetilde{\varphi}(g) = g \circ \varphi$. Clearly, $g \circ \varphi \in \Omega(A)$. A subbasis element of $\Omega(A)$ is of the form $U_x := \{f \in \Omega(A) : f(x) \in U\}$ for some open subset $U$ of $\mathbb{C}$. Notice that $\widetilde{\varphi}^{-1}(U_x) = \{g \in \Omega(B) : g \circ \varphi(x) \in U\} = U_{\varphi(x)}$ which is a subbasis element of $\Omega(B)$. Finally, if $C$ is a unital commutative $C^*$-algebra and $\psi : B \to C$ be a $*$-homomorphism, then $\widetilde{\psi \circ \varphi} = \widetilde{\varphi} \circ \widetilde{\psi}$; moreover, $\widetilde{id_A} = id_{\Omega(A)}$ which proves that if $\varphi : A \to B$ is a $*$-isomorphism, then $\widetilde{\varphi} : \Omega(B) \to \Omega(A)$ is a homeomorphism.

$(b)$ Straightforward.

$(c)$ Notice that by 2.3.18, $\Omega(A)$ and $\Omega(B)$ are compactifications of the compact spaces $X$ and $Y$, respectively. Apply $(a)$ and $(b)$. $\square$

**Corollary 2.3.22.** *$Y$ is a compactification of $X$ if and only if $Y$ is the maximal ideal space of the unital commutative $C^*$-subalgebra $C_Y := \{f \in C_b(X, \mathbb{C}) : f$ extends continuously to $Y\}$ of $C_b(X, \mathbb{C})$.*

*Proof.* Since $\Omega(C_b(Y, \mathbb{C}))$ is a compactification of the compact space $Y$, so $Y = \Omega(C_b(Y, \mathbb{C}))$. On the other hand, applying part $(d)$ of 2.3.7 and part $(a)$ of 2.3.21, we have $Y = \Omega(C_Y)$ as wanted. $\square$

We conclude this section by describing the corona of a compactification of a locally compact space. We need the following lemma to that end.

**Lemma 2.3.23.** *Let $X$ be a locally compact space and $Y$ be a compactification of $X$. A continuous function $f : X \to \mathbb{C}$ vanishes at infinity if and only if the extension $\widetilde{f} : Y \to \mathbb{C}$ given by:*

$$\widetilde{f} = \begin{cases} f(x) & if x \in X, \\ 0 & if x \in Y \setminus X \end{cases}$$

*is a continuous function.*

*Proof.* Assume that $f : X \to \mathbb{C}$ vanishes at infinity. By 2.2.25, $\widetilde{f} : Y \to \mathbb{C}$ is continuous. Conversely, assume that $\widetilde{f} : Y \to \mathbb{C}$ is a continuous extension of $f : X \to \mathbb{C}$. For $\varepsilon > 0$, and $y \in Y \setminus X$, choose a neighborhood $U_y$ of $y$ such that $\left\|\widetilde{f}(x) - \widetilde{f}(y)\right\| = \left\|\widetilde{f}(x)\right\| < \varepsilon$. Since $Y \setminus X$ is compact, it can be covered by finitely many such neighborhoods, say $U_{y_1}, \ldots, U_{y_n}$. Put $U := \bigcup_{i=1}^{n} U_{y_i}$, then $K := Y \setminus U$ is a compact subset of $X$, with $\|f(x)\| = \left\|\widetilde{f}(x)\right\| < \varepsilon$, for all $x \in X \setminus K = Y \setminus K$. Thus, $f \in C_0(X, \mathbb{C})$ as desired. □

**Theorem 2.3.24.** *Let $Y$ be a compactification of a locally compact space $X$, and let $\partial_Y X := Y \setminus X$ be the corona of $Y$. Then $\partial_Y X$ is the maximal ideal space of unital commutative $C^*$-subalgebra ${C_Y}\big/{C_0(X, \mathbb{C})}$.*

*Proof.* As $\partial_Y X$ is a compact space, so it is the maximal ideal space of $C(\partial_Y X, \mathbb{C})$ by 2.3.18. Define $\varphi : C(Y, \mathbb{C}) \to C(\partial_Y X, \mathbb{C})$ by $\varphi(f) = f|_{\partial_Y X}$. It is straightforward to see that $\varphi : C(Y, \mathbb{C}) \to C(\partial_Y X, \mathbb{C})$ is a $*$-homomorphism. Given a continuous function $f : \partial_Y X \to \mathbb{C}$, since $\partial_Y X$ is a closed subset of the normal space $Y$, by Tietze extension theorem, it admits a continuous extension $\widetilde{f} : Y \to \mathbb{C}$. Hence, $\varphi(\widetilde{f}) = \widetilde{f}|_{\partial_Y X} = f$, which proves that $\varphi : C(Y, \mathbb{C}) \to C(\partial_Y X, \mathbb{C})$ is surjective. Notice that $Ker(\varphi) = \{f \in C(Y, \mathbb{C}) : f|_{\partial_Y X} \equiv 0\} = C_0(X, \mathbb{C})$ by 2.3.23. Applying the First Isomorphism Theorem, we conclude that $C(\partial_Y X, \mathbb{C}) = {C(Y, \mathbb{C})}\big/{C_0(X, \mathbb{C})} = {C_Y}\big/{C_0(X, \mathbb{C})}$. □

**Corollary 2.3.25.** *Let $(X, d)$ be a proper metric space, then $\partial_h X$ is the maximal ideal space of ${C_h(X, \mathbb{C})}\big/{C_0(X, \mathbb{C})}$.*

*Proof.* Apply 2.3.24. □

# Chapter 3

# The Stone's Representation Theorems and compactifications of a discrete space

Stone's spaces are central objects in this thesis. The Stone's Representation Theorems show that there is a duality between the category of Boolean algebras and the category of Stone's spaces. Compactifications of a discrete space can be constructed as the Stone's spaces of some sub-Boolean algebra of its power set. This chapter is based on the preprint of the following paper: H. Rashed, The Stone's Representation Theorems and compactifications of a discrete space, arXiv:2112.07821 [math.GN].

## 3.1 Boolean algebras and their homomorphisms

**Definition 3.1.1.** A **Boolean algebra** $(B, +, \cdot, \neg)$ is a set $B$ equipped with two binary operations $+$, $\cdot$ and a unary operation $\neg$ called the complement, where the following are satisfied:

(1) $+$ and $\cdot$ are **commutative** and **associative**, that is:

• $x + y = y + x$ and $x \cdot y = y \cdot x$, for all $x, y \in B$.

• $x + (y + z) = (x + y) + z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, for all $x, y, z \in B$.

(2) $+$ is **distributive over** $\cdot$, and $\cdot$ is distributive over $+$, that is:

• $x + (y \cdot z) = (x + y) \cdot (x + z)$, for all $x, y, z \in B$.

• $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, for all $x, y, z \in B$.

(3) $+$ and $\cdot$ are **connected by the absorption law**, that is:

• $x + (x \cdot y) = x \cdot (x + y) = x$, for all $x, y \in B$.

(4) There exist two elements $0, 1 \in B$ satisfying:

• $x + (\neg x) = 1$, for all $x \in B$;

• $x \cdot (\neg x) = 0$, for all $x \in B$.

**Example 3.1.2.** *Consider the power set $\mathcal{P}(X)$ of $X$; then $(\mathcal{P}(X), +, \cdot, \neg)$ is a Boolean algebra, where $+ := \cup$, $\cdot := \cap$, $0 := \emptyset$, $1 := X$ and $\neg A := X \setminus A$ for all $A \in \mathcal{P}(X)$. The Boolean algebra $(\mathcal{P}(X), \cup, \cap, \neg)$ is called the* ***power set Boolean algebra****.*

**Definition 3.1.3.** Let $(B, +, \cdot, \neg)$ be a Boolean algebra, a subset $A \subset B$ is called a **sub-Boolean algebra** if $(A, +, \cdot, \neg)$ is itself a Boolean algebra.

**Example 3.1.4.** *Let $X$ be a topological space. The family of all clopen subsets $Clop(X) := \{U \subset X : U \text{ is clopen}\}$ is a sub-Boolean algebra of $\mathcal{P}(X)$. The Boolean algebra $(Clop(X), +, \cdot, \neg)$ is called the* ***clopen Boolean algebra*** *associated with $X$.*

**Example 3.1.5.** *Let $G$ be a group acts on $X$; a subset $A \subset X$ is called $G$**-commensurated** or $G$**-almost invariant** if $g \cdot A \Delta A$ is a finite subset of $X$ for all $g \in G$. The family of all $G$-commensurated subsets of $X$ is denoted by $Comm_G(X)$. $Comm_G(X)$ is a sub-Boolean algebra of $\mathcal{P}(X)$ (7).*

**Example 3.1.6.** *Let $G$ be a group acts on $X$; define a relation $\sim$ on $Comm_G(X)$ by: for all $A, B \in Comm_G(X)$, $A \sim B$ if and only if $A \Delta B$ is finite. Then:*
*(a) The relation $\sim$ is an equivalence relation on $Comm_G(X)$.*
*(b) The family of all equivalence classes $Comm_G(X)/\sim := \{[A] : A \in Comm_G(X)\}$ of $Comm_G(X)$ is a Boolean algebra with operations:*

• $[A] + [B] := [A \cup B]$;

• $[A] \cdot [B] := [A \cap B]$;

• $\neg[A] := [X \setminus A]$.

The fact that the binary relations $+$ and $\cdot$ being connected by the absorption law gives us the following properties.

**Proposition 3.1.7.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, the following are satisfied:*

$(a)$ $\cdot$ *is* ***idempotent****, that is $x^2 := x \cdot x = x$, for all $x \in B$.*

$(b)$ $+$ *is* ***idempotent****, that is $2x := x + x = x$ for all $x \in B$.*

$(c)$ *$1 \cdot x = x$, for all $x \in B$.*

$(d)$ *$1 + x = 1$, for all $x \in B$.*

$(e)$ *$0 + x = x$, for all $x \in B$.*

$(f)$ *$0 \cdot x = 0$, for all $x \in B$.*

*Proof.* $(a)$ Let $y \in B$, then $x = x + (x \cdot y)$ and therefore $x^2 = x \cdot (x + (x \cdot y)) = x$.

$(b)$ By part $a)$ we have $2x = x^2 + x^2 = x \cdot (x + x) = x$.

$(c)$ $x \cdot 1 = x \cdot (x + (\neg x)) = x$.

$(d)$ $1 + x = (x + (-x)) + x = 2x + (\neg x) = x + (\neg x) = 1$.

$(e)$ $0 + x = x + 0 = x + (x \cdot (\neg x)) = x$.

$(f)$ $0 \cdot x = x \cdot 0 = x \cdot (x \cdot (\neg x)) = x^2 \cdot (\neg x) = x \cdot (\neg x) = 0$. □

**Proposition 3.1.8.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra and $x, y \in B$. Then:*

$(a)$ *If $x \cdot y = 0$ and $x + y = 1$, then $y = \neg x$.*

$(b)$ *$\neg(\neg x) = x$.*

$(c)$ *$\neg 1 = 0$ and $\neg 0 = 1$.*

*Proof.* Notice that part $(b)$ and $(c)$ follow from $(a)$. Let $x, y \in B$, such that $x \cdot y = 0$ and $x + y = 1$. On one hand, one has $\neg x = \neg x \cdot 1 = \neg x \cdot (x + y) = (\neg x) \cdot x + (\neg x) \cdot y = 0 + (\neg x) \cdot y = \neg x \cdot y$. On the other hand, $\neg x = \neg x + 0 = (\neg x \cdot y) + (x \cdot y) = (\neg x + x) \cdot y = 1 \cdot y = y$. □

**Proposition 3.1.9.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and define a relation $\leqslant$ on $B$ by: $x \leqslant y$ if and only if $x + y = y$ for all $x, y \in B$. Then the relation $\leqslant$ is a partial order relation on $B$.*

*Proof.* • Reflexivity: for all $x \in B$, $x + x = x^2 + x^2 = x \cdot (x + x) = x$ and hence $x \leqslant x$.

• Antisymmetry: for all $x, y \in B$, if $x \leqslant y$ and $y \leqslant x$ then $x + y = y$ and $x + y = x$; thus, $x = y$.

• Transitivity: for all $x, y, z \in B$, if $x \leqslant y$ and $y \leqslant z$, then $x + y = y$ and $y + z = z$. Thus, $x + z = x + (y + z) = (x + y) + z = y + z = z$ which implies that $x \leqslant z$. □

**Proposition 3.1.10.** *Let* $(B, +, \cdot, \neg)$ *be a Boolean algebra and* $x, y, x', y' \in B$. *Then:*

$(a)$ $0 \leqslant x \leqslant 1$.

$(b)$ $x \cdot y \leqslant x \leqslant x + y$ *and* $x \cdot y \leqslant y \leqslant x + y$.

$(c)$ *If* $x \leqslant y$ *and* $x' \leqslant y'$, *then* $x \cdot x' \leqslant y \cdot y'$.

$(d)$ *If* $x \leqslant y$ *and* $x' \leqslant y'$, *then* $x + x' \leqslant y + y'$.

$(e)$ $\sup\{x, y\} = x + y$.

$(f)$ $\inf\{x, y\} = x \cdot y$.

*Proof.* $(a)$ Let $x \in B$, since $x + 0 = x$ and $x + 1 = 1$, so $0 \leqslant x \leqslant 1$.

$(b)$ Let $x, y \in B$, since $x + (x \cdot y) = x$ and $x + (x + y) = 2x + y = x + y$, hence $x \cdot y \leqslant x \leqslant x + y$. Similarly, $x \cdot y \leqslant y \leqslant x + y$.

$(c)$ Let $x, y, x', y' \in B$ and assume that $x \leqslant y$ and $x' \leqslant y'$. Now, $y \cdot y' \leqslant x \cdot x' + y \cdot y' \leqslant x \cdot x' + x \cdot y' + y \cdot x' + y \cdot y' = x \cdot (x' + y') + y \cdot (x' + y') = (x + y) \cdot (x' + y') = y \cdot y'$ and hence $x \cdot x' + y \cdot y' = y \cdot y'$ as desired.

$(d)$ Let $x, y, x', y' \in B$ and assume that $x \leqslant y$ and $x' \leqslant y'$, then $(x + x') + (y + y') = (x + y) + (x' + y') = y + y'$.

$(e)$ Let $x, y, z \in B$ such that $x \leqslant z$ and $y \leqslant z$; then by part $(b)$, $x + y \leqslant 2z = z$.

$(f)$ Let $x, y, z \in B$ such that $z \leqslant x$ and $z \leqslant y$; then by part $(c)$, $z = z^2 \leqslant x \cdot y$.

□

3.1.10 tells us that every Boolean algebra is a distributive lattice.

**Definition 3.1.11.** Let $(B, +, \cdot, \neg)$ and $(B', +', \cdot', \neg')$ be Boolean algebras. A map $\varphi : B \to B'$ is called a **homomorphism** if for all $x, y \in B$:

- $\varphi(x + y) = \varphi(x) +' \varphi(y)$,
- $\varphi(x \cdot y) = \varphi(x) \cdot' \varphi(y)$,
- $\varphi(0_B) = 0_{B'}$ and $\varphi(1_B) = 1_{B'}$.

Notice that for $x \in B$, since $0_{B'} = \varphi(0_B) = \varphi(x \cdot (\neg x)) = \varphi(x) \cdot' \varphi(\neg x)$ and $1_{B'} = \varphi(1_B) = \varphi(x + (\neg x)) = \varphi(x) +' \varphi(\neg x)$, hence by 3.1.8, $\varphi(\neg x) = \neg' \varphi(x)$.

**Example 3.1.12.** *The field* $\mathbb{Z}_2$ *is a Boolean algebra where the complement* $\neg$ *of its elements is given by:* $\neg 0 := 1$ *and* $\neg 1 := 0$. *Moreover,* $\mathbb{Z}_2$ *is unique up to isomorphism; that is,* *any Boolean algebra with cardinality* $2$ *is isomorphic to* $\mathbb{Z}_2$. $\mathbb{Z}_2$ *is called the* ***two-element Boolean algebra****.*

Now we associate a Boolean algebra to any given topological space with the property that any continuous map between two topological spaces induces a homomorphism between their associated Boolean algebras; and if the two topological spaces are homeomorphic, their associated Boolean algebras are isomorphic.

**Proposition 3.1.13.** *Let* $X$ *be a topological space. Then:*
*(a) If* $f : X \to Y$ *is a continuous map between topological spaces, it induces a homomorphism* $\widetilde{f} : Clop(Y) \to Clop(X)$.
*(b) If* $f : X \to Y$ *is a homeomorphism between topological spaces, it induces an isomorphism* $\widetilde{f} : Clop(Y) \to Clop(X)$.

*Proof.* $(a)$ Let $f : X \to Y$ be a continuous map between topological spaces, and define $\widetilde{f} : Clop(Y) \to Clop(X)$ by $\widetilde{f}(U) = f^{-1}(U)$ for all $U \in Clop(Y)$. Clearly, the map $\widetilde{f} : Clop(Y) \to Clop(X)$ is a homomorphism.
$(b)$ Let $g : Y \to X$ be a continuous map such that $g \circ f = id_X$ and $f \circ g = id_Y$. By part $(b)$, $\widetilde{g} : Clop(X) \to Clop(Y)$ is a homomorphism. Moreover, $\widetilde{f} \circ \widetilde{g} = id_{Clop(X)}$ and $\widetilde{g} \circ \widetilde{f} = id_{Clop(Y)}$. □

The clopen Boolean algebra associated with a topological space $X$ is the two-elements Boolean algebra if and only if $X$ is connected. The more components $X$ has, the more interesting the Boolean algebra $Clop(X)$ is. Hausdorff compact spaces that are totally disconnected are of a particular interests as every Boolean algebra is the clopen Boolean algebra associated with a such topological space.

## 3.2 Stone's space of a Boolean algebra

In this section, out of a given Boolean algebra $B$, we construct a compact Hausdorff totally disconnected topological space called the **Stone's space associated with** the Boolean algebra $B$ such that every homomorphism between two Boolean algebras induces a continuous map between their associated Stone's spaces. One way to construct a such space is by taking the set of all ultrafilters of $B$ as the underlying set of the Stone's space associated with a Boolean algebra $B$. We first give a characterization of a totally disconnected space $X$, when $X$ is locally compact.

## 3.3 Characterization of locally compact totally disconnected spaces

**Definition 3.3.1.** Let $X$ a Hausdorff topological space:

• $X$ is called **totally disconnected** if the connected components of $X$ are exactly the singletons. Obviously, every discrete topological space is totally disconnected; however, the converse does not hold. $\mathbb{Q}$ as a subspace of $\mathbb{R}$ serves as an example of a totally disconnected space that is not discrete.

• $X$ is called a **Stone's space** if it is compact and totally disconnected.

• $X$ is called **zero-dimensional** if $X$ admits a basis consists of clopen subsets.

**Definition 3.3.2.** Let $X$ be a non-empty set and $\mathcal{A} \subset \mathcal{P}(X)$. $\mathcal{A}$ is said to **satisfy the finite intersection property** if the intersection of any finitely many elements of $\mathcal{A}$ is non-empty. It is well-known that a topological space is compact if and only if $\bigcap\limits_{A \in \mathcal{A}} A \neq \emptyset$ for every family of closed subsets $\mathcal{A}$ that satisfies the finite intersection property.

**Proposition 3.3.3.** *Let $X$ be a topological space. If $X$ is zero-dimensional then it is totally disconnected.*

*Proof.* Let $A \subset X$ and $x, y \in A$ be distinct elements. Since $X$ is a $T_1$ space, we can find a clopen subset $U$ containing $x$ but not $y$. Let $V := X \setminus U$, then $U$ and $V$ are disjoint open subsets containing $x$ and $y$, respectively, and $X = U \cup V$. $\square$

When $X$ is locally compact, the concept of being totally disconnected coincide with the concept of being zero-dimensional.

**Lemma 3.3.4.** *Let $X$ be a Hausdorff compact space. For $x \in X$, let $C_x$ be the component of $X$ containing $x$, and $\mathcal{A}_x$ be the family of all clopen subsets of $X$ containing $x$. Then, $C_x = \bigcap_{A \in \mathcal{A}_x} A$.*

*Proof.* Notice that every component of $X$ that intersects a clopen subset of $X$ is contained in that clopen subset, so $C_x \subset \bigcap_{A \in \mathcal{A}_x} A$. To prove that $\bigcap_{A \in \mathcal{A}_x} A \subset C_x$, it suffices to show that $\bigcap_{A \in \mathcal{A}_x} A$ is connected. Put $O_x := \bigcap_{A \in \mathcal{A}_x} A$ and assume that $U_1$ and $U_2$ are disjoint closed subsets of $O_x$ such that $O_x = U_1 \cup U_2$. We aim to show that either $U_1 = \emptyset$ or $U_2 = \emptyset$. Since $X$ is normal, there exist disjoint open subsets $V_1$ and $V_2$ of $X$ such that $U_1 \subset V_1$ and $U_2 \subset V_2$. Put $V := V_1 \cup V_2$; we claim that there exist finitely many elements $A_1, \ldots, A_n \in \mathcal{A}_x$ such that $\bigcap_{i=1}^{n} A_i \subset V$. Seeking contradiction, assume that the intersection of any finitely many elements of $\mathcal{A}_x$ non-trivially intersects the closed subset $X \setminus V$. The family $\mathcal{A}_x \cup \{X \setminus V\}$ consists of closed subsets of the compact space $X$ and it satisfies the Finite Intersection Property, and hence $O_x \cap (X \setminus V) \neq \emptyset$ which contradicts the fact that $O_x \subset V$. Let, $A_1, \ldots, A_n \in \mathcal{A}_x$ such that $\bigcap_{i=1}^{n} A_i \subset V = V_1 \cup V_2$. Without loss of generality, assume that $x \in V_1$, and notice that $A := V_1 \cap \bigcap_{i=1}^{n} A_i$ is an open subset of $X$. Furthermore, as $A^c = (\bigcap_{i=1}^{n} A_i)^c \cup (V_1^c \cap V_2) = (\bigcap_{i=1}^{n} A_i)^c \cup V_2$ is open, $A = V_1 \cap \bigcap_{i=1}^{n} A_i$ is a clopen subset of $X$ containing $x$, and hence $A \in \mathcal{A}_x$. This shows that $O_x \subset A \subset V_1$ which implies that $V_2 = \emptyset$ and, a fortiori, $U_2 = \emptyset$. Thus, $O_x$ is connected as desired. $\square$

**Proposition 3.3.5.** *Let $X$ be a compact totally disconnected space. Then $X$ is zero-dimensional.*

*Proof.* Let $C_x$ and $\mathcal{A}_x$ be as in 3.3.4. Notice that $C_x = \bigcap_{A \in \mathcal{A}_x} A = \{x\}$. Let $U$ be an open neighborhood of $x \in X$. We can find finitely many elements $A_1, \ldots, A_n \in \mathcal{A}_x$ such that $\bigcap_{i=1}^{n} A_i \subset U$. Since $\bigcap_{i=1}^{n} A_i$ is a clopen subset containing $x$, $X$ admits a basis consists of clopen subsets. $\square$

**Theorem 3.3.6.** *Let $X$ be a Hausdorff locally compact space. $X$ is totally disconnected if and only if $X$ is zero-dimensional.*

*Proof.* Assume that $X$ is totally disconnected. For $x \in X$, let $U$ be an open neighborhood of $x \in X$. Since $X$ is a Hausdorff locally compact space, we can find a relatively compact open neighborhood $V$ of $x$ such that $V \subset cl(V) \subset U$. Now, $cl(V)$ itself is compact totally disconnected as a subspace of $X$, and by 3.3.5, $cl(V)$ is zero-dimensional. Let $A$ be a clopen subset of $cl(V)$ containing $x$, then as $A \subset V$, $A$ is clopen subset of $X$ containing $x$ and contained in $U$. Hence, $X$ is zero-dimensional. The other direction follows from 3.3.3. □

**Corollary 3.3.7.** *Let $X$ be a Hausdorff compact space. $X$ is a Stone's space if and only if $X$ is zero-dimensional.*

*Proof.* Apply 3.3.6. □

One way to construct a totally disconnected space $Y$ from a given topological space $X$ is to contract each component of $X$ to a singleton.

**Lemma 3.3.8.** *If $q : X \to Y$ is a quotient map with the property that the fiber of each $y \in Y$ is contained in some component of $X$. Then:*
*(a) $q^{-1}(D)$ is connected for all $D$ component of $Y$.*
*(b) $q : X \to Y$ maps each component of $X$ to a component of $Y$.*

*Proof.* $(a)$ Seeking contradiction, assume $q^{-1}(D) = U \cup V$ where $U$ and $V$ are non-empty disjoint open subsets of $q^{-1}(D)$, then $q(U)$ and $q(V)$ are non-empty disjoint subsets of $D$ with $D = q(U) \cup q(V)$. We claim that $q(U)$ and $q(V)$ are open subsets of $D$. To this end, notice that if $x \in q^{-1}(q(U))$, then $q(x) = q(a)$ for some $a \in U$. But, $q(x) = q(a) = y \in D$ (since $q$ is onto) and hence $a, x \in q^{-1}(y) \subset U$ as $U$ is clopen subset of $q^{-1}(D)$ and $q^{-1}(y)$ is contained in a connected subset. Thus, $U = q^{-1}(q(U))$ and similarly $V = q^{-1}(q(V))$. Since $q^{-1}(D)$ is a closed subset, $q : q^{-1}(D) \to q(q^{-1}(D)) = D$ is a quotient map which implies that $q(U)$ and $q(V)$ are open, a contradiction.
$(b)$ Assume that $C$ is a component of $X$, then $q(C)$ is contained in a component $D$ of $Y$. Now, $q(C) \subset D$ and thus $C \subset q^{-1}(q(C)) \subset q^{-1}(D)$. By part $(a)$, $C = q^{-1}(D)$ and hence $q(C) = D$. □

**Proposition 3.3.9.** *Let $X$ be a Hausdorff space. Define a relation $\sim$ on $X$ by $x \sim y$ if and only if $x$ and $y$ are in the same component of $X$. The relation $\sim$ on $X$ is an equivalence relation and the quotient space $X/_\sim$ is totally disconnected.*

*Proof.* Clearly the relation $\sim$ on $X$ is an equivalence relation on $X$. For all $x \in X$, let $[x] := \{y \in X : x \sim y\}$ be the equivalence class of $x$ and consider the quotient map $q : X \to X/_\sim$ that maps each $x$ to its equivalence class $[x]$. Notice that the fiber of each $[x]$ is contained in some component of $X$. Let $D$ be a component of $X/_\sim$ and let $[x], [y] \in D$. By above lemma, $q^{-1}(D)$ is connected and $x, y \in q^{-1}(D)$, and thus $[x] = [y]$. This proves that $D = [x]$ is a singleton and hence $X/_\sim$ is totally disconnected. $\square$

## 3.4 Stone's space of a Boolean algebra as the family of its ultrafilters

Now we construct the Stone's space that is associated with a Boolean algebra using its ultrafilters as the building blocks.

**Definition 3.4.1.** Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $F$ be a nonempty subset of $B$; $F$ is called a **filter** of $B$ if:

• For all $x, y \in F$, $x \cdot y \in F$.

• For all $x \in F$ and $y \in B$, if $x \leqslant y$, then $y \in F$.

• $F$ is called a **proper filter** if it is a filter contained properly in $B$. Clearly, a filter $F$ is proper if and only if $0 \notin F$.

**Definition 3.4.2.** Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $A \subset B$, $A$ is said to **satisfy the finite product property** if for any finitely many elements $x_1, \ldots, x_n \in A$, then $x_1 \cdot \ldots \cdot x_n \neq 0$.

In the Boolean algebra $(\mathcal{P}(X), \cup, \cap, \neg)$, a subset of $\mathcal{P}(X)$ satisfies the finite product property if and only if it satisfies the finite intersection property. Furthermore, $F \in \mathcal{P}(X)$ is a filter of the Boolean algebra $\mathcal{P}(X)$ if and only if it is a filter in the usual sense.

It can be evidently seen that every subset of a proper filter must satisfy the finite product property. Moreover, every subset of of a Boolean algebra that satisfies the finite product property is contained in a proper filter.

**Proposition 3.4.3.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra and $A$ be a nonempty subset of $B$ that satisfies the finite product property. Then, $A$ is contained in a proper filter $F_A$. Furthermore, if $F$ is a filter containing $A$, then $F_A \subset F$.*

*Proof.* Set $F_A := \{x \in B : x_1 \cdot \ldots \cdot x_n \leqslant x$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in A\}$. Then:

• $F_A \neq \emptyset$ as $1 \in F_A$.

• If $x, y \in F_A$, then there exist $n, m \in \mathbb{N}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$ such that $x_1 \cdot \ldots \cdot x_n \leqslant x$ and $y_1 \cdot \ldots \cdot y_m \leqslant y$. By 3.1.10, $x_1 \cdot \ldots \cdot x_n \cdot y_1 \cdot \ldots \cdot y_m \leqslant x \cdot y$ and hence $x \cdot y \in F_A$.

• By the definition of $F_A$, if $x \in F_A$ and $y \in B$ such that $x \leqslant y$, then $y \in F_A$.

Finally, it is clear that if $F_A$ is contained in every filter containing $A$. □

**Definition 3.4.4.** Let $(B, +, \cdot, \neg)$ be a Boolean algebra and $A$ be a nonempty subset of $B$ that satisfies the finite product property. The proper filter $F_A$ is called the **filter generated by** $A$. When $A = \{x\}$ is a singleton, $F_x$ is called the **principal filter generated by** $\{x\}$.

**Definition 3.4.5.** Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $F$ be a proper filter of $B$;

• $F$ is called a **prime filter** if for all $x, y \in B$ with $x + y \in F$, then either $x \in F$ or $y \in F$.

• $F$ is called a **maximal filter** if it is maximal with respect to the inclusion.

• $F$ is called an **ultrafilter** if for all $x \in B$, either $x \in F$ or $\neg x \in F$ and not both.

• The family of all ultrafilters of $B$ is denoted by $\mathcal{U}(B)$.

**Lemma 3.4.6.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $F'$ be a proper filter of $B$. For every $x \in B \setminus F$, there exists a maximal filter $M$ of $B$ containing $F'$ such that $x \notin M$.*

*Proof.* The family $\mathcal{F} := \{F \subset B :$ F is a proper filter containing $F'$ and $x \notin F$ $\}$ is partially ordered by the inclusion. Let $\mathcal{C} \subset \mathcal{F}$ be a chain; then $E = \bigcup_{F \in \mathcal{C}} F$ is an upper bound of $\mathcal{C}$ in $\mathcal{F}$. By Zorn's Lemma, $\mathcal{F}$ contains a maximal element $M$. □

**Lemma 3.4.7.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $F$ be a proper filter of $B$. $F$ is an ultrafilter if and only if it is a prime filter.*

*Proof.* Assume that $F$ is an ultrafilter of $B$. Let $x, y \in B \setminus F$. Since $F$ is an ultrafilter, one has $\neg x, \neg y \in F$. Seeking contradiction, assume that $x + y \in F$, then $(\neg x) \cdot (x + y) = (\neg x) \cdot y \in F$ and hence $((\neg x) \cdot y) \cdot (\neg y) = 0 \in F$, a contradiction.

Conversely, If $F$ is prime and $x \in B \setminus F$; then as $x + (\neg x) = 1 \in F$, $\neg x$ must be in $F$ and hence $F$ is an ultrafilter. □

**Lemma 3.4.8.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $F$ be a proper filter of $B$; $F$ is an ultrafilter if and only if it is a maximal filter.*

*Proof.* Assume that $F$ is an ultrafilter of $B$. Seeking contradiction, assume that $F$ is not a maximal filter. Let $M$ be a maximal filter containing $F$ and let $x \in M \setminus F$. Since $F$ is an ultrafilter, $\neg x \in F \subset M$. Thus, $x, \neg x \in M$ and in particular $x \cdot \neg x = 0 \in M$, a contradiction.

Conversely, let $F$ be maximal and $x \in B \setminus F$; notice that if $a \cdot x \neq 0$ for all $a \in F$, then $A := F \cup \{x\}$ satisfies the finite product property. By 3.4.3, $F$ is contained in a proper filter $F_A$. Since $F$ is maximal, $F = F_A$ which contradicts that fact that $x \in B \setminus F$. Therefore, there exists $a \in F$ such that $a \cdot x = 0$. Now, $a = a \cdot 1 = a \cdot (x + (\neg x)) = a \cdot (\neg x) \in F$, and since $a \cdot (\neg x) \leqslant \neg x$, $\neg x$ must be in $F$ and hence $F$ is an ultrafilter. $\square$

**Corollary 3.4.9.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $F$ be an ultrafilter of $B$; if $x \in B$ such that $x \cdot a \neq 0$ for all $a \in F$, then $x \in F$.*

*Proof.* Notice that $A := F \cup \{x\}$ satisfies the finite product property. Hence $F_A$ is a proper filter containing $F$. By the maximality of $F$, one has $F = F_A$ as wanted. $\square$

**Example 3.4.10.** *Consider the power set Boolean algebra $(\mathcal{P}(X), \cup, \cap, \neg)$ and $x \in X$; the principal filter $F_x$ generated by $x$ is an ultrafilter. Moreover, as ultrafilters are maximal, if $F$ is an ultrafilter containing $\{x\}$, then $F_x = F$.*

**Lemma 3.4.11.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $F$ be a proper filter of $B$; $F$ is an ultrafilter if and only if $F = \varphi_F^{-1}\{1\}$ for some homomorphism $\varphi_F : B \to \mathbb{Z}_2$.*

*Proof.* Assume that $F$ is an ultrafilter, and define $\varphi_F : B \to \mathbb{Z}_2$ by:

$$\varphi_F(x) = \begin{cases} 0 & \text{if } x \notin F \\ 1 & \text{if } x \in F \end{cases}$$

We claim that $\varphi_F : B \to \mathbb{Z}_2$ is a homomorphism.

• Clearly, $\varphi_F(0_B) = 0$ and $\varphi_F(1_B) = 1$ as $1_B \in F$ and $0_B \notin F$.

• Let $x, y \in B$; on one hand, if $x + y \in F$ then either $x \in F$ or $y \in F$. Without loss of generality assume that $x \in F$, then $\varphi_F(x) = 1$ and $\varphi_F(x) + \varphi_F(y) = 1 + \varphi_F(y) = 1$. Hence, $1 = \varphi_F(x + y) = \varphi_F(x) + \varphi_F(y)$. On the other hand, if $x + y \notin F$ then $x \notin F$ and $y \notin F$. Hence, $0 = \varphi_F(x + y) = \varphi_F(x) + \varphi_F(y)$.

• Let $x, y \in B$; on one hand, if $x \cdot y \in F$, then $x \in F$ and $y \in F$. Hence, $1 = \varphi_F(x \cdot y) =$

$\varphi_F(x) \cdot \varphi_F(y)$. On the other hand, if $x \cdot y \notin F$, then either $x \notin F$ or $y \notin F$. Without loss of generality assume that $x \notin F$, then $0 = \varphi_F(x \cdot y) = 0 \cdot \varphi_F(y) = \varphi_F(x) \cdot \varphi_F(y)$. Therefore, $\varphi_F : B \to \mathbb{Z}_2$ is a homomorphism and $F = \varphi_F^{-1}\{1\}$.

Conversely, let $\varphi_F : B \to \mathbb{Z}_2$ is a homomorphism, and let $F := \varphi_F^{-1}\{1\}$. Clearly, $F$ is a proper filter. If $x \in B \setminus F$, then $\varphi_F(\neg x) = \neg\varphi_F(x) = \neg 0 = 1$ and hence $\neg x \in F$. □

**Definition 3.4.12.** Let $(B, +, \cdot, \neg)$ be a Boolean algebra; and $I \subset B$:

• $I$ is called an **ideal** if for all $a, b \in I$ and $x \in R$, then $x \cdot a, a + b \in I$.

• A proper ideal $I$ is called a **prime ideal** if for all $x, y \in B$ with $x \cdot y \in I$, then either $x \in I$ or $x \in I$.

• A proper ideal $I$ is called a **maximal ideal** if it is maximal with respect the inclusion.

**Lemma 3.4.13.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $I$ be a proper ideal of $B$; $I$ is a prime ideal if and only if $I = \varphi_I^{-1}\{0\}$ for some homomorphism $\varphi_I : B \to \mathbb{Z}_2$.*

*Proof.* Assume that $I$ is a prime ideal, and define $\varphi_I : B \to \mathbb{Z}_2$ by:

$$\varphi_I(x) = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I \end{cases}$$

We claim that $\varphi_I : B \to \mathbb{Z}_2$ is a homomorphism.

• Clearly, $\varphi_I(0_B) = 0$ and $\varphi_I(1_B) = 1$ as $1_B \notin I$ and $0_B \in I$.

• Let $x, y \in B$; notice that $x + y \in I$ if and only if $x, y \in I$. Indeed, if $x \notin I$ and $x + y \in I$, then by the absorption law one has $x \cdot (x + y) = x + x \cdot y = x \in I$, a contradiction. Now, if $x + y \in I$ then $x, y \in I$ and hence $0 = \varphi_I(x + y) = \varphi_I(x) + \varphi_I(y)$. If $x + y \notin I$ then $x, y \notin I$ and hence $1 = \varphi_I(x + y) = \varphi_I(x) + \varphi_I(y)$ as $1 + 1 = 1$.

• Let $x, y \in B$; if either $x \in I$ or $y \in I$, then $x \cdot y \in I$ and hence $0 = \varphi_I(x \cdot y) = \varphi_I(x) \cdot \varphi_I(y)$. If $x, y \notin I$, then as $I$ is prime $x \cdot y \notin I$ and hence $1 = \varphi_I(x \cdot y) = \varphi_I(x) \cdot \varphi_I(y)$.

Conversely, let $\varphi_I : B \to \mathbb{Z}_2$ is a homomorphism such that $I := \varphi_I^{-1}\{0\}$. Clearly, $I$ is a proper ideal. Seeking contradiction, assume that $x, y \in B \setminus I$ such that $x \cdot y \in I$, then $0 = \varphi_I(x \cdot y) = \varphi_I(x) \cdot \varphi_I(y) = 1$, a contradiction. □

**Corollary 3.4.14.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $F \subset B$. $F$ is an ultrafilter of $B$ if and only if $B \setminus F$ is a prime ideal.*

*Proof.* Apply 3.4.11 and 3.4.13. $\square$

From 3.4.7, 3.4.8, 3.4.11 and 3.4.13, we have:

**Corollary 3.4.15.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra, and $F \subset B$ be a proper filter. The following are equivalent:*

*(a) $F$ is an ultrafilter.*

*(b) $F$ is a prime filter.*

*(c) $F$ is a maximal filter.*

*(d) $B \setminus F$ is a prime ideal.*

**Notation 3.4.16.** *Let $B$ be a Boolean algebra and $x \in B$, we put:*

$$U_x = \{F \in \mathcal{U}(B) : x \in F\}.$$

**Lemma 3.4.17.** *Let $(B, +, \cdot, \neg)$ be a Boolean algebra. Then:*

*(a) For all $x, y \in B$, $U_x \cap U_y = U_{x \cdot y}$.*

*(b) If $x_1, \ldots, x_n \in B$, then $\bigcap_{i=1}^{n} U_{x_i} = \emptyset$ if and only if $x_1 \cdot \ldots \cdot x_n = 0$.*

*(c) For all $x, y \in B$, $U_x \cup U_y = U_{x+y}$.*

*(d) For all $x \in B$, $\mathcal{U}(B) \setminus U_x = U_{\neg x}$.*

*(e) The family $\{U_x : x \in B\}$ is a basis for a topology $\mathcal{T}_{\mathcal{U}(B)}$ on $\mathcal{U}(B)$ in which $U_x$ is a clopen subset for all $x \in B$.*

*Proof.* $(a)$ Let $x, y \in B$, then $F \in U_x \cap U_y$ if and only if $x \cdot y \in F$.
$(b)$ By 3.4.3, 3.4.6 and 3.4.8, $U_x = \emptyset$ if and only if $x = 0$. Applying part $(a)$, $\bigcap_{i=1}^{n} U_{x_i} = \emptyset$ if and only if $x_1 \cdot \ldots \cdot x_n = 0$.
$(c)$ Let $x, y \in B$. If $F \in U_x \cup U_y$, then either $x \in F$ or $y \in F$; in either case, $x + y$ must be in $F$. Conversely, if $F \in U_{x+y}$, then $x + y \in F$ and hence either $x \in F$ or $y \in F$ as $F$ is prime.
$(d)$ Let $x \in B$, as $F$ is an ultrafilter then $x \notin F$ if and only if $\neg x \in F$.
$(e)$ The family $\{U_x : x \in B\}$ being basis follows from part $(a)$ and the fact that $U_1 = \mathcal{U}(B)$. Part $(c)$ shows that $U_x$ is a clopen subset for all $x \in B$. $\square$

**Theorem 3.4.18.** *Let $B$ be a Boolean algebra, the topological space $(\mathcal{U}(B), \mathcal{T}_{\mathcal{U}(B)})$ is a Stone's space.*

*Proof.* • $\mathcal{U}(B)$ is Hausdorff. Indeed, let $F_1, F_2 \in \mathcal{U}(B)$ be distinct ultrafilters and let $x \in F_1 \setminus F_2$, then $\neg x \in F_2$ and hence $F_1$ and $F_2$ are contained in the disjoint open sets $U_x$ and $U_{\neg x}$, respectively.

• $\mathcal{U}(B)$ is compact. To this end, it suffices to show that for arbitrary collection $\mathcal{C}$ of closed sets of $\mathcal{U}(B)$ that satisfies the finite intersection property, then $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$. Notice that for every $C \in \mathcal{C}$, there exists $A_C \subset B$ such that $C = \bigcap_{x\in A_C} U_x$. Put $A := \bigcup_{C\in\mathcal{C}} A_C$, and notice that as $\mathcal{C}$ satisfies the finite intersection property, the family $\{U_x : x \in A\}$ satisfies the finite intersection property and hence, by 3.4.17 part $(b)$, $A$ satisfies the finite product property. Let $F_A$ be an ultrafilter of $B$ containing $A$. Therefore, $F_A \in \bigcap_{x\in A} U_x = \bigcap_{C\in\mathcal{C}} C$.

• Since $(\mathcal{U}(B), \mathcal{T}_{\mathcal{U}(B)})$ is locally compact and it admits a basis consists of clopen subsets, by 3.3.6 it is totally disconnected. □

It is worth mentioning that if $G$ is a group that acts on $X$; the Stone's space of the Boolean algebra ${Comm_G(X)}/{\sim}$, given in 3.1.6, is called **the space of ends** of $X$ (see (7)).

**Notation 3.4.19.** *For $A \subset X$, the family of all principal filters $F_a$ such that $a \in A$ is denoted by $Pr(A)$; that is $Pr(A) := \{F_a \in \mathcal{U}(\mathcal{P}(X)) : a \in A\}$.*

**Proposition 3.4.20.** *Let $F \in \mathcal{U}(\mathcal{P}(X))$ and $A \in \mathcal{P}(X)$. $A \in F$ if and only $F \in cl(Pr(A))$ where the closure is taking in the topological space $\mathcal{U}(\mathcal{P}(X))$.*

*Proof.* Assume that $A \in F$, and let $U_B$ be a neighborhood of $F$. Since $A, B \in F$, $A \cap B \neq \emptyset$. Pick $a \in A \cap B$, then $F_a \in U_B \cap Pr(A)$.

Conversely, assume that $F \in cl(Pr(A))$. By 3.4.9, to show that $A \in F$, it suffices to show that for every $B \in F$, $A \cap B \neq \emptyset$. Indeed, if $B \in F$, then $F \in U_B$ and hence $U_B \cap Pr(A) \neq \emptyset$. Therefore, there exists $a \in A$ such that $F_a \in U_B$, and hence $a \in A \cap B$ as desired. □

**Corollary 3.4.21.** *Let $B$ be a Boolean algebra and $C$ be a clopen subset of $\mathcal{U}(B)$. There exists $x \in B$ such that $C = U_x$, where $U_x = \{F \in \mathcal{U}(B) : x \in F\}$.*

*Proof.* On one hand, $C$ is an open subset of $\mathcal{U}(B)$, so $C$ there exists a subset $A \subset B$ such that $C = \bigcup_{x\in A} U_x$. On the other hand, $C$ is a closed subset of the Hausdorff compact space $\mathcal{U}(B)$

and hence it is compact. In particular, there exist $x_1, \ldots, x_n \in A$ such that $C = \bigcup_{i=1}^{n} U_{x_i}$. Put $x = x_1 + \ldots + x_n$, then $C = U_x$ as desired. $\square$

**Theorem 3.4.22.** *Every homomorphism $\varphi : B_1 \to B_2$ between Boolean algebras induces a continuous map $\widetilde{\varphi} : \mathcal{U}(B_2) \to \mathcal{U}(B_1)$ between their associated Stone's spaces. If, moreover, $\varphi : B_1 \to B_2$ is an isomorphism, then its induced map $\widetilde{\varphi} : \mathcal{U}(B_2) \to \mathcal{U}(B_1)$ is a homeomorphism.*

*Proof.* First notice that if $F$ is an ultrafilter of $B_2$, then $\varphi^{-1}(F)$ is an ultrafilter of $B_1$. Consider the map $\widetilde{\varphi} : \mathcal{U}(B_2) \to \mathcal{U}(B_1)$ that sends $F \in \mathcal{U}(B_2)$ to $\varphi^{-1}(F)$. Now, if $U_x$ is a clopen subset of $\mathcal{U}(B_1)$, then $\widetilde{\varphi}^{-1}(U_x) = U_{\varphi(x)}$ as $F \in \widetilde{\varphi}^{-1}(U_x)$ if and only if $\widetilde{\varphi}(F) \in (U_x)$ if and only if $\varphi^{-1}(F) \in (U_x)$ if and only if $\varphi(x) \in F$, an hence $\widetilde{\varphi} : \mathcal{U}(B_2) \to \mathcal{U}(B_1)$ is continuous. It is straightforward to see that $\widetilde{\varphi} : \mathcal{U}(B_2) \to \mathcal{U}(B_2)$ is a homeomorphism when $\varphi : B_1 \to B_2$ is an isomorphism. $\square$

### 3.4.1 The Stone's representation theorems

Every Boolean algebra (up to isomorphism) is the clopen Boolean algebra associated with some Stone's space; and conversely, every Stone's space (up to homeomorphism) is the space of all ultrafilters of some Boolean algebra. These tow facts are the core of the Stone's representation theorems.

**Theorem 3.4.23.** ***(Stone's representation theorem for Boolean algebras)*** *Up to isomorphism, every Boolean algebra is the clopen Boolean algebra for some Stone's space. More specifically, if $B$ is a Boolean algebra, then it is isomorphic to the Boolean algebra $Clop(\mathcal{U}(B))$.*

*Proof.* Let $(B, +, \cdot, \neg)$ be a Boolean algebra; consider the map $\varphi : B \to Clop(\mathcal{U}(B))$ that sends each $x \in B$ to $U_x \in Clop(\mathcal{U}(B))$. It is straightforward to see that $\varphi : B \to Clop(\mathcal{U}(B))$ is a homomorphism. It is, moreover, surjective by 3.4.21. Now, let $x, y \in B$ be distinct elements, then either $x \nleq y$ or $y \nleq x$. Without loss of generality, assume that $x \nleq y$, then $y \notin F_x$, where $F_x$ is the principal filter generated by $x$. Let $M_x$ be a maximal filter containing $F_x$ such that $y \notin M_x$. Notice that $M_x \in U_x$ and $M_x \notin U_y$; hence $U_x \neq U_y$. Consequently, $\varphi : B \to Clop(\mathcal{U}(B))$ is an isomorphism as desired. $\square$

**Lemma 3.4.24.** *Let $X$ be a Stone's space. Then:*

*(a) For every $x \in X$, the family $F^x := \{U \in Clop(X) : x \in U\}$ is an ultrafilter of the clopen Boolean algebra $Clop(X)$.*

*(b) If $F$ is an ultrafilter of $Clop(X)$, then $\bigcap_{U \in F} U = \{x_F\}$ for some $x_F \in X$.*

*Proof.* $(a)$ Since $X$ is totally disconnected, $Clop(X)$ is a basis for $X$. Since $X$ is Hausdorff $F^x$ is a proper subset of $Clop(X)$. Evidently, $F^x$ is a filter; and if $V \in Clop(X) \setminus F^x$, then $U := X \setminus V$ is a clopen subset containing $x$ and hence it is contained in $F^x$. Thus, $F^x$ is an ultrafilter.

$(b)$ Let $F$ be an ultrafilter of $Clop(X)$. Since $F$ is a family of closed sets of the compact space $X$ and that it satisfies the finite intersection property; hence $\bigcap_{U \in F} U$ is non-empty. Seeking contradiction, let $x, y \in \bigcap_{U \in F} U$ be distinct elements. Let $U_x$ be a clopen set containing $x$ such that $y \notin U_x$. Notice that $U_y := X \setminus U_x$ is a clopen subset containing $y$. Since $F$ is an ultrafilter, either $U_x \in F$ or $U_y \in F$. Without loss of generality, assume that $U_x \in F$, then as $y \notin U_x$, $y \notin \bigcap_{U \in F} U$, a contradiction. □

**Theorem 3.4.25.** ***(Stone's representation theorem for Stone's spaces)*** *Up to homeomorphism, every Stone's space is the ultrafilter space of some Boolean algebra. More specifically, if $X$ is a Stone's space then it is homeomorphic to $\mathcal{U}(Clop(X))$.*

*Proof.* Let $X$ be a Stone's space. Exploiting 3.4.24$(a)$, we define the map $\psi : X \to \mathcal{U}(Clop(X))$ that sends each $x \in X$ to $F^x$. To show that $\psi : X \to \mathcal{U}(Clop(X))$ is homeomorphism, it suffices to show that it is continuous bijective map as $X$ and $\mathcal{U}(Clop(X))$ are compact Hausdorff spaces.

• $\psi$ is injective. Indeed, let $x, y \in X$ be distinct elements. Since $X$ is Hausdorff and $Clop(X)$ is a basis for $X$, there exists $U_x \in Clop(X)$ containing $x$ such that $y \notin U_x$. Hence, $U_x \in F^x \setminus F^y$.

• $\psi$ is surjective. Indeed, let $F$ be an ultrafilter of $Clop(X)$, then $\bigcap_{U \in F} U = \{x_F\}$, by 3.4.24$(b)$. Now, by 3.4.24$(a)$, $F^{x_F}$ is an ultrafilter and it is obvious that $F \subset F^{x_F}$. Since ultrafilters are maximal, $F = F^{x_F}$.

• $\psi$ is continuous. To this end, notice that a basis element of $\mathcal{U}(Clop(X))$ is of the form $U_C = \{F \in \mathcal{U}(Clop(X)) : C \in F\}$ for some a clopen subset $C$ of $X$. Now, $\psi^{-1}(U_C) = \{x \in X : F^x \in U_C\} = \{x \in X : C \in F^x\} = C$ and thus $\psi$ is continuous. □

**Corollary 3.4.26.** *(a) Let $B_1$ and $B_1$ be Boolean algebras; then $B_1$ and $B_1$ are isomorphic if and only if their Stone's spaces $\mathcal{U}(B_1)$ and $\mathcal{U}(B_2)$ are homeomorphic.*
*(b) Let $X$ and $Y$ be Stone's spaces; then $X$ and $Y$ are homeomorphic if and only if $Clop(X)$ and $Clop(Y)$ are isomorphic.*

*Proof.* $(a)$ Assume that the Stone's spaces $\mathcal{U}(B_1)$ and $\mathcal{U}(B_2)$ are homeomorphic. By 3.1.13, $Clop(\mathcal{U}(B_1))$ and $Clop(\mathcal{U}(B_2))$ are isomorphic. By 3.4.23, $B_1$ and $B_1$ are isomorphic. The other direction follows from 3.4.22.
$(b)$ If $X$ and $Y$ are homeomorphic, then by 3.1.13, $Clop(X)$ and $Clop(Y)$ are isomorphic. Conversely, assume that $Clop(X)$ and $Clop(Y)$ are isomorphic, then by 3.4.22, $\mathcal{U}(Clop(X))$ and $\mathcal{U}(Clop(Y))$ are homeomorphic. Therefore, by 3.4.25, $X$ and $Y$ are homeomorphic. □

## 3.5 Compactifications of a discrete space

In the case of discrete spaces, every compactification of a discrete space $X$ is determined by a sub-Boolean algebra $\mathcal{B} \subset \mathcal{P}(X)$ that separates points of $X$; and conversely, every sub-Boolean algebra $\mathcal{B} \subset \mathcal{P}(X)$ that separates points of $X$ determines a compactification of $X$. Every compactification of $X$ is totally disconnected. The domination relation between any two compactifications of $X$ is determined by the relation between their corresponding sub-Boolean algebras.

**Lemma 3.5.1.** *Let $X$ be a non-empty set and $\mathcal{B} \subset \mathcal{P}(X)$ be a sub-Boolean algebra. The family $\{F_x : x \in X\}$, where $F_x := \{A \in \mathcal{B} : x \in A\}$, is a dense discrete subspace of $\mathcal{U}(\mathcal{B})$.*

*Proof.* Notice that a basis element of $\mathcal{U}(\mathcal{B})$ is of the form $U_A = \{F \in \mathcal{U}(\mathcal{B}) : A \in F\}$, and that for $x \in X$, $F_x := \{A \in \mathcal{B} : x \in A\}$ is an ultrafilter of $\mathcal{B}$. Now, for any $a \in A$, $F_a \in U_A \cap \{F_x : x \in X\}$ and hence $\{F_x : x \in X\}$ is a dense subset of $\mathcal{U}(\mathcal{B})$. Moreover, for any $x \in X$, the singleton $\{F_x\}$ is an open set as $\{F_x\} = U_{\{x\}}$ is a singleton. Consequently, $\{F_x : x \in X\}$ is a discrete subspace of $\mathcal{U}(\mathcal{B})$. □

**Definition 3.5.2.** Let $X$ be a non-empty set and $\mathcal{B} \subset \mathcal{P}(X)$ be a subset. $\mathcal{B}$ is said to **separate points** of $X$ if for all $x, y \in X$ such that $x \neq y$, there exists $A \in \mathcal{B}$ such that $x \in A$ and $y \in X \setminus A$.

**Theorem 3.5.3.** *Let $X$ be a non-empty set and $\mathcal{B} \subset \mathcal{P}(X)$ be a sub-Boolean algebra. If $\mathcal{B}$ separates points of $X$, then $\mathcal{U}(\mathcal{B})$ is a compactification of $X$.*

*Proof.* Define $\alpha : X \to \mathcal{U}(\mathcal{B})$ that sends each $x \in X$ to $F_x \in \mathcal{U}(\mathcal{B})$.

• $\alpha : X \to \mathcal{U}(\mathcal{B})$ is continuous as $X$ is discrete.

• It is injective since $\mathcal{B}$ separates points of $X$. Indeed, if $x, y \in X$ are distinct, then there exists $A \in \mathcal{B}$ such that $x \in A$ and $y \in X \setminus A$. Because $A \in F_x$ and $X \setminus A \in F_y$, $\alpha(x) \neq \alpha(y)$.

• $\alpha : X \to \mathcal{U}(\mathcal{B})$ is open as for all $x \in X$, $\alpha(\{x\}) = \{F_x\} = U_{\{x\}}$ by 3.5.1.

• $\alpha(X) = \{F_x : x \in X\}$ which is a dense subset of $\mathcal{U}(\mathcal{B})$. Therefore, $\mathcal{U}(\mathcal{B})$ is a compactification of $X$. □

**Example 3.5.4.** *Let $G$ be a group acts on $X$; then $Comm_G(X)$ is a sub-Boolean algebra of $\mathcal{P}(X)$ that separates points of $X$. The compactification $\mathcal{U}(Comm_G(X))$ of $X$ is called the **end compactification**.*

**Example 3.5.5.** *Let $X$ be a non-empty set, the family $\mathcal{B}_0 := \{A \in \mathcal{P}(X) : A$ is finite or $X \setminus A$ is finite$\}$ is a sub-Boolean algebra of $\mathcal{P}(X)$ that separates points of $X$. We will prove that the compactification $\mathcal{U}(\mathcal{B}_0)$ is the one-point compactification of $X$ (7).*

### 3.5.1 The one-point compactification of a discrete space

Let $Y := X \cup \{\infty\}$ be the one-point compactification of the discrete space $X$. It can be easily seen that a subset $A \subset Y$ is clopen if and only if either $A$ is a finite subset of $X$ or $Y \setminus A$ is finite. We aim to show that the one-point compactification of $X$ is the Stone's space of the Boolean algebra $\mathcal{B}_0 := \{A \in \mathcal{P}(X) : A$ is finite or $X \setminus A$ is finite$\}$.

**Theorem 3.5.6.** *Given a non-empty set $X$, the family $\mathcal{B}_0 := \{A \in \mathcal{P}(X) : A$ is finite or $X \setminus A$ is finite$\}$ is a sub-Boolean algebra of $\mathcal{P}(X)$ that separates points of $X$. Moreover, as a discrete space $\mathcal{U}(\mathcal{B}_0)$ is the one-point compactification of $X$.*

*Proof.* The first part is straightforward. Let $Y := X \cup \{\infty\}$ be the one-point compactification of $X$. As noticed a subset $A \subset Y$ is clopen if and only if either $A$ is a finite subset of $X$ or $Y \backslash A$ is finite. In particular, if $A \in Clop(Y)$, then $A \backslash \{\infty\} \in \mathcal{B}_0$. Define the map $\varphi : Clop(Y) \to \mathcal{B}_0$ that sends $A \in Clop(Y)$ to $A \backslash \{\infty\}$. It is straightforward to see that $\varphi : Clop(Y) \to \mathcal{B}_0$ is an isomorphism and $Y$ is a Stone's space. Hence, by 3.4.26, $Y = \mathcal{U}(Clop(Y))$ is homeomorphic to the Stone's space $\mathcal{U}(\mathcal{B}_0)$. $\square$

### 3.5.2 The Stone-$\breve{C}$ech compactification of a discrete space

The compactification $Y$ is called the **Stone-$\breve{C}$ech compactification** if for any compact Hausdorff space $K$, every continuous map $f : \alpha(X) \to K$ admits a continuous extension $\widetilde{f} : Y \to K$. For the discrete space $X$, the topological space $\mathcal{U}(\mathcal{P}(X))$ is a compactification of $X$; we aim to show that $\mathcal{U}(\mathcal{P}(X))$ is the Stone-$\breve{C}$ech compactification of $X$. We first give a characterization for the continuity of a map between topological spaces in terms of convergence of filters.

**Definition 3.5.7.** Let $X$ be a topological space, and $F$ be a filter of $\mathcal{P}(X)$. We say that $F$ **converges** to $x \in X$ if for every neighborhood $U$ of $X$, there exists $A \in F$ such that $A \subset U$.

**Example 3.5.8.** *Let $X$ be a topological space, and $x \in X$. The family $\mathcal{N}(x)$ of all neighborhoods of $x$ is a filter called the **neighborhood filter** of $x$. Clearly, $\mathcal{N}(x)$ converges to $x$. Moreover, a filter $F$ converges to $x$ if and only if $\mathcal{N}(x) \subset F$.*

A function $f : X \to Y$ between sets maps filters (ultrafilters) of $\mathcal{P}(X)$ to filters (ultrafilters) of $\mathcal{P}(Y)$.

**Lemma 3.5.9.** *Let $f : X \to Y$ be a function. Then:*
*(a) If $F$ is a filter of $\mathcal{P}(X)$, then $F_f := \{f(A) : A \in F\}$ is a filter of $\mathcal{P}(Y)$.*
*(b) If $F$ is an ultrafilter of $\mathcal{P}(X)$, then $F_f := \{f(A) : A \in F\}$ is an ultrafilter of $\mathcal{P}(Y)$.*

*Proof.* $(a)$ Let $F$ be a filter of $\mathcal{P}(X)$; observe that for any $B \in \mathcal{P}(Y)$, $f(f^{-1}(B)) = B$. Now:
- Let $B_1, B_2 \in F_f$, then there exist $A_1, A_2 \in F$ such that $f(A_1) = B_1$ and $f(A_2) = B_2$. Since $A_1 \cap A_2 \in F$ and $A_1 \cap A_2 \subset f^{-1}(B_1 \cap B_2)$ and hence $f^{-1}(B_1 \cap B_2) \in F$. Therefore, $f(f^{-1}(B_1 \cap B_2)) = B_1 \cap B_2 \in F_f$.

• $B \in F_f$ and $C \subset Y$ such that $B \subset C$. Let $A \in F$ such that $f(A) = B$; since $A \subset f^{-1}(C)$, then $f^{-1}(C) \in F$ and hence $f(f^{-1}(C)) = C \in F_f$.

$(b)$ Let $F$ be an ultrafilter of $\mathcal{P}(X)$, then $F_f$ is a filter of $\mathcal{P}(Y)$. Moreover, if $B \subset Y$, then either $f^{-1}(B) \in F$ or $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B) \in F$ and therefore either $B = f(f^{-1}(B)) \in F_f$ or $Y \setminus B = f(f^{-1}(Y \setminus B)) \in F_f$. □

**Lemma 3.5.10.** *A map $f : X \to Y$ between topological spaces is continuous if and only if for every $x \in X$ and every filter $F$ of $\mathcal{P}(X)$ that converges to $x$, the filter $F_f$ converges to $f(x)$.*

*Proof.* Assume that $f : X \to Y$ is continuous, and $F$ is a filter of $\mathcal{P}(X)$ that converges to $x$. Let $V$ be a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of $x$ and consequently $f^{-1}(V) \in F$. Hence, $V = f(f^{-1}(V)) \in F_f$ which implies that $F_f$ converges to $f(x)$.

