

Different Aspects on Aspects

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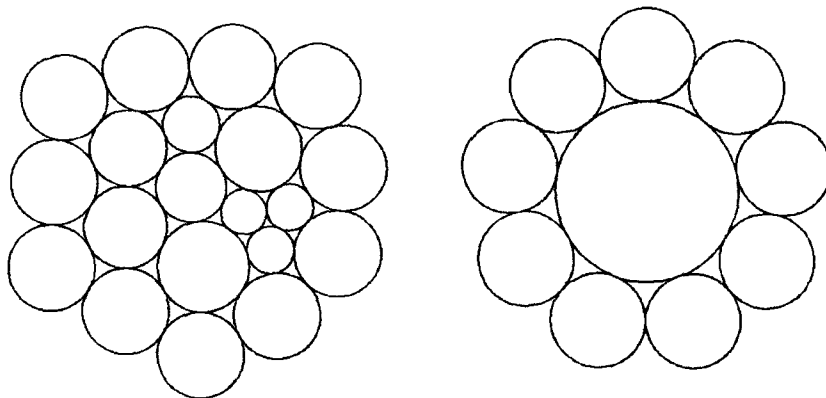
INTRODUCTION

There are many different ways to look at patterns of circles in the plane. This paper looks at different flowers and the patterns confirmed and contradicted by experiment. Before beginning, one must have some background information on the subject. The basic unit of a circle packing is called a **flower**, which is one center circle tangent to and surrounded by **petals**.

A **circle packing** is a collection of circles with a prescribed pattern of tangencies. According to Beardon, "A collection P of circles in the plane is a circle packing if P is a union of flowers satisfying the following conditions (40):

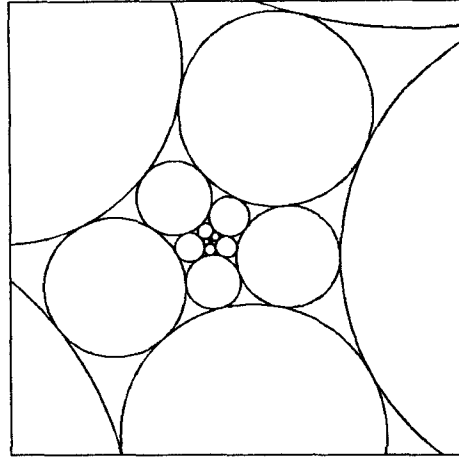
1. Every circle is the center of a designated flower.
2. Whenever B_1, B_2, B_3 are successive petals of the flower with center C , then B_3, C, B_1 must be successive petals of the flower with center B_2 .
3. Any two circles that can be connected by a finite chain of circles in which each is a petal of its predecessor."

Circle packings are the base for this paper and the definition is essential for understanding the content. Here are a few examples of circle packings:

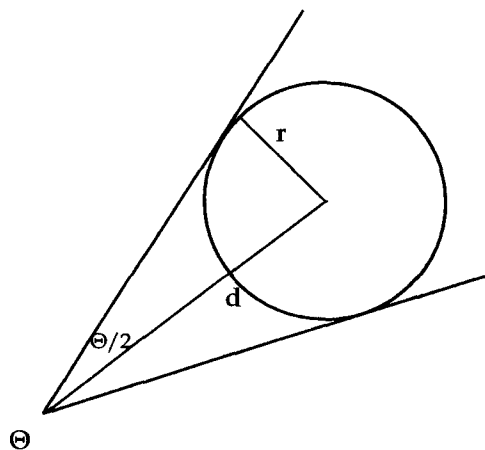


A more specific circle packing is a Doyle Spiral. In a **Doyle Spiral** every circle has a Doyle flower pattern. A **Doyle flower** is formed by a center circle of

radius 1 and petals with aspects $a, b, b/a, 1/a, a/b, 1/b$ for any $a, b > 0$. Below is an example of an infinite coherent Doyle Spiral, in which no circles overlap.



It has previously been found that for Doyle Spirals, there is always some point left uncovered, and from this point every circle in the packing has the same **aspect** when that particular point is transformed to the origin. The **aspect** of a circle, C , relative to a point, z , is the ratio of the radius of C to the distance of its center from z . The **aspect angle** of C is the angle formed by the two lines tangent to C and intersecting at z . Note that the aspect of a circle is the sine of half of the aspect angle of that circle (i.e. $\sin \frac{\theta}{2} = \frac{r}{d}$).



