To the Graduate Council:

I am submitting herewith a dissertation written by Arlene Kay Sprague entitled "An Interview-Based Analysis of Sixth Graders' Cognition of Fractions in Contexts of Subdivision, Comparison, and Partitioning." I have examined the final copy of the dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Education.

Major Professor

We have read this dissertation
and recommend its acceptance:

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and Dean of The Graduate School

AN INTERVIEW-BASED ANALYSIS OF SIXTH GRADERS' COGNITION OF FRACTIONS IN CONTEXTS OF SUBDIVISION, COMPARISON, AND PARTIITIONING

A Dissertation

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Degree

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Arlene Kay Sprague

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Abstract

The research questions prompting this study called for exploration of sixth grade participant's knowledge of fractions in terms of the meaning of fractions and fraction symbols, ordering and equivalence of fractions, and operations on fractions; investigation of their use of conceptual and procedural knowledge in operating on fractions; contrast of more successful and less successful math students' understanding of fractions and operations on fractions; and examination of students' cognition of fractions in contexts of comparing, subdividing and partitioning.

Think-aloud protocols and clinical interviews with twelve participants were conducted and video-taped. Seven of the participants were identified as successful in their sixth grade math classes, and five were experiencing difficulty with math, while all had average or above verbal scores, thus permitting a contrast of cognition of fractions in high and low performing math students. Analysis of transcripts involved segmenting, episodes into statements, which were categorized utilizing the context of the interview episodes.

Findings indicated that the sixth graders had developed a part-whole concept of fractions and that one-half was a relatively strong intuitive concept for all participants. Among the less proficient math students, a whole number interpretation of fraction symbols was common, and the reasoning behind errors revealed whole number interference.

Successful students had more complete and well integrated concepts of fraction, and displayed greater conceptual knowledge than their counterparts.

Students were not limited to procedural knowledge, but utilized physical referents to represent fractional situations, and produced high frequencies of conceptual knowledge statements in conjunction with verbally presented problems. Furthermore those students who generated the highest frequencies of conceptual knowledge and automaticity statements were also the students who produced the fewest errors. However, a dissociation between conceptual knowledge was suggested by differential reasoning in response to symbolically and verbally presented problems.

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CHAPTER 1

Introduction

Fractions are introduced to children as early as kindergarten and remain in the mathematics curriculum throughout the child's school career. As early as fourth grade, students begin learning operations on fractions. The introduction of this new set of numbers which appear to be composed of the now familiar whole numbers, but which at the same time seem to behave quite differently, presents middle-grade children with the often confusing challenges of new terminology, new rules, and new algorithms. Success in understanding and dealing with fractions is apparently not the experience of most students. Results of the third and fourth National Assessment of Educational Progress (NAEP) reveal low scores on fractional computations accompanied by an apparent lack of understanding by nine, thirteen, and seventeen-year-olds (Lindquist, et al., 1983; Dossey et al., 1988). Fractions present one of the most difficult instructional tasks for both teachers and students in the elementary and middle grades. Difficulties with fractions are known to continue and sometimes worsen as students continue through the years of public school.

Statement of the Problem

It has been suggested that students' understanding of fractions might be enhanced if new ideas associated with

fractions were presented in a manner that exploited children's existing conceptual understandings, that is, if connections were made between what children understood conceptually and new material being presented. It is therefore important to explore and describe what it is that children are understanding, what concepts they have developed, what connections they have made between their perceived world and the symbols of fractions as well as the manipulations of those symbols. Therefore, this study was designed to investigate knowledge of fractions in a group of sixth graders, to examine concept development, to search for connections participants have made between abstract symbols and concrete experience embedded in the situations of subdividing, comparing, and partitioning.

Purpose of the Study

This study was designed to explore and describe knowledge of fractions in a group of sixth graders. It examined the concepts the participants had developed related to fractions as they were experienced in subdividing of continuous quantities, partitioning of sets of discrete objects, and comparisons of quantities. The study examined the children's sense of meaning and underlying cognitive processes related to fractions and operations on fractions. The understanding of more successful math students was contrasted with that of less successful students to reveal potential expert-novice differences in the development of conceptual and procedural knowledge of fractions.

Rationale

Development of mathematical understanding in general begins in early childhood with basic number sense and counting schemas, and continues through the entire school experience, leading a relatively few students to advanced mathematics and possible related careers, but leading numerous others to skills fixed at early elementary school levels, with a corresponding sense of incompetence. Therefore, studies contributing to a greater understanding of how children construct mathematical knowledge would be potentially very beneficial. It seems logically advantageous for educators to learn as much as possible about children's knowledge of mathematics and about the misunderstandings they may tend to develop (Dahlberg, 1989). The majority of prior cognitive research has focused on simpler mathematical skills such as counting, addition, and subtraction. There is not yet a similarly extensive body of research reflecting attempts to unravel the construction of complex concepts such as those involved in knowledge of fractions.

Research Questions

The questions that prompted the undertaking of this study were broad and open-ended, calling for a descriptive approach and necessitating qualitative methodology. Because of the breadth of the questions and the relative paucity of research in this area, a pilot study was conducted prior to the current research (See Chapter 3). The pilot study

revealed a) that the sixth graders interviewed did not connect fractions with the world outside of school, b) that both the interference of whole number knowledge and the relativity of the size and meaning of fractions were problematic for the sixth-grade participants, c) that their knowledge of fractions was largely syntactic procedural knowledge, and d) that for five of the six participants, subdivision of wholes was the only source of conceptual knowledge of fractions. Based on the outcomes of the pilot study and on the review of literature, the research questions were formulated as follows:

Research Question I

What is the content of sixth graders' knowledge of fractions, specifically regarding the meaning of fractions and fraction symbols, ordering and equivalence of fractions, and operations on fractions?

The pilot study, conducted prior to the current research, and the research literature had indicated that children's knowledge of fractions is limited in large portion to knowledge of rules and algorithms, disconnected from their sense of the real world. This study was designed to analyze sixth graders' conceptual knowledge of fractions related to definitions and relationships as well as their procedural knowledge of rules and algorithms. While it was anticipated that procedures or algorithms would dominate children's' responses, the analysis was designed to reveal how often and in what areas childrens' responses

indicated conceptual knowledge of definitions and relationships or connections with the world at large.

Research Question II

How do sixth graders utilize conceptual and procedural knowledge related to operations on fractions?

In the pilot study it appeared that participants treated symbolically presented problems differently from verbal problems solved with physical referents. This study was designed to analyze students' statements elicited in solving parallel problems, containing similar or identical numerical content, under both of the aforementioned conditions. Although the pilot study and prior research had noted a dissociation between symbol-manipulations and work with physical representations, this study sought to reveal any previously unrecognized links that children might have constructed between concepts and rules, between symbols and meanings.

Research Question III

How do successful sixth grade math students think about, understand and work with fractions and operations on fractions in contrast with sixth graders displaying less proficiency with math?

While it was anticipated that all of the sixth grade participants would be relative novices in the field of mathematics, within the realm of fractions the more successful students might have exhibited some movement toward expertise in their approach to fraction problems. This study contrasted the thinking of less versus more

successful math students when confronted with fractions and operations on fractions, especially with regard to use of procedural and conceptual knowledge.

Research Question IV

How do sixth graders think about and understand fractions when encountered in the contexts of comparisons between physical objects, subdividing of continuous quantities, or partitioning of discrete quantities?

Prior research has indicated that partitioning may be more difficult for students than subdivision as a basis for representing fractions. Comparing has not been explored as a source of fraction knowledge separate from the concepts of ratio and proportion. This study analyzed participants' statements related to knowledge of fractions elicited by work with physical referents which embodied each of the three sources of fractions: comparing, subdividing, and partitioning. It was anticipated that the sixth grade participants would exhibit the greatest levels of understanding of fractions within the subdividing framework. However, the study was designed to provide additional information regarding the knowledge which participants possessed related to subdivision and comparison.

Limitations, Assumptions and Definitions

Limitations

Participation in the study was limited to a relatively small sample of twelve sixth graders. During the pilot study it became apparent that drawing all participants from

one school resulted in some patterns that were indicative of the participants' common teaching history. Therefore, the current study selected participants from three middle schools, the first in a rural area of east Tennessee, the second in a small town in east Tennessee with a population of 1700, the third in a small city in southeast Kentucky, with a population of 12,300. Both males and females as well as African-American, eastern Indian and Caucasian students were included in the study. While a certain amount of diversity therefore was built into the sample, it is nevertheless difficult to generalize information regarding the knowledge of fractions exhibited by the participants to other population groups.

Assumptions

In designing this study the researcher assumed that sixth graders would be able to verbalize their thinking about fractions. It was further assumed that participants would be willing to answer questions in a series of four interviews, and to do so while being video taped. Finally, it was assumed that within the confines of the interview questions asked and the manipulative materials made available in the study, participants would be able to give expression to the fraction concepts they had developed.

Definitions

A set of definitions of terms which are often-used and significant to the study been constructed and is available to the reader in Appendix A. Definitions are also embedded in the body of the text.

CHAPTER 2

Literature Review

Mathematics and Cognition

How do children think about mathematical problems? How do they develop the ability to think mathematically? How do they analyze mathematical situations or approach problems of various types? These are the kinds of questions that necessitate studies in cognition and mathematics. Children's mathematical understanding develops over a long period of time and includes informally developed concepts of number, such as counting and simple problem solving, as well as formal school arithmetic with its algorithms and rules. The constructivist perspective suggests that as children encounter a variety of mathematical situations, they develop hypotheses about the way numbers behave and should be treated under various conditions. However, children's mental representations of mathematical ideas are not totally consonant with reality. Davis & McKnight (1980) proposed that the degree of correspondence of real situations with children's representations of those situations is their level of understanding. Such understanding is the result of experience which provides the basis for the construction of mathematical schemas and for modification of those schemas through accommodation.

In the area of elementary school mathematics and cognition, the majority of studies have focused on basic

concepts and skills, such as concepts of number and skills of addition and subtraction. Studies have included analysis of error patterns, inferences regarding the conceptual structures necessary for understanding of procedures, and proposed theory regarding concepts and procedures (Brown & Burton, 1978; Brown & VanLehn, 1980; Davis, 1983; Ohlssen & Rees, 1991). Regardless of the differences in emphasis from one study to another, there appears to be consensus among cognitive scientists that children's mathematical behaviors are basically, although not perfectly, consistent, and that children's mathematical behaviors represent their best attempts at proceeding in meaningful ways.

Procedural and Conceptual Knowledge

In the study of mathematics and cognition, a distinction has been made between children's procedural and conceptual knowledge. Procedural knowledge is goal specific knowledge of how to achieve a particular end result. Procedural knowledge is not true or false, but can be more or less effective. In mathematics, it is knowledge of the syntactic system of symbols and algorithms, and consists largely of rules for manipulating symbols. Conceptual knowledge, in contrast, is principled rather than merely factual, and is concerned with meaning and intent. Concepts have been defined as mental representations, relatively stable wholes, having internal structure which include intention, functions, properties, relationships, and the conditions under which they apply (Nelson, 1983). Anderson,

Greeno, Kline and Neves (1981) identified three sources of knowledge students access in solving geometric proof problems. The highest level in power and sophistication they referred to as meaningful learning, in which schemas represent relationships among objects, networked with possible appropriate procedures and salient contextual features. Conceptual knowledge is highly networked. Rowan, Payne, and Towsley (1990) suggested that conceptual knowledge is by definition connected to many ideas and experiences. It has been proposed that conceptual knowledge is necessary for developing mathematical understanding (Heibert & LeFevre, 1986).

Importance of a Conceptual Knowledge Base

Among investigators who have noted the importance of conceptual understanding in the learning of mathematics, the notion of the interconnectedness of knowledge and of resulting flexibility of thought are common themes (Rowan, Payne & Towsley, 1990; Streefland, 1982; Wearne & Heibert, 1986; Wearne & Hiebert, 1988). Connections between symbols and physical referents, between operations and manipulation of a variety of physical embodiments have been shown to increase richness of associations and flexibility of conceptual understanding (Wearne & Heibert, 1988).

Other researchers have pointed out that success rates in solving problems may be enhanced by conceptual understanding, as opposed to rule-based learning (Hardiman & Mestre, 1989; Harrison, Brindley & Bye, 1989; Wearne & Hiebert, 1989). Students with adequate conceptual

knowledge, who are able to deal with fractions as flexible units or multiples of units were found to produce "superior solutions" in fractions tasks (Hunting, 1980). In approaching novel problems, successful performance was found to be strongly related to conceptual analysis and processes (Wearne & Hiebert, 1988).

Errors May Reflect Lack of Conceptual Knowledge

Researchers have proposed that learning mathematical procedures without supporting conceptual knowledge may set the stage for the development of many of the procedural errors children demonstrate (Fuson, 1990; Ohlsson & Rees, 1991). Observing students as they interacted with drill and practice computer programs, Vinner (1984) noted that children often used trial and error to identify operations or processes that worked, rather than seeking to understand the meaning of the questions asked or the relationships inferred. Wood (1988) stated that the construction of meaning cannot be separated from social commerce, that meanings are "negotiated" in social and instructional interaction. Mathematical meanings, however are often impoverished because they have not been negotiated during the teaching process. Wood also argued that requiring children to practice procedures that rest on an incomplete or non-existent conceptual base is not efficacious.

Experts' and Novices' Use of Conceptual and Procedural Knowledge

Research on novices versus experts in mathematics and physics has revealed that experts tend to use the semantic

aspects of problem situations, classifying problems based on underlying principles, and extracting relevant information, while novices are forced to rely on syntactic features of problems (Anzai, 1991; Glover, Ronning & Bruning, 1990; LeFevre, 1986). In addition, experts possess proceduralized knowledge which allows them to solve problems within a particular domain efficiently, while novices' procedures tend to be more general and weak (Anzai, 1991). A study of middle school students' knowledge of decimals concluded that the path from novice to expert includes the use of a series of incrementally improved procedural rules leading to an integrated sense of knowledge of not only decimals, but also fractions and whole numbers (Nesher & Peled, 1986). Expert-style integration of knowledge of rational numbers with knowledge of proportional rates was not found to be common among seventh and eighth grade students. Rather students appeared to use different reasoning on verbal proportional rate problems than on "context free" fraction problems, even though the numerical values involved were identical and fraction knowledge could have been directly applied to the rate problems (Heller, Post, Behr, & Lesh, 1990). Others have noted similar lack of integration of procedural and conceptual knowledge in elementary and middle grades mathematics students (Davis & McKnight, 1980; Graber & Tirosh, 1990; Hart, 1981; Hunting, 1984b).