Conversely, assume that for every $x \in X$ and every filter $F$ of $\mathcal{P}(X)$ that converges to $x$, the filter $F_f$ converges to $f(x)$. Since the neighborhood filter $F = \mathcal{N}(x)$ of $x$ converges to $x$, the filter $F_f$ converges to $f(x)$. Let $V \subset Y$ be a neighborhood of $f(x)$, then $V \in F_f$ and therefore there exists $U \in F$ such that $f(U) = V$ which proves that $f : X \to Y$ is continuous. □

**Theorem 3.5.11.** *Let $X$ be a non-empty set equipped with the discrete topology. The compactification $\mathcal{U}(\mathcal{P}(X))$ of $X$ is the Stone-Čech compactification.*

*Proof.* By 3.5.3, $\mathcal{U}(\mathcal{P}(X))$ is a compactification of $X$. Let $K$ be a compact Hausdorff space, and $f : \alpha(X) \to K$ be a continuous map, where $\alpha : X \to \mathcal{U}(\mathcal{P}(X))$ is the embedding defined in the proof of 3.5.3.

**Claim 1:** For every $F \in \mathcal{U}(\mathcal{P}(X))$, $\bigcap\limits_{A \in F} cl(f(\alpha(A)))$ is a singleton.

**Proof of claim 1:** Put $g := f \circ \alpha$. Notice that the family $F$ satisfies the finite intersection property and hence the family $\{g(A) : A \in F\}$ satisfies the finite intersection property. Therefore, the family of the closed sets $\{cl(g(A)) : A \in F\}$ satisfies the finite intersection property. Consequently, $\bigcap\limits_{A \in F} cl(g(A)) \neq \emptyset$, as $K$ is compact. Since $F$ is an ultrafilter, for every $O \subset X$, either $O \in F$ or $X \setminus O \in F$; in particular, if $O_1, O_2 \subset X$ are disjoint subsets of $X$, then there exists $A \in F$ such that either $A \cap O_1 = \emptyset$ and $A \cap O_2 \neq \emptyset$ or $A \cap O_1 \neq \emptyset$ and $A \cap O_2 = \emptyset$. Seeking contradiction assume that $a, b \in \bigcap\limits_{A \in F} cl(g(A))$ are distinct, and choose two disjoint neighborhoods $U_a$ and $U_b$ of $a$ and $b$, respectively. Since $g^{-1}(U_a)$ and $g^{-1}(U_b)$ are disjoint, we may choose, without loss of generality, $A \in F$ such that $A \cap g^{-1}(U_a) \neq \emptyset$ and $A \cap g^{-1}(U_b) = \emptyset$. Hence, $g(A) \cap U_a \neq \emptyset$ and $g(A) \cap U_b = \emptyset$, and thus $b \notin cl(g(A))$, a contradiction. Therefore, there exists $a_F \in K$ such that $\bigcap_{A \in F} cl(g(A)) = \{a_F\}$.

**Claim 2:** Define $\widetilde{f} : \mathcal{U}(\mathcal{P}(X)) \to K$ by $\widetilde{f}(F) = a_F$. Then $\widetilde{f}$ is an extension of $f$.

**Proof of claim 2:** For $x \in X$, then $\widetilde{f}(F_x) = a_{F_x}$. On the other hand, $F_x \in \alpha(A)$ for all $A \in F_x$ and hence $f(F_x) \in \bigcap_{A \in F} f(\alpha(A))) \subset \bigcap_{A \in F} cl(f(\alpha(A))) = \{a_{F_x}\}$.

**Claim 3:** $cl(\widetilde{f}(U_A)) = cl(g(A))$ for all $A \in \mathcal{P}(X)$; where:

$$U_A = \{F \in \mathcal{U}(\mathcal{P}(X)) : A \in F\}.$$

**Proof of claim 3:** Let $a \in \widetilde{f}(U_A)$, then there exists $F \in U_A$ such that $a = a_F$ where $\{a_F\} = \bigcap_{A \in F} cl(g(A)) \subset cl(g(A))$, and hence $cl(\widetilde{f}(U_A)) \subset cl(g(A))$. On the other hand, since $F_a \in U_A$ for all $a \in A$, thus $g(a) = f(F_a) = \widetilde{f}(F_a) \in \widetilde{f}(U_A)$. Therefore, $cl(g(A)) \subset cl(\widetilde{f}(U_A))$.

**Claim 4:** $\widetilde{f} : \mathcal{U}(\mathcal{P}(X)) \to K$ is continuous.

**Proof of claim 4:** By 3.5.10, it suffices to show that if $\mathcal{F}$ is a filter of $\mathcal{P}(\mathcal{U}(\mathcal{P}(X)))$ that converges to $F \in \mathcal{U}(\mathcal{P}(X))$, then $\mathcal{F}_{\widetilde{f}}$ converges to $\widetilde{f}(F)$. Notice that for all $A \in F$, $F \in U_A$, and since $\mathcal{F}$ converges to $F$ and $U_A$ is a neighborhood of $F$, $U_A$ must be in $\mathcal{F}$. Consequently, $\{\widetilde{f}(U_A) : A \in F\} \subset \mathcal{F}_{\widetilde{f}}$. Since $\widetilde{f}(U_A) \subset cl(\widetilde{f}(U_A)) = cl(g(A))$ and $\mathcal{F}_{\widetilde{f}}$ is a filter, so $\{cl(g(A)) : A \in F\} \subset \mathcal{F}_{\widetilde{f}}$. Now, let $O$ be a neighborhood of $\widetilde{f}(F) = a_F$. Choose a relatively compact neighborhood $V$ of $a_F \in \bigcap_{A \in F} cl(g(A))$ such that $V \subset cl(V) \subset O$. As $V \cap g(A) \neq \emptyset$ for all $A \in F$ and that $F_g$ is an ultrafilter, hence, by 3.4.9, $V \in F_g$ and therefore there exists $B \in F$ such that $V = g(B)$. Thus, $cl(V) \in \mathcal{F}_{\widetilde{f}}$ and $cl(V) \subset O$ which implies that $\mathcal{F}_{\widetilde{f}}$ converges to $\widetilde{f}(F)$. As a result, $\mathcal{U}(\mathcal{P}(X))$ is the Stone-Čech compactification of $X$. □

**Corollary 3.5.12.** *Let $X$ be a non-empty set equipped with the discrete topology, and $Y$ be a compactification of $X$, then $Y$ is totally disconnected.*

*Proof.* Let $\beta X$ be the Stone-Čech compactification of $X$, then $Y$ is a quotient space of the totally disconnected space $\beta X$, and therefore $Y$ is totally disconnected. □

When $X$ is infinite, its Stone-$\breve{C}$ech compactification is non-metrizable.

**Theorem 3.5.13.** *Let $X$ be an infinite discrete space. The Stone-$\breve{C}$ech compactification $\beta X$ is non-metrizable.*

*Proof.* Seeking contradiction, assume that $\beta X$ is metrizable. Choose $\omega \in \beta X \setminus X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct elements of $X$ that converges to $\omega$. Let $A := \{x_{2n} : n \in \mathbb{N}\}$, and $\chi_A : X \to \{0,1\}$ be the characteristic function of $A$. Since $\chi_A$ is continuous, it admits a continuous extension $\widetilde{\chi_A} : \beta X \to \{0,1\}$. On one hand, we have $\widetilde{\chi_A}(cl_{\beta X} A) = \{1\}$, and hence $\widetilde{\chi_A}(\omega) = 1$. On the other hand, we have $\widetilde{\chi_A}(x_{2n-1}) = 0$ for all $n \in \mathbb{N}$ and $(x_{2n-1})_{n\in\mathbb{N}}$ converges to $\omega$, a contradiction. □

Now we prove the converse of 3.5.3.

**Theorem 3.5.14.** *Let $Y$ be a compactification of the discrete space $X$; there exists a sub-Boolean algebra $\mathcal{B} \subset \mathcal{P}(X)$ that separates points of $X$ such that $Y$ is homeomorphic to $\mathcal{U}(\mathcal{B})$.*

*Proof.* Without loss of generality we may assume that $X \subset Y$. Put $\mathcal{B} := \{A \cap X : A \in Clop(Y)\}$. It is straightforward to see that $\mathcal{B} \subset \mathcal{P}(X)$ is a sub-Boolean algebra that separates points of $X$. Therefore, $\mathcal{U}(\mathcal{B})$ is a compactification of $X$. Consider the a map $\varphi : Clop(Y) \to \mathcal{B}$ that sends every $A \in Clop(Y)$ to $A \cap X$. It is evident to see that $\varphi : Clop(Y) \to \mathcal{B}$ is a surjective homomorphism. Furthermore, let $A, B \in Clop(Y)$ be distinct. Without loss of generality assume that $A \setminus B$ is non-empty. Since $A \setminus B$ is an open subset of $Y$ and hence $(A \setminus B) \cap X \neq \emptyset$. Thus, $A \cap X \neq B \cap X$, which shows that $\varphi : Clop(Y) \to \mathcal{B}$ is injective. By 3.4.26, $Y$ is homeomorphic to $\mathcal{U}(\mathcal{B})$. □

**Corollary 3.5.15.** *Let $X$ be a non-empty set equipped with the discrete topology. A topological space $Y$ is a compactification of $X$ if and only if there exists a sub-Boolean algebra $\mathcal{B} \subset \mathcal{P}(X)$ that separates points of $X$ such that $Y$ is homeomorphic to $\mathcal{U}(\mathcal{B})$.*

*Proof.* Apply 3.5.3 and 3.5.14. □

**Theorem 3.5.16.** *Let $Y := \mathcal{U}(\mathcal{B}_Y)$ and $Z := \mathcal{U}(\mathcal{B}_Z)$ be compactifications of the discrete space $X$, where $\mathcal{B}_Y$ and $\mathcal{B}_Z$ sub-Boolean algebras of $\mathcal{P}(X)$ that separate points of $X$. $Y$ dominates $Z$ if and only if $\mathcal{B}_Z$ is isomorphic to a sub-Boolean algebra of $\mathcal{B}_Y$.*

*Proof.* By 3.1.13, a continuous function $f : \mathcal{U}(\mathcal{B}_Y) \to \mathcal{U}(\mathcal{B}_Z)$ induces a homomorphism $\widetilde{f} : Clop(\mathcal{U}(\mathcal{B}_Z)) \to Clop(\mathcal{U}(\mathcal{B}_Y))$ given by $\widetilde{f}(A) = f^{-1}(A)$ for all $A \in Clop(\mathcal{U}(\mathcal{B}_Z))$. It can be easily seen that if $f : \mathcal{U}(\mathcal{B}_Y) \to \mathcal{U}(\mathcal{B}_Z)$ is surjective, then $\widetilde{f} : Clop(\mathcal{U}(\mathcal{B}_Z)) \to Clop(\mathcal{U}(\mathcal{B}_Y))$ is injective. On the other hand, we may assume that $\mathcal{B}_Z \subset \mathcal{B}_Y$. Now the inclusion $\varphi : \mathcal{B}_Z \to \mathcal{B}_Y$ is a homomorphism. By 3.4.22, it induces a continuous map $\widetilde{\varphi} : \mathcal{U}(\mathcal{B}_Y) \to \mathcal{U}(\mathcal{B}_Z)$ that sends $F \in \mathcal{U}(\mathcal{B}_Y)$ to $\varphi^{-1}(F) = F \cap \mathcal{B}_Z$. Notice that every ultrafilter $F_Z$ of $\mathcal{B}_Z$ is contained in an ultrafilter $F_Y$ of $\mathcal{B}_Y$ with $F_Z = F_Y \cap \mathcal{B}_Z$ which shows that $\widetilde{\varphi} : \mathcal{U}(\mathcal{B}_Y) \to \mathcal{U}(\mathcal{B}_Z)$ is surjective. Moreover, since $X$ is homeomorphic to $\{F_x^Y : x \in X\}$ and homeomorphic to $\{F_x^Z : x \in X\}$ where $F_x^Y = \{A \in \mathcal{B}_Y : x \in A\}$ and $F_x^Z = \{A \in \mathcal{B}_Z : x \in A\}$, so $\widetilde{\varphi} : \mathcal{U}(\mathcal{B}_Y) \to \mathcal{U}(\mathcal{B}_Z)$ is a domination. $\square$

# Chapter 4

# Freudenthal compactification and ends of finitely generated groups

This chapter is meant to study the construction of the Freudenthal compactification and its corona for Freudenthal spaces in general and for finitely generated groups in particular, and their basic properties that motivate our work. Although, this is an expository chapter, we observed few things that may consider original. Namely, as an adaptation of the construction of the explicit metric of the space of ends of a locally finite graph given in (26), we define a metric on the space of ends of a Freudenthal space that induces its topology, and prove that if the space of ends is infinite then it is indeed homeomorphic to a Cantor space. Another observation that does not seem to appear in the literature is that the Freudenthal ends of a proper geodesic metric space are in one-to-one correspondence with the components of its Higson corona. Its proof is an adaptation of the fact that the Freudenthal ends are in one-one correspondence with the components of the Stone $\check{C}$ech corona of $X$ given originally in (18).

We conclude by adopting Specker's approach of defining the space of ends of groups to define the space of ends of any metric space. In case if the metric space is proper and geodesic, we prove that the space of ends using the Specker's approach coincide with its Freudenthal corona. In the first three sections, we follow (18), whereas (11) and (3) cover the basic facts of section 4 and section 5 of this chapter.

## 4.1 Internal description of the Freudenthal Compactification

**Definition 4.1.1.** Let $X$ be a Hausdorff topological space:

• $X$ is **$\sigma$-compact** if it is a countable union of compact subsets of $X$.

• $X$ is **locally connected** (**locally path-connected**) if it admits a bases of connected (path-connected) subsets. Equivalently, every component (path component) of an open subset of $X$ is itself open.

• $X$ is a **Freudenthal space** if it is a $\sigma$-compact, locally compact, connected and locally connected space. A proper geodesic metric is an important example of a Freudenthal space.

• $X$ admits an **exhaustion by compact sets** if there exists a sequence of compact subsets $(K_n)_{n\in\mathbb{N}}$ that covers $X$ such that $K_n \subset int(K_{n+1})$ for all $n \in \mathbb{N}$. The sequence of compact subsets $(K_n)_{n\in\mathbb{N}}$ is called an **exhausting sequence**.

• A subset $Y \subset X$ is called **bounded** if it is relatively compact. Otherwise, $Y$ is called **unbounded**. The collection of all components of $Y$ is denoted by $\mathcal{C}(Y)$. The collection of all unbounded components of $Y$ is denoted by $\mathcal{UC}(Y)$.

**Proposition 4.1.2.** *If $X$ is a $\sigma$-compact and locally compact space; then $X$ admits an exhaustion by compact sets.*

*Proof.* Let $X = \bigcup\limits_{n\in\mathbb{N}} K_n$, where each $K_n$ is a compact subset of $X$. Inductively, we construct a sequence $(U_n)_{n\geq 0}$ of open relatively compact subsets such that $cl(U_n) \subset U_{n+1}$ for all $n \geq 0$. Put $D_0 = U_0 = \emptyset$ and assume that $U_1, \ldots, U_n$ are open relatively compact subsets such that $cl(U_i) \subset U_{i+1}$ for all $1 \leq i \leq n-1$. Consider the compact subset $K'_n := cl(U_n) \cup K_{n-1}$. Since $X$ is locally compact, $K'_n$ can be covered by finitely many open relatively compact subsets $V^n_{n_1}, \ldots, V^n_{n_m}$. Set $U_{n+1} := \bigcup\limits_{i=1}^{m} V^n_{n_i}$; $U_{n+1}$ is an open relatively compact subset containing $cl(U_n)$. The sequence $(L_n)_{n\in\mathbb{N}}$, where $L_n := cl(U_n)$, is an exhausting sequence. $\square$

**Lemma 4.1.3.** *Let $C$ be a connected subset of the topological space $X$ and $U$ be an open subset of $X$ such that $C \cap U \neq \emptyset$ and $C \cap U^c \neq \emptyset$; then $C \cap \partial U \neq \emptyset$.*

*Proof.* Seeking contradiction assume that $C \cap \partial U = \emptyset$ and set $V := X \setminus cl(U)$. Then, $V_C = V \cap C$ and $U_C = U \cap C$ are non-empty, open and disjoint subsets of $C$ that cover $C$, a contradiction. $\square$

The following lemma is a key player in the theory of ends. It says that if removing a bounded subset $K$ of a nice connected topological space $X$ disconnects $X \setminus K$, then we would have at most finitely many unbounded components of $X \setminus K$, and that the union of all bounded components of $X \setminus K$ is relatively small, in the sense that it is compact, and a fortiori, bounded.

**Lemma 4.1.4.** *Let $X$ be connected, locally connected and locally compact. If $K \subset X$ is a compact subset contained in an open subset $U \subset X$. Then:*
*(a)* $|\{C \in \mathcal{C}(X \setminus K) : C \nsubseteq U\}| < \infty$.
*(b)* $|\mathcal{UC}(X \setminus K)| < \infty$. *If moreover $X$ is unbounded, then* $1 \leq |\mathcal{UC}(X \setminus K)|$.
*(c) The subset* $X \setminus (\bigcup\limits_{C \in \mathcal{UC}(X \setminus K)} C)$ *is compact.*

*Proof.* $(a)$ Let $\mathcal{A} = \{C \in \mathcal{C}(X \setminus K) : C \nsubseteq U\}$. Choose an open relatively compact neighborhood $V$ of $K$ such that $K \subset V \subset cl(V) \subset U$. Set $\mathcal{B} = \{C \in \mathcal{C}(X \setminus K) : C \nsubseteq V\}$ and notice that $|\mathcal{A}| \leq |\mathcal{B}|$. The desired result can be obtained by showing that $\mathcal{B}$ is finite. To this end, notice that every component $C$ of $X \setminus K$ is an open subset of $X$. Moreover, since $X$ is connected, $C$ cannot be closed subset of $X$; in particular, if $cl(C) \subset X \setminus K$, then $C = cl(C)$, a contradiction. Therefore, for all $C \in \mathcal{C}(X \setminus K)$, $cl(C) \cap K \neq \emptyset$. If $C \cap V = \emptyset$, then $C$ would be contained in the closed subset $X \setminus V$, and hence $cl(C) \subset X \setminus V \subset X \setminus K$, a contradiction. For arbitrary component $C \in \mathcal{B}$, one must have $C \cap V \neq \emptyset$ and $C \cap V^c \neq \emptyset$ and hence by 4.1.3, $C \cap \partial V \neq \emptyset$. The subset $\partial V \subset cl(V)$ is a compact subset contained in $\bigcup\limits_{C \in \mathcal{B}} C$, so there exist $C_1, \ldots, C_n \in \mathcal{B}$ that cover $\partial V$ which implies that $|\mathcal{B}| \leq n$.

$(b)$ Without loss of generality, we may assume that $U$ is an open relatively compact neighborhood of $K$. Notice that $|\mathcal{UC}(X \setminus K)| \leq |\mathcal{UC}(X \setminus U)|$. by part $(a)$, $|\mathcal{UC}(X \setminus U)| \leq |\{C \in \mathcal{C}(X \setminus K) : C \nsubseteq U\}| < \infty$. Finally, let $C_1, \ldots, C_n$ be the all components of $X \setminus K$ that are not contained in $U$, then $X \setminus U \subset \bigcup\limits_{i=1}^{n} C_i$. In particular, since $X \setminus U$ is unbounded, at least one of the components $C_1, \ldots, C_n$ is unbounded and thus $1 \leq |\mathcal{UC}(X \setminus K)|$.

$(c)$ Assume that $U$ is a relatively compact neighborhood of $K$. By part $(a)$, there are finitely many bounded components $C_1, \ldots, C_n$ of $X \setminus K$ not contained in $U$. Put $L := U \cup (\bigcup\limits_{i=1}^{n} C_i)$. Notice that $X \setminus (\bigcup\limits_{C \in \mathcal{UC}(X \setminus K)} C)$ is a closed subset of $cl(L)$ and hence compact as wanted. $\square$

Unless otherwise stated, we restrict ourselves to unbounded Freudenthal spaces. Given a Freudenthal space $X$, it admits an exhaustion by compact sets $(K_n)_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$, there exists at least one unbounded component $C_n$ of $X \setminus K_n$. An end of $X$ is defined to be a decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of unbounded subsets, where each $C_n$ is unbounded component of $X \setminus K_n$. The following lemma asserts the existence of ends of $X$.

**Lemma 4.1.5.** *Let $X$ be an unbounded Freudenthal space and $(K_n)_{n\in\mathbb{N}\cup\{0\}}$ be an exhausting sequence, with $K_0 = \emptyset$. Every unbounded component $C_n$ of $X \setminus K_n$ contains an unbounded component $C_{n+1}$ of $X \setminus K_{n+1}$ and is contained in exactly one unbounded component $C_{n-1}$ of $X \setminus K_{n-1}$.*

*Proof.* Let $C_n$ be an unbounded component of $X \setminus K_n$.

• Notice that $C_n \cap X \setminus K_{n+1} = C_n \setminus K_{n+1} = \bigcup\limits_{D\in\mathcal{A}} C$, where $\mathcal{A} \subset \mathcal{C}(X \setminus K_{n+1})$. By 4.1.4 $(a)$, $int(K_{n+2})$ contains all but finitely many components of $X \setminus K_{n+1}$. In particular, there exist $D_1, \ldots, D_m \in \mathcal{A}$ such that $C_n \setminus K_{n+1} \subset int(K_{n+2}) \cup (\bigcup\limits_{i=1}^{m} D_i)$. Since $C_n \setminus K_{n+1}$ is unbounded, at least one of the $D_i$'s, say $C_{n+1}$, must be unbounded.

• $C_n$ is an unbounded connected subset of $X \setminus K_n$, hence it contained in an unbounded component $C_{n-1}$ of $X \setminus K_{n-1}$. □

**Definition 4.1.6.** Let $X$ be an unbounded Freudenthal space and $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. An **end** of $X$ is a decreasing sequence $(C_n)_{n\in\mathbb{N}}$ such that each $C_n$ is an unbounded component of $X \setminus K_n$. Such a sequence exists by 4.1.4 and 4.1.5. The **set of all ends** of $X$ is denoted by $ends_{\mathcal{K}}(X)$.

We aim to equip the set $X \cup ends_{\mathcal{K}}(X)$ with a Hausdorff topology that extends the topology on $X$ such that $X \cup ends_{\mathcal{K}}(X)$ as a topological space is a compactification of $X$. To this end, we introduce the following notations:

**Notation 4.1.7.** *Given a Freudenthal space $(X, \mathcal{T})$. If $U \in \mathcal{T}$, we define:*

• $U_{end} := \{(C_n) \in ends_{\mathcal{K}}(X) : C_n \subset U \text{ for some } n \in \mathbb{N}\}$,

• $\widetilde{U}_{end} := U_{end} \cup U$.

**Theorem 4.1.8.** *Let $(X, \mathcal{T})$ be a Freudenthal space and $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. The family $\widetilde{B}_{end} := \{\widetilde{U}_{end} : U \in \mathcal{T}\}$ is a basis for a Hausdorff topology $\mathcal{T}_{end}$ on $X \cup ends_{\mathcal{K}}(X)$ that extends $\mathcal{T}$. Moreover, $(X, \mathcal{T})$ is an open dense subset $(X \cup ends_{\mathcal{K}}(X), \mathcal{T}_{end})$.*

*Proof.* • $\widetilde{B}_{end}$ is a basis for a topology on $X \cup ends_{\mathcal{K}}(X)$ as $\widetilde{X}_{end} = X \cup ends_{\mathcal{K}}(X)$ and for all $U, V \in \mathcal{T}$, $\widetilde{U}_{end} \cap \widetilde{V}_{end} = (\widetilde{U \cap V})_{end}$. Let $\mathcal{T}_{end}$ be the topology on $X \cup ends_{\mathcal{K}}(X)$ induced by $\widetilde{B}_{end}$. Notice that for any relatively compact subset $U \in \mathcal{T}$, then $\widetilde{U}_{end} = U \in \mathcal{T}_{end}$ and as $(X, \mathcal{T})$ is locally compact, $\mathcal{T} \subset \mathcal{T}_{end}$. Clearly, $(X, \mathcal{T})$ is an open dense subspace of $(X \cup ends_{\mathcal{K}}(X), \mathcal{T}_{end})$.

• $(X \cup ends_{\mathcal{K}}(X), \mathcal{T}_{end})$ is a Hausdorff space. To this end, we consider two cases:

**Case 1:** Let $(C_n)_{n\in\mathbb{N}}, (D_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$ be distinct ends of $X$, then there exists $n \in \mathbb{N}$ such that $C_n \cap D_n = \emptyset$. Take the open neighborhoods $\widetilde{U}_{end}$ and $\widetilde{V}_{end}$ of $(C_n)_{n\in\mathbb{N}}$ and $(D_n)_{n\in\mathbb{N}}$, respectively, where $U = C_n$ and $V = D_n$. Obviously, $\widetilde{U}_{end} \cap \widetilde{V}_{end} = \emptyset$.

**Case 2:** Let $x \in X$ and $(C_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$. Take $m \in \mathbb{N}$ such that $x \in K_m$. Set $U := int(K_{m+1})$ and $V := C_{m+1}$. The open subsets $\widetilde{U}_{end}$ and $\widetilde{V}_{end}$ are disjoint neighborhoods of $x \in X$ and $(C_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$, respectively. □

**Corollary 4.1.9.** *Let $(X, \mathcal{T})$ be a Freudenthal space and $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. The subspace $ends_{\mathcal{K}}(X)$ of $X \cup ends_{\mathcal{K}}(X)$ is second countable.*

*Proof.* It is obvious that $\mathcal{B}_{end} := \{U_{end} : U \in \mathcal{T}\}$ is a basis for the topology of $ends_{\mathcal{K}}(X)$ as a subspace of $X \cup ends_{\mathcal{K}}(X)$. The family $\{U_{end} : U \in \bigcup_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)\}$ is a countable basis for $ends_{\mathcal{K}}(X)$. □

So far we have seen that $(X, \mathcal{T})$ is an open dense subspace of the Hausdorff space $(X \cup ends_{\mathcal{K}}(X), \mathcal{T}_{end})$. It remains to show that $(X \cup ends_{\mathcal{K}}(X), \mathcal{T}_{end})$ is compact. Before doing so, we show first that the subspace $ends_{\mathcal{K}}(X)$ is compact.

**Theorem 4.1.10.** *Let $(X, \mathcal{T})$ be an unbounded Freudenthal space and $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. The space of ends $ends_{\mathcal{K}}(X)$ is compact.*

*Proof.* Notice that for all $n \in \mathbb{N}$, the finite set $\mathcal{UC}(X \setminus K_n)$ equipped with the discrete topology is compact, and hence $\prod_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)$ is compact.

**Claim:** The map $\varphi : ends_{\mathcal{K}}(X) \to \prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)$ defined by:
$\varphi((C_n)_{n\in\mathbb{N}}) = (C_1, C_2, \ldots)$ is an embedding and $\varphi(ends_{\mathcal{K}}(X)) \subset \prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)$ is a closed subset of the compact space $\prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)$.
**Proof of the claim:** Notice that $\varphi : ends_{\mathcal{K}}(X) \to \prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)$ is injective and the basis of each $\mathcal{UC}(X \setminus K_n)$ consists of singletons.
• $\varphi : ends_{\mathcal{K}}(X) \to \prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)$ is continuous. Indeed, let $\prod\limits_{n\in\mathbb{N}} V_n$ be a basis element of $\prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X\setminus K_n)$. Then, $V_n = \mathcal{UC}(X\setminus K_n)$ for all but finitely many natural numbers $n_1, \ldots, n_r$, $V_{n_i} = \{C_{n_i}\}$ where $C_{n_i} \in \mathcal{UC}(X \setminus K_{n_i})$ for some $1 \le i \le r$. Put $U^{n_i} := C_{n_i}$ for all $1 \le i \le r$, then $\varphi^{-1}(\prod\limits_{n\in\mathbb{N}} V_n) = \bigcap\limits_{i=1}^{r} U^{n_i}_{end}$.
• $\varphi : ends_{\mathcal{K}}(X) \to \prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)$ is an open map. To that end, pick a basis element $U_{end}$ where $U \in \mathcal{UC}(X \setminus K_m)$ for some $m \in \mathbb{N}$. Notice $\varphi(U_{end}) = \prod\limits_{n\in\mathbb{N}} X_n$ where $X_n = \mathcal{UC}(X \setminus K_n)$ for all $n \in \mathbb{N} \setminus \{m\}$ and $X_n = U$ when $n = m$; and therefore $\varphi(U_{end})$ is open.
• $\varphi(ends_{\mathcal{K}}(X))$ is a closed subset of $\prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)$. Indeed, let $(C_1, C_2, \ldots)$ be an element of $\prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n) \setminus \varphi(ends_{\mathcal{K}}(X))$. We can find $m \in \mathbb{N}$ such that $C_{m+1} \not\subseteq C_m$. Consider the open subset $U := \prod\limits_{n\in\mathbb{N}} X_n \subset \prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)$, where $X_n = \mathcal{UC}(X \setminus K_n)$ for all $n \in \mathbb{N} \setminus \{m, m+1\}$ and $X_n = C_n$ when $n \in \{m, m+1\}$. Clearly, $(C_1, C_2, \ldots)$ is an element of $U \subset (\prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X\setminus K_n)\setminus\varphi(ends_{\mathcal{K}}(X))$ and thus $\varphi(ends_{\mathcal{K}}(X))$ is closed subset of $\prod\limits_{n\in\mathbb{N}} \mathcal{UC}(X\setminus K_n)$.
This shows that $ends_{\mathcal{K}}(X)$ is compact. □

The following observation serves two purposes. The first purpose is that it shows that the space of ends is a Stone's space. The second purpose is that it motivates us to define the space of ends in a more general setting (the setting of coarse spaces as we will see later in Chapter 7).

**Observation 4.1.11.** *Let $X$ be an unbounded Freudenthal space and $\mathcal{K} = (K_n)_{n\in\mathbb{N}\cup\{0\}}$ be an exhausting sequence, where $K_0 := \emptyset$. Consider the family $\mathcal{B}$ where $A \in \mathcal{B}$ if and only $A$ is a finite union of components $C_{n_1}, \ldots, C_{n_r}$ such that $C_{n_i} \in \mathcal{UC}(X \setminus K_{n_i})$ for all $1 \le i \le r$, or $A = \emptyset$. Define the complement $\neg A := (\bigcup\limits_{C\in\mathcal{UC}(X\setminus K_{n_i})} C) \setminus A$, where $n_i := \max\{n_1, \ldots, n_r\}$. Then:*
*(a) $(\mathcal{B}, \cup, \cap, \neg)$ is a Boolean algebra with $1_{\mathcal{B}} := X$ and $0_{\mathcal{B}} := \emptyset$.*
*(b) Every end $(C_n)_{n\in\mathbb{N}}$ is contained in an ultrafilter of $\mathcal{B}$ and every ultrafilter of $\mathcal{B}$ contains*

*exactly one end.*

*(c) The Stone's space of $\mathcal{B}$ is homeomorphic to $ends(X)$.*

*Proof.* $(a)$ It is straightforward to see that $(\mathcal{B}, \cup, \cap, \neg)$ is a Boolean algebra.

$(b)$ Let $e = (C_n)_{n\in\mathbb{N}}$ be an end, for fixed $n \in \mathbb{N}$, $C_n$ is contained in some ultrafilter $F_e$. Therefore, $C_m \in F_e$ for all $0 \leq m \leq n$. For $N > n$, if $C_N \notin F_e$, then $\bigcup\limits_{C\in\mathcal{UC}(X\setminus K_N)\setminus\{C_N\}} C \in F_e$, but $(\bigcup\limits_{C\in\mathcal{UC}(X\setminus K_N)\setminus\{C_N\}} C) \cap C_n = C_n \setminus C_N \notin F_e$, a contradiction. Thus, $(C_n)_{n\in\mathbb{N}} \subset F_e$. Clearly, no ultrafilter contains two distinct ends. Let $F$ be an ultrafilter of $\mathcal{B}$. Inductively, we find an end contained if $F$. If $n = 0$, then $C_0 = X \in F$. Assume that $C_n \in F$, for some $n \in \mathbb{N}$. Let $\mathcal{UC}(X \setminus K_{n+1}) = \{C^1_{n+1}, \ldots, C^s_{n+1}\}$; and seeking contradiction assume that $C^j_{n+1} \notin F$, for all $1 \leq j \leq s$. Put $D^j_{n+1} = \neg C^j_{n+1}$, then $D^j_{n+1} \in F$, for all $1 \leq j \leq s$, and hence $\bigcap\limits_{j=1}^{s} D^j_{n+1} = \emptyset \in F$, a contradiction. Therefore, there is $C_{n+1} \in \mathcal{UC}(X \setminus K_{n+1})$ such that $C_{n+1} \in F$. We must have $C_{n+1} \subset C_n$.

$(c)$ By part $(b)$, we can assign for each ultrafilter $F \in \mathcal{U}(\mathcal{B})$ of $\mathcal{B}$, an end $e_F$. Define the map $\varphi : \mathcal{U}(\mathcal{B}) \to ends(X)$ that sends each ultrafilter $F$ to the end $e_F$. Let $e_F = (C_n)_{n\in\mathbb{N}}$ and consider a basis element $U_{end}$, where $U = C_n$, for some $n \in N$. Clearly that $\varphi^{-1}(U_{end}) = U_{C_n} = \{F \in \mathcal{U}(\mathcal{B}) : C_n \in F\}$ is a basis element of the Stone's space $\mathcal{U}(\mathcal{B})$. Thus, $\varphi : \mathcal{U}(\mathcal{B}) \to ends(X)$ is a homeomorphism. $\square$

A direct proof to the fact that $ends(X)$ is totally disconnected is given in the following proposition.

**Proposition 4.1.12.** *Let $X$ be an unbounded Freudenthal space, the subspace $ends(X)$ is totally disconnected.*

*Proof.* Let $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence, and $(C_n)_{n\in\mathbb{N}}, (D_n)_{n\in\mathbb{N}} \in ends(X)$ be two distinct ends of $X$ and let $n \in \mathbb{N}$ such that $C_n \cap D_n = \emptyset$. Take $U := C_n$ and $V = (X \setminus K_n) \setminus C_n$. Since, $C_n$ is a clopen subset of $X \setminus K_n$, $V$ is an open subset of $X$. In particular, $U_{end}$ and $V_{end}$ are neighborhoods of $(C_n)_{n\in\mathbb{N}}$ and $(D_n)_{n\in\mathbb{N}}$, respectively. Moreover, $U_{end}$ and $V_{end}$ are disjoint and $ends(X) = U_{end} \cup V_{end}$. Therefore, $ends(X)$ is totally disconnected. $\square$

Urysohn's metrization theorem asserts that the space of ends $ends_{\mathcal{K}}(X)$ is metrizable. We aim to give an explicit metric on $ends_{\mathcal{K}}(X)$ that induces its topology. For the ends

$e = (C_n)_{n\in\mathbb{N}}, e' = (D_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$, we define:

$$n(e,e') = \begin{cases} \min\{n \in \mathbb{N} : C_n \cap D_n = \emptyset\} & \text{if } e \neq e' \\ \infty & \text{if } e = e' \end{cases}$$

**Lemma 4.1.13.** *Let $(X, \mathcal{T})$ be an unbounded Freudenthal space and $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. Let $f : \mathbb{N} \to [0,\infty)$ be the function given by $f(n) = \exp(n)$ and define the function $d : ends_{\mathcal{K}}(X) \times ends_{\mathcal{K}}(X) \to [0,\infty)$ by:*

$$d(e,e') := f(-n(e,e')).$$

*Then $(ends_{\mathcal{K}}(X), d)$ is an ultrametric space.*

*Proof.* Clearly the function $d : ends_{\mathcal{K}}(X) \times ends_{\mathcal{K}}(X) \to [0,\infty)$ is non-negative symmetric and $d(e,e') = 0$ if and only if $e = e'$. Let $e_1 = (C_n)_{n\in\mathbb{N}}, e_2 = (D_n)_{n\in\mathbb{N}}, e_3 = (E_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$. We claim that:

$$d(e_1,e_3) \leq \max\{d(e_1,e_2), d(e_2,e_3)\}.$$

Without loss of generality, assume that $e_1 \neq e_3$, $\max\{d(e_1,e_2), d(e_2,e_3)\} \leq d(e_1,e_3)$ and $d(e_1,e_2) < d(e_1,e_3)$, and we aim to show that $d(e_2,e_3) = d(e_1,e_3)$. To this end, it suffices to show that $d(e_1,e_3) \leq d(e_2,e_3)$ which is satisfied if and only if $n(e_2,e_3) \leq n(e_1,e_3)$. Since $d(e_1,e_2) < d(e_1,e_3)$, one has $n(e_1,e_3) < n(e_1,e_2)$ and hence $C_{n(e_1,e_3)} = D_{n(e_1,e_3)}$. However, $C_{n(e_1,e_3)} \cap E_{n(e_1,e_3)} = \emptyset$ and therefore $D_{n(e_1,e_3)} \cap E_{n(e_1,e_3)} = \emptyset$ which implies that $n(e_2,e_3) \leq n(e_1,e_3)$. □

**Theorem 4.1.14.** *Let $(X, \mathcal{T})$ be an unbounded Freudenthal space and $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. The ultrametric $d : ends_{\mathcal{K}}(X) \times ends_{\mathcal{K}}(X) \to [0,\infty)$ given by $d(e,e') := f(n(e,e')) = \exp(-n(e,e'))$ induces the topology of $ends_{\mathcal{K}}(X)$ as a subspace of $X \cup ends_{\mathcal{K}}(X)$.*

*Proof.* By 4.1.9, the family $\{U_{end} : U \in \bigcup_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)\}$ is a basis for $ends_{\mathcal{K}}(X)$. Let $e = (C_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$, and $r > 0$. We claim that the open ball $B_r(e) = \{e' \in ends_{\mathcal{K}}(X) : d(e,e') < r\}$ is open subset of $ends_{\mathcal{K}}(X)$ as a subspace of $X \cup ends_{\mathcal{K}}(X)$. Choose $n \in \mathbb{N}$

such that $f(n) < r$. Notice that $U_{end}$, where $U = C_n$, is an open set containing $e$ and if $e'' = (D_n)_{n\in\mathbb{N}} \in U_{end}$, then $n(e, e'') > n$ as $C_n = D_n$; and therefore, $d(e, e'') = f(n(e, e'')) < f(n) < r$. Thus, $e \in U_{end} \subset B_r(e)$ as desired.

Conversely, pick a basis element $U_{end}$ of $ends_{\mathcal{K}}(X)$. Then, $U = C_n$ for some $C_n \in \mathcal{UC}(X \setminus K_n)\}$. Let $e = (C_n)_{n\in\mathbb{N}} \in U_{end}$, and choose $m \in \mathbb{N}$ such that $m > n$. Put $r := f(m)$ and let $e' = (D_n)_{n\in\mathbb{N}} \in B_r(e)$, then $f(n(e, e')) = d(e, e') < r = f(m)$ which implies that $m < n(e, e')$. Consequently, $C_m = D_m$ and as $n < m$, we must have $C_n = D_n$ and hence $B_r(e) \subset U_{end}$. $\square$

The Brouwer's Theorem says that a second countable totally disconnected compact Hausdorff space with no isolated points is homeomorphic to the Cantor middle-third set. Utilizing the Brouwer's Theorem, we prove that the space of ends $ends(X)$ is homeomorphic to the Cantor middle-third set.

**Corollary 4.1.15.** *Let $X$ be a Freudenthal space and $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of $X$. If $|ends_{\mathcal{K}}(X)| = \infty$, then $ends_{\mathcal{K}}(X)$ is homeomorphic to the Cantor middle-third set.*

*Proof.* We aim to show that every $e \in ends_{\mathcal{K}}(X)$ is an accumulation point. Indeed, let $e = (C_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$, and for $r > 0$ choose $n_r \in \mathbb{N}$ such that $\exp(-n_r) < r$. Since $ends(X)$ is infinite, we can find $N \in \mathbb{N}$ such that $n_r \leq N$, and an end $e' = (D_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$ such that $C_N \cap D_N = \emptyset$ and $C_{N-1} \cap D_{N-1} \neq \emptyset$. Now, as $n(e, e') = N \geq n_r$, one has $d(e, e') \leq \exp(-n_r) < r$ and therefore $e' \in B_r(e) \cap (ends_{\mathcal{K}}(X) \setminus \{e\})$. $\square$

**Theorem 4.1.16.** *Let $(X, \mathcal{T})$ be a Freudenthal space and $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. The space $X \cup ends_{\mathcal{K}}(X)$ is a compactification of $X$.*

*Proof.* All we need to prove is that $X \cup ends_{\mathcal{K}}(X)$ is compact. Without loss of generality we may assume that $X$ is unbounded. Let $\{V_\alpha : \alpha \in I\}$ be an open cover of $X \cup ends_{\mathcal{K}}(X)$. Since $ends_{\mathcal{K}}(X)$ is a compact subset and that the family $\{U_{end} : U \in \bigcup_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)\}$ is a basis for $ends_{\mathcal{K}}(X)$, we can find a finite subset $\{\alpha_1, \ldots, \alpha_r\} \subset I$ such that $ends_{\mathcal{K}}(X) \subset \bigcup_{i=1}^{r} V_{\alpha_i}$ where $V_{\alpha_i} = \widetilde{U^i}_{end}$ and $U^i = C_{n_i} \in \mathcal{UC}(X \setminus K_{n_i})$. In particular, $ends_{\mathcal{K}}(X) = \bigcup_{i=1}^{r} U^i{}_{end}$. Choose $N \in \mathbb{N}$ large enough such that $N \geq \max\{n_1, \ldots, n_r\}$ and that all components of

$X \setminus K_N$ are unbounded. We claim that $X \setminus K_N \subset \bigcup_{i=1}^{r} U^i$. Seeking contradiction, assume that $D_N \in \mathcal{UC}(X \setminus K_N)$ such that $D_N \cap U^i = \emptyset$ for all $1 \leq i \leq r$. Notice that for all $n \in \mathbb{N}$ any unbounded component $D_{N+n}$ of $X \setminus K_{N+n}$ contained in $D_N$ then $D_{N+n} \cap U^i = \emptyset$ for all $1 \leq i \leq r$. Inductively, we can construct an end $(D_n)_{n\in\mathbb{N}} \notin \bigcup_{i=1}^{r} {U^i}_{end}$, a contradiction. Since $K_N$ is a compact subset of $X \cup ends_{\mathcal{K}}(X)$, the open cover $\{V_\alpha : \alpha \in I\}$ contains a finite sub-cover for $X \cup ends_{\mathcal{K}}(X)$ as desired. $\square$

We are one step further from defining the Freudenthal compactification of a Freudenthal space. All we need is to show that the topology $\mathcal{T}_{end}$ on $X \cup ends_{\mathcal{K}}(X)$ is independent from the choice of the exhausting sequence $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$.

**Theorem 4.1.17.** *Let $(X, \mathcal{T})$ be an unbounded Freudenthal space. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are exhausting sequence; then $X \cup ends_{\mathcal{K}_1}(X)$ is homeomorphic to $X \cup ends_{\mathcal{K}_2}(X)$.*

*Proof.* Let $Com(X)$ be the family of all nonempty compact subsets of $X$ and $\mathcal{P}(X)$ be the power set of $X$. Both $\mathcal{P}(X)$ and $Com(X)$ are partially ordered by the inclusion. By 4.1.4 $(b)$, for each $K \in Com(X)$, there exists $C_K \in \mathcal{UC}(X \setminus K)$. By the axiom of choice define an decreasing function $f : Com(X) \to \mathcal{P}(X)$ mapping each compact subset $K$ to an unbounded component $C_K \in \mathcal{UC}(X \setminus K)$. Notice that for all $K, L \in Com(X)$ such that $K \subset L$, then $f(L) = C_L \subset C_K = f(K)$. Let $\mathcal{F}$ be the collection of all such decreasing functions that map each compact subset $K \in Com(X)$ to $C_K \in \mathcal{UC}(X \setminus K)$. For $U \in \mathcal{T}$ define the set:

- $U_{\mathcal{F}} := \{f \in \mathcal{F} : f(K) \subset U \text{ for some } K \in Com(X)\} \cup U$.
- The family $\mathcal{B}_{\mathcal{F}} = \{U_{\mathcal{F}} : U \in \mathcal{T}\}$ is a basis for a topology $\mathcal{T}_{\mathcal{F}}$ on $X \cup \mathcal{F}$. Indeed, $X_{\mathcal{F}} = X \cup \mathcal{F}$ and if $U_{\mathcal{F}}, V_{\mathcal{F}} \in \mathcal{B}_{\mathcal{F}}$ and $f \in U_{\mathcal{F}} \cap V_{\mathcal{F}}$, then there exist $K, L \in Com(X)$ such that $f(K) \subset U$ and $f(L) \subset V$. Now, $K \cup L \in Com(X)$ and $f(K \cup L) \subset U \cap V$ leading to the equality $U_{\mathcal{F}} \cap V_{\mathcal{F}} = (U \cap V)_{\mathcal{F}}$.

Let $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an arbitrary exhausting sequence; and notice that for all $f \in \mathcal{F}$, $(f(K_n))_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$. On the other hand, for every $e = (C_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$ we assign a function $f_e$ in $\mathcal{F}$ by provoking the axiom of choice as follows:

For each $K \in Com(X)$, choose $n \in \mathbb{N}$ such that $K \subset K_n$, and let $C_K \in \mathcal{UC}(X \setminus K)$ be the component containing $C_n$. Define $f_e : Com(X) \to \mathcal{P}(X)$ by $f_e(K) = C_K$. Clearly, $f_e$ is decreasing and hence belongs to $\mathcal{F}$.

**Claim:** $X \cup ends_{\mathcal{K}}(X) \cong X \cup \mathcal{F}$.

**Proof of the claim:** • The map $\varphi : X \cup \mathcal{F} \to X \cup ends_{\mathcal{K}}(X)$ that fixes every element of $X$ and maps every $f \in \mathcal{F}$ to $(f(K_n))_{n\in\mathbb{N}}$ is continuous. To this end, let $f \in \mathcal{F}$ and $(f(K_n))_{n\in\mathbb{N}} \in \widetilde{V}_{end}$. Notice that $f(K_n) = C_n \subset V$ for some $n \in \mathbb{N}$. Take $U = C_n$, then $f \in U_{\mathcal{F}}$. Moreover, if $g \in U_{\mathcal{F}}$, then $g(K) \subset U$ for some $K \in Com(X)$. Choose $m \in \mathbb{N}$ such that $K \subset K_m$. We have $g(K_m) \subset g(K) \subset U \subset V$, and hence $\varphi(U_{\mathcal{F}}) \subset \widetilde{V}_{end}$.

• The map $\psi : X \cup ends_{\mathcal{K}}(X) \to X \cup \mathcal{F}$ that fixes every element of $X$ and maps every $e = (C_n)_{n\in\mathbb{N}} \in ends(X)$ to the function $f_e$ is continuous. Indeed, let $e = (C_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$ and $f_e \in V_{\mathcal{F}}$. There is a compact subset $K$ such that $f_e(K) = C_K \subset V$, where $C_K \in \mathcal{UC}(X \setminus K)$ and it contains an unbounded component $C_m \in \mathcal{UC}(X \setminus K_m)$. Take $U = C_m$, then $(C_n)_{n\in\mathbb{N}} \in \widetilde{U}_{end}$, and $\psi(\widetilde{U}_{end}) \subset V_{\mathcal{F}}$.

• Let $f \in \mathcal{F}$ and $e = (C_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$. Notice that if $e' = (D_n)_{n\in\mathbb{N}} = (f(K_n))_{n\in\mathbb{N}}$ and $K \in Com(X)$, then both $f(K)$ and $f_{e'}(K)$ are unbounded components of $X \setminus K$ and that $D_m \subset f(K) \cap f_{e'}(K)$, for some $m \in \mathbb{N}$. In particular, $\psi \circ \varphi(f) = f$. On the other hand, if $e = (C_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$, then for any $n \in \mathbb{N}$, $f_e(K_n) = C_n$ and therefore $\varphi \circ \psi((C_n)_{n\in\mathbb{N}}) = (C_n)_{n\in\mathbb{N}}$. □

**Definition 4.1.18.** Let $(X, \mathcal{T})$ be an unbounded Freudenthal space. The **Freudenthal compactification of** $X$ is $\bar{X} := X \cup ends_{\mathcal{K}}(X)$ for some exhausting sequence $\mathcal{K} := (K_n)_{n\in\mathbb{N}}$ of $X$. The corona of $\bar{X}$ is called **space of ends of** $X$ and is dented by $ends(X)$.

**Corollary 4.1.19.** *Let $(X, \mathcal{T})$ be an unbounded Freudenthal space and $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. Then $|ends(X)| = \sup\{|\mathcal{UC}(X \setminus K_n)| : n \in \mathbb{N}\}$.*

*Proof.* By 4.1.17, the space $ends(X)$ is independent of the choice of the exhausting sequence. Moreover, $|\mathcal{UC}(X \setminus K_n)| \le |\mathcal{UC}(X \setminus K_m)|$, for all $n, m \in \mathbb{N}$ such that $n \le m$ and that $|\mathcal{UC}(X \setminus K_n)| \le |ends(X)|$, for all $n \in \mathbb{N}$. □

**Example 4.1.20.** *It is straightforward to see that $|ends(\mathbb{R}^n)| = 2$ if $n = 1$; otherwise, $|ends(\mathbb{R}^n)| = 1$.*

Recall that a function $f : X \to Y$ between topological spaces is called **proper** if the pre-image $f^{-1}(K)$ is compact for every compact subset $K \subset Y$. We conclude this section with two facts about the Freudenthal compactification and its corona:

1. Continuous proper maps between Freudenthal spaces extends continuously to their respective Freudenthal compactification.

2. If $X$ and $Y$ are Freudenthal spaces, then $|ends(X \times Y)| = 1$ if both $X$ and $Y$ are unbounded; and $ends(X \times Y) = ends(Y)$ if $X$ is bounded.

**Proposition 4.1.21.** *Let $f : X \to Y$ be a proper continuous function between topological spaces.*

*(a) If $(L_n)_{n\in\mathbb{N}}$ is an exhausting sequence of compact subsets of $Y$, then $(f^{-1}(L_n))_{n\in\mathbb{N}}$ is an exhausting sequence of compact subsets of $X$.*

*(b) If $X$ and $Y$ are Freudenthal spaces then $f : X \to Y$ induces a continuous function between their ends spaces. In particular, induces a continuous function between their Freudenthal compactifications.*

*Proof.* For $(a)$, notice that $(f^{-1}(L_n))_{n\in\mathbb{N}}$ is an increasing sequence of compact subsets of $X$. Moreover, $f^{-1}(L_n) \subset f^{-1}(int(L_{n+1})) \subset int(f^{-1}(L_{n+1}))$.

For part $(b)$, consider the exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of $X$, where $K_n := f^{-1}(L_n)$, for all $n \in \mathbb{N}$. Now, let $(C_n)_{n\in\mathbb{N}}$ be an end of $X$. Since $f$ is continuous and proper, $f(C_n)$ is an unbounded connected subset of $Y \setminus L_n$ and hence it is contained in some unbounded component $C_n^f$ of $Y \setminus L_n$. In particular, there exists a unique end $(C_n^f)_{n\in\mathbb{N}}$ of $Y$ such that $f(C_n) \subset C_n^f$ for all $n \in \mathbb{N}$. Define the function:

$f_{end} : ends(X) \to ends(Y)$ by $f_{end}((C_n)_{n\in\mathbb{N}}) = (C_n^f)_{n\in\mathbb{N}}$. Let $V = C_n^f$, then $V_{end}$ a basis set containing the end $(C_n^f)_{n\in\mathbb{N}}$. Let $U := f^{-1}(V)$, then $U_{end}$ is an open neighborhood of $(C_n)_{n\in\mathbb{N}}$. Moreover, if $(C'_n)_{n\in\mathbb{N}} \in U_{end}$, then $C'_m \subset U = f^{-1}(C_n^f)$. Therefore, $f(C'_m) \subset C_n^f$ which implies that $C_n^f$ contains the component containing $f(C'_m)$, so $f_{end}(U_{end}) \subset V_{end}$. Thus, $f_{ends} : ends(X) \to ends(Y)$ is continuous. Finally, the function $\widetilde{f_{end}} := f_{end} \cup f$ : $ends(X) \cup X \to ends(Y) \cup Y$ is continuous. $\square$

**Proposition 4.1.22.** *Let $X$ and $Y$ be Freudenthal spaces, then $X \times Y$ is Freudenthal space. Moreover:*

*(a) $X$ is bounded if and only if $ends(X) = \emptyset$.*

*(b) If $X$ is bounded, then $ends(X \times Y) = ends(Y)$.*

*(c) If $X$ and $Y$ are unbounded, then $|ends(X \times Y)| = 1$.*

*Proof.* $(a)$ Apply 4.1.3.

$(b)$ Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of $Y$; notice that $(L_n)_{n\in\mathbb{N}}$, where $L_n := X \times K_n$, is an exhausting sequence of $X \times Y$. On one hand, if $C_n$ is an unbounded component of $Y \setminus K_n$, then $D_n := X \times C_n$ is an unbounded component of $X \times Y \setminus L_n$. On the other hand, if $D_n$ is an unbounded component of $X \times Y \setminus L_n$, it has the form $D_n := X \times C_n$ for some $C_n$ an unbounded component of $Y \setminus K_n$. In particular, the map $\varphi : ends(Y) \to ends(X \times Y)$ that maps the end $(C_n)_{n\in\mathbb{N}}$ to the end $(D_n)_{n\in\mathbb{N}}$, where $D_n := X \times C_n$, is a bijective map between Hausdorff compact spaces. Moreover, for a basis element $V_{end}$ of $ends(X \times Y)$ where $V := X \times C_n$ for some $n \in \mathbb{N}$, then $\varphi^{-1}(V_{end}) = U_{end}$ where $U = C_n$. Therefore, $\varphi : ends(Y) \to ends(X \times Y)$ is a homeomorphism.

$(c)$ Let $(K_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ be an exhausting sequence of $X$ and $Y$, respectively. Notice $(K_n \times L_n)_{n\in\mathbb{N}}$ is an exhausting sequence of $X \times Y$. We aim to show that $X \times Y \setminus (K_n \times L_n)$ has exactly one unbounded component for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, fix $a \in X \setminus K_n$ and $b \in Y \setminus L_n$. Then:

$$X \times Y \setminus (K_n \times L_n) = \Big(\bigcup_{x\in X\setminus K_n} \{x\} \times Y\Big) \cup \Big(\bigcup_{y\in Y\setminus L_n} Y \times \{y\}\Big)$$

$$= \Big(\bigcup_{x\in X\setminus K_n} (\{x\} \times Y \cup X \times \{b\}\Big) \cup \Big(\bigcup_{y\in Y\setminus L_n} (X \times \{y\} \cup \{a\} \times Y)\Big)$$

which is a union of connected spaces with non-empty intersection, so the subset $X \times Y \setminus (K_n \times L_n)$ is connected. Thus, $|ends(X \times Y)| = 1$ as desired. $\square$

## 4.2 Maximally of the Freudenthal Compactification among compactifications of totally disconnected corona

Our goal in this section is to show that the Freudenthal compactification is maximal among all compactification whose corona is totally disconnected (zero-dimensional) in the sense that any compactification $Y$, whose corona is totally disconnected, is dominated by the the Freudenthal compactification.

**Lemma 4.2.1.** *Let $X$ be an unbounded Freudenthal space, $(C_n)_{n\in\mathbb{N}} \in ends(X)$ and $Y$ be a compactification of $X$. Then:*
*(a) $\bigcap\limits_{n\in\mathbb{N}} cl_X(C_n) = \emptyset$;*
*(b) $\bigcap\limits_{n\in\mathbb{N}} cl_Y(C_n)$ is non-empty connected subset of $Y \setminus X$.*
*(c) $Y \setminus X = \bigcup\limits_{(C_n)_{n\in\mathbb{N}}\in ends(X)} (\bigcap\limits_{n\in\mathbb{N}} cl_Y(C_n))$.*

*Proof.* $(a)$ Note $\bigcap\limits_{n\in\mathbb{N}} cl_X(C_n) \subset \bigcap\limits_{n\in\mathbb{N}} (X \setminus int(K_n)) \subset X \setminus \bigcup\limits_{n\in\mathbb{N}} int(K_n) = \emptyset$.
$(b)$ Let $(C_n)_{n\in\mathbb{N}} \in ends(X)$ and notice $\bigcap\limits_{n\in\mathbb{N}} (cl_Y(C_n) \cap X) = \bigcap\limits_{n\in\mathbb{N}} (cl_X(C_n)) = \emptyset$ and therefore $\bigcap\limits_{n\in\mathbb{N}} cl_Y(C_n) \subset (Y \setminus X)$ which implies that $\bigcup\limits_{(C_n)_{n\in\mathbb{N}}\in ends(X)} (\bigcap\limits_{n\in\mathbb{N}} cl_Y(C_n))$ is contained in $(Y \setminus X)$. Since $X$ is connected, $C_n$ is not closed. So, $cl_Y(C_n) \cap (Y \setminus X) \neq \emptyset$. Now, the family $\{cl_Y(C_n) \cap (Y \setminus X) : n \in \mathbb{N}\}$ satisfies the Finite Intersection Property (FIP) and by the compactness of $Y \setminus X$, $\bigcap\limits_{n\in\mathbb{N}} cl_Y(C_n) \neq \emptyset$. Finally, each $C_n$ is connected and hence $cl_Y(C_n)$ is connected in $Y$ for all $n \in \mathbb{N}$ implying that $\bigcap\limits_{n\in\mathbb{N}} cl_Y(C_n)$ is connected as desired.
$(c)$ Let $y \in Y \setminus X$ and let $\mathcal{U}$ be the family of all open sets in $Y$ containing $y$ and let $\mathcal{U}_X := \{U \cap X : U \in \mathcal{U}\}$. Notice that $U \cap X \neq \emptyset$ for all $U \in \mathcal{U}$ as $X$ is dense in $Y$. Let $\bar{X}$ be the Freudenthal compactification. Since $\mathcal{U}_X$ has the finite intersection property, the family $\{cl_{\bar{X}}(V) : V \in \mathcal{U}_X\}$ has the finite intersection property, and hence $\bigcap\limits_{V\in\mathcal{U}_X} cl_{\bar{X}}(V) \neq \emptyset$. On the other hand, for any $x \in X$, by the Hausdorffess of $Y$, we can find disjoint open neighborhood $O$ and $U \in \mathcal{U}$ of $x$ and $y$, respectively, and hence $O \cap (U \cap X) = \emptyset$. In particular, $\bigcap\limits_{V\in\mathcal{U}_X} cl_X(V) = \emptyset$. Therefore, $\bigcap\limits_{V\in\mathcal{U}_X} cl_{\bar{X}}(V) \subset ends(X)$. Let $(C_n)_{n\in\mathbb{N}} \in \bigcap\limits_{V\in\mathcal{U}_X} cl_{\bar{X}}(V) \cap ends(X)$. Notice, for any $U \in \mathcal{U}$ and $n \in \mathbb{N}$, as $(C_n)_{n\in\mathbb{N}} \in cl_{\bar{X}}(U \cap X)$ and $\widetilde{W^n}_{end}$ is an open neighborhood of $(C_n)_{n\in\mathbb{N}}$ where $W^n := C_n$ and hence $\widetilde{W^n}_{end} \cap (U \cap X) = C_n \cap (U \cap X) \neq \emptyset$, and a fortiori $C_n \cap U \neq \emptyset$. Hence, $y \in \bigcap\limits_{n\in\mathbb{N}} cl_Y(C_n)$ which implies that $\bigcup\limits_{(C_n)_{n\in\mathbb{N}}\in ends(X)} (\bigcap\limits_{n\in\mathbb{N}} cl_Y(C_n)) \supset (Y \setminus X)$. □

**Corollary 4.2.2.** *Let $X$ be an unbounded Freudenthal space. Then:*
*(a) If $e = (C_n)_{n\in\mathbb{N}} \in ends(X)$, then $\bigcap\limits_{n\in\mathbb{N}} cl_{\bar{X}}(C_n) = \{x_e\}$ is a singleton.*
*(b) If $(K^1_n)_{n\in\mathbb{N}}$ and $(K^2_n)_{n\in\mathbb{N}}$ are exhausting sequences, and if $(C^i_n)_{n\in\mathbb{N}}$ is decreasing sequence*

*such that* $C_n^i \in \mathcal{UC}(X \setminus K_n^i)$*, for i=1,2, and* $C_n^1 \cap C_n^2 \neq \emptyset$ *for all* $n \in \mathbb{N}$*; then* $\bigcap_{n\in\mathbb{N}} cl_{\bar{X}}(C_n^1) = \bigcap_{n\in\mathbb{N}} cl_{\bar{X}}(C_n^2)$.