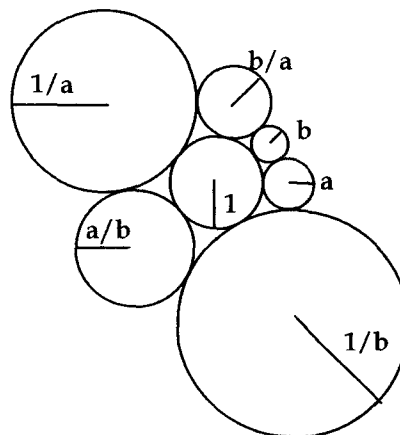
One last definition that will be important in certain proofs is that of **angle sum**. The angle sum of a set of circles around a center circle is the sum of the angles

formed by connecting the center of the center circle to the centers of its petals. Note that if the petals fit perfectly around the center circle, the angle sum will be 2π .

This paper aims to clarify four important facts concerning aspects of circles. The first is a proof of a certain problem dealing with the ratios of radii and a particular Doyle Spiral. The next is an attempt to experiment with functions that relate the aspect of a center circle as a function of the number of petals with constant petal aspects. Then there is the question of the locus of points from which two tangent circles have the same aspect. Lastly, the question of whether it can be proven that for any aspect A between 0 and 1, a hexagonal flower can be formed with all aspects equal to A .

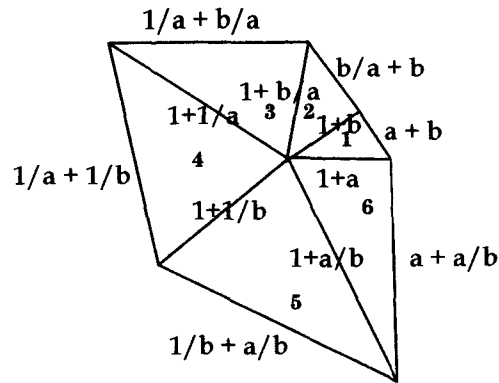
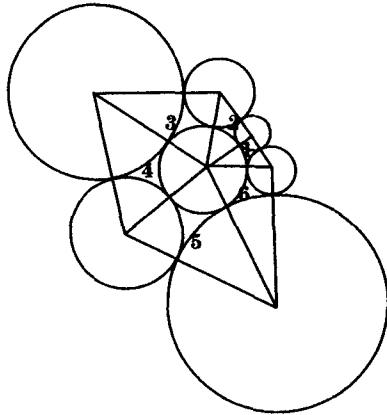
A PROOF

Problem: Choose ANY two positive numbers a and b . Use these numbers to grow a hex flower as follows: Place a central circle of radius 1 at the origin. Now add the first petal of radius a tangent to it; then add the second petal of radius b tangent to the first and counterclockwise with these radii (in order): b/a , $1/a$, a/b , $1/b$.

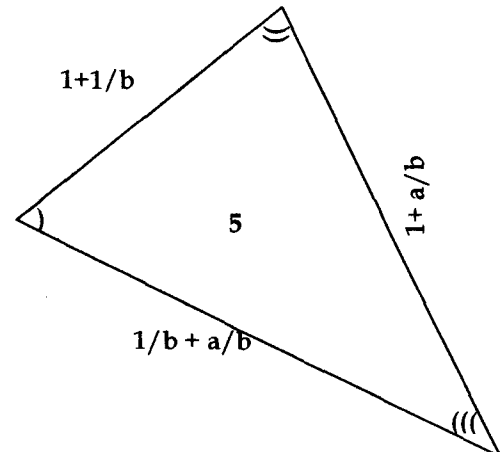
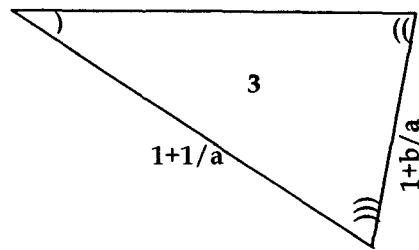
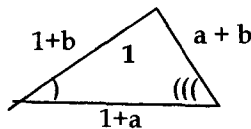