Relationship of Conceptual and Procedural Knowledge

Resnick (1982) cautioned that, while the lack of conceptual knowledge may result in procedural errors, the

development of conceptual knowledge does not guarantee competence. Evidence suggests that the development of conceptual knowledge is a necessary but not sufficient condition for the development of useful procedural knowledge. The relationship between developing conceptual and procedural knowledge is clearly interactive (Peterson, Ridenour, & Somers, 1990). Recognizing the contributions of both conceptual and procedural knowledge to the foundations of mathematical competence, Hiebert and Wearne (1986) proposed that both the development of and connectedness between conceptual and procedural knowledge were essential.

In recent research, strong emphasis has been placed on the development of conceptual knowledge in students' understanding of decimals, ratios, and fractions. However, while it may be important that conceptual knowledge precede and undergird procedural knowledge, both are essential to successful mathematical endeavors.

Conceptual Understanding of Fractions

In recent years, cognitive scientists have begun exploration of complex areas of elementary mathematical knowledge such as fractions and related areas such as decimals and ratios. Although Tierney, in a case-study of two students, found that the first student, who utilized memorized rule-based algorithms, was equally successful and had similar difficulties as the second student, who utilized physical representations and invented methods to solve problems, numerous other authors have argued the necessity

of concept development in the instruction of fractions (Hasemann, 1981; Rowan, Payne & Towsley, 1990; Wearne & Hiebert, 1988, 1989).

Importance of a Conceptual Knowledge Base

In considering NCTM's standards related to the teaching and learning of fractions, Rowan, Payne and Towsley (1990) noted that "the most critical part of instruction in fractions is the careful development of sound concepts. . . prior to any instructional work" (p. 24). Wearne & Heibert (1988, 1989) found that when students were given adequate opportunities to develop semantic or conceptually-based quantitative reasoning related to decimal fractions, they experienced higher success both on measures of direct instruction and on measures of transfer learning. Working with real problems in a meaningful context, in fact, may allow students to think and behave mathematically beyond anticipated levels (Kieren & Southwell, 1979).

Errors May Reflect Inadequacy of Conceptual Knowledge

In studies of children's understanding of fractions, researchers have noted that students' difficulties with fractions most often reflect problems with the concepts involved. Hasemann (1981) concluded that the major underlying cause of difficulties with fractions for poorer middle-grades math students was their "lack of an idea of fractions" (p.83). Carpenter et al. (1981) stated that the results of failure to allow students to develop concepts adequately are especially disastrous in the learning of fractions.

> The assessment results certainly reiterate what has been known about students' ability to work with fractions: they have difficulty with many of the concepts, and their computational work often appears to be a nonmeaningful routine. (p. 39)

Others have noted that children's work with fractions often consists of non-meaningful routines bounded by algorithms (Hardiman & Mestre, 1989; Carpenter, et al., 1981). The power of children's sense that mathematics is algorithm-bound is demonstrated by their willingness to accept algorithm-obtained answers as the "right" answers, over their own informally obtained answers (Mack, 1990; Maher & Alston, 1989). Davis and McKnight (1980) gave children the opportunity to improve on their (incorrect) algorithm-based performances, allowing them to change their answers as a result of their own correct thinking-through of the situation informally, without algorithmic computations. Typical responses were of the "that's the way you have to do it" type, and the participants stuck with their incorrect answers. They trusted the syntactic rule-based algorithms over their own semantic, conceptual understanding of the mathematical situation.

Other problems with fraction concepts have also been recorded. In a series of interview studies, Peck and Jenks (1981) found that less than ten percent of the sixth graders they interviewed had a strong enough concept of fraction to "make sense of addition or comparison of fractions" (p. 347). Larson (1980) found that many seventh grade students

did not associate equivalent fractions as naming the same point on the number line, and that 25% of their sample responded to fraction-value number-line problems as though the entire number line segment were being considered as the unit, regardless of the line segment length. Students classified as "remedial" math students had special problems in dealing with fractions in word problems, and were found to approach solutions with rigid criteria and little sense of the interrelatedness of operations (Hardiman & Mestre, 1989).

Dissociation Between Procedures and Concepts

Hunting (1984a) asked fourth, sixth and eighth grade children to construct equivalent fractions and then to verify their results utilizing counters. He noted that students' actions with the counters were often unrelated to the algorithms they had used to derive the equivalent fractions initially. Similarly, Mack (1990) found a dissociation between sixth graders' solutions to real world fraction problems and similar problems presented symbolically. A study which presented perceptual distracters in diagrams accompanying fraction tasks revealed a tendency for students, when dealing with discrete quantities, to rely on syntactic symbolic-level strategies and to avoid restructuring the diagrams. When confronted with a conflict between the meaning of the problem and the structure of the diagrams of discrete quantities, children opted for a syntactic procedural approach to seeking a solution (Cramer, Post & Behr, 1989).

Aspects of Concept Development

Development of concepts with relation to ratios has been shown to occur in a series of qualitatively differentiable stages (Noelting, 1980). Similarly, stages have been identified in preschool and primary school children's constructions of fractions (Piaget, Inhelder & Szeminska, 1960; Hiebert & Tonnessen, 1978). Two common observations noted regarding children's concepts of fractions are that halves and repeated halving are intuitive and dominant in children's constructions of fractions (Piaget, et al., 1960; Pothier & Sawada, 1983; Streefland, 1984), and that subdivision of wholes and part/whole relationships are treated differently than partitioning of discrete objects and set/subset relationships (Hiebert & Tonnessen, 1978; Piaget, et al., 1960).

Studies of the development of children's fraction knowledge to date have emphasized identifying and constructing fractions, and order and equivalence of fractions. Operations on fractions and underlying cognitive mechanisms associated with operations have not been explored in depth.

Flaws in Children's Concept of Fractions

Two major categories of difficulty in understanding fractions appear in research studies: dependence on rules and algorithms, and interference of whole number knowledge. It has been noted that prior instruction in rule-based algorithms may interfere with attempts to encourage children

to adopt a semantic or conceptual approach, and that rule-based learning creates an over-reliance on recall of rules which appear to be disconnected from the quantities and actions they represent (Wearne & Hiebert, 1988; Wearne & Hiebert, 1989). A discontinuity between the approach and outcomes of children using algorithms versus solving problems using their own methods has been noted (Hart, 1981; Hunting, 1983a). Problems presented in words, problems with diagrams, and problems solved with manipulatives were treated differently than the same types of problems presented with symbols only. Greater success rates in the former led Hart (1981) to the conclusion that children see these problems as two different kinds of mathematics, one based on recall of a rule, the other based on "common sense." Errors often result when algorithms are applied at inappropriate times (Kelly, Gersten & Carnine, 1990). Rule-based operations seem to be devoid of mental referents, disconnected from the real world (Bezuk & Cramer, 1989; Graber & Tirosh, 1990; Streefland, 1978).

It has been argued that a language-experience approach to teaching fractions, which emphasizes real-world experiences and the language aspects associated with developing concepts, may help to bridge the terminology gap between the real world of the child and the mathematical world of symbols (Curcio, 1990; Langford & Sarullo, 1993). However, fractions confront students with many different "personalities" in real world problems (eg. part-whole, quotient, ratio, operator, set/subset) and it is not yet

clear what real-world experiences are most efficacious in helping children construct knowledge of fractions (Behr, Harel, Post & Lesh, 1992; Behr, Lesh & Post, 1981; Kieren & Southwell, 1979).

Children have been observed to have difficulties in moving from a system of whole numbers, with which they have become somewhat comfortable by the time operations on fractions are introduced, to a system of rational numbers. Whole numbers do not require the flexibility of thought that fractions demand (Hunting, 1984b). The quality of "fourness" does not change whether one is counting apples or alligators. The relative nature of "one-fourthness", in contrast, becomes apparent when one confronts the very different values indicated by, for example, a quarter of a glass of water, a quarter of an hour, or a quarter of a million dollars. While whole numbers have a constant quality for children, fractions seem to change depending on what it is one is "taking a fraction <u>of</u>." Children are also required to utilize terminology to refer to fractional numbers which they have previously learned in the context of ordinal numbers (third, fourth, etc.), another contributing factor to possible confusion in the meaning of fractions (Streefland, 1978).

In learning about fractions, children are required to adopt new rules which conflict with well-established ideas about whole numbers (Bezuk & Cramer, 1989; LeFevre, 1986). Resnick, Nesher, Leonard, Magoen, Omanson and Peled (1989) confirmed that children's errors are often the result of

their attempts at integrating new information with prior knowledge. Aspects of whole number knowledge that children appear to apply to fractions are the notions that multiplication makes bigger, that division makes smaller, and that a smaller number cannot be divided by a larger one (Graber & Tirosh, 1990; Hart, 1981). There is strong resistance to giving up whole number schemas which compete with new fraction schemas, and whole number schemas inappropriately applied to fraction problems result in inconsistent outcomes (Hunting, 1986; Post, et al., 1985). Children often treat fractions as two whole numbers which can be dealt with separately. This is apparent when children are asked to make judgments about the comparative size of two fractions, and they use the size of the numerators and/or denominators separately as the basis of their decision-making (Hart, 1981).

Physical Referents and Conceptual Understanding of Fractions

Active use of a variety of concrete embodiments or physical referents may help students construct adequate fractions concepts (Columba, 1989; Hasemann, 1981). Translating among and between various modes of representation, including manipulative, verbal and written forms, fosters meaningful concepts of rational numbers (Behr, Lesh, & Post, 1981). Wearne & Hiebert (1988) found that, in developing competence with decimal fractions, students passed through four stages, the first two of which

were: connecting symbols with physical referents, and developing symbol-manipulation processes, in which actions on referents were extended and reflected onto symbols. Kieren (1988) proposed a concentric-circle model of rational number knowledge which places the child's own experiences with the environment at the center. Intuitive knowledge, that is, school learning related back to personal experiences, is in the second concentric ring. Symbolic knowledge comes after and builds upon such intuitive knowledge. Post, et al. (1985) listed six levels of "abilities" in making ordering and equivalence judgments, which moved in a sequence of steps from those tied to immediate experience with physical embodiments to judgments based on relationships.

Harrison, Brindley & Bye (1989) attributed gains in ability to work with fractions and ratios made by transitional thinkers, (ie. students who were ready to move from concrete into more formal modes of thinking), to the use of concrete materials in conjunction with realistic problems and systematic experimentation. Wearne (1990) reported that students who had developed semantic quantitative reasoning about decimal fractions through the use of physical referents retained and used their reasoning abilities at the same level or better, one year later. She concluded that the students were using the meanings developed from experience with referents to justify processes and to solve novel tasks. Streefland (1982) recorded students' success in extending patterns found in

meaningful, physically-based contexts to develop an algorithm for subtracting fractions. He believed that meaningful contexts provide a means of representation and visualization that aid in the construction of conceptual understanding.

Phenomenological Sources of Fractions

The variable meanings or "personalities" of fractions arise not only from the differing functions a fraction may serve in a mathematical situation, but also from the experiential contexts in which fractions may be encountered. It has been suggested that the range and uses of varying forms of examples (contexts) employed in instruction may have an impact on instructional outcomes (Kelly, Gersten & Carnine, 1990). Having overviewed 100 articles on fractions published in the Arithmetic Teacher since 1954, Streefland (1982) lamented the fact that the literature on instruction of fractions had neglected "meaningful contexts both as sources and as domains for the application of fractions" (p. 235). The notion of the fraction as a means to organize phenomena experienced in everyday life has not been common to educational thought. The course of instruction has aimed directly at teaching algorithms and practicing with them rather than contributing to children's constructions of fraction concepts. The few fraction models which have been utilized have been static and isolated and have not been extended to serve as bases for developing conceptually-based algorithms. Further, such models have been derived almost

exclusively from the part-whole or subdivision situation (Streefland, 1982; Streefland, 1978).

Potential phenomenological sources of fractions include subdivision of wholes or continuous quantities into equivalent parts, comparisons of relative sizes or amounts, and partitioning of sets of discrete objects into equivalent subsets (Hiebert & Tonnessen, 1988; Hunting, 1986; Pothier & Sawada, 1983; Streefland, 1978). Although partitioning and subdividing have been explored in some research, comparing as a source of fraction knowledge (not yet formalized into ratio and proportion) has not.

Methodology for Studying Children's Fraction Concepts

The representation of mathematical knowledge is viewed by constructionists not as the recall of static stored information, but rather as a dynamic, real-time processing of information and construction of representations. Students are recognized as knowing beings who have the ability to organize complexity, to attend to meaning rather than surface elements, and to construct knowledge by combining or creating mental structures or schemas. Further, they are self-aware and are capable of commenting on their own actions and experiences (Harre & Secord, 1972). To understand children's mathematical behavior one must attempt to investigate their internal operational systems (Hunting, 1983b). Within the constructivist paradigm, task-based interviews and think-aloud protocols are appropriate

methods for probing children's cognitive processes (Davis, 1983; Thompson, 1992; Steffe & Cobb, 1983). Utilizing the clinical method of interviewing, that is, presenting children with tasks or problems to complete, asking them to think aloud, and probing with follow-up questions to gain further insight into their thinking, the researcher is able to examine underlying cognitive processes (Hunting, 1983a). In arguing the efficacy of task-based interviews and think-aloud protocols, Ericsson & Simon (1984) stated that the structure of cognitive processes is not significantly affected by a researcher's instructions to think aloud while completing a task. Ginsburg et al. (1983) suggested that clinical interviews and think-aloud protocols do elicit representative behaviors, provide access to complex thought processes, and allow for exploration of internal mechanisms underlying mathematical behavior. Although verbal reports can only be accurate within domains to which participants have access, it has been argued that mathematics is a domain in which subjects are able to report on their activities and their thinking (Ginsburg, et al., 1983).

Summary and Connections with the Current Research

Children's mathematical behaviors have been found to be largely consistent, reflecting their understandings of the mathematical situation at hand, and their best attempts at proceeding in meaningful ways. The meanings that children assign to fractions and fraction symbols and their attempts

at meaningful operations on fractions were of interest in the current research.

A distinction has been made between conceptual and procedural knowledge, and recent research has emphasized the importance of conceptual knowledge in mathematical understanding. Concepts have been shown to aid in problem solving; experts versus novices have been shown to make use of semantic as opposed to syntactic information; and procedures learned without the support concepts have been blamed for procedural errors. However, the interrelatedness of concepts and procedures and the necessity of both conceptual and procedural knowledge must be recognized. The current research was designed to investigate how students who are successful in math versus students who are experiencing difficulty with math utilize both conceptual and procedural knowledge. In addition questions were posed that gave students opportunities to reveal connections between procedures and concepts.