*Proof.* Apply 4.1.12 and 4.2.1. □

**Theorem 4.2.3.** *Let* $X$ *be an unbounded Freudenthal space and* $Y$ *be a compactification of* $X$ *whose corona is totally disconnected; then it is dominated by the Freudenthal compactification.*

*Proof.* We aim to find a continuous function $f : \bar{X} \to Y$ such that $f|_X = id_X$. Notice that, by 4.2.1, for any $e = (C_n)_{n\in\mathbb{N}} \in ends(X)$, $\bigcap_{n\in\mathbb{N}} cl_Y(C_n)$ is non-empty connected subset of $Y \setminus X$ and since $Y \setminus X$ is totally disconnected, $\bigcap_{n\in\mathbb{N}} cl_Y(C_n) = \{x_e\}$. Define $f : \bar{X} \to Y$ by $f|_X = id_X$ and $f(e) := x_e$ for all $e \in ends(X)$. Let $U$ be an open neighborhood of $x_e$ in $Y$. Since $(cl_Y(U) \setminus U) \cap U = \emptyset$, $\bigcap_{n\in\mathbb{N}} cl_Y(C_n) \cap (cl_Y(U) \setminus U) = \emptyset$. Since, $(cl_Y(U) \setminus U)$ is compact, the family $\{cl_Y(C_n) \cap (cl_Y(U) \setminus U) : n \in \mathbb{N}\}$ does not satisfy the finite intersection property. Hence, there is $N \in \mathbb{N}$ such that $cl_Y(C_N) \cap (cl_Y(U) \setminus U) = \emptyset$. Now, as $cl_Y(C_N)$ is connected and $cl_Y(C_N) \subset U \cup (Y \setminus cl_Y(U))$, so $C_N \subset cl_Y(C_N) \subset U$. Let $V := C_N$, then $V_{end}$ is an open neighborhood of $e$ in $ends(X)$ and $f(V_{end}) \subset U$ which shows that $f : \bar{X} \to Y$ is continuous and therefore $Y$ is dominated by $\bar{X}$. □

**Corollary 4.2.4.** *Let* $X$ *be an unbounded Freudenthal space and* $Y$ *be a compactification of* $X$ *whose corona is totally disconnected.* $Y$ *is the Freudenthal compactification if and only if for all* $y \in Y \setminus X$*,* $y$ *is contained in an open subset* $U \subset Y$ *such that* $U \cap X$ *is connected.*

*Proof.* Assume that $Y$ is the Freudenthal compactification. If $e = (C_n)_{n\in\mathbb{N}}$ is an end; fix $n \in \mathbb{N}$. Then $\widetilde{U}_{end}$, where $U = C_n$, is an open neighborhood of $e$ and $\widetilde{U}_{end} \cap X = C_n$ which is connected.

Conversely, assume that for all $y \in Y \setminus X$, $y$ is contained in an open subset $U \subset Y$ such that $U \cap X$ is connected; we claim that the domination $f : \bar{X} \to Y$ given in 4.2.3 is injective. Indeed, let $e_1 = (C_n)_{n\in\mathbb{N}}, e_2 = (D_n)_{n\in\mathbb{N}} \in ends(X)$ such that $f(e_1) = f(e_2)$ and thus $\bigcap_{n\in\mathbb{N}} cl_Y(C_n) = \bigcap_{n\in\mathbb{N}} cl_Y(D_n) = \{x_e\}$. Let $n \in \mathbb{N}$, then $x_e \in cl_Y(C_n) \cap cl_Y(D_n) \cap (Y \setminus X)$. Pick an open neighborhood $U$ of $x_e$ such that $U \cap X$ is connected. Now, $(U \cap X) \cap C_n = U \cap C_n \neq \emptyset$, and therefore $(U \cap X) \subset C_n$. Similarly, $(U \cap X) \subset D_n$ and hence $C_n = D_n$.

In particular, $(C_n)_{n\in\mathbb{N}} = (D_n)_{n\in\mathbb{N}}$. Since $X$ is dense in $\bar{X}$ and $Y$, $f$ is surjective. Hence, $f$ is homeomorphism as it is bijective continuous between compact Hausdorff topological spaces. □

## 4.3 External description of the Freudenthal Compactification

The following theorem tells us that the Freudenthal ends are in one-to-one correspondence with the components of $Y \setminus X$ where $Y$ is the Stone Čech compactification of $X$. In section 6, a similar result will be established between the ends of proper geodesic metric spaces and the components of their Higson coronas.

**Theorem 4.3.1.** *Let $X$ a non-compact Freudenthal space and $Y$ be the Stone Čech compactification of $X$, then every component $D$ of $Y \setminus X$ has the form $\bigcap_{n\in\mathbb{N}} cl_Y(C_n)$ for some $(C_n)_{n\in\mathbb{N}} \in ends(X)$. In particular, there is a one-to-one correspondence between the elements of $ends(X)$ and the components of $Y \setminus X$.*

*Proof.* We know that for all $(C_n)_{n\in\mathbb{N}} \in ends(X)$, $\bigcap_{n\in\mathbb{N}} cl_Y(C_n)$ is non-empty connected subset of $Y$, and that $Y\setminus X = \bigcup_{(C_n)_{n\in\mathbb{N}}\in ends(X)} (\bigcap_{n\in\mathbb{N}} cl_Y(C_n))$. Let $D$ be a component of $Y\setminus X$ containing $(\bigcap_{n\in\mathbb{N}} cl_Y(C_n))$ for some $(C_n)_{n\in\mathbb{N}} \in ends(X)$. Seeking contradiction, assume that $(\bigcap_{n\in\mathbb{N}} cl_Y(C_n))$ is properly contained in $D$. There exist an end $(D_n)_{n\in\mathbb{N}} \in ends(X)$ distinct from $(C_n)_{n\in\mathbb{N}}$ such that $D\cap \bigcap_{n\in\mathbb{N}} cl_Y(D_n) \neq \emptyset$. Choose $n \in \mathbb{N}$ such that $C_n \cap D_n = \emptyset$. Note that $X$ is locally compact and $\sigma$-compact Hausdorff space and hence it is paracompact Hausdorff space which implies that $X$ is normal. Now, $X\setminus int(K_{n+1})$ is a normal space, and as $D_n$ is a clopen subset of $X\setminus K_n$, the set $A := D_n \cap X\setminus int(K_{n+1})$ is a non-empty clopen subset of $X\setminus int(K_{n+1})$, thus, by Urysohn's Lemma, there exists a continuous function $f : X \setminus int(K_{n+1}) \to \{0, 1\}$ such that $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \in (X \setminus int(K_{n+1})) \setminus A = D_n \setminus (int(K_{n+1}))$. By Tietze extension theorem, the continuous function $f : X \setminus int(K_{n+1}) \to \{0, 1\}$ admits a continuous extension $\widetilde{f} : X \to \{0, 1\}$; let $g : Y \to \{0, 1\}$ be the continuous extension of $\widetilde{f} : X \to \{0, 1\}$. Notice $D_{n+1} \subset A$ and $C_{n+1} \subset A^c = (X \setminus int(K_{n+1})) \setminus A$. Also, $g(cl_Y(C_{n+1})) \subset cl_{\{0,1\}}(f(C_{n+1})) = \{0\}$ and $g(cl_Y(D_{n+1})) \subset cl_{\{0,1\}}(f(D_{n+1})) = \{1\}$. However, the pair $U := g^{-1}(\{0\}) \cap (Y \setminus X)$ and $V := g^{-1}(\{1\}) \cap (Y \setminus X)$ is a separation of $Y \setminus X$. Moreover, $D \cap U \neq \emptyset$ and $D \cap V \neq \emptyset$, a contradiction. □

**Corollary 4.3.2.** *Let $X$ be a unbounded Freudenthal space. If $|ends(X)| = 1$ and $Y$ is an arbitrary compactification of $X$ then its corona $Y \setminus X$ is a connected subspace. In particular:*
*(a) If $|ends(X)| = 1$, then the only possible finite-point compactification is the 1-point compactification.*
*(b) If $X_1, \ldots, X_n$ are Freudenthal spaces and at least two of them are unbounded, then the corona of any compactification of $X_1 \times \ldots \times X_n$ is connected.*

As a result of 4.3.1, we can obtain the Freudenthal compactification by collapsing each component of the Stone Čech corona $Y \setminus X$ into a singleton. This gives us an external description of the Freudenthal corona or the end space.

**Corollary 4.3.3.** *Let $X$ be a unbounded Freudenthal space and $Y$ be the Stone Čech compactification of $X$. Define a relation on $Y$ by $x \sim x$ for all $x \in X$ and for all $x, y \in Y \setminus X$, $x \sim y$ if and only if $x$ and $y$ are in the same component of $Y \setminus X$. Then:*
*(a) $\sim$ is an equivalence relation on $Y$.*
*(b) The quotient space $Y/_\sim$ is a compactification of $X$ with totally disconnected corona.*
*(c) The space $Y/_\sim$ coincide with the Freudenthal compactification $\bar{X}$ of $X$.*

*Proof.* Part($a$) and ($b$) are straightforward. Notice that $\bar{X}$ dominates $Y/_\sim$. Let $f : \bar{X} \to Y/_\sim$ be a domination. $f : \bar{X} \to Y/_\sim$ is a quotient map and $Y/_\sim = \bar{X}/_{\sim_f}$, where for all $x, y \in \bar{X}$, $x \sim_f y$ if and only if $f(x) = f(y)$. Now, if $e_1, e_2 \in ends(X)$ are distinct ends, then they belong to different components $C_{e_1}$ and $C_{e_2}$ of $Y \setminus X$. Now, $C_{e_1}$ and $C_{e_2}$ collapse to different points in the corona of $Y/_\sim = \bar{X}/_{\sim_f}$, and therefore $f(e_1) \neq f(e_2)$. $\square$

The following theorem gives us another way to externally describe the Freudenthal compactification.

**Theorem 4.3.4.** *(Compare with (9),Theorem 4.6) If $X$ is a Freudenthal space, then its Freudenthal compactification $\bar{X}$ is the supremum of finite-point compactifications of $X$.*

*Proof.* Let $Y$ be the supremum of finite-point compactifications of $X$. Since the corona of each finite-point compactification of $X$ is discrete, $\bar{X}$ dominates every finite-point compactification of $X$ and consequently dominates $Y$. Let $f : \bar{X} \to Y$ be the domination. We claim that $f : \bar{X} \to Y$ is indeed injective. Seeking contradiction, assume that there exist distinct elements $x, y \in \bar{X}$ such that $f(x) = f(y)$. Clearly, $x, y \in \bar{X} \setminus X$. Let $U$ be a clopen subset of the Freudenthal corona such that $x \in U$ and $y \notin U$. Let $Z$ be the quotient space of $\bar{X}$ obtained by contracting the clopen set to a point $\bar{X}$ and the clopen set $(\bar{X} \setminus X) \setminus U$ to a point $\overline{y}$. Clearly, $Z$ is a finite-point compactifications of $X$. Let $h : \bar{X} \to Z$ be the quotient map and $g : Y \to Z$ is a domination. Notice that $g \circ f : \bar{X} \to Z$ is a domination. By 2.2.8, $g \circ f = h$. But, $g \circ f(x) = g \circ f(y)$ and $h(x) \neq h(y)$, a contradiction. Thus, $f : \bar{X} \to Y$ is a homeomorphism as desired. $\square$

**Corollary 4.3.5.** *Let $X$ be a Freudenthal space. If $X$ is $n$-ended and $Y$ is a finite-point compactification, then $|Y \setminus X| \leq n$.*

**Definition 4.3.6.** If $X$ is a locally compact space; its **generalized Freudenthal compactification** $GF(X)$ is the supremum of finite-point Hausdorff compactifications of $X$. By 4.3.4, when $X$ is a Freudenthal space, its Freudenthal compactification coincide with its generalized Freudenthal compactification.

It is well-known that an infinite discrete space admits $n$-point compactification for all $n \in \mathbb{N}$, see (28) on p.1081; and according to 4.3.6, its generalized Freudenthal corona is infinite. While the generalized Freudenthal compactification in the above definition generalize the Freudenthal compactification, and it can be defined for a larger class of topological spaces, namely, locally compact spaces; its corona does not serve our purpose, namely, generalizing the concept of ends of groups. For example, the corona of $GF(\mathbb{Z}^2)$ is infinite, whereas $\mathbb{Z}^2$ as a group is 1-ended. What makes the corona of the generalized Freudenthal compactification not a good candidate to generalize the concept of ends of groups is the fact that the corona of the generalized Freudenthal compactification is not a coarse invariant.

## 4.4 Space of ends of proper geodesic metric spaces

**Definition 4.4.1.** Given a topological space $X$;

- A continuous map $\alpha : [0, \infty) \to X$ is called a **proper ray** if the pre-image $\alpha^{-1}(K)$ is bounded for all $K \subset X$ bounded.
- Two proper rays $\alpha, \beta : [0, \infty) \to X$ are said to **represent the same end** of $X$ if for all

$K \subset X$ bounded, there exist $t_K > 0$ such that $\alpha([t_K, \infty))$ and $\beta([t_K, \infty))$ are in the same component of $X \setminus K$. In that case we write $\alpha \sim \beta$.

• The relation $\sim$ on the family of all proper rays in $X$ is an equivalence relation. The equivalence class of the proper ray $\alpha : [0, \infty) \to X$ is denoted by $end(\alpha)$ and the set of all equivalence classes is denoted by $\epsilon(X)$.

• For an open subset $U \subset X$, we put:

- $U_\epsilon := \{end(\alpha) \in \epsilon(X) : \alpha([t, \infty)) \subset U \text{ for some } t > 0\}$, and

- $\widetilde{U}_\epsilon := U_\epsilon \cup U$.

**Proposition 4.4.2.** *Let $(X, \mathcal{T})$ be a Hausdorff, path-connected, locally path-connected, locally compact and $\sigma$-compact. The family $\widetilde{\mathcal{B}}_\epsilon := \{\widetilde{U}_\epsilon : U \in \mathcal{T}\}$ is a basis for a Hausdorff topology $\mathcal{T}_\epsilon$ on $X \cup \epsilon(X)$ that extends $\mathcal{T}$. Moreover, as a subspace, $X$ is an open dense subspace of $X \cup \epsilon(X)$.*

*Proof.* Without loss of generality, we may assume that $X$ is non-compact. Clearly $\widetilde{\mathcal{B}}_\epsilon$ covers $X \cup \epsilon(X)$ as $\widetilde{X_\epsilon} = X \cup \epsilon(X)$. Also, if $\widetilde{U}_\epsilon, \widetilde{V}_\epsilon \in \widetilde{\mathcal{B}}_\epsilon$, then $(\widetilde{U \cap V})_\epsilon = \widetilde{U}_\epsilon \cap \widetilde{V}_\epsilon$, and therefore is a basis for a topology $\mathcal{T}_\epsilon$ on $X \cup \epsilon(X)$. Moreover, since $X$ is locally compact, it admits a basis consists of bounded sets. Notice that $\mathcal{T}_\epsilon$ contains every bounded open subset as if $U \in \mathcal{T}$ is bounded, then $U \subset X$, $\widetilde{U}_\epsilon = U$ and hence $\mathcal{T} \subset \mathcal{T}_\epsilon$.

Now we show that $(X \cup \epsilon(X), \mathcal{T}_\epsilon)$ is indeed a Hausdorff space. To this end, we discuss tow cases:

**Case1:** Let $end(\alpha), end(\beta) \in \epsilon(X)$ be distinct. There exists a compact subset $K \subset X$ such that for all $t > 0$, $\alpha([t, \infty))$ and $\beta([t, \infty))$ are in different components of $X \setminus K$. Choose $t > 0$ such that $\alpha([t, \infty)) \cap K = \emptyset$ and $\beta([t, \infty)) \cap K = \emptyset$. Let $C$ and $D$ be distinct components of $X \setminus K$ containing $\alpha([t, \infty))$ and $\beta([t, \infty))$, respectively. Notice $\widetilde{C_\epsilon}$ and $\widetilde{D_\epsilon}$ are disjoint neighborhoods of $end(\alpha)$ and $end(\beta)$, respectively.

**Case2:** Let $x \in X$ and $end(\alpha) \in \epsilon(X)$. Choose an open neighborhood $O$ of $x \in X$ such that $K := cl(O)$ is compact. Pick $t > 0$ such that $\alpha([t, \infty)) \cap K = \emptyset$ and let $C$ be a component of $X \setminus K$ containing $\alpha([t, \infty))$, then $O$ and $\widetilde{C_\epsilon}$ are disjoint neighborhoods of $x$ and $end(\alpha)$, respectively. $\square$

**Corollary 4.4.3.** *Let $(X, \mathcal{T})$ be a Hausdorff, path-connected, locally path-connected, locally compact and $\sigma$-compact. The subspace $\epsilon(X)$ is second countable.*

*Proof.* Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. The family

$$\{U_\epsilon : U \in \bigcup_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)\}$$

is a countable basis for $\epsilon(X)$. $\square$

**Theorem 4.4.4.** *Let $(X, \mathcal{T})$ be a Hausdorff, path-connected, locally path-connected, locally compact and $\sigma$-compact. The topological space $(X \cup \epsilon(X), \mathcal{T}_\epsilon)$ is a compactification of $X$ and it coincide with its Freudenthal compactification.*

*Proof.* Let $\bar{X}$ be the Freudenthal compactification of $X$ induced by the exhausting sequence $(K_n)_{n\in\mathbb{N}}$. Given an end $e = (C_n)_{n\in\mathbb{N}}$ of $X$, choose $x_n \in C_n \setminus C_{n+1}$ for all $n \in \mathbb{N}$. Notice that the sequence $(x_n)_{n\in\mathbb{N}}$ diverges to infinity, as otherwise $(x_n)_{n\in\mathbb{N}} \subset K_N$ for some $N \in \mathbb{N}$ and hence $x_{N+1} \notin C_{N+1}$, a contradiction. Choose a path $\alpha_n : [a_n, a_{n+1}] \to C_n$ in $C_n$ connecting $x_n$ and $x_{n+1}$. The concatenation of all these paths $\alpha_e : [0, \infty) \to X$ is a proper ray. Indeed, for any bounded subset $K \subset X$, then $K \subset K_N$ for some $N \in \mathbb{N}$. In particular, $x_n \notin K_N$ for all $n \geq N$. Therefore, $\alpha_e^{-1}(K) \subset [0, a_{N+1}]$ which implies that $\alpha_e^{-1}(K)$ is bounded. Furthermore, if $(y_n)_{n\in\mathbb{N}}$ is a sequence chosen such that $y_n \in C_n \setminus C_{n+1}$ for all $n \in \mathbb{N}$, and if $\beta_e : [0, \infty) \to X$ is the concatenation of $\beta_n : [b_n, b_{n+1}] \to C_n$ where each $\beta_n$ is a path in $C_n$ connecting $y_n$ and $y_{n+1}$; then $end(\alpha) = end(\beta)$.

Consider the function $\varphi : X \cup ends(X) \to X \cup \epsilon(X)$ that maps each end $e \in ends(X)$ to $end(\alpha_e) \in \epsilon(X)$ and that $\varphi|_X = id_X$.

• $\varphi : X \cup ends(X) \to X \cup \epsilon(X)$ is surjective. Indeed, let $end(\alpha) \in \epsilon(X)$. Choose $t > 0$ such that $\alpha([t, \infty)) \cap K_1 = \emptyset$. Without loss of generality, we may assume that $\alpha([t, \infty)) \cap (C_1 \setminus C_2) \neq \emptyset$. Choose $x_n \in C_n \setminus C_{n+1}$ for all $n \in \mathbb{N}$, and let $\alpha_e : [0, \infty) \to X$ be the concatenation of $\alpha_n : [a_n, a_{n+1}] \to C_n$ for all $n \in \mathbb{N}$, where $\alpha_n : [a_n, a_{n+1}] \to C_n$ is a path in $C_n$ connecting $x_n$ and $x_{n+1}$. Then $end(\alpha_e) \in \epsilon(X)$ and $end(\alpha_e) = end(\alpha)$. Moreover, $\varphi(e) = end(\alpha_e)$.

• $\varphi : X \cup ends(X) \to X \cup \epsilon(X)$ is continuous. To this end, let $e = (C_n)_{n\in\mathbb{N}}$ be an end of $X$, and $V_\epsilon$ be a basis element containing $end(\alpha_e)$, where $V = C_n$ for some $n \in \mathbb{N}$. The subset $U_{end}$ where $U = C_n$ is a neighborhood of $e$ and $\varphi(\varphi) \subset V_\epsilon$.

Now, $\varphi : X \cup ends(X) \to X \cup \epsilon(X)$ is a continuous bijective function from a compact space to a Hausdorff space and hence it is a homeomorphism as desired. $\square$

The next goal is to show that the Freudenthal corona of proper geodesic metric spaces is quasi-isometry invariant.

**Definition 4.4.5.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces:

• A map $f : X \to Y$ is called a **quasi-isometric embedding** if there exist real numbers $\lambda, \mu > 0$ such that:

$$\lambda^{-1} \cdot d_X(x, y) - \mu < d_Y(f(x), f(y)) < \lambda \cdot d_X(x, y) + \mu,$$

for all $x, y \in X$.

• The maps $f, h : X \to Y$ is called a **close** if there exists $r > 0$ such that:

$$d_Y(f(x), h(x)) < r$$

for all $x \in X$.

• A map $f : X \to Y$ is called a **quasi-isometry** if it is quasi-isometric embedding and there exists a quasi-isometric embedding $g : Y \to X$ such that $g \circ f$ is close to $id_X$ and $f \circ g$ is close to $id_Y$. In this case, $g : Y \to X$ is called **quasi-inverse** of $f : X \to Y$.

• $X$ is called **quasi-isometric** to $Y$ if there exists a quasi-isometry $f : X \to Y$.

**Example 4.4.6.** • *Let $(X, d)$ be a metric space and $A \subset X$ be a subspace; the inclusion map $\imath : A \to X$ is is a quasi-isometric embedding.*

• *The map $\jmath : \mathbb{R} \to \mathbb{Z}$ sending each $x \in \mathbb{R}$ to $\lfloor x \rfloor \in \mathbb{Z}$, where $\lfloor x \rfloor$ is the largest integer less than or equal to $x$, is a quasi-isometry where the inclusion map $\imath : \mathbb{Z} \to \mathbb{R}$ is a quasi-inverse of $\jmath : \mathbb{R} \to \mathbb{Z}$. In particular, $\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric.*

We have seen that a proper continuous function between topological spaces $f : X \to Y$ extends continuously to their Freudenthal compactifications 4.1.21. What makes a proper continuous function $f : X \to Y$ extends to the coronas is the fact that we can find an exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of $X$ and an exhausting sequence $(L_n)_{n\in\mathbb{N}}$ of $Y$ such that it maps every unbounded component of $X \setminus K_n$ to an unbounded component of $Y \setminus L_n$. Quasi-isometric maps between proper geodesic spaces has that property and that what makes the Freudenthal corona preserved by the quasi-isometry.

**Lemma 4.4.7.** *Let $f : X \to Y$ be a quasi-isometry between proper geodesic metric spaces. There exist exhausting sequences $(K_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ of $X$ and $Y$, respectively, with the property that: for any $n \in \mathbb{N}$ and $C \in \mathcal{UC}(X \setminus K_n)$, there exists $D \in \mathcal{UC}(Y \setminus L_n)$ such that $f(C) \subset D$.*

*Proof.* Choose $\lambda, \mu \in \mathbb{N}$ such that $f : X \to Y$ is $(\lambda, \mu)$-quasi-isometry. For $n \in \mathbb{N}$, set $N_n := (n+2)\lambda^2 + 2\mu\lambda$ and $M_n := n\lambda + \mu$. Fix $x_0 \in X$, and set $K_n := \overline{B(x_0, N_n)}$ and $L_n := \overline{B(f(x_0), M_n)}$. Clearly that $(K_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ are indeed exhausting sequences of $X$ and $Y$, respectively. Notice that for $n \in \mathbb{N}$ and $C \in \mathcal{UC}(X \setminus K_n)$, then $\overline{B(f(C), \lambda + \mu)} \cap L_n = \emptyset$. Indeed, if $y \in \overline{B(f(C), \lambda + \mu)} \cap L_n$, then there is $a \in C$ such that $d_Y(f(a), f(x_0)) \leq (n+1)\lambda + 2\mu$. On the other hand, $d_Y(f(a), f(x_0)) \geq \lambda^{-1} d_X(a, x_0) - \mu > \lambda^{-1} N_n - \mu > (n+1)\lambda + \mu$, a contradiction.

To conclude, we show that $f(C)$ is contained in some component $D \in \mathcal{UC}(Y \setminus L_n)$. Let $y \in f(C)$ and $D \in \mathcal{UC}(Y \setminus L_n)$ be a component containing $y$. It is straightforward to see that if $d \in D$ and $y' \in Y$ such that $d_Y(y, y') \leq \lambda + \mu$, then $y' \in D$. Let $z \in f(C)$, then there exist $a, b \in C$ such that $y = f(a)$ and $z = f(b)$. As $X$ is a geodesic space and $C$ is a path component, we can find a sequence $a = x_1, \ldots, x_p = b \in C$ such that $d_X(x_i, x_{i+1}) \leq 1$, for all $1 \leq i < p$. In particular, $d_Y(f(x_i), f(x_{i+1})) \leq \lambda + \mu$, which inductively implies that $z = f(x_p) \in D$. □

**Theorem 4.4.8.** *Let $X$, $Y$ and $Z$ be proper geodesic metric spaces.*
*(a) Every $f : X \to Y$ quasi-isometry induces a continuous map $f_{end} : ends(X) \to ends(Y)$ between their spaces of ends.*
*(b) If $h : X \to X$ is close to the identity map $id_X : X \to X$, then $h_{end} = id_{ends(X)}$.*
*(c) If $g : Y \to Z$ is quasi-isometry, then $(g \circ f)_{end} = g_{end} \circ f_{end}$.*

*Proof.* Appealing to 4.4.7, there exist exhausting sequences $(K_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ of $X$ and $Y$, respectively, such that for all $n \in \mathbb{N}$ and $C \in \mathcal{UC}(X \setminus K_n)$, $f(C)$ is contained in some component $C^f \in \mathcal{UC}(Y \setminus L_n)$. By 4.2.2, if $e \in ends(X)$ then there exists a decreasing sequence $(C_n)_{n\in\mathbb{N}}$ where

$$C_n \in \mathcal{UC}(X \setminus K_n)$$

for all $n \in \mathbb{N}$ such that $\bigcap\limits_{n\in\mathbb{N}} cl_{\bar{X}}(C_n) = \{e\}$. Notice that $\bigcap\limits_{n\in\mathbb{N}} cl_{\bar{Y}}(C_n^f) = \{e_f\}$ is an end of $Y$. Define the function $f_{end} : ends(X) \to ends(Y)$ that maps each end of $X$ $e \in \bigcap\limits_{n\in\mathbb{N}} cl_{\bar{X}}(C_n)$ to the end $e_f \in \bigcap\limits_{n\in\mathbb{N}} cl_{\bar{Y}}(C_n^f) = \{e_f\}$, where $\bar{Y}$ is the Freudenthal compactification of $Y$. Clearly, $f_{end} : ends(X) \to ends(Y)$ is continuous.

For part $(b)$, notice first that $h : X \to X$ is quasi-isometry. Let $r > 0$ such that $d_X(h(x), x) < r$ for all $x \in X$. By 4.4.7, we can find exhausting sequences $(K_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ of $X$ and $Y$, respectively, such that for all $n \in \mathbb{N}$ and $C \in \mathcal{UC}(X \setminus K_n)$, $h(C)$ is contained in some component $C^h \in \mathcal{UC}(X \setminus L_n)$ and that if $x \in C^h$ and $y \in X$ such that $d_X(x, y) < n$, then $y \in C^h$. Now, let $e = (C_n)_{n\in\mathbb{N}} \in ends(X)$ and $N \in \mathbb{N}$ such that $N > r$. Then, $h_{end}(e) = e_h \in \bigcap\limits_{n\geq N} cl_{\bar{X}}(C_n^h) = \{e_h\}$. Since $h(C_n) \subset C_n^h$ and for all $x \in C_n$, $d_X(x, h(x)) < r < n$, so $C_n \subset C_n^h$ for all $n \geq N$; and hence, by 4.2.2, $\{e\} = \bigcap\limits_{n\geq N} cl_{\bar{X}}(C_n) = \bigcap\limits_{n\geq N} cl_{\bar{X}}(C_n^h) = \{e_h\}$.

$(c)$ By 4.2.2, one can observe that if $(K_n^1)_{n\in\mathbb{N}}$ and $(K_n^2)_{n\in\mathbb{N}}$ are exhausting sequences, and if $(C_n^i)_{n\in\mathbb{N}}$ is decreasing sequence such that $C_n^i \in \mathcal{UC}(X \setminus K_n^i)$, for i=1,2, and $C_n^1 \cap C_n^2 \neq \emptyset$ for all $n \in \mathbb{N}$; then $(C_n^1 \cap C_n^2)_{n\in\mathbb{N}}$ is decreasing sequence such that $C_n^1 \cap C_n^2 \in \mathcal{UC}(X \setminus K_n^1 \cup K_n^2)$ and $\bigcap\limits_{n\in\mathbb{N}} cl_{\bar{X}}(C_n^1 \cap C_n^2) = \bigcap\limits_{n\in\mathbb{N}} cl_{\bar{X}}(C_n^1) = \bigcap\limits_{n\in\mathbb{N}} cl_{\bar{X}}(C_n^2)$.

Now, choose $(K_n)_{n\in\mathbb{N}}$ an exhausting sequence of $X$, $(L_n^1)_{n\in\mathbb{N}}$ and $(L_n^2)_{n\in\mathbb{N}}$ exhausting sequences of $Y$ and $(M_n^1)_{n\in\mathbb{N}}$ and $(M_n^2)_{n\in\mathbb{N}}$ exhausting sequences of $Z$ such that:

- for each $C \in \mathcal{UC}(X \setminus K_n)$, $f(C)$ is contained in some component $C^f \in \mathcal{UC}(Y \setminus L_n^1)$ and $(g \circ f)(C)$ is contained in some component $C^{g\circ f} \in \mathcal{UC}(Z \setminus M_n^1)$,
- for each $D \in \mathcal{UC}(X \setminus L_n^2)$, $g(D)$ is contained in some component $D^g \in \mathcal{UC}(Z \setminus M_n^2)$.

Let $e$ be an end of $X$, then there exists a decreasing sequence $(C_n)_{n\in\mathbb{N}}$ where $C_n \in \mathcal{UC}(X \setminus K_n)$ for all $n \in \mathbb{N}$ such that $\bigcap\limits_{n\in\mathbb{N}} cl_{\bar{X}}(C_n) = \{e\}$. Then, $\bigcap\limits_{n\in\mathbb{N}} cl_{\bar{Y}}(C_n^f) = \{e_f\}$, where $C_n^f \in$

$\mathcal{UC}(Y \setminus L_n^1)$ for all $n \in \mathbb{N}$. Choose, a decreasing sequence $(D_n)_{n\in\mathbb{N}}$ where $D_n \in \mathcal{UC}(Y \setminus L_n^2)$ for all $n \in \mathbb{N}$ and $\bigcap\limits_{n\in\mathbb{N}} cl_{\bar{Y}}(D_n) = \{e_f\}$. On one hand, we have $g_{end}(e_f) \in \bigcap\limits_{n\in\mathbb{N}} cl_{\bar{Z}}(D_n^g)$ where $D_n^g \in \mathcal{UC}(Z \setminus M_n^2)$, for all $n \in \mathbb{N}$. On the other hand, $(g \circ f)_{end}(e) \in \bigcap\limits_{n\in\mathbb{N}} cl_{\bar{Z}}(C_n^{g\circ f})$, where $C_n^{g\circ f} \in \mathcal{UC}(Z \setminus M_n^1)$ for all $n \in \mathbb{N}$. Notice that $C_n^f \cap D_n \in \mathcal{UC}(Y \setminus L_n^1 \cup L_n^2)$ for all $n \in \mathbb{N}$, and $g(C_n^f \cap D_n) \subset C_n^{g\circ f} \cap D_n^g$, for all $n \in \mathbb{N}$. Hence, $\bigcap\limits_{n\in\mathbb{N}} cl_{\bar{Z}}(C_n^{g\circ f}) = \bigcap\limits_{n\in\mathbb{N}} cl_{\bar{Z}}(D_n^g)$. This proves that $(g \circ f)_{end}(e) = g_{end} \circ f_{end}(e)$, as desired. □

**Corollary 4.4.9.** *Let $X$ and $Y$ be proper geodesic metric spaces. If $X$ and $Y$ are quasi-isometric, then $ends(X) \cong ends(Y)$.*

## 4.5 Ends of finitely generated groups

Given a group $G$ (not necessarily finitely generated) and a symmetric subset $K \subset G$ generating $G$; $G$ can naturally be equipped with the norm $|.|_K : G \to [0, \infty)$ given by:

$$|x|_K = \begin{cases} 0 & \text{iff } x = 1_G \\ \inf\{n \in \mathbb{N} : x = x_1 \cdot \ldots \cdot x_n \text{ and } x_1, \ldots, x_n \in K \setminus \{1_G\}\} & \text{if } x \in G \setminus \{1_G\} \end{cases}$$

• The norm $|.|_K : G \to [0, \infty)$ on $G$ induces a left-invariant metric $d_K : G \times G \to [0, \infty)$ given by $d_K(x, y) := |x^{-1}y|$ and it is called the **Cayley** metric with respect to the subset $K$.

• The Cayley metric $d_K$ induces a graph $Cay(G, K) = (V, E)$ called the **Cayley graph of** $G$ **with respect to the subset** $K$ where $V = G$ and $\{x, y\} \in E$ if and only if $d_K(x, y) = 1$ if and only if $y = a \cdot x$ for some $a \in K \setminus \{1_G\}$.

• Passing to the geometric realization $Geom(G, K)$, we can think of the Cayley graph as a geodesic metric space.

**Proposition 4.5.1.** *Let $G$ be a group and $K \subset X$ be a symmetric subset generating $G$. Then:*
*(a) $(G, d_K)$ is quasi-isometric to $Geom(G, K)$.*
*(b) If $K, L \subset G$ are finite symmetric subsets that generate $G$, then $Geom(G, K)$ and $Geom(G, L)$ are quasi-isometric proper geodesic spaces.*

*Proof.* $(a)$ Let $\imath : G \to Geom(G,K)$ be the inclusion map. Clearly, the map $\imath : G \to Geom(G,K)$ is is a quasi-isometric embedding. Moreover, for all $x \in Geom(G,K)$, $x$ belongs to a segment of length one connecting $a_x$ and $b_x$ for some $a_x, b_x \in G$ and hence $d_{Geom(G,K)}(x, a_x) \leq 1$, for all $x \in Geom(G,K)$ which shows that the map $\imath : G \to Geom(G,K)$ is a quasi-isometry.
$(b)$ Let $K, L \subset G$ be finite symmetric subsets that generate $G$. Clearly, $Geom(G,K)$ and $Geom(G,L)$ are proper geodesic metric spaces. Consider the identity map $id_G : (G, d_K) \to (G, d_L)$. For $x, y \in G$, let $d_K(x,y) = n$, $d_L(x,y) = m$ and $M := \max\{d_K(1_G, b), d_L(1_G, a) : (a,b) \in K \times L\}$. So, $x^{-1} \cdot y = a_1 \cdot \ldots \cdot a_n$ and $x^{-1} \cdot y = b_1 \cdot \ldots \cdot b_m$ where $a_i \in K$ and $b_j \in L$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Notice that $d_K(x,y) = d_K(1_G, x^{-1} \cdot y) = d_K(1_G, b_1 \cdot \ldots \cdot b_m)$ and that for $1 \leq j < m$, we have $d_K(b_1 \cdot \ldots \cdot b_j, b_1 \cdot \ldots \cdot b_{j+1}) \leq d_K(1_G, b_1 \cdot \ldots \cdot b_j) + d_K(1_G, b_{j+1})$ and therefore we obtain $d_K(x,y) \leq \sum_{j=1}^{m} d_K(1_G, b_j) \leq m \cdot M$; in particular, $d_K(x,y) \leq M \cdot d_L(x,y)$. Similarly, $d_L(x,y) \leq M \cdot d_K(x,y)$. By part $(a)$, $Geom(G,K)$ and $Geom(G,L)$ are quasi-isometric. □

A consequence of 4.5.1, finitely generated groups can be viewed as proper geodesic spaces and that their Cayley graphs are independent of the choice of the finite generating set. Hence, when $G$ is finitely generated group, there is no ambiguity in writing $Geom(G)$ to denote the geometric realization of a Cayley graph of $G$ with respect to some finite symmetric generating subset of $G$.

**Definition 4.5.2.** Let $G$ be a group finitely generated; the space of ends $ends(G)$ of $G$ is the space of ends of the geometric realization of its Cayley graph. By 4.5.1 and 4.4.8, the space of ends $ends(G)$ is independent of the choice of the finite generating set, and hence is well-defined.

**Lemma 4.5.3.** *Let $G$ be finitely generated group.*
*(a) If $H$ is a subgroup of finite index in $G$, then $ends(G)$ and $ends(H)$ are homeomorphic.*
*(b) If $G$ has finitely many ends then it has a subgroup of finite index acts trivially on on $ends(G)$.*

*Proof.* $(a)$ The inclusion map $\imath : H \to G$ is a quasi-isometric embedding. Suppose that $[G : H] = n$ and let $\{1_G = g_1, g_2, \ldots, g_n\}$ be a subset of $G$ such that $G$ is the disjoint union

of the right cosets $H, H\cdot g_2, \ldots, H\cdot g_n$. Define the map $\jmath : G \to H$ by $\jmath(x\cdot g_i) = x$ for all $x \in H$ and $1 \le i \le n$. Then $\jmath : G \to H$ is a quasi-isometric embedding. Furthermore, $\jmath\circ\imath = id_H$ and $d(\imath\circ\jmath(x\cdot g_i), x\cdot g_i) \le M$ for all $x \in H$ and $1 \le i \le n$, where $M = \max\{d(g_i, 1_G) : 1 \le i \le n\}$.
$(b)$ Let $K \subset G$ be a finite symmetric subset generating $G$. Notice that for all $x \in G$, the map $\sigma^x : (G, d_K) \to (G, d_K)$ given by $\sigma^x(g) = x\cdot g$ is an isomrty. By 4.4.8, the isometry $\sigma^x : (G, d_K) \to (G, d_K)$ induces a homeomorphism $\sigma^x_{end} : ends(G) \to ends(G)$. Consider the group of all homeomorphisms $Homeo(ends(G))$ from $ends(G)$ to itself. The map $\varphi : G \to Homeo(ends(G))$ defined by $\varphi(x) := \sigma^x_{end}$ is a homomorphism with $H := Ker(\varphi)$ is a subgroup of finite index in $G$ that acts trivially on $ends(G)$. □

The rest of this section is dedicated to the Freudenthal-Hopf's theorems for finitely generated groups.

**Theorem 4.5.4.** *Let $G$ be a finitely generated group. $G$ either has infinitely many ends or at most it has two ends. Furthermore, $|ends(G)| = 0$ if and only if $G$ is finite.*

*Proof.* The later part of the theorem is straightforward. Assume that $G$ is infinite, then $Geom(G)$ has at least one end. Seeking contradiction assume that $Geom(G)$ there exist three proper rays $\alpha_0, \alpha_1, \alpha_2 : [0,\infty) \to Geom(G)$ that represent three different ends of $Geom(G)$. By concatenating $1_G$ and the initial points $\alpha_1(0)$, $\alpha_2(0)$, we may assume that $\alpha_1(0) = \alpha_2(0) = 1_G$. Choose $N \in \mathbb{N}$ such that $r_0((N,\infty))$, $\alpha_1((N,\infty))$ and $\alpha_2((N,\infty))$ are in different components of $Geom(G) \setminus B(1_G, N)$. Choose a subgroup $H$ of finite index in $G$ that acts trivially on $ends(G)$. Choose $n \in \mathbb{N}$ large enough such that $\alpha_0(n) = h \in H$ and $d(h, B(1_G, N)) > 2N$. Notice that for all $t, s \in [2N, \infty)$, $d(\alpha_1(s), \alpha_2(t)) > 2N$. Now, $end(h\cdot\alpha_1) = end(\alpha_1)$ and $end(h\cdot\alpha_2) = end(\alpha_2)$ since $H$ acts trivially on $ends(G)$. As, $h\cdot\alpha_1(0) = h\cdot\alpha_2(0) = h$ and $h$ is in a different component than $\alpha_1((N,\infty))$ and $\alpha_2((N,\infty))$, the proper rays $h\cdot\alpha_1$ and $h\cdot\alpha_2$ must pass through the ball $B(1_G, N)$. In particular, there exist $s, r > 2N$ such that $h\cdot\alpha_1(s), h\cdot\alpha_1(t) \in B(1_G, N)$ and hence $d(h\cdot\alpha_1(s), h\cdot\alpha_2(t)) = d(\alpha_1(s), \alpha_2(t)) \le 2N$ which contradicts the fact that for all $t, s \in [2N, \infty)$, $d(\alpha_1(s), \alpha_2(t)) > 2N$. □

**Theorem 4.5.5.** *Let $G$ be a finitely generated group. $G$ has two ends if and only it contains an infinite cyclic subgroup of finite index.*

*Proof.* If $G$ contains an infinite cyclic subgroup of a finite index it is quasi-isometric to $\mathbb{Z}$ and hence it has two ends by 4.1.20 and 4.4.6.

Conversely, assume that $G$ has two ends. Let $\alpha_l, \alpha_r : [0, \infty) \to Geom(G)$ be two proper rays that represent two different ends of $Geom(G)$ with $\alpha_l(0) = \alpha_r(0) = 1_G$. Choose $N \in \mathbb{N}$ such that $Geom(G) \backslash B(1_G, N)$ has exactly two disjoint components, the left component $C_l$ and the right component $C_r$ containing $\alpha_l((N, \infty))$ and $\alpha_r((N, \infty))$, respectively. Let $H \leqslant G$ be of finite index and $H$ acts trivially on $ends(G)$. Choose $n > 2N$ such that $\alpha_r(n) = h \in H$. Note that for all $t > N$, $d(\alpha_l(t), \alpha_r(t)) > 2N$. Moreover, $end(h \cdot \alpha_l) = end(\alpha_l)$ and $h \cdot \alpha_l(0) = h$ and therefore $h$ moves $\alpha_l : [0, \infty) \to Geom(G)$ to the right implying that it moves every element to the right, and that $h^{-1}$ moves every element to the left. Inductively, one has:

- $h^n \in C_r$ and $h^{-n} \in C_l$,
- $h^{n+1} \cdot B(1_G, N) \subset h^n \cdot C_r$ and $h^{n+1} \cdot C_r \subset h^n \cdot C_r$,
- $h^n \cdot C_l \subset h^{n+1} \cdot C_l$ and $h^{n+1} \cdot B(1_G, N) \cap h^n \cdot B(1_G, N) = \emptyset$ for all $n \in \mathbb{N} \cup \{0\}$.

Consequently, the subgroup $H' :=< h >$ is infinite cyclic subgroup and we claim that $H'$ is of finite index in $G$. To that end, notice that:

$K := (B(1_G, N) \cup h \cdot C_l \setminus C_l) \cap G$ is a finite subset of $G$, and we aim to prove that $G = H' \cdot K$. Let $x \in G$, and choose $m \in \mathbb{N} \cup \{0\}$ such that:

$d(x, h^m \cdot B(1_G, N)) \leq d(x, h^n \cdot B(1_G, N))$ for all $n \in \mathbb{N}$. If $d(x, h^m \cdot B(1_G, N)) = 0$, then $x \in K \subset H' \cdot K$. Assume that $d(x, h^m \cdot B(1_G, N)) > 0$; then either $x \in h^m \cdot C_l$ or $x \in h^m \cdot C_r$.

- If $x \in h^m \cdot C_l$, then $x \in h^m \cdot C_l \setminus h^{m-1} \cdot C_l$. Indeed, if $x \in h^{m-1} \cdot C_l$, then as $h^m \cdot B(1_G, N) \subset h^{m-1} \cdot C_r$, so any path connecting $x$ to $h^m \cdot B(1_G, N)$ must pass through $h^{m-1} \cdot B(1_G, N)$ and hence $d(x, h^{m-1} \cdot B(1_G, N)) < d(x, h^m \cdot B(1_G, N))$, a contradiction. Thus, $x \in h^m \cdot C_l \backslash h^{m-1} \cdot C_l$ and hence $x \in H' \cdot K$.
- If $x \in h^m \cdot C_r$, then $x \in h^m \cdot C_r \setminus h^{m+1} \cdot C_r$. Indeed, if $x \in h^{m+1} \cdot C_r$, then as $h^m \cdot B(1_G, N) \subset h^{m+1} \cdot C_l$, then any path from $x$ to $h^m \cdot B(1_G, N)$ must pass through $h^{m+1} \cdot B(1_G, N)$, and hence $d(x, h^{m+1} \cdot B(1_G, N)) < d(x, h^m \cdot B(1_G, N))$, a contradiction. Thus, $x \in h^m \cdot C_r \setminus h^{m+1} \cdot C_r \subset (h^{m+1} \cdot C_l \setminus h^m \cdot C_l) \cup h^{m+1} \cdot B(1_G, N)$ and hence $x \in H' \cdot K$. Thus, $G = H' \cdot K$, as desired. □

As a result of 9.2.9, if $G$ is a finitely generated group with $|ends(G)| = \infty$, then $ends(G)$ is the Cantor middle-third set (compare with (3), Theorem 8.32).

## 4.6 Higson corona and the space of ends of a proper geodesic metric space

In this section we show that, when $X$ is a geodesic proper metric space, the Freudenthal compactification is a coarse compactification; namely, it is dominated by the Higson compactification. Moreover, the Freudenthal compactification is the quotient space of the Higson compactification obtained by collapsing each component of the Higson corona to a singleton. Consequently, one can generalizes the Freudenthal compactification to be the supremum of finite-point compactifications that are dominated by its Higson compactification. We begin this section with the concept of coarsely disjoint subsets of a metric space first introduced in (12) (see Definition 2.1 of (12)).

**Definition 4.6.1.** Let $A$ and $C$ be subsets of a metric space $X$, $A$ and $C$ are **coarsely disjoint** if for any $r > 0$, the subset $B(A, r) \cap B(C, r)$ is bounded. If $A \subset X$, $A$ is called **coarsely clopen** if $A$ and $A^c$ are coarsely disjoint. $A$ is **non-trivial coarsely clopen** if $A$ is coarsely clopen with both $A$ and $A^c$ are unbounded.

**Proposition 4.6.2.** *Let $A$ and $C$ be subsets of a metric space $X$, the following conditions are equivalent:*

*(a) $A$ and $C$ are coarsely disjoint,*

*(b) for all $r > 0$, the subset $B(A, r) \cap C$ is bounded,*

*(c) for all $r > 0$, the subset $A \cap B(C, r)$ is bounded,*

*(d) for all $r > 0$, there is a bounded subset $K_r$ of $X$ such that $B(A \setminus K_r, r) \cap B(C, r)$ is empty,*

*(e) for all $r > 0$, there is a bounded subset $K_r$ of $X$ such that $B(A, r) \cap B(C \setminus K_r, r)$ is empty.*

*Proof.* Straightforward. □

**Proposition 4.6.3.** *Let $(X, d)$ be a proper geodesic metric space, $a \in X$ and $n \in \mathbb{N}$.*

*(a) Every component $C_n$ of $X \setminus B(a, n)$ is coarsely clopen.*

*(b) $|ends(X)| \geq 2$ if and only if it has a non-trivial coarsely clopen subset.*

*Proof.* $(a)$ For $r > 0$, let $A := C_n$ and $K := B(a, n + r)$. Notice that $B(A \setminus K, r) \cap A^c \setminus B(a, n) = \emptyset$. This shows that $C_n$ is coarsely clopen.
$(b)$ Assume that $|ends(X)| \geq 2$, and let $n \in \mathbb{N}$ such that $C_n$ and $D_n$ are unbounded distinct components of $X \setminus B(a, N)$. Let $A = C_n$, then by part$(a)$ $A$ is a non-trivial coarsely clopen subset.
Conversely, let $A$ be a non-trivial coarsely clopen subset. Choose $N \in \mathbb{N}$ large enough such that $X \setminus K_n$, where $K_n := cl(B(a, n))$, has at least two unbounded components and all components of $X \setminus K_n$ are unbounded, say, these components are $C_N^1, \dots, C_N^m$. For $r > 0$, the set $B(A, r)$ is a non-trivial coarsely clopen and open subset. Note that $B(A, r) \setminus K_n$ is unbounded clopen subset of $X \setminus K_n$, and hence any component intersects $B(A, r) \setminus K_n$ is contained in $B(A, r) \setminus K_n$. In particular, at least a component of $X \setminus K_n$ is contained in $B(A, r) \setminus K_n$ and at least a component is contained in $B(A, r)^c \setminus K_n$. $\square$

A subset $A$ of a group $G$ is called **right $G$-commensurated** or **almost right invariant** if the symmetric difference $A \Delta A \cdot g$ is finite for all $g \in G$. In finitely generated groups equipped with a Cayley metric, or more generally in countable groups equipped with left-invariant proper metric, coarsely clopen subsets of $G$ are exactly those subsets that are right $G$-commensurated.

**Proposition 4.6.4.** *Let $G$ be a countable group equipped with a left-invariant proper metric $d : G \times G \to [0, \infty)$. A subset $A \subset G$ is coarsely clopen if and only if $A \Delta A \cdot g$ is finite for all $g \in G$.*

*Proof.* Notice first, as $d : G \times G \to [0, \infty)$ is left-invariant, for arbitrary subset $A \subset G$, an element $g \in G$ and $r > 0$, one has $B(A \cdot g, r) = A \cdot B(g, r)$. Now, assume that $A$ is a coarsely clopen subset of $G$. Let $g \in G$ and pick $r > 0$ such that $d(g, 1_G) < r$. Notice that $A \cdot g \cap (X \setminus A) \subset A \cdot B(1_G, r) \cap (X \setminus A) = B(A, r) \cap (X \setminus A)$ which is bounded and hence finite as $d : G \times G \to [0, \infty)$ is is proper. Similarly, $A \cap (X \setminus A \cdot g) = A \cap (X \setminus A) \cdot g \subset A \cap (X \setminus A) \cdot B(1_G, r) = A \cap B((X \setminus A), r)$ which is finite. Therefore, the symmetric difference $A \Delta A \cdot g$ is finite.
Conversely, assume that $A \Delta A \cdot g$ is finite for all $g \in G$. Let $r > 0$, since $d : G \times G \to$

$[0, \infty)$ is is proper, there exist $g_1, \ldots, g_n \in G$ such that $B(1_G, r) = \{g_1, \ldots, g_n\}$. Now, $B(A, r) \cap (X \setminus A) = A \cdot B(1_G, r) \cap (X \setminus A) = \bigcup_{i=1}^{n} (A \cdot g_i \cap (X \setminus A))$ which is finite. $\square$

**Proposition 4.6.5.** *A subset $A$ of a metric space $X$ is coarsely clopen if and only if the characteristic function $\chi_A$ of $A$ is slowly oscillating on $X$.*

*Proof.* Assume that $A \subset X$ is coarsely clopen. If either $A$ or $A^c$ is bounded, then $\chi_A : X \to \{0, 1\}$ is obviously slowly oscillating on $X$. Assume that both $A$ and $A^c$ are unbounded. For $r > 0$, take the bounded set $K_r := B(A, r) \cap B(A^c, r)$, and notice that for all $x, y \in X \setminus K_r$ either $x, y \in A$ or $x, y \in A^c$, and thus $|\chi_A(x) - \chi_A(y)| = 0$. This shows that $\chi_A : X \to \{0, 1\}$ is slowly oscillating on $X$.

Conversely, assume that $\chi_A : X \to \{0, 1\}$ is slowly oscillating on $X$. Then $A = \chi_A^{-1}(\{1\})$ and $A^c = \chi_A^{-1}(\{0\})$. Let $r > 0$ and $0 < \varepsilon < 1$, we can find a bounded subset $K_r \subset X$ such that for all $x, y \in X \setminus K_r$ and $d(x, y) < r$, then $|\chi_A(x) - \chi_A(y)| < \varepsilon$. Therefore, for all $x, y \in X \setminus K_r$, either $x, y \in A$ or $x, y \in A^c$, and thus $B(A, r) \cap A^c \setminus K_r = \emptyset$. $\square$

**Definition 4.6.6.** A function between metric spaces $f : X \to Y$ is called **coarsely constant** if there exists a bounded subset $K \subset X$ such that $f|_{X \setminus K}$ is constant. Otherwise, $f : X \to Y$ is called **coarsely non-constant**.

**Corollary 4.6.7.** *A subset $A$ of a metric space $X$ is non-trivial coarsely clopen if and only if the characteristic function $\chi_A$ of $A$ is coarsely non-constant and is slowly oscillating on $X$.*

**Proposition 4.6.8.** *Let $Y$ be a compactification of a metric space $(X, d)$. $Y$ is a coarse compactification of $X$ if and only if for all sequences $(x_n), (y_n) \subset X$ with $d(x_n, y_n) < r$ for all $n \in \mathbb{N}$, for some $r > 0$ and $\omega \in (Y \setminus X) \cap cl_Y(\{x_n : n \in \mathbb{N}\})$; then $\omega \in cl_Y(\{y_n : n \in \mathbb{N}\})$.*

*Proof.* Suppose that $Y$ is a coarse compactification of $X$ and $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subset X$ with $d(x_n, y_n) < r$ for some $r > 0$, for all $n \in \mathbb{N}$. Suppose $\omega \in (Y \setminus X) \cap cl_Y(\{x_n : n \in \mathbb{N}\})$ but $\omega \notin cl_Y(\{y_n : n \in \mathbb{N}\})$. Choose a continuous function $f : Y \to [0, 1]$ sending $\omega$ to 0 and sending $cl_Y(\{y_n : n \in \mathbb{N}\})$ to 1. Notice that for each $m \in \mathbb{N}$, $f^{-1}[0, \frac{1}{m})$ is an open neighborhood of $\omega$. By choosing $x_{n_m} \in f^{-1}[0, \frac{1}{m}) \cap \{x_n : n \in \mathbb{N}\}$ for all $m \in \mathbb{N}$, we may reduce $(x_n)$ to a subsequence $(x_{n_m})_{m \in \mathbb{N}}$ diverging to infinity in $X$ with $f(x_{n_m}) \to 0$ as $m \to \infty$.

Since $f|_X$ is slowly oscillating, so $|f(x_{n_m}) - f(y_{n_m})| \to 0$. Thus, $f(y_{n_m}) \to 0$ as $m \to \infty$ contradicting $f(cl_Y(\{y_n : n \in \mathbb{N}\})) = 1$.

Conversely, let $f : Y \to [0,1]$ be a continuous function, and we aim to show that $f|_X$ is slowly oscillating. Seeking contradiction, assume that there exist $(x_n), (y_n) \subset X$ such that $(x_n)$ diverges to infinity and $d(x_n, y_n) < r$ for some $r > 0$, for all $n \in \mathbb{N}$, with $|f(x_n) - f(y_n)|$ does not converge to 0. Passing to subsequences if necessary, we may assume that $f(x_n) \to a \in [0,1]$ and $f(y_n) \to b \in [0,1]$ as $n \to \infty$ and $a \neq b$. Pick $U_a$ and $U_b$ open neighborhoods in $[0,1]$ of $a$ and $b$, respectively, such that $cl_{[0,1]}(U_a) \cap cl_{[0,1]}(U_b) = \emptyset$. Notice that $f^{-1}(cl_{[0,1]}(U_a))$ is a closed subset contains all but finitely many elements of $(x_n)$; in particular, $(Y \setminus X) \cap cl_Y(\{x_n : n \in \mathbb{N}\}) \subset f^{-1}(cl_{[0,1]}(U_a))$. Similarly, $(Y \setminus X) \cap cl_Y(\{y_n : n \in \mathbb{N}\}) \subset f^{-1}(cl_{[0,1]}(U_b))$, but $f^{-1}(cl_{[0,1]}(U_a)) \cap f^{-1}(cl_{[0,1]}(U_b)) = \emptyset$, a contradiction.

□

When $(X, d)$ is a proper metric space, the coarsely disjoint subsets of $X$ are exactly those subsets with disjoint boundaries where the closures are taken in its Higson compactification $hX$.

**Proposition 4.6.9.** *If $(X, d)$ is a proper metric space and $A$ is a subset of its Higson compactification $hX$ and the Higson corona $\partial_h X := hX \setminus X$, then $A \cap X$ is coarsely clopen in $X$ if and only if $cl_h(A) \cap \partial_h X$ and $cl_h(A^c) \cap \partial_h X$ are disjoint, where the closures are taken in $hX$.*

*Proof.* If $C := A \cap X$ is coarsely clopen, then $cl_X(C) \cap cl_X(X \setminus C)$ is a bounded subset. Choose a bounded open subset $U \subset X$ containing $cl_X(C) \cap cl_X(X \setminus C)$. Notice that $cl_X(C) \setminus U$ and $cl_X(X \setminus C)$ are closed disjoint subsets of $X$. By Urysohn's lemma, the characteristic function $\chi_{cl_X(C) \setminus U}$ is continuous on $X \setminus U$. By Tietze extension theorem, it extends to a continuous function $f : X \to \{0,1\}$. Clearly, the function $f : X \to \{0,1\}$ is slowly oscillating, and hence it extends continuously to the function $g : hX \to \{0,1\}$. Notice that $g(cl_h(cl_X(C) \setminus U)) = \{1\}$ and $g(cl_h(cl_X(X \setminus C)) = \{0\}$. Therefore, $cl_h(A) \cap \partial_h X \subset g^{-1}\{1\}$ and $cl_h(A^c) \cap \partial_h X \subset g^{-1}\{1\}$, which implies that $cl_h(A) \cap \partial_h X$ and $cl_h(A^c) \cap \partial_h X$ are disjoint.

Conversely, assume that $cl_h(A) \cap \partial_h X$ and $cl_h(X \setminus A) \cap \partial_h X$ are disjoint, and $C := A \cap X$ is not coarsely clopen. There exist $r > 0$, $S_1 := \{x_n : n \in \mathbb{N}\}$ in $C$ and $S_2 := \{y_n : n \in \mathbb{N}\}$ in

$C^c$ such that $x_n \to \infty$ (that means each bounded subset $K$ of $X$ contains only finitely many members of the sequence) and $d(x_n, y_n) < r$ for all $n \in \mathbb{N}$. Notice that $cl_h(S_1) \cap \partial_h X$ and $cl_h(S_2) \cap \partial_h X$ are disjoint, which contradicts 4.6.8. $\square$

**Theorem 4.6.10.** *Let $X$ be a proper geodesic metric space. There is a one-one correspondence between the the ends of $X$ and the components of $\partial_h X$.*

*Proof.* Let $e = (C_n)_{n\in\mathbb{N}}$ be an end of $X$, then, by 4.2.1, $\bigcap\limits_{n\in\mathbb{N}} cl_h(C_n)$ is a connected subset of $\partial_h X$. Let $C_e$ be the component of $\partial_h X$ containing $\bigcap\limits_{n\in\mathbb{N}} cl_h(C_n)$, and $\mathcal{C}(\partial_h X)$ be the family of all components of $\partial_h X$. Define the map $\varphi : ends(X) \to \mathcal{C}(\partial_h X)$ sending each end $e = (C_n)_{n\in\mathbb{N}}$ to the component $C_e$. It is surjective by 4.2.1. Let $e_1 = (C_n)_{n\in\mathbb{N}}$ and $e_2 = (D_n)_{n\in\mathbb{N}}$ be two different ends. Let $n \in \mathbb{N}$ such that $C_n \cap D_n = \emptyset$, and put $A := C_n$. By 4.6.3, $A$ is a non-trivial coarsely clopen subset. Moreover, by 4.6.9, $cl_h(A) \cap \partial_h X$ and $cl_h(A^c) \cap \partial_h X$ are disjoint clopen subsets of $\partial_h X$. Since $C_{e_1} \cap cl_h(A) \cap \partial_h X \neq \emptyset$ and $C_{e_2} \cap cl_h(A^c) \cap \partial_h X \neq \emptyset$; hence $C_{e_1} \subset cl_h(A) \cap \partial_h X$ and $C_{e_2} \subset cl_h(A^c) \cap \partial_h X$ and therefore $C_{e_1}$ and $C_{e_2}$ are distinct components of $\partial_h X$. In particular, the map $\varphi : ends(X) \to \mathcal{C}(\partial_h X)$ is bijective. $\square$

As a result of that correspondence, we can obtain the Freudenthal compactification by collapsing each component of the Higson corona $\partial_h X$ into a Singleton. This gives us an external description of the Freudenthal corona or the end space.

**Corollary 4.6.11.** *Let $X$ be an unbounded proper metric space and $hX$ be the Higson compactification of $X$. Define a relation on $hX$ by $x \sim x$ for all $x \in X$ and for all $x, y \in \partial_h X$, $x \sim y$ if and only if $x$ and $y$ are in the same component of $\partial_h X$. The quotient space $hX/_\sim$ is a coarse compactification of $X$ with totally disconnected corona. If, moreover, $X$ is geodesic, then the compactification $hX/_\sim$ coincides with its Freudenthal compactification $\bar{X}$ of $X$.*

*Proof.* Similar to 4.3.3. $\square$

**Theorem 4.6.12.** *If $X$ is a proper geodesic metric space, then its Freudenthal compactification $\bar{X}$ is the supremum of finite-point compactifications of $X$ that are dominated by the Higson compactification of $X$.*

*Proof.* Similar to 4.3.4. $\square$

## 4.7 Specker's approach to the space ends

Given a topological space $X$, the family $Clop(X) := \{U \subset X : U \text{ is clopen}\}$ is a sub-Boolean algebra of $\mathcal{P}(X)$. If $B$ is a Boolean algebra, the family of all ultrafilters of $B$ is denoted by $\mathcal{U}(B)$. The family $\{U_x : x \in B\}$, where $U_x = \{F \in \mathcal{U}(B) : x \in F\}$, is a basis for a topology $\mathcal{T}_{\mathcal{U}(B)}$ on $\mathcal{U}(B)$ in which $U_x$ is a clopen subset for all $x \in B$, and $(\mathcal{U}(B), \mathcal{T}_{\mathcal{U}(B)})$ is a compact Hausdorff totally disconnected space (Stone's space). The Stone's representation theorems for Boolean algebras and Stone's spaces assert us the following:

• Every Boolean algebra $B$ is isomorphic to $Clop(\mathcal{U}(B))$, and two Boolean algebras $B_1$ and $B_2$ are isomorphic if and only if their Stone's spaces $\mathcal{U}(B_1)$ and $\mathcal{U}(B_2)$ are homeomorphic.

• Every compact Hausdorff totally disconnected space (Stone's space) $X$ is homeomorphic to $\mathcal{U}(Clop(X))$, and two Stone's spaces are homeomorphic if and only if their corresponding Boolean algebras $Clop(X)$ and $Clop(Y)$ are isomorphic.