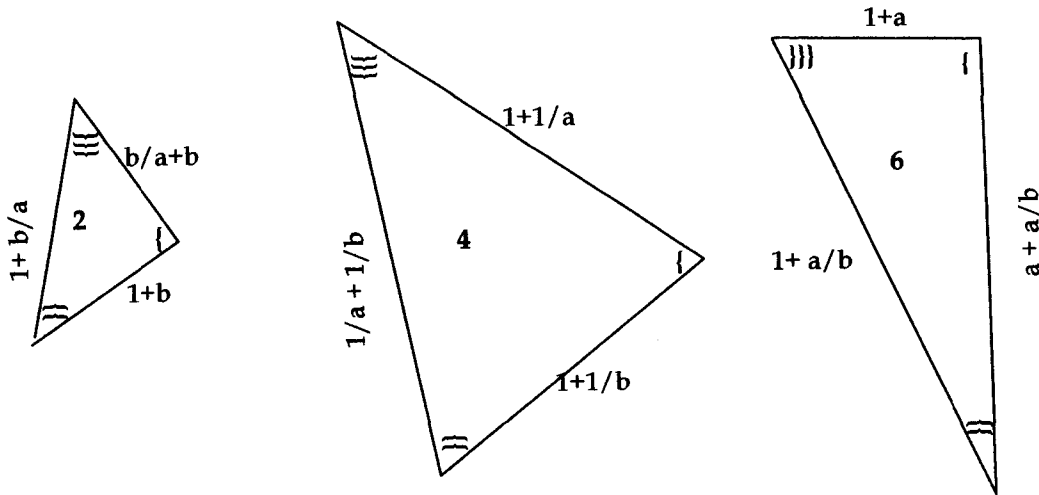
Prove that the sixth petal you place will always land so that it is precisely tangent to the first petal.

Proof: By looking at the triangles formed by connecting the centers of the circles, the proof becomes quite simple. Below, there is a drawing of the triangles within the circles, and then the triangles alone with the lengths of each side noted.

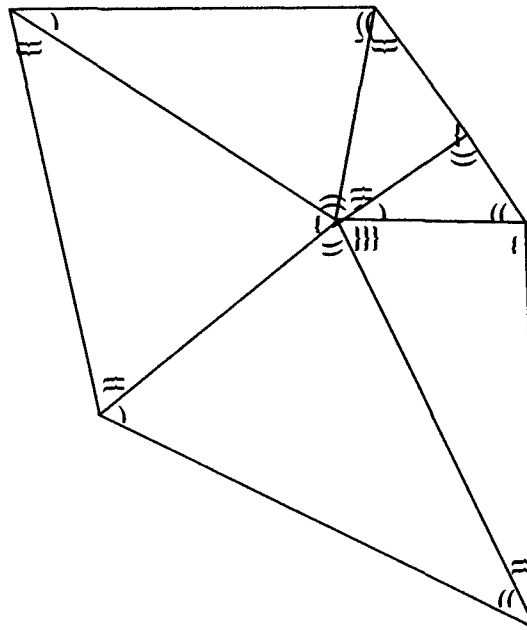


If one simply draws each triangle separately, it is easy to see that all the triangles that are not adjacent to each other are similar (in other words, every other triangle is similar). Triangle 1 is equal to triangle 3 divided by a , and triangle 3 is equal to triangle 5 times a/b . Therefore triangles 1, 3, and 5 are similar triangles and have equal angles. Below equal angles are marked equally with a $\cup$, $\cup\cup$, or $\cup\cup\cup$. The same is true of triangles 2, 4 and 6. $T_2 = T_4/b$, and $T_4 = T_6 \cdot a$, making each of these three triangles similar and their angles equal. In these three, equal angles are denoted with $\cup$, $\cup\cup$, or $\cup\cup\cup$.





So if one looks at the angles of the vertices that are at the center of the flower, one will notice that each angle represents a different one of the six possible angles.



We know that since angles $\angle$, $\angle\angle$, and $\angle\angle\angle$ (similarly with $\angle$) are the three angles of a triangle, then their sum is 180° . Therefore, the three angles represented by the center vertices of triangles 1, 3, and 5 add up to 180° and the angles represented by the center vertices of triangles 2, 4 and 6 add up to 180° . This gives a total angle sum of 360° , which proves that the six petals close up around the center circle tangently and without overlapping.