Fractions in particular have posed difficulties for students and work with fractions in several studies has been shown to consist largely of non-meaningful syntactic routines. Flaws in fraction knowledge also include interference of whole number knowledge which has not been integrated with knowledge of fractions. A dissociation has been noted between work with physical referents and real world problems versus problems presented symbolically. The current research presented parallel forms of many fractions problems in both verbal form with physical referents, and in

symbolic form, in order to observe possible differences or similarities in students' approaches. In addition, an error analysis was included to further investigate the flaws in children's thinking about fractions.

Various "personalities" of fractions have been noted which arise in part from differing phenomenological sources of fractions. However, fractions have been presented to children typically in a static part-whole model derived from a subdivision context. Other possibilities, in addition to subdivision of continuous quantities, include comparison of size or amount and partitioning of discrete sets. The current research utilized physical embodiments in exploring children's conceptual knowledge of fractions and their application of conceptual and procedural knowledge in ordering fractions and operating on fractions within the contexts of subdivision of continuous quantities into equivalent parts, comparison of length of rods or paper strips, and partitioning of discrete sets into equivalent subsets.

The literature with regard to cognition of mathematics and methodology has suggested that task-based clinical interviews and think aloud protocols are appropriate methods for investigating children's cognitive processes related to mathematics. The current research employed clinical interviews and think aloud protocols to investigate children's thinking regarding the concept of fraction, ordering and equivalence of fractions and operations on fractions.

CHAPTER 3

The Pilot Study

Necessity for the Study

In the initial stages of planning for the current research, it was determined that a pilot study was required. The literature was not replete with studies in the area of cognition and children's understanding of fractions; there was no specific theory to directly guide the research; and the research questions had not yet been composed. Therefore the pilot study was designed to explore the area of interest in an open-ended fashion, and to aid in the formulation of the research questions

Research Question

Because this research was designed as a pilot study, and because the methodology was qualitative in nature, the research question was very broad and open-ended. Initially the research question for this pilot study was: How do sixth graders conceptualize fractions? However, as the study progressed, it became clear that much of the participants' knowledge of fractions was not conceptual knowledge, but syntactic knowledge of rules and algorithms. Therefore the research question became the following: What is the content of sixth graders' knowledge of fractions?

Method

Participants

Sixth grade students were chosen as the target population for the pilot study, because by sixth grade, children have been introduced to the concept of fraction, reducing and building equivalent fractions, as well as carrying out the four operations on fractions. A sample size of six was deemed adequate for a pilot study of this nature. Six sixth grade students were selected from a middle school in rural east Tennessee. Selection of three students who were doing well in mathematics and three who were having trouble with mathematics was based upon teacher evaluation. There were two males and four females in the sample selected.

Interviews

The first interview schedule was designed to give the interviewer and participants an opportunity to become acquainted and at the same time to explore childrens' concepts of what a fraction is and when and how fractions might be used. Because the primacy of 1/2 as an intuitive concept and the importance of developing a sense of the ordering of fractions have been discussed in the literature, a few questions were included to begin looking at participants' awareness of the ordering of fractions in relation to 1/2.

The second interview schedule was designed to observe participants' use of physical referents in comparing, subdividing, and partitioning tasks. Equivalence and

ordering (greater than, less than) were also included. In order to engage the children in considering fractions produced by comparing, they were presented with Cuisenaire rods with an added 12-rod, introduced as the base unit. To encourage thinking of fractions as subdivided quantities, circles with fractional parts were utilized. Partitioning tasks were designed around creating subsets from a bag or bowl of candies.

The third interview schedule was designed to elicit think-aloud protocols based on fraction problems presented in symbol form. Manipulatives in the form of circles with fractional parts and counters were available to participants who were told the materials could be used at any time they might be helpful. Participants were asked to write down their responses and to think aloud as they worked problems. Problems were presented one at a time on 3 X 5 cards, and follow-up questions were asked after each problem was answered, to probe participants' underlying cognitive processes.

Results

Fractions in everyday life outside school

The children in this study did not connect fractions with people or activities in the larger world. Answering a question related to who might use fractions outside of school, three children thought bankers or cashiers used fractions because they handle money, two referred in a non-specific way to scientists or "college people," and one

didn't know of anyone who might use fractions. When asked if they, themselves, ever used fractions outside of school, all but one said no. Only Betty said she used fractions while baking with her grandmother. Four prompts were used to get the children to think about possible situations in which they might have used fractions: sharing, measuring, cooking, and telling time. When asked if they used fractions when sharing something with friends or siblings, three said no and three said they shared candy bars. Only two mentioned specific fractions: halves and fourths. The prompt regarding measuring elicited only one clear instance of measuring. When asked to consider cooking, all of the participants were aware that fractions were used in cooking, and four gave specific instances of fractions that might be encountered. The time prompt resulted in only one immediate response of "quarter 'til." Others initially said no, but then after reflection produced the "quarter 'til" phrase. Betty was further asked what her answer of "quarter 'til six" meant. She responded, "25 minutes 'til six," apparently confusing a quarter of an hour with a quarter of a dollar. The relativity of the value of a fraction depending on its base unit seems to have caused confusion.

Syntactic Versus Semantic Content of Fraction Knowledge

Evidence that participants' understanding of fractions was largely syntactic in nature was provided by their responses to a question in the first interview which asked them simply to recall something they had learned about fractions. Rules and algorithms dominated their answers.

Four participants cited a rule regarding treatment of fractions during one of the four operations, add, subtract, multiply or divide.

Tess: "You only add the top numbers."

Int: "Why?"

Tess: "I don't know. I just remember you don't add the bottom ones."

Carl: "The dividing, that some way or another you flip the first one over, then multiply by the second, or something like that."

Only one student, Carrie, analyzed any aspect of the meaning of a fraction. She defined the terms numerator and denominator by explaining the relationship between them.

Carrie: "Well, the top number is the numerator and the bottom is the denominator. The bottom number shows how many there are in the entire group and the top number shows how many you're dealing with."

Further evidence of the lack of semantic content in children's concepts of fractions is derived from noting that it was difficult for them to put into words a statement of what a fraction is. Asked to explain to an alien from outer space (who could speak English very well, but who knew little about math), what a fraction is, only two participants gave a definition, and it was limited to the interpretation of a fraction as a part of a subdivided whole.

Carrie: "It means a part of a whole."

Carl: "A portion of a whole object."

Three others resorted to giving specific examples of fractions, two in drawings and one in words, and the sixth said she could not give an answer.

During the third interview, the dominance of syntactic content in participants' fraction concepts was also reflected in their explanations as to why they did what they did when solving problems or in telling why they were unable to solve a problem. The most common response when difficulty with a problem was experienced was a statement indicating inability to recall an algorithm.

Carrie: "I don't remember how we did that."

Tess: "I forgot. We done this last year. I forgot how to do it."

Fourteen instances of responses of this nature were recorded in the third interview. In five more instances, rules or algorithms were applied in inappropriate situations.

Tess: (In response to 4 ÷ 1/8), "4 into 8 is 2. 2 times 1 is 2. And you gotta use the same bottom number. (In solving this division problem, Tess employed an algorithm for equivalent fractions, and a rule from addition of fractions.)

Additionally, six times when algorithmic procedures were employed and successfully completed, students expressed concern or uncertainty regarding the obtained result, not because of any conflict with their conceptual sense of the problem and probable answers, but because they were not sure they had remembered the algorithm correctly. Apparently

they had no semantic, conceptual basis for judging the correctness of the obtained solutions.

> Carl: (In response to 2/3 + 3/4), "...I was trying to make that into twelfths and this is what I got, total [17/12]. I don't think that's the way to do it though."

Participants, in the twenty-five instances cited above, used descriptions of algorithmic procedures or cited rules in explaining why they had answered as they did, or stated their inability to recall an algorithmic procedure when they could not produce an answer. Other strategies utilized or cited in explaining answers were: a) drawing subdivided circles, used seven times; b) use of manipulative circles and segments, used only three times; and c) visualizing parts of a pizza, cited once.

Interference of Whole Number Knowledge with Fractional Concepts

Errors made by participants in ordering fractions and in operating on fractions reflected the interference of whole number knowledge with fractional concepts. For participants, knowledge of the whole number system has been developed during six years of prior school experience, and natural number experiences probably go back another three or four years. Children working with a new number set, the rational numbers, were observed to revert to the use of whole number knowledge when faced with apparent dissonance between the two. During the third interview, in ten instances, participants operated separately on numerators

and denominators when adding, subtracting or dividing fractions.

Betty: (In response to 3/4 + 1/8), "Well, you add 1 to 3 that's 4, and then 8 plus 4 is 12."

Jessica: (In response to 2/3 ÷ 1/3), "You divide the 1 into the 2 and the 3 into the 3."

The same behavior in multiplication mimics the algorithm learned by students, and results in the correct answer. It was, therefore, not treated as inappropriate. However, in the case of the students who consistently operated on numerators and denominators as though they were separate whole numbers, multiplication is most likely another example of the same schema in action.

In seven judgments of the relative size of two fractions (greater than or less than), participants used separate whole number values of numerators and/or denominators to support their decisions.

Jessica: (Asked which is greater, 2/3 or 2/5), "I think, yeah, it's 2/5. The top number is the same; bottom number is different."

Adam used a somewhat idiosyncratic method of multiplying the numerator times the denominator and comparing the products to determine the relative size of the fractions in question.

Adam: (Asked which is greater, 2/3 or 2/5), "On this one you could say, 2 fives make 10 and 2 threes equal 6."

Even during the second interview, in which manipulatives were not only available, but were utilized

throughout as prompted by the interviewer, three instances of whole number interference were noted. Asked to find a rod that was 1/3 the length of the base unit (12-rod), three students initially chose the 7-rod which is one cm. longer than the 6-rod (1/2 of the base unit). After being questioned, two of the three remained convinced that the 7-rod was 1/3 of the 12-rod.

Jessica: "This stick is longer than this one."

Int: "Why is 1/3 longer?

Jessica: "Because it has larger numbers.

Adam: "That'd be a little more than half."

Int: "How many thirds does it take to make a whole mater?" ('Mater' was Adam's name for the base unit.)

Adam: "Just one. Two would be too much."

Later, presented with three pieces of a circle, each 1/8 of the circle, Adam was asked what part of a circle they would be. He placed the 3/8 on the whole circle and stated that since it was bigger than a third, it was "maybe 1/4 or 1/5." Adam was observed to vacillate between recognition of fractional values and interference of whole number values throughout the second interview. An insight would seem to occur, but would later be obscured by whole number knowledge once again. This was evident within one response when Adam was asked to identify the value of a piece of a circle (1/6).

Adam: "It's a little bigger than these [indicating eighths]."

Int: "How many make a whole?"

Adam: [Places 3 sixths on circle] "Probably about 6."

Int: "What would one piece be called?"

Adam: "1/6."

Int: "What if you had five of those pieces?"

Adam: "1/5."

The Primacy of 1/2 as an Intuitive Concept

In observing participants' drawings during the first interview, the investigator was struck by the similarity of schemas in four of the six children. Asked to draw a picture that showed what a fraction looked like, all four drew circles and immediately subdivided them by halving. Three of the four then used repeated halving to create fourths or eighths. The fourth child, even though she also halved her circle again and added a third diameter creating unequal sixths, completed her diagram by shading and talking about 1/2 of the circle.

The facility children have with the concept of 1/2 was also noted in the number of times various fractions were mentioned, without having been specifically prompted, throughout the first interview session. They were as follows: 1/2, half or halves, fourteen times; 1/4 or fourths, nine times; 3/4, six times; 1/8, four times; 1/5, three times; 2/3, twice; 1/3, 3/5, and 4/10, each once. Since some of the instances of fractions other than 1/2 (especially 1/4 and 3/4) were elicited in response to two questions asking for fractions greater or less than 1/2, the numbers of times fractions other than 1/2 were recorded are

somewhat inflated in relation to the number of times 1/2 was mentioned.

During the second interview, every child responded quickly and correctly, "half" to the question asking them to identify the length of the 6-rod compared to the base unit (12-rod). It was noted that all participants in answering this question used the term "half" rather than "one-half." The concept of half appears to be so well integrated that it is different in kind than other fractions. In responding to the question asking children to identify a rod that was 1/4 of the unit, one participant responded with "half of a half." The same child, presented with 1/8 of a circle referred to it as "half of a half of a half." One participant utilized his concept of 1/2 in attempting to determine which is the greater of the fractions 2/5 and 5/8. He puzzled with this briefly, saying, "You can't reduce them." Then he brightened and said, "5/8, because its greater than 1/2."

Results from the third interview also supported the primacy of half and halving as intuitive concepts. When confronted with the two problems, "What is 1/2 of 8?" and "What is 1/4 of 20?," four of the six gave the answer 4 to the former question, three of the four responding immediately with no readily perceptible pause for thought. However, only two gave the response of 5 to the latter question, and one of the two worked through an algorithm "from last year" to arrive at the answer.

Evidence of the tendency to utilize the procedure of halving was noted in the work of Tess. On five of six comparison problems asking which of two fractions is greater or less, and on one of two equivalence problems, Tess used drawings of subdivided circles to make her judgments. In beginning her drawings of every fraction except thirds, she started with halving and then halving again (producing fourths). For fractions with more than four segments, she would then add a diameter or radius and count segments, add another and count, and so on until the desired number of segments was present. Even sixths, with their obvious relationship to thirds (which Tess did have a schema for producing), were produced by halving, halving again, and then adding a third diameter. Although she was at times concerned about the obvious inequality of the resulting fractional segments, and made erasures and adjustments, the initial halving procedure was not altered in attempting to create equal segments.

Order and Equivalence of Fractions

Although it has been suggested that it is important for children to develop a sense of the order of fractions prior to beginning computations with fractions (Bezuk & Cramer, 1989), responses to questions regarding fractions greater or less than 1/2 indicated that participants in this study have not established such a sense. Asked to name any fraction less than 1/2, only two of the six participants were able to produce an appropriate example, and only one could explain how he knew that his fraction, 1/8, was less than 1/2.

Carl: "You divide the circle in half, you got two big pieces, but as you add the other pieces, they get smaller so that means the larger the number, the smaller the pieces."

None of the other participants noted the compensatory relationship between the number of pieces, the denominator of the fraction, and the corresponding size of each piece. Asked to name any fraction larger than 1/2, three students responded with 3/4. No other fractions greater than 1/2 were cited. Only one of those four could explain his response and did so by drawing diagrams of two circles subdivided respectively into halves and quarters and shading 1/2 and 3/4.