Taking advantage of the fact that the space of ends $ends(G)$ of a finitely generated group $G$ is a Stone's space, and therefore it can be determined by the Boolean algebra $Clop(ends(G))$, Specker(40) defined the space of ends for arbitrary groups as follows:

• Consider the family $Comm(G)$ of all right $G$-commensurated subsets of the group $G$; the relation $\sim$ on $Comm(G)$ given by $A \sim B$ if and only if $A\Delta B$ is bounded (equivalently finite in discrete spaces), is an equivalence relation. The equivalence class of $A \in Comm(G)$ is denoted by $[A]$, and the family of the equivalence classes of $Comm(G)$ is denoted by $Comm(G)/_{\sim}$.

• The 4-tuple $(Comm(G)/_{\sim}, +, \cdot, \neg)$ is a Boolean algebra, where $0 := [\emptyset]$, $1 := [X]$ and the operations are defined as follows:

* $[A] + [B] = [A \cup B]$;

* $[A] \cdot [B] = [A \cap B]$;

* $\neg[A] = [X \setminus A]$; for all $[A], [B] \in Comm(G)/_{\sim}$.

• The space of of ends of $G$ is the Stone's space $\mathcal{U}(Comm(G)/_{\sim})$.

Our goal is to show that if $G$ is finitely generated then $ends(G)$ is indeed homeomorphic to $\mathcal{U}(Comm(G)/_\sim)$. Let $d : G \times G \to [0, \infty)$ be a proper left-invariant metric on $G$; as noticed in 4.6.4, $Comm(G) = \mathcal{CC}(G)$, where $\mathcal{CC}(G)$ is the family of all coarsely clopen subsets of $(G, d)$. Moreover, the Freudenthal compactification of a proper geodesic metric is metrizable and dominated by its Higson compactification and its corona is totally disconnected. To accomplish our goal, we will show that if $(X, d)$ is a proper geodesic metric space and $Y$ is the Freudenthal compactification, then $\partial X = Y \setminus X$ is homeomorphic to $\mathcal{U}(\mathcal{CC}(X)/_\sim)$.

**Definition 4.7.1.** Let $X$ be a coarse space and $A, B \subset X$. We say that $A$ and $B$ are called **coarsely identical** and we write $A \sim B$ if their symmetric difference $A\Delta B$ is a bounded subset.

**Lemma 4.7.2.** *Let $X$ be a coarse space; the relation $\sim$ on the power set $\mathcal{P}(X)$ of $X$ is an equivalence relation. The equivalence class of $A \in P(X)$ is denoted by $[A]$.*

*Proof.* Notice that $A$ and $B$ are coarsely identical if and if there exist bounded subsets $K_A, K_B \subset X$ such that $A = (A \cap B) \cup K_A$ and $B = (A \cap B) \cup K_B$. Assume that $B$ and $C$ are coarsely identical and let $K'_B, K_C \subset X$ be bounded such that $B = (B \cap C) \cup K'_B$ and $C = (B \cap C) \cup K_C$. Now, $A \setminus C = (A \cap B) \cup K_B \setminus C \subset (B \cup K_B) \setminus C \subset (B \setminus C) \cup K_B$ and $C \setminus A = ((B \cap C) \cup K_C) \setminus A \subset (B \cup K_C) \setminus A \subset (B \setminus A) \cup K_C$. Therefore, $A\Delta C$ is a bounded and thus $\sim$ is a transitive relation. □

**Notation 4.7.3.** *Let $X$ be a coarse space; the family of all coarsely clopen subsets of $X$ is denoted by $\mathcal{CC}(X)$. The family of all equivalence classes $[A]$ where $A \in \mathcal{CC}(X)$ is denoted by $\mathcal{CC}(X)/_\sim$.*

**Lemma 4.7.4.** *The 4-tuple $(\mathcal{CC}(X)/_\sim, +, \cdot, \neg)$ is a Boolean algebra, where $0 := [\emptyset]$, $1 := [X]$ and the operations are defined as follows:*

- $[A] + [B] = [A \cup B]$;
- $[A] \cdot [B] = [A \cap B]$;
- $\neg[A] = [X \setminus A]$; *for all* $[A], [B] \in \mathcal{CC}(X)/_\sim$.

*Proof.* Notice that if $A, B \in \mathcal{CC}(X)$, then $A \cap B, A \cup B, X \setminus A \in \mathcal{CC}(X)$. □

When $X$ is a proper geodesic metric space, then $\mathcal{U}(\mathcal{CC}(X)/_\sim)$ is the space of ends of $X$.

**Lemma 4.7.5.** *Let $\bar{X}$ be the Freudenthal compactification of the proper geodesic metric space $(X, d)$ and $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be an exhausting sequence. If $A \subset X$ is a coarsely clopen subset and $e = (C_n)_{n\in\mathbb{N}} \in ends_{\mathcal{K}}(X)$ is an end, then there exists $N \in \mathbb{N}$, such that either $C_N \subset A$ or $C_N \subset X \setminus A$.*

*Proof.* Let $r > 0$, and choose $N \in \mathbb{N}$ such that $B_r(A) \cap B_r(X \setminus A) \subset K_N$. Put $A' := A \setminus int(K_N)$, and notice that:

- $A'$ is a closed subset of $X \setminus K_N$ as $A' = (X \setminus cl_X(X \setminus A)) \cap (X \setminus K_N)$.
- $A'$ is an open subset of $X \setminus K_N$ as $A' = (X \setminus int(X \setminus A)) \cap (X \setminus K_N)$.

Therefore, $A'$ is a clopen subset of $X \setminus K_N$, and hence the component $C_N$ of $X \setminus K_N$ is either contained entirely in $A'$ or entirely in $X \setminus (A \cup K_N)$. Since $C_N \cap K_N = \emptyset$, $C_N$ is either contained entirely in $A$ or entirely in $X \setminus A$ as claimed. $\square$

**Theorem 4.7.6.** *Let $\bar{X}$ be the Freudenthal compactification of a proper geodesic metric space $X$. Then:*

*(a) If $U \subset X$ is coarsely clopen, then $C = cl_{\bar{X}}(U) \cap \partial X$ is clopen subset of $\partial X$ and $(cl_{\bar{X}}(U) \cap cl_{\bar{X}}(X \setminus U)) \cap \partial X = \emptyset$.*

*(b) If a subset $C \subset \partial X$ is clopen then there exists a coarsely clopen subset $U_C \subset X$ such that $C = cl_{\bar{X}}(U_C) \cap \partial X$. Furthermore, if $V_C \subset X$ is coarsely clopen such that $C = cl_{\bar{X}}(V_C) \cap \partial X$, then $U_C$ and $V_C$ are coarsely identical.*

*Proof.* $(a)$ Assume that $U$ is a coarsely clopen subset of $X$. Pick $r > 0$, and choose a compact subset $K \subset X$ such that $B(U, r) \cap B(X \setminus U, r) \subset int(K)$. Put $A := U \setminus int(K)$, and notice that:

- $cl(A) \cap \partial X = cl(U) \cap \partial X$.
- $cl(X \setminus A \cup int(K)) \cap \partial X = cl_{\bar{X}}(X \setminus U) \cap \partial X$.
- $A$ is a closed subset of $X \setminus int(K)$ as $A = (X \setminus cl_X(X \setminus U)) \cap (X \setminus int(K))$.
- $A$ is an open subset of $X \setminus int(K)$ as $A = (X \setminus int(X \setminus U)) \cap (X \setminus int(K))$.

Now, since $A$ is a clopen subset of $X \setminus int(K)$, the characteristic function of $A$ $\chi_A : X \setminus int(K) \to \{0, 1\}$ is continuous. Moreover, since $X \setminus int(K)$ is a closed subset of the normal space $X$, the continuous function $\chi_A : X \setminus int(K) \to \{0,1\}$, by Tietze extension theorem, admits a continuous extension $f : X \to [0,1]$. By 4.7.5, we can define a function $g : \bar{X} \to [0,1]$ as follows: $g|_X = f$ and for $e = (C_n)_{n\in\mathbb{N}} \in ends(X)$ let $g(e) = 1$ if $C_n \subset U_C$ for some $n \in \mathbb{N}$, and $g(e) = 0$ if $C_n \not\subset U$ for all $n \in \mathbb{N}$. Clearly, $g(\partial X) \subset \{0,1\}$ and $g : \bar{X} \to [0,1]$ is continuous. Indeed, let $e = (C_n)_{n\in\mathbb{N}} \in ends(X)$, we discuss two cases:

**Case 1:** If $g(e) = 1$, then $C_n \subset U_C$ for some $n \in \mathbb{N}$. Put $O = C_n$, then $\widetilde{O}_{end} := O_{end} \cup O$ is a neighborhood of $e$ on which $g$ is constant.

**Case 2:** If $g(e) = 0$, then by 4.7.5, we can find $n \in \mathbb{N}$ such that $C_n \subset X \setminus U_C$. Put $O = C_n$, then $\widetilde{O}_{end} := O_{end} \cup O$ is a neighborhood of $e$ on which $g$ is constant. This shows that $g : \bar{X} \to [0,1]$ is a continuous extension of $f : X \to [0,1]$. Moreover, $g^{-1}(\{1\}) \cap \partial X = cl_{\bar{X}}(U) \cap \partial X$, and $g^{-1}(\{0\}) \cap \partial X = cl_{\bar{X}}(X \setminus U) \cap \partial X$ which shows that $C = cl_{\bar{X}}(U) \cap \partial X$ is clopen subset of $\partial X$ and $(cl_{\bar{X}}(U) \cap cl_{\bar{X}}(X \setminus U)) \cap \partial X = \emptyset$.

$(b)$ Assume that $C \subset \partial X$ is a clopen subset. Since $\bar{X}$ is normal, we can choose disjoint neighborhoods $U$ and $V$ in $\bar{X}$ of $C$ and $\partial X \setminus C$ such that $cl_{\bar{X}}(U) \cap cl_{\bar{X}}(V) = \emptyset$. Notice that $C = cl_{\bar{X}}(U) \cap \partial X$. Indeed, if $\omega \in (cl_{\bar{X}}(U) \cap \partial X) \setminus C$, then $\omega \in V$, a contradiction. Put $U_C := U \cap X$, and notice that if $O \subset \bar{X}$ is a neighborhood of $\omega \in cl_{\bar{X}}(U) \cap \partial X$, then $O \cap U \neq \emptyset$. As $X$ is dense subset of $\bar{X}$, $(O \cap U) \cap X = O \cap U_C \neq \emptyset$. Therefore, $C = cl_{\bar{X}}(U) \cap \partial X = cl_{\bar{X}}(U_C) \cap \partial X$.

**Claim:** $U_C$ is coarsely clopen.

**Proof of the claim:** since $\bar{X} \setminus U$ and $C$ are disjoint closed subsets, there exists, by Urysohn's lemma, a continuous function $f : \bar{X} \to [0,1]$ such that $f(\bar{X} \setminus U) = \{0\}$ and $f(C) = \{1\}$. Choose $a, b \in (0,1)$ such that $a < b$, and notice that $K := f^{-1}([a,b])$ is a compact subset of $\bar{X}$. Moreover, as $\partial X \subset (\bar{X} \setminus U) \cup C$, $K$ is a compact subset of $X$. Seeking contradiction, assume that $U_C$ is not coarsely clopen; hence, $U_C \setminus K$ and $X \setminus (U_C \cup K)$ are not coarsely disjoint. There exists $r > 0$, such that $B(U_C \setminus K, r) \cap X \setminus (U_C \cup K)$ is unbounded. In particular, we can find sequences $(x_n)_{n\in\mathbb{N}} \subset U_C \setminus K$ and $(y_n)_{n\in\mathbb{N}} \subset X \setminus (U_C \cup K)$ converging to infinity points $\omega_x \in \partial X$ and $\omega_y \in \partial X$, respectively, such that $d(x_n, y_n) < r$ for all $n \in \mathbb{N}$. Notice that $(y_n)_{n\in\mathbb{N}} \subset X \setminus (U_C \cup K) \subset \bar{X} \setminus U$ and hence $f(y_n) = 0$ for all $n \in \mathbb{N}$. Also, since $\omega_x \in cl_{\bar{X}}(U_C \setminus K) \cap \partial X \subset C$, $f(\omega_x) = 1$. Choose $N \in \mathbb{N}$ such that $f(x_n) \in (b, 1]$ for all

$n > N$, then $|f(x_n) - f(y_n)| > b$ for all $n > N$, which contradicts the fact that $f : X \to [0, 1]$ is slowly oscillating function. Therefore, $U_C$ must be coarsely clopen as claimed.

Finally, assume that $V_C \subset X$ is coarsely clopen such that $C = cl_{\bar{X}}(V_C) \cap \partial X$. Notice that by part $(a)$, $(cl_{\bar{X}}(V_C) \cap cl_{\bar{X}}(X \setminus V_C)) \cap \partial X = \emptyset$. Seeking contradiction, assume that $U_C \setminus V_C$ is unbounded; so there is an unbounded sequence $(x_n)_{n\in\mathbb{N}} \subset U_C \setminus V_C$. Without loss of generality, assume that $(x_n)_{n\in\mathbb{N}}$ converges to $\omega \in \partial X$. On one hand, $\omega \in cl_{\bar{X}}(U_C) \cap \partial X = cl_{\bar{X}}(V_C) \cap \partial X$. On the other hand, $\omega \in cl_{\bar{X}}(X \setminus V_C) \cap \partial X$, which contradicts the fact that $(cl_{\bar{X}}(V_C) \cap cl_{\bar{X}}(X \setminus V_C)) \cap \partial X = \emptyset$. Thus, $U_C \setminus V_C$ is bounded; and similarly, $V_C \setminus U_C$ is bounded. □

**Theorem 4.7.7.** *Let $X$ be a proper geodesic metric space, and $\bar{X}$ be the Freudenthal compactification. The space of ends $\partial X = \bar{X} \setminus X$ is homeomorphic to $\mathcal{U}(\mathcal{CC}(X)/_\sim)$.*

*Proof.* By the Stone's representation theorems, it suffices to show that the Boolean algebras $Clop(\partial X)$ and $\mathcal{CC}(X)/_\sim$ are isomorphic. By 4.7.6$(b)$, for every $C \in Clop(\partial X)$ there exists $U_C \in \mathcal{CC}(X)$ such that $(cl_{\bar{X}}(U_C) \cap cl_{\bar{X}}(X \setminus U_C)) \cap \partial X) = \emptyset$ and $C = cl_{\bar{X}}(U_C) \cap \partial X$. Define the map $\varphi : Clop(\partial X) \to \mathcal{CC}(X)/_\sim$ that sends each $C \in Clop(\partial X)$ to $[U_C] \in \mathcal{CC}(X)/_\sim$.

- $\varphi$ is well-defined by 4.7.6$(b)$.
- $\varphi$ is injective. Indeed, let $C, D \in Clop(\partial X)$ such that $\omega \in C \setminus D$. In particular, $\omega$ is not a limit point of $U_D$, and hence there exists unbounded sequence $(x_n)_{n\in\mathbb{N}} \subset U_C \setminus U_D$ converging to $\omega$. Therefore, $U_C \setminus U_D$ is unbounded.
- By 4.7.6$(a)$, $\varphi$ is surjective.
- $\varphi$ is a homomorphism. To that end, notice that $\varphi(\emptyset) = [\emptyset]$ and $\varphi(\partial X) = [X]$. To show it is a homomorphism it suffices to show that for $C, D \in Clop(\partial X)$, then $[U_{C\cup D}] = [U_C \cup U_D]$ and $[U_{C\cap D}] = [U_C \cap U_D]$. Notice that $C \cup D = (cl_{\bar{X}}(U_C) \cup cl_{\bar{X}}(U_D)) \cap \partial X = cl_{\bar{X}}(U_C \cup U_D) \cap \partial X$, and hence $[U_{C\cup D}] = [U_C \cup U_D]$. Seeking contradiction, assume that $[U_{C\cap D}] \neq [U_C \cap U_D]$. One has $C \cap D = cl_{\bar{X}}(U_{C\cap D}) \cap \partial X = (cl_{\bar{X}}(U_C) \cap cl_{\bar{X}}(U_D)) \cap \partial X$. We consider two cases:

**Case 1:** Assume that $U_C \cap U_D \setminus U_{C\cap D}$ contains an unbounded sequence $(x_n)_{n\in\mathbb{N}}$ converging to $\omega \in \partial X$. Then $\omega \in cl_{\bar{X}}(U_C \cap U_D) \cap \partial X$ and as $cl_{\bar{X}}(U_C \cap U_D) \subset cl_{\bar{X}}(U_C) \cap cl_{\bar{X}}(U_D)$, $\omega \in cl_{\bar{X}}(U_C) \cap cl_{\bar{X}}(U_D) \cap \partial X = cl_{\bar{X}}(U_{C\cap D}) \cap \partial X$ which contradicts the fact that $(cl_{\bar{X}}(U_{C\cap D}) \cap cl_{\bar{X}}(X \setminus U_{C\cap D})) \cap \partial X = \emptyset$.

**Case 2:** Assume that $U_{C\cap D} \setminus U_C \cap U_D$ contains an unbounded sequence $(x_n)_{n\in\mathbb{N}}$ converging to $\omega \in \partial X$. Notice that $\omega \in cl_{\bar{X}}(X \setminus (U_C \cap U_D)) \cap \partial X = (cl_{\bar{X}}(X \setminus U_C) \cap \partial X) \cup (cl_{\bar{X}}(X \setminus U_D) \cap \partial X)$. Without loss of generality, we may assume that $\omega \in cl_{\bar{X}}(X \setminus U_C) \cap \partial X$. Therefore, $\omega \notin cl_{\bar{X}}(U_C) \cap \partial X$, which contradicts the fact that $\omega \in cl_{\bar{X}}(U_C \cap U_D) \cap \partial X \subset cl_{\bar{X}}(U_C) \cap \partial X$. $\square$

# Chapter 5

# Coarse Geometry

Throughout this paper, all topological spaces are assumed to be Hausdorff spaces. The basic reference for this chapter is (36).

## 5.1 Definitions and examples

**Definition 5.1.1.** The subset $\Delta = \Big\{(x,x) : x \in X\Big\}$ of $X \times X$ is called the **diagonal** of $X$. For $E, F \subset X \times X$, the **product** of $E$ and $F$ is denoted by $E \circ F$ and defined by $E \circ F = \Big\{(x,z) : \exists y \in X, (x,y) \in E, (y,z) \in F\Big\}$. The **inverse** of $E$ is $E^{-1} = \Big\{(y,x) : (x,y) \in E\Big\}$. A **coarse structure** on $X$ is a collection $\mathcal{E}$ of subsets $E \subset X \times X$ satisfying the following conditions:

(1) $\Delta \in \mathcal{E}$,

(2) If $E \subset F \in \mathcal{E}$, then $E \in \mathcal{E}$,

(3) If $E, F \in \mathcal{E}$, then $E \cup F, E \circ F, E^{-1} \in \mathcal{E}$.

An element of $\mathcal{E}$ is called an **entourage** or a **controlled** set. The set $X$ equipped with a coarse structure $\mathcal{E}$ is called a **coarse space** and denoted by $(X, \mathcal{E})$. We say that $(X, \mathcal{E})$ is (coarsely) **connected** if $\{(x,y)\}$ is an entourage for all $x, y \in X$.

**Example 5.1.2.** *Let $(X,d)$ be a metric space and $r \geq 0$. We put:*

$$E_r := \{(x,y) \in X \times X : d(x,y) \leq r\}.$$

*The family $\mathcal{E}_d := \{E \subset X \times X : E \subset E_r$ for some $r \geq 0\}$ is a coarsely connected coarse structure called the* ***bounded coarse structure*** *induced by the metric $d$.*

**Example 5.1.3.** *Let $X$ be a non-empty set and $G$ be a group that acts on $X$ from the right. For a subset $A \subset G$ and a subset $Y \subset X$, put $E_A^Y := \{(x, x \cdot g) : x \in Y, g \in A\}$. Let $\mathcal{G}_{fin}$ be the family of all finite subsets of $G$, then the family:*

$$\mathcal{E}_G := \{E \subset E_A^Y : Y \in \mathcal{P}(X), A \in \mathcal{G}_{fin}\}$$

*is a coarse structure on $X$ called the* ***coarse structure induced by the right action*** *of $G$ on $X$. If the action of $G$ on $X$ is transitive, then $(X, \mathcal{E}_G)$ is coarsely connected space.*

*Proof.* Notice that $\Delta = E_{\{1_G\}}^X$. If $Y, Z \in \mathcal{P}(X)$ and $A, B \in \mathcal{G}_{fin}$, then $E_A^Y \cup E_B^Z \subset E_{A \cup B}^{Y \cup Z}$, and $E_A^Y \circ E_B^Z \subset E_{A \cdot B}^Z$. For $g \in G$, $Y \cdot g = \{x \cdot g : x \in Y\}$. Let $Y' := \bigcup_{g \in A} Y \cdot g$, then $(E_A^Y)^{-1} \subset E_{A^{-1}}^{Y'}$. Finally, suppose that the action of $G$ on $X$ is transitive, and $x, y \in X$, then there exists $g \in G$ such that $y = x \cdot g$, and hence $\{(x, y)\} = E_{\{g\}}^{\{x\}}$. $\square$

**Definition 5.1.4.** Let $\mathcal{E}$ and $\mathcal{F}$ be two coarse structures on $X$:

- We say that $\mathcal{E}$ is **finer** than $\mathcal{F}$ or $\mathcal{F}$ is **coarser** than $\mathcal{E}$ if $\mathcal{E} \subset \mathcal{F}$.
- A subset $\mathcal{B} \subset \mathcal{E}$ is called a **basis** for $\mathcal{E}$ if for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{B}$.

**Example 5.1.5.** *Given a metric space $(X, d)$, the family $\{E_n : n \in \mathbb{N}\}$ is a basis for $\mathcal{E}_d$.*

## 5.2 Metrizability of a coarse structure

**Definition 5.2.1.** A coarse structure $\mathcal{E}$ on $X$ is called **metrizable** if there exists a metric $d : X \times X \to [0, \infty)$ on $X$ such that $\mathcal{E} = \mathcal{E}_d$

**Lemma 5.2.2.** *If a coarse structure $\mathcal{E}$ on $X$ admits a countable basis, then it admits an ascending basis consists of symmetric entourages.*

*Proof.* Let $\mathcal{A} := \{A_n : n \in \mathbb{N}\}$ be a countable basis for $\mathcal{E}$, then $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$, where $B_n = A_n \cup A_n^{-1}$, is a countable basis consists of symmetric entourages. Put $C_n = \bigcup_{i=1}^{n} B_i$; then $\mathcal{C} := \{C_n : n \in \mathbb{N}\}$ is a countable basis consists of symmetric entourages such that $C_n \subset C_{n+1}$ for all $n \in \mathbb{N}$. $\square$

**Theorem 5.2.3.** *A coarse structure $\mathcal{E}$ on $X$ is metrizable if and only if $\mathcal{E}$ admits a countable basis.*

*Proof.* If $\mathcal{E}$ is metrizable by a metric $d$, then the countable family $\{E_n : n \in \mathbb{N}\}$, where $E_n = \{(x, y) \in X \times X : d(x, y) \leq n\}$, is basis for $\mathcal{E}_d = \mathcal{E}$.
Conversely, by 5.2.2, we can choose a countable basis $\mathcal{C} := \{C_n : n \in \mathbb{N} \cup \{0\}\}$ consists of symmetric entourages such that $C_n \subset C_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, and $C_0 = \Delta$. Define a map $d : X \times X \to [0, \infty]$ by:

$$d(x,y) = \begin{cases} \infty & \text{if } (x,y) \notin \bigcup\limits_{n \in \mathbb{N} \cup \{0\}} C_n \\ \min\{n \in \mathbb{N} \cup \{0\} : (x,y) \in C_n\} & \text{if } (x,y) \in \bigcup\limits_{n \in \mathbb{N} \cup \{0\}} C_n. \end{cases}$$

$d : X \times X \to [0, \infty]$ defines a metric on $X$ and $E_n = \{(x, y) \in X \times X : d(x, y) \leq n\} = C_n$, for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $\mathcal{E} = \mathcal{E}_d$, as wanted. $\square$

**Definition 5.2.4.** Let $(X, \mathcal{E})$ be a coarse space, and $A$ be a set, the maps $f, g : A \to X$ are **close** if $\Big\{(f(a), g(a)) : a \in A\Big\} \in \mathcal{E}$.

**Notation 5.2.5.** *Let $\mathcal{U}$ be a collection of subsets of $X$; we write $E_{\mathcal{U}} = \bigcup_{U \in \mathcal{U}} U \times U$. For a subset $A \subset X$, the star of $A$ with respect to $\mathcal{U}$ is denoted by $St(A,\mathcal{U})$ and defined by $St(A,\mathcal{U}) =: \bigcup\limits_{\substack{U \in \mathcal{U} \\ U \cap A \neq \emptyset}} U$. Notice that $E_{\mathcal{U}}[A] = St(A,\mathcal{U})$.*

**Definition 5.2.6.** Let $(X, \mathcal{E})$ be a coarse space. A subset $K \subset X$ is called **coarsely bounded** or just **bounded** if $K \times K \in \mathcal{E}$. In general, a collection $\mathcal{U}$ of subsets of $X$ is said to be uniformly bounded if $E_{\mathcal{U}} = \bigcup_{U \in \mathcal{U}} U \times U \in \mathcal{E}$. Clearly, every element of $\mathcal{U}$ is a bounded subset of $X$. If, moreover, $\mathcal{U}$ is a cover of $X$, $\mathcal{U}$ is said to be **uniformly bounded cover** of $X$.

**Definition 5.2.7.** Let $X$ be a set, with $A \subset X$, and a subset $E \subset X \times X$; we define the $E$**-ball around** $A$ by:

$$E[A] = \Big\{x \in X : \exists a \in A, (x, a) \in E\Big\}$$

.

**Lemma 5.2.8.** *Let $(X, \mathcal{E})$ be a coarse space and $K \subset X$. The following are equivalent:*

*(a) $K$ is bounded;*

*(b) $K \times \{p\} \in \mathcal{E}$ for some $p \in X$;*

*(c) $K = E[p]$ for some $p \in X$ and $E \in \mathcal{E}$;*

*(d) The inclusion map $K \to X$ is to a constant map $c : K \to X$.*

*Proof.* Assume that $K = E[p]$ for some $p \in X$ and $E \in \mathcal{E}$. Notice that $E \circ E^{-1} \in \mathcal{E}$ and $E \circ E^{-1} = K \times K$. The rest of the implications are straightforward. □

**Proposition 5.2.9.** *Let $(X, \mathcal{E})$ be a coarse space. Then:*

*(a) For all $p \in X$, $\{p\}$ is a bounded subset of $X$.*

*(b) If $K$ is a bounded subset of $X$ and $L \subset K$, then $L$ is a bounded subset of $X$.*

*(c) If $K \subset X$ is a bounded subset and $E$ is an entourage then $E[K]$ is a bounded subset.*

*(d) If $L, K \subset X$ are bounded subsets of $X$ such that $L \cap K \neq \emptyset$, then $L \cup K$ is a bounded subset of $X$.*

*(e) If $K_0, K_1, \ldots, K_n$ are all bounded subsets of $X$ such that $K_i \cap K_0 \neq \emptyset$ for all $i = 1, \ldots, n$, then $\bigcup_{i=1}^{n} K_i$ is bounded subset of $X$.*

*(f) If $K_1, \ldots, K_n$ are all bounded subsets of $X$ such that $K_i \cap K_{i+1} \neq \emptyset$ for all $i = 1, \ldots, n-1$, then $\bigcup_{i=1}^{n} K_i$ is bounded subset of $X$.*

*(g) If $L$ and $K$ are bounded subsets $X$ and $L \cap K \neq \emptyset$, then $L \times K$ is is an entourage.*

*(h) If $L$ and $K$ are bounded subsets $X$ and $(X, \mathcal{E})$ is coarsely connected; then $L \times K$ is an entourage.*

*(i) Finite union of bounded sets of a coarsely connected coarse space $X$ is a bounded subset.*

*Proof.* $(a)$ and $(b)$ are obvious.

$(c)$ Notice that if $K \subset X$, and $E \subset X \times X$, then $E[K] \times E[K] \subset E \circ (K \times K) \circ E^{-1}$.

$(d)$ Assume that $L, K \subset X$ are bounded subsets of $X$ and that $p \in L \cap K$. Notice that $E = L \times \{p\}$ and $F = K \times \{p\}$ are entourages and therefore, by part $(c)$, $(E \cup F)[L]$ is bounded. Since $L \cup K \subset (E \cup F)[L]$, $L \cup K$ is bounded.

$(e)$ and $(f)$ follow from part $(d)$ along with the induction.

$(g)$ Assume that $L$ and $K$ are bounded subsets $X$ and $p \in L \cap K$. Notice that $(L \times \{p\}) \circ$

$(\{p,p\}) \circ (K \times \{p\}) \in \mathcal{E}$. Clearly, $L \times K \subset (L \times \{p\}) \circ (\{p,p\}) \circ (K \times \{p\})$.

$(h)$ Let $L$ and $K$ be bounded subsets $X$ and choose $a \in L$ and $b \in K$. Notice that as $X$ is coarsely connected, $(L \times \{a\}) \circ (\{a,b\}) \circ (K \times \{b\}) \in \mathcal{E}$. Clearly, $L \times K \subset (L \times \{a\}) \circ (\{a,b\}) \circ (K \times \{b\})$, and hence $L \times K \in \mathcal{E}$ as desired.

$(i)$ By induction, it suffices to show that if $L$ and $K$ are bounded, then $L \cup K$ is bounded. Indeed, by part $(h)$, we have $L \times K, K \times L \in \mathcal{E}$, and therefore $(L \cup K) \times (L \cup K) \in \mathcal{E}$. □

**Remark 5.2.10.** Let $(X, \mathcal{E})$ be a coarse space. If $X$ is bounded then $\mathcal{E} = \mathcal{P}(X \times X)$ the power set of $X \times X$. In which follows, unless otherwise stated, $X$ is always assumed to be coarsely unbounded.

## 5.3 Maps between coarse spaces

Chapter 2 of (38) is the main reference of this section.

**Definition 5.3.1.** Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces, and $\phi : X \to Y$ be a map.

- The map $\phi$ is called **coarsely proper** if for each bounded subset $B \subset Y$, then $\phi^{-1}(B)$ is bounded subset of $Y$.
- The map $\phi$ is called **controlled** or **bornologous** if for each $E \in \mathcal{E}_X$, then $(\phi \times \phi)(E) \in \mathcal{E}_Y$.
- The map $\phi$ is called **expanding** if for each $F \in \mathcal{E}_Y$, then $(\phi \times \phi)^{-1}(F) \in \mathcal{E}_X$.
- The map $\phi$ is called **coarse** if it is coarsely proper and controlled.
- The coarse spaces $X$ and $Y$ are called **coarsely equivalent** if there exist coarse maps $\phi : X \to Y$ and $\psi : Y \to X$ such that $\phi \circ \psi$ is close to $id_Y$ and $\psi \circ \phi$ is close to $id_X$. The coarse map $\phi$ is called a **coarse inverse** of the map $\psi$.
- The coarse map $\phi : X \to Y$ is called a **coarse equivalence** if it has a coarse inverse.
- The map $\phi$ is called **coarsely surjective** if there exists $F \in \mathcal{E}_Y$ such that $F[\phi(X)] = Y$.

**Proposition 5.3.2.** *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces and let $\varphi : X \to Y$ be a map. $\varphi : X \to Y$ is controlled if and only if for all $E \in \mathcal{E}_X$, there exists $F \in \mathcal{E}_Y$ such that $\varphi(E[x]) \subset F[\varphi(x)]$ for all $x \in X$.*

*Proof.* Assume that $\varphi : X \to Y$ is controlled and that $E \in \mathcal{E}_X$; then we have $F := \varphi \times \varphi(E) \in \mathcal{E}_Y$. Moreover, $\varphi(E[x]) \subset F[\varphi(x)]$ for all $x \in X$.

Conversely, assume that $E \in \mathcal{E}_X$ and choose $F \in \mathcal{E}_Y$ such that $\varphi(E[x]) \subset F[\varphi(x)]$ for all $x \in X$. Notice that $E = \bigcup_{x\in X} E[x] \times \{x\}$ and $F = \bigcup_{y\in Y} F[y] \times \{y\}$. Now, $\varphi \times \varphi(E) = \bigcup_{x\in X} \varphi(E[x]) \times \{\varphi(x)\} \subset \bigcup_{x\in X} F[\varphi(x)] \times \{\varphi(x)\} \subset \bigcup_{y\in Y} F[y] \times \{y\} = F$. Hence, $\varphi : X \to Y$ is controlled. □

**Lemma 5.3.3.** *Let $\varphi : X \to Y$ and $\psi : Y \to X$ be maps, and consider the subsets $E_{\varphi\circ\psi} := \{(\varphi \circ \psi(y), y) : y \in Y\}$ and $E_{\psi\circ\varphi} := \{(\psi \circ \varphi(x), x) : x \in X\}$. Then:*

*(a) $E_{\varphi\circ\psi} \subset (\psi \times \psi)^{-1}(E_{\psi\circ\varphi})$ and $E_{\psi\circ\varphi} \subset (\varphi \times \varphi)^{-1}(E_{\varphi\circ\psi})$.*

*(b) For all $E \subset X \times X$ and $F \subset Y \times Y$, $(\varphi \times \varphi)^{-1}(F) \subset E^{-1}_{\psi\circ\varphi} \circ (\psi \times \psi)(F) \circ E_{\psi\circ\varphi}$ and $(\psi \times \psi)^{-1}(E) \subset E^{-1}_{\varphi\circ\psi} \circ (\varphi \times \varphi)(E) \circ E_{\varphi\circ\psi}$.*

*(c) For all $E \subset X \times X$ and $F \subset Y \times Y$, $(\varphi \times \varphi)(E) \subset (\psi \times \psi)^{-1}(E_{\psi\circ\varphi} \circ E \circ E^{-1}_{\psi\circ\varphi})$ and $(\psi \times \psi)(F) \subset (\varphi \times \varphi)^{-1}(E_{\varphi\circ\psi} \circ F \circ E^{-1}_{\varphi\circ\psi})$.*

*Proof.* Straightforward calculations. □

**Lemma 5.3.4.** *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces and let $\varphi : X \to Y$ and $\psi : Y \to X$ be maps such that $\varphi \circ \psi$ is close to $id_Y$. Then:*

*(a) If $\varphi : X \to Y$ is controlled, then $\psi : Y \to X$ is expanding.*

*(b) If $\varphi : X \to Y$ is expanding if and only if $\psi : Y \to X$ is controlled and $\psi \circ \varphi$ is close to $id_X$.*

*Proof.* $(a)$ Assume that $\varphi : X \to Y$ is controlled, and let $E \in \mathcal{E}_X$, then by 5.3.3 $(b)$, we have $(\psi \times \psi)^{-1}(E) \subset E^{-1}_{\varphi\circ\psi} \circ (\varphi \times \varphi)(E) \circ E_{\varphi\circ\psi}$, and hence $(\psi \times \psi)^{-1}(E) \in \mathcal{E}_Y$.

$(b)$ Assume that $\varphi : X \to Y$ is expanding, and let $F \in \mathcal{E}_Y$, then by 5.3.3 $(c)$, $(\psi \times \psi)(F) \subset (\varphi \times \varphi)^{-1}(E_{\varphi\circ\psi} \circ F \circ E^{-1}_{\varphi\circ\psi}) \in \mathcal{E}_X$. Moreover, by 5.3.3 $(a)$, $E_{\psi\circ\varphi} \subset (\varphi \times \varphi)^{-1}(E_{\varphi\circ\psi}) \in \mathcal{E}_X$. □

**Corollary 5.3.5.** *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces and $\varphi : X \to Y$ be a map. The following are equivalent:*

*(a) $\varphi : X \to Y$ is a coarse equivalence.*

*(b) $\varphi : X \to Y$ is controlled and there exists a controlled map $\psi : Y \to X$ such that $\psi \circ \varphi$ close to $id_X$ and $\varphi \circ \psi$ is close to $id_Y$.*

*(c) $\varphi : X \to Y$ is controlled, expanding and $\varphi \circ \psi$ is close to $id_Y$ for some map $\psi : Y \to X$.*

*(d) $\varphi : X \to Y$ is controlled, expanding and coarsely surjective.*

*Proof.* $(c) \implies (d)$ : Since $\varphi \circ \psi$ is close to $id_Y$, $E^{-1}_{\varphi\circ\psi} \in \mathcal{E}_Y$; and as $E^{-1}_{\varphi\circ\psi}[\varphi(X)] = Y$, $\varphi : X \to Y$ is coarsely surjective.

$(d) \implies (a)$ : Choose $F \in \mathcal{E}_Y$ such that $F[\varphi(X)] = Y$. Notice that for every $y \in Y$, there exists $x_y \in X$ such that $(y, \varphi(x_y)) \in F$. Define a map $\psi : Y \to X$ by $\psi(y) := x_y$. Notice that for all $y \in Y$, $(y, \varphi \circ \psi(y)) = (y, \varphi(x_y)) \in F$, and therefore, $\varphi \circ \psi$ is close to $id_Y$. By 5.3.4 $(a)$, $\psi : Y \to X$ is expanding, and a fortiori, coarsely proper. By 5.3.4 $(b)$, $\psi : Y \to X$ is controlled and $\psi \circ \varphi$ is close to $id_X$. The other implications follow directly from 5.3.4. □

**Example 5.3.6.** *If $G$ is a countable group equipped with two proper, left-invariant metrics $d_1$ and $d_2$; then $(G, d_1)$ and $(G, d_2)$ are coarsely equivalent.*

*Proof.* The identity map $id : (G, d_1) \to (G, d_2)$ is a coarse equivalence. Indeed, notice that for $r > 0$, $B_{d_1}(1_G, 2r)$ is finite and therefore we can find $s > 0$ such that $B_{d_1}(1_G, 2r) \subset B_{d_2}(1_G, s)$. Now, for all $x, y \in G$, if $d_1(x, y) < r$, then $d_1(1_G, x^{-1} \cdot y) < r$ and hence $x^{-1} \cdot y \in B_{d_1}(1_G, 2r) \subset B_{d_2}(1_G, s)$ which implies that $d_2(x, y) < s$. Thus, the identity map $id : (G, d_1) \to (G, d_2)$ is coarse continuous and consequently a coarse equivalence. □

**Example 5.3.7.** *Let $G$ be a group equipped with a left-invariant metric $d$. If $H$ is a subgroup of finite index in $G$, then $G$ and $H$ are coarsely equivalent.*

*Proof.* The inclusion map $\varphi : H \to G$ is clearly a coarse map. Suppose that $[G : H] = n$ and let $\{g_1 = 1_G, g_2, \ldots, g_n\}$ be a subset of $G$ such that $G$ is the disjoint union of the right cosets $H, H \cdot g_2, \ldots, H \cdot g_n$. Define the map $\psi : G \to H$ by $\psi(x \cdot g_i) = x$ for all $x \in H$ and $1 \leq i \leq n$. Then $\psi : G \to H$ is a coarse map. Furthermore, $\psi \circ \varphi = id_H$ and $d(\varphi \circ \psi(x \cdot g_i), x \cdot g_i) \leq M$ for all $x \in H$ and $1 \leq i \leq n$, where $M = \max\{d(g_i, 1_G) : 1 \leq i \leq n\}$. □

## 5.4 Coarse components and coarse path components

**Lemma 5.4.1.** *Let $(X, \mathcal{E})$ be a coarse space and define a relation $\sim$ on $X$ by $x \sim y$ if and only if $(x, y) \in E$ for some $E \in \mathcal{E}$. The relation $\sim$ on $X$ is an equivalence relation.*

*Proof.* Straightforward. □

**Notation 5.4.2.** *If $(X, \mathcal{E})$ is a coarse space, the equivalence class of $x \in X$ is called the* ***coarse connected component*** *of $x$ and denoted by $C_x$. Clearly, $X$ is coarsely connected if and only if $X = C_x$ for some $x \in X$ .*

**Definition 5.4.3.** Let $(X, \mathcal{E})$ be a coarse space and $E \in \mathcal{E}$ be a symmetric entourage, and $x, y \in X$:

• $x$ and $y$ are said to be $E$**-related** if $(x, y) \in E$.

• $x$ and $y$ are said to be $E$**-connected** and we write $x \sim_E y$ if there exists a finite sequence $x_1, \ldots, x_n \in X$ such that $x = x_1$, $y = x_n$ and $(x_i, x_{i+1}) \in E$ for all $1 \leq i \leq n-1$. Clearly, $x \sim_E y$ if and only if there exists a finite sequence $x_1, \ldots, x_n \in X$ such that $x = x_1$, $y = x_n$ and $E[x_i] \cap E[x_{i+1}] \neq \emptyset$ for all $1 \leq i \leq n-1$ if and only if $y \in E^n[x]$ for some $n \in \mathbb{N}$.

• $X$ is said to be $E$**-connected** if for all $x, y \in X$, $x$ and $y$ are $E$-connected. It is obvious that if $X$ is $E$-connected then it is coarsely connected.

**Lemma 5.4.4.** *Let $(X, \mathcal{E})$ be a coarse space and $E \in \mathcal{E}$ be a symmetric entourage containing $\Delta$. The relation $\sim_E$ on $X$ is an equivalence relation.*

*Proof.* Straightforward. □

**Notation 5.4.5.** *Let $(X, \mathcal{E})$ be a coarse space and $E \in \mathcal{E}$ be a symmetric entourage containing $\Delta$. The equivalence class of $x \in X$ is denoted by $C_x^E$ and called the* ***coarse $E$-path component*** *or just the* ***$E$-component*** *of $x \in X$. It can be easily seen that $C_x^E = \bigcup_{n \in \mathbb{N}} E^n[x]$.*

In the special case when $(X, d)$ is a metric space equipped with the bounded coarse structure induced by $d$, for $r > 0$, we say that $x, y \in X$ are $r$-connected instead of $E_r$-connected and the $E_r$-component is referred to as the $r$-component.

**Definition 5.4.6.** Let $(X, \mathcal{E})$ be a coarse space, $E \in \mathcal{E}$ and $A \subset X$:

• $A$ is called $E$**-coarsely dense** if $E[A] = X$ and it is called **coarsely dense** if it is $E$-coarsely dense for some $E \in \mathcal{E}$.

• $A$ is called $E$**-separated** if $(a, b) \notin E$ for all distinct elements $a, b \in A$.

• $A$ is called **extremely** $E$**-separated** if $E[a] \cap E[b] = \emptyset$ for all distinct elements $a, b \in A$.

**Proposition 5.4.7.** *(a) If $(X, \mathcal{E})$ is a coarse space, a subset $A \subset X$ is coarsely dense if and only if $A$ is coarsely equivalent to $X$.*

*(b) Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces. The map $\varphi : X \to Y$ is coarsely surjective if and only if $\varphi(X)$ is coarsely dense subset of $Y$.*

*Proof.* Straightforward. $\square$

Given a coarse space $(X, \mathcal{E})$ and $E \in \mathcal{E}$ we aim to find subsets $A$ and $B$ such that $A$ is coarsely dense $E$-separated, and $B$ is coarsely dense extremely $E$-separated.

**Proposition 5.4.8.** *Let $(X, \mathcal{E})$ be a coarse space, and $E \in \mathcal{E}$ be symmetric entourage containing $\Delta$. Then:*
*(a) $X$ contains a coarsely dense $E$-separated subset $A$.*
*(b) $X$ contains a coarsely dense extremely $E$-separated subset $A$.*

*Proof.* $(a)$ Let $\mathcal{A} := \{Y \subset X : Y \text{ is } E\text{-separated}\}$. As every singleton is vacuously $E$-separated, $\mathcal{A} \neq \emptyset$. Partially order $\mathcal{A}$ by inclusion. Let $\mathcal{C} \subset \mathcal{A}$ be a chain and put $B := \bigcup\limits_{C \in \mathcal{C}} C$. Clearly, $B$ is an upper bound of $\mathcal{C}$ and $B \in \mathcal{A}$. Indeed, if $x, y \in B$ are distinct elements, then there is $C \in \mathcal{C}$ such that $x, y \in C$ and hence $(x, y) \notin E$. By Zorn's lemma, $\mathcal{A}$ contains a maximal element $A$. We claim that $E[A] = X$. To this end, assume that there exists $x \in X \setminus E[A]$, then $x \notin A$ and for all $a \in A$, $(x, a) \notin E$. Hence, $A \cup \{x\}$ is an $E$-separated subset containing $A$ properly, a contradiction. Therefore, $E[A] = X$ as wanted.
$(b)$ Let $\mathcal{A} := \{Y \subset X : Y \text{ is extremely } E\text{-separated}\}$. As every singleton is vacuously extremely $E$-separated, $\mathcal{A} \neq \emptyset$. As in part $(a)$, $\mathcal{A}$ contains a maximal element $A$. We need to show that $A$ is coarsely dense. Take $F = E \circ E$, then $F[A] = X$. Indeed, if $x \in X \setminus F[A]$, then $x \notin E[A]$ and hence for all $a \in A$, $(x, a) \notin E$. Notice that $E[x] \cap E[a] = \emptyset$ as if $y \in E[x] \cap E[a]$, then $(x, y), (y, a) \in E$ and hence $x \in E[A]$, a contradiction. Now, as $x \notin A$, $A \cup \{x\}$ is extremely $E$-separated containing $A$ properly, a contradiction. Thus, we must have $F[A] = X$. $\square$

## 5.5 Proper coarse spaces

A proper coarse space $X$ is set that equipped with a coarse structure and a topology where the coarse structure and the topology are compatible in some sense. The concept of being a proper coarse space generalizes the concept of being a proper metric space.

### 5.5.1 Topological coarse spaces

**Definition 5.5.1.** A topology on $X$ is **compatible** with the coarse structure on $X$ if there is a uniformly bounded cover of $X$ consisting of open subsets of $X$.

**Observation 5.5.2.** *The simplest non-trivial topology compatible with a coarse structure is the discrete topology.*

**Definition 5.5.3.** A set $X$ that is equipped with a topology $\mathcal{T}$ and a coarse structure $\mathcal{E}$ simultaneously. We say that $X$ is a **topological coarse space** if $\mathcal{T}$ is **compatible** with $\mathcal{E}$.

**Example 5.5.4.** *Any $(X, \mathcal{E})$ coarse space equipped with the discrete topology is a topological coarse space.*

The topology of a topological coarse space admits a basis consists of coarsely bounded subsets.

**Lemma 5.5.5.** *Let $(X, \mathcal{T})$ be a topological space equipped with a coarse structure $\mathcal{E}$. Then:*
*(a) If $X$ has a uniformly bounded open cover if and only if $X$ has a uniformly bounded cover that is a basis for its topology.*
*(b) $X$ has a uniformly bounded open cover if and only if the diagonal $\Delta$ has an open neighborhood in $\mathcal{E}$.*

*Proof.* $(a)$ Let $\mathcal{U}$ be a uniformly bounded open cover of $X$. That is, $E_{\mathcal{U}} = \bigcup\limits_{U \in \mathcal{U}} U \times U$ is an entourage, $X = \bigcup\limits_{U \in \mathcal{U}} U$ and every $U \in \mathcal{U}$ is coarsely bounded and open subset of $X$. For every $O \in \mathcal{T}$ and $U \in \mathcal{U}$, put $O_U = O \cap U$; and notice that $\bigcup\limits_{O \in \mathcal{T}} O_U = U \times U$. Put $\mathcal{B} := \{O_U : O \in \mathcal{T} \text{ and } U \in \mathcal{U}\}$. Clearly, $\mathcal{B}$ is a basis and it is uniformly bounded cover as $E_{\mathcal{B}} = E_{\mathcal{U}}$ is an entourage.
$(b)$ Let $\mathcal{U}$ be a uniformly bounded open cover of $X$. Clearly, $E_{\mathcal{U}} = \bigcup\limits_{U \in \mathcal{U}} U \times U$ is an open entourage containing $\Delta$.

Conversely, let $E$ be an open entourage containing $\Delta$. A basis element of $X \times X$ is of the form $U \times V$ for some $U, V \in \mathcal{T}$. In particular, there exists a subfamily $\mathcal{T}' \subset \mathcal{T}$ such that $E = \bigcup\limits_{U,V \in \mathcal{T}'} U \times V$. Put $\mathcal{U} := \{U \cap V : U, V \in \mathcal{T}'\}$, then $\mathcal{U}$ is a uniformly bounded open cover of $X$. $\square$

**Proposition 5.5.6.** *Let $(X, \mathcal{T})$ be a topological space equipped with a coarse structure $\mathcal{E}$. $X$ is a topological coarse space if and only if its coarse structure $\mathcal{E}$ admits a basis consists of open subsets of $X \times X$.*

*Proof.* Assume that $X$ is a topological coarse space, and $F \in \mathcal{E}$, be an open neighborhood of $\Delta$. For every $E \in \mathcal{E}$, put $E' = F \circ E \circ F$. Obviously, $E \subset E'$. It is left to show that $E'$ is an open subset of $X \times X$. Indeed, let $(x, y) \in E'$, then there exist $a, b \in X$ such that $(x, a) \in F$, $(a, b) \in E$ and $(b, y) \in F$. Pick open neighborhoods $U_x, U_a, U_b, U_y \subset X$ of $x$, $a$, $b$ and $y$, respectively; such that $(x, a) \in U_x \times U_a \subset F$ and $(b, y) \in U_b \times U_y \subset F$. Then $(x, y) \in U_x \times U_y \subset E'$, which shows that $E'$ is an open entourage containing $E$. The family $\{E' : E \in \mathcal{E}\}$ is a basis consists of open subsets of $X \times X$. □

**Proposition 5.5.7.** *If $(X, \mathcal{E})$ is a topological coarse space, then it contains a closed discrete coarsely dense subset $D$.*

*Proof.* By 5.5.5, we can choose a uniformly bounded cover $\mathcal{U}$ of $X$ that is a basis for its topology. Put $E_{\mathcal{U}} = \bigcup\limits_{U \in \mathcal{U}} U \times U$. By 5.4.8, choose $D$ to be a coarsely dense extremely $E_{\mathcal{U}}$-separated subsets of $X$. By construction, each $U \in \mathcal{U}$ contains at most one element of $D$. Indeed, if $a, b \in D$ are distinct contained in some $U \in \mathcal{U}$, then $a \in E_{\mathcal{U}}[a] \cap E_{\mathcal{U}}[b]$, a contradiction. Now, for $x \in D$, choose a basis element $U \in \mathcal{U}$ containing $x$, then $U \cap D = \{x\}$, and hence $D$ is discrete. If $x \in X \setminus D$, choose a basis element $U \in \mathcal{U}$ containing $x$. Without loss of generality, assume that $U \cap D = \{a\}$, then $V = U \setminus \{a\}$ is an open subset of $X$ containing $x$ and $V \cap D = \emptyset$. Thus, $D$ is a closed subset of $X$. □

Notice that coarse connectedness of a topological coarse space dose not imply that it is topologically connected; for instance, take a non-empty set $X$ and equip it with the coarse structure $\mathcal{E} = \mathcal{P}(X)$ and the discrete topology. Then $X$ is coarsely connected but not topologically connected. However, if a topological coarse space is topologically connected, then it is coarsely connected. To prove that fact, we need the following lemma.

**Lemma 5.5.8.** *Let $(X, \mathcal{E})$ is a topological coarse space, then the coarse connected component $C_x$ of $x \in X$ is an open subset of $X$ for all $x \in X$.*

*Proof.* Notice that if $E \subset X \times X$ is an open subset and $A \subset X$, then $E[A]$ is an open subset of $X$. Indeed, if $x \in E[A]$, then $(x, a) \in E$ for some $a \in A$. Let $U_x$ and $U_a$ be open neighborhoods of $x$ and $a$, respectively, such that $U_x \times U_a \subset E$. Then $U_x \subset E[A]$ which proves that $E[A]$ is an open subset of $X$. Now, notice that if $E \in \mathcal{E}$, then $E[C_x] \subset C_x$. In particular, if $E$ is an entourage containing the the diagonal $\Delta$, then $E[C_x] = C_x$. Now, let $F$ be an open entourage containing $\Delta$, then as $C_x = F[C_x]$, $C_x$ is an open subset of $X$ as wanted. $\square$

**Corollary 5.5.9.** *If $X$ is a topological coarse space that is topologically connected, then it is coarsely connected.*

*Proof.* It follows from the fact that coarse connected components are open and partition $X$. $\square$

**Lemma 5.5.10.** *Let $(X, \mathcal{E})$ be a topological coarse space and let $\mathcal{U}$ be a uniformly bounded open cover of $X$. Let $E = \bigcup_{U \in \mathcal{U}} U \times U$, then the $E$-connected component $C_x^E$ of $x \in X$ is a clopen (closed and open) subset of $X$ for all $x \in X$.*

*Proof.* Since $C_x^E = \bigcup_{n \in \mathbb{N}} E^n[x]$, $C_x^E$ is open. Let $y \in cl(C_x^E)$ and $U_y \in \mathcal{U}$ be a neighborhood of $y$. Then $U_y$ intersects $E^n[x]$ for some $n \in \mathbb{N}$; hence $U_y \subset E^{n+1}[x]$, which implies that $C_x^E$ is closed.

$\square$

**Corollary 5.5.11.** *Let $(X, \mathcal{E})$ be a topological coarse space. If $X$ is topologically connected, then $X$ is $E_{\mathcal{U}}$-connected for any uniformly bounded open cover $\mathcal{U}$ of $X$, where $E_{\mathcal{U}} = \bigcup_{U \in \mathcal{U}} U \times U$.*

*Proof.* The result follows from the fact $C_x^{E_{\mathcal{U}}}$ is a non-empty clopen subset of $X$ for any $x \in X$. $\square$

### 5.5.2 Proper coarse spaces

**Definition 5.5.12.** Let $(X, \mathcal{T})$ be a topological space equipped with a coarse structure $\mathcal{E}$. $X$ is called **coarsely locally compact** space if every coarsely bounded subset of $X$ is a relatively compact subset.

**Definition 5.5.13.** A topological coarse space $(X, \mathcal{E})$ is called a **proper coarse space** if it is coarsely locally compact.

**Example 5.5.14.** *Let $(X, d)$ be a metric space. For $r > 0$, set $E_r = \Big\{(x, y) \in X \times X : d(x, y) < r\Big\}$. The collection $\mathcal{E} = \Big\{E_r : r > 0\Big\}$ is a coarse structure which is called the* ***bounded coarse structure****. Furthermore, the coarse space $(X, \mathcal{E})$ is proper coarse space if and only if the metric space $(X, d)$ is proper metric space.*

**Proposition 5.5.15.** *If $(X, \mathcal{E})$ is a proper coarse space, then $X$ is locally compact.*

*Proof.* Since $(X, \mathcal{E})$ is a topological coarse space, then by 5.5.5, its topology admits a basis $\mathcal{B}$ consists of coarsely bounded subsets. As $(X, \mathcal{E})$ is coarsely locally compact, each element of $\mathcal{B}$ is relatively compact. Hence, $X$ is locally compact. □

**Proposition 5.5.16.** *Let $(X, \mathcal{E})$ be a proper coarse space. If $X$ is coarsely connected, then a subset of $X$ is coarsely bounded if and only if it is relatively compact.*

*Proof.* Since $(X, \mathcal{E})$ is a topological coarse space, then by 5.5.5, its topology admits a basis $\mathcal{B}$ consists of coarsely bounded subsets. Now, every relatively compact subset is covered by finitely many elements of $\mathcal{B}$, then by 5.2.9 $(h)$, it is coarsely bounded. □

Roe, in his book "Lectures On Coarse Geometry" ((36)), defines a proper coarse space $X$ to be a Hausdorff paracompact topological space equipped with a coarse structure such that every bounded subset is relatively compact and the diagonal is contained in an open entourage. In this section, we show that the paracompactness hypothesis is redundant by showing that every proper coarse space in the sense of Definition 5.5.13 is indeed paracompact. Moreover, if the proper coarse space is $\mathcal{U}$-connected for some uniformly bounded open cover $\mathcal{U}$ of $X$, then $X$ is $\sigma$-compact.

**Lemma 5.5.17.** *If $(X, \mathcal{E})$ is a proper coarse space with $\mathcal{U}$ is a uniformly bounded open cover, the $E_{\mathcal{U}}$-connected component $C_x^{E_{\mathcal{U}}}$ of $x \in X$ is $\sigma$-compact.*

*Proof.* Since $E_{\mathcal{U}}^n[x])$ is bounded for all $n \in \mathbb{N}$, $E_{\mathcal{U}}^n[x])$ is relatively compact for all $n \in \mathbb{N}$ and hence $cl(E_{\mathcal{U}}^n[x])$ is compact for all $n \in \mathbb{N}$. Now, as $C_x^{E_{\mathcal{U}}} = \bigcup\limits_{n \in \mathbb{N}} E_{\mathcal{U}}^n[x]$ is a closed subset of $X$, we have $C_x^{E_{\mathcal{U}}} = \bigcup\limits_{n \in \mathbb{N}} cl(E_{\mathcal{U}}^n[x])$, which proves that $C_x^{E_{\mathcal{U}}}$ is $\sigma$-compact. □

**Definition 5.5.18.** Given a topological space $X$ and collections $\mathcal{U}$, $\mathcal{V}$ of its subsets:

• $\mathcal{U}$ is called **locally finite** if for every $x \in X$, there exist a neighborhood $U_x$ of $x$ that intersects only finitely many elements of $\mathcal{U}$.

• $\mathcal{U}$ is said to be a refinement of $\mathcal{V}$ if for every $U \in \mathcal{U}$, there exists $V \in \mathcal{V}$ such that $U \subset V$.

• $X$ called **paracompact** if every open cover $\mathcal{U}$, there is a locally finite open cover $\mathcal{V}$ that refines $\mathcal{U}$.

Proper coarse spaces are locally compact. It is well-known that if a locally compact space is $\sigma$-compact, then it is paracompact. Moreover, disjoint union of open paracompact subspaces of a topological space is paracompact.

**Corollary 5.5.19.** *Proper coarse spaces are paracompact.*

*Proof.* Let $(X, \mathcal{E})$ be a proper coarse space and let $\mathcal{U}$ be a uniformly bounded open cover of $X$, and $C_x^{E_\mathcal{U}}$ be the $E_\mathcal{U}$-connected component of $x \in X$. Then $X$ can be written as a disjoint union of open sets $X = \bigcup_{x\in X} C_x^{E_\mathcal{U}}$ each of which is $\sigma$-compact, and hence $X$ is paracompact. $\square$

A topological space is called **Lindelöf** if every open cover of $X$ admits a countable subcover. $\sigma$-compact space is an example of a Lindelöf space.

**Corollary 5.5.20.** *Let $(X, \mathcal{E})$ be a proper coarse space. If $X$ is $E_\mathcal{U}$-connected for some uniformly bounded open cover $\mathcal{U}$ of $X$, then $X$ admits a countable uniformly bounded open cover.*

*Proof.* For $x \in X$, $X = C_x$ and hence $X$ is $\sigma$-compact. Thus, $X$ is Lindelöf. $\square$

### 5.5.3 Coarse structures induced by a compactification

**Notation 5.5.21.** *Let $Y$ be a compactification of $X$:*

• *For $E \subset X \times X$ let $\overline{E}$ denote the closure of $E$ in $Y \times Y$.*

• *The corona of $Y$ and its diagonal are denoted by $\partial_Y X$ and $\Delta_{\partial_Y}$, respectively.*

• $\mathcal{E}_Y := \{E \subset X \times X : \overline{E} \subset X \times X \cup \Delta_{\partial_Y}\}$.

We aim to show that for a topological space $X$ and a compactification $Y$, the family $\mathcal{E}_Y$ is a coarse structure on $X$. The following lemma will be used to that end.

**Lemma 5.5.22.** *Let $Y$ be a compactification of $X$, and $E \subset X \times X$. The following are equivalent:*
*(a) $E \in \mathcal{E}_Y$.*
*(b) For every net $(x_\alpha, y_\alpha)_{\alpha\in I} \in E$, if either $(x_\alpha)_{\alpha\in I}$ or $(y_\alpha)_{\alpha\in I}$ converges to an infinity point $\omega \in \partial_Y X$ then the other net converges to the same infinity point $\omega \in \partial_Y X$.*
*(c) If $\omega \in \partial_Y X$, then every neighborhood $V \subset Y$ of $\omega$ contains a smaller neighborhood $U \subset Y$ of $\omega$ such that $E \cap (U \times (X \setminus V)) = \emptyset$.*

*Proof.* $(a) \iff (b)$: Obvious.
$(b) \implies (c)$: Let $E \subset X \times X$ such that every net $(x_\alpha, y_\alpha)_{\alpha\in I} \in E$ and $\omega \in \partial_Y X$; $(x_\alpha)_{\alpha\in I}$ converges to $\omega \in \partial_Y X$ if and only if $(y_\alpha)_{\alpha\in I}$ converges to $\omega \in \partial_Y X$. Seeking contradiction, assume that there exists a neighborhood $V \subset Y$ of $\omega \in \partial_Y X$ such that $E \cap (U \times (X \setminus V)) \neq \emptyset$ for every neighborhood $U \subset V$ of $\omega \in \partial_Y X$. For every neighborhood $U \subset V$ of $\omega \in \partial_Y X$, choose $(x_U, y_U) \in E \cap (U \times (X \setminus V))$. Note that the set $\mathcal{N}_x^V := \{U \subset X : U$ is a neighborhood of $x$ and $U \subset V\}$ ordered by reverse inclusion (i.e. $U \leq U'$ if and only if $U' \subset U$) is a directed set. Thus, $(x_U, y_U)_{U\in\mathcal{N}_x^V}$ is a net. However, $(x_U)_{U\in\mathcal{N}_x^V}$ converges to $\omega$ and $(y_U)_{U\in\mathcal{N}_x^V}$ does not converges to $\omega$, a contradiction.
$(c) \implies (b)$: Let $(x_\alpha, y_\alpha)_{\alpha\in I} \in E$ be a net with $(x_\alpha)_{\alpha\in I}$ converges to $\omega \in \partial_Y X$. Pick an arbitrary neighborhood $V \subset Y$ of $\omega$. We aim to find $\lambda \in I$ such that $y_\alpha \in V$ for all $\alpha \geq \lambda$. To this end, choose a neighborhood $U \subset V$ of $\omega$ such that $E \cap (U \times (X \setminus V)) = \emptyset$ and pick $\lambda \in I$ such that $x_\alpha \in U$ for all $\alpha \geq \lambda$, then $y_\alpha \in V$ for all $\alpha \geq \lambda$, as desired. □

**Theorem 5.5.23.** *Let $Y$ be a compactification of $X$; then $\mathcal{E}_Y$ is a coarse structure on $X$ and the coarse space $(X, \mathcal{E}_Y)$ is coarsely locally compact and coarsely connected.*

*Proof.* **Claim 1:** $\mathcal{E}_Y$ is a coarse structure on $X$.
**Proof of Claim 1:** It is not hard to see that $\Delta_X \in \mathcal{E}_Y$. Furthermore, if $E, F \in \mathcal{E}_Y$ and $E' \subset E$, then $E', E^{-1}, E \cup F \in \mathcal{E}_Y$. We need to show that $E \circ F \in \mathcal{E}_Y$. To this end, let $(x_\alpha, z_\alpha)_{\alpha\in I} \in E \circ F$ be a net such that $(x_\alpha)_{\alpha\in I}$ converges to an infinity point $\omega \in \partial_Y X$. Let $(y_\alpha)_{\alpha\in I}$ be a net such that $(x_\alpha, y_\alpha) \in E$ and $(y_\alpha, z_\alpha) \in F$ for all $\alpha \in I$. On one hand, since

$(x_\alpha, y_\alpha)_{\alpha\in I} \in E$ and $(x_\alpha)_{\alpha\in I}$ converges to $\omega \in \partial_Y X$; then, by 5.5.22 $(b)$, $(y_\alpha)_{\alpha\in I}$ converges to $\omega \in \partial_Y X$. On the other hand, since $(y_\alpha, z_\alpha)_{\alpha\in I} \in F$ and $(y_\alpha)_{\alpha\in I}$ converges to $\omega \in \partial_Y X$, $(z_\alpha)_{\alpha\in I}$, by 5.5.22 $(b)$, must converge to $\omega \in \partial_Y X$. By 5.5.22, $E \circ F \in \mathcal{E}_Y$.