THE LOCUS OF POINTS WHERE ASPECTS ARE EQUAL

If we are trying to find the point relative to which all circles in a spiral have the same aspect, we can start by finding the locus of points where two circles have the same aspect. Since aspect depends only on the radius of a circle and distance, it is equivalent for pairs of circles under scaling, translation and rotation. Therefore, a simple example can help find the locus for any pair of circles. This example will be that of a circle centered at the origin with radius 1 and a petal with radius r centered at the point $(1+r, 0)$. With these two circles and the ability to scale, translate and rotate, the locus for any two tangent circles can be found.

In order for two circles to have the same aspect with respect to a point $z=x+yi$, then the ratio of their radii to their distance from that point z must be equal. Therefore we begin with the equation:

$$\frac{1}{|z-(0,0)|} = \frac{r}{|z-(1+r,0)|}$$

Cross-multiplying, we have $|z-(1+r,0)| = r |z-(0,0)|$.

Replacing z by $x+yi$ and using basic algebra on complex numbers, the equation for the locus of points where the aspects are the same is a circle defined by the equation:

$$\left(x - \frac{1}{1-r}\right)^2 + (y-0)^2 = \left(\frac{r}{r-1}\right)^2$$

With this equation, we can find the locus of points of equal aspect for any two tangent circles. For example, if scaling the pair of circles by $q>0$ (so the new radii are q and qr), then the locus is given by the equation:

$$\left(x - \frac{q}{1-r}\right)^2 + (y-0)^2 = \left(\frac{qr}{r-1}\right)^2$$

This locus will be important in proving our next point: that for any A between 0 and 1, there is a hexagonal flower with all circle aspects equal to A .

ALL ASPECTS POSSIBLE

The proof above relating to the hex flower with petals of radii $a, b, \dots$ will be useful in proving that there is a hexagonal flower with equal aspects for any aspect between 0 and 1. This is because this proof holds for *any* positive a and b .

Therefore, if it can be proven that there is a group of three petals with the same aspect for any aspect between 0 and 1, it follows that there is a hexagonal flower with equal petal aspects. To begin this proof, we will also use the locus of points found above. We will replace the radius r by a . Therefore, we know that the locus of points where the circle with center $1+a$ and the center circle of radius 1 centered at the origin have the same aspect is:

$$\left(x - \frac{1}{1-a}\right)^2 + y^2 = \left(\frac{a}{a-1}\right)^2$$

We also know that the aspect of both the circle with radius 1 and that with radius a

is given by $A = \frac{a}{\sqrt{(x - (1+a))^2 + y^2}}$. Through basic calculus (and through visually

looking), we find that the maximum of A occurs at the point $(1,0)$, or the point on the locus where the two circles are tangent to each other. This maximum for A is 1, since a/a is 1. The minimum occurs when y is zero and is given by the equation

$$A_{\min} = \frac{a}{\frac{1+a}{1-a} - a}$$

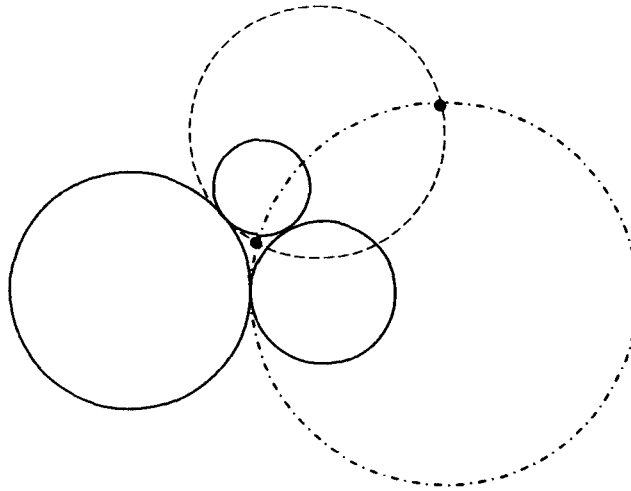
This minimum obviously depends on the value of a . Note that as a approaches 1, the value of $A_{\min}$ approaches 0, so the universal minimum value for A is 0. Also note that when $a = 1$, the locus extends to infinity, and that is point where the aspect would be 0. Now, by the intermediate value theorem, since A is a continuous function on $(0,1)$, then for every number c between 0 and 1 there exists an x and a y

between the two extremes such that $A(x,y) = c$. Therefore, for any value c between 0 and 1, the radius a can be manipulated to find a point on the locus of equal aspects such that the aspect with respect to that point is c .