Equivalence was also poorly developed as a concept for most of the participants. Asked which is greater, 4/8 or 2/4, Betty responded, 4/8. Asked why, she stated that 2/4 would be like half; 4/8 would be like the whole circle plus more. All six students were given the problem, 3/4 = ?/8. Two students stated that they couldn't remember "how to do that." Carrie who did follow the algorithm through correctly said, "this is probably wrong." Betty, who talked her way through the relationship also arriving at 6/8, was also dubious of the correctness of her thinking. Only two of the six children seemed at all sure that 3/4 = 6/8.

Dominance of Subdivision as a Conceptual Source for fractions

Subdivision of wholes appeared to be the basis of the fraction concept in five of the six (if not in all six)

participants in this study. Of all the drawings illustrating fractions produced by the children during the first interview, 21 diagrams were of circles (17) or rectangles (4) subdivided most often by repeated halving. Only one child, Carrie, used partitioning of sets to illustrate fractions on two occasions, showing 10 small circles with 4 shaded to illustrate the fraction 4/10, and attempting to use two sets of 4 small circles to show why 1/4 was less than 1/2. However, she was unsuccessful in her attempt, and she reverted to 1/4 being 1 out of 4, and 1/2 being 1 out of 2. This same unit-based thinking was displayed when Carrie was asked to show 1/2 of a set of 12 candies. She separated 3 candies, picked up one, and said, "Well, one out of three -- ." Her thinking in both cases was limited to partitioning when the cardinal number of the set is the same as the denominator of the fraction. Although she appeared to recognize partitioning of sets as a source for fractions, she was not successful in using partitioning to justify an ordering judgment, nor was she able to execute partitioning of a set of items. Therefore, partitioning did not seem to be present as a well-developed schema, even for Carrie. Comparing, as a source of possible constructions of fractions was never mentioned by any of the participants. Their conceptual framework for construction of fractions seemed to be largely limited to subdivision of continuous quantities.

Dissociation Between the Concept of Fraction as Symbol and Fraction as Physical Construction

A comparison of the results of the second and third interview suggests that students treat problems written in symbol form differently than problems connected to physical referents. In the second interview, children used physical referents throughout the interview as prompted by the interviewer. In the third interview students were presented with problems in symbol form and had the option to use physical referents whenever they might be helpful. However, students confronted with problems in symbol form seldom referred to the available manipulatives. Even when they found themselves unable to solve a problem, they did not utilize the physical materials. Only three instances of utilizing physical referents to answer symbolically presented problems were observed. Asked if they had used Cuisenaire rods before, participants said no. Asked if they had used other manipulatives, like the circles with fractional parts, two mentioned having played a fraction game (completing a pizza) in fifth grade. None related having used manipulatives to solve problems. Perhaps this may partially explain why participants did not seem to connect the experience of working problems in symbol form with the experiences of having used the physical referents to discuss questions about fractions in our previous interview.

Two types of problems that were found in both the second and third interview were stating the value of an

improper fraction in another way, and finding a fraction of a set of items. During the second interview, three students were asked for another way to say 4/3 of a pizza. Using circles and fractional pieces, all three determined that 4/3 is the same as 1 1/3. However when all six students were presented with the symbols 7/3 = ? and asked to write the fraction another way, only two, Carrie and Carl, were able to produce the answer, 2 1/3, and both used the algorithm of dividing numerator into denominator. Carrie was not at all sure that her answer was correct, stating that she knew it had a 3, 2, and 1 in it, but that's all she was sure of. Betty, who was one of the students who had worked out the 4/3 of a pizza problem, was convinced that 7/3 was meaningless and that "you just couldn't do it." In attempting to make sense of this problem, not one of the students picked up the circles and pieces lying on the table in front of them.

Finding a fraction of a set of items on the second interview came in the form of a question asking participants what 1/3 of a bowl of (12) candies would be. In response, three of the six students partitioned out three sets of four candies each. However, in interview three, while four students could tell that 1/2 of 8 was four, only two could determine that 1/4 of 20 is five. Throughout interview three, the participants cited rules and algorithms repeatedly. The task of remembering so many rules without integrating them in a conceptual framework becomes oppressive as noted by Carl when trying to remember how to

complete 3/4 X 1/2.

Carl: "You turn that one upside down. No you leave that one the same. As Ms. P_ would say, that one gets all excited and does a flip. No, that's in division. Crap. [Chuckles.] 3/8.

Int: "Could you explain how you got that?"

Carl: "4 X 2 is 8 and 3 X 1 is 3. There's too much stuff in this. It's hard to remember.

It should also be noted that Carl clearly had the most flexible thinking and greatest conceptual knowledge of any of the six participants. How much more frustrating all these rules and procedures must seem to the others.

Implications for Dissertation Research

While prior research has begun the process of analyzing students' concepts of fractions, further research is needed to add to that analysis. Therefore, the following broad question is still of value: What is the content of sixth graders' knowledge of fractions?

The apparent dissociation between syntactic procedural knowledge of fractions utilized when students were working symbolically-presented problems versus conceptual knowledge evidenced when students were working with physical referents suggests a need to systematically explore the question: How do students utilize conceptual and procedural knowledge when operating on fractions? A series of problems presented in verbal form with physical referents and parallel problems

presented symbolically would provide a window through which to observe this phenomenon.

The dominance of subdivision as the major source of conceptual knowledge of fractions for five of the six students in the pilot study leads to the question: How do sixth graders' understandings of fractions differ in situations involving comparing, subdividing and partitioning? Prior research has indicated that young children presented with "sharing" tasks are better at partitioning sets than subdividing wholes. However, middle school children have been observed to have problems representing fractions via partitioning of sets. Comparing should also be explored as a third source of fraction knowledge.

In addition to the above three questions being carried into the dissertation research, two other aspects of the dissertation study were influenced by the pilot study: the number of questions in the interview schedule, and the nature of probing questions following the main questions. The number of questions asked was increased to include variations on each problem type. It appeared that small differences in otherwise parallel problems sometimes made unexpected differences in individual responses (eg. one participant could deal, apparently quite easily, with building equivalent fractions when the required factor was two, but could not work a parallel problem requiring a factor of four). The probing questions, which follow participants' initial responses and which are directed at

finding out in detail what the child is thinking while working problems or considering fraction situations, were augmented to derive as much information as possible from each problem situation with each child.

CHAPTER 4

Methods and Procedures

Design

The purpose of the research was to explore the development of fraction knowledge in a group of sixth graders, to examine fraction concepts they had developed related to embodiments of subdivision, partitioning and comparing, to probe participants' underlying sense of meaning of fractions and operations on fractions, and to note differences in the thinking of successful math students as compared with less successful students. The research questions led to a qualitative approach, and the desire to gain access to children's thinking related to fractions led directly to the clinical interview and think-aloud protocol techniques.

Description of Techniques

The clinical interview and think-aloud protocol processes allow for both flexibility and sensitivity in exploring children's thinking. In a clinical interview, the researcher begins with a common set of questions, often supported by manipulative materials. The unique characteristics of the method arise from the probing questions which follow each participant's response to the primary interview question, and are based upon the individual's initial rejoinder. The purpose of probing questions is not to establish whether answers are correct or

incorrect, but rather to ascertain the reasoning that lies behind the answers. Probing questions allow the interviewer to gain insights into the cognitive processes underlying individual participants' initial responses.

Think-aloud protocols present participants with a task or problem accompanied by a request to think out loud, to verbalize their thoughts while completing the task. Following, or if deemed appropriate, during task completion, probing questions may be asked which are based upon initial responses, and which require participants to reflect further on their thinking processes during task completion.

Participants

Rationale for Selection

Children are introduced to fractions very early in their public school experience, often in kindergarten or first grade. However, this introduction tends to be brief and is limited most often to pictorial and/or physical representations of a few basic fractions. It is not until the middle grades, often fourth or fifth grade, that operations on fractions are included in mathematics curricula. For this study, sixth grade was chosen as the year during which most students would have at least begun work in reducing fractions, building equivalent fractions, and carrying out the four operations (add, subtract, multiply and divide) on fractions. Sixth graders, therefore, were the major population for this study.

Number of Participants

Because this was a qualitative study designed to examine and describe cognitive processes, and to explore meaning in a somewhat flexible manner, the sample size was limited. Initially twelve sixth graders were engaged in the interview process. However, while all twelve students completed the first three interviews, one of the twelve did not participate in the fourth interview. The sample was large enough to produce redundancy in responses, suggesting adequacy of data for analysis. The twelve participants were drawn from three different middle schools in east Tennessee and southeast Kentucky, one rural, one in a mid-sized town with a population of 1700, and one in a small city with a population of 12,300. African-American, eastern Indian and Caucasian as well as male and female students were included in the study.

Criteria for Selection

One of the research questions required contrasting the thinking of students of high versus low math performance. In order to achieve a contrast between high and low achievers in mathematics rather than observing differences in overall ability, it was deemed necessary to hold verbal ability relatively constant within an average to above average range. Therefore, students were selected based on two criteria. All participants selected had scores of fiftieth percentile or above in total reading and/or total language on their most recent achievement test. Five of the participants selected displayed low math performance based

on average math grade for the current year of C- or below, and the remaining participants, high math performance with an average of B+ or above during the same period. However, none of the participants had been identified as eligible for services under any category of special education. The student who was involved in only three of the four interviews was one of the lower-performing math students. There were, then, seven students who were doing well in math classes and five whose performance was below average.

Descriptions of Participants

The five students who had experienced difficulties in math classes, as reflected by below average grades in math during the current school year, will be described first.

Ann. Although she had obvious difficulty with mathematics, Ann was always positive and willing to engage in any discussion or activity posed during the interview. Ann had a marvelous smile that lit up her entire face. She never seemed to be discouraged by the frustration of being unable at times to produce answers to interview questions, although on occasion she would talk around a question rather than say directly, "I don't know."

Bea. Bea's most striking quality throughout the four interviews was her shyness. She was the only participant who ever seemed bothered by the video camera, stating in the middle of one interview, "I hate being on cameras." She had a very quiet voice and tended to put her hands in front of her face or lean her head on one hand. However, she was

very pleasant, and she was the only participant who seemed interested in and asked questions about the study.

Dan. Dan was a cheerful sixth grader who was not afraid to tackle the tasks presented. He seemed genuinely interested in pleasing adults, sometimes asking directly for feedback or directions. He worked at a relatively slow pace, taking his time especially while working with the manipulative materials.

Tes. Tes was a reflective child whose answers came after numerous long pauses. She was not a risk taker. She did seem perplexed at times and expressed consternation at having skipped several items on the fourth interview. However, she was generally happy in interview sessions, and was especially excited on the day she was to try out for a coveted position on a school sports team.

Nel. Nel was a reticent child who only occasionally warmed to the tasks or discussions in the interview sessions. She seemed to be practiced at only revealing small glimpses of her real self. In the hallway with her friends, Nel was considerably more affable and energetic than when she came in for interviews. Nel chose not to participate in the fourth interview.

The seven students whose descriptions follow were those who were generally experiencing success in math classes as indicated by above average math grades.

Sam. Sam was highly energetic and seemed to enjoy the interview sessions. He had a tendency to begin his statements with 'uh' or 'well' which seemed to be his way to

begin answering immediately but still take a moment for thought. Sam was not one for long silent pauses.

Lee. Lee, like Sam was energetic and enthusiastic. She virtually bubbled her way through the interview sessions, and she gave the impression that sitting still was not high on her list of favorite pastimes. Lee used her hands to illustrate and emphasize what she was saying, even if there were no manipulatives in use. She was energy in motion.

Jay. On some interview days, Jay seemed more interested in getting to P.E. class than in discussing fractions, but his demeanor was friendly and positive during all of the interview sessions. In spite of himself, Jay became more engrossed in the discussions and tasks than most of the other participants, at times expressing surprise or satisfaction and at other times becoming involved in explaining things in great detail.

Pat. Pat was a thoughtful child who gave the impression of being considerably older than her twelve years. She took the interviews quite seriously and answered questions very thoroughly. However, a smile crept across her face now and then, and she seemed relaxed and unhurried in her responses. Pat was a great storyteller who sometimes gave elaborate answers to questions that others answered with one-liners.

Tad. Tad had a large voice with which he seemed uncomfortable and seldom used above half volume, sometimes trailing off into a barely audible whisper. For a sixth grader, Tad had a unique approach to explaining things.

During the first interview he explained fractions to a hypothetical uninformed second grader in a manner one might expect of a pedantic teacher with twenty years experience under his belt, interspersing rhetorical questions with his own ready answers.

Mel. Mel gave the impression of being quite sure of himself and his understanding of fractions. He sometimes seemed very involved in the discussion and in problem solving, all of which he did in his head. However, during one interview he busied himself with arranging the manipulative materials in neat stacks and organizing them from largest to smallest, stopping only to ponder a question occasionally and then returning to his self-directed project. Mel had a ready smile and sometimes preceded what, for him, were obvious answers to questions with a chuckle, suggesting that the questions had seemed just a little silly to him.

Ron. Ron also had a ready smile. In addition he had an impish sense of humor that he unfortunately only liberated on one or two occasions; it was delightful to observe. He kept the tell-tale sparkle in his eyes stifled most of the time as well. He seemed to enjoy working with the manipulative materials and was generally cooperative and friendly.

Interviews

In order to gain insight into the internal symbolic mechanisms underlying participant's mathematical behavior,

clinical interview methods and think-aloud protocols were employed in conducting the study. Clinical interviews consisted of verbal questioning of participants and verbal presentation of tasks or problems. Follow-up or probing questions were asked which were composed in response to the meanings in children's initial rejoinders. Think-aloud protocols involved presenting participants with problems, and asking them to verbalize their thoughts while working toward solutions. As in clinical interviews, follow-up questions were asked to further probe participants' underlying cognitive processes.

During the current research project, four interviews were conducted with each participant. The first three were clinical interviews during which concrete materials were utilized to elicit intellectual behaviors with the intent of examining underlying cognitive processes. During the fourth session, think-aloud protocols were utilized. Fractions problems were presented in symbol form, and participants were encouraged to "think out loud," verbalizing their thoughts while solving the problems. Manipulatives were available to participants, along with paper and pencil. Probing questions followed students' answers to each problem.