**Claim 2:** $(X, \mathcal{E}_Y)$ is coarsely locally compact and coarsely connected.

**Proof of Claim 2:** Notice that for every $(x, y) \in X \times X$, then $\overline{\{(x, y)\}} = \{(x, y)\}$ which show that $(X, \mathcal{E}_Y)$ is coarsely connected. Now, let $B \subset X$ be bounded. We aim to show that $B$ is relatively compact, i.e. $cl_X(B)$. Since $cl_Y(B)$ is compact subset of $Y$, it suffices to show that $cl_Y(B) \subset X$. Seeking contradiction, assume that $\omega \in cl_Y(B) \setminus X$. As $B \subset X$ is bounded, $B \times \{b\} \in \mathcal{E}_Y$, where $b \in B$. Since $V = Y \setminus \{b\}$ is an open neighborhood of $\omega$, so by 5.5.22 $(c)$, there exists an open neighborhood $U \subset V$ of $\omega$ such that $B \times \{b\} \cap (U \times \{b\}) = \emptyset$. Notice that $X \setminus V = \{b\}$ and therefore $B \times \{b\} \cap (U \times \{b\}) = \emptyset$ implying that $U \cap B = \emptyset$, which contradicts the fact that $\omega \in cl_Y(B)$. Hence, $cl_Y(B) \subset X$ and thus $cl_X(B) = cl_Y(B)$ is compact. This proves that the coarse space $(X, \mathcal{E}_Y)$ is coarsely locally compact. □

**Definition 5.5.24.** If $Y$ is a compactification of $X$, the coarse structure $\mathcal{E}_Y$ is called the **coarse structure induced by** the compactification $Y$. In (36), $\mathcal{E}_Y$ is referred to as the topological coarse structure associated to the compactification $Y$.

If $Y$ is a metrizable compactification of a locally compact space $X$, then the coarse space $(X, \mathcal{E}_Y)$ is a proper coarse space. We need the following fact.

**Lemma 5.5.25.** *Let $A$ be a subset of a metric space $(X, d)$. The function $f_A : X \to [0, \infty)$ given by $f_A(x) = d(x, A)$ where:*

$$d(x, A) := \inf_{a\in A}\{d(x, a)\},$$

*is uniformly continuous and $cl(A) = \{x \in X : f_A(x) = 0\}$. In particular, if $A$ is a closed subset, then $f_A$ vanishes exactly on $A$ and takes positive value outside $A$; namely, $f_A(A) = \{0\}$ and $f_A(x) > 0$ for all $x \in X \setminus A$.*

*Proof.* Let $\varepsilon > 0$ and take $\delta = \varepsilon$ and assume that $x, y \in X$ such that $d(x,y) < \delta$. For arbitrary $a \in A$, we have $d(x,a) - d(y,a) \leq d(x,y)$ and therefore

$$\inf_{a \in A}\{d(x,a)\} - \inf_{a \in A}\{d(y,a)\} = \inf_{a \in A}\{d(x,a) - d(y,a)\} \leq d(x,y)$$

which implies that $f_A(x) - f_A(y) \leq d(x,y) < \delta$. Similarly, $f_A(y) - f_A(x) < \delta$. Now, put $Z = \{x \in X : f_A(x) = 0\}$, and assume that $x \in Z$. If $x \notin cl(A)$, then we can find $r > 0$ such that $B(x,r) \cap A = \emptyset$ and hence $d(x,a) \geq r$ for all $a \in A$ which implies that $\inf\limits_{a \in A}\{d(x,a)\} \geq r$, a contradiction. Conversely, let $x \in cl(A)$ and choose a sequence $(x_n)_{n \in \mathbb{N}} \subset A$ converging to $x$. One one hand we have $d(x_n, A) = 0$ for all $n \in \mathbb{N}$; on the other hand, as $f_A$ is continuous, the zero sequence $d(x_n, A)$ converges to $d(x, A)$. Thus, $d(x, A) = 0$. □

**Proposition 5.5.26.** *If $Y$ is a metrizable compactification of a locally compact space $X$, then the coarse structure $\mathcal{E}_Y$ induced by $Y$ is a proper coarse structure.*

*Proof.* By 5.5.23, $(X, \mathcal{E}_Y)$ is coarsely locally compact. To show that $(X, \mathcal{E}_Y)$ is a topological coarse space, we need to find an open entourage $E$ containing $\Delta_X$. Notice that $A = \Delta_Y$ is a closed subset of $Y \times Y$ as $Y$ is Hausdorff; also, as $X$ is locally compact, $\partial_Y X$ is a closed subset of $Y$, and thus $B = \partial_Y X \times Y \cup Y \times \partial_Y X$ is a closed subset of $Y \times Y$. By 5.5.25, we may choose continuous functions $f_A, f_B : Y \times Y \to [0, \infty)$ that vanish exactly on $A$ and $B$, respectively; and $f_A, f_B : Y \times Y \to [0, \infty)$ are positive outside $A$ and $B$, respectively. Now, the subset $E := \{(x,y) \in X \times X : f_A(x,y) < f_B(x,y)\}$ is open and $\Delta_X \subset E$. Moreover, if $(x_n, y_n)_{n \in \mathbb{N}}$ is a sequence in $E$ such that $(x_n)_{n \in \mathbb{N}}$ converges to $\omega \in \partial_Y X$. Since, $f_B : Y \times Y \to [0, \infty)$ is continuous and $f_B(\omega, y_n) = 0$ for all $n \in \mathbb{N}$, and hence $f_B(\omega, \lim\limits_{n \to \infty} y_n) = 0$. Since $0 \leq \lim\limits_{n \to \infty} f_A(\omega, y_n) \leq \lim\limits_{n \to \infty} f_B(\omega, y_n) = 0$, so $f_A(\omega, \lim\limits_{n \to \infty} y_n) = 0$. As $f_A : Y \times Y \to [0, \infty)$ vanishes exactly on $A = \Delta_Y$, we must have $\omega = \lim\limits_{n \to \infty} y_n$, and therefore $E$ is an entourage by 5.5.22. □

## 5.6 Coarse compactifications of proper coarse spaces

### 5.6.1 Higson compactification of proper coarse spaces

The concept of a complex-valued function $f : X \to \mathbb{C}$, on a metric space, to vanish at infinity and to be slowly oscillating can be generalized for complex-valued functions $f : X \to \mathbb{C}$ defined on arbitrary coarse space. When the coarse space is proper and hence locally compact, the Higson compactification can be defined in a similar fashion to 2.3.10 and 2.3.20.

**Definition 5.6.1.** Let $(X, \mathcal{E})$ be a proper coarse space, and let $f : X \to \mathbb{C}$ be a bounded function:

• $f : X \to \mathbb{C}$ is said to **vanish at infinity** if for all $\varepsilon > 0$ there exists a coarsely bounded subset of $K \subset X$ such that $\|f(x)\| < \varepsilon$ for all $x \in X \setminus K$.

• The collection of all continuous bounded functions vanishing at infinity is a $C^*$-subalgebra of $C_b(X, \mathbb{R})$ denoted by $C_0(X, \mathbb{C})$, while the collection of all bounded functions vanishing at infinity is a $C^*$-subalgebra of $B(X, \mathbb{C})$ denoted by $B_0(X, \mathbb{C})$, where $B(X, \mathbb{C})$ is the $C^*$-algebra of all bounded functions.

• $f : X \to \mathbb{C}$ is called a **Higson function** if for each $E \in \mathcal{E}$ and $\varepsilon > 0$ there exists a (coarsely) bounded subset of $K_E \subset X$ such that $\|f(x) - f(y)\| < \varepsilon$ for all $x, y \in X \setminus K$ and $(x, y) \in E$.

• The collection of all bounded Higson functions is a $C^*$-subalgebra of $B(X, \mathbb{C})$ denoted by $B_h(X, \mathbb{C})$, while the collection of all continuous Higson functions is a $C^*$-subalgebra of $C_b(X, \mathbb{R})$ denoted by $C_h(X, \mathbb{C})$.

• The **Higson compactification** of $X$ is defined to be the maximal ideal space of $C_h(X, \mathbb{C})$ and it is denoted by $hX$. The corona of $hX$ is called the **Higson corona** of $hX$ and denoted by $\partial_h X$.

**Lemma 5.6.2.** *Let $(X, \mathcal{E})$ be a coarsely connected proper coarse space. A continuous function $f : X \to \mathbb{C}$ is a Higson function if and only if $\|f(x_\alpha) - f(y_\alpha)\| \to 0$ for any net $(x_\alpha, y_\alpha)_{\alpha \in I}$ contained in some entourage $F$ with either $(x_\alpha)_{\alpha \in I}$ or $(y_\alpha)_{\alpha \in I}$ converges to an infinity point.*

*Proof.* Assume that $\|f(x_\alpha) - f(y_\alpha)\| \to 0$ for any net $(x_\alpha, y_\alpha)_{\alpha \in I}$ contained in some entourage $F$ with either $(x_\alpha)_{\alpha \in I}$ or $(y_\alpha)_{\alpha \in I}$ converges to an infinity point. Seeking contradiction, assume that there exists $E \in \mathcal{E}$ and $\varepsilon > 0$ such that for every bounded subset $K \subset X$, there exists

$(x, y) \in E \cap (X \times X \setminus K \times K)$ such that $\|f(x) - f(y)\| \geq \varepsilon$. Notice that as $(X, \mathcal{E})$ be a coarsely connected proper coarse space, a subset is bounded if and only if it is relatively compact. Fix a bounded subset $K_1 \subset X$ and choose $(x_1, y_1) \in E \cap (X \times X \setminus K_1 \times K_1)$ with $\|f(x_1) - f(y_1)\| \geq \varepsilon$. Since $X$ is locally compact, we can find a relatively compact neighborhood $K_2$ properly containing $cl_X(K_1 \cup \{x_1, y_1\})$. Inductively, we construct a strictly increasing sequence $(K_n)_{n\in\mathbb{N}}$ of open relatively compact subsets and a sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset E$ such that $(x_n, y_n) \in E \cap (X \times X \setminus K_n \times K_n)$, $\|f(x_n) - f(y_n)\| \geq \varepsilon$ and $cl_X(K_n \cup \{x_n, y_n\}) \subset K_{n+1}$ for all $n \in \mathbb{N}$. Notice that the sequence $(x_n)_{n\in\mathbb{N}}$ is unbounded. Indeed, notice that $\bigcup\limits_{n\in\mathbb{N}} K_n$ is a clopen subset as $cl_X(\bigcup\limits_{n\in\mathbb{N}} K_n) = \bigcup\limits_{n\in\mathbb{N}} cl_X(K_n) = \bigcup\limits_{n\in\mathbb{N}} K_n$; now, if $(x_n)_{n\in\mathbb{N}}$ is bounded, then $cl_X(\{x_n : n \in \mathbb{N}\})$ is compact and it is covered by $\bigcup\limits_{n\in\mathbb{N}} K_n$, and hence there exists $N \in \mathbb{N}$ such that $cl_X(\{x_n : n \in \mathbb{N}\}) \subset K_N$, a contradiction. Without loss of generality, we may assume that $(x_n)_{n\in\mathbb{N}}$ converges to an infinity point $\omega \in \partial_h X$, but $\|f(x_n) - f(y_n)\| \not\to 0$, a contradiction. Hence, $f$ is a Higson function as wanted. □

### 5.6.2 Coarse compactifications of proper coarse spaces

**Definition 5.6.3.** A compactification $Y$ of the proper coarse space $(X, \mathcal{E})$ is called a **coarse compactification** if it is dominated by the Higson compactification associated to $\mathcal{E}$.

**Theorem 5.6.4.** *A compactification $Y$ of a coarsely connected proper coarse space $(X, \mathcal{E})$ is a coarse compactification if and only if $\mathcal{E} \subset \mathcal{E}_Y$, where $\mathcal{E}_Y$ is the coarse structure induced by $Y$.*

*Proof.* Assume that $Y$ is a coarse compactification of $(X, \mathcal{E})$ and $E \in \mathcal{E}$. We claim that $E \in \mathcal{E}_Y$. To this end, notice first that as $Y$ is dominated by $hX$, one has $C(Y, \mathbb{C}) \subset C(hX, \mathbb{C})$. Now, assume that $(x_\alpha, y_\alpha)_{\alpha\in I} \in E$ is a net such that $(x_\alpha)_{\alpha\in I}$ converges to an infinity point $\omega \in \partial_Y X$. For arbitrary continuous function $f \in C(Y, \mathbb{C})$, as $f|_X$ is a Higson function, for every $\varepsilon$, we can find a bounded subset $K_E$ such that $\|f(x) - f(y)\| < \varepsilon$ for all $(x, y) \in E \cap (X \times X \setminus K_E \times K_E)$. In particular, $f(\omega) = f(\lim x_\alpha) = f(\lim y_\alpha)$, for all $f \in C(Y, \mathbb{C})$. Since $C(hX, \mathbb{C})$ separates points of $Y$, we must have $\lim y_\alpha = \omega$ and hence, by 5.5.22, $E \in \mathcal{E}_Y$.

Conversely, assume that $\mathcal{E} \subset \mathcal{E}_Y$ and $f \in C(Y, \mathbb{C})$. We aim to show that $f|_X$ is a Higson function. Indeed, assume that $E \in \mathcal{E}$, and pick a net $(x_\alpha, y_\alpha)_{\alpha \in I} \in E$ with $(x_\alpha)_{\alpha \in I}$ converges to an infinity point $\omega \in \partial_Y X$. Since $E \in \mathcal{E}_Y$, hence $(y_\alpha)_{\alpha \in I}$ must converge to $\omega \in \partial_Y X$ and thus $\|f(x_\alpha) - f(y_\alpha)\| \to 0$. Therefore, by 5.6.2, $f|_X$ is a Higson function. □

**Theorem 5.6.5.** *Let $Y$ be a metrizable compactification of a locally compact space $X$. The Higson compactification of the proper coarse space $(X, \mathcal{E}_Y)$ is $Y$ itself.*

*Proof.* To that end, it suffices to prove that $Y$ dominates and is dominated by the Higson compactification $hX$ of the proper coarse space $(X, \mathcal{E}_Y)$. By 5.6.4, $Y$ is dominated by the Higson compactification $hX$ of the proper coarse space $(X, \mathcal{E}_Y)$. Let $f X \to \mathbb{C}$ be a continuous Higson function (slowly oscillating), and we aim to find a continuous extension $\widetilde{f} : Y \to \mathbb{C}$ of $f$. For $\omega \in \partial_Y X$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ converging to $\omega \in \partial_Y X$. Let $r_\omega$ be an accumulation point of the bounded sequence $(f(x_n))_{n \in \mathbb{N}}$. Define $\widetilde{f} : Y \to \mathbb{C}$ by:

$$\widetilde{f}(y) = \begin{cases} f(y) & \text{if } y \in X \\ r_y & \text{if } y \in \partial_Y X \end{cases}$$

• $\widetilde{f}$ is well-defined. Indeed, assume that $(y_n)_{n \in \mathbb{N}}$ is another sequence converging to $\omega \in \partial_Y X$, then $E := \{(x_n, y_n) : n \in \mathbb{N}\} \in \mathcal{E}_Y$, and hence, by 5.6.2, $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(y_n)$.
• $\widetilde{f}$ is a continuous extension by construction. Therefore, $Y$ dominates the Higson compactification $hX$ of the proper coarse space $(X, \mathcal{E}_Y)$. □

**Definition 5.6.6.** Given a topological space $X$:
• The **support** of a real valued function $f : X \to \mathbb{R}$ is defined to be $supp(f) := cl_X(f^{-1}(\mathbb{R} \setminus \{0\})$. Notice that, $a \notin supp(f)$ if and only if there exists a neighborhood $U_a$ of $a$ such that $U_a \cap (f^{-1}(\mathbb{R} \setminus \{0\}) = \emptyset$ if and only if $f(U_a) = \{0\}$.
• A family of continuous functions $\{\rho_\alpha : X \to [0,1] : \alpha \in I\}$ is called a **partition of unity** if:
1. For every $x \in X$, there exists a neighborhood $U_x$ of $x$ such $\rho_\alpha\big|_{U_x} \equiv 0$ for all but finitely many $\alpha \in I$.
2. $\sum_{\alpha \in I} \rho_\alpha(x) = 1$, for all $x \in X$. In particular, if $\{\rho_\alpha : X \to [0,1] : \alpha \in I\}$ is a partition of unity, then $\{supp(\rho_\alpha) : \alpha \in I\}$ is locally finite cover of $X$.

• If $\mathcal{U}$ is a cover of $X$ and $\{\rho_{\alpha} : X \to [0,1] : \alpha \in I\}$ is a partition of unity, we say that $\{\rho_{\alpha} : X \to [0,1] : \alpha \in I\}$ **subordinate to** $\mathcal{U}$ if $\{supp(\rho_{\alpha}) : \alpha \in I\}$ is a refinement of $\mathcal{U}$.

We utilize the well-known fact, that a topological space is paracompact if and only if every open cover admits a subordinate partition of unity, to prove that every bounded Higson function can be approximated by a continuous Higson function up to a bounded function that vanishes at infinity, namely $B_h(X,\mathbb{C}) = C_h(X,\mathbb{C}) + B_0(X,\mathbb{C})$.

**Lemma 5.6.7.** *Let $(X,\mathcal{E})$ be a proper coarse space. For every $f \in B_h(X,\mathbb{C})$, there exists $g \in C_h(X,\mathbb{C})$ such that $f - g \in B_0(X,\mathbb{C})$. In particular, $B_h(X,\mathbb{C}) = C_h(X,\mathbb{C}) + B_0(X,\mathbb{C})$.*

*Proof.* Let $\mathcal{U} := \{U_\alpha : \alpha \in I\}$ be a locally finite uniformly bounded basis of $X$. Choose a partition of unity $\{\rho_{\alpha} : X \to [0,1] : \alpha \in I\}$ subordinate to $\mathcal{U}$. For each $\alpha \in I$, choose $x_\alpha \in U_\alpha$ and define the function $g(x) = \sum\limits_{\alpha\in I} \rho_{\alpha}(x) f(x_\alpha)$. Notice that $\rho_{\alpha}(x) = 0$ for all but finitely many $\alpha \in I$, and hence for all $x \in X$, $g(x)$ is a finite linear combination of elements of $\{\rho_{\alpha} : X \to [0,1] : \alpha \in I\}$, thus $g : X \to \mathbb{C}$ is bounded continuous. Let $E = (\bigcup\limits_{U\in\mathcal{U}} U \times U) \circ (\bigcup\limits_{U\in\mathcal{U}} U \times U)$. Since $f \in B_h(X,\mathbb{C})$ and $E \in \mathcal{E}$, so for any $\varepsilon > 0$, there exists a bounded subset $K_\varepsilon \subset X$ such that $\|f(x) - f(y)\| < \varepsilon$ for all $x \in X \setminus K_\varepsilon$ and $(x,y) \in E$. Moreover, for any $x \in X$, there exists $U_\beta \in \mathcal{U}$ a neighborhood of $x$ that intersects finitely many elements of $\mathcal{U}$, say $U_{\alpha_1}, \dots, U_{\alpha_n}$. Notice that $(x, x_{\alpha_i}) \in E$, for all $1 \le i \le n$, and hence if $x \in X \setminus K_\varepsilon$, then $\|f(x) - f(x_{\alpha_i})\| < \varepsilon$, for all $1 \le i \le n$. Now, since $\sum\limits_{\alpha\in I} \rho_{\alpha}(x) = 1$ for all $x \in X$, hence $f(x) - g(x) = \sum\limits_{\alpha\in I} \rho_{\alpha}(x)(f(x) - f(x_\alpha))$. Therefore, for any $\varepsilon > 0$ and $x \in X \setminus K_\varepsilon$, then $\|f(x) - g(x)\| \le (\sum\limits_{\alpha\in I} \rho_{\alpha}(x)) \|f(x) - f(x_\alpha)\| < \varepsilon$, which implies that $f - g \in B_0(X,\mathbb{C})$. □

**Corollary 5.6.8.** *Let $(X,\mathcal{E})$ be a proper coarse space. The Higson corona $\partial_h X$ is the maximal ideal space of $B_h(X,\mathbb{C})/B_0(X,\mathbb{C})$.*

*Proof.* It is obvious that $C_0(X,\mathbb{C}) = C_h(X,\mathbb{C}) \cap B_0(X,\mathbb{C})$, and $C_h(X,\mathbb{C}) + B_0(X,\mathbb{C}) \subset B_h(X,\mathbb{C})$. By 5.6.7, we have $B_h(X,\mathbb{C}) = C_h(X,\mathbb{C}) + B_0(X,\mathbb{C})$. By 2.3.24, we know that $\partial_h X$ is the maximal ideal space of $C_h(X,\mathbb{C})/C_0(X,\mathbb{C})$. Notice that $\frac{C_h(X,\mathbb{C})}{C_0(X,\mathbb{C})} = \frac{C_h(X,\mathbb{C})}{C_h(X,\mathbb{C}) \cap B_0(X,\mathbb{C})}$. By the Second Isomorphism Theorem, we have $\frac{C_h(X,\mathbb{C})}{C_h(X,\mathbb{C}) \cap B_0(X,\mathbb{C})} \cong \frac{C_h(X,\mathbb{C}) + B_0(X,\mathbb{C})}{B_0(X,\mathbb{C})} = \frac{B_h(X,\mathbb{C})}{B_0(X,\mathbb{C})}$. Therefore, by 2.3.21, $\partial_h X$ is the maximal ideal space of $B_h(X,\mathbb{C})/B_0(X,\mathbb{C})$. □

As a result, the Higson corona can be defined for arbitrary coarse space $X$ as the maximal ideal space of $B_h(X,\mathbb{C})/B_0(X,\mathbb{C})$.

**Lemma 5.6.9.** *Let $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ be proper coarse spaces, and $\varphi : X \to Y$ be a coarse map. Then:*

*(a) $g \circ \varphi \in B_0(X,\mathbb{C})$, for all $g \in B_0(Y,\mathbb{C})$.*

*(b) $g \circ \varphi \in B_h(X,\mathbb{C})$, for all $g \in B_h(Y,\mathbb{C})$.*

*(c) The coarse map $\varphi : X \to Y$ induces a $*$-homomorphism $\varphi^* : B_h(Y,\mathbb{C}) \to B_h(X,\mathbb{C})$ mapping each $g \in B_h(Y,\mathbb{C})$ to $g \circ \varphi$.*

*(d) If $(Z,\mathcal{G})$ is a proper coarse space and $\gamma : Y \to Z$ is a coarse map, then $(\gamma\circ\varphi)^* = \varphi^* \circ \gamma^*$.*

*(e) If $\lambda : X \to Y$ is another coarse map that is close to $\varphi : X \to Y$, then $\varphi^*$ and $\lambda^*$ agree up to $B_0(X,\mathbb{C})$.*

*Proof.* $(a)$ Follows from the fact that $g \in B_0(Y,\mathbb{C})$ and the fact that $\varphi : X \to Y$ is coarsely proper.

$(b)$ Let $E \in \mathcal{E}$ and $\varepsilon > 0$, then $\varphi \times \varphi(E) \in \mathcal{F}$, and hence there is a bounded subset $K_Y$ such that $\|g(a) - g(b)\| < \varepsilon$ for all $a \in Y \setminus K_Y$ and $(a,b) \in \varphi \times \varphi(E)$. Let $K_X = \varphi^{-1}(K_Y)$, then $\|g \circ \varphi(x) - g \circ \varphi(y)\| < \varepsilon$ for all $x \in X \setminus K_X$ and $(x,y) \in E$.

$(c)$ Straightforward.

$(d)$ Straightforward.

$(e)$ Let $\lambda : X \to Y$ be a coarse map close to $\varphi : X \to Y$ and let $F := \{(\varphi(x),\lambda(x)) : x \in X\}$. For $g \in B_h(Y,\mathbb{C})$, we aim to show that $\varphi^*(g) - \lambda^*(g) \in B_0(X,\mathbb{C})$. To this end, for $\varepsilon > 0$, choose a bounded subset $K_Y$ of $Y$ such that whenever $a \in Y \setminus K_Y$, then $\|g(a) - g(b)\| < \varepsilon$ for all $b \in Y$ with $(a,b) \in F$. Notice that $K_X := \varphi^{-1}(K_Y)$ is a bounded subset of $X$ and that for all $x \in X \setminus K_X$, $\varphi(x) \in Y \setminus K_Y$ and hence $\|g(\varphi(x)) - g(\lambda(x))\| < \varepsilon$. Thus, $\varphi^*(g) - \lambda^*(g) \in B_0(X,\mathbb{C})$ as desired. □

**Corollary 5.6.10.** *If $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ are coarsely equivalent proper coarse spaces; then, their Higson coronas are homeomorphic.*

*Proof.* Let $\varphi : X \to Y$ be a coarse equivalence with a coarse inverse $\psi : Y \to X$. By 5.6.9, they induces homomorphisms $\varphi^* : B_h(Y,\mathbb{C})/B_0(Y,\mathbb{C}) \to B_h(X,\mathbb{C})/B_0(X,\mathbb{C})$ and $\psi^* : B_h(X,\mathbb{C})/B_0(X,\mathbb{C}) \to B_h(Y,\mathbb{C})/B_0(Y,\mathbb{C})$ defined by $\varphi^*(g + B_0(Y,\mathbb{C})) = g \circ \varphi + B_0(X,\mathbb{C})$

and $\psi^*(f + B_0(X,\mathbb{C})) = f \circ \psi + B_0(Y,\mathbb{C})$, respectively. By 5.6.9 part ($d$) and ($e$), $\varphi^* : B_h(Y,\mathbb{C})/B_0(Y,\mathbb{C}) \to B_h(X,\mathbb{C})/B_0(X,\mathbb{C})$ is an isomorphism. By 2.3.21, $\partial_h X$ and $\partial_h Y$ are homeomorphic. □

**Theorem 5.6.11.** *Suppose $\bar{X}$ is the Freudenthal compactification of a proper, connected, and locally connected metric space $X$. $\bar{X}$ is a coarse compactification of $X$ if and only if for each $r > 0$ and each bounded subset $K$ of $X$ there is a bounded subset $L \supset K$ of $X$ such that for every $x \in X \setminus L$ the $r$-ball $B(x,r)$ is contained in a component of $X \setminus K$.*

*Proof.* Assume that $\bar{X}$ is a coarse compactification of $X$ and let $(K_n)$ be an exhausting sequence of $X$. Seeking contradiction, assume that there exists $r > 0$ and a bounded subset $K$ of $X$ such that for each bounded subset $L \supset K$ of $X$ there exists $x \in X \setminus L$ such that for any component $C$ of $X \setminus L$, $B(x,r) \setminus C$ is not empty. Choose $m \in \mathbb{N}$ such that $K \subset K_m$ and let $U$ be a relatively compact neighborhood of $K_m$. Notice $U$ contains all but finitely many connected components of $X \setminus K_m$. Without loss of generality assume that $U$ contains all bounded components of $X \setminus K_m$, and choose $N \in \mathbb{N}$ such that $cl(U) \subset K_N$. Now, for all $n \geq N$, and $x \in X \setminus K_n$, let $C_n^x$ be the component of $x \in X \setminus K_n$ containing $x$. For all $n \geq N$, let $D_n$ be a component of $X \setminus K_n$ distinct from $C_n^x$, where $x_n \in X \setminus K_n$, and choose $y_n \in B(x_n,r) \cap D_n$. Notice the components $C_n^x$ and $D_n$ are contained in components of $X \setminus K_N$, and since $X \setminus K_N$ has finitely many components, we can find two distinct components $C$ and $D$ of $X \setminus K_N$ containing all but finitely many elements of $(x_n)_{n \geq N}$ and $(y_n)_{n \geq N}$, respectively. Passing to subsequences, if necessary, we may assume that $(x_n)_{n \geq N} \subset C$ and $(y_n)_{n \geq N} \subset D$. However, $d(x_n, y_n) < r$ for all $n \geq N$ with $(x_n)_{n \geq N} \subset C$ and $(y_n)_{n \geq N} \subset D$ converge to two different points of the corona (as $\bar{X}$ is metrizable), a contradiction in view of 4.6.8.

Conversely, let $f : \bar{X} \to \mathbb{C}$ be continuous and we aim to show that $f|_X$ is slowly oscillating. Let $r > 0$ and $\varepsilon > 0$, since $\bar{X} \setminus X$ is compact totally disconnected, there exist clopen subsets $U_1, \ldots, U_n$ of $\bar{X}$ that cover $\bar{X} \setminus X$ with $\operatorname{diam}(f(U_i)) < \varepsilon$, for all $1 \leq i \leq n$. Without loss of generality we may assume that $U_j \setminus \bigcup_{i \neq j} U_i \neq \emptyset$. Notice the mutually disjoint family $V_j := U_j \setminus \bigcup_{i \neq j} U_i$, $1 \leq j \leq n$, is an open cover of $\bar{X} \setminus X$ and $\operatorname{diam}(f(V_i)) < \varepsilon$, for all $1 \leq i \leq n$. Put $O_i := V_i \cap X$, and notice that $\operatorname{diam}(f(O_i)) < \varepsilon$ and $O_i \neq \emptyset$, for all $1 \leq i \leq n$, as $X$

is dense in $\bar{X}$. Moreover, $K := X \setminus \bigcup_{i=1}^{n} O_i = \bar{X} \setminus \bigcup_{i=1}^{n} V_i$ is a compact subset of $X$ and hence bounded. Therefore, we can find a bounded subset $L$ of $X$ containing $K$ such that for every $x \in X \setminus L$ the $r$-ball $B(x, r)$ is contained in a component of $X \setminus K = \bigcup_{i=1}^{n} O_i$. In particular, for all $X \setminus L$, $x, y \in O_i$ for some $1 \leq i \leq n$, and thus $\|f(x) - f(y)\| < \varepsilon$. This shows that $f|_X$ is slowly oscillating. □

The above theorem provides an alternative proof to the fact that the Freudenthal compactification of a geodesic metric space is a coarse compactification that was implicitly given in 4.6.11.

**Corollary 5.6.12.** *If $X$ is a proper geodesic space, then its Freudenthal compactification is a coarse compactification of $X$.*

*Proof.* Given a bounded subset $K$ of $X$ and given $r > 0$, put $L := B(K, r)$ and notice $B(x, r)$ is a subset of $X \setminus K$ if $x \notin L$. Therefore $B(x, r)$ is a subset of a component of $X \setminus K$ if $x \notin L$. □

### 5.6.3 Coarsely disjoint and coarsely clopen subsets

The basic reference for this section is (34).

**Definition 5.6.13.** Let $(X, \mathcal{E})$ be a coarse space, and $A, B, U \subset X$.

• The subsets $A$ and $B$ are called **coarsely disjoint** if for all $E \in \mathcal{E}$, the subset $E[A] \cap E[B]$ is a coarsely bounded subset of $X$.

• $U$ is called a **coarse neighborhood** of $A$ if for all $E \in \mathcal{E}$, the subset $E[A] \setminus U$ is bounded.

**Lemma 5.6.14.** *Let $(X, \mathcal{E})$ be a coarse space and $A, B \subset X$. Suppose that $\mathcal{E}' \subset \mathcal{E}$ is a basis. The subsets $A$ and $B$ are coarsely disjoint if and only if for all $E \in \mathcal{E}'$, the subset $E[A] \cap E[B]$ is bounded.*

*Proof.* Assume that for all $E \in \mathcal{E}'$, the subset $E[A] \cap E[B]$ is bounded. Let $F \in \mathcal{E}$, there exists $E \in \mathcal{E}'$ such that $F \subset E$. Since $F[A] \cap F[B] \subset E[A] \cap E[B]$, $F[A] \cap F[B]$ is bounded. □

A special basis of the coarse space $(X, \mathcal{E})$, is the collection $\mathcal{E}' = \{E_\Delta : E \in \mathcal{E}\}$, where $E_\Delta = E \cup E^{-1} \cup \Delta$.

**Proposition 5.6.15.** *Let $(X, \mathcal{E})$ be a coarse space and $A, B \subset X$. The following are equivalent:*
*(a) $A$ and $B$ are coarsely disjoint.*
*(b) For all $E \in \mathcal{E}$, there exists a bounded subset $K_E \subset X$, such that:*

$$E[A \setminus K_E] \cap E[B] = \emptyset.$$

*(c) For all $E \in \mathcal{E}$, there exists a bounded subset $K_E \subset X$, such that:*

$$(E[A] \setminus K_E) \cap E[B] = \emptyset.$$

*(d) $E[A] \cap F[B]$ is coarsely bounded for all $E, F \in \mathcal{E}$.*
*(e) For all $E \in \mathcal{E}$, there exists a bounded subset $K_E \subset X$, such that:*

$$E[A \setminus K_E] \cap B = \emptyset.$$

*(f) For all $E \in \mathcal{E}$, there exists a bounded subset $K_E \subset X$, such that:*

$$(E[A] \setminus K_E) \cap B = \emptyset.$$

*(g) $E[A] \cap B$ is coarsely bounded, for all $E \in \mathcal{E}$.*
*(h) $X \setminus B$ is a coarse neighborhood of $A$.*
*(i) For all $E \in \mathcal{E}$, there exists a bounded subset $K_E \subset X$, such that:*

$$E[A \setminus K_E] \subset X \setminus B.$$

*Proof.* $(a) \iff (b)$: Assume that $A$ and $B$ are coarsely disjoint and $E \in \mathcal{E}$. Then $E_\Delta[A] \cap E_\Delta[B] = L$ is bounded. Set $K_E = E_\Delta[L]$. Seeking contradiction assume that $x \in E[A \setminus K_E] \cap E[B]$, then $x \in L$ and $(x, a) \in E$ for some $a \in A \setminus K_E$. Since $(a, x) \in E_\Delta$, we have $a \in K_E$, a contradiction. Conversely, let $E \in \mathcal{E}$ and choose a bounded subset $K_E \subset X$ such that

$E[A\setminus K_E]\cap E[B]=\emptyset$. Notice that $E[A]\cap E[B]=(E[A\setminus K_E]\cup E[K_E])\cap E[B]=E[K_E]\cap E[B]$ which is bounded.

$(b) \implies (c)$: Let $E\in\mathcal{E}$ and choose $K\subset X$ is bounded such that $E[A\setminus K]\cap E[B]=\emptyset$. Take $K_E=E_\Delta[K]$, and notice that $E[A]\setminus K_E\subset E[A\setminus K]$.

$(c) \implies (a)$: For $E\in\mathcal{E}$ and choose a bounded subset $K_E\subset X$ such that $(E[A]\setminus K_E)\cap E[B]=\emptyset$. Notice that $E[A]\cap E[B]=((E[A]\setminus K_E)\cup K_E)\cap E[B]=K_E\cap E[B]$ which is bounded.

$(a) \iff (d)$: Assume that $A$ and $B$ are coarsely disjoint and $E,F\in\mathcal{E}$; then $E\cup F\in\mathcal{E}$ and hence $(E\cup F)[A]\cap(E\cup F)[B]$ is coarsely bounded and containing $E[A]\cap F[B]$. The converse is straightforward.

$(a) \iff (e)$: Let $A$ and $B$ be coarsely disjoint and $E\in\mathcal{E}$. By part $(b)$, we can find a bounded subset $K_{E_\Delta}\subset X$, such that $E_\Delta[A\setminus K_{E_\Delta}]\cap E_\Delta[B]=\emptyset$. Notice that $E_\Delta[A\setminus K_{E_\Delta}]\cap B=\emptyset$, and a fortiori, $E[A\setminus K_{E_\Delta}]\cap B=\emptyset$. Conversely, let $E\in\mathcal{E}$, and put $F=E_\Delta\circ E_\Delta$. Choose a bounded subset $K_F\subset X$, such that $F[A\setminus K_F]\cap B=\emptyset$. We claim that $E[A\setminus K_F]\cap E[B]=\emptyset$. Indeed, seeking contradiction assume that $x\in E[A\setminus K_F]\cap E[B]$, then $(x,a),(x,b)\in E\subset E_\Delta$ for some $a\in A\setminus K_F$ and $b\in B$ which implies that $(b,a)\in F$ and hence $b\in F[A\setminus K_F]\cap B$, a contradiction.

$(a) \iff (f)$: Follows by a similar argument to $(a) \iff (e)$ and $(b) \implies (c)$.

$(a) \iff (h)$: Follows from part $(d)$ and $(f)$.

$(a) \iff (h)$: Straightforward.

$(h) \iff (i)$: Straightforward. □

**Proposition 5.6.16.** *Let $(X,\mathcal{E})$ and $(X,\mathcal{E})$ be coarse spaces, $X'\subset X$ be a coarse subspace, and $\varphi: X\to Y$ be a map. Then:*

*(a) If $A,B\subset X$ are coarsely disjoint in $X$, then $A\cap X'$ and $B\cap X'$ are coarsely disjoint in $X'$.*

*(b) Let $C,D\subset Y$ be coarsely disjoint in $Y$, and $\varphi: X\to Y$ be a coarse map; then $\varphi^{-1}(C)$ and $\varphi^{-1}(D)$ are coarsely disjoint in $X$.*

*Proof.* $(a)$ Follows from the fact that if $K\subset X$ is bounded, then $K\cap X'$ is bounded subset of $X'$.

$(b)$ Let $C,D\subset X$ be coarsely disjoint in $Y$, and $\varphi: X\to Y$ be a coarse map. Seeking contradiction, assume that there exists an entourage $E \in \mathcal{E}$ such that $E[\varphi^{-1}(C)] \cap \varphi^{-1}(D)$ is unbounded. By 5.3.2, there exists $F \in \mathcal{E}_Y$ such that $\varphi(E[x]) \subset F[\varphi(x)]$ for all $x \in X$. Since $\varphi : X \to Y$ is coarsely proper, $\varphi(E[\varphi^{-1}(C)] \cap \varphi^{-1}(D))$ is unbounded. However, $\varphi(E[\varphi^{-1}(C)] \cap \varphi^{-1}(D)) \subset \varphi(E[\varphi^{-1}(C)]) \cap \varphi(\varphi^{-1}(D)) \subset F[\varphi(\varphi^{-1}(C))] \cap D \subset F[C] \cap D$ which implies that $F[C] \cap D$ is unbounded, a contradiction. □

**Definition 5.6.17.** Let $(X, \mathcal{E})$ be a coarse space, and $A \subset X$. We say that $A$ is **coarsely clopen** if $A$ and $X \setminus A$ are coarsely disjoint. If, in addition, both $A$ and $X \setminus A$ are unbounded, we say that $A$ is **non-trivial coarsely clopen**. A function $f : X \to Y$ is called **coarsely constant** if there exists a bounded subset $K \subset X$ such that $f\big|_{X \setminus K}$ is constant; otherwise, $f : X \to Y$ is called **coarsely non-constant**.

**Proposition 5.6.18.** *Let $(X, \mathcal{E})$ be a coarse space, and $A \subset X$. Then,*
*(a) $A$ is coarsely clopen if and only if the characteristic function of $\chi_A : X \to \{0, 1\}$ is a Higson function.*
*(b) $A$ is non-trivial coarsely clopen if and only if the characteristic function $\chi_A : X \to \{0, 1\}$ of $A$ is a Higson function.*

*Proof.* $(a)$ Suppose that $A$ is coarsely clopen. For $E \in \mathcal{E}$ and $\varepsilon > 0$, take the bounded subset $K := E_\Delta[A] \cap E_\Delta[X \setminus A]$. Then, for all $x, y \in X \setminus K$ such that $(x, y) \in E$, either $x, y \in A$ or $x, y \in X \setminus A$; hence, $\|\chi_A(x) - \chi_A(y)\| = 0 < \varepsilon$ for all $x, y \in X \setminus K$ such that $(x, y) \in E$.
Conversely, assume that $\chi_A : X \to \{0, 1\}$ of $A$ is a Higson function, and let $E$ be an entourage. For $0 < \varepsilon < 1$, there exists a bounded subset $K \subset X$ such that for all $x, y \in X \setminus K$ such that $(x, y) \in E$ then $\|\chi_A(x) - \chi_A(y)\| < \varepsilon$ and therefore we must have either $x, y \in A$ or $x, y \in X \setminus A$. Thus, $E[A] \cap E[X \setminus A] = \emptyset$.
$(b)$ Straightforward. □

**Proposition 5.6.19.** *Let $(X, \mathcal{E})$ be a coarsely connected coarse space. If $A$ and $C$ are coarsely clopen subsets of $X$, then so are $A \cup C$, $A \cap C$ and $A \setminus C$.*

*Proof.* Let $E \in \mathcal{E}$, and notice that $E[A \cup C] \cap (A \cup C)^c = (E[A] \cup E[C]) \cap (A^c \cap C^c)$ which is bounded and thus $A \cup C$ is coarsely clopen. By symmetry, $A \cap C$ is also coarsely clopen. Finally, as $E[A \setminus C] \cap (A \setminus C)^c = (E[A \setminus C] \cap (A^c \cup C) \subset (E[A] \cap A^c) \cup (E[C^c] \cap C)$ which is coarsely bounded and hence $A \setminus C$ is coarsely clopen as desired. □

**Definition 5.6.20.** Let $X$ be a coarse space. Two subsets $A, B \subset X$ are said to be **coarsely identical** if their symmetric difference $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is a bounded subset of $X$.

**Lemma 5.6.21.** *Let $(X, \mathcal{E})$ be a coarsely connected coarse space, and $A \subset X$ be coarsely clopen subset of $X$.*
*(a) If $F \in \mathcal{E}$ is containing the diagonal $\Delta$, then, $F[A]$ is coarsely clopen, and the subsets $A$ and $F[A]$ are coarsely identical.*
*(b) If $F$ is containing $\Delta$ and $A \subset X$ is a non-trivial coarsely clopen, then $F[A]$ is a non-trivial coarsely clopen.*

*Proof.* $(a)$ Notice first that $if x \in X \setminus F[A]$, then $x \in X \setminus A$ and therefore $x \in F[X \setminus A]$. Now, if $E \in \mathcal{E}$, then $E[F[A]] \cap X \setminus F[A] \subset (E \circ F)[A] \cap F[X \setminus A]$ which is bounded by 5.6.14. Since $A \setminus F[A] \subset A \cap F[X \setminus A]$ and $F[A] \setminus A = F[A] \cap X \setminus A$ which are bounded, then $A$ and $F[A]$ are coarsely identical.
$(b)$ We aim to show that $X \setminus F[A]$ is unbounded. Since $A$ and $F[A]$ are coarsely identical, there exists a bounded subset $K \subset X$ such that $F[A] \subset A \cup K$. Hence $X \setminus (A \cup K) \subset X \setminus F[A]$ which is unbounded as $X \setminus (A \cup K)$ is unbounded. □

# Chapter 6

# Coarse Freudenthal compactification and ends of countable groups

As observed in 4.6.11, for a proper metric space $X$, the quotient space $CF(X)$ of its Higson compactification obtained by collapsing each component of the Higson corona to a point is a coarse compactification of $X$ that coincide with its Freudenthal compactification when $X$ is geodesic. In this chapter, we define that coarse compactification in a way similar to the Higson compactification as a maximal ideal space of some $C^*$-subalgebra of $C_b(X, \mathbb{C})$, and we call it the Coarse Freudenthal compactification of $X$. The corona of the Coarse Freudenthal compactification is a coarse invariant; that fact leads to defining the space of ends for countable groups as the corona of their Coarse Freudenthal compactification.

## 6.1 Glacial oscillations

In this section we define a concept in the spirit of slowly oscillating functions and we use it to introduce Coarse Freudenthal compactifications later on.

**Definition 6.1.1.** A **glacial scale** on a metric space $X$ is a sequence of pairs $\{(K_i, n_i)\}_{i=1}^{\infty}$ of bounded subsets of $X$ and natural numbers such that for each pair $(K, r)$ consisting of a bounded subset of $X$ and $r > 0$ there is $i$ such that $K \subset K_i$ and $n_i > r$.

Given a glacial scale $\mathcal{S} = \{(K_i, n_i)\}_{i=1}^{\infty}$, a chain of points $x_1, \dots, x_n$ in $X$ is called an $\mathcal{S}$**-chain** if for each $i \leq n-1$ there is $m \geq 1$ such that $x_i, x_{i+1} \notin K_m$ and $d(x_i, x_{i+1}) \leq n_m$.

**Definition 6.1.2.** A function $f : X \to \mathbb{C}$ is **glacially oscillating** if for each $\varepsilon > 0$ there is a glacial scale $\mathcal{S}$ with the property that $\|f(x_1) - f(x_n)\| < \varepsilon$ for each $\mathcal{S}$-chain $x_1, \ldots, x_n$.

**Observation 6.1.3.** *One can introduce the concept of a subset $A$ of $X$ to be $\mathcal{S}$-connected and reword the above definition as requiring that the diameter of $f(A)$ is less than $\varepsilon$ for each $\mathcal{S}$-connected subset $A$ of $X$.*

We give a characterization of coarsely clopen subsets of a metric space.

**Proposition 6.1.4.** *Given a subset $A$ of a metric space $X$ the following conditions are equivalent:*
*(a) The characteristic function $\chi_A$ of $A$ is glacially oscillating,*
*(b) $A$ is coarsely clopen,*
*(c) There is a glacial scale $\mathcal{S}$ with the property that any $\mathcal{S}$-chain starting at a point of $A$ is completely contained in $A$.*

*Proof.* $(a) \implies (b)$: is clear as $\chi_A$ of $A$ is glacially oscillating implies $\chi_A$ of $A$ is slowly oscillating which is equivalent to $A$ being coarsely clopen by 5.6.18.

$(b) \implies (c)$: Fix $x_0 \in X$. For each $n \in \mathbb{N}$ choose a bounded subset $K_n$ containing $B(x_0, n)$ such that given two points $x, y \in X \setminus K_n$ at distance less than $n$, $x \in A$ implies $y \in A$. Put $\mathcal{S} = \{(K_n, n)\}_{n \in \mathbb{N}}$.

$(c) \implies (a)$: Given $\varepsilon > 0$, notice that for any $\mathcal{S}$-chain $x_1, \ldots, x_n$ in $X$ one has $\chi_A(x_1) = \chi_A(x_n)$. □

**Corollary 6.1.5.** *Given finitely many coarsely clopen subsets $A_1, \ldots, A_n$ of a metric space $X$, there is a glacial scale $\mathcal{S}$ with the property that any $\mathcal{S}$-chain starting at a point in $A_j$, for some $1 \leq j \leq n$, is completely contained in $A_j$.*

*Proof.* For each $j \leq n$ pick a glacial scale $\mathcal{S}_j = \{(K_i^j, k_i^j\}_{i \geq 1}$ with the property that any $\mathcal{S}_j$-chain starting at a point of $A_j$ is completely contained in $A_j$. Define $K_i = \bigcup_{j=1}^{n} K_i^j$ and $k_i = \min(k_i^1, \ldots, k_i^n)$. Notice that for $\mathcal{S} = \{(K_i, k_i)\}_{i \geq 1}$ any $\mathcal{S}$-chain starting at a point of some $A_j$ is completely contained in $A_j$. □

**Corollary 6.1.6.** *If $(X, d)$ is a metric space, then any slowly oscillating function $f : X \to \mathbb{C}$ whose image is finite is glacially oscillating.*

*Proof.* Let $f(X) = \{z_1, \ldots, z_n\}$, and $A_i := f^{-1}(z_i)$, for all $1 \leq i \leq n$. Notice $X = \bigcup_{i=1}^{n} A_i$. We claim that for all $1 \leq i \leq n$, $A_i$ is coarsely clopen. To this end, take $\varepsilon := \min\{\|z_i - z_j\| : i \neq j$ and $1 \leq i, j \leq n\}$. Since $f : X \to \mathbb{C}$ is slowly oscillating, for any $r > 0$ there exits a bounded subset $K_r$ of $X$ such that for all $x, y \in X \setminus K_r$, $d(x, y) < r$ implies $\|f(x) - f(y)\| < \varepsilon$. In particular, for any $r > 0$, $B(A, r) \cap (X \setminus A) \setminus K_r = \emptyset$ and hence $A_i$ is coarsely clopen as claimed. By 6.1.4, there is a glacial scale $\mathcal{S} = \{(K_j, n_j)\}_{j \geq 1}$ such that any $\mathcal{S}$-chain starting at a point of some $A_i$ is completely contained in $A_i$. Notice that for any $\mathcal{S}$-chain $x_1, \ldots, x_m$, then $\|f(x_1) - f(x_m)\| = 0$, proving that $f : X \to \mathbb{C}$ is glacially oscillating. $\square$

In the case of geodesic spaces there is a simple characterization of glacially oscillating functions.

**Proposition 6.1.7.** *Let $X$ be a geodesic space. The function $f : X \to \mathbb{C}$ is glacially oscillating if and only if for each $\varepsilon > 0$ there is a bounded subset $K$ of $X$ such that for every component $C$ of $X \setminus K$ the diameter of $f(C)$ is at most $\varepsilon$.*

*Proof.* Assume that $f : X \to \mathbb{C}$ is glacially oscillating. Given $\varepsilon > 0$ pick a glacial scale $\mathcal{S} = \{(K_i, n_i)\}_{i \in \mathbb{N}}$ with the property that $\|f(x_1) - f(x_n)\| < \varepsilon/2$ for every $\mathcal{S}$-chain $x_1, \ldots, x_n$. Put $K = B(K_1, n_1)$ and notice that every two points $x, y$ in a component $C$ of $X \setminus K$ can be connected by a geodesic and therefore by an $n_1$-chain in $X \setminus K_1$. That chain is also an $\mathcal{S}$-chain, so $\|f(x) - f(y)\| < \varepsilon/2$ and $\operatorname{diam}(f(C)) \leq \varepsilon$.

Conversely, for each $n \in \mathbb{N}$ pick a bounded set $K_n$ containing such that the diameter of $f(C)$ is at most $1/n$ for each component $C$ of $X \setminus K_n$. Given $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Fix $x_0 \in X$ and a strictly increasing sequence of natural numbers $(m_i)_{i \in \mathbb{N}}$ such that $K_n \subset B(x_0, m_n)$ and $K_N \subset B(x_0, m_1)$. Put $L_n := B(x_0, m_n)$, for all $n \in \mathbb{N}$. Then, $\mathcal{S} = \{(L_i, m_i)\}_{i \in \mathbb{N}}$ is a glacial scale and each $\mathcal{S}$-chain in entirely contained in a component of $X \setminus L_n$, for some $n \in \mathbb{N}$, and hence is contained in a component of $X \setminus K_n$. $\square$

**Lemma 6.1.8.** *Suppose $\{K_i\}_{i \geq 1}$ is an increasing sequence of bounded subsets of a metric space $X$, $n_i$ is a strictly increasing sequence of natural numbers, and $A_i$ is a union of $n_i$-components of $X \setminus K_i$ for each $i \geq 1$. If $A_i \setminus K_{i+1} \subset A_{i+1}$ for each $i \geq 1$, then $A := \bigcup_{i=1}^{\infty} A_i$ is a coarsely open subset of $X$.*

*Proof.* Notice $A_i$ equals $B(A_i, n_i)$ in $X \setminus K_i$ for each $i \geq 1$. Let $r > 0$ and $j \in \mathbb{N}$ such that $n_j > r$. We claim that $B(A \setminus K_j, r) \cap A^c = \emptyset$. To this end, take $i \in \mathbb{N}$, we discuss two cases:
**Case 1:** If $i \leq j$, then $A_i \setminus K_j \subset A_j$, and hence $B(A_i \setminus K_j, r) \cap A^c \subset A_j \cap A^c = \emptyset$.
**Case 2:** If $i > j$, then $n_i > r$ and $B(A_i \setminus K_j, r) = A_i \cap A^c = \emptyset$. This shows that $B(A \setminus K_j, r) \cap A^c = \emptyset$. $\square$

Now we give a characterization of glacially oscillating functions making use of **The Lebesgue Number Lemma** that says that for every open cover of a compact metric space, there exists a number $\delta > 0$ such that every subset of that metric space of a diameter less than $\delta$ is contained in an element of that cover. $\delta$ is called a **Lebesgue number** for that cover.

**Proposition 6.1.9.** *Given a bounded function $f : X \to \mathbb{C}$ on a metric space $X$, the following conditions are equivalent:*
*(a) $f$ is glacially oscillating.*
*(b) For every $\varepsilon, M > 0$ there exist a bounded subset $K$ of $X$ such that* $\operatorname{diam}(f(U)) < \varepsilon$ *for every $M$-component $U$ of $X \setminus K$.*
*(c) For every $\varepsilon > 0$ there exist a bounded subset $K$ of $X$ and finitely many coarsely clopen subsets $U_i$, $i \leq n$, covering $X \setminus K$ such that* $\operatorname{diam}(f(U_i)) < \varepsilon$ *for each $i \leq n$.*

*Proof.* $(a) \implies (b)$: Let $f$ be glacially oscillating. For $\varepsilon > 0$, choose a glacial scale $\mathcal{S} = \{(K_i, n_i)\}_{i \in \mathbb{N}}$ such that $\|f(x_1) - f(x_n)\| < \varepsilon/2$ for each $\mathcal{S}$-chain $x_1, \ldots, x_n$. There is $j \geq 1$ such that $n_j > M$. Put $K = K_j$, and notice that if $U$ is an $M$-component of $X \setminus K$ then any two points of $U$ can be connected by an $\mathcal{S}$-chain. Therefore $\operatorname{diam}(f(U)) < \varepsilon$.

$(b) \implies (c)$: Assume that for every $\varepsilon, M > 0$ there exist a bounded subset $K$ of $X$ such that $\operatorname{diam}(f(U)) < \varepsilon$ for every $M$-component $U$ of $X \setminus K$. Choose $x_0 \in X$ and fix $\varepsilon > 0$. Since $cl_{\mathbb{C}} f(X)$ is compact, it can be covered by finitely many open balls $I_1, \ldots, I_n$ of diameter less than or equal to $\varepsilon/3$. Choose a strictly increasing sequence of positive numbers $a_i$ converging to $\varepsilon/3$ and put $b_{i+1} = a_{i+1} - a_i$ with $b_1 = a_1$. Let $\delta_i > 0$ be a Lebesgue number of the cover $\{B(I_1, a_i), \ldots, B(I_n, a_i)\}$ of the compact metric space $cl_{\mathbb{C}} f(X)$. For every $i \in \mathbb{N}$, put $\varepsilon_i := \min(\delta_i, b_i)$. By the hypothesis, for each pair $\varepsilon_i, i > 0$, there exists a bounded subset

$L_i$ of $X$ such that $\operatorname{diam}(f(U)) < \varepsilon_i$, for every $i$-component $U$ of $X \setminus L_i$. Choose a strictly increasing sequence of natural numbers $(n_i)_{i\in\mathbb{N}}$ such that $L_i \subset B(x_0, n_i)$ for all $i \in \mathbb{N}$. Put $K_i := B(x_0, n_i)$ and notice that $\operatorname{diam}(f(U)) < \varepsilon_i$ for every $i$-component $U$ of $X \setminus K_i$.

For each $1 \le j \le n$ and $i \in \mathbb{N}$, let $U_j^i$ be the union of all $i$-components $U$ of $X \setminus K_i$ such that $f(U)$ is contained in $B(I_j, a_i)$. Notice $U_j^i \setminus K_{i+1} \subset U_j^{i+1}$. Indeed, let $x \in U_j^i \setminus K_{i+1}$ and $V$ be an $(i+1)$-component of $X \setminus K_{i+1}$ containing $x$. Since $\operatorname{diam}(f(V)) < b_{i+1}$, one has $f(V) \subset B(I_j, a_i + b_{i+1}) = B(I_j, a_{i+1})$.

Put $U_j = \bigcup\limits_{i=1}^{\infty} U_j^i$. By 6.1.8 each $U_j$ is a coarsely clopen subset of $X$. Notice that $\operatorname{diam}(f(U_j)) < \varepsilon$ as $f(U_j) \subset B(I_j, \varepsilon/3)$ and $U_1, \ldots, U_n$ cover $X \setminus K_1$.

$(c) \implies (a)$. Given $\varepsilon > 0$ choose a bounded subset $K$ of $X$ and finitely many coarsely clopen subsets $U_i$, $i \le n$, covering $X \setminus K$ such that $\operatorname{diam}(f(U_i)) < \varepsilon$ for each $i \le n$. We may assume $U_i$, $i \le n$, are mutually disjoint by applying 5.6.19. By 6.1.5 there is a glacial scale $\mathcal{S}$ with the property that any $\mathcal{S}$-chain starting at a point of some $U_j$ is completely contained in $U_j$. We may assume the first bounded set of $\mathcal{S}$ contains $K$. Therefore, for any two points $x, y \in X$ joinable by an $\mathcal{S}$-chain one has $\|f(x) - f(y)\| < \varepsilon$ as they are contained in one of sets $U_i$, $i \le n$. Thus $f$ is glacially oscillating. $\square$

**Corollary 6.1.10.** *Every bounded glacially oscillating function $f : X \to \mathbb{C}$ on a metric space $X$ is slowly oscillating.*

*Proof.* Let $f : X \to \mathbb{C}$ be a bounded glacially oscillating function and $\varepsilon > 0$. By 6.1.9, there exists a bounded subset $K$ of $X$ and finitely many mutually disjoint coarsely clopen subsets $U_i$, $i \le n$, covering $X \setminus K$ such that $\operatorname{diam}(f(U_i)) < \varepsilon$ for each $i \le n$. Let $r > 0$, for $j \le n$, $U_j$ and $\bigcup\limits_{i\neq j} U_i$ are coarsely disjoint; in particular, there exists a bounded subset $K_j$ of $X$ such that $B(U_j, r) \cap (\bigcup\limits_{i\neq j} U_i) \setminus K_j = \emptyset$. Take $L := (\bigcup\limits_{j=1}^{n} K_j) \cup K$. Note that for all $x, y \in X \setminus L$ and $d(x, y) < r$ implies that $x, y \in U_j$ for some $j \le n$ and hence $\|f(x) - f(y)\| < \varepsilon$. $\square$

When the metric space $(X, d)$ is ultrametric, the converse of 6.1.10 holds in the sense that every slowly oscillating is glacially oscillating and therefore the family of the bounded slowly oscillating functions coincide with the family of the bounded glacially oscillating functions.

**Proposition 6.1.11.** *If $(X, d)$ is an ultrametric space, then every slowly oscillating function $f : X \to \mathbb{C}$ is glacially oscillating.*

*Proof.* Recall that $(X, d)$ is an ultrametric space if every triangle in $X$ is isosceles and the lengths of two equal sides are at least the size of the third side. Equivalently, $d(x, y) \leq \max(d(x, z), d(y, z))$ for all points $x, y, z \in X$.