This is of course, only for two circles. Now we must apply a third circle tangent to these first two to the problem. We will say that this circle will have radius b . The locus of points B relative to which this circle and the center circle have the same aspect (once rotated and translated) is given by

$$B = \frac{b}{\sqrt{(x - (1+b))^2 + y^2}}.$$

B will intersect A in two points: one inside the tangencies of the center circle, a , and b and one outside of the area formed by the tangencies (see picture below).



It has been observed that the next circle's locus of points with equal aspects will intersect the locus for the first two at the outer point, so we will only concern ourselves with this point from now on (as it is the point relative to the proof). Now, we can observe that as b approaches 0, then the locus of points B draws closer to the circle with radius b and the intersection of A and B draws closer to the point where $A=1$ (and therefore $B=1$). As b approaches 1, then the intersection of A and B approaches the point $A_{\min}$. Therefore, since we already can have any A, by choosing

b large or small enough, we can intersect B with any part of A and can make a triple whose petals have all the same aspect for any aspect between 0 and 1.

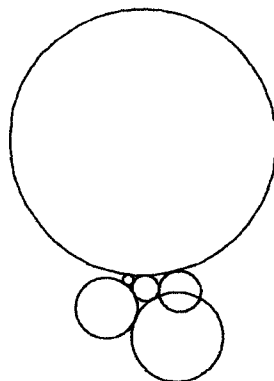
Note that this proof only looks at the radii between 0 and 1, but the proof is similar for radii between 1 and infinity. The only difference is that the loci would be on the other side of the center circle.

EXPERIMENTAL RESULTS

During this project, an important hypothesis given was that for any flower with six petals, the aspect of the center circle is always less than or equal to the average of the aspects of the petals. We have shown above that for hexagonal flowers with any aspect between 0 and 1, the center aspect is equal to the average. This experiment tested both constant and differing petal aspects. One important counterexample to the hypothesis was found from the following data:

Center aspect	0.8482
1 st tangency angle	0
petal 1	0.42176
petal 2	0.91574
petal 3	0.79221
petal 4	0.95949
petal 5	0.65574
petal 6	0.035712
Meanaspect	0.63011
Angsum	2Π

Notice that the center aspect is larger than the average of the petal aspects, although the angle sum is still 2Π . Also notice that the aspect of petal 6 is quite small (in fact it is unnoticeable when the picture is a reasonable size), which causes petal 5 and petal 1 to overlap.