Interview One

The interview and protocol sessions were organized around the four research questions developed out of the literature search and pilot study. The first interview session was a clinical interview designed to gain

information about participants' knowledge of fractions, specifically related to: a) their concept of what a fraction is, b) when and how fractions might be used, and c) how the system of rational numbers is ordered. To explore conceptual understanding, students were asked to describe in their own words what a fraction is; to explain the meaning of particular fractions, such as one-third and three-fourths, to a hypothetical younger child; to construct illustrations of particular fractions such as two-thirds and five-eighths; and then to tell stories about the diagrams they had produced. In order to access participants' awareness of possible uses of fractions, they were asked about ways they or others might use fractions in the real world. In order to gain insights into the sixth graders' knowledge of the ordering and equivalence of fractions, questions of greater than, less than, in between and equal to were posed. (See Interview Schedule #1 in Appendix B.)

Interviews Two and Three

In the second and third interview sessions, concrete materials were utilized to observe possible differences in participants' understanding of fractions within the contexts of comparing amounts, subdivision of continuous quantities, and partitioning of discrete sets. Cuisenaire rods and paper strips were utilized in asking children to make comparisons of length relative to an arbitrary unit. At one point during the second interview, in order to explore students' sense of the relative nature of fractional values, a second arbitrary unit of comparison was introduced, and a

"think out loud" as they worked. Follow-up questions were asked after each question was answered to further probe participants' thinking. Problems in this protocol session paralleled those presented in the second and third interviews related to meaning, equivalence, and ordering of fractions, and operations on fractions. (See Interview Schedule #4 in Appendix B.)

Order of Presentation

The order of the interviews and of the questions within interviews was arranged so that students would have the opportunity to display their knowledge to the fullest extent possible. To this end, the first interview was a relaxed discussion of the meaning of fractions followed by a few open-ended questions about order and equivalence of fractions. In interviews two and three, verbal problems modeled with manipulatives were organized from least to most complex, and partitioning of discrete sets, demonstrated in prior research to be more difficult than subdivision for students, was presented last. Finally, symbolically-presented problems were not introduced to participants until the fourth interview, so that any possible transfer might be made. This order of presentation allowed for either a maximized opportunity to observe connections between concepts and syntactic procedures, or strong evidence of dissociation of conceptual and procedural knowledge, since students were given the opportunity to use transfer of knowledge from concrete experiences and verbal tasks in

defined as either a complete sentence, or a clause or phrase which is indicative of a complete thought. Ericsson and Simon (1984) concluded that segmenting protocols into statements is not difficult, and is a reliable procedure (p. 266). Once transcripts were segmented into statements, each statement was identified as a statement of: a) action (AC), b) procedure/algorithm (PR), c) sub-procedure (SP), d) unelaborated answer (AN), e) example (EX), f) concept (CN), g) automaticity (AT), h) evaluation (EV), i) clarification/restatement (CL), j) comment (CM), or k) error (ER). A description of each category and one to three exemplars of each category taken from transcripts follow:

Action

A statement of an action refers to a statement describing a behavior being carried out, or an action may be implied or described but not actually carried out. This category also includes behaviors related to responses that were noted by the researcher in sessions or in transcribing video tapes.

See, I marked from here to here. (AC)

It'd fit twice, yeah; it'd fit in there twice. (AC)

[Draws circle; subdivides into fourths by repeated halving.] (AC)

Procedure/algorithm

A statement of a procedure or algorithm refers to manipulations of symbols in carrying out an operation or other process. Procedure statements include descriptions of

such manipulations as well as statements of rules stipulating procedures or subprocedures. When students rely on a procedure or algorithm as the only means to solve a problem, or express awareness that an algorithm exists but that it has been forgotten, this will also be coded as a procedure/algorithm statement.

I don't remember why, but I just know you flip one of them. (PR)

Three'll go into 6, 2 times, and 2 times 1 is 2. (PR)

We done something like this last year, but I forgot how to do it. (PR)

Subprocedure

Subprocedures are the individual steps taken to complete a procedure. When used correctly within an adequately functioning procedure, subprocedures will not be separately noted. However, subprocedures will be noted if they are applied inappropriately, omitted, or otherwise result in an error.

Three'll go into 9, 3 times, (PR)

and then your 3 on the bottom (SP-ER)

(kept denominator, as in addition)

Answer

An unelaborated result given in response to a problem, usually in the form of a number with no additional explanation will be coded as an answer.

About 1/2. (AN)

She would have 1/2 as much. (AN)

Example

An example is a particular member (or list of members) of a category, given in an otherwise unelaborated response. However, vague, non-specific, or indefinite examples will be coded as comments.

3/6, 4/10, 5/8. (EX)

Like if I got 2/3 of a pizza, . . . (EX)

Concept

The concept category includes those statements that reveal semantic understanding of fractions, symbols and/or relationships. Concepts include statements of the meaning of symbols words or phrases. Concepts may be implied in action sequences.

5/8 would be larger than 1/2 because 4/8 would be equal to a half; and 2/5 would be smaller than a half, because you can't get half of 5. (CN)

I saw that [circle with 3/4 shaded], and thought since there's 4, there must be two in each one, because 4 is half of 8. (CN)

[Draws rectangle; subdivides into 5 segments; shades 2] (AC)

Tell me about your drawing.

It's 2/5. (CN)

Automaticity

Automaticity will be coded when participants produce a result with no apparent time for thought or when they describe the experience of producing a result without consciously thinking through procedures.

I just recognize the shape. (AT)

I just see the answer. (AT)

I just knew it. (AT)

Evaluation

Evaluation statements are judgments regarding the appropriateness or correctness of a procedure, an action, or one's line of thinking. Evaluation statements may indicate metacognitive awareness of being unsure or feeling confused.

I don't think that's the way to do it though. (EV)

This is not a good example. (EV)

Yeah that'll work, yeah. (EV)

Clarification/restatement

When a participant asks for further explanation, or simply restates all or part of a question, clarification will be coded.

What do you mean like? (CL)

Less than 1/2? (CL)

Comment

Comments are other statements verbalized by participants which are not exemplars of the previously defined categories, and which may or may not be related to the problem or question at hand and efforts at solving or answering. Included in this category will be simple yes or no responses or similar statements without further explanation attached, as well as incomplete statements or statements whose meaning is unclear.

My aunt, she used to be a school teacher. (CM)

Everybody uses fractions. (CM)

[Pause, shakes head], no, not really. (CM)

You could put another one of --; you could use --. (CM)

Error

Errors in answers, processes, or reasoning will be coded with a parenthetical note describing the nature of the error.

The whole thing is one six, 1/6, I mean. (CN-ER)
(entire array seen as unit fraction)

Derivation of Statement Categories

Seven statement types were identified as preliminary categories derived from the pilot study data and based on Ericsson and Simon's (1984) discussion regarding the coding of segments in problem solving protocols (pp. 267-270). The seven categories were: goal, action, evaluation, comment, procedure, subprocedure, and error. Six additional categories of statements were added during the early stages of data analysis of the current research: conceptual knowledge, unelaborated answer, example, relationship, automaticity, and clarification. However, goal statements were seldom produced (<20 across all episodes), and those that were generated related directly to, and therefore overlapped, the category of procedures. Similarly, relationship was a seldom-coded category (<40 statements), and such statements tended to overlap the conceptual knowledge category. Because these two statement types were infrequent and because they were not clearly separate from more commonly represented categories, they were eliminated

early in the data coding process. Thus the remaining categories were the eleven listed and described in the preceding several pages.

Coding of Statements

The coding of statements involved considering each statement within the context of an entire episode, that is, the entire set of responses to one interview question or protocol problem along with any follow-up probes. Responses were first considered within the context of the individual's collected transcripts. Further analysis required inspecting episodes related to each interview question across participants. For this purpose, responses were re-organized so that participants' responses to common questions could be considered together. Responses also were re-organized so that all items bearing upon a particular aspect of the concept of fraction or a particular process with fractions might be viewed as a group. A third set of re-organized responses juxtaposed responses of successful math students with those of students experiencing difficulties in math classes.

Response patterns were analyzed based on percents of the types of statements made. Because clarification and comment statements revealed little about students' understanding of fractions, they were not tallied in the data analysis process. Error patterns were evaluated within the context of the problems presented and the participants' statements in the episode associated with the errors. Regularities in patterns of responses and errors were accepted as evidence

of the content of sixth graders knowledge of fractions, their use of conceptual and procedural knowledge, and the manner in which the contexts of partitioning, subdividing and comparing related to students' thinking about fractions.

Example of the Coding Process

In order to describe the data reduction process, a sample of three consecutive episodes has been selected from the transcript of Mel's second interview. They were as follows:

(Interview 2, question 7)

I: Let's go back to our original unit (12-rod). You named this an "aaknid." What length was this rod then (4-rod)?

L: Um, I think this one was a third.

I: Can you think of any other ways to make 1/3 of an aaknid, using any of the other rods?

L: [Places 2 sixths on top of 1/3]. Hmm, one-third.

I: What's the name of that fraction?

L: 2/6.

I: Any other ways to make 1/3 of an aaknid?

L: Hm, let's try these (1-rods). [Places 4 1-rods beside aaknid]. 1/3.

I: Okay, do you have any idea what fraction that would be?

L: M-m, 4/8.

I: 4/8?

L: Um, not eighths, twelfths.

I: Do you want to check it out or do you know that for sure, already ?

L: Ah, I think I might want to check it out. [Places 12 1-rods beside aaknid]. Yeah, it's --, yeah.

(Interview 2, question 8)

I: How many ways could you show 1/2 of an aaknid?

L: I could use most of these things. [Places 6 twelfths beside unit]. Here that would be half of an aaknid.

I: How many of those would it take?

L: Six. Ah, one more of those. [Places 3 sixths beside aaknid].

I: What would you call that?

L: Three, um, three-sixths. That's half an aaknid.

I: Okay.

L: I might be able to use these. [Places 2 fourths beside unit]. That's half an aaknid.

I: What are those, what fraction is that?

L: 2/4. That's about it.

(Interview 2, question 9)

I: If these are whole pizzas, what's this (1/3)?

L: Ah, 1/3 of a pizza.

I: Want to show me why that's true?

L: [places 3 thirds on circle].

I: So how many thirds does it take to make a whole pizza?

L: Three.

I: How about these green ones, what size would you say they are (fourths)?

L: Ah, 1/4. [Places 4 fourths on whole]. Yup, 1/4.

I: How about if you had three of those green pieces, what would you have?

L: You'd have 3/4. [Places 3 fourths on circle].

The first step in data analysis, after transcribing tapes verbatim, including all actions observed, was to segment the protocols into statements. Following segmenting, each statement was coded utilizing the eleven statement categories that had evolved from the work of Ericsson and Simon, the pilot study, and the early stages of data analysis in the current study (see pp. 57-62).

The three episodes from Mel's transcript reproduced above, after segmenting and coding were in the following form:

(Interview 2, question 7)

I: Let's go back to our original unit (12-rod). You named this an "aaknid." What length was this rod then (4-rod)?

L: Um, I think this one was a third. AN

I: Can you think of any other ways to make 1/3 of an aaknid, using any of the other rods?

L: [Places 2 sixths on top of 1/3]. AC
Hmm, one-third. CN

I: What's the name of that fraction?

L: 2/6. CN

I: Any other ways to make 1/3 of an aaknid?

L: Hm, let's try these (1-rods). CM
[Places 4 1-rods beside aaknid]. AC
1/3. CN

I: Okay, do you have any idea what fraction that would be?

L: M-m, 4/8. CN-ER
(whole number interference)
(4 1-rods in place, 8 spaces left)

I: 4/8?

L: Um, not eighths, twelfths. EV CN

I: Do you want to check it out or do you know that for sure, already ?

L: Ah, I think I might want to check it out. CM
[Places 12 1-rods beside aaknid]. AC
Yeah, it's --, yeah. EV

(Interview 2, question 8)

I: How many ways could you show 1/2 of an aaknid?

L: I could use most of these things. CM
[Places 6 twelfths beside unit]. AC
Here that would be half of an aaknid. CN

I: How many of those would it take?

L: Six. AN
Ah, one more of those. CM
[Places 3 sixths beside aaknid] AC

I: What would you call that?

L: Three, um, three-sixths. CN
That's half an aaknid. CN

I: Okay.

L: I might be able to use these. CM
[Places 2 fourths beside unit]. AC
That's half an aaknid. CN

I: What are those, what fraction is that?

L: 2/4. CN
That's about it. CM

(Interview 2, question 9)

I: If these are whole pizzas, what's this (1/3).

L: Ah, 1/3 of a pizza. AN

I: Want to show me why that's true?

L: [places 3 thirds on circle]. AC

I: So how many thirds does it take to make a whole pizza?

L: Three. AN

I: How about these green ones, what size would you say they are (fourths)?

L: Ah, 1/4. AN
[places 4 fourths on whole]. AC
Yup, 1/4. CN

I: How about if you had three of those green pieces, what would you have?

L: You'd have 3/4. AN
[Places 3 fourths on circle]. AC

Reliability Issues

Early in the research process, the investigator was active in a qualitative research group and accessed that group for both criticism and verification as the research process was initiated. In addition, a colleague served as a second rater to validate the process of coding statements. Using a random number table, thirty questions were randomly selected from all interview transcripts. Of those thirty questions selected, one was a repeat and one had not been asked in the particular interview selected. The twenty-eight remaining questions provided 104 statements which were utilized in establishing inter-rater reliability. The researcher and second rater categorized each of the 104 statements, producing seventeen "misses" and resulting in 84% inter-rater agreement, supporting Ericsson and Simon's (1984) claim that coding of statements embedded in context is a reliable procedure at .8 or .9 levels. A review of the "misses" did not reveal any obvious bias. Seven different categories were represented: conceptual knowledge, comments,

and answers were most often missed (five, five, and three misses respectively), while action, example, evaluation and procedures each contributed one miss.

CHAPTER 5

Results: Case Studies 1-5

In order to distill patterns that are similar among participants from thousands of statements in hundreds of episodes, it is necessary to categorize and summarize verbal reports. Patterns are of interest because they reveal commonalities in the way participants approach, work with, understand, and talk about fractions. However, in the data reduction process, specific behaviors and individual differences are lost. For this reason, a brief case study of each of the participants, based on responses to all four interviews, has been constructed. The first five case reviews describe the five participants who were selected as students performing at below-average levels in mathematics class during their current sixth grade year.

Ann

Concept of Fraction

Ann had a rudimentary sense of the meaning of fraction; she was able to posit a pragmatic part-whole definition of what a fraction is.

Ann: Well, fractions are something like dividing things up to get an equal amount of something.

However, Ann was not able to name her own drawing of the fraction 2/3, referring to it as 1/2 or 3/2. Her limited

understanding of fraction symbols did not permit her to translate the diagram into symbol form.