If $f : X \to \mathbb{C}$ is slowly oscillating and $\varepsilon > 0$, then we can choose an increasing sequence $\{K_n\}_{n\in\mathbb{N}}$ of non-empty bounded subsets of $X$ such that $B(K_n, n) \subset K_{n+1}$ for each $n \in \mathbb{N}$ and $\|f(x) - f(y)\| < \varepsilon$ if $x, y \notin K_n$ and $d(x, y) \leq n$. Let $\mathcal{S} := \{(B(K_n, n), n)\}_{n\in\mathbb{N}}$ and suppose $x_1, \ldots, x_n$ is an $\mathcal{S}$-chain. Pick $j \leq n-1$ such that $d(x_j, x_{j+1})$ is the maximum of all $d(x_i, x_{i+1})$, $i \leq n-1$. Let $M$ be the smallest integer satisfying $d(x_j, x_{j+1}) \leq M$. Notice $x_j, x_{j+1} \notin B(K_M, M)$, the distance from $x_1$ to either $x_j$ or $x_{j+1}$ is at most $M$, the distance from $x_n$ to either $x_j$ or $x_{j+1}$ is at most $M$, hence $d(x_1, x_n) \leq M$. That implies $x_1, x_n \notin K_M$, in particular $\|f(x_1) - f(x_n)\| < \varepsilon$. That proves $f$ is glacially oscillating. □

**Proposition 6.1.12.** *Suppose $\varphi : X \to X$ is close to $id_X$ and $f : X \to \mathbb{C}$ is slowly oscillating. If $f \circ \varphi$ is glacially oscillating, then so is $f$.*

*Proof.* $\varphi : X \to X$ being close to $id_X$ means there is $r > 0$ such that $d_X(\varphi(x), x) < r$ for all $x \in X$. Given $\varepsilon > 0$, since $f$ is slowly oscillating, there is a bounded subset $K \subset X$ such that for all $x, y \in X \setminus K$ and $d(x, y) < r$ implies $\|f(x) - f(y)\| < \varepsilon/3$. On the other hand, choose a glacial scale $\mathcal{S} = \{(K_i, n_i)\}_{i\geq 1}$ so that $\|f \circ \varphi(x) - f \circ \varphi(y)\| < \varepsilon/3$ for any $x, y \in X$ that can be connected by an $\mathcal{S}$-chain in $X$. We may assume that $K \subset K_i$ for all $i \geq 1$ and that $n_1 > r$ by truncating $\mathcal{S}$. Let $\mathcal{S}' = \{(B(K_i, r), n_i)\}_{i\geq 1}$. Now, given any $\mathcal{S}'$-chain $x_1, \ldots, x_n$ notice $x_1, \varphi(x_1)$ and $x_n, \varphi(x_n)$ are $\mathcal{S}$-chains. Therefore, $\|f(x_1) - f(x_n)\| \leq \|f(x_1) - f(\varphi(x_1))\| + \|f(\varphi(x_1)) - f(\varphi(x_n))\| + \|f(\varphi(x_n)) - f(x_n)\| < \varepsilon$. □

**Proposition 6.1.13.** *Suppose $\varphi : (X, d_X) \to (Y, d_Y)$ is a coarse map and $f : Y \to \mathbb{C}$ is a function.*
*(a) If $f$ is glacially oscillating, then so is $f \circ \varphi$.*
*(b) If $f \circ \varphi$ is glacially oscillating and $\varphi$ is a coarse equivalence, then $f$ is glacially oscillating.*

*Proof.* $(a)$ Given $\varepsilon > 0$ choose a glacial scale $\mathcal{S} = \{(K_i, n_i)\}_{i\geq 1}$ so that $\|f(y_1) - f(y_2)\| < \varepsilon$ for any $y_1, y_2 \in Y$ that can be connected by an $\mathcal{S}$-chain in $Y$. Put $C_i = \varphi^{-1}(K_i)$ and let $m_i$ be the maximum of natural numbers such that $d_X(x_1, x_2) \le m_i$ implies $d_Y(\varphi(x_1), \varphi(x_2)) \le n_i$. Notice $\mathcal{C} = \{(C_i, m_i)\}_{i\ge 1}$ is a glacial scale in $X$ and the image under $\varphi$ of any $\mathcal{C}$-chain is an $\mathcal{S}$-chain. Therefore, if $x$ and $y$ can be connected by a $\mathcal{C}$-chain, $\varphi(x)$ and $\varphi(y)$ can be connected by an $\mathcal{S}$-chain and $\|f(\varphi(x)) - f(\varphi(y))\| < \varepsilon$.

$(b)$ Let $\psi : Y \to X$ be a coarse inverse of $\varphi : (X, d_X) \to (Y, d_Y)$. By $(a)$, $f \circ \varphi \circ \psi$ is glacially oscillating and by 6.1.12, so is $f$. $\square$

**Corollary 6.1.14.** *If $(X, d)$ is a metric space of asymptotic dimension $0$, then every slowly oscillating function $f : X \to \mathbb{C}$ is glacially oscillating.*

*Proof.* As shown in (4) there is an ultrametric space coarsely equivalent to $(X, d)$. Apply 6.1.13. $\square$

Knowing that the asymptotic dimension of a proper metric space of a finite asymptotic dimension is exactly the topological dimension of its Higson Corona(cf. (10)); 6.2.13 provides alternative prove to the above fact.

## 6.2 Coarse Freudenthal compactification

**Notation 6.2.1.** *Let $(X, d)$ be a metric space. The family of all bounded glacially oscillating functions $f : X \to \mathbb{C}$ is denoted by $B_F(X, \mathbb{C})$, and the family of all bounded continuous glacially oscillating functions is denoted by $C_F(X, \mathbb{C})$. Clearly, $C_F(X, \mathbb{C}) = B_F(X, \mathbb{C}) \cap C_b(X, \mathbb{C})$.*

**Proposition 6.2.2.** *Let $(X, d)$ be a metric space. Then:*
*(a) $B_F(X, \mathbb{C})$ is unital $C^*$-subalgebra of $B(X, \mathbb{C})$ containing $B_0(X, \mathbb{C})$ and contained in $B_h(X, \mathbb{C})$.*
*(b) $C_F(X, \mathbb{C})$ is unital $C^*$-subalgebra of $C_b(X, \mathbb{C})$ containing $C_0(X, \mathbb{C})$ and contained in $C_h(X, \mathbb{C})$.*

*Proof.* $(a)$ $B_F(X, \mathbb{C})$ is unital as it contains all constant functions. Let $f, g \in B_F(X, \mathbb{C})$ and $\varepsilon > 0$. Set $M := \max\{\underset{x\in X}{sup}\|f(x)\|, \underset{x\in X}{sup}\|g(x)\|, 1\}$ and choose glacial scales $\mathcal{S}_f = \{(K_i, m_i)\}_{i=1}^{\infty}$ and $\mathcal{S}_g = \{(L_i, n_i)\}_{i=1}^{\infty}$ for $f$ and $g$ and $\varepsilon/2M$, respectively. Consider the

glacial scale $\mathcal{S} = \{(R_i, r_i)\}_{i\geq 1}$ where $R_i = K_i \cup L_i$ and $r_i = \min\{n_i, m_i\}$, for all $i \geq 1$. Notice any an $\mathcal{S}$-chain is an $\mathcal{S}_f$-chain and an $\mathcal{S}_g$-chain. Hence, for any two points $x, y \in X$ joinable by an $\mathcal{S}$-chain one has $\|f(x) - f(y)\| < \varepsilon/2M$ and $\|g(x) - g(y)\| < \varepsilon/2M$. In particular, $\|(f+g)(x) - (f+g)(y)\| < \varepsilon$ and $\|(f \cdot g)(x) - (f \cdot g)(y)\| < \varepsilon$. To show that $C_F(X, \mathbb{C})$ is complete, we show that $B_F(X, \mathbb{C})$ is a closed subspace of $B(X, \mathbb{C})$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $B_F(X, \mathbb{C})$ converging to $f \in B(X, \mathbb{C})$. For $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\|f_N - f\|_\infty < \varepsilon/3$. Since $f_N \in B_F(X, \mathbb{C})$, there exists a glacial scale $\mathcal{S}$ with the property that for every $\mathcal{S}$-chain $x_1, \ldots, x_n$, then $\|f_N(x_1) - f_N(x_n)\| < \varepsilon/3$. Therefore, $\|f(x_1) - f(x_n)\| \leq \|f(x_1) - f_N(x_1)\| + \|f_N(x_1) - f_N(x_n)\| + \|f_N(x_n) - f(x_n)\| < \varepsilon$. This shows that $B_F(X, \mathbb{C})$ is a unital $C^*$-subalgebra of $B(X, \mathbb{C})$. By 6.1.10, $B_F(X, \mathbb{C}) \subset B_h(X, \mathbb{C})$. To see that $B_F(X, \mathbb{C})$ contains the ideal $B_0(X, \mathbb{C})$, let $f : X \to \mathbb{C}$ be a bounded function that vanishes at infinity and $\varepsilon > 0$. Choose $K$ to be a bounded subset of $X$ such that $\|f(x)\| < \varepsilon/2$ for all $x \in X \setminus K$. Take the glacial scale $\mathcal{S} = \{(B(K, i), i)\}_{i=1}^{\infty}$ and notice that for any two points $x, y \in X$ joinable by an $\mathcal{S}$-chain one has $\|f(x) - f(y)\| \leq \|f(x)\| + \|f(y)\| < \varepsilon$.
$(b)$ Similar to $(a)$. □

**Definition 6.2.3.** The **coarse Freudenthal compactification** $CF(X)$ of a proper metric space $X$ is the maximal ideal space of the $C^*$-algebra $C_F(X, \mathbb{C})$. Its corona $\partial_{CF} X := CF(X) \setminus X$ is a compact subset of $CF(X)$. Moreover, $\partial_{CF} X$ is the maximal ideal space of the $C^*$-algebra $C_F(X, \mathbb{C})/C_0(X, \mathbb{C})$.

By 6.1.10, the Coarse Freudenthal compactification is a coarse compactification. Our next goal is to show that the corona $\partial_{CF} X$ is a coarse invariant of a space.

**Proposition 6.2.4.** *Let $(X, d)$ be a proper metric space. Then:*
*(a) $C_0(X, \mathbb{C}) = C_F(X, \mathbb{C}) \cap B_0(X, \mathbb{C})$.*
*(b) $B_F(X, \mathbb{C}) = C_F(X, \mathbb{C}) + B_0(X, \mathbb{C})$.*

*Proof.* The first part is clear. Let $f \in B_F(X, \mathbb{C}) \subset B_h(X, \mathbb{C})$. By 5.6.7, we have $B_h(X, \mathbb{C}) = C_h(X, \mathbb{C}) + B_0(X, \mathbb{C})$; so $f = g + h$ for some $g \in C_h(X, \mathbb{C})$ and $h \in B_0(X, \mathbb{C})$. Since $g$ is continuous and $f, h \in B_F(X, \mathbb{C})$, $g = f - h \in C_F(X, \mathbb{C})$. Therefore, $B_F(X, \mathbb{C}) = C_F(X, \mathbb{C}) + B_0(X, \mathbb{C})$. □

**Corollary 6.2.5.** *Let $(X, d)$ be a proper metric space. The corona of the Coarse Freudenthal compactification is the maximal ideal space of the $C^*$-algebra $B_F(X, \mathbb{C})/B_0(X, \mathbb{C})$.*

*Proof.* By 2.3.21, it suffices to show that $B_F(X, \mathbb{C})/B_0(X, \mathbb{C})$ and $C_F(X, \mathbb{C})/C_0(X, \mathbb{C})$ as $C^*$-algebras are $*$-isomorphic. Using the Second Isomorphism Theorem along with 6.2.4, we have $\frac{C_F(X,\mathbb{C})}{C_0(X,\mathbb{C})} = \frac{C_F(X,\mathbb{C})}{B_0(X,\mathbb{C}) \cap C_F(X,\mathbb{C})} \cong \frac{C_F(X,\mathbb{C})+B_0(X,\mathbb{C})}{B_0(X,\mathbb{C})} = \frac{B_F(X,\mathbb{C})}{B_0(X,\mathbb{C})}$. □

A consequence of 6.2.5, the Coarse Freudenthal corona can be defined for arbitrary metric spaces. Moreover, defining $\partial_{CF} X$ as the maximal ideal space of the $C^*$-algebra $B_F(X, \mathbb{C})/B_0(X, \mathbb{C})$ is more suitable when working with coarse spaces as, in contrast to boundedness of functions, the continuity of a function is a small scale property and it is not preserved when taking the composition of a coarse map with a continuous function. The following proposition demonstrates the usefulness of defining $\partial_{CF} X$ as the maximal ideal space of the $C^*$-algebra $B_F(X, \mathbb{C})/B_0(X, \mathbb{C})$.

**Lemma 6.2.6.** *Let $\varphi : X \to Y$ be a coarse map between metric spaces; it induces a $*$-homomorphism $\varphi^* : B_F(Y, \mathbb{C}) \to B_F(X, \mathbb{C})$ defined by mapping each $f \in B_F(Y, \mathbb{C})$ to $f \circ \varphi$. Furthermore, $\varphi^*(B_0(Y, \mathbb{C})) \subset B_0(X, \mathbb{C})$ and if $\lambda : X \to Y$ is another coarse map close to $\varphi : X \to Y$, then $\varphi^*$ and $\lambda^*$ agree up to $B_0(X, \mathbb{C})$.*

*Proof.* It is straightforward to see that $\varphi^*(B_0(Y, \mathbb{C})) \subset B_0(X, \mathbb{C})$, and by 6.1.13 we have $\varphi^*(B_F(Y, \mathbb{C})) \subset B_F(X, \mathbb{C})$. Therefore, $\varphi^*$ is a well-defined $*$-homomorphism. Let $\lambda : X \to Y$ be a coarse map close to $\varphi : X \to Y$, and let $r > 0$ such that $d_Y(\varphi(x), \lambda(x)) < r$ for all $x \in X$. For $f \in B_F(Y, \mathbb{C})$, we aim to show that $\varphi^*(f) - \lambda^*(f) \in B_0(X, \mathbb{C})$. To this end, for $\varepsilon > 0$, choose a bounded subset $K_Y$ of $Y$ such that whenever $a \in Y \setminus K_Y$, then $\|f(a) - f(b)\| < \varepsilon$ for all $b \in Y$ with $d_Y(a, b) < r$. Notice that $K_X := \varphi^{-1}(K_Y)$ is a bounded subset of $X$ and that for all $x \in X \setminus K_X$, $\varphi(x) \in Y \setminus K_Y$ and hence $\|f(\varphi(x)) - f(\lambda(x))\| < \varepsilon$. Thus, $\varphi^*(f) - \lambda^*(f) \in B_0(X, \mathbb{C})$ as desired. □

**Theorem 6.2.7.** *Coarsely equivalent metric spaces have homeomorphic Coarse Freudenthal coronas.*

*Proof.* Let $\varphi : X \to Y$ be a coarse equivalent map between metric spaces and $\psi : Y \to X$ is its coarse inverse. Define $\varphi^* : B_F(Y, \mathbb{C})/B_0(Y, \mathbb{C}) \to B_F(X, \mathbb{C})/B_0(X, \mathbb{C})$ by mapping

$f+B_0(Y,\mathbb{C})$ to $f\circ\varphi+B_0(X,\mathbb{C})$. By 6.2.6, $\varphi^*$ is a well-defined $*$-homomorphism. Moreover, the induced $*$-homomorphism $\psi^* : B_F(X,\mathbb{C})/B_0(X,\mathbb{C}) \to B_F(Y,\mathbb{C})/B_0(Y,\mathbb{C})$ that maps $g+B_0(X,\mathbb{C})$ to $g\circ\psi+B_0(Y,\mathbb{C})$ is an inverse of $\varphi^*$. By 2.3.21, the coronas $\partial_{CF}X$ and $\partial_{CF}Y$ are homeomorphic. $\square$

The following lemmas along with 6.2.10 tell us that there a correspondence between a finite family of mutually disjoint non-trivial coarsely clopen subsets of $X$ and a finite family of mutually disjoint clopen subsets of $\partial_{CF}X$. In particular, they give us a way to introduce the number of ends of proper metric spaces (see 6.2.12).

**Lemma 6.2.8.** *Let $Y$ be a coarse compactification of a proper metric space $(X,d)$, and $\partial_Y X = Y\setminus X$ be its corona.*
*(a) If $U$ is an open subset of $Y$ such that $C = cl_Y(U)\cap\partial_Y X$ is a clopen subset of $\partial_Y X$, then $U\cap X$ is coarsely clopen in $X$.*
*(b) If $C_1,\dots,C_n$ are mutually disjoint clopen subsets of $\partial_Y X$ covering $\partial_Y X$, then there exist mutually disjoint open and coarsely clopen subsets $U_1,\dots,U_n$ of $X$ such that $C_i = cl_Y(U_i)\cap\partial_Y X$. In such a case $X\setminus\bigcup_{i=1}^{n} U_i$ is always bounded.*

*Proof.* $(a)$ Choose open neighborhoods $V$ of $C$ and $W$ of $\partial_Y X\setminus C$ in $Y$ with disjoint closures. We may assume $V\subset U$ by replacing $V$ with $V\cap U$. Choose a continuous function $f:Y\to[0,1]$ such that $f(cl_Y(V))\subset\{0\}$ and $f(cl_Y(W))\subset\{1\}$. Notice $K:=Y\setminus(V\cup W)$ is a compact subset of $X$. Seeking contradiction, assume that $U\cap X$ is not coarsely clopen, then there exists $r>0$ and unbounded sequences $(x_n)_{n\in\mathbb{N}}\subset U\cap X\setminus K$ and $(y_n)_{n\in\mathbb{N}}\subset X\setminus(U\cup K)$ diverging to infinity with $d(x_n,y_n)<r$ for all $n\in\mathbb{N}$. Notice that $(x_n)_{n\in\mathbb{N}}\subset V\cap X$ and $(y_n)_{n\in\mathbb{N}}\subset X\cap W$, and hence $f(x_n)=0$ for all $n\in\mathbb{N}$, and $f(y_n)\to 1$, which contradicts the fact that $f|_X : X\to[0,1]$ is slowly oscillating.

$(b)$ Fix $i\le n$. For any $j\le n$ such that $j\ne i$, choose open neighborhoods $A_i^j$ and $B_j^i$ in $Y$ of $C_i$ and $C_j$, respectively, such that $cl(A_i^j)$ and $cl(B_j^i)$ are disjoint. Fix $i\le n$, and put $O_i=\bigcap_{j\ne i}(A_i^j\cap B_i^j)$. Notice that $O_1,\dots,O_n$ are mutually disjoint open subsets of $Y$ and that $C_i = cl(O_i)\cap\partial_Y X$ for all $i\le n$. Put $U_i = O_i\cap X$, then $U_1,\dots,U_n$ are mutually disjoint open and $cl(U_i)=cl(O_i)$ for all $i\le n$. Therefore, $C_i = cl(U_i)\cap\partial_Y X$ for all $i\le n$. By part $(a)$, $U_i$ is coarsely clopen for all $i \leq n$ as desired. Finally, since $O_1, \ldots, O_n$ cover $\partial_Y X$, $X \setminus \bigcup_{i=1}^{n} U_i = Y \setminus \bigcup_{i=1}^{n} O_i$ is compact and hence bounded. $\square$

**Lemma 6.2.9.** *Let $Y = CF(X)$ be the Coarse Freudenthal compactification of the proper metric space $(X, d)$.*

*(a) If $U$ is a coarsely clopen subset of $X$, then $(cl_Y(U) \cap cl_Y(U^c)) \cap \partial_{CF} X = \emptyset$, and therefore $C = cl_Y(U) \cap \partial_{CF} X$ is a clopen subset of $\partial_{CF} X$.*

*(b) If $U_1, \ldots, U_n$ are mutually disjoint non-trivial coarsely clopen subsets of $X$, then $C_i := cl_Y(U_i) \cap \partial_{CF} X$ are non-empty mutually disjoint clopen subsets $C_1, \ldots, C_n$ of $\partial_{CF} X$ forming its cover.*

*Proof.* $(a)$ Assume that $U$ is a coarsely clopen subset of $X$. Pick $r > 0$, and choose a compact subset $K \subset X$ such that $B(U, r) \cap B(U^c, r) \subset int(K)$. Put $A := U \setminus int(K)$, and notice that:

- $cl_Y(A) \cap \partial_{CF} X = cl_Y(U) \cap \partial_{CF} X$.
- $cl_Y(X \setminus A \cup int(K)) \cap \partial_{CF} X = cl_Y(U^c) \cap \partial_{CF} X$.
- $A$ is a closed subset of $X \setminus int(K)$ as $A = (X \setminus int(U^c)) \cap (X \setminus int(K))$.
- $A$ is an open subset of $X \setminus int(K)$ as $A = (X \setminus cl_X(U^c)) \cap (X \setminus int(K))$.

Now, since $A$ is a clopen subset of $X \setminus int(K)$, the characteristic function of $A$ $\chi_A : X \setminus int(K) \to \{0, 1\}$ is continuous. Moreover, since $X \setminus int(K)$ is a closed subset of the normal space $X$, the continuous function $\chi_A : X \setminus int(K) \to \{0, 1\}$, by Tietze extension theorem, admits a continuous extension $f : X \to [0, 1]$. Obviously, $f : X \to [0, 1]$ is glacially oscillating and therefore it admits a continuous extension $g : Y \to [0, 1]$. Notice that $g(cl_Y(U) \cap \partial_{CF} X) \subset \{1\}$ and $g(cl_Y(U^c) \cap \partial_{CF} X) \subset \{0\}$, so $(cl_Y(U) \cap cl_Y(U^c)) \cap \partial_{CF} X = \emptyset$, and hence $cl_Y(U) \cap \partial_{CF} X$ is a clopen subset in $\partial_{CF} X$ as its complement in $\partial_{CF} X$ is $cl_Y(U^c) \cap \partial_{CF} X$.

$(b)$ Let $U_1, \ldots, U_n$ be mutually disjoint non-trivial coarsely clopen subsets of $X$, by part $(a)$, each $C_i = cl(U_i) \cap Y \setminus X$ is clopen subset of $\partial_{CF} X$. Moreover, each $C_i$ is non-empty subset as each $U_i$ is unbounded. Notice that for all $1 \leq i \neq j \leq n$, $U_i \subset U_j^c$ and hence by $(a)$, $C_i \cap C_j = \emptyset$. $\square$

**Theorem 6.2.10.** *Let $Y = CF(X)$ be the Coarse Freudenthal compactification of the proper metric space $(X, d)$. Then, its corona $\partial_{CF}X$ is of dimension $0$.*

*Proof.* We aim to show that $\partial_{CF}X$ is totally disconnected. To this end, let $x, y \in \partial_{CF}X$ be two distinct points, and we want to find a clopen subset $C$ of $\partial_{CF}X$ such that $x \in C$ and $y \in C^c$. Let $f : Y \to [0, 1]$ be a continuous function such that $f(x) = 0$ and $f(y) = 1$. By 6.1.9 and 6.1.5, for $\varepsilon = 1/4$ we can choose a glacial scale $\mathcal{S}$ with the property that any $\mathcal{S}$-chain in $X$ is mapped by $f$ to a subset of diameter less than $1/4$. Put $A_1 = f^{-1}[0, 1/4] \cap X$ and $A_2 = f^{-1}(3/4, 1] \cap X$. Let $U_i$, $i = 1, 2$, be the set of all points of $X$ can be connected to $A_i$ by an $\mathcal{S}$-chain. Notice each $U_i$ is coarsely clopen (see 6.1.4) and $U_1 \cap U_2 = \emptyset$. Indeed, if $a \in U_1 \cap U_2$, then there exists $x_a \in A_1$ such that $a$ and $x_a$ are joinable by a $\mathcal{S}$-chain. Similarly, there exists $y_a \in A_2$ such that $a$ and $y_a$ are joinable by a $\mathcal{S}$-chain. In particular, $|f(a) - f(x_a)| < 1/4$ and $|f(a) - f(y_a)| < 1/4$. Since, $0 \leq f(x_a) \leq 1/4$ and $3/4 < f(y_a) \leq 1$, we have $1/2 < f(y_a) - 1/4 < f(a) < f(x_a) + 1/4 \leq 1/2$, a contradiction. Therefore $C_1 := cl(U_1) \cap \partial_{CF}X$ and $C_2 := cl(U_2) \cap \partial_{CF}X$ are disjoint clopen in $\partial_{CF}X$ containing $x$ and $y$, respectively (see 6.2.9). In particular, we can take $C = C_1$, and that proves $\partial_{CF}X$ is is totally disconnected and hence of dimension 0. □

**Definition 6.2.11.** The Coarse Freudenthal corona $\partial_{CF}X$ of a proper metric space $(X, d)$ is called the **space of ends Ends(X)**, and its cardinality is called the **number of ends**.

**Proposition 6.2.12.** *Let $(X, d)$ be a proper metric space. The number of ends of $X$ is the supremum of $n \geq 0$ such that there are $n$ mutually disjoint non-trivial coarsely clopen subsets of $X$.*

*Proof.* If $\partial_{CF}X$ has at least $n$ points, then it contains at least $n$ non-empty clopen sets $C_i$ that are mutually disjoint. By 6.2.9, we can find mutually disjoint open coarsely clopen subsets $U_i$ of $X$ such that $C_i = cl(U_i) \cap (\partial_{CF}X)$ for each $i \leq n$.

Conversely, the existence of $n$ mutually disjoint non-trivial coarsely clopen subsets of $X$ implies that $\partial_{CF}X$ contains at least $n$ points. □

The Freudenthal compactification of a Freudenthal space $X$ is universal among all compactifications of $X$ whose corona is of dimension 0, in the sense that it dominates every compactification of $X$ whose corona is of dimension 0. Analogous to the Freudenthal compactification, the corona of the Coarse Freudenthal compactification is of dimension 0, and that the Coarse Freudenthal compactification is universal among all coarse compactifications whose corona is of dimension 0. In particular, when the Freudenthal compactification of a Freudenthal space $X$ is a coarse compactification, the Freudenthal compactification and the Coarse Freudenthal compactification of $X$ coincide.

**Theorem 6.2.13.** *If $Y$ is a coarse compactification of a proper metric space $X$ such that the corona $\partial_Y X$ is of dimension 0, then $Y$ is dominated by $CF(X)$.*

*Proof.* Given a continuous function $f : Y \to \mathbb{C}$ suppose $\varepsilon > 0$. Since $\partial_Y X$ is a compact Hausdorff space of dimension 0, it is totally disconnected. Consider mutually disjoint clopen subsets $C_i$, $i \leq n$, covering $\partial_Y X$ such that $\operatorname{diam}(f(C_i)) < \varepsilon/2$ for each $i \leq n$. Moreover, for each $i \leq n$, choose $A_i$ and $B_i$ open neighborhoods with disjoint closures in $Y$ of $C_i$ and $\partial_Y X \setminus \bigcup_{j \neq i} C_j$, respectively. Let $I_i$ be a ball containing $f(C_i)$ with $\operatorname{diam}(I_i) < \varepsilon/2$; and take $A'_i := f^{-1}(B(I_i, \varepsilon/4))$. Notice that each $U_i := A_i \cap A'_i \cap O_i$, where $O_i = \bigcap_{j \neq i} B_j$, is an open subset of $Y$ with $C_i = cl_(U_i) \cap \partial_Y X$ and $C_i$ is a clopen subset of $\partial_Y X$, for all $i \leq n$. By 6.2.8, each $V_i := U_i \cap X$ is coarsely clopen. Moreover, $\operatorname{diam}(f(V_i)) < \varepsilon$ as $\operatorname{diam}(f(A'_i)) < \varepsilon$, for all $i \leq n$. Notice that as $U_1, \ldots, U_n$ cover $\partial_Y X$, and $K := X \setminus \bigcup_{i=1}^{n} V_i$ is bounded. By 6.1.9 (c), $f|_X$ is glacially oscillating.

□

**Corollary 6.2.14.** *The Coarse Freudenthal compactification of a proper metric space $X$ is the maximal coarse compactification whose corona is of dimension 0.*

*Proof.* By 6.2.10 the corona of the Coarse Freudenthal compactification is of dimension 0. By 6.2.13 any coarse compactification whose corona is of dimension 0 is dominated by the Coarse Freudenthal compactification. □

The Coarse Freudenthal compactification is the quotient space of the Higson compactification obtained by collapsing each component of its Higson corona to a singleton.

**Corollary 6.2.15.** *Let $X$ be a proper metric space and $hX$ be its Higson compactification. Define a relation on $hX$ by $x \sim x$ for all $x \in X$ and for all $x, y \in \partial_h X$, $x \sim y$ if and* *only if $x$ and $y$ are in the same component of $\partial_h X$. The quotient space $hX/_\sim$ is a coarse compactification of $X$ with totally disconnected corona. In particular, $hX/_\sim = CF(X)$.*

*Proof.* Similar to 4.3.3. □

Another way to define the Coarse Freudenthal compactification of a proper metric $X$ is as the supremum of finite-point coarse compactifications of $X$.

**Corollary 6.2.16.** *The Coarse Freudenthal compactification of a proper metric $X$ is the the supremum of finite-point coarse compactifications of $X$.*

*Proof.* Similar to 4.3.4. □

The Coarse Freudenthal compactification of a proper geodesic metric space is its Freudenthal compactification.

**Corollary 6.2.17.** *If $(X, d)$ is a proper geodesic space, then its Freudenthal compactification is the Coarse Freudenthal compactification.*

*Proof.* By 5.6.12, the Freudenthal compactification of $X$ is a coarse compactification, and its corona is of dimension 0. Therefore, it is dominated by the Coarse Freudenthal compactification. On the other hand, the Freudenthal compactification dominates every compactification of $X$ whose corona is of dimension 0. □

## 6.3 Ends of countable groups

By 5.3.6, if a countable group $G$ is equipped with two proper, left-invariant metrics $d_1$ and $d_2$; then $(G, d_1)$ and $(G, d_2)$ are coarsely equivalent. By 6.2.7, the proper metric space $(G, d_1)$ and $(G, d_2)$ have homeomorphic Coarse Freudenthal coronas. We know also, by 6.2.17, that the Coarse Freudenthal compactification of a proper metric space $X$ coincide with its Freudenthal compactification when the metric space $X$ is proper geodesic. Therefore, the natural way to define the space of ends of a countable as the Coarse Freudenthal corona of $(G, d)$ for some proper left-invariant metric $d : G \times G \to [0, \infty)$. All we need is to show that a such metric always exists.

**Proposition 6.3.1.** *Let $G$ be a countable group. There exists a proper left-invariant metric $d$ on $G$.*

*Proof.* One way to introduce a proper left-invariant metric $d$ on a countable group $G$ that is not necessarily finitely generated is as follows:

1. Choose a map $\psi : G \to \mathbb{N} \cup \{0\}$ with the following properties:

i. $\psi(g) = 0$ if and only if $g = 1_G$.

ii. $\psi^{-1}(\{n\})$ is finite for all $n \in \mathbb{N} \cup \{0\}$.

iii. $g \in \psi^{-1}(\{n\})$ if and only if $g^{-1} \in \psi^{-1}(\{n\})$.

2. Equip $G$ with the norm $|.| : G \to \mathbb{N} \cup \{0\}$ given by:

$$|g| := min\{\sum_{i=1}^{n} \psi(g_i) | g = g_1 \cdots g_n \in G\},$$

3. Define the metric $d(g, h) := |g^{-1} \cdot h|$, then $d$ is a proper left-invariant metric on $G$. □

**Definition 6.3.2.** The **number of ends** of a countable group $G$ is the cardinality of the space of its ends $Ends(G)$, where $G$ is equipped with some proper left-invariant metric.

Clearly that if a countable group $G$ is finitely generated, then the space of ends $Ends(G)$ as the Coarse Freudenthal corona coincide with the space of ends $ends(G)$ as the Freudenthal corona where $G$ is equipped with the work metric (Cayley metric) with respect to a symmetric finite subset $K \subset G$ generating $G$. Properties of countable groups will be discuss in a more general setting as a special case of coarse groups. We first introduce ends in coarse space in general in the next section, then we restrict ourselves to coarse groups.

# Chapter 7

# Freudenthal large scale compactification and coarse ends of coarse spaces

In this chapter we generalize the concept of Freudenthal ends to arbitrary coarse spaces. See (14) for other ways to introduce ends in coarse spaces. All coarse spaces are assumed to be coarsely connected. This chapter is based on Section 2 of (16).

## 7.1 Construction and properties of space of coarse ends

**Definition 7.1.1.** A **topological coarse space** is a set equipped with coarse structure and with a compatible topology. Additionally, we assume that the coarse structure is **coarsely connected**, i.e. the union of two bounded subsets of $X$ is bounded.

**Definition 7.1.2.** A **coarse end** of a coarse space $X$ is a family $e$ of unbounded and coarsely clopen subsets of $X$ that is maximal with respect to the property of all finite intersections being unbounded. Equivalently, $e$ is a coarse end of the coarse space $X$ if it is an ultrafilter of the quotient Boolean algebra of $\mathcal{CC}(X)$ by the ideal of all bounded subsets, where $\mathcal{CC}(X)$ is the family of all coarsely clopen subsets of $X$. The family of all coarse ends of $X$ is denoted by $Ends(X)$.

**Notation 7.1.3.** *Let $X$ be a coarse space. If $U$ is a coarsely clopen subset of $X$, we define $U_{end} := \{e \in Ends(X) : U \in e\}$ and $\widetilde{U}_{end} := U_{end} \cup U$.*

**Proposition 7.1.4.** *Let $\mathcal{T}$ be the topology of a topological coarse space $X$. The collection $\widetilde{\mathcal{B}} = \{\widetilde{U}_{end} : U$ open coarsely clopen subset of $X\}$ is a basis for a topology $\mathcal{T}_{end}$ on $X \cup Ends(X)$ that extends the topology $\mathcal{T}$.*

*Proof.* Clearly $\widetilde{\mathcal{B}}$ covers $X \cup Ends(X)$ as either $X$ is unbounded open coarsely clopen and $\widetilde{X}_{end} = X \cup Ends(X)$ or $X$ is bounded and $Ends(X) = \emptyset$. Now, let $\widetilde{U}_{end}, \widetilde{V}_{end} \in \widetilde{\mathcal{B}}$. Notice $\widetilde{U}_{end} \cap \widetilde{V}_{end} = (\widetilde{U \cap V})_{end} \in \widetilde{\mathcal{B}}$. Since $X$ is topological coarse space, by 5.5.5, it admits a basis consists of coarsely bounded subsets. Finally, every bounded element of $\mathcal{T}$ is contained in $\widetilde{\mathcal{B}}$ which implies that $\mathcal{T} \subset \mathcal{T}_{end}$. $\square$

**Corollary 7.1.5.** *Let $\mathcal{T}$ be the topology of a topological coarse space $X$. The collection $\mathcal{B}_{\mathcal{T}} = \{U_{end} : U$ open coarsely clopen subset of $X\}$ is a basis for the topology of $Ends(X)$ as a subspace of $X \cup Ends(X)$.*

The topology on $Ends(X)$ is independent of the compatible topology on the coarse space $X$.

**Lemma 7.1.6.** *Let $X$ be a topological coarse space and $A$ belongs to a coarse end $e$ of $X$, then there is an open set $A' \in e$ contained in $A$.*

*Proof.* Let $\mathcal{U}$ be a uniformly bounded open cover of $X$, and put $E_{\mathcal{U}} := \bigcup_{U \in \mathcal{U}} U \times U$. Notice that for any bounded subset $K \subset X$, $cl(K) \subset E_{\mathcal{U}}[K]$, and hence bounded. The subset $K := E_{\mathcal{U}}[A] \cap E_{\mathcal{U}}[X \setminus A]$ is bounded. Therefore, $A' := A \setminus cl(K)$ is coarsely clopen and it is in $e$. Moreover, if $x \in A'$, then $E_{\mathcal{U}}[x] \setminus cl(K)$ is an open neighborhood of $x$ and $E_{\mathcal{U}}[x] \setminus cl(K) \subset A'$. Indeed, if $y \in (E_{\mathcal{U}}[x] \setminus cl(K)) \setminus A$, then there exists $U \in \mathcal{U}$ such that $x, y \in U$. Hence $x \in E_{\mathcal{U}}[X \setminus A] \cap A \subset K$, a contradiction. This shows that $A'$ is open. $\square$

**Corollary 7.1.7.** *Given two compatible topologies on a coarse space $X$, the induced topologies on $Ends(X)$ coincide.*

*Proof.* Let $\mathcal{T}$ be a compatible topology of a coarse space $X$. Let $A$ be unbounded coarsely clopen subset of $X$. We claim that $A_{end}$ is open in the topology of $Ends(X)$ induced by the basis $\mathcal{B}_{\mathcal{T}}$. Let $e \in A_{end}$, then $A \in e$ and hence by 7.1.6, there exists $A' \in e$ open such that $A' \subset A$. Hence $e \in A'_{end} \subset A_{end}$ which shows that $A_{end}$ is open in the topology of $Ends(X)$. $\square$

**Lemma 7.1.8.** *Let $e \in Ends(X)$ and $A \in e$, then $int(A), cl(A) \in e$.*

*Proof.* By 7.1.6, there exists $A' \in e$ open and coarsely identical to $A$ such that $A' \subset A$. Hence, $A' \subset int(A) \subset A$, which implies that $int(A) \in e$. To show that $cl(A) \in e$, it suffices to show that $int(A)$ and $cl(A)$ are coarsely identical. Let $\mathcal{U}$ be a uniformly bounded open cover of $X$, and notice that $cl(A) \setminus int(A) \subset E_{\mathcal{U}}[A] \cap E_{\mathcal{U}}[X \setminus A]$ is bounded as $A$ is coarsely clopen. □

**Proposition 7.1.9.** *Let $X$ be a topological coarse space. If $X$ is Hausdorff, then the topological space $X \cup Ends(X)$ is Hausdorff.*

*Proof.* We consider two cases:

**Case 1:** Let $x \in X$ and $e \in Ends(X)$ and let $\mathcal{U}$ be a uniformly bounded open cover of $X$. Choose $U \in \mathcal{U}$ such that $x \in U$. Notice that $cl(U)$ is a bounded subset and hence $V := X \setminus cl(U)$ is open and coarsely clopen contained in $e$. Hence, $U$ and $\widetilde{V}_{end}$ are disjoint open neighborhood of $x$ and $e$, respectively.

**Case 2:** If $e_1, e_2 \in Ends(X)$ are two distinct coarse ends of $X$, then we can find $A \in e_1$ and $B \in e_2$ such that $A \cap B$ is a bounded subset of $X$. Without loss of generality, we may assume that $A \cap B = \emptyset$. By 7.1.8, $U = int(A) \in e_1$ and $V = int(B) \in e_2$. Clearly, $U$ and $V$ are disjoint and hence $\widetilde{U}_{end}$ and $\widetilde{V}_{end}$ are disjoint neighborhoods of $e_1$ and $e_2$, respectively. □

$X$ is an open dense subspace of $X \cup Ends(X)$. Moreover, $(X \cup Ends(X), \mathcal{T}_{end})$ is Hausdorff whence $(X, \mathcal{T})$ is Hausdorff. The question then arises: is $(X \cup Ends(X), \mathcal{T}_{end})$ a compactification of $(X, \mathcal{T})$? The answer is positive in a coarse sense!

**Definition 7.1.10.** Let $Y$ be a topological space and $X \subset Y$ is a topological subspace equipped with a coarse structure compatible with its topology. $Y$ is **large scale compact** if for any open cover $\{U_s\}_{s \in S}$ of $Y$, there is a finite subset $F$ of $S$ such that $Y \setminus \bigcup_{s \in F} U_s$ is a bounded subset of $X$. $Y$ is a **large scale compactification** of $X$ if in addition to being large scale compact, $Y$ is Hausdorff and $X$ is an open dense subspace of $Y$(see (15)).

**Observation 7.1.11.** *When $X$ is a proper metric space every large scale compactification is indeed a compactification of $X$.*

**Lemma 7.1.12.** *For any family $\{U^s\}_{s\in I}$ of coarsely clopen subsets of $X$ such that $Ends(X) \subset \bigcup\limits_{s\in I} U^s_{end}$, there is a finite subset $F$ of $I$ such that $X \setminus \bigcup\limits_{s\in F} U^s$ is a bounded subset of $X$.*

*Proof.* Let $\mathcal{F}$ be the collection of all finite subsets of $I$. Seeking contradiction assume that for any $F \in \mathcal{F}$, $A_F = X \setminus \bigcup\limits_{s\in F} U^s$ is unbounded. The collection $\{A_F : F \in \mathcal{F}\}$ is contained in some coarse end $e \in Ends(X)$. Hence, $\{A_F : F \in \mathcal{F}\} \subset e \in U^s_{end}$ for some $s \in I$ which implies that $U^s \in e$ and $X \setminus U^s \in e$, a contradiction. □

**Theorem 7.1.13.** *Let $X$ be a Hausdorff topological coarse space, the topological space $X \cup Ends(X)$ is a large scale compactification of $X$. Furthermore, $Ends(X)$ is a compact totally disconnected subspace of $X \cup Ends(X)$.*

*Proof.* Let $\{O^s\}_{s\in I}$ be an open cover $X \cup Ends(X)$, we can find a subset $J \subset I$ such that $\{\widetilde{U^s}_{end}\}_{s\in J}$ is a sub-cover of $\{O^s\}_{s\in I}$ that covers $Ends(X)$. In particular, $Ends(X) \subset \bigcup\limits_{s\in J} U^s_{end}$. By the above lemma, there is a finite subset $F$ of $J$ such that $X\setminus \bigcup\limits_{s\in F} U^s$ is a bounded subset of $X$. We claim that $Ends(X) \subset \bigcup\limits_{s\in F} U^s_{end}$. To this end, notice that if $U \subset X$ is an arbitrary coarsely clopen subset of $X$ and $V = X \setminus U$, then $Ends(X) = U_{end} \cup V_{end}$. Indeed, if $e \in Ends(X) \setminus (U_{end} \cup V_{end})$, then we can find $A, B \in e$ such that both $A \cap U$ and $B \cap V$ are bounded. Hence, there exist $K, L$ bounded subsets of $X$ such that $A \subset V \cup K$ and $B \subset U \cup L$. In particular, $A \cap B$ is bounded, a contradiction. Now, seeking contradiction assume that $e \in Ends(X) \setminus \bigcup\limits_{s\in F} U^s_{end}$, then $e \notin U^s_{end}$ for all $s \in F$. Therefore, by above observation, $X \setminus U^s \in e$ for all $s \in F$ which implies that $\bigcap\limits_{s\in F}(X \setminus U^s) = X \setminus \bigcup\limits_{s\in F} U^s \in e$, a contradiction. This shows that $Ends(X)$ is compact and $X \cup Ends(X)$ is large scale compact as $(X \cup Ends(X)) \setminus \bigcup\limits_{s\in F} \widetilde{U^s}_{end}$ is bounded.
Finally, we show that $Ends(X)$ is totally disconnected. Let $e_1, e_2 \in Ends(X)$ be two distinct coarse ends of $X$, then we can find $A \in e_1$ and $B \in e_2$ such that $cl(A) \cap B$ is a bounded subset of $X$. Without loss of generality, we may assume that $cl(A) \cap B = \emptyset$. Clearly, $U = int(A) \in e_1$ and $V = X \setminus cl(A) \in e_2$. Moreover, $Ends(X) = U_{end} \cup V_{end}$ and $U_{end} \cap V_{end} = \emptyset$. □

Alternatively, one can show that the space of coarse ends $Ends(X)$ is compact totally disconnected Hausdorff space by noticing that $Ends(X)$ is actually the Stone's space (space of ultrafilters) of the quotient Boolean algebra of $\mathcal{CC}(X)$ by the ideal of all bounded subsets.

**Definition 7.1.14.** Let $X$ be a Hausdorff topological coarse space. We call the space $X \cup Ends(X)$ the **Freudenthal large scale compactification**.

In (7), the space of ends of $X$ on which the group $G$ acts is defined as the Stone's space of the quotient Boolean algebra of all $G$-commensurated subsets ($G$-almost invariant) of $X$ by the ideal of all finite subsets. We show that there exists a connected coarse structure on $X$ such that the space of ends of $X$, in the sense of (7), is exactly the space of coarse ends of $X$ with respect to that coarse structure.

**Proposition 7.1.15.** *Let $X$ be a non-empty set on which the group $G$ acts transitively. Equip $X$ with the coarse structure $\mathcal{E}_G$ induced by the right action as in 5.1.3. A subset $A \subset X$ is coarsely clopen if and only if it is $G$-almost invariant, that is, $A \cdot g \Delta A$ is finite for all $g \in G$.*

*Proof.* Observe that for all $g \in G$, $Y \subset X$, then $E^Y_{\{g\}} \subset A \cdot g^{-1}$. □

Let us show that $Ends(X)$ is a coarse invariant of a space.

**Lemma 7.1.16.** *Suppose $f : X \to Y$ is a coarse map between topological coarse spaces and $e \in Ends(X)$. Then, using the closure operation $cl$ in $Y \cup Ends(Y)$, the following hold:*
*(a) If $e' \in \bigcap\limits_{A \in e} cl(f(A))$ is a coarse end of $Y$ and $V \in e'$ is an open and coarsely clopen subset of $Y$, then $f^{-1}(V) \in e$.*
*(b) $\bigcap\limits_{A \in e} cl(f(A))$ is a singleton belonging to $Ends(Y)$.*

*Proof.* By 5.6.16, $f^{-1}(D)$ is coarsely clopen in $X$ if $D$ is coarsely clopen in $Y$.

$(a)$ If $f^{-1}(V) \notin e$, then there is $A \in e$ such that $A \cap f^{-1}(V)$ is bounded. Observe that $f(A \cap f^{-1}(V)) = f(A) \cap V$. Put $K := f(A) \cap V$; since $K$ is a bounded subset of $Y$, $f^{-1}(K)$ is a bounded subset of $X$. In particular, $A' = A \setminus f^{-1}(K) \in e$ and hence $e' \in cl(f(A'))$. On the other hand, notice that $f(A') \cap V = \emptyset$ and therefore $f(A') \cap \widetilde{V}_{end} = \emptyset$, contradicting the fact that $e' \in cl(f(A'))$. Thus $f^{-1}(V) \in e$, as desired.

$(b)$ First assume that $y \in \bigcap\limits_{A \in e} cl(f(A)) \cap Y$, and choose an open bounded neighborhood $K$ of $y$ in $Y$. Then, $f^{-1}(K)$ is bounded in $X$. Hence $A := X \setminus f^{-1}(K) \in e$. But, $f(A) \cap K = \emptyset$, which contradicts the assumption that $y \in cl(f(A))$. Therefore, $\bigcap\limits_{A \in e} cl(f(A)) \subset Ends(Y)$.

Secondly, assume that $\bigcap_{A\in e} cl(f(A)) = \emptyset$. Put $B_A = (Y \cup Ends(Y)) \setminus cl(f(A))$, then clearly $\{B_A : A \in e\}$ is an open cover of $Y \cup Ends(Y)$. Since $Y \cup Ends(Y)$ is a large scale compactification of $Y$, there exist finitely many $A_i \in e$, $i \leq n$, such that $(Y \cup Ends(Y)) \setminus (\bigcup_{i=1}^{n} B_{A_i}) = \bigcap_{i=1}^{n} cl(f(A_i))$ is a bounded subset of $Y$, and a fortiori, $\bigcap_{i=1}^{n} f(A_i)$ is bounded. However, $A := \bigcap_{i=1}^{n} A_i \in e$, and $f(A) \subset \bigcap_{i=1}^{n} f(A_i)$ is bounded, a contradiction. Finally, if $e_1, e_2 \in \bigcap_{A\in e} cl(f(A))$ and $e_1 \neq e_2$, then there exists two disjoint open and coarsely clopen subsets $V_1 \in e_1$ and $V_2 \in e_2$. By $(a)$, $f^{-1}(V_1) \in e$ and $f^{-1}(V_2) \in e$, a contradiction. □

**Corollary 7.1.17.** *If $f : X \to Y$ is a coarse map between topological coarse spaces and $e \in Ends(X)$, then it induces a continuous function $f_{end} : Ends(X) \to Ends(Y)$ defined by $f_{end}(e) \in \bigcap_{A\in e} cl(f(A))$.*
*(a) If $f, g : X \to Y$ are close, then $f_{end} = g_{end}$.*
*(b) If $f$ is continuous, then $f \cup f_{end} : X \cup Ends(X) \to Y \cup Ends(Y)$ is continuous.*

*Proof.* Given an open, coarsely clopen subset $V$ of $Y$, notice that $f_{end}^{-1}(V_{end}) \subset U_{end}$, where $U = f^{-1}(V)$. Indeed, if $f_{end}(e) \in V_{end}$ then $e \in U_{end}$ by $(a)$ of 7.1.16. That proves continuity of $f_{end}$ and of $f \cup f_{end}$ if $f$ is continuous.

Assume that $f, g : X \to Y$ are close, then $F := \{(f(x), g(x)) : x \in X\}$ is an entourage of $Y$. If there is $e \in Ends(X)$ such that $f_{end}(e) \neq g_{end}(e)$, then we can choose open subsets $U \in f_{end}(e), V \in g_{end}(e)$ so that $U \cap F[V] = \emptyset$. Notice $f^{-1}(U) \in e$ and $g^{-1}(V) \in e$ by $(a)$ of 7.1.16. Now, there is $x \in f^{-1}(U) \cap g^{-1}(V)$, so $f(x) \in U$, $g(x) \in V$ contradicting $U \cap F[V] = \emptyset$. □

**Corollary 7.1.18.** *If two topological coarse spaces $X$ and $Y$ are coarsely equivalent, then $Ends(X)$ is homeomorphic to $Ends(Y)$.*

*Proof.* It suffices to show that if $f : X \to Y$ and $g : Y \to Z$ are coarse maps between topological coarse spaces, then $(g \circ f)_{end} = f_{end} \circ g_{end}$. Indeed, let $e \in Ends(X)$ such that $(g \circ f)_{end}(e) \neq f_{end} \circ g_{end}(e)$. There exist open disjoint subsets $U \in (g \circ f)_{end}(e)$ and $V \in f_{end} \circ g_{end}(e)$. Now, $(g \circ f)^{-1}(U) \in e$ and $f^{-1}(V) \in g_{end}(e)$, by $(a)$ of 7.1.16. In particular, $(g \circ f)^{-1}(U), (g \circ f)^{-1}(V) \in e$, a contradiction. □

## 7.2 Freudenthal large scale compactification vs Coarse Freudenthal compactification

The aim of this section is to show that when $(X, d)$ is a proper metric space, the compactifications $CF(X)$ and $X \cup Ends(X)$ of $X$ are identical, where $Ends(X)$ the space of coarse ends with respect to the bounded coarse structure induced by its metric $d$.

**Proposition 7.2.1.** *Let $(X, d)$ be a proper metric space. The compactification $X \cup Ends(X)$ of $X$ is dominated by $CF(X)$.*

*Proof.* By 6.2.13, since the corona $Ends(X)$ of $X \cup Ends(X)$ is of dimension 0, it suffices to show that $X \cup Ends(X)$ is a coarse compactification of $X$.

Suppose $f : X \cup Ends(X) \to \mathbb{C}$ is a continuous function but $f|_X$ is not slowly oscillating. There exists $r > 0$, and sequences $(x_n)$ and $(y_n)$ in $X$ diverging to infinity such that $d(x_n, y_n) < r$, but $f(x_n) \to z$, $f(y_n) \to w$ and $z \neq w \in \mathbb{C}$. Put $\varepsilon = \|z - w\| /3$ and cover $Ends(X)$ with a finite family of mutually disjoint sets $\widetilde{U}^i_{end}$, $i \le m$, where each $U^i$ is open in $X$ and $\operatorname{diam}(f(U^i)) < \varepsilon$ for each $i$. Let $V = U^i$ for some $i \le m$ such that $V$ contains infinitely many $x_n$'s. Without loss of generality we may assume $x_n \in V$ for each $n \in \mathbb{N}$. As $V$ is coarsely clopen, there exists a bounded subset $K_r \subset X$ such that $B(V, r) \cap V^c \setminus K_r = \emptyset$. In particular, infinitely many $y_n$'s must belong to $V$. There exists $N \in \mathbb{N}$ such that $x_N, y_N \in V$ and $\|f(x_N) - f(y_N)\| > \varepsilon$, a contradiction to $\operatorname{diam}(f(V)) < \varepsilon$. This shows that $X \cup Ends(X)$ is indeed a coarse compactification and hence dominated by $CF(X)$. $\square$

**Lemma 7.2.2.** *If $(X, d)$ is a metric space and $f : X \to \mathbb{C}$ is glacially oscillating, then:*
*(a) For each compact subset $C$ of an open set $U \subset \mathbb{C}$ there is a coarsely clopen subset $A$ of $X$ such that $f^{-1}(C) \subset A \subset f^{-1}(U)$.*
*(b) If, moreover, $f : X \to \mathbb{C}$ is bounded, then for any coarse end $e \in Ends(X)$ of $X$ the subset $\bigcap\limits_{A \in e} cl(f(A))$ consists of one point.*

*Proof.* (a) Choose $\varepsilon > 0$ such that $B(C, \varepsilon) \subset U$. Then choose a glacial scale $\mathcal{S}$ such that $|f(x) - f(y)| < \varepsilon$ if $x$ and $y$ can be connected by an $\mathcal{S}$-chain. Define $A$ as all points in $X$ that can be connected to $f^{-1}(C)$ by an $\mathcal{S}$-chain. Clearly, $f^{-1}(C) \subset A \subset f^{-1}(U)$ and $A$ is coarsely clopen by 6.1.4.

($b$) Choose $r > 0$ such that $f(X) \subset \overline{B}(0, r)$. Let $e$ be a coarse end of $X$; the set $D := \bigcap_{A \in e} cl(f(A))$ is non-empty by the Finite Intersection Property. Suppose there are two different points $z, w \in D$ and put $\varepsilon = \|z - w\| / 4$. By part ($a$), there is a coarsely clopen subset $A'$ of $X$ such that $f^{-1}(\overline{B}(z, \varepsilon)) \subset A' \subset f^{-1}(\overline{B}(z, 2\varepsilon))$. We claim that $A' \notin e$. Indeed, assume that $A' \in e$, and notice that the inclusion $A' \subset f^{-1}(\overline{B}(z, 2\varepsilon))$ implies that $f(A') \subset f(f^{-1}(\overline{B}(z, 2\varepsilon))) \subset (\overline{B}(z, 2\varepsilon))$, so $w \notin cl(f(A'))$, a contradiction. Now, let $A \in e$ such that $A \cap A' = \emptyset$. Notice that as $A = A \setminus A' \subset f^{-1}(\overline{B}(0, r) \setminus B(z, \varepsilon))$, so $cl(f(A)) \subset \overline{B}(0, r) \setminus B(z, \varepsilon)$ and thus $z \notin cl(f(A))$, a contradiction. Therefore, $D := \bigcap_{A \in E} cl(f(A))$ is a singleton as desired. □

**Proposition 7.2.3.** *If $(X, d)$ is a proper metric space, then the coarse compactification $X \cup Ends(X)$ dominates the coarse Freudenthal compactification $CF(X)$.*

*Proof.* Let $f : X \to \mathbb{C}$ be a continuous and glacially oscillating function. For any coarse end $e \in Ends(X)$ of $X$, put $\bigcap_{A \in e} cl(f(A)) = \{a_e\}$, and extend $f : X \to \mathbb{C}$ to $\bar{f} : X \cup Ends(X) \to \mathbb{C}$ that maps each coarse end $e \in Ends(X)$ to $a_e$. Let $\varepsilon > 0$ and $B(a_e, \varepsilon)$ be a neighborhood of $a_e$, $e$ being a fixed coarse end. There exists a coarsely clopen subset $A'$ of $X$ such that $f^{-1}(\overline{B}(a_e, \varepsilon/4)) \subset A' \subset f^{-1}(B(a_e, \varepsilon/2))$. Notice that $A'$ is unbounded as otherwise we would have $A \setminus A' \in e$ for any $A \in e$, which is a contradiction as $a_e \notin cl(f(A \setminus A'))$. Let $O = int(cl(A'))$, then $O$ is an open set belonging to $e$.

Choose disjoint open neighborhoods $\widetilde{U}_{end}$ and $V$ of $e$ and $Ends(X) \setminus \widetilde{O}_{end}$, respectively. Now, $\widetilde{O}_{end} \cap \widetilde{U}_{end}$ is a neighborhood of $e$ and $\bar{f}(\widetilde{O}_{end} \cap \widetilde{U}_{end}) \subset (B(a_e, \varepsilon))$. Hence, $\bar{f}$ is a continuous extension of $f : X \to \mathbb{C}$ and therefore $X \cup Ends(X)$ dominates the coarse Freudenthal compactification $CF(X)$. □

**Corollary 7.2.4.** *Let $(X, d)$ be a proper metric space equipped with the bounded coarse structure induced by $d$. Then, $X \cup Ends(X) = CF(X)$. If, moreover, $(X, d)$ be a geodesic metric, then $X \cup Ends(X) = CF(X) = \bar{X}$, where $\bar{X}$ is the Freudenthal compactification of $X$.*

*Proof.* Follows from 7.2.1, 7.2.3 and 6.2.17. □

We next show that the the space of Freundenthal ends of a Freundenthal space $X$ is actually the space of coarse ends $Ends(X)$ with respect to some coarse structure $\mathcal{E}$ on $X$.

## 7.3 The space of Freundenthal ends as the space of coarse ends

Th aim of this section is to construct a coarse structure $\mathcal{E}_{\mathcal{F}}$ on a Freudenthal space $X$ such that the space of Freudenthal ends and the space of coarse ends are topologically homeomorphic. As observed in 4.1.11, we aim to construct a coarse structure $\mathcal{E}_{\mathcal{F}}$ on a Freudenthal space $X$ such that each Freudenthal end is contained exactly in one coarse end and every coarse end contains exactly one Freudenthal end.

**Lemma 7.3.1.** *Let $X$ be a Freudenthal space. $X$ admits an exhausting sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets such that both $K_n$ and $int(K_n)$ are connected.*

*Proof.* Since $X$ is $\sigma$-compact, it can be written as a countable union of compact subsets, say $X := \bigcup\limits_{n\in\mathbb{N}} L_n$. As $X$ is locally compact and locally connected, each $L_n$ can be covered by finitely many open connected and relatively compact subsets. Therefore $X$ is a countable union of open connected relatively compact subsets, say $X := \bigcup\limits_{n\in\mathbb{N}} O_n$. Now, we inductively construct an increasing sequence $(U_n)_{n\in\mathbb{N}\cup\{0\}}$ of open connected relatively compact subsets satisfying $cl_X(U_n) \subset U_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$, and $X := \bigcup\limits_{n\in\mathbb{N}} U_n$. Put $U_0 := \emptyset$, and assume that $U_1, \ldots, U_n$ are open connected relatively compact subsets such that $cl_X(U_i) \subset U_{i+1}$ for all $1 \leq i < n$. Since $cl_X(U_n)$ is compact, there exist $O_{n_1}, \ldots, O_{n_m}$ open connected relatively compact subsets such that $cl_X(U_n) \subset \bigcup\limits_{i=1}^{m} O_{n_i}$ and $cl_X(U_n) \cap O_{n_i} \neq \emptyset$ for all $1 \leq i \leq m$. Put $U_{n+1} := \bigcup\limits_{i=1}^{m} O_{n_i}$; since $X$ is connected, $cl_X(U_n)$ is properly contained in $U_{n+1}$. Moreover, $U_{n+1}$ is open connected relatively compact subset. Finally, since $cl_X(\bigcup\limits_{n\in\mathbb{N}} U_n) = \bigcup\limits_{n\in\mathbb{N}} cl_X(U_n) = \bigcup\limits_{n\in\mathbb{N}} U_n$, and $X$ is connected, so $X = \bigcup\limits_{n\in\mathbb{N}} U_n$.

Now, put $K_n := cl_X(U_n)$ for all $n \in \mathbb{N}$. Notice that for any open subset $U \subset X$, one has $U \subset int(cl_X(U))$. Notice that:

$$U_n \subset int(cl_X(U_n)) = int(K_n) \subset K_n = cl_X(U_n) \subset U_{n+1} \subset int(cl_X(U_{n+1})) = int(K_{n+1}),$$

and therefore, $int(K_n)$ is connected for all $n \in \mathbb{N}$, and $(K_n)_{n\in\mathbb{N}}$ is an exhausting sequence consists of compact connected subsets. $\square$

**Lemma 7.3.2.** *Let $X$ be a Freudenthal space, and $\mathcal{F}$ be the family of all covers of $X$ that consists of of open connected subsets of $X$ such that for every relatively compact subset $K \subset X$, $E_{\mathcal{V}}[K]$ is relatively compact for all $\mathcal{V} \in \mathcal{F}$. The family $\mathcal{E}_{\mathcal{F}} := \{E \subset X \times X : E \subset E_{\mathcal{V}}$ for some $\mathcal{V} \in \mathcal{F}\}$ is a proper coarse structure.*

*Proof.* Put $K_0 := \emptyset$. By 7.3.1, we can find an exhausting sequence $(K_n)_{n\in\mathbb{N}\cup\{0\}}$ of compact subsets such that both $K_n$ and $int(K_n)$ are connected. Without loss of generality, we may assume that every component of $X \setminus K_1$ is unbounded component. Put $U_n := int(K_{n+2}) \setminus K_n$ and $\mathcal{U} := \{U \subset X : U$ is a component of $U_n$, $n \in \mathbb{N}\}$. Since $X$ is locally connected, every $U \in \mathcal{U}$ is open connected and relatively compact subset. Moreover, every relatively compact subset $K \subset X$ is covered by finitely many open connected relatively compact subsets of the form $U_n$. Hence, $E_{\mathcal{U}}[K]$ is relatively compact. Thus, $E_{\mathcal{U}} \in \mathcal{E}_{\mathcal{F}}$, and therefore $\Delta \in \mathcal{E}_{\mathcal{F}}$ as $\Delta \subset E_{\mathcal{U}}$. It is straightforward to see that $\mathcal{E}_{\mathcal{F}}$ is a coarse structure and since $E_{\mathcal{U}}$ is an open entourage containing the diagonal, $\mathcal{E}_{\mathcal{F}}$ is a topological coarse structure. Finally, if $B \subset X$ is a coarsely bounded subset of $X$, then there exists $x \in X$ and $\mathcal{V} \in \mathcal{F}$ such that $B \subset E_{\mathcal{V}}[x]$; thus, $B$ is relatively compact as $E_{\mathcal{V}}[x]$ is. This shows that the coarse structure $\mathcal{E}_{\mathcal{F}}$ is a proper coarse structure. $\square$

We call the proper coarse structure $\mathcal{E}_{\mathcal{F}}$ on the Freudenthal space $X$, the **Freudenthal coarse structure**.