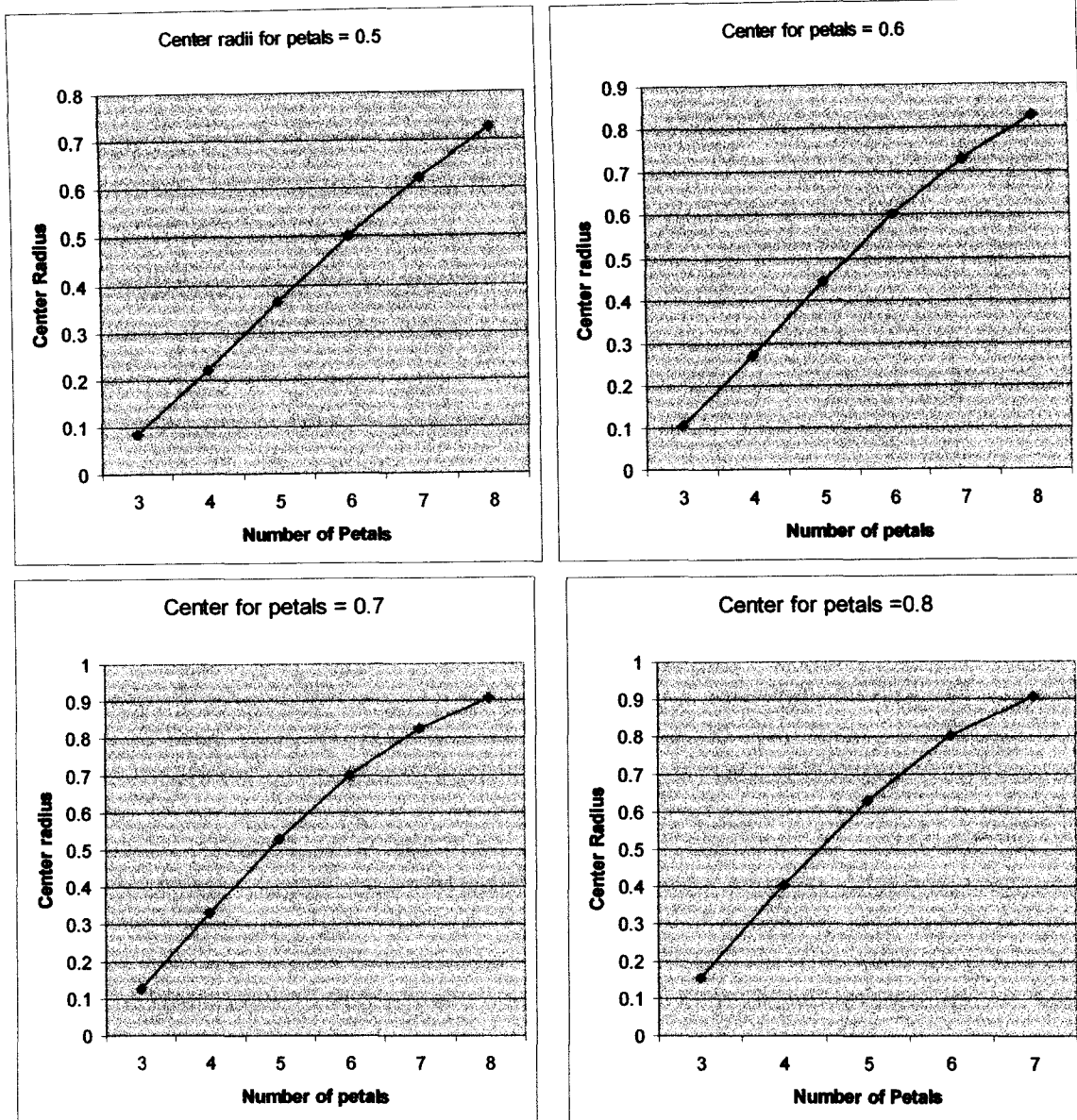


Because of the extreme size of petal 6, the hypothesis no longer stands. A new hypothesis might be drawn that for any flower with six or less petals that do not overlap, the aspect of the center circle is always less than or equal to the average of the aspects of the six petals. This hypothesis has been tested, but not yet proven.

The last experiment done was with the center circle's aspect with relation to the number of petals. This experiment was done with constant petal aspects. Although the ability to test the accuracy of the functions created was limited due to problems with MATLAB files that had previously been created, approximate polynomial functions were created through Jump. However, later it was realized that the functions were inappropriate for the data because the aspect of a circle can never exceed 1. Below are the different aspects and center radii for each number of petals from 3 to 8.

Number of Petals	Petal Aspects	Center R	Petal Aspects	Center R	Petal Aspects	Center R	Petal Aspects	Center R
3	0.5	0.0828	0.6	0.103	0.7	0.1262	0.8	0.1548
4	0.5	0.2195	0.6	0.2718	0.7	0.331	0.8	0.4029
5	0.5	0.3632	0.6	0.4438	0.7	0.5307	0.8	0.6279
6	0.5	0.5	0.6	0.6	0.7	0.7	0.8	0.8
7	0.5	0.6222	0.6	0.7293	0.7	0.8255	0.8	0.9072
8	0.5	0.7255	0.6	0.8278	0.7	0.9075		

The graphs associated with these different relationships are shown below.



It is an obvious observation, but important to notice that as the petal aspects increase, the center radius approaches 1 quicker. Therefore, the graph curves more for larger petal radii. Although there is surely a relationship between these different center radii and the number of petals, it is still unclear what this might be for sure. These approximations at least give an idea of what a center radius might be given these specific petal aspects and a variable number of petals.

CONCLUSIONS

There is still much to be learned about aspects and how tangent circles relate to each other with respect to their aspects. However, through this research we now better understand how to find the single point where all circles in a Doyle Spiral have the same aspect, as well as understanding why a certain flower (that with petals of radius a , b , b/a , $1/a$, a/b and $1/b$) closes perfectly around a center circle with radius 1. Although there are not yet concrete conclusions about the way that number of petals relates to a center circle's aspect, progress has been made in the research. Additionally, we now understand better how any aspect between 0 and 1 is possible for a hexagonal flower with each petal of equal aspect. This paper has touched the basic elements relating to aspects of circles in a flower. Hopefully it has set a beginning for further research into the topic.

REFERENCES

The principal references are Beardon, Dubejko, and Stephenson (1994) and Stephenson's appendix (2004).