Ann's shading schema. All of Ann's explanations of the meaning of specific fractions had to do with shading circles, and she did not connect her diagrams with anything from her own experiences. In considering ordering and equivalence of fractions, she also visualized shading circles in order to think about the relative size of fractions. The shading schema seemed to be very powerful for her, especially when she was not involved with physical referents. Shading was apparently her way of bringing the abstract into the more familiar realm of the concrete.

(Interview 4, Question 9)

Qstn 9: Which is greater, 2/5 or 3/5

Ann: 3/5

Int: How come 3/5 would be greater?

Ann: Because if you had five you'd shade in three of those, but if you had five you'd be only shading in 2 of those.

Deficiencies in Ann's fraction concept. One indication of the incomplete nature of Ann's understanding of the meaning of fraction was her response when asked to draw a picture of the fraction 2/3. She constructed her "two-thirds" within less than half of the circle (see figure 5-1)

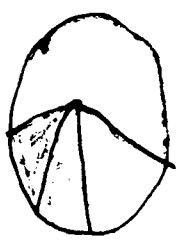

Figure 5-1: Ann's Drawing of 2/3.

Ann's concept of fraction as subdivision of a continuous quantity, at least in this case, did not include termination of the whole into equivalent parts. However, it was only within the realm of subdivision that she correctly identified the fractions 1/4 and 1/6.

Concept of 1/2. One-half was a fairly strong intuitive concept for Ann which contrasted with her otherwise incomplete meaning of fraction. While working with Cuisenaire rods and circles with fractional pieces, 1/2 was the one embodied fraction she recognized and named with ease. When confronted with a rod 1/3 the length of the unit, in comparing lengths, she stated that it was less than half but could not name it. The relative conceptual strength of 1/2 and Ann's use of visualization surfaced in working with circles and pieces when she was asked how many 1/2's were in 3 whole pizzas.

(Interview 3, question 12)

Int: How many 1/2's are there in three whole pizzas?

Ann: Three, in three whole pizzas? Are you saying like two of these on both pizzas? [Places 1/2 on circle].

Int: Uh-huh, on all three pizzas, how many halves would there be?

Ann: [Looking at 1/2 on circle], two, four, six.

However, even Ann's concept of 1/2 was incomplete. In partitioning of discrete sets, Ann recognized 8 of 16 candies only as 8/16 and could not identify another way to name the fraction, nor did she recognize without utilizing

manipulatives that 2/2 would make a whole. In addition, there were times when 1/2 seemed to be a generalized term indicating any size piece of something.

(Interview 2, question 9)

Int: Tell me about the orange pieces (thirds).

Ann: [Places 1/3 on circle.] Two halves.

Int: How many of those would it take to make a whole pizza?

Ann: [Places 3 thirds on circle.] Might take three.

Int: Takes three of them to make a whole pizza, so what would you call one of them?

Ann: Uh, 1/2.

Int: 1/2?

Ann: Yeah, 3--; yeah, 3/2 or 1/2.

Ordering and Equivalence

Ann had only a partial sense of what made one fraction equal to, greater than, or less than another. Employing her shading schema, she depended largely on the value of the numerator, which indicated to her how many parts would be shaded, to make relative size judgments. She often appeared to use the information provided by the numerator in isolation, disregarding the total number of parts indicated by the denominator.

(Interview 1, question 13)

Int: Could you give me an example of a fraction that's greater than 1/2?

Ann: 1/5.

Int: What is it about 1/5 that tells you that it's

greater than 1/2?

Ann: Because if you had --; no, yeah, because if you had five --. No, it wouldn't be greater. Because you'd only be shading one. Maybe 2/5, because you'd be shading two, you'd be taking two of something.

Operations on Fractions

Operations with fractions did not seem to be connected with conceptual understanding for Ann. She depended upon syntactic procedures which she often misapplied, usually because of whole number knowledge interference.

(Interview 3, question 4)

Int: What if you started with 3/4 of a pizza, and someone gave you 1/8 more?

Ann: [Places 2 eighths on circle]. 4/12.

Int: How did you decide that?

Ann: I added 4 and 8, and 3 and 1.

(Interview 4, question 16)

Qstn 16: Add: 1/2 + 1/3

Ann: Be 2/5. It can't be reduced.

Int: Tell me what you did on that one.

Ann: Added top and bottom.

Spatial-Perceptual Difficulties

One final observation regarding Ann's individual behavior that was not seen in any of the other participants was her difficulty with spatial tasks. Specifically, in working with circles and fractional pieces, Ann was often unable to place the wedge-shaped fraction pieces on the

circle and align the curved edges. Working with fourths, she managed to place two fourths on a circle and gradually slide them into position, but could not place the other two in alignment to complete the whole. When asked to identify the size of the white pieces (sixths), she again placed only two pieces on the circle, experiencing the same difficulty with aligning the curved versus straight edges. Although she is not currently involved in special classes, it seems likely that Ann's perceptual-spatial difficulties may have provided a clue to one possible contributing factor to her lack of success in math class.

Bea

Concept of Fraction

Bea did not readily convey a sense of the meaning of fraction. Asked to tell what a fraction is, Bea seemed bound by her procedural knowledge, describing the subprocedures of a division algorithm and giving an example of solving a division problem with fractions. Asked to draw a picture of the fraction 1/3, Bea drew the diagram in Figure 5-2 and gave the following explanation:

Figure 5-2: Bea's drawing of 1/3

Bea: That right there would be the 1 [pointing to the one white space], and that would be the 3 [pointing

to the three shaded segments].

This same sense of the meaning of fraction symbols emerged when Bea was considering partitioning of discrete sets in the third interview, and when she was asked to name a fraction represented by a diagram in interview four.

(Interview 3, question 16)

Int: What would 1/3 of this bowl of candy (12 candies) look like?

Bea: [Sets aside 3; places 1 piece below the 3.]

Int: How did you decide that?

Bea: Take three pieces out, and one by itself, and I suppose that makes 1/3.

Int: Tell me again now, tell me what you did.

Bea: I took three out, you take three pieces out; that'd be the third; [moves one piece away from 3], and that'd be the one.

(Interview 4, question 1)

Qstn 1: What part is shaded (2/3)?

Bea: Two [laughs].

Int: What fraction would that be?

Bea: Probably 2 over 1.

At other times, in comparing lengths of rods and in subdivision of circles, Bea displayed a slightly modified version of her cognitive frame for the meaning of fraction symbols. In these situations, an entire array represented the unit fraction, but separate slots for the numerator and denominator as whole numbers were not obvious.

(Interview 2, question 5)

Int: Tell me how long this (2-rod) is compared to a square ("square" was Bea's name for the 12-rod unit).

Bea: [Places 6 sixths beside unit; counts].

Int: And how many of them did it take?

Bea: Six.

Int: So what did you decide about one of these, what would each one of these be?

Bea: Just one? Hmm. [Removes 5 sixths, leaving 1/6 beside unit; long pause.]

Int: Or what did you decide about the whole thing?

Bea: The whole thing is one six; one sixth, I mean.

In these and similar episodes, it was clear that Bea's understanding of the fraction symbol consisted of a whole number interpretation of the numerator and denominator.

Concept of 1/2. Like Ann, Bea had a relatively strong intuitive sense of 1/2. She recognized and named 1/2 of a circle, 1/2 of a rod, and 1/2 of a paper strip. However, she did not appear to have made connections between her informal sense of 1/2 and her meanings for other fractions. She could not name any fraction less than 1/2 or greater than 1/2, nor could she name any fraction equal to 1/2.

Operations on Fractions

Bea's procedural knowledge of fractions consisted mainly of inflexible algorithms for addition and subtraction which she had apparently overlearned and which she generalized to multiplication problems. Several participants displayed

this same type of generalization. All addition and subtraction problems were written in column form, rewritten with common denominators if necessary, and added or subtracted. Even 1/5 + 2/5 and 2/3 - 1/3 were treated in this fashion. Bea did not have adequate conceptual knowledge to inform her procedures and she appeared to be constrained by memorized rote algorithms.

(Interview 4, question 20)

Qstn 20: Subtract 1/2 - 3/8

Bea: [Writes in column form; changes to 4/8 and 3/8; writes 1/8].

Int: Tell me a little bit about what you're doing when you're changing these.

Bea: Well, the teacher taught us how to do that a long time ago, and --.

Int: Why are you changing them like that; why can't you just subtract the 1/2 and the 3/8?

Bea: Because, uh, that won't work, cause, um, I don't know.

None of the students from either group could adequately explain what goes on in the subprocedure of finding common denominators, or in the inverting subprocedure in division, or why those subprocedures were necessary.

Dan

Concept of Fraction

Dan's concept of fraction was not well-developed, nor was it integrated with his procedural knowledge. Dan

illustrated, named, and verbalized part-whole explanations for the fraction symbols 5/8 and 3/4. However, he did not succeed in doing the same for 1/3. Using repeated halving seemed to be his only schema for producing diagrams of fractions, and the schema broke down when he was confronted with 1/3.

Concept of 1/2. Like Ann, Dan recognized and named 1/2 readily in situations of comparing and subdivision, but not in partitioning. He also knew and expressed with automaticity (stating his answer quickly with no apparent time for thought), that 2/2 would be equal to a whole. Dan seemed to have an intuitive concept of 1/2. However, like Ann, he seemed to sometimes use the name 1/2 to apply to any size fractional piece.

(Interview 1, question 5)

Int: Can you draw me a picture of what a fraction would look like?

Dan: [Draws circle; subdivides by halving three times creating six uneven segments.]

Int: Tell me about what you've drawn.

Dan: It's a pie or something, and break it up into 6, into 6 pieces. Instead of everybody just getting a whole pie, break it down into as many pieces as you can.

Int: If you broke it into six pieces, what would each piece be?

Dan: Half.

Ordering and Equivalence

Equivalence and ordering were not connected with Dan's fledgling fraction concept. While comparing two fractions, faced with a conflict between his conceptual sense of fifths supported by physical referents and an inappropriately applied syntactic rule, Dan opted for the incorrect rule-based answer.

(Interview 2, question 16)

Int: Which is greater, 2/5 or 3/5?

Dan: I think it'd be 3/5.

Int: Why would you think that?

Dan: I think that these would be the 3/5. [Places 1/6 on top of 1/5]. That would be, the 2/5 --. Oh, [speaks faster, with more confidence], it would be 2/5, because 3/5 is littler, littler than 2/5.

Int: Why would it be littler?

Dan: Like the smaller you get, the bigger it is. [apparently remembering that smaller fractions have "bigger numbers"].

Displaying the inverse of a tendency observed in Ann's protocols, Dan, in making relative size judgments, sometimes used the information in the denominators of fractions while ignoring the numerators.

(Interview 2, question 19)

Int: Which is greater, 5/8 or 3/4?

Dan: 3/4.

Int: How did you decide that?

Dan: Cause the 3/4 is just half the size of the 5/8.

Operations on Fractions

When presented with fraction problems in symbol form, Dan became a textbook case of rigid application of syntactic algorithms. Every addition and subtraction problem he worked in the fourth interview he immediately wrote in column form, going through the subprocedure of changing to a common denominator whether the denominators were already the same or not, reducing answers, and then "dividing up" if the numerator turned out to be larger than the denominator. Even working such problems as 1/5 + 2/5, or 2/3 - 1/3, or 4/5 - 4/5, Dan went through the entire script of subprocedures. Like some of his cohorts, Dan generalized this overlearned algorithm to a multiplication problem and also to a division problem.

The unevenness of Dan's performance overall, at times displaying a sense of the meaning of fraction symbols, and other times seeming to have entirely dismissed his conceptual sense of fraction, suggested that Dan was vacillating between making sense -- utilizing his rudimentary conceptual knowledge of fractions -- and falling back on conflicting whole number knowledge or relying on syntactic rules and algorithms.

Tes

Concept of Fraction

Like most of her confederates, Tes did not connect fractions with her own or others' real-world experiences. Tes did not verbalize a general definition of fraction. She did, however, produce drawings of several fractions.

Tes consistently treated the numerator and denominator of fractions as two separate whole numbers. Even though she drew part-whole diagrams, her verbalizations indicated that she viewed her rectangular strips as strings of boxes of which so many were shaded, much as Ann shaded circles. Whole number knowledge interfered with Tes's concept of fraction on many occasions.

Tes had a unique tendency to use "a whole" to stand for any fractional part she could not otherwise identify. It seemed as though she translated parts into wholes in themselves to avoid the difficulties inherent in dealing with fractions.

(Interview 2, question 2)

Int: What part of a stick (12-rod) would that (4-rod) be?

Tes: I don't know.

Int: Would you like to take a guess?

Tes: Is it about --; [long pause], I can't tell.

Int: How many of those would it take to make a whole stick?

Tes: Three. [Places 3 thirds on unit].

Int: Okay, if it takes three of those to make a stick,

what would just one of them be?

Tes: One whole.

Like several of her compatriots, Tes referred to 3/4 as 1/3 on at least one occasion. The tendency to see that one part was missing and three parts were present and to refer to the figure as 1/3 was noted in the protocols of all members in this group of lower-performing math students, indicating the tendency toward a cognitive frame that has separate whole number slots for the numerator and denominator of a fraction.

Concept of 1/2. Also in alignment with her fellow group members, Tes's strongest concept of fraction was related to 1/2. She recognized and named one-half in comparing and in subdivision situations, although not in partitioning. Like Dan, reflecting automaticity in her immediate response, Tes knew that 2/2 would be the same as a whole.

Ordering and Equivalence

Like most of the members of her group, the clearest deficiencies in Tes's concept of fraction were revealed in her difficulties with ordering and equivalence.

(Interview 1, question 12)

Int: Can you tell me any fraction that would be less than 1/2?

Tes: No, one whole, but --.

Int: If you look at the pictures you've drawn, are any of those less than 1/2?

Tes: [Scrutinizes paper; long pause.] I'd say this

one. [Points to 1/3].

Int: Okay, what is it about 1/3 that let's you know it's less than 1/2?

Tes: I think I was wrong.

(Interview 1, question 13)

Int: Could you give me an example of a fraction that's greater than 1/2?

Tes: [Pause.] Hm-um.

(Interview 1, question 14)

Int: Let's go back to 1/2 and 1/3. Do you think there are any fractions between 1/3 and 1/2?

Tes: One whole.

Tes utilized a whole number approach to evaluate the relative size of fractions. Asked to compare two fractions she consistently compared numerators and denominators as separate pairs of whole numbers.

(Interview 4, question 10)

Qstn 10: Which is greater: 1/3 or 1/2

Tes: [Writes 1/3].

Int: How come you decided that?

Tes: Cause the three is larger than that [two].

(Interview 4, question 11)

Qstn 11: Which is greater 2/5 or 5/8?

Tes: [Writes 5/8].