**Lemma 7.3.3.** *Let $X$ be a Freudenthal space equipped with the Freudenthal coarse structure $\mathcal{E}_{\mathcal{F}}$. Let $E_{\mathcal{U}}$ as in 7.3.2. Then:*

*(a) The coarse space $(X, \mathcal{E}_{\mathcal{F}})$ is $E_{\mathcal{U}}$-connected.*

*(b) For every coarsely bounded subset $B \subset X$, $X \setminus B$ has finitely many $E$-components, where $E = E_{\mathcal{U}} \circ E_{\mathcal{U}}$.*

*Proof.* $(a)$ Notice first that $U$ is a component of $U_n := int(K_{n+2}) \setminus K_n$ if and only if $U = C \cap int(K_{n+2})$ for some $C \in \mathcal{UC}(X \setminus K_n)$, for all $n \in \mathbb{N}$. Indeed, notice that:

$$int(K_{n+2}) \setminus K_n = int(K_{n+2}) \cap (\bigcup_{C \in \mathcal{UC}(X \setminus K_n)} C).$$

Since $int(K_{n+2})$ is connected, $int(K_{n+2}) \cap C$ is connected for all $C \in \mathcal{UC}(X \setminus K_n)$. Now, fix $x_0 \in K_1$, and let $x \in X \setminus K_1$. Since, $X := \bigcup_{n \in \mathbb{N}} U_n$, there exists $n \in \mathbb{N}$, such that $x \in U_n$. In particular, $x \in C_n \cap int(K_{n+2})$, for some $C_n \in \mathcal{UC}(X \setminus K_n)$. Notice that $C_n$ is contained in the component $C_{n-1}$ of $\mathcal{UC}(X \setminus K_{n-1})$; and that $(C_n \cap int(K_{n+2})) \cap (C_{n-1} \cap int(K_{n+1})) = C_n \cap int(K_{n+1}) \neq \emptyset$. Let $x_{n+2} = x$, and choose $x_i \in C_i \cap int(K_{i+1})$, for all $1 \leq i \leq n$. Clearly, $x = x_{n+2}, x_{n+1}, \ldots, x_1, x_0$ is an $E_{\mathcal{U}}$-chain.
$(b)$ Let $B \subset X$ be a coarsely bounded subset; then $B$ is relatively compact and hence it can be covered by finitely many open connected subsets $U_1, \ldots, U_m \in \mathcal{U}$. Fix $b \in B$, and $x, y \in X \setminus B$. Choose $x = x_0, \ldots, x_s = b$ and $y = y_0, \ldots, y_r$ $E_{\mathcal{U}}$-chains connecting $x$ with $b$, and $y$ with $b$, respectively. There exist $O_1, \ldots, O_s \in \mathcal{U}$ and $W_1, \ldots, W_r \in \mathcal{U}$ such that $x_{i-1}, x_i \in O_i$ and $y_{j-1}, y_j \in W_j$, for all $1 \leq i \leq s$ and $1 \leq j \leq r$. Notice that the $E_{\mathcal{U}}$-chains $x = x_0, \ldots, x_s = b$ and $y = y_0, \ldots, y_r$ must penetrate $\bigcup_{k=1}^{m} U_k$. If for some $1 \leq k \leq m$, $1 \leq i \leq s$ and $1 \leq j \leq r$, one has $x_i, y_j \in U_k \setminus B$, then $x_i, y_j \in E_{\mathcal{U}}[U_k \setminus B]$ and hence $x$ and $y$ belong to the same $E$-component of $X \setminus B$, where $E = E_{\mathcal{U}} \circ E_{\mathcal{U}}$. In particular, $X \setminus B$ has at most $m$ $E$-components. ◻

**Lemma 7.3.4.** *Let $X$ be a Freudenthal space equipped with the Freudenthal coarse structure $\mathcal{E}_{\mathcal{F}}$. Let $E_{\mathcal{U}}$ as in 7.3.2. Then:*
*(a) For every $n \in \mathbb{N}$, if $C_n \in \mathcal{UC}(X \setminus K_n)$ is an unbounded component of $X \setminus K_n$, then $C_n$ is coarsely clopen. In particular, every Freudenthal end $\{C_n\}_{n \in \mathbb{N}}$ is contained in a coarse end.*
*(b) If $e$ is a coarse end and $A \in e$, and $F \in \mathcal{E}_{\mathcal{F}}$ is a symmetric entourage containing the diagonal; then there exists a coarsely bounded subset $B$ such that for every coarsely bounded subset $B'$ containing $B$, $A \setminus B'$ is a finite union of $F$-components of $X \setminus B'$.*
*(c) For every $n \in \mathbb{N}$, if $C_n \in \mathcal{UC}(X \setminus K_n)$ is an unbounded component of $X \setminus K_n$, then $C_n$ is contained in an $E_{\mathcal{U}}$-component.*
*(d) If $A$ is an unbounded $E$-component of $X \setminus K_n$, for some $n \in \mathbb{N}$, where $E = E_{\mathcal{U}} \circ E_{\mathcal{U}}$;*

*then $A$ contains an unbounded component $C_n \in \mathcal{UC}(X \setminus K_n)$.*

*(e) Every Freudenthal end is contained in exactly one coarse end.*

*(f) Every coarse end contains exactly one Freudenthal end.*

*Proof.* ($a$) Let $n \in \mathbb{N}$ and $C_n \in \mathcal{UC}(X \setminus K_n)$ be an unbounded component of $X \setminus K_n$, then $C_n$ is coarsely clopen. To this end, it suffices to show that if $\mathcal{V} \in \mathcal{F}$, then $E_{\mathcal{V}}[C_n] \cap (X \setminus C_n) \subset E_{\mathcal{V}}[K_n]$. If $x \in E_{\mathcal{V}}[C_n] \cap (X \setminus C_n)$, then there exists $a \in C_n$ such that $(x, a) \in E_{\mathcal{V}}$ and $x \in X \setminus C_n$. Choose $V \in \mathcal{V}$ such that $x \in V$. If $V \subset \bigcup_{C \in \mathcal{UC}(X \setminus K_n)} C$, then as $V$ is connected, it must be contained in the component $C_n$, which contradicts the fact that $x \in V \cap (X \setminus C_n)$. Hence, $V \cap K_n \neq \emptyset$ which implies that $x \in E_{\mathcal{V}}[K_n]$, and therefore $E_{\mathcal{V}}[C_n] \cap (X \setminus C_n) \subset E_{\mathcal{V}}[K_n]$. Thus, $E_{\mathcal{V}}[C_n] \cap (X \setminus C_n)$ is coarsely bounded as desired. Therefore, $C_n$ is contained in some coarse end $e \in Ends(X)$. If $(C_n)_{n \in \mathbb{N}}$ is a Freudenthal end, then as for any $m \in \mathbb{N}$, $C_n$ and $C_m$ are coarsely identical, so $C_m \in e$, for all $m \in \mathbb{N}$, which proves that $(C_n)_{n \in \mathbb{N}}$ is entirely contained in $e$.

($b$) Let $e$ be a coarse end, $A \in e$, and $F \in \mathcal{E}_{\mathcal{F}}$ be a symmetric entourage containing the diagonal. Consider the coarsely bounded subset $B := F[A] \cap F[X \setminus A]$. Now, assume that $B'$ is coarsely bounded containing $B$. Clearly, $A \setminus B' \in e$, and we claim that $A \setminus B'$ is a finite union of $F$-components of $X \setminus B'$. Put $A' =: A \setminus B'$; we claim that every $F$-chain starts at a point in $A'$ is completely contained in $A'$. To this end, it suffices to show that if $x \in A'$ and $y \in X$ such that $(x, y) \in F$, then $y \in A'$. Seeking contradiction, assume that $y \in (X \setminus A') = (X \setminus A) \cup B'$, then $y \in X \setminus A$, and hence $x \in F[A] \cap F[X \setminus A] \subset B'$, a contradiction.

($c$) Similar to part ($a$) of 7.3.3.

($d$) Let $A$ be an unbounded $E$-component of $X \setminus K_n$, for some $n \in \mathbb{N}$, where $E = E_{\mathcal{U}} \circ E_{\mathcal{U}}$. Since $A$ is unbounded, it must unboundedly intersect some $C_n$. But, $C_n$, by part ($c$), is contained in $E_{\mathcal{U}}$-component of $X \setminus K_n$, and a fortiori in $E$-component of $X \setminus K_n$. Hence, $C_n \subset A$.

($e$) Let $(C_n)_{n \in \mathbb{N}}$ be a Freudenthal end. Seeking contradiction, assume that $e, e' \in Ends(X)$ are two different coarse ends both containing $(C_n)_{n \in \mathbb{N}}$. Choose $A \in e$ and $A' \in e'$ such that $A \cap A' = \emptyset$, and $N \in \mathbb{N}$ such that $A \setminus K_N$ and $A' \setminus K_N$ are union of $E$-components of $X \setminus K_N$. Since there are only finitely many $E$-components of $X \setminus K_N$, both $A \setminus K_N$ and $A' \setminus K_N$

contain at least one unbounded $E$-component. By part $(c)$, $C_N$ is contained in both $A \setminus K_N$ and $A' \setminus K_N$, a contradiction.

$(f)$ Straightforward. $\square$

**Theorem 7.3.5.** *Let $X$ be a Freudenthal space equipped with the Freudenthal coarse structure $\mathcal{E}_{\mathcal{F}}$. The space of coarse ends $Ends(X)$ with respect to the Freudenthal coarse structure is homeomorphic to the space of Freudenthal ends $ends(X)$.*

*Proof.* By 4.1.9, $\mathcal{B}_{\mathcal{F}end} := \{U^F_{end} : U \in \bigcup_{n\in\mathbb{N}} \mathcal{UC}(X \setminus K_n)\}$ is a basis for the topology of the space of Freudenthal ends $ends(X)$, where $U^F_{end} := \{(C_n)_{n\in\mathbb{N}} \in ends(X) : C_n \subset U$ for some $n \in \mathbb{N}\}$; and by 7.1.5, $\mathcal{B}_{\mathcal{C}end} = \{A^C_{end}$ :$A$ unbounded coarsely clopen subset of $X\}$ is a basis for the topology of the space of coarse ends $Ends(X)$, where $A^C_{end} := \{e \in Ends(X) : A \in e\}$. For each coarse end $e \in Ends(X)$, let $(C^e_n)_{n\in\mathbb{N}}$ be the Freudenthal end contained in $e$. Define the map $\varphi : Ends(X) \to ends(X)$ that sends each $e \in Ends(X)$ to the Freudenthal end $(C^e_n)_{n\in\mathbb{N}}$. By part $(e)$ and $(f)$ of 7.3.4, one can see that $\varphi : Ends(X) \to ends(X)$ is bijective. Moreover, for a basis element $U^F_{end}$ of the space of the Freudenthal ends $ends(X)$, where $U = C_N$ for some $N \in \mathbb{N}$ and $C_N \in \mathcal{UC}(X \setminus K_N)$, then $\varphi^{-1}(U^F_{end}) = U^C_{end}$ which a basis element of $Ends(X)$, and therefore $\varphi : Ends(X) \to ends(X)$ is continuous. Thus, $Ends(X)$ and $ends(X)$ are homeomorphic. $\square$

# Chapter 8

# Coarse groups

This chapter is concerned with studying groups as coarse spaces equipped with coarse structures induced by generating sets. The class of such groups include:

1. Metrizable groups,
2. Countable groups with proper left-invariant metrics,
3. Topological groups. This chapter is mainly based on Section 5, Section 7 and Section 8 of (16).

## 8.1 Definitions and basic properties of coarse groups

Most of the material of this section can be found in (2) and (38).

**Definition 8.1.1.** Let $G$ be a group and $\mathcal{G} \subset \mathcal{P}(G) \setminus \{\emptyset\}$ be a family of non-empty subsets of $G$. $\mathcal{G}$ is called a **generating set** of $G$ if:

- $A \cup B, A \cdot B, A^{-1} \in \mathcal{G}$, for all $A, B \in \mathcal{G}$.
- If $A \in \mathcal{G}$ and $A' \subset A$, then $A' \in \mathcal{G}$.

**Proposition 8.1.2.** *Let $G$ and $K$ be groups, and $\mathcal{G}$ be a generating set of $G$. Then:*
*(a) If $H \leqslant G$ is a subgroup, then $\mathcal{G}_{|H} := \{A \cap H : A \in \mathcal{G}\}$ is a generating set of $H$.*
*(b) If $\varphi : G \to K$ is a group homomorphism, then $\varphi(\mathcal{G}) := \{\varphi(A) : A \in \mathcal{G}\}$ is a generating set of $K$.*

*Proof.* Straightforward. □

**Proposition 8.1.3.** *Let $G$ be a group equipped with a left-invariant metric $d$ on $G$; for every $r > 0$, put $B_r := \{g \in G : d(1_G, g) \leq r\}$. The family $\mathcal{G}_d := \{A \subset G : A \subset B_r$ for some $r > 0\}$ is a generating set of $G$.*

*Proof.* For $r, s > 0$, one has $B_r \cup B_s, B_r \cdot B_s \subset B_{r+s}$, and $B_r^{-1} = B_r$. □

**Example 8.1.4.** *Let $G$ be arbitrary group:*

1. *The family of all finite subsets $\mathcal{G}_{fin}$ of $G$ is a generating set of $G$.*
2. *The family of all countable subsets $\mathcal{G}_{coun}$ of $G$ is a generating set of $G$.*

**Example 8.1.5.** *Let $G$ be a topological group and let $\mathcal{G}_{com}$ be the family of all relatively compact subsets of $G$. Then, $\mathcal{G}_{com}$ is a generating set of $G$.*

**Notation 8.1.6.** *Given a group $G$, a subset $A \subset G$, and a subset $E \subset G \times G$, we put:*

$$G_A := \{(g \cdot a, g \cdot b) : a, b \in A, g \in G\},$$

*and*

$$A_E := \{y^{-1} \cdot x : (x, y) \in E\}.$$

**Lemma 8.1.7.** *Given a group $G$ and subsets $A, B \subset G$, then $G_A[B] = B \cdot A^{-1} \cdot A$.*

*Proof.* Let $p \in G_A[B]$, then $(p, b) \in G_A$ for some $b \in B$. There exist $g \in G$ and $x, y \in A$ such that $p = g \cdot x$ and $b = g \cdot y$. So, $g = b \cdot y^{-1}$ and hence $p = b \cdot y^{-1} \cdot x \in B \cdot A^{-1} \cdot A$.

Conversely, if $p \in B \cdot A^{-1} \cdot A$, then $p = b \cdot y^{-1} \cdot x$, for some $b \in B$ and $x, y \in A$. Put $g = b \cdot y^{-1}$, then $(p, b) = (g \cdot x, g \cdot y) \in G_A$. □

**Proposition 8.1.8.** *Let $G$ be a group and $\mathcal{G}$ be a generating set. The family: $\mathcal{E}_{\mathcal{G}} := \{E \subset G \times G : E \subset G_A$, for some $A \in \mathcal{G}\}$ is a coarse structure on $G$.*

*Proof.* It is easy to see that $\Delta \in \mathcal{E}_{\mathcal{G}}$ as for $A \in \mathcal{G}$ containing $1_G$, then $\Delta \in G_A$. Let $E, F \in \mathcal{E}_{\mathcal{G}}$, then there exist $A, B \in \mathcal{G}$ such that $E \subset G_A$ and $F \subset G_B$. Now,

• $E \cup F \subset G_A \cup G_B \subset G_{A \cup B}$;

• $E \circ F \subset G_A \circ G_B \subset G_{A \cdot B^{-1} \cdot B}$. Indeed, if $(x, z) \in G_A \circ G_B$, then there is $y \in G$ such that $(x, y) \in G_A$ and $(y, z) \in G_B$. Thus, there exist $a, a' \in A$, $b, b' \in B$ and $g, h \in G$ such that $(x, y) = (g \cdot a, g \cdot a')$ and $(y, z) = (h \cdot b, h \cdot b')$. In particular, $g \cdot a' = h \cdot b$ and hence $h = g \cdot a' \cdot b^{-1}$ implying that $(x, z) = (g \cdot a, g \cdot a' \cdot b^{-1} \cdot b') \in G_{A \cdot B^{-1} \cdot B} \in \mathcal{E}_{\mathcal{G}}$ as $A \cdot B^{-1} \cdot B \in \mathcal{G}$. This proves that $\mathcal{E}_{\mathcal{G}}$ is a coarse structure on $G$. $\square$

The coarse structure $\mathcal{E}_{\mathcal{G}}$ on $G$ is referred to as the **coarse structure induced by the generating set** $\mathcal{G}$.

**Lemma 8.1.9.** *Let $G$ be a group, $\mathcal{G}$ be a generating set, and $E \subset G \times G$. The set $A_E = \{y^{-1} \cdot x : (x, y) \in E\} \in \mathcal{G}$ if and only if $E \in \mathcal{E}_{\mathcal{G}}$.*

*Proof.* Assume that $E \subset G \times G$ such that $A_E = \{y^{-1} \cdot x : (x, y) \in E\} \in \mathcal{G}$. Without loss of generality, assume that $1_G \in A_E$. Notice that $F := \{g \cdot a, g) : a \in A_E, g \in G\} \subset G_{A_E} \in \mathcal{E}_{\mathcal{G}}$. Moreover, if $(x, y) \in E$, then $y^{-1} \cdot x \in A_E$, and hence $(x, y) = (y \cdot y^{-1} \cdot x, y) \in F$. Thus $E \subset F \in \mathcal{E}_{\mathcal{G}}$, as wanted.

Conversely, assume that $E \in \mathcal{E}_{\mathcal{G}}$, and choose $A \in \mathcal{G}$ such that $E \subset G_A$. We claim that $G_{A_E} \subset G_A \circ G_A^{-1}$. Indeed, if $(a, b) \in G_{A_E}$, then there exist $(x_1, y_1), (x_2, y_2) \in E$ and $g \in G$ such that $a = g \cdot y_1^{-1} \cdot x_1$ and $b = g \cdot y_2^{-1} \cdot x_2$. Since $(x_1, y_1), (x_2, y_2) \in E \subset G_A$ and $G_A$ is $G$-invariant, then $(y_1^{-1} \cdot x_1, 1_G), (y_2^{-1} \cdot x_2, 1_G) \in G_A$ and hence $(g \cdot y_1^{-1} \cdot x_1, g), (g \cdot y_2^{-1} \cdot x_2, g) \in G_A$. Therefore, $(a, b) \in G_A \circ G_A^{-1}$ as claimed. $\square$

**Proposition 8.1.10.** *Let $G$ be a group equipped with a coarse structure $\mathcal{E}_{\mathcal{G}}$ induced by a generating set $\mathcal{G}$. Then:*

*(a) Every $A \in \mathcal{G}$ is coarsely bounded.*

*(b) A subset $B \subset G$ is coarsely bounded if and only if $B^{-1} \cdot B \in \mathcal{G}$.*

*Proof.* $(a)$ Straightforward.

$(b)$ Assume that $B$ is coarsely bounded, and choose $A \in \mathcal{G}$ and $p \in G$ such that $B \subset G_A[p]$. By 8.1.7, $G_A[p] = p \cdot A^{-1} \cdot A$. Hence, $B^{-1} \cdot B \subset (p \cdot A^{-1} \cdot A)^{-1} \cdot (p \cdot A^{-1} \cdot A) = A^{-1} \cdot A \cdot A^{-1} \cdot A \in \mathcal{G}$ and hence $B^{-1} \cdot B \in \mathcal{G}$. Conversely, assume that $B^{-1} \cdot B \in \mathcal{G}$. Notice that $A_{G_B} = B^{-1} \cdot B \in \mathcal{G}$, hence by 8.1.9, $G_B \in \mathcal{E}_{\mathcal{G}}$. Pick $b \in B$, then $B \subset G_B[b] = b \cdot B^{-1} \cdot B$, and therefore $B$ is coarsely bounded as $G_B[b]$ is coarsely bounded. $\square$

**Proposition 8.1.11.** *Let $G$ be a group equipped with a coarse structure $\mathcal{E}_{\mathcal{G}}$ induced by a generating set $\mathcal{G}$. Then:*

*(a)* $(G,\mathcal{E}_\mathcal{G})$ *is coarsely connected if and only if* $G=\bigcup\limits_{A\in\mathcal{G}} A$.
*(b) If* $(G,\mathcal{E}_\mathcal{G})$ *is coarsely connected, then a subset* $B\subset G$ *is coarsely bounded if and only if* $B\in\mathcal{G}$.

*Proof.* $(a)$ Assume that $(G,\mathcal{E}_\mathcal{G})$ is coarsely connected, and $x\in G$. We aim to find $B\in\mathcal{G}$ such that $x\in B$. Indeed, as $(G,\mathcal{E}_\mathcal{G})$ is coarsely connected, $\{(x,1_G)\}\in\mathcal{E}_\mathcal{G}$. Thus, there exists $A\in\mathcal{G}$ such that $(x,1_G)\in G_A$. Therefore, $(x,1_G)=(g\cdot a,g\cdot b)$ for some $a,b\in A$ and $g\in G$, and hence $x=b^{-1}\cdot a\in B=A^{-1}\cdot A\in\mathcal{G}$, as wanted. Conversely, assume that $G=\bigcup\limits_{A\in\mathcal{G}} A$ and $(x,y)\in G\times G$. There exist $A,B\in\mathcal{G}$ such that $x\in A$ and $y\in B$. Put $C:=A\cup B$, then $\{(x,y)\}\subset G_C$ and hence $\{(x,y)\}\in\mathcal{E}_\mathcal{G}$.
$(b)$ Since $(G,\mathcal{E}_\mathcal{G})$ is coarsely connected, $G=\bigcup\limits_{A\in\mathcal{G}} A$. In particular, $\{x\}\in\mathcal{G}$ for all $x\in G$. Assume that $B\subset G$ is coarsely bounded. By part $(h)$ of 5.2.9, $E:=B\times\{x\}\in\mathcal{E}_\mathcal{G}$ for any $x\in G$. By 8.1.9, $A_E=x^{-1}\cdot B\in\mathcal{G}$. Since $\{x\}\in\mathcal{G}$, $x\cdot A_E=B\in\mathcal{G}$ as wanted. The other implication is obvious. □

The following provides an alternative description of $\mathcal{E}_\mathcal{G}$.

**Notation 8.1.12.** *Given a group* $G$ *and a subset* $A\subset G$, *we put:*

$$E_A:=\{(x,y): x^{-1}\cdot y\in A\}.$$

**Lemma 8.1.13.** *Let* $G$ *be a group equipped with a coarse structure* $\mathcal{E}_\mathcal{G}$ *induced by a generating set* $\mathcal{G}$, *and* $A\subset G$. *Then,* $A\in\mathcal{G}$ *if and only if* $E_A\in\mathcal{E}_\mathcal{G}$. *Moreover, the family* $\{E_A: A\in\mathcal{G}\}$ *is a basis for* $\mathcal{E}_\mathcal{G}$.

*Proof.* Assume that $A\in\mathcal{G}$. Without loss of generality, assume that $1_G\in A$. We claim that $E_A\subset G_A$. Indeed, if $(x,y)\in E_A$, then $x^{-1}\cdot y\in A$ and therefore $(1_G,x^{-1}\cdot y)\in G_A$ which implies that $(x,y)\in G_A$. Conversely, suppose that $E_A\in\mathcal{E}_\mathcal{G}$, then by 8.1.9, $A_{E_A}\in\mathcal{G}$. Since $A_{E_A}=A^{-1}$ and hence $A\in\mathcal{G}$.
Finally, we know that $\{E_A: A\in\mathcal{G}\}$ is a basis for $\mathcal{E}_\mathcal{G}$. It suffices to show that for every $A\in\mathcal{G}$, there exists $B\in\mathcal{G}$ such that $G_A\subset E_B$. To this end, let $A\in\mathcal{G}$ and let $(x,y)\in G_A$, then there exist $a,b\in A$ and $g\in G$ such that $x=g\cdot a$ and $y=g\cdot b$. Thus, $x^{-1}\cdot y=a^{-1}\cdot b\in A^{-1}\cdot A$. Take $B=A^{-1}\cdot A$, then $B\in\mathcal{G}$ and $G_A\subset E_B$. □

The significance of above lemma lies in the fact that sets of the form $E_A$ are easier to work with as the following lemma asserts us.

**Lemma 8.1.14.** *Let $G$ and $H$ be groups, $A, B \subset G$, $C \subset H$, and $\varphi : G \to H$ be a group homomorphism Then:*
*(a)* $E_A^{-1} = E_{A^{-1}}$.
*(b)* $E_A \circ E_B = E_{A \cdot B}$. *In particular,* $E_A^n = E_{A^n}$, *for all* $n \in \mathbb{N}$.
*(c)* $E_A[B] = B \cdot A^{-1}$.
*(d)* $A \subset B$ *if and only if* $E_A \subset E_B$.
*(e)* $\varphi \times \varphi(E_A) \subset E_{\varphi(A)}$.
*(f)* $(\varphi \times \varphi)^{-1}(E_C) = E_{\varphi^{-1}(C)}$.

*Proof.* Straightforward. □

The above Lemma 8.1.14 gives us a simple characterization of coarsely disjoint and coarsely clopen subsets of a coarse group.

**Proposition 8.1.15.** *Let $G$ be a group equipped with a coarse structure $\mathcal{E}_{\mathcal{G}}$ induced by a generating set $\mathcal{G}$, and let $A, B \subset G$ be subsets. Then:*
*(a) $A$ and $B$ are coarsely disjoint if and only if $(A \cdot K) \cap (B \cdot K)$ is bounded for all $K \in \mathcal{G}$.*
*(b) $A$ is coarsely clopen if and only if $(A \cdot K) \cap (A^c \cdot K)$ is bounded for all $K \in \mathcal{G}$.*

*Proof.* Apply 8.1.14 ($c$). □

**Corollary 8.1.16.** *Let $G$ be a group equipped with a coarse structure $\mathcal{E}_{\mathcal{G}}$ induced by a generating set $\mathcal{G}$, and let $A, B \subset G$ be subsets. Then:*
*(a) $A$ and $B$ are coarsely disjoint if and only if $(A \cdot K) \cap B$ is bounded for all $K \in \mathcal{G}$.*
*(b) $A$ is coarsely clopen if and only if $(A \cdot K) \cap A^c$ is bounded for all $K \in \mathcal{G}$.*

*Proof.* Apply 5.6.15 ($g$) and 8.1.14 ($c$). □

**Proposition 8.1.17.** *Let $G$ be a group equipped with the coarse structure induced by the generating set $\mathcal{G}_{fin}$ of all finite subsets of $G$. A subset $A \subset G$ is coarsely clopen if it is almost right invariant.*

*Proof.* Apply 8.1.16. □

**Observation 8.1.18.** *If $G$ is a group equipped with a left-invariant metric $d$, then $\mathcal{E}_{\mathcal{G}_d} = \mathcal{E}_d$ is the bounded coarse structure induced by $d$. Moreover, if $d$ is a proper metric then $\mathcal{E}_{\mathcal{G}_d} = \mathcal{E}_d = \mathcal{E}_{\mathcal{G}_{fin}}$.*

**Proposition 8.1.19.** *Let $G$ and $H$ be groups with the coarse structures $\mathcal{E}_{\mathcal{G}}$ and $\mathcal{E}_{\mathcal{H}}$ induced by the generating sets $\mathcal{G}$ and $\mathcal{H}$ of $G$ and $H$, respectively. If $\varphi : G \to H$ is a group homomorphism, then:*
*(a) $\varphi : G \to H$ is controlled if and only if $\varphi(A) \in \mathcal{H}$, for all $A \in \mathcal{G}$.*
*(b) $\varphi : G \to H$ is coarsely proper if and only if $\varphi^{-1}(B) \in \mathcal{G}$, for all $B \in \mathcal{H}$.*

*Proof.* Apply $(e)$ and $(f)$ of 8.1.14. □

**Example 8.1.20.** *Given a coarse group $(G, \mathcal{E}_{\mathcal{G}})$, and an element $x \in G$; the automorphism $\sigma_x : G \to G$ is a coarse equivalent with a coarse inverse $\sigma_{x^{-1}} : G \to G$.*

**Definition 8.1.21.** Let $G$ be a group equipped with the coarse structure $\mathcal{E}_{\mathcal{G}}$ induced by the generating set $\mathcal{G}$. A subgroup $H$ is called of **bounded index** if $G = A \cdot H$ for some coarsely bounded subset $A \subset G$.

**Observation 8.1.22.** *If $H$ is a subgroup of $G$, and $A \subset G$, then $G = A \cdot H$ if and only if $G = H \cdot A^{-1}$. Indeed, if $G = A \cdot H$ and $x \in G$, then $x^{-1} \in G$. So, $x^{-1} = a \cdot h$ for some $a \in A$ and $h \in H$. Thus, $a^{-1} \cdot x^{-1} \in H$ and hence $x \cdot a \in H$; say $x \cdot a = h'$, for some $h' \in H$. Therefore, $x = h' \cdot a^{-1} \in H \cdot A^{-1}$.*

**Proposition 8.1.23.** *Let $G$ be a group equipped with the coarse structure $\mathcal{E}_{\mathcal{G}}$ induced by the generating set $\mathcal{G}$. A subgroup $H$ is coarsely equivalent to $G$ if and only if $G = A \cdot H$ for some $A \in \mathcal{G}$.*

*Proof.* Assume that $H$ coarsely equivalent to $G$. By 5.4.7, $H$ is $E$-coarsely dense in $G$. Choose $A \in \mathcal{G}$ such that $E \subset E_A$. Then, $G = E_A[H] = H \cdot A^{-1}$ and hence $G = A \cdot H$ as desired.

Conversely, if $G = A \cdot H$ for some $A \in \mathcal{G}$, then $G = H \cdot A^{-1}$ and hence $G = E_A[H]$. Therefore, $H$ is coarsely dense in $G$, and by 5.4.7, it must be coarsely equivalent to $G$. □

**Corollary 8.1.24.** *Let $G$ be a group equipped with the coarse structure $\mathcal{E}_{\mathcal{G}}$ induced by the generating set $\mathcal{G}$. If $(G, \mathcal{E}_{\mathcal{G}})$ is coarsely connected, then a subgroup $H$ is coarsely equivalent to $G$ if and only if it is of bounded index in $G$.*

*Proof.* Apply 8.1.23, and 8.1.10. □

## 8.2 Connectivity in coarse groups

In this section we introduce concepts needed to generalize being finitely generated to being boundedly generated.

**Definition 8.2.1.** A coarse group $G$ is **boundedly generated** if there is a symmetric bounded set $K$ that generates $G$.

**Proposition 8.2.2.** *Let $K$ be a symmetric bounded subset of a coarse group $G$ containing $1_G$. The coarse group $G$ is boundedly generated by $K$ if and only if it is $E_K$-connected.*

*Proof.* Assume that $G =< K >$ and $x \in G$. Then, there exist $a_1, \ldots, a_n \in K$ such that $x = a_1 \cdot \ldots \cdot a_n$. Notice that $(a_i, 1_G) \in E_K$ for all $1 \leq i \leq n$ and hence $G$ is $E_K$-connected.

Conversely, assume that $G$ is $E_K$-connected and that $x \in G$. Choose a sequence $1_G = x_0, \ldots, x_n = x$ such that $(x_i, x_{i+1}) \in E_K$ for all $1 \leq i < n$. Notice that $x_i^{-1} \cdot x_{i+1} \in K$ for all $1 \leq i < n$, and $x = (x_0^{-1} \cdot x_1) \cdot \ldots \cdot (x_{n-2}^{-1} \cdot x_{n-1}) \cdot (x_{n-1}^{-1} \cdot x_n)$. □

**Proposition 8.2.3.** *Suppose $K$ is a symmetric bounded subset of a group coarse $G$. If, for some bounded subset $B$ of $G$, $G \setminus B$ has finitely many unbounded $E_K$-components and the union of all bounded $E_K$-components is bounded, then $G$ is boundedly generated.*

*Proof.* Let $L$ be the union of $K \cup B$ and of the following:

1. The union of all bounded $E_K$-components of $G \setminus B$,
2. One point from each non-empty unbounded $E_K$-component of $G \setminus B$.

Put $M = L \cup L^{-1}$ and notice $G$ is $E_M$-connected. Indeed, the $E_M$-component of $1_G$ contains $B$ and all $E_K$-components of $G \setminus B$. Therefore, $G =< M >$ as wanted. □

**Observation 8.2.4.** *Let $(G, \mathcal{E}_{\mathcal{G}})$ be a coarse group. If $G$ is boundedly generated by a symmetric bounded subset $K \subset G$ that contains $1_G$; then:*

*(a) For all $g \in G$ and $n \in \mathbb{N}$, the subset $g \cdot K^n$ is $E_K$-connected.*

*(b) Consider the Cayley metric $d_K : G \times G \to [0, \infty)$ on $G$, then $B(g, n) = g \cdot K^n$, for all $g \in G$ and $n \in \mathbb{N}$.*

*(c) $\mathcal{E}_{d_K} \subset \mathcal{E}_{\mathcal{G}}$; in particular, $\mathcal{G}_{d_K} \subset \mathcal{G}$.*

*Proof.* $(a)$ Notice first that $K^n$ is $E_K$-connected, as if $x, y \in K^n$, then there exists $g_1, \ldots, g_n, h_1, \ldots, h_n \in K$ such that $x = g_1 \cdot \ldots \cdot g_n$ and $y = h_1 \cdot \ldots \cdot h_n$. Then $g_1 \cdot \ldots \cdot g_n, g_1 \cdot \ldots \cdot g_{n-1}, \ldots, g_1 \cdot g_2, g_1, 1_G$ and $h_1 \cdot \ldots \cdot h_n, h_1 \cdot \ldots \cdot h_{n-1}, \ldots, h_1 \cdot h_2, h_1, 1_G$ are $E_K$-chains in $K^n$ connecting $x$ and $1_G$; and $y$ and $1_G$, respectively. Since, $K^n$ is symmetric, we can find $c$ an $E_K$-chain in $K^n$ connecting $x$ and $y$. Hence, $g \cdot c$ an $E_K$-chain in $g \cdot K^n$ connecting $g \cdot x$ and $g \cdot y$.

$(b)$ Straightforward.

$(c)$ Straightforward. □

**Example 8.2.5.** *Consider the topological group of the rational numbers $(\mathbb{Q}, +)$, equipped with the coarse structure induced from the generating set of all relatively compact subsets of $G$. $\mathbb{Q}$ is boundedly generated by the compact $K := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.*

**Example 8.2.6.** *Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices over $\mathbb{R}$. Topologically, $M_n(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{n^2}$. The special linear group:*

$$SL_2(\mathbb{R}) := \{A \in M_2(\mathbb{R}) : \det(A) = 1\},$$

*is a topological subgroup of $M_2(\mathbb{R})$ and it is compactly generated by:*

$$K := \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} : r \in \mathbb{R}, |r| \leq 1 \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

*Therefore, $SL_2(\mathbb{R})$, equipped with the coarse structure induced from the generating set of all relatively compact subsets, is boundedly generated.*

## 8.3 Coarse groups of bounded geometry

In this section we introduce the concept of bounded geometry for coarse groups. Typically, bounded geometry is defined for metric spaces by requiring that for each $r > 0$ there is $N_r \in \mathbb{N}$ such that every $r$-ball contains at most $N_r$ elements. We want a coarse invariant, so we extend this definition to arbitrary coarse spaces as follows:

**Definition 8.3.1.** A coarse space $(X, \mathcal{E}_X)$ has **bounded geometry** if it is coarsely equivalent to a coarse space $(Y, \mathcal{E}_Y)$ with the property that for each $F \in \mathcal{E}_Y$ there is $N_F \in \mathbb{N}$ such that every element of $F[y]$ contains at most $N_F$ elements, for all $y \in Y$.

**Observation 8.3.2.** *If $(Y, \mathcal{E}_Y)$ is a coarse space with the property that for each entourage $F \in \mathcal{E}_Y$ there is $N_F \in \mathbb{N}$ such that every element of $F[y]$ contains at most $N_F$ elements, for all $y \in Y$; then every bounded subset $B \subset Y$ is finite. Indeed, let $F = B \times B$ and choose $N_F \in \mathbb{N}$ such that every element of $F[y]$ contains at most $N_F$ elements, for all $y \in Y$. Notice that $B \subset F[y]$, where $y \in B$. Moreover, if $Y' \subset Y$, then, as a subspace, $(Y', \mathcal{E}'_Y)$ has also the property that for each entourage $F \in \mathcal{E}'_Y$ there is $N_F \in \mathbb{N}$ such that every element of $F[y]$ contains at most $N_F$ elements, for all $y \in Y'$.*

**Proposition 8.3.3.** *Suppose $X$ and $Y$ are coarse spaces and $Y$ has bounded geometry. If there is a coarse map $f : X \to Y$, then $X$ has bounded geometry. In particular, a subspace of a coarse space of bounded geometry is also of bounded geometry.*

*Proof.* Pick a coarse equivalence $g : Y \to Z$, where $(Z, \mathcal{E}_Z)$ is a coarse space that has the property that for each entourage $F \in \mathcal{E}_Z$ there is $N_F \in \mathbb{N}$ such that every element of $F[z]$ contains at most $N_F$ elements, for all $z \in Z$. Observe that $X$ is coarsely equivalent to $g(f(X))$. $\square$

**Theorem 8.3.4.** *A coarse group $G$ is of bounded geometry if and only if there is a bounded set $K$ such that for every bounded subset $B$ of $G$ there are elements $g_i \in G$, $i \leq n$, so that $B \subset \bigcup_{i=1}^{n} g_i \cdot K$.*

*Proof.* Suppose $f : G \to Y$ is a coarse equivalent, where $(Y, \mathcal{E}_Y)$ is a coarse space that has the property that for each entourage $F \in \mathcal{E}_Y$ there is $N_F \in \mathbb{N}$ such that every element of

$F[y]$ contains at most $N_F$ elements, for all $y \in Y$. Since $f : G \to Y$ is expanding, the subset $(f \times f)^{-1}(\Delta_Y)$ is an entourage of $G$. Choose a symmetric bounded subset $K$ of $G$ such that $(f \times f)^{-1}(\Delta_Y) \subset E_K$. Notice that for every $y \in Y$, and $g_y \in G \cap f^{-1}(\{y\})$, then such that $f^{-1}(\{y\}) \subset E_K[g_y] = g_y \cdot K$. Given a bounded subset $B$ of $G$, $f(B)$ is bounded, hence it is finite and $B \subset \bigcup_{y \in f(B)} f^{-1}(y) \subset \bigcup_{y \in f(B)} g_y \cdot K$.

Conversely, suppose there is a bounded set $K$ such that for every bounded subset $B$ of $G$ there are finitely many elements $g_i \in G$, $i \leq n$, so that $B \subset \bigcup_{i=1}^{n} g_i \cdot K$. We may assume $K$ is symmetric containing $1_G$. By 5.4.8, since $E_{K^2}$ is symmetric entourage containing the diagonal, $G$ has a coarsely dense $E_{K^2}$-separated subset $Y \subset G$. By 5.4.7, $Y$ is coarsely equivalent to $G$. To conclude, we prove that $Y$, as a coarse space of $G$, has the property that for each $F \in \mathcal{E}_Y$ there is $N_F \in \mathbb{N}$ such that every element of $F[y]$ contains at most $N_F$ elements, for all $y \in Y$. Indeed, $F \in \mathcal{E}_Y$, then there exists a symmetric bounded subset $B \subset G$, such that $F \subset E_B$, and there exist finitely many elements $g_i \in G$, $i \leq n$, so that $B \subset \bigcup_{i=1}^{n} g_i \cdot K$. Put $N_F = n$, and we claim that $|F[y]| \leq N_F$, for all $y \in Y$. Notice that $F[y] \subset E_B[y]$, and that $E_B[y] = y \cdot B \subset \bigcup_{i=1}^{n} y \cdot g_i \cdot K$. Now, if $a, b \in Y \cap y \cdot B$ are distinct such that $a, b \in y \cdot g_i \cdot K$ for some $1 \leq i \leq n$, then $a^{-1} \cdot b \in K^2$ and hence $(a, b) \in E_{K^2}$ which contradicts the fact that $Y$ is $E_{K^2}$-separated. This shows that each $g_i \cdot K$ has at most one element of $F[y]$, and thus $|F[y]| \leq N_F$. □

**Corollary 8.3.5.** *The following coarse groups are of bounded geometry:*

1. *Groups with generating set consisting of finite sets,*
2. *Locally compact topological groups.*

*Proof.* In case 2), any relatively compact neighborhood $K$ of $1_G$ works. □

**Example 8.3.6.** *Consider the special linear group $SL_n(\mathbb{R})$ and the general linear group $GL_n(\mathbb{R})$. They are both locally compact $\sigma$-compact groups as the determinant function $det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous with $det^{-1}(\mathbb{R} \setminus \{0\}) = GL_n(\mathbb{R})$ and $det^{-1}(\{1\}) = SL_n(\mathbb{R})$. Therefore, both $SL_n(\mathbb{R})$ and $GL_n(\mathbb{R})$, equipped with the coarse structure induced from the generating set of all relatively compact subsets, are of bounded geometry.*

**Proposition 8.3.7.** *Let $G$ be boundedly generated coarse group of bounded geometry. There exist a symmetric bounded subset $K$ of $G$ such that for every bounded subset $L$ of $G$ its complement $G \setminus L$ has finitely many $E_K$-components.*

*Proof.* Choose a bounded symmetric subset $M \subset G$ such that $1_G \in M$, $G =< M >$ and every bounded subset of $G$ is covered by finitely many subsets of the form $g \cdot M$. For arbitrary bounded subset $L \subset G$, choose $m \in \mathbb{N}$ such that $E_M[L] \subset \bigcup_{i=1}^{m} g_i \cdot M$. If $L = \emptyset$, then $G \setminus L$ has exactly one $E_M$-component. Fix $a \in L$ and for each $x \in G \setminus L$ choose a $E_K$-chain $c_x$ joining $x$ to $a$. Let $i_x \leq m$ be the first index of $c_x$ such that $x_{i_x} \in \bigcup_{i=1}^{m} g_i \cdot M \setminus L$. For $y \in G \setminus L$, if $x_{i_x}, y_{i_y} \in g_i \cdot M$, then $x_{i_x}^{-1} \cdot y_{i_y} \in M^2$ and hence $(x_{i_x}, y_{i_y}) \in E_{M^2}$. Since $x$ and $x_{i_x}$ are $E_M$-connected, and $y$ and $y_{y_x}$ are $E_M$-connected, so $x$ and $y$ are $E_M^2$-connected. Let $K = M^2$, then $G \setminus L$ has at most $m$ $E_K$-components. □

**Proposition 8.3.8.** *Let $G$ be a coarse group metrizable by the Cayley metric $d_K$ for some symmetric bounded set $K$ containing $1_G$. The following conditions are equivalent:*
*(a) $G$ is of bounded geometry.*
*(b) There are elements $g_i \in G$, $1 \leq i \leq n$, so that $K \cdot K \subset \bigcup_{i=1}^{n} g_i \cdot K$.*
*(c) For each $s \geq 1$ there is $N_s \geq 1$ such that for each $m \geq 1$ the set $K^{m+s}$ can be covered by at most $N_s$ sets of the form $g \cdot K^m$.*

*Proof.* $(a) \iff (b)$: If $G$ is of bounded geometry, then as $K \cdot K$ is bounded, and therefore $K \cdot K \subset \bigcup_{i=1}^{n} g_i \cdot K$ for some $g_i \in G$, $1 \leq i \leq n$. Conversely, notice that $(K \cdot K) \cdot K \subset \bigcup_{i=1}^{n} g_i \cdot K \cdot K \subset \bigcup_{i,j=1}^{n} g_i \cdot g_j \cdot K$. Apply induction to get that for every $m \geq 1$ there are elements $h_i \in G$, $i \leq m$, so that $K^m \subset \bigcup_{i=1}^{n} h_i \cdot K$. Since every bounded subset $B$ of $G$ is contained in some $K^n$, $G$ is of bounded geometry.

$(b) \iff (c)$: By induction, one can see that For each $s \geq 1$ there is $N_s \geq 1$ such that $K^{s+1}$ can be covered by at most $N_s$ sets of the form $g \cdot K$. Therefore $K^{m+s} = K^{s+1} \cdot K^{m-1}$ can be covered by at most $N_s$ sets of the form $g \cdot K \cdot K^{m-1} = g \cdot K^m$. The converse is trivial. □

## 8.4 Metrizability of the coarse structure of a group

In this section we discuss coarse groups whose coarse structure is metrizable or coarsely equivalent to a geodesic space.

**Definition 8.4.1.** Let $\mathcal{G}$ be a generating set of $G$. A subset $\mathcal{B} \subset \mathcal{G}$ is a called a **basis** for $\mathcal{G}$ if for every $A \in \mathcal{G}$, there exists $B \in \mathcal{B}$ such that $A \subset B$.

When $\mathcal{E}_{\mathcal{G}}$ is metrizable, it is induced by a left-invariant metric.

**Theorem 8.4.2.** *Let $G$ be a group equipped with a coarsely connected coarse structure $\mathcal{E}_{\mathcal{G}}$ induced by the generating set $\mathcal{G}$. The following are equivalent:*
*(a) $\mathcal{E}_{\mathcal{G}}$ is induced by a metric $d : G \times G \to [0, \infty)$.*
*(b) $\mathcal{G}$ has a countable basis.*
*(c) $\mathcal{E}_{\mathcal{G}}$ is induced by a left-invariant metric.*

*Proof.* $(a) \implies (b)$: Follows from 8.1.14.
$(b) \implies (c)$: Let $\mathcal{B} := \{B_n : n \in \mathbb{N} \cup \{0\}\}$ be a basis for $\mathcal{G}$ such that $B_0 := \{1_G\}$ and $(B_n)_{n \in \mathbb{N}_0}$ is a strictly increasing sequence of symmetric elements of $\mathcal{G}$. Define the function $|.| : G \to \mathbb{N}$ by $|g| = \min\{n \in \mathbb{N} : g \in B_n\}$. It induces a left-invariant metric $d : G \times G \to [0, \infty)$ given by: $d(g, h) := |g^{-1} \cdot h|$. We claim that $d$ induces $\mathcal{E}_{\mathcal{G}}$. Indeed, notice that for $g \in G$ and $n \in \mathbb{N}$, then $B(g, n) = \{h \in G : g^{-1} \cdot h \in B_m \text{ and } m \leq n\} \subset g \cdot B_n \in \mathcal{G}$. On the other hand, for arbitrary $n \in \mathbb{N}$, choose $g \in B_n$ and $N > n^2$; then $B(g, N) = \bigcup_{i=1}^{N} g \cdot B_i = B_N$ which contains $B_n$. Hence, $\mathcal{G}_d = \mathcal{G}$ as desired. $\square$

**Definition 8.4.3.** A coarse group $G$ is **$\sigma$-bounded** if it is a countable union of bounded subsets.

When $G$ is equipped with a metrizable coarse structure $\mathcal{E}_{\mathcal{G}}$, it is $\sigma$-bounded. The converse is true when $G$ is of bounded geometry.

**Proposition 8.4.4.** *If $G$ is a $\sigma$-bounded coarse group of bounded geometry, then $\mathcal{E}_{\mathcal{G}}$ is metrizable. Consequently, $G$ has a countable basis of boundedly generated subgroups.*

*Proof.* Pick a bounded symmetric set $K$ containing $1_G$ such that for every bounded subset $B$ of $G$ there are elements $g_i \in G$, $i \leq k$, so that $B \subset \bigcup_{i=1}^{k} g_i \cdot K$. Suppose $G = \bigcup_{i=1}^{\infty} B_i$, where each $B_i$ is bounded and $B_i \subset B_j$ if $i < j$. Each $g \in G$ has an index $n(g)$ such that $g \in B_{n(g)}$. Now, if $B$ is bounded and $B \subset \bigcup_{i=1}^{k} g_i \cdot K$, then for $m \geq n(g_i)$ for all $i \leq n$ one has $B \subset B_m \cdot K$. Therefore, $\{B_n \cdot K : n \in \mathbb{N}\}$ is a countable basis for $\mathcal{G}$. □

**Definition 8.4.5.** Let $X$ and $Y$ be coarse spaces. $X$ is coarsely dominated by $Y$ if there exist $\varphi : X \to Y$ and $\psi : Y \to X$ controlled maps such that $\psi \circ \varphi$ is close to $id_X$.

**Theorem 8.4.6.** *Let $G$ be a group equipped with a coarse structure $\mathcal{E}_{\mathcal{G}}$ induced by the generating set $\mathcal{G}$. The following are equivalent:*

*(a) $G$ is boundedly generated by a symmetric subset $K \in \mathcal{G}$ and $\{K^n\}_{n\in\mathbb{N}}$ is a basis for $\mathcal{G}$.*

*(b) $\mathcal{E}_{\mathcal{G}}$ induced by a Cayley metric.*

*(c) $G$ is coarsely equivalent to $Cay(G, K)$ for some $K \in \mathcal{G}$.*

*(d) $G$ is coarsely dominated by a geodesic metric space.*

*Proof.* $(a) \implies (b)$: Assume that $G =< K >$ for some symmetric subset $K \in \mathcal{G}$ such that $\{K^n\}_{n\in\mathbb{N}}$ is a basis for $\mathcal{G}$. Consider the Cayley metric $d_K$ induced by $K$. We aim to show that $\mathcal{G} = \mathcal{G}_{d_K}$. Notice that for $g \in G$ and $n \in \mathbb{N}$, then $B(g, n) \subset g \cdot K^n$, and $K^n \subset B(x, n+1)$ for any $x \in K$.

$(b) \implies (c)$: The inclusion $\imath : G \to Cay(G, K)$ is controlled, expanding and coarsely surjective.

$(c) \implies (d)$: Obvious.

$(d) \implies (a)$: Let $(X, d)$ be a geodesic metric space coarsely dominates $G$, and let $\varphi : G \to X$ and $\psi : X \to G$ controlled maps such that $\psi \circ \varphi$ is close to $id_G$. Notice that $\psi \times \psi(E_2) \in \mathcal{E}_{\mathcal{G}}$ where $E_2 = \{(x, y) \in X \times X : d(x, y) \leq 2\}$; and $E = \{(\psi \circ \varphi(g), g) : g \in G\} \in \mathcal{E}_{\mathcal{G}}$. By 8.1.13, There exists a symmetric subset $K \in \mathcal{G}$ such that $E \cup E^{-1} \cup \psi \times \psi(E_2) \subset E_K$. We claim that $\{K^n\}_{n\in\mathbb{N}}$ is a basis for $\mathcal{G}$. Let $L \in \mathcal{G}$ containing $1_G$. Choose $r > 0$ such that $\varphi(L) \subset B(x_0, r)$, where $x_0 = \varphi(1_G)$. Choose a boundary point $x$ of the ball $B(x_0, r)$. Pick $N \in \mathbb{N}$ and a chain $x_0, \ldots, x_N$ on the geodesic connecting $x_0$ to $x$ such that $d(x_i, x_{i+1}) < 1$ for all $1 \leq i < N$. Let $g \in L$ and choose a chain $x_0, \ldots, x_n = \varphi(g)$ such that $d(x_i, x_{i+1}) < 1$ for all $1 \leq i < n$. Clearly, $n \leq N$ and $(x_i, x_{i+1}) \in E_2$, for all $1 \leq i < n$. Therefore, $\psi(x_i)^{-1} \cdot \psi(x_{i+1}) \in K$. Notice that $\psi(\varphi(g)) = \psi(x_n) = \psi(x_0) \cdot (\psi(x_0)^{-1} \cdot \psi(x_1)) \cdot (\psi(x_1)^{-1} \cdot \psi(x_2)) \cdots (\psi(x_{n-1})^{-1} \cdot \psi(x_n))$. Since $(1_G, \psi(x_0)) \in E_K$, $\psi(x_0) \in K$, and hence $\psi(\varphi(g)) \in K^{n+1}$. Moreover, as $(\psi(\varphi(g)), g) \in E_K$,

$\psi(\varphi(g))^{-1} \cdot g \in K$ and hence $g \in \psi(\varphi(g)) \cdot K \subset K^{n+2} \subset K^{N+2}$. In particular, $L \subset K^{N+2}$ which shows that $\{K^n\}_{n\in\mathbb{N}}$ is a basis for $\mathcal{G}$. $\square$

A coarse group $(G, \mathcal{E}_{\mathcal{G}})$ that satisfies one of the equivalent properties of 8.4.6 is referred to as a **coarsely geodesic group**.

**Proposition 8.4.7.** *If $G$ is a boundedly generated coarse group of bounded geometry, then it is coarsely geodesic.*

*Proof.* Pick a bounded symmetric set $K$ containing $1_G$ such that for every bounded subset $B$ of $G$ there are elements $g_i \in G$, $i \leq k$, so that $B \subset \bigcup_{i=1}^{k} g_i \cdot K$. We may assume $K$ generates $G$. Each $g \in G$ has an index $n(g)$ such that $g \in K^{n(g)}$. Now, if $B$ is bounded and $B \subset \bigcup_{i=1}^{k} g_i \cdot K$, then for $m \geq n(g_i)$ for all $i \leq n$ one has $B \subset K^{m+1}$. By 8.4.6, $G$ is coarsely geodesic. $\square$

**Example 8.4.8.** *$SL_2(\mathbb{R})$, equipped with the coarse structure induced from the generating set of all relatively compact subsets, is coarsely geodesic; whereas, the coarse structure of $GL_n(\mathbb{R})$ induced from the generating set of all relatively compact subsets is metrizable.*

**Definition 8.4.9.** A sequence of subgroups $\{G_n\}_{n\in\mathbb{N}}$ of a coarse group $G$ is a **basis for boundedly generated subgroups** of $G$ if every boundedly generated subgroup $H$ of $G$ is contained in some $G_n$. Equivalently, for any bounded subset $B$ of $G$ there is $n \in \mathbb{N}$ such that $B \subset G_n$.

Clearly, if a coarse group $G$ has a metrizable coarse structure, then it can be expressed as the union of an increasing sequence of boundedly generated subgroups $\{G_n\}_{n\in\mathbb{N}}$ such that $\{G_n\}_{n\in\mathbb{N}}$ is a basis for boundedly generated subgroups. The converse is partially true; that is if $\{G_n\}_{n\in\mathbb{N}}$ is a basis for boundedly generated subgroups such that each $G_n$ is coarsely geodesic then the coarse structure of $G$ is metrizable.

**Proposition 8.4.10.** *Let $G$ be a coarse group that has a basis $\{G_n\}_{n\in\mathbb{N}}$ for boundedly generated subgroups consisting of unbounded subgroups that are coarsely geodesic; then its coarse structure is metrizable.*

*Proof.* Notice for each $G_n$ there exists a symmetric bounded subset $K_n \subset G$ containing $1_G$ such that $G_n =< K_n >$ and that $(K_n^m)_{m\geq 1}$ is a basis for $G_n$. In particular, $\{K_n^m : n, m \in \mathbb{N}\}$ is a countable basis for $G$. $\square$

**Definition 8.4.11.** A coarse group $G$ is **locally bounded** if for every bounded subset $B$ of $G$ the subgroup $< B >$ of $G$ generated by $B$ is bounded.

**Lemma 8.4.12.** *Let $\varphi : G \to H$ be a group homomorphism between coarse groups. If $\varphi : G \to H$ is a coarse equivalence and $H$ is locally bounded, then $G$ is locally bounded.*

*Proof.* It straightforward to see that, as $\varphi : G \to H$ is a group homomorphism, for every subset $A \subset G$, $\varphi(< A >) =< \varphi(A) >$. Notice that as $\varphi : G \to H$ is coarsely proper, if $A \subset G$ is unbounded, then $\varphi(A)$ is an unbounded subset of $H$. Seeking contradiction, assume that $K \subset G$ is coarsely bounded such that $< K >$ is unbounded subset of $G$. On one hand, $\varphi(< K >) =< \varphi(K) >$ in unbounded. On the other hand, since $\varphi : G \to H$ is controlled, $\varphi(K)$ is a coarsely bounded subset of $H$, and hence, as $H$ is locally bounded, $< \varphi(K) >$ is coarsely bounded; a contradiction. □

**Proposition 8.4.13.** *An unbounded coarse group $G$ is locally bounded and its coarse structure is metrizable if and only if it admits a strictly increasing sequence $\{G_n\}_{n\in\mathbb{N}}$ of bounded subgroups that is a basis for boundedly generated subgroups.*

*Proof.* Assume that the coarse group $G$ is locally bounded and its coarse structure is metrizable. For each $n \in \mathbb{N}$, put $B_n := B(1_G, n)$. We inductively define the strictly increasing sequence of bounded subgroups as follows:
$G_1 :=< B_1 >$, and $G_n :=< B_n \cup \{x_n\} >$ for all $n \geq 2$, where $x_n \in G \setminus G_{n-1}$. Then, $\{G_n\}_{n\in\mathbb{N}}$ is the desired sequence.

Conversely, assume that $\{G_n\}_{n\in\mathbb{N}}$ is strictly increasing sequence of bounded subgroups that is a basis for boundedly generated subgroups. It is a basis for its bounded sets, so the coarse structure of $G$ is metrizable. Moreover, every boundedly generated subgroup is contained in some $G_n$, and hence bounded, so $G$ is locally bounded. □

**Proposition 8.4.14.** *Let $G$ be a coarse group $G$ that is non-locally bounded and is not boundedly generated. Its coarse structure is metrizable if and only if $G$ admits a strictly increasing sequence $\{G_n\}_{n\in\mathbb{N}}$ of unbounded boundedly generated subgroups whose coarse structures as coarse subspace of $G$ are metrizable, and that $\{G_n\}_{n\in\mathbb{N}}$ is a basis for boundedly generated subgroups.*

*Proof.* Assume that $G$ is non-locally bounded and its coarse structure is metrizable. For each $n \in \mathbb{N}$, put $B_n := B(1_G, n)$. We inductively define the strictly increasing sequence of the subgroups as follows:

$G_1 :=< B_1 >$, and $G_n :=< B_n \cup \{x_n\} >$ for all $n \geq 2$, where $x_n \in G \setminus G_{n-1}$. Notice for any bounded subset $B \subset G$, there exists $n \in \mathbb{N}$ such that $B \subset B_n$ and hence $< B >\subset G_n$. If each $G_n$ is bounded then $G$ is locally bounded, a contradiction. Therefor, there exists $N \geq 1$ such that $G_n$ is unbounded for all $n \geq N$. Without loss of generality, we may assume that $N = 1$. It is clear that for each $n \geq N$, the coarse structure of $G_n$ is metrizable coarse structure as a coarse subspace of $G$.

Conversely, assume that $G$ admits a strictly increasing sequence $\{G_n\}_{n\in\mathbb{N}}$ of unbounded boundedly generated subgroups whose coarse structures as coarse subspace of $G$ are metrizable, and that $\{G_n\}_{n\in\mathbb{N}}$ is a basis for boundedly generated subgroups. Let $\mathcal{B}_n$ be a countable basis for the bounded subsets of $G_n$, for all $n \in \mathbb{N}$. Then $\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is a countable basis for the bounded subsets of $G$. Clearly, $G$ is non-locally bounded. $\square$

# Chapter 9

# Space of coarse ends of coarse groups

In this chapter we apply the theory of coarse ends of coarse spaces to coarse groups. In particular, we generalize 1.1.5, 1.1.6, and partially generalize 1.1.8; and give a proof for 1.1.7, in the language of coarse geometry. This chapter is based on Section 9 of (16).

## 9.1 Space of coarse ends

We begin with a coarse version of the first part of 1.1.8.

**Proposition 9.1.1.** *Let $G$ be a coarse group equipped with a metrizable coarse structure. If $G$ is an unbounded and locally bounded group, then its number of coarse ends is infinite.*

*Proof.* It suffices to show that $G$ has infinitely many unbounded coarsely clopen subsets of $G$ that are mutually disjoint. By 8.4.13, $G$ can be expressed as a union of a strictly increasing sequence $\{G_n\}_{n\in\mathbb{N}}$ of bounded subgroups that is a basis for boundedly generated subgroups. Choose $g_n \in G_{n+1} \setminus G_n$ for each $n \in \mathbb{N}$. Given an infinite subset $I$ of natural numbers define $A_I$ as $\bigcup\limits_{n\in\ I} g_n \cdot G_n$. We claim that $A_I$ is unbounded coarsely clopen. Indeed, $A_I$ is unbounded as for any $N \in \mathbb{N}$, there exists $n \in I$ such that $n > N$ and hence $g_n \in A_I \backslash G_N$. Since, $\{G_n\}_{n\in\mathbb{N}}$ is a basis for the bounded sets of $G$, $A_I$ must be unbounded. To show $A_I$ is coarsely clopen assume $B \subset G$ is bounded and choose $n \in I$ so that $B \subset G_n$, and therefore $\{G_n : n \in I\}$ is a basis for the bounded sets of $G$. Notice for any $i \in I$, if $i < n$, then $(g_i \cdot G_i) \cdot G_n = G_n$; and if $i \geq n$, then $(g_i \cdot G_i) \cdot G_n = g_i \cdot G_i$. In particular, $A_I \cdot G_n = G_n \cup \bigcup\limits_{i\in I, i\geq n} g_i \cdot G_i$, so $(A_I \cdot G_n)\Delta A_I$

is bounded, and hence $A_I$ is coarsely clopen. To complete the proof notice $A_I \cap A_J = \emptyset$ if $I \cap J = \emptyset$. Indeed, if $g \in A_I \cap A_J$, then there exist $i < j$ such that $g_i \cdot h = g_j \cdot h'$, where $h \in G_i$ and $h' \in G_j$, with $i \in I$ and $j \in J$. Therefore $g_j \in G_j$, a contradiction. Finally, since $\mathbb{N}$ can be expressed as an infinite union of mutually disjoint infinite subsets of $\mathbb{N}$, $G$ has infinitely many mutually disjoint unbounded coarsely clopen subsets and hence infinitely many coarse ends. □

**Definition 9.1.2.** Let $G$ be a coarse group and $A$ is a subset of $G$. We say that $G$ **acts trivially on** $A$ if $A$ and $x \cdot A$ are coarsely identical for all $x \in G$; that is, if $x \cdot A \Delta A$ is coarsely bounded for all $x \in G$.