Int: And why did you decide 5/8?

Tes: Because both of these are larger than that.

Int: So you're looking at each number separately?

Tes: Um-hm.

Operations on Fractions

Tes's procedural knowledge was displayed mainly during the fourth interview. Her error rate was very high on this interview largely because of whole number knowledge interference. Tes added and subtracted by dealing with numerator and denominators as separate pairs of whole numbers, just as she had dealt with them in ordering of fractions.

(Interview 4, question 14)

Qstn 14: Add 1/5 + 2/5

Tes: Um, 3/10.

Int: I think I can see what you did, but tell me anyway.

Tes: I added the 1 and the 2, and 5 and 5.

Only in multiplication did her straight-across whole number procedures result in "correct" answers. Tes's interviews provided a cogent reminder that right answers do not always reflect adequate conceptual understanding, and that wrong answers often reflect thinking that may be flawed but is generally consistent.

Nel

Nel participated in the first three of the four interviews. Of the five participants in the group of less-successful math students, Nel seemed to have the most complete concept of fraction. She gave a brief definition of what a fraction is, similar to many given by students in the contrast group.

Nel: It's part of a whole.

Nel successfully drew and named 1/4, 1/3, 2/3, and 5/8. Her responses in comparing the lengths of various Cuisenaire rods to a 12-rod unit were also essentially indistinguishable from those of many of the more successful math students. Her rate of errors (20 in all three interviews in which she took part) was considerably lower than her group's mean rate of 66 errors per interviewee over four interview sessions.

One series of episodes that revealed weaknesses in Nel's concept of fraction was the partitioning task sequence in which a dish of candies, some with red and some with yellow wrappers, was used as a physical referent. Nel had difficulty identifying 8 of 12 candies as 2/3 of the candy, and was unwilling or unable to determine what 1/3 of 12 red candies would be. These results were similar to those of the other members in Nel's group.

Ordering and Equivalence

Although Nel on occasion displayed more conceptual knowledge than most of the other members of her group, her fraction concept was not as complete as that of most members of the high-performing group. One area of questioning within the three interviews in which Nel took part revealed a serious chink in her concept of fraction. In the sections related to ordering of the rational numbers in interviews one and three, Nel seemed to shut down. Whether this was uncharted territory for her and she was refusing to take any risks, or whether Nel could not connect less-than and

greater-than questions with her available fraction schemas could only be conjecture. However, one sequence of two episodes points to the latter.

(Interview 1, question 12)

Int: Can you tell me a fraction that would be less than 1/2?

Nel: No.

Int: You were looking at your pictures you've drawn, are there any fractions there that are less than a half? (One of her three diagrams was a circle with 1/4 shaded.)

Nel: No.

(Interview 1, question 13)

Int: Could you give me an example of a fraction that's greater than 1/2?

Nel: 1/3.

Int: What is it about 1/3 that lets you know that it's greater than 1/2?

Nel: Cause it's like 1/2 with one more little block in it.

Nel's final response in the above episode suggests a whole number definition of fraction symbols with numerator and denominator each representing so many blocks, much like that held by other members of her group.

One other error in Nel's verbal reports linked her thinking with that of her cohorts. In subdivision, when asked what 3 green pieces (3/4) would be, Nel first responded 1/3. As previously indicated, this same whole

number based response -- noting that one piece had been removed and three pieces remained, so the resulting fraction must be 1/3 -- was found in all other protocols of members of Nel's group.

Operations on Fractions

Because Nel did not participate in the final interview in which problems were presented in symbol form, it is difficult to speak to her procedural knowledge. Most interviewees displayed more procedural knowledge in the fourth interview than anywhere else. In three interviews Nel made only one statement categorized as procedure, and it was connected with an error, involving an incomplete method for comparing fractions.

Nel's relatively strong grasp of fraction concepts in some situations, especially when working with physical referents, made her stand out as somewhat unusual in her group. Viewing tallies of responses made in order to contrast successful and less successful math students' protocols, Nel's tally mark several times stood apart from those of her cohorts as one correct among four incorrect answers. Success in math class hinges on many factors. Nel's interviews which suggested greater knowledge than she may be revealing in her math classes presented a reminder of that fact.

Summary

Concept of Fraction

Definitions. A simple definition of what a fraction is was stated by only two of the five participants in this group. A third gave an example of 1/2, another gave an explanation of a division algorithm, and the remaining member indicated that she could not explain what a fraction is.

Connections with experiences. The fractions concepts of people in this group generally were not interconnected with their experiences outside of school. Only after prompting did four members of this group each produced one example of fraction use related to cooking or sharing food. Dan generated three examples, all related to cooking and sharing food. These results contrast sharply with those of the higher-performing group of math students, who each indicated several connections between their understanding of fractions and their everyday experiences.

Meanings of symbols. Four of the five members of this group drew at least one diagram of a fraction, linking the parts of the diagram with the meanings of the numerator and denominator of the fraction at hand. However, thirds were especially troublesome for this group. Dan and Bea tried without success to diagram 1/3, and Ann could not illustrate 2/3. The use of repeated halving was a common schema for subdividing circles, and that schema may have interfered with attempts to create thirds.

Concept of 1/2. Because half and halving, and perhaps to a lesser degree quarters or fourths, are experienced in everyday life in ways that few other fractions are, it is not surprising that the concept of 1/2 was apparently an intuitive concept even for for students experiencing difficulty with mathematics. One-half was the one fraction that all participants in this group named successfully both in settings of fractions as comparisons of length, and in subdivision of continuous quantities. Two of the five named 1/2 in the context of partitioning of discrete sets as well, even though the cardinal numbers of the sets conflicted with the set-subset information in the task. Nel and Bea recognized 1/2 in the two subsets of eight candies each. In the same episode, other responses included 1/8 and 8/8, reflecting the interference of the whole cardinal number information. Ann's correct response of 8/16 similarly reflected whole number interference, because her attempts to "reduce" or rename the fraction as 1/2 were unsuccessful.

Ordering and Equivalence

Some of the most striking contrasts between groups occurred in the area of ordering and equivalence. The members of this group of five were unable to name fractions greater than or less than 1/2. In making less than or greater than judgments, all five displayed whole number knowledge interference, most often making decisions based on separate whole number values of numerators and/or denominators.

Operations on Fractions

Considering operations with fractions, two major patterns emerged from the protocols of the five members of the lower-performing math group. The first was a tendency to utilize highly practiced syntactic algorithms on every symbolically presented problem, regardless of the relative efficiency of the approach, and even though the same concrete materials were available that had been used to solve verbally-presented problems with the same numerical content in previous interviews. Dan, Bea and on occasion Ann all displayed this behavior, and both Dan and Bea overgeneralized subprocedures such as finding common denominators to multiplication and/or division problems. This pattern of behavior was not unique to members of this group, however.

The second pattern was a continuation of the tendency to treat numerators and denominators as separate whole numbers. Both Tes and Ann completed addition problems by adding numerators and adding denominators separately. When feasible, subtraction and division were carried out in the same manner, while problems such as 1/2 - 3/8 or 3 ÷ 1/2 resulted in such responses as, "I don't know how to do that one." This pattern of whole number interference in operations on fractions was unique to this group, not appearing in any of the protocols of the contrast group.

CHAPTER 6

Results: Case Studies 6-8

In this study, the three major areas of difference noted between the group of students who were performing well in math and the group of students who were performing at below average levels were their rates of production of statements in the categories of conceptual knowledge, automaticity, and errors. Within the high-performing group of math students, Tad, Ron and Mel were the three members who exhibited a pattern, when compared with their other four compatriots, of producing slightly fewer statements reflecting conceptual knowledge (mean rate of 71.7 statements per participant versus 84.3) or automaticity (19.7 versus 28.5) and at the same time generating somewhat higher rates of errors (18.3 versus 8.5). The case studies of Tad, Ron and Mel follow.

Tad

Concept of Fraction

Tad's conceptual knowledge of fractions included a simple definition, "part of a number." A part-whole concept was also indicated by his explanations and diagrams of 1/3, 3/4, 5/8, and 2/3. His examples like those of many of his cohorts were either shading segments of a subdivided circle or rectangle, or part-whole examples related to dividing food.

(Interview 1, question 7)

Int: (Posing as an uninformed second grader) Could you tell me what 3/4 means?

Tad: [Points to drawing of circle subdivided into fourths; shades 3/4.] Say you had these right here, and how many of em's shaded in? Three. [Writes 3.] And how many is there? There's four. [Writes 4 under 3 in fraction form.]

Int: Explain one more for me. How about 5/8?

Tad: Say you have an orange cut in eight pieces, and you eat five of em. How many of em did you eat? You ate five and there was eight so that would be 5--, 5/8.

Ordering and Equivalence

While his basic definition of fractions seemed sound, and his conceptual knowledge included the ability to interpret fraction symbols, Tad did not appear to have a sense of the ordering of fractions. Even though physical referents were present and had been used to answer preceding questions during the same interview, Tad's whole number knowledge interfered with his conceptual knowledge of fractions in making ordering decisions. He compared numerators and denominators as two separate pairs of whole numbers in making judgments of relative size.

(Interview 2, question 17)

Int: Which is greater 1/3 or 1/2?

Tad: Mm, 1/3.

Int: Can you tell me why 1/3 is greater than 1/2?

Tad: Let's see, two is smaller than three.

Int: (18) Which is greater 2/5 or 5/8?

Tad: 5/8

Int: How come?

Tad: Cause five is bigger than two and eight is bigger than five.

Int: So you're comparing the numerators and denominators?

Tad: Yeah.

However, in interview four Tad answered the same questions posed in symbol form correctly by following a syntactic procedure requiring common denominators. It is clear that this syntactic procedure was not connected to his conceptual knowledge.

Tad was also unable to name a fraction less than 1/2, and when asked to cite a fraction greater than 1/2, he listed 3/4, 1/4, and 5/8. The whole number comparison rule would predict the inclusion of 1/4 as a possible error in this situation, because four is greater than two. Asked to list any fractions between 1/2 and 3/4, Tad listed 2/3 and 2/5, apparently again using a whole number rule, but considering only the whole number value of the numerators.

Operations on Fractions

Operations on fractions were generally not problematic for Tad. His approach was flexible in that he sometimes used physical referents, in other episodes he produced answers without any apparent working-out, and when needed, he utilized syntactic procedures, writing problems in column

form, changing to common denominators, and adding or subtracting. However, in one episode while considering multiplication of fractions in terms of pizzas, Tad developed a unique problem. He thought of the pizza being cut into a particular number of pieces unrelated to the denominators of the fractions in question.

(Interview 3, question 10)

Int: What if you had 1/6 of a pizza, four times?

Tad: One-sixth of a pizza, four times. How many pieces are in the pizza?

Int: You can think about it any way you want to.

Tad: If there was twelve, it would be two; and if there was ten, it would be two and 2/3.

Int: What are you doing in your head when you're thinking about that?

Tad: I'm just, I'm dividing by twelve, or I'm dividing by ten; 24 divided by ten would be two and 4/10.

Tad did not use this problematic approach in considering division and was able to use the physical referents or procedural knowledge to successfully solve division questions. On interview four, Tad solved both multiplication and division problems successfully, following syntactic procedures. Like his cohorts, Tad executed subprocedures, such as inverting the divisor in division, without any conceptual sense of why or of what happens in such processes.

Mel

Mel verbalized his part-whole concept of fraction in a succinct definition:

Mel: I'd say a piece of something.

He also communicated his understanding of fraction symbols by drawing part-whole diagrams and giving part-whole explanations or examples of 1/3, 3/4, 5/8 and 2/3. Mel produced several examples of how fractions might be used in real life, including one unique example, batting averages, which he explained quite well.

(Interview 1, question 10)

Int: Could you explain how fractions work in batting averages? (Mel had introduced batting averages as an example of fraction use.)

Mel: It's how many, it's like if you, if your batting average is about two-fifty [250], then it'd be like 1/4. And for every four times you, um, for every four times you go up to bat you hit one ball.

Int: So 250 out of 1000?

Mel: It's decimal numbers.

Int: So "two-fifty" is 250/1000?

Mel: Yeah.

Mel exhibited no patterns of difficulty in considering fractions as comparisons of length, subdivision of continuous quantities, or partitioning of discrete sets, and he used physical referents to make sense of both judgments of equivalence or ordering and operations on fractions in all three settings.

Ordering and Equivalence

Mel stated a rule for comparison of unit fractions that he later modified in the face of non-unit fraction comparisons.

(Interview 1, question 12)

Int: Can you tell me a fraction that would be less than 1/2?

Mel: [Pause.] A third.

Int: What lets you know that 1/3 is less than 1/2?

Mel: Our math teacher told us that the larger the bottom number, the smaller the fraction.

Int: Does that ever not work, or does that always seem to hold true?

Mel: Always works for me.

(Interview 1, question 13)

Int: How about a fraction that's greater than 1/2?

Mel: Can't think of any.

Int: How about when you think about the pictures you've drawn?

Mel: Oh, yeah, there are kind of; I can think of some that are larger than half, like 2/3 or 5/6.

Int: Why are those greater if the bottom number is bigger? You told me before that if the bottom number is bigger, then the fraction is smaller.

Mel: The reason that is is because the top number is larger.

Int: So you have to think about both numbers at the same time?

Mel: Mm-hm.

Mel's sense of the relative size of fractions included a rule that Ron also expressed -- if a numerator and denominator are close together in size, the fraction will be "large", or close to one.

(Interview 1, question 15)

Int: Tell me a fraction that's close to one.

Mel: [Pause.] 7/8, um, 9/10.

Int: What is it about those numbers that lets you know they're close to one?

Mel: Because the top number is one under the bottom number.

Int: What does that tell you about the fraction?

Mel: That it's pretty close to a whole.

This rule was conceptually linked for Mel, as was evident in the remainder of the same episode.

Int: Which is closer to a whole, 7/8 or 9/10?

Mel: 9/10.

Int: Why?

Mel: Because the bottom number, the ten is larger than eight, so that makes the 9/10, um makes a tenth smaller than the eighths, so 9/10 is closer to a whole.

Operations on Fractions

Similarly with many other participants, Mel generally approached fraction problems in the fourth interview, problems presented in symbol form, with automaticity or with algorithms. He was more flexible in his approach than some

of his cohorts in that he utilized the available physical materials briefly in two of the 28 episodes in this interview. His protocols, though, reveal high numbers of procedure statements and a relatively suppressed rate of conceptual statements during this interview. This same pattern was evident to varying degrees in the protocols of all interviewees.

Mel overgeneralized the common denominator subprocedure of addition to operations of multiplication and division resulting in his only concentrated pattern of errors, and supporting the argument that errors may occur when syntactic procedures are applied without semantic understanding.