**Lemma 9.1.3.** *Let $G$ be a coarsely geodesic coarse group. If $G$ contains three mutually disjoint unbounded coarsely clopen subsets, then it acts trivially on at most one of them.*

*Proof.* Let $K \subset G$ be symmetric bounded subset containing $1_G$ and generating $G$ such that the Clayey metric $d_K$ induces its coarse structure. Let $A_1, A_2$ and $A_3$ be mutually disjoint unbounded coarsely clopen subsets. Seeking contradiction, assume that $G$ acts trivially on $A_1$ and $A_2$. Put $A_4 := G \setminus \bigcup_{i=1}^{3} A_i$. Clearly that $A_4$ is coarsely clopen disjoint from $A_1, A_2$ and $A_3$. Choose $N \in \mathbb{N}$ large enough so that for all $x \in A_i \setminus B(1_G, N)$ and $y \in (G \setminus A_i) \setminus B(1_G, N)$ then $d(x, y) > 2$, for all $1 \leq i \leq 4$. If $A_4$ is coarsely bounded, we may assume that $A_4 \subset B(1_G, N)$. Choose $m \in \mathbb{N}$ such that $(A_i \setminus B(1_G, m)) \cap E_{2 \cdot N+2}[G \setminus A_i] = \emptyset$, for all $1 \leq i \leq 3$. In particular, $m > N$ and if $x \in A_i$ such that $d_K(x, 1_g) \geq m$, then $B(x, 2 \cdot N + 2) \subset A_i$, for all $1 \leq i \leq 3$. If $A_4$ is unbounded, we assume also that $(A_4 \setminus B(1_G, m)) \cap E_{2 \cdot N+2}[G \setminus A_4] = \emptyset$. Notice that if $x \in A_i$ such that $d_K(x, 1_G) \geq m$, then $x \cdot B(1_G, N) = B(x, N) \subset B(x, 2 \cdot N + 2) \subset A_i$, for all $1 \leq i \leq 3$.

Now, pick $x \in A_3$ such that $d_K(x, 1_G) \geq m$. On one hand, as $d_K(x^{-1}, 1_G) \geq m$, so $x^{-1} \in G \setminus B(1_G, N)$ and hence $B(x^{-1}, N)$ is contained in exactly one coarsely clopen $A_i$ for some $1 \leq i \leq 3$, if $A_4$ is bounded; and $B(x^{-1}, N)$ is contained in exactly one coarsely clopen $A_i$ for some $1 \leq i \leq 4$, if $A_4$ is unbounded. As $G$ acts trivially on $A_1$, $x \cdot A \Delta A$ is coarsely bounded, and thus we can choose $g_1 \in A_1$ such that $d_K(g_1, 1_G) \geq m$ and $x \cdot g_1 \in A_1$. Since $G =< K >$, $G$ is $E_K$-connected, and hence we can find an $E_K$-chain $g_1 = a_0, a_1, \ldots, a_s = 1_G$ connecting $g_1$ and $1_G$. Notice that $d_K(a_i, a_{i+1}) = 1$ for all $0 \leq i \leq s - 1$. Let $a_s$ be the first element of the $E_K$-chain that hits $B(1_G, N)$. Then, as $d_K(a_{s-1}, a_s) = 1$, $a_s \in A_1 \cap B(1_G, N)$ and hence $x \cdot a_s \in A_3$. Since $d_K$ is left-invariant, the sequence $x \cdot g_1 = x \cdot a_0, x \cdot a_1, \ldots, x \cdot a_s$ is still a $E_K$-chain connecting $x \cdot g_1 \in A_1$ to $x \cdot a_s \in A_3$ and $d_K(x \cdot a_i, x \cdot a_{i+1}) = 1$ for all $0 \leq i \leq s-1$. Thus, $x \cdot g_1 = x \cdot a_0, x \cdot a_1, \ldots, x \cdot a_s$ must hit $B(1_G, N)$, as otherwise, we would have two consecutive terms $x \cdot a_i$ and $x \cdot a_{i+1}$ of $x \cdot g_1 = x \cdot a_0, x \cdot a_1, \ldots, x \cdot a_s$ such that $d_K(x \cdot a_i, x \cdot a_{i+1}) > 2$, a contradiction. Therefore, there exists $x_1 \in A_1$ such that $x \cdot x_1 \in B(1_G, N)$ which implies that $x_1 \in B(x^{-1}, N)$, and hence $B(x^{-1}, N)$ must be contained in $A_1$. On the other hand, as $G$ acts trivially on $A_2$, we can find $x_2 \in A_2$ such that $x \cdot x_2 \in B(1_G, N)$ which implies that $x_2 \in B(x^{-1}, N)$ and hence $B(x^{-1}, N)$ intersects $A_2$, which contradicts the fact that $B(x^{-1}, N) \subset A_1$. Consequently, $G$ can act trivially on at most one of the coarsely clopen subsets $A_1, A_2$ and $A_3$. $\square$

It can be seen that if $A$ is a coarsely clopen subset of the coarse group $G$, then for any $x \in G$, $x \cdot A$ is coarsely clopen because if $K \subset G$ is bounded $((x \cdot A) \cdot K) \cap (x \cdot A)^c \cdot K) = x \cdot ((A \cdot K) \cap (A^c \cdot K))$ which is bounded. If, moreover, $A$ is unbounded, then $x \cdot A$ is unbounded. In particular, if $e \in Ends(G)$, then $x \cdot e = \{x \cdot A : A \in e\}$ is also a coarse end; and therefore, the action of $G$ on itself by left multiplication extends to $Ends(G)$. $G$ **acts trivially on** $Ends(G)$ if $x \cdot e = e$ for all $x \in G$ and $e \in Ends(G)$.

**Lemma 9.1.4.** *A coarse group $G$ acts trivially on $Ends(G)$ if and only if for all $e \in Ends(G)$ and $A \in e$, $G$ acts trivially on $A$.*

*Proof.* Assume that $G$ acts trivially on $Ends(G)$, $e \in Ends(G)$ and $A \in e$. For $x \in G$, we aim to show that $A$ and $x \cdot A$ are coarsely identical. Put $Ends(G) = \{e_\alpha : \alpha \in I\}$, for some index set $I$. Assume that $e = e_\beta$ for some $\beta \in I$. As $G$ acts trivially on $Ends(G)$, $A, x \cdot A \in e_\beta$. For all $\alpha \neq \beta$, we can find $A^\alpha \in e_\alpha$ such that $x \cdot A \cap A^\alpha = \emptyset$. Since $Ends(G) = A_{end} \cup (\bigcup_{\alpha \in I \setminus \{\beta\}} A^\alpha_{end})$. By 7.1.12, there exists a coarsely bounded subset $K \subset G$ such that $G = A \cup (\bigcup_{\alpha \in I \setminus \{\beta\}} A^\alpha) \cup K$. Therefore, $(x \cdot A) \setminus A \subset (x \cdot A) \cap K$ which is coarsely bounded. Similarly, $A \setminus (x \cdot A)$ is coarsely bounded and therefore $A$ and $x \cdot A$ are coarsely identical. Consequently, $G$ acts trivially on $A$.

Conversely, assume that for all $e \in Ends(G)$ and $A \in e$, $G$ acts trivially on $A$. It is obvious that $x \cdot e = e$, for all $x \in G$. $\square$

**Theorem 9.1.5.** *Let $G$ be a coarsely geodesic coarse group. The number of the coarse ends of $G$ is either infinite or at most $2$.*

*Proof.* If $G$ is bounded, then $Ends(G)$ is empty. If $G$ is locally bounded and unbounded, then $Ends(G)$ is infinite by 9.1.1.

Assume $G$ is unbounded, non-locally bounded, and its number of the coarse ends $m \geq 3$ is finite. Notice that, by 7.1.18, for any $x \in G$, the map $\sigma^x : G \to G$ that maps each $g \in G$ to $x \cdot g$ is a coarse equivalence, it induces a homeomorphism $\sigma^x_{end} : Ends(G) \to Ends(G)$. Let $Homeo(Ends(G))$ be the finite group of all homeomorphisms from $Ends(G)$ to itself. The map $\rho : G \to Homeo(Ends(G))$ given by: $\rho(x) = \sigma^x_{end}$ is a group homomorphism with $H := Ker(\rho)$ is a subgroup of finite index in $G$. By 8.1.23, $G$ and $H$ are coarsely equivalent and hence $Ends(G)$ and $Ends(H)$ are homeomorphic by 7.1.18. Moreover, $H$ acts trivially on $Ends(H)$, with $|Ends(H)| \geq 3$. Now, let $e_1, e_2, e_3 \in Ends(H)$ be three distinct ends, and choose $A_i \in e_i$ for all $1 \leq i \leq 3$ such that $A_1, A_2$ and $A_3$ are mutually disjoint. By 9.1.4, $H$ acts trivially on $A_1, A_2$ and $A_3$. This contradicts Lemma 9.1.3. □

As a result, a locally compact compactly generated group $G$ equipped with the coarse structure $\mathcal{E}_{\mathcal{G}_{com}}$ induced by the generating set $\mathcal{G}_{com}$ of all relatively compact subsets of $G$ has either infinitely many coarse ends or at most 2. For example, the topological group $SL_2(\mathbb{R})$ has either infinitely many coarse ends or at most 2 coarse ends. We aim to generalize 9.1.8 to a larger class of coarse groups that includes:

• Locally compact $\sigma$-compact groups equipped with the coarse structure induced by the generating set $\mathcal{G}_{com}$ of all relatively compact subsets.

• Infinite countable groups equipped with the generating set of all finite subsets.

Although these two classes of coarse groups are not necessarily coarsely geodesic, they can be expressed as a union of a sequence of coarsely geodesic unbounded subgroups where that sequence is a basis for boundedly generated subgroups.

**Lemma 9.1.6.** *Let $G$ be a coarse group that has a basis $\{G_n\}_{n \in \mathbb{N}}$ for boundedly generated subgroups consisting of unbounded subgroups. If $A$ is an unbounded coarsely clopen subset of $G$, then there is $n \in \mathbb{N}$ such that $A \cap G_n$ is unbounded.*

*Proof.* Let $G_1$ be generated by a symmetric bounded subset $B$ of $G$. Since $(A \cdot B)\Delta A$ is bounded, there is $m \in \mathbb{N}$ such that $(A \cdot B)\Delta A \subset G_m$. Notice that $(A \cdot B) \setminus G_m = A \setminus G_m$. If $A \setminus G_m = \emptyset$, we are done so assume $A \setminus G_m \neq \emptyset$. Now, for each $g \in A \setminus G_m$ and $b \in B$, we have $g \cdot b \in (A \cdot B) \setminus G_m = A \setminus G_m$. By induction, that implies $g \cdot G_1 \subset A \setminus G_m$. Choose $n \in \mathbb{N}$ such that $< B \cup g > \subset G_n$, then $g \cdot G_1 \subset A \cap G_n$ and hence $A \cap G_n$ is unbounded. □

As a consequence, a coarse group $G$ that can be expressed as a union of a sequence of boundedly generated subgroups where that sequence is a basis for boundedly generated unbounded subgroups, then $|Ends(G)| \leq \sup\limits_{n \in \mathbb{N}} |Ends(G_n)|$.

**Corollary 9.1.7.** *Let $G$ be a coarse group that has a basis $\{G_n\}_{n\in\mathbb{N}}$ for boundedly generated subgroups consisting of unbounded subgroups. If $m \leq \infty$ and each $G_n$ has at most $m$ ends, then the number of ends of $G$ is at most $m$.*

*Proof.* The case $m = \infty$ is clear, so assume $m < \infty$.

If $G$ has $(m+1)$ mutually disjoint unbounded coarsely clopen subsets $A_i$, $1 \leq i \leq m+1$, then by 9.1.6 we can find an index $n$ such that each $A_i \cap G_n$ is an unbounded coarsely clopen subset of $G_n$. Hence $|Ends(G_n)| \geq m+1$, a contradiction. □

Combining 9.1.6 and 9.1.7, we generalize 9.1.8.

**Theorem 9.1.8.** *Let $G$ be a coarse group that admits a basis $\{G_n\}_{n\in\mathbb{N}}$ for boundedly generated subgroups consisting of coarsely geodesic subgroups. The number of the coarse ends of $G$ is either infinite or at most 2.*

*Proof.* Notice for each $G_n$ there exists a symmetric bounded subset $K_i \subset G$ containing $1_G$ such that $G_i =< K_i >$ and that $(K_i^n)_{n\in\mathbb{N}}$ is a basis for the bounded subsets $G_i$. In particular, $\{K_i^n : i, n \in \mathbb{N}\}$ is a countable basis for the bounded subsets of $G$, and hence its coarse structure is metrizable. If $G$ is bounded, then $Ends(G)$ is empty. If $G$ is locally bounded and unbounded, then $Ends(G)$ is infinite by 9.1.1.

Assume $G$ is unbounded, non-locally bounded and its number of coarse ends is finite. Notice that the boundedly generated groups cannot be all bounded. Choose $N \in \mathbb{N}$ such that $G_n$ unbounded for all $n \geq N$. Let $H_1 :=< \bigcup\limits_{i=1}^{N} K_i >$, and $H_i := G_{N+i}$, for all $i \in \mathbb{N}$. Clearly, $\{H_n\}_{n\in\mathbb{N}}$ is a basis for boundedly generated subgroups consisting of unbounded coarsely geodesic subgroups. Seeking contradiction, assume that $G$ has $m \geq 3$ coarse ends. Without loss of generality, we may assume that $G$ acts trivially on $Ends(G)$. Notice that $G$ contains three disjoint unbounded coarsely clopen subsets $A_1, A_2$ and $A_3$, on which, by 9.1.4, $G$ acts trivially. By 9.1.6, there exists $n \in \mathbb{N}$ such that $A_i \cap G_n$ is unbounded for all $1 \leq i \leq 3$. Moreover, $G_n$ acts trivially on $A_i \cap G_n$, for all $1 \leq i \leq 3$, which contradicts 9.1.3. Hence, $|Ends(G)| \leq 2$. □

**Corollary 9.1.9.** *Let $G$ be a locally compact $\sigma$-compact topological group equipped with the coarse structure induced by the generating set of all relatively compact subsets. The number of ends of $G$ is either infinite or at most 2.*

*Proof.* Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of $G$. The sequence $\{G_n\}_{n\in\mathbb{N}}$, where $G_n :=< K_n >$ is a basis for boundedly generated subgroups consisting of subgroups that are coarsely geodesic. □

**Example 9.1.10.** *Consider the topological groups $G = SL_n(\mathbb{R})$ and $H = GL_n(\mathbb{R})$ equipped with the coarse structure induced from the generating set of all relatively compact subsets. Then, $G$ and $H$ have either infinitely many coarse ends or at most 2.*

**Corollary 9.1.11.** *Let $G$ be a countable coarse group equipped with the coarse structure induced by the generating set of all finite subsets. The number of ends of $G$ is either infinite or at most 2.*

*Proof.* $G$ can be expressed as an increasing sequence of finitely generated subgroups. □

9.1.11 can be generalized to arbitrary groups equipped with the coarse structure induced by the generating set of all finite subsets. We first prove that if an uncountable locally finite group has finitely many coarse ends with respect to the coarse structure induced by the generating set of all finite subsets, then it has exactly one coarse end.

**Lemma 9.1.12.** *Let $G$ be a locally finite uncountable group equipped with the coarse structure induced by the generating family of finite subsets. For every infinitely countable $C \subset G$ and non-trivial coarsely clopen subset $A \subset G$, there exists a countable subgroup $H$ containing $C$ such that*

$$A \cdot H \setminus A \subset H$$

*and*

$$A^c \cdot H \setminus A^c \subset H.$$

*Proof.* Inductively, one can construct a strictly increasing sequence of finite subgroups $(H_n)_{n\in\mathbb{N}\cup\{0\}}$ such that:

1. $H_n$ contains at least $n$ elements of $C$.

2. $A \cdot H_n \setminus A \subset H_{n+1}$.

3. $A^c \cdot H_n \setminus A^c \subset H_{n+1}$. Indeed, put $H_0 := \{1_G\}$, and $H_1 :=< a_1 >$ where $a_1 \in C \setminus \{1_G\}$.

Now, for $n \in \mathbb{N}$, assume that $H_n$ is strictly in the finite subgroup $H_{n+1}$ and that:

1. $H_n$ contains at least $n$ elements of $C$, and $H_{n+1}$ contains at least $n+1$ elements of $C$.

2. $A \cdot H_n \setminus A \subset H_{n+1}$.

3. $A^c \cdot H_n \setminus A^c \subset H_{n+1}$.

Let $H_{n+2}$ be the subgroup of $G$ generated by $(A \cdot H_{n+1} \setminus A) \cup (A^c \cdot H_{n+1} \setminus A^c) \cup \{a_{n+2}\}$, where $a_{n+2} \in C \setminus H_{n+1}$. Since $A$ is coarsely clopen and $G$ is locally finite, $H_{n+2}$ is finite. Let $H := \bigcup\limits_{n\in\mathbb{N}} H_n$. □

**Theorem 9.1.13.** *Let $G$ be a locally finite uncountable group equipped with the coarse structure induced by the generating family of finite subsets. If $G$ has finitely many coarse ends, then it has exactly one coarse end.*

*Proof.* By 8.4.12, there is no loss of generality in assuming that $G$ acts trivially on its coarse ends. Seeking contradiction assume that $G$ has more than one coarse end. Then, $G$ has a non-trivial coarsely clopen subset $A \subset G$ such that $G$ acts trivially on both $A$ and $A^c$, by 9.1.4. Let $C$ be a countably infinite subset that contains countably infinite elements of $A$ and countably infinite elements $A^c$. By 9.1.12, we can choose a countably infinite subgroup $H$ that contains $C$ such that:

$$A \cdot H \setminus A \subset H$$

and

$$A^c \cdot H \setminus A^c \subset H.$$

Now, let $g \in G \setminus H$, then either $g \in A$ or $g \in A^c$. Without loss of generality assume that $g \in A$. Then, $g \cdot H \subset A$. Indeed, if $g \cdot h \in A^c$ for some $h \in H$, then $g = (g \cdot h) \cdot h^{-1} \in A^c \cdot H \setminus A^c \subset H$, a contradiction. Since $g \cdot H \subset A$, so $H \subset g^{-1} \cdot A$ and therefore $H \cap A^c \subset g^{-1} \cdot A \cap A^c$. But, $H \cap A^c$ is infinite, and $g^{-1} \cdot A \cap A^c$ is finite because $G$ acts trivially on $A$, a contradiction. Hence, either $A$ is infinite and $A^c$ is finite or $A^c$ is infinite and $A$ is finite, which shows that $G$ has exactly one coarse end as desired. $\square$

**Theorem 9.1.14.** *Let $G$ be group equipped with the coarse structure induced by the generating set of all finite subsets. If $G$ has finitely many ends, then it has at most 2 ends.*

*Proof.* Without loss of generality, assume that $G$ is non-locally finite uncountable group that has finitely many coarse ends. $G$ has a subgroup $H$ of finite index that has the same number of coarse ends as $G$; and $H$ acts on its coarse ends trivially. Suppose $H$ has at least 3 coarse ends. In that case we can find three mutually disjoint coarsely clopen subsets $A_i$ of $H$, $1 \leq i \leq 3$. Let $B_i := A_i^c$, and choose countable unbounded subsets $C_i$ and $D_i$ of each $A_i$ and $B_i$, respectively. Since $G$ is a non-locally finite group; $H$ is a non-locally finite group, by 8.4.12. Choose a finite subset $K$ of $H$ such that $< K >$ is infinite. Let $H'$ be the subgroup of $H$ generated by $K \cup \bigcup_{i=1}^{3}(C_i \cup D_i)$. Notice that $H'$ is countably generated and hence it is countable. Moreover, each $A_i' = A_i \cap H'$ is an unbounded coarsely clopen subset of $H'$ on which $H'$ acts trivially. Now, since $H'$ is non-locally finite, $H'$ can be expressed as an increasing sequence of infinite finitely generated subgroups $\{H_n\}_{n\in\mathbb{N}}$. By 9.1.6, there exists $n \in \mathbb{N}$ such that each $A_i' \cap H_n$, $1 \leq i \leq 3$, is unbounded coarsely clopen subset of $H_n$ on which $H_n$ acts trivially, contradicting 9.1.3. $\square$

Now we generalize the Specker's theorem 1.1.7 that says that $|ends(G)| = 2$ if and only if $G$ contains an infinite cyclic subgroup of finite index; by showing that any group of bounded geometry of two ends contains an unbounded cyclic subgroup of bounded index.

**Lemma 9.1.15.** *Suppose $K$ is a symmetric bounded subset of a coarse group $G$ containing $1_G$, $B$ is a bounded subset of $G$ contained in a bounded $E_K$-connected subset $Q$. If $G \setminus B$ has two unbounded $E_K$-components $L$ and $R$ on which $G$ acts trivially, then $G$ has an infinite cyclic subgroup of bounded index.*

*Proof.* By 8.2.3, $G$ is boundedly generated. It is indeed generated by $K$. To this end, it suffices to show that $G$ is $E_K$-connected. Choose $a_L \in L$ and $a_R \in R$. Put $g := a_L \cdot a_R^{-1}$.

Since $g^{-1} \cdot R \Delta R$ is bounded, there is $x \in R$ such that $g^{-1} \cdot x \in R$. Choose an $E_K$-chain $c$ in $R$ connecting $a_R$ and $g^{-1} \cdot x$. Notice $g \cdot c$ is an $E_K$-chain connecting $a_L$ and $x \in R$. Therefore $G$ is $E_K$-connected and $B \neq \emptyset$. By switching to $b^{-1} \cdot B$, $b^{-1} \cdot L$, and $b^{-1} \cdot R$ for some $b \in B$, we may assume $1_G \in B \subset Q$.

**Claim 1:** There exists $g \in R$ such that $g \cdot B \subset R$, $L \subset g \cdot L$, and $g \cdot R \subset R$.

**Proof of the Claim 1:** Choose $g \in R \setminus (Q \cdot Q^{-1} \cup Q^{-1} \cdot Q)$, then $g \cdot Q \cap Q = \emptyset$, and hence $g \cdot Q \cap B = \emptyset$. In particular, $g \cdot Q$ is an $E_K$-connected subset of $G \setminus B$ with $g \in R \cap g \cdot Q$ which implies that $g \cdot Q \subset R$. To see that $L \subset g \cdot L$, notice that $L \cap g \cdot B = \emptyset$, and thus $L$ is an unbounded $E_K$-connected subset of $G \setminus g \cdot B$ and therefore either $L \subset g \cdot L$ or $L \subset g \cdot R$. Since $G$ acts trivially on $L$, we have $L \subset g \cdot L$. Finally, to prove that $g \cdot R \subset R$. Indeed, notice that $g^{-1} \cdot Q \cap Q = \emptyset$ and $g^{-1} \cdot B \subset g^{-1} \cdot Q$. So either $g^{-1} \cdot B \subset L$, or $g^{-1} \cdot B \subset R$. If $g^{-1} \cdot B \subset L$, then $B \subset g \cdot L$ which implies that $g \cdot R \subset G \setminus g \cdot L \subset G \setminus (L \cup B) = R$, and we are done. Seeking contradiction, assume that $g^{-1} \cdot B \subset R$. Then, $B \subset g \cdot R$ and therefore $g \cdot L \cap B = \emptyset$. Now, $g \cdot L$ is an $E_K$-connected subset of $G \setminus B$, and $g \cdot L \cap L \neq \emptyset$, so $g \cdot L \subset L$ leading to $g \cdot L = L$. Since $G$ is $E_K$-connected, we can find an $E_K$-chain $a_R = a_0, a_1, \ldots, a_k = a_L$ in $G$ connecting $a_R \in R$ to $a_L \in L$. There exists $1 \leq i \leq k-1$ such that $a_i \in B$ and $a_{i+1} \in L$. Since $(g \cdot a_i, g \cdot a_{i+1}) \in E_K$, and $g \cdot a_{i+1} \in L$, then $g \cdot a_i \in L$, as $g \cdot a_i \in G \setminus B$ and $L$ is an $E_K$-component of $G \setminus B$. In particular, $g \cdot Q \cap L \neq \emptyset$, and hence $g \in g \cdot Q \subset L$, which contradicts the fact that $g \in R$. Therefore, $g \cdot R \subset R$ as claimed.

Inductively, we have:

- $g^n \cdot L \subset g^{n+1} \cdot L$,
- $g^n \cdot B \subset g^{n+1} \cdot L$,
- $g^{n+1} \cdot R \subset g^n \cdot R$,
- $g^{n+1} \cdot B \subset g^n \cdot R$,
- $g^n \cdot B \cap g^{n+1} \cdot B = \emptyset$, for all $n \in \mathbb{Z}$.

The last equality tells us that $g$ is of infinite order.

**Claim 2:** $G$ has a cyclic subgroup of bounded index.

**Proof of the Claim 2:** Put $H := < g >$, and $K_0 := (g \cdot L \setminus L) \cup g \cdot B$. Clearly, $K_0$ is a coarsely bounded subset of $G$. We claim that $G = H \cdot K_0$. Given $x \in G$ find $m \in \mathbb{N} \cup \{0\}$ such that $d_K(x, g^m \cdot B) \leq d_K(x, g^n \cdot B)$, for all $n \in \mathbb{N}$. If $d_K(x, g^m \cdot B) = 0$, then $x \in g^m \cdot B \subset H \cdot K_0$.

Assume that $d_K(x, g^m \cdot B) > 0$, then $x \notin g^m \cdot B$, and hence either $x \in g^m \cdot R$ or $x \in g^m \cdot L$. We consider two cases:

**Case 1:** If $x \in g^m \cdot R$, then $x \in g^m \cdot R \setminus g^{m+1} \cdot R$. Indeed, seeking contradiction, assume that $x \in g^{m+1} \cdot R$, then as $g^m \cdot B \subset g^{m+1} \cdot L$, any $E_K$-chain connecting $x \in g^{m+1} \cdot R$ to $x \in g^m \cdot B$ must pass through $g^{m+1} \cdot B$ and hence $d_K(x, g^{m+1} \cdot B) \leq d_K(x, g^m \cdot B)$. Moreover, as $g^m \cdot B \cap g^{m+1} \cdot B = \emptyset$, $d_K(x, g^{m+1} \cdot B) < d_K(x, g^m \cdot B)$, a contradiction. Thus, $x \in g^m \cdot R \setminus g^{m+1} \cdot R \subset (g^{m+1} \cdot L \setminus g^m \cdot L) \cup g^{m+1} \cdot B = g^m \cdot (g \cdot L \setminus L \cup g \cdot B) \subset H \cdot K_0$.

**Case 2:** If $x \in g^m \cdot L$, then $x \in g^m \cdot L \setminus g^{m-1} \cdot L$. Indeed, seeking contradiction, assume that $x \in g^{m-1} \cdot L$, then as $g^m \cdot B \subset g^{m-1} \cdot R$, any $E_K$-chain connecting $x \in g^{m-1} \cdot L$ to $x \in g^m \cdot B$ must pass through $g^{m-1} \cdot B$ and hence $d_K(x, g^{m-1} \cdot B) \leq d_K(x, g^m \cdot B)$. Moreover, as $g^{m-1} \cdot B \cap g^m \cdot B = \emptyset$, $d_K(x, g^{m-1} \cdot B) < d_K(x, g^m \cdot B)$, a contradiction. Thus, $x \in g^m \cdot L \setminus g^{m-1} \cdot L = g^{m-1} \cdot (g \cdot L \setminus L) \subset H \cdot K_0$. Therefore, $G = H \cdot K_0$ as desired. □

**Lemma 9.1.16.** *If $A$ is a coarsely clopen subset of a coarsely geodesic coarse group $G$ with a coarse structure metrizable by a Cayley metric $d$; then there exists a bounded subset $L$ of $G$ such that every 1-component of $G \setminus L$ intersecting $A \setminus L$ is completely contained in $A \setminus L$. In particular, $A$ is a union of 1-components of $G \setminus L$.*

*Proof.* Let $L := (A \cdot K) \cap (A^c \cdot K)$. If $g \in A \setminus L$, then $g \notin E_K[G \setminus A]$. Now, let $h \in G$ such that $(g, h) \in E_K$, then $h \in A$. Indeed, if $h \in G \setminus A$, then $g \in E_K[G \setminus A]$, a contradiction. □

**Lemma 9.1.17.** *Let $G$ be a boundedly generated coarse group of bounded geometry. $G$ acts trivially on a subset $A$ and its complement $A^c$, for some non-trivial coarsely clopen subset $A \subset G$, if and only if it contains an unbounded cyclic subgroup of bounded index that has two coarse ends.*

*Proof.* It is obvious that $G$ is unbounded. If $G$ is cyclic, then we have nothing to prove. Without loss of generality, we may assume that $G$ is non-cyclic and it acts on its coarse ends trivially. By 8.4.7, $G$ is coarsely geodesic group. Let $d_K : G \times G \to [0, \infty)$ be a Cayley metric induces its coarse structure, where $K$ is a coarsely bounded symmetric subset of $G$ containing $1_G$. Let $A \subset G$ be a non-trivial coarsely clopen subset. By 8.3.7 and 9.1.16, we can find a coarsely connected subset $L \subset G$ such that both $A \setminus L$ and $A^c \setminus L$ are union of unbounded 1-components (which are $E_K$-components) on which $G$ acts trivially. As seen in

8.2.4, $K^n$ is $E_K$-connected for all $n \in \mathbb{N}$. Choose $r \in \mathbb{N}$ such that $r > 1$ and $L \subset K^r$. Pick $g \in A \setminus K^{2r}$. Assume that $d_K(g, 1_G) = m$. Notice that $m > 2r$ and for any $a \in K$, then $d_K(g \cdot a, 1_G) \geq 2r - 1$ and hence $g \cdot a \in A \setminus L$. Moreover, as $(g \cdot a, g) \in E_K$, we have $g \cdot a$. Put $g := a_1 \cdot \ldots \cdot a_m$, where $a_1, \ldots, a_m \in K$, and let $F(K)$ be the free group generated by $K$; we discuss two cases:

**Case 1:** If $g$ is cyclically reduced in the sense that $a_1 \cdot a_m \neq 1_G$, then $(a_1 \cdot \ldots \cdot a_m)^n = (a_1 \cdot \ldots \cdot a_m) \cdot \ldots \cdot (a_1 \cdot \ldots \cdot a_m)$ is a reduced word, for all $n \in \mathbb{N}$. Since every element of $G$ is uniquely represented as a reduced word in $F(K)$, we have $d_K(g^n, 1_G) = m \cdot n$. Put $H :=< g >$, then as seen in 9.1.15, $g^n \in A$, for all $n \in \mathbb{N}$, and $H$ is infinite cyclic subgroup of bounded index in $G$. Furthermore, $g^n \notin K^n$, for all $n \in \mathbb{N}$, which proves that $H$ is unbounded.

**Case 2:** If $g = a_1 \cdot \ldots \cdot a_m$ is not cyclically reduced; that is $a_1 \cdot a_m = 1_G$; then as $G =< K >$ is not cyclic, there exists $a \in K$ such that $a_1 \cdot a \neq 1_G$. In particular, $g \cdot a \in A \setminus K^r$ and $g \cdot a$ is cyclically reduced. Hence, $H :=< g \cdot a >$ is unbounded cyclic subgroup of bounded index in $G$.

Finally, we prove that $H =< g >$ has exactly tow coarse ends. Without loss of generality, assume that $g$ cyclically reduced and $d_K(g, 1_G) = m$. Then, $d_K(g^n, 1_G) = n \cdot m$, for all $n \in \mathbb{N}$. Clearly, $A_H = A \cap H$ is non-trivial coarsely clopen. Without loss of generality, assume that $A_H = \{g^n : n \in \mathbb{N}\}$ and $A_H^c = \{g^n : n \in \mathbb{Z} \setminus \mathbb{N}\}$. Choose $e_1, e_2 \in Ends(H)$ such that $A_H \in e_1$ and $A_H^c \in e_2$. We claim that $Ends(H) = \{e_1, e_2\}$. To this end, we prove that for arbitrary unbounded coarsely clopen subset $C \subset H$ of $H$, then either $C \in e_1$ or $C \in e_2$. We may assume that $C$ is non-trivial coarsely clopen.

**Claim:** If $C \cap A_H$ is unbounded, then $C \cap A_H$ and $A_H$ are coarsely identical. Similarly, if $C \cap A_H^c$ is unbounded, then $C \cap A_H^c$ and $A_H^c$ are coarsely identical.

**Proof of the Claim:** Assume that $C \cap A_H$ is unbounded. Seeking contradiction, assume that there exists an infinite subset $I \subset \mathbb{N}$ such that $\mathbb{N} \setminus I$ is infinite and $C \cap A_H = \{g^n : n \in I\}$. Clearly, we can write $C \cap A_H = \{g^{n_i} : n_i \in \mathbb{N}\}$ as $I$ is infinite countable subset of $\mathbb{N}$. Notice that there are infinitely many non-consecutive pairs $(n_i, n_{i+1})$ in the sense that $n_i < n_i + 1 < n_{i+1}$. Take $N > m + 1$, then $B(C, N) \cap C^c$ is infinite and hence unbounded, as $K^r$ contains only finitely many elements of $H$ for all $r \in \mathbb{N}$; a contradiction. Hence, $C \cap A_H$

and $A_H$ are coarsely identical. Similarly, if $C \cap A_H^c$ is unbounded, then $C \cap A_H^c$ and $A_H^c$ are coarsely identical.
If both $C \cap A_H$ and $C \cap A_H^c$ are unbounded, then $H \setminus C$ is bounded which contradicts the fact that $C$ is non-trivial coarsely clopen. Thus, either $C \in e_1$ or $C \in e_2$ which implies that $H$ has exactly two coarse ends.

The converse is obvious. □

**Corollary 9.1.18.** *Let $G$ be a $\sigma$-bounded coarse group of bounded geometry. $G$ has two coarse ends if and only if it acts trivially $A$ and $A^c$ for some non-trivial coarsely clopen subset $A \subset G$.*

*Proof.* If $G$ has two coarse ends, then it acts trivially $A$ and $A^c$ for some non-trivial coarsely clopen subset $A \subset G$, by 9.1.4. Assume that $G$ acts trivially $A$ and $A^c$ for some non-trivial coarsely clopen subset $A \subset G$. We consider two cases:
**Case 1:** If $G$ is boundedly generated, then $G$ has two coarse ends by and 9.1.17.
**Case 2:** Assume that $G$ is not boundedly generated. By 8.4.14, $G$ can be expressed as the union of a strictly increasing sequence $\{G_n\}_{n \in \mathbb{N}}$ of unbounded boundedly generated subgroups and that $\{G_n\}_{n \in \mathbb{N}}$ is a basis for boundedly generated subgroups. By 9.1.6, we may assume that $G_n \cap A$ and $G_n \cap A^c$ are unbounded for all $n \in \mathbb{N}$. By **Case 1**, each $G_n$ has exactly two coarse ends. Applying 9.1.7, we obtain that $|Ends(G)| = 2$. □

**Theorem 9.1.19.** *Let $G$ be a boundedly generated coarse group of bounded geometry. If $G$ has two ends, then it contains an unbounded cyclic subgroup of bounded index.*

*Proof.* Apply 9.1.4 and 9.1.17. □

**Lemma 9.1.20.** *Let $G$ be a coarse group containing two unbounded cyclic subgroups $H$ and $K$. If $H$ is of bounded index in $G$, then so is $K$.*

*Proof.* Since there is a coarse equivalence $f : G \to H$ inverse to the inclusion $i : H \to G$, $f|K : K \to H$ is a coarse embedding, hence a coarse equivalence. Consequently, the inclusion $K \to G$ is a coarse equivalence and $K$ is of bounded index in $G$. □

**Theorem 9.1.21.** *Let $G$ be a $\sigma$-bounded coarse group of bounded geometry. If $G$ has two coarse ends, then it is boundedly generated.*

*Proof.* Without loss of generality, assume that $G$ acts trivially on its ends. By 8.4.4, its coarse structure is metrizable. Since $G$ has finitely many ends, it is non-locally bounded by 9.1.1. Seeking contradiction, assume that $G$ is not boundedly generated, then by 8.4.14 and 8.4.7, $G$ admits a strictly increasing sequence $(G_n)_{n\in\mathbb{N}}$ of unbounded coarsely geodesic subgroups that is a basis for boundedly generated subgroups. Let $A$ be a non-trivial coarsely clopen subset of $G$. By 9.1.6, there exists $n_0 \in \mathbb{N}$ such that $A \cap G_{n_0}$ and $A^c \cap G_{n_0}$ are unbounded. Without loss of generality, assume that $n_0 = 1$. Consequently, $A \cap G_n$ and $A^c \cap G_n$ are unbounded, for all $n \in \mathbb{N}$. By 9.1.15, each $G_n$ has an unbounded cyclic subgroup $H_n$ of bounded index.

**Claim :** For all $n \in \mathbb{N}$, there exists $N_n \in \mathbb{N}$ such that $N_n > n$, $A \cap G_{N_n} \setminus G_n \neq \emptyset$ and $A^c \cap G_{N_n} \setminus G_n \neq \emptyset$.

**Proof of the Claim :** It suffices to prove that $A$ and $A^c$ cannot be contained in any $G_n$. Seeking contradiction, assume that $A \subset G_n$ for some $n \in \mathbb{N}$. Choose $z \in G_{n+1} \setminus G_n$. On one hand, if $A \cap A \cdot z \neq \emptyset$, then we can find $a, b \in A$ such that $z = a^{-1} \cdot b \in A^{-1} \cdot A \subset G_n$, a contradiction. On the other hand, if $A \cap A \cdot z = \emptyset$, then that contradicts the fact that $A$ is unbounded coarsely clopen. Therefore, for all $n \in \mathbb{N}$, there exists $N_n \in \mathbb{N}$ such that $N_n > n$ and $A \cap G_{N_n} \setminus G_n \neq \emptyset$. Similarly, for all $n \in \mathbb{N}$, there exists $N_n \in \mathbb{N}$ such that $N_n > n$ and $A^c \cap G_{N_n} \setminus G_n \neq \emptyset$.

Now, put $H_1 :=< t >$; as $A \Delta A \cdot t$ and $A^c \Delta A^c \cdot t$ are bounded subsets, there exists $m \in \mathbb{N}$ such that $A \Delta A \cdot t, A^c \Delta A^c \cdot t \subset G_m$. Choose $N_m \in \mathbb{N}$ large enough such that $A \cap G_{N_m} \setminus G_m \neq \emptyset$ and $A^c \cap G_{N_m} \setminus G_m \neq \emptyset$; and pick $x \in A \cap G_{N_m} \setminus G_m$. Notice that $x \cdot t^n \in (G_{N_m} \setminus G_m) \cdot t^n = G_{N_m} \setminus G_m$, for all $n \in \mathbb{Z}$, as $< t > \subset G_m$. We claim that $x \cdot t^n \in A \cap G_{N_m} \setminus G_m$, for all $n \in \mathbb{Z}$. If $x \cdot t \notin A$, then $x \cdot t \in A \cdot t \setminus A \subset G_m$, a contradiction. Assume that $x \cdot t^n \in A \cap G_{N_m} \setminus G_m$, then $x \cdot t^{n+1} \in A \cdot t$. If $x \cdot t^{n+1} \notin A$, then $x \cdot t^{n+1} \in A \cdot t \setminus A \subset G_m$, a contradiction. Therefore, $x \cdot t^n \in A \cap G_{N_m} \setminus G_m$, for all $n \in \mathbb{Z}$ as claimed. Put $B := \{x \cdot t^n : n \in \mathbb{Z}\}$; then $B \subset A \cap G_{N_m}$ which implies that $H_1 = x^{-1} \cdot B \subset x^{-1} \cdot A \cap G_{N_m}$. Since $G$ acts trivially on $A$, $x^{-1} \cdot A \setminus A$ is bounded, and hence

$H_1 \cap A$ is unbounded and $H_1 \cap A^c$ is bounded. One other hand, we can find $y \in A^c \cap G_{N_m} \setminus G_m$ such that $y \cdot t^n \in A^c \cap G_{N_m} \setminus G_N$. Put $C := \{y \cdot t^n : n \in \mathbb{Z}\}$; then $C \subset A^c \cap G_{N_m}$ which implies that $H_1 = y^{-1} \cdot C \subset y^{-1} \cdot A^c \cap G_{N_m}$. Since $G$ acts trivially on $A^c$, $y^{-1} \cdot A^c \setminus A^c$ is bounded, and hence $H_1 \cap A^c$ is unbounded and $H_1 \cap A$ is bounded; a contradiction. □

**Theorem 9.1.22.** *If $(G, \mathcal{E}_{\mathcal{G}})$ is a coarse group of bounded geometry. Then, the following conditions are equivalent:*

*(a) $G$ is boundedly generated with 2 coarse ends.*

*(b) $G$ is coarsely geodesic with 2 coarse ends.*

*(c) $\mathcal{E}_{\mathcal{G}}$ is metrizable and $G$ has 2 coarse ends.*

*(d) $G$ is $\sigma$-bounded with 2 coarse ends.*

*(e) $G$ contains an unbounded cyclic subgroup of bounded index that has two coarse ends.*

*Proof.* $(a) \implies (b)$ follows from 8.4.7.

$(d) \implies (e)$ follows from 9.1.19.

The other implications straightforward. □

**Corollary 9.1.23.** *Let $G$ be a locally compact $\sigma$-compact topological group equipped with the coarse structure induced by the generating set of all relatively compact subsets. If $G$ is not compactly generated, then either $G$ is 1-ended or it has infinitely many ends.*

*Proof.* Apply 9.1.22. □

**Corollary 9.1.24.** *Let $G$ be a countable coarse group equipped with the coarse structure induced by the generating set of all finite subsets. If $G$ has two ends, then it is finitely generated and contains an infinite cyclic subgroup of finite index.*

*Proof.* Apply 9.1.22. □

**Example 9.1.25.** *The group of the rationals $(\mathbb{Q}, +)$ has exactly one coarse end.*

*Proof.* Every finitely generated subgroup of $\mathbb{Q}$ is cyclic, and $\mathbb{Q}$ is not finitely generated. Therefore, $\mathbb{Q}$ can be written as a union of strictly increasing sequence of cyclic subgroups. From 9.1.7, and 9.1.24, $|Ends(\mathbb{Q})| = 1$. □

**Theorem 9.1.26.** *Let $G$ be a group equipped with the coarse structure induced by the generating set of all finite subsets. $G$ has tow coarse ends if and only if $G$ is finitely generated and contains an infinite cyclic subgroup of finite index.*

*Proof.* If $G$ is countable, then we are done. Seeking contradiction assume that $G$ is uncountable. If $G$ is locally finite, then, by 9.1.14, it has one coarse end, a contradiction. Without loss of generality, we may assume that $G$ acts on its coarse ends trivially. Since $G$ is non-locally finite, there exists a finite subset $K \subset G$ such that $< K >$ is infinite subgroup. Let $A$ be a non-trivial coarsely clopen subset of $G$. Let $B$ and $C$ be countably infinite subsets of $A$ and $A^c$, respectively. Let $H$ be arbitrary countable subgroup of $G$; then $H' :=< K \cup B \cup C \cup H >$ is a countable non-locally finite subgroup of $G$. Moreover, $H'$ acts trivially on $A \cap H'$ and $A^c \cap H'$, with are unbounded coarsely clopen subsets of $H'$. By 9.1.18, $H'$ has exactly tow coarse ends. By 9.1.22, $H'$ is a finitely generated subgroup of $G$ containing $H$. In particular, every countable subgroup $H$ of $G$ is contained in a finitely generated subgroup $H'$ that has two coarse ends.

Let $H_1 = H'$, and choose $x_1 \in G \setminus H_1$, and a finitely generated subgroup $H_2$ of two coarse ends containing $< H_1 \cup \{x_1\} >$. Inductively, we construct a strictly increasing sequence $H_n$ of finitely generated subgroups of $G$ each having two coarse ends. Put $G' := \bigcup\limits_{n \in \mathbb{N}} H_n$. Clearly, $G'$ is a countable subgroup of $G$. Moreover, $G'$ has two coarse ends as $|Ends(G')| \leq 2$, by 9.1.7; and hence $|Ends(G')| = 2$ as $A \cap G'$ and $A^c \cap G'$ are unbounded coarsely clopen subsets of $G'$ that belong to two distinct coarse ends of $G'$. Therefore, by 9.1.22, $G'$ is a finitely generated subgroup of $G$. However, any finite subset $K' \subset G'$ is contained in $H_n$, for some $n \in \mathbb{N}$, which contradicts the fact that $G'$ is finitely generated. Hence, $G$ must be countable. $\square$

## 9.2 Metrizablity of the space of coarse ends of coarsely geodesic coarse groups

We give the the necessary and sufficient conditions for the space of coarse ends $Ends(G)$ of a coarsely geodesic coarse group $G$ to be topologically metrizable. Then, similar to 4.1.14, we construct a metric on $Ends(G)$ that induces its topology. When $|Ends(G)| = \infty$, we show, in similar manner to 9.2.9, that the space of coarse ends is a Cantor space.

**Lemma 9.2.1.** *If $A$ is a coarsely clopen subset of a coarsely geodesic coarse group $G$ with a coarse structure metrizable by a Cayley metric $d$. Then:*
*(a) Every union of 1-components of $A$ is coarsely clopen.*
*(b) There is a bounded subset $L$ of $G$ such that every 1-component of $G \setminus L$ intersecting $A \setminus L$ is completely contained in $A \setminus L$.*

*Proof.* $(a)$ Let $K = \{g \in G : d(g, 1_G) \leq 1\}$. Notice that $\{E_{K^n}\}_{n\in\mathbb{N}}$ is a basis for $\mathcal{E}_d$, so it suffices to show that $E_{K^n}[C] \cap G \setminus C$ is bounded for all $n \in \mathbb{N}$. Since $E_K$ is an entourage containing the diagonal, and $A$ is coarsely clopen, then $A$ and $E_K[A]$ are coarsely identical. Without loss of generality, we may assume that $E_K[A] = A$. Since $C \subset A$ is a union of 1-components of $A$, $E_K[C] = C \cdot K = C$. Therefore $E_{K^n}[C] = C \cdot K^n = C$ for all $n \in \mathbb{N}$ and hence $E_{K^n}[C] \cap G \setminus C = \emptyset$. Thus, $C$ is coarsely clopen.
$(b)$ Let $L := (A \cdot K) \cap (A^c \cdot K)$. If $g \in A \setminus L$, then $g \notin E_K[G \setminus A]$. Now, let $h \in G$ such that $(g, h) \in E_K$, then $h \in A$. Indeed, if $h \in G \setminus A$, then $g \in E_K[G \setminus A]$, a contradiction. □

**Proposition 9.2.2.** *Suppose $G$ is a coarsely geodesic coarse group metrizable by a Cayley metric $d$. $Ends(G)$ is metrizable if both of the following conditions are satisfied:*
*1. For each bounded subset $B$ of $G$ the union of bounded 1-components of $G \setminus B$ is bounded.*
*2. For each bounded subset $B$ of $G$ its complement $G \setminus B$ has finitely many unbounded 1-components.*
*Moreover, $Ends(G)$ can be described as the family of decreasing sequences $\{A_n\}_{n\in\mathbb{N}}$ of unbounded 1-components of $G \setminus K_i$, where $\{K_n\}_{n\in\mathbb{N}}$ is an increasing sequence of bounded subsets of $G$ that is a basis of bounded subsets of $G$.*

*Proof.* Suppose $\{K_n\}_{n\in\mathbb{N}}$ is an increasing sequence of bounded subsets of $G$ that is a basis of bounded subsets of $G$. By 9.2.1 every unbounded 1-component $A_n$ of $G\setminus K_n$ is coarsely clopen. Therefore each decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of unbounded 1-components of $G\setminus K_n$ is contained in an end $e$ of $G$. It cannot be contained in two different ends $e$ and $e'$. Indeed, in that case we can pick disjoint coarsely clopen subsets $C\in e$ and $D\in e'$. By 9.2.1 there are bounded subsets $B_C$ and $B_D$ of $G$ such that $C\setminus B_C$ is a union of 1-components of $G\setminus B_C$ and $D\setminus B_D$ is a union of 1-components of $G\setminus B_D$. Find $n\in\mathbb{N}$ such that $B_C\cup B_D\subset K_n$. Notice $A_n$ does not intersect at least one of $C\setminus B_C$ or $D\setminus B_D$, a contradiction.

Suppose $e\in Ends(G)$. By 9.2.1 there is a bounded subset $B$ of $G$ such that $A\setminus B$ is a union of 1-components of $G\setminus B$. Find $n\in\mathbb{N}$ such that $B\subset K_n$. Now, $A\setminus K_n$ is the union of 1-components of $G\setminus K_n$, so $A$ must contain exactly one unbounded 1-component of $G\setminus K_n$. That shows $Ends(G)$ has a countable basis, hence it is metrizable. □

In particular, the space of coarse ends of a group that belongs to the class of coarsely geodesic coarse groups of bounded geometry is topologically metrizable.

**Corollary 9.2.3.** *The space of coarse ends of a boundedly generated coarse group of bounded geometry is topologically metrizable. In particular, The space of coarse ends of a coarsely geodesic coarse group of bounded geometry is topologically metrizable*

*Proof.* Apply 9.2.2 and 8.3.7. □

Our next goal is to prove that the converse of 9.2.2 holds. To that end, we use the well-known facts that the number of ultrafilters of an infinite set $X$ is $2^{|\mathcal{P}(X)|}$, and that a first countable separable Hausdorff space $X$ has cardinality at most $2^{|\mathbb{N}|}$.

**Lemma 9.2.4.** *Let $X$ be a coarse space and $\{A_n\}_{n\in\mathbb{N}}$ be a family of mutually disjoint coarsely clopen subsets of $X$. If for each infinite subset $P$ of naturals $\mathbb{N}$ the union $\bigcup\limits_{n\in P} A_n$ is an unbounded coarsely clopen subset of $X$, then the space of ends $Ends(X)$ of $X$ is not metrizable.*

*Proof.* Let $\mathcal{B}$ be the Boolean quotient algebra of $\mathcal{P}(\mathbb{N})$ by the ideal of all finite subsets of $\mathbb{N}$. Given an ultrafilter $\mathcal{F}$ of $\mathcal{B}$ consisting of infinite sets, notice that as $\mathcal{F}$ satisfies the Finite

Intersection Property, the family $\{Ends(\bigcup_{n\in P} A_n) : P \in \mathcal{F}\}$ of closed subsets of the compact space $Ends(X)$ satisfies the Finite Intersection Property, and hence $\bigcap_{P\in\mathcal{F}} Ends(\bigcup_{n\in P} A_n) \neq \emptyset$. For each ultrafilter $\mathcal{F}$ of $\mathcal{B}$, choose a coarse end $e_{\mathcal{F}} \in \bigcap_{P\in\mathcal{F}} Ends(\bigcup_{n\in P} A_n)$, and define the map $\varphi : \mathcal{U}(\mathcal{B}) \to Ends(X)$ sending each ultrafilter $\mathcal{F}$ of $\mathcal{B}$ to $e_{\mathcal{F}}$. Given two different ultrafilters $\mathcal{F}_1$ and $\mathcal{F}_2$ there are disjoint $P \in \mathcal{F}_1$ and $Q \in \mathcal{F}_2$ resulting in $e_{\mathcal{F}_1} \neq e_{\mathcal{F}_2}$. That means $Ends(X)$ contains at least $2^{|\mathcal{P}(\mathcal{B})|}$ points. Since $2^{|\mathcal{P}(\mathcal{B})|} > 2^{|\mathbb{N}|}$, $Ends(X)$ cannot be metrizable. □

**Proposition 9.2.5.** *Suppose $G$ is a coarse group metrizable by a Cayley metric $d$. $Ends(G)$ is non-metrizable if and only if one of the following conditions is satisfied:*

1. *There is a bounded subset $B$ of $G$ such that the union of bounded 1-components of $G \setminus B$ is unbounded.*
2. *There is a bounded subset $B$ of $G$ such that of $G \setminus B$ has infinitely many unbounded 1-components.*

*Proof.* In case of 1) choose a sequence $A_n$ of bounded 1-components of $G \setminus B$ such that $g_n \in A_n$ and $d(g_n, 1_G) \to \infty$. Notice that for each infinite subset $P$ of naturals $\mathbb{N}$ the union $\bigcup_{n\in P} A_n$ is an unbounded coarsely clopen subset of $G$ by 9.2.1. Therefore, by 9.2.4, $Ends(G)$ is non-metrizable. In case of 2) choose a sequence $A_n$ of unbounded 1-components of $G \setminus B$ that are mutually disjoint. Apply 9.2.1 and 9.2.4. □

**Corollary 9.2.6.** *Suppose $G$ is a coarsely geodesic coarse group metrizable by a Cayley metric $d$. The space of ends $Ends(G)$ is topologically metrizable if and only if both of the following conditions are satisfied:*

1. *For each bounded subset $B$ of $G$ the union of bounded 1-components of $G \setminus B$ is bounded.*
2. *For each bounded subset $B$ of $G$ its complement $G \setminus B$ has finitely many unbounded 1-components.*

*Moreover, $Ends(G)$ can be described as the family of decreasing sequences $\{A_n\}_{n\in\mathbb{N}}$ of unbounded 1-components of $G \setminus K_n$, where $\{K_n\}_{n\in\mathbb{N}}$ is an increasing sequence of bounded subsets of $G$ that is a basis of bounded subsets of $G$.*

*Proof.* Apply 9.2.2 and 9.2.5. □

Given a coarsely geodesic coarse group $G$ metrizable by a Cayley metric $d$. If $Ends(G)$ is topologically metrizable, then there exists an increasing sequence $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of bounded subsets of $G$ that is a basis of bounded subsets of $G$, such that every decreasing sequences $(A_n)_{n\in\mathbb{N}}$ of unbounded 1-components of $G \setminus K_n$ determines exactly one coarse end, and that every coarse end is determined by a such sequence. We utilize these facts to construct an explicit metric on $Ends(G)$ that induces its topology. For the coarse ends $e, e' \in Ends(G)$ determined by the decreasing sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ of of unbounded 1-components of $G \setminus K_n$, respectively; we define:

$$n_{\mathcal{K}}(e, e') = \begin{cases} \min\{n \in \mathbb{N} : A_n \cap B_n = \emptyset\} & \text{if } e \neq e' \\ \infty & \text{if } e = e' \end{cases}$$

**Lemma 9.2.7.** *Let $G$ be a coarsely geodesic coarse group metrizable by a Cayley metric $d$. Assume that $Ends(G)$ is topologically metrizable. Fix an increasing sequence $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of bounded subsets of $G$ that is a basis of bounded subsets of $G$, such that every decreasing sequences $(A_n)_{n\in\mathbb{N}}$ of unbounded 1-components of $G \setminus K_n$ determines exactly one coarse end, and that every coarse end is determined by a such sequence. Define the function $f : \mathbb{N} \to [0, \infty)$ by:*

$$f(n) = \exp(n)$$

*and the function $d_{\mathcal{K}} : Ends(G) \times Ends(G) \to [0, \infty)$ by:*

$$d_{\mathcal{K}}(e, e') := f(-n_{\mathcal{K}}(e, e')).$$

*Then $(Ends(G), d_{\mathcal{K}})$ is an ultrametric space.*

*Proof.* Clearly the function $d_{\mathcal{K}} : Ends(G) \times Ends(G) \to [0, \infty)$ is non-negative symmetric and $d_{\mathcal{K}}(e, e') = 0$ if and only if $e = e'$. Let $e_1, e_2, e_3 \in Ends(G)$ be coarse ends determined by the decreasing sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$, respectively, where each $A_n, B_n$ and $C_n$ is an unbounded 1-component of $G \setminus K_n$. We claim that:

$$d_{\mathcal{K}}(e_1, e_3) \leq \max\{d_{\mathcal{K}}(e_1, e_2), d_{\mathcal{K}}(e_2, e_3)\}.$$

Without loss of generality, assume that $e_1 \neq e_3$, $\max\{d_{\mathcal{K}}(e_1, e_2), d(e_2, e_3)\} \leq d_{\mathcal{K}}(e_1, e_3)$ and $d_{\mathcal{K}}(e_1, e_2) < d_{\mathcal{K}}(e_1, e_3)$. We aim to show that $d_{\mathcal{K}}(e_2, e_3) = d_{\mathcal{K}}(e_1, e_3)$. To this end, it suffices to show that $d_{\mathcal{K}}(e_1, e_3) \leq d_{\mathcal{K}}(e_2, e_3)$ which is satisfied if and only if $n_{\mathcal{K}}(e_2, e_3) \leq n_{\mathcal{K}}(e_1, e_3)$. Since $d_{\mathcal{K}}(e_1, e_2) < d_{\mathcal{K}}(e_1, e_3)$, one has $n_{\mathcal{K}}(e_1, e_3) < n_{\mathcal{K}}(e_1, e_2)$ and hence $A_{n_{\mathcal{K}}(e_1,e_3)} = B_{n_{\mathcal{K}}(e_1,e_3)}$. However, $A_{n_{\mathcal{K}}(e_1,e_3)} \cap C_{n_{\mathcal{K}}(e_1,e_3)} = \emptyset$ and therefore $B_{n_{\mathcal{K}}(e_1,e_3)} \cap C_{n_{\mathcal{K}}(e_1,e_3)} = \emptyset$ which implies that $n_{\mathcal{K}}(e_2, e_3) \leq n_{\mathcal{K}}(e_1, e_3)$. $\square$

**Theorem 9.2.8.** *Let $G$ be a coarsely geodesic coarse group metrizable by a Cayley metric $d$. Assume that $Ends(G)$ is topologically metrizable. Fix an increasing sequence $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of bounded subsets of $G$ as in 9.2.7. The ultrametric $d_{\mathcal{K}} : Ends(G) \times Ends(G) \to [0, \infty)$ given by $d_{\mathcal{K}}(e, e') := f(n_{\mathcal{K}}(e, e')) = \exp(-n_{\mathcal{K}}(e, e'))$ induces the topology of $Ends(G)$ as a subspace of $G \cup Ends(G)$.*

*Proof.* Notice that the family:
$\{U_{end} : U$ is an unbounded 1-components of $G \setminus K_n$ for some $n \in \mathbb{N}\}$ is a basis for $Ends(G)$. Let $e \in Ends(G)$ be a coarse end determined by the decreasing sequence $(A_n)_{n\in\mathbb{N}}$, where each $A_n$ is an unbounded 1-component of $G \setminus K_n$. For $r > 0$, we claim that the open ball $B_{d_{\mathcal{K}}}(e, r) = \{e' \in Ends(G) : d_{\mathcal{K}}(e, e') < r\}$ is open subset of $Ends(G)$ as a subspace of $G \cup Ends(G)$. Choose $n \in \mathbb{N}$ such that $f(n) < r$. Notice that $U_{end}$, where $U = A_n$, is an open set containing $e$ and if $e'' \in U_{end}$ is a coarse end determined by the decreasing sequence $(B_n)_{n\in\mathbb{N}}$, where each $B_n$ is an unbounded 1-component of $G \setminus K_n$, then $n_{\mathcal{K}}(e, e'') > n$ as $A_n = B_n$; and therefore, $d_{\mathcal{K}}(e, e'') = f(n_{\mathcal{K}}(e, e'')) < f(n) < r$. Thus, $e \in U_{end} \subset B_{d_{\mathcal{K}}}(e, r)$ as desired.

Conversely, pick a basis element $U_{end}$ of $Ends(G)$. Then, $U = A_i$ for some an unbounded 1-component $A_i$ of $G \setminus K_i$. Let $e \in U_{end}$ be a coarse end determined by the decreasing sequence $(A_n)_{n\in\mathbb{N}}$, where each $A_n$ is an unbounded 1-component of $G \setminus K_n$, and choose $m \in \mathbb{N}$ such that $m > i$. Put $r := f(m)$ and let $e' \in B_{d_{\mathcal{K}}}(e, r)$ be a coarse end determined by the decreasing sequence $(B_n)_{n\in\mathbb{N}}$, where each $B_n$ is an unbounded 1-component of $G \setminus K_n$, then $f(n_{\mathcal{K}}(e, e')) = d_{\mathcal{K}}(e, e') < r = f(m)$ which implies that $m < n_{\mathcal{K}}(e, e')$. Consequently, $A_m = B_m$ and as $i < m$, we must have $A_i = B_i$ and hence $B_{d_{\mathcal{K}}}(e, r) \subset U_{end}$. $\square$

Using the Brouwer's Theorem which states that a second countable totally disconnected compact Hausdorff space with no isolated points is homeomorphic to the Cantor middle-third set, we will show that if $|Ends(G)| = \infty$, then $Ends(G)$ is homeomorphic to the Cantor middle-third set.

**Corollary 9.2.9.** *Let $G$ be a coarsely geodesic coarse group metrizable by a Cayley metric $d$. Assume that $Ends(G)$ is topologically metrizable. Fix an increasing sequence $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of bounded subsets of $G$ as in 9.2.7. If $|Ends(G)| = \infty$, then $Ends(G)$ is homeomorphic to the Cantor middle-third set.*

*Proof.* We aim to show that every $e \in Ends(G)$ is an accumulation point. Indeed, let $e \in Ends(G)$ be a coarse end determined by the decreasing sequence $(A_n)_{n\in\mathbb{N}}$, where each $A_n$ is an unbounded 1-component of $G \setminus K_n$. For $r > 0$ choose $n_r \in \mathbb{N}$ such that $\exp(-n_r) < r$. Since $Ends(G)$ is infinite, we can find $N \in \mathbb{N}$ such that $n_r \leq N$, and a coarse end $e' \in Ends(G)$ determined by the decreasing sequence $(B_n)_{n\in\mathbb{N}}$, where each $B_n$ is an unbounded 1-component of $G \setminus K_n$ such that $A_N \cap B_N = \emptyset$ and $A_{N-1} \cap B_{N-1} \neq \emptyset$. Now, as $n_{\mathcal{K}}(e, e') = N \geq n_r$, one has $d_{\mathcal{K}}(e, e') \leq \exp(-n_r) < r$ and therefore $e' \in B_{d_{\mathcal{K}}}(e, r) \cap (Ends(G) \setminus \{e\})$. $\square$

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# Vita

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