Ron

Concept of Fraction

Ron's concept of fraction was not expressed in the form of a definition. However, he did understand the meaning of fraction symbols, at least in terms of translating into diagram form. Ron drew diagrams of 1/4, 2/3 and 5/8. Ron had no difficulty considering the meaning of fractions as comparison of length or subdivision of continuous quantity.

The one area of questioning which revealed a deficiency in Ron's concept of fraction was partitioning of discrete sets.

(Interview 3, question 15)

Int: What part of this bowl of candies is pink (8 of 12)?

Ron: Uh, l'see, 3/8.

Int: How did you decide that?

Ron: Uh, well, a minute ago half of it was --. It's divide; you've got four here and that'd be one eighth, and this'd be two eighths and this'd be three eighths, [separating candies into three groups of 4].

Int: So you said what part of the candies was pink?

Ron: 2/8.

Partitioning presents a difficult cognitive task, because students must think in terms of numbers of sets while confronted with conflicting whole number information from the cardinal numbers of the sets and subsets. Ron's use of the denominator eight reflected the cardinal number of the subset of pink candies, even though he had clearly divided the candies into three sets.

Ordering and Equivalence

Like Pat, Ron used his concept of 1/2 to easily construct numbers greater than, less than, and equal to 1/2.

(Interview 1, question 13)

Int: Could you give me an example of a fraction that's greater than 1/2?

Ron: 6/8, 6/10, 2/3, 3/4.

Int: How do you know that those are greater than 1/2?

Ron: I'm thinking of something that's half of the number and then going up.

For fractions other than 1/2, Ron used a syntactic method unique among the participants for making greater than and less than judgments. He used cross-multiplication, placing

the cross product beside the numerators of each fraction in question, and then comparing the size of the resultant products. However, he did not use this method in total isolation from his conceptual knowledge as apparent in the following episode:

(Interview 4, question 12)

Qstn 12: Which is greater: 5/8 or 3/4

Ron: Uh, 3/4.

Int: And did you do the same thing again? (Ron had explained his cross-multiplication procedure in the previous episode.)

Ron: Yup. Or sometimes you can tell by your denominator in some of 'em.

Int: Is that what you did on this one, or did you do the cross-multiply thing?

Ron: Well, I did it first, and then I did the cross-multiply thing to check.

Int: So, what did you do first, before you cross multiplied?

Ron: Um, I looked at the denominator, and then I looked at how close your numerator was to your denominator. And then to make sure I used cross-multiply.

Int: So if the numerator is close to the denominator, what does that tell you?

Ron: That most of the time it'll be greater.

Operations on Fractions

Throughout the sections of interviews related to subdivision and comparison, it was obvious that Ron could use physical referents to make sense of operations on fractions. However, when fraction problems were presented in symbol form during the fourth interview, even though all of the same referents were available, Ron consistently chose to utilize syntactic algorithms, which were apparently disconnected from concepts.

(Interview 4, question 17)

Qstn 16: Add: 1/2 +1/3

Ron: L'see, you would have to change your denominator to the same, to 6. [Writes in column form]. Two will go into six three times, three times one is three; three will go into six two times, three plus two is 5/6.

(Interview 4, question 20)

Qstn 20: Subtract: 1/2 - 3/8

Ron: L'see. [Writes in column form. Changes to 4/8 - 3/8.] 1/8.

Int: Tell me about these where you changed the denominators, how come you have to do that?

Ron: Cause your denominator has to be the same number, where it'll come out equally.

The problems in interview four for which Ron did not use algorithms were mainly problems which elicited automaticity, those for which he produced an answer with no apparent time for thought.

(Interview 4, question 18)

Qstn 18: Subtract: 2/3 - 1/3

Ron: 1/3.

Int: Did you have to think about that at all?

Ron: No. [Shakes head; smiles.]

Subprocedure of Regrouping. Ron executed the typical addition and subtraction algorithm -- writing in column form, changing to common denominators and adding or subtracting numerators -- with aplomb, until the subprocedure of regrouping was required.

(Interview 4, question 21)

Qstn 21: Subtract: 2 - 2/5

Ron: Oh, let's see, you can think of it as one and, I think it's 9/9. Minus 2/5. [Writes in column form; 1 9/9 - 2/5.] That you would have to change to, would be 25, 30, no, 35, 40, 45. [Changes to 1 45/45 - 18/45.]

Int: I'm curious. Tell me when you started out, why did you decide that you should call that one and 9/9?

Ron: So, uh; you can't subtract a fraction from a whole number so you have to change your fra--, uh, your whole number to a fraction.

Int: So you just chose 9/9 as a place to start from?

Ron: Yeah.

Ron overgeneralized the subtraction procedure with the subprocedural error intact, to the multiplication problem 4 X 1/5, and to the division problem 3 ÷ 1/2. He also overgeneralized the more typical column addition algorithm

to almost all of the remaining multiplication and division problems in the interview. The resulting errors were produced by two very consistent procedural and subprocedural bugs.

Summary

Concept of Fraction

This group of three, although similar in rates of responding in categories of conceptual knowledge, automaticity and error, presented a mixed bag in terms of the nature of their individual responses. Seldom did all three young men present a unified front in approaching any aspect of the interviews.

Definition. Asked to explain what a fraction is, a simple part-whole definition of fraction was proposed by Tad and Mel. Ron, instead, gave the example of "reducing" a whole number to halves or fourths.

Meaning of symbols. Asked to explain the meaning of particular fractions such as 1/3 and 3/4, Tad and Mel gave part-whole examples of sharing food and/or subdividing some unidentified quantity (eg. "You divide something in three pieces and you get one of those three pieces.") Ron, on the other hand, explained the relative size of 1/3 and 3/4 as compared with 1/2 and 1, respectively.

Contexts of comparing, subdividing, and partitioning. Tad and Mel were comfortable discussing the meaning of fractions in all three settings of comparison of length, subdivision of continuous quantities, and partitioning of

discrete sets. Ron, however, displayed whole number interference in the partitioning setting, with the cardinal number of the subsets masking the number of sets Ron had created.

Ordering and Equivalence

All three of the participants described in this chapter had different approaches to making less-than, greater-than, and equal-to judgments. Tad's protocols revealed whole number interference as he considered the numerators and denominators as two separate pairs of whole numbers, an atypical approach among his six cohorts in the mathematically more proficient group, and more typical of members in the contrast group. Ordering questions elicited the highest concentration of errors from Tad in interviews one through three. However, when the same ordering questions were posed in symbol form in interview four, Tad used a syntactic procedure which produced consistently correct answers. Obviously this procedure was not connected with Tad's conceptual sense of the ordering of fractions.

Mel had the most conceptual approach to making ordering decisions, using two cognitive rules which functioned in concert. If a fraction were a unit fraction, Mel used a rule that a larger denominator indicates a smaller fraction. But he was also aware, when considering non-unit fractions, that both numerator and denominator had to be accounted for, with the denominator indicating the size of the pieces and the numerator the number of pieces.

Ron used a syntactic cross-multiplication procedure to judge the relative size of fractions. However, this procedure did not appear to be isolated from Ron's conceptual knowledge. Both Ron and Mel expressed awareness that fractions whose numerator and denominator were close together would be large fractions, that is, close to one, and Ron used this knowledge in conjunction with his cross-multiplication algorithm.

Operations on Fractions

In approaching operations on fractions as subdivisions of continuous quantities, all three students used the physical referents available to make sense of the operations of addition and subtraction. Mel also dealt successfully with operations on fractions as partitioning of discrete sets. Tad and Ron, however, both indicated whole number interference from the cardinal numbers of sets and subsets when asked to consider operations on fractions in the partitioning setting.

When problems involving addition and subtraction of fractions were presented in symbol form, all three participants generated more procedure statements and fewer conceptual statements than in previous interviews, as did all participants. However, Tad and Mel were somewhat flexible in their approaches to addition and subtraction, sometimes answering with automaticity, infrequently referring to physical referents, and often utilizing syntactic algorithms. Ron, in contrast was more rigid in his application of the column addition algorithm. Both Mel

and Ron overgeneralized the addition algorithm including, the common denominator subprocedure, to multiplication and/or division, resulting in their largest concentrations of errors in any area of the interviews.

Although this group was similar in rates of particular categories of responses, their individual differences in actual mathematical behaviors were striking.

CHAPTER 7

Results: Case studies 9-12

Among the seven students in the high-performing group, the four students who displayed the highest levels of conceptual knowledge coupled with relatively high levels of automaticity and who produced the fewest error statements were Sam, Lee, Jay and Pat. Their case studies follow.

Sam

Concept of Fraction

Sam's conceptual knowledge of fractions included a clear sense of what fraction symbols mean. Although Sam did not give a verbal definition of what a fraction is, he did give examples, explain, and/or draw diagrams of 1/3, 3/4, 2/3, and 5/8.

His conceptual knowledge of fractions was interconnected with his own experiences, as illustrated by his examples of the ways fractions might be used. In discussing one example, Sam also revealed his understanding of the relative nature of the value of a fraction.

(Interview 1, question 11)

Int: How about telling time; have you ever used fractions when you tell time?

Sam: Yeah, like a quarter past. That's 1/4 past.

Int: What does that mean, one-quarter or 1/4 past?

Sam: Um, that's 1/4 of an hour, 1/4 of the 60 minutes.

Int: So how many minutes would that be?

Sam: Uh, 15.

Int: That confuses me a little, because, tell me about 1/4 of a dollar, what does that mean?

Sam: A quarter is 25 cents; that's 1/4 of 100. And this [time] is 1/4 of 60.

Only when working with comparison of lengths did Sam's responses reveal any weakness in his concept of fraction. Comparing lengths of paper strips, he once reversed the sense of the situation referring to a strip 2/3 the length of the other as 1 1/2 of the other.

Ordering and Equivalence

Sam's concept of fraction extended to equivalence and ordering of fractions. At times he used physical referents to make judgments, especially when working with subdivision of circles.

(Interview 3, question 14)

Int: How many fourths are the same as 6/8?

Sam: [Places 6 eighths on circle.] That'd be 3/4.

Int: How'd you know that?

Sam: Well, because the fourths each take up two of these -- these were the fourths -- the fourths each take up two of these. [Places 3/4 on top of 6/8.]

In other episodes, Sam called upon his procedural knowledge.

(Interview 4, question 7)

Qstn 7: Write another way: 9/3 =

Sam: Um, 3.

Int: How did you decide that?

Sam: Well, it's a --; what do you call it? [Laughs], where the big number is over the little number, I'll say that. And the 3 went into the 9, 3 times, so it'd be 3.

However, in comparing lengths of rods as a source of fractions, even though he was able to construct equivalent lengths, Sam did not understand the question when asked to give another name for the fractions he had created.

(Interview 2, question 8)

Int: How many ways could you build 1/2 of a J-rod? (A J-rod was Sam's name for the 12-rod unit.)

Sam: [Places 2 fourths beside 1/2.] L'see, two of those or three of these. [Places 3 sixths beside 2/4.]

Int: What would you call each of the fractions that you've built?

Sam: They're each 1/2.

Int: What else could you call them?

Sam: Um, what do you mean?

Int: What other name could you give to the red ones?

Sam: I don't know; I can't see what you're asking.

Automaticity

Automaticity was displayed in Sam's responses when he fired back answers with no apparent pause for thought.

(Interview 2, question 17)

Int: Which is greater, 1/3 or 1/2?

Sam: 1/2.

Int: How did you know that?

Sam: Well, because you think, like, here'd be 1/2 and

l'see, where's 1/3? That would be 1/3, so you know that'd be greater [1/2].

Int: So, you just knew what they looked like?

Sam: Yeah.

Operations on Fractions

Working with the symbolically presented problems in interview four, Sam made no errors. He was flexible in his use of both syntactic procedures and conceptual knowledge. At times he would use typical column form addition or subtraction; on other occasions he would simply produce an answer without paperwork. He did not, however, employ physical referents during this interview.

The strength of Sam's conceptual knowledge and his ability to make use of it was apparent in his rejoinder to what was probably the most challenging question in the series of interviews, a partitioning question at the end of interview three.

(Interview 3, question 19)

Int: If this (9 pieces of candy) is 3/5 of a box of candy, what would the full box of candy hold?

Sam: [Counts; separates into 3 groups of 3]. It would be --; 3/5; [pause.] Um, 15.

Int: I can see what you did, but tell me about it.

Sam: Well, I kinda --, I split this into, since you said 3/5, I split this into three groups, and there's three in each group. And you said 3/5 so that'd take two more groups and I just took the nine and added two more groups, which would be six, to it.

Lee

Concept of Fraction

Lee's conceptual knowledge of fractions included a basic sense of what a fraction is and what fraction symbols mean. Lee defined a fraction as follows:

Lee: A unit of a whole number.

Int: What does "unit" mean to you?

Lee: A part or region.

Lee also described, diagramed and/or gave examples of 2/5, 1/3, 3/4, 2/3 and 5/8.

The interconnected nature of Lee's conceptual knowledge was exhibited not only in her several examples of uses of fractions, but also in her realization during our discussion that the note values she had been working with in band were really fractions with the same relationships between values. However, she did get confused and lost track of those relationships as she described her band director's chart.

(Interview 1, question 11)

Lee: And in band -- I just now thought of that -- we have like eighth notes, quarter notes, half notes, whole notes, sixteenth notes and thirty-second notes.

Int: How are those related to fractions?

Lee: Um, [pause].

Int: Tell me about an eighth note; what does that mean?

Lee: It's 1/2 of a quarter note. She [band director] always did this chart; it started out with a whole note. It takes two half notes to make a quarter

note; then it take four quarter notes to make two half notes; then it takes eight sixteenth notes to equal four quarter notes and then so on. It just keeps multiplying.

Ordering and Equivalence

Lee's fraction concept included a sense of equivalence and ordering. However, like many of her cohorts, she sometimes depended on syntactic procedures that seemed to be disconnected from concepts.

(Interview 4, question 5)

Qstn 5: 1/2 = ?/8

Lee: Five, no, four [smiles].

Int: How did you do that?

Lee: Two times four is eight; one times --; you have to bring the 4 up; one times 4 is 4.

Int: Why do you have to multiply the top one by four too?

Lee: You just do.

Operations on Fractions

Like Sam, in dealing with symbolically presented problems, Lee was flexible in her choice of approaches. If she could use conceptual knowledge to think her way directly to an answer she would do so.

(Interview 4, question 25)

Qstn 25: Divide: 3 ÷ 1/2

Lee: [Writes 6.]

Int: How did you decide that?

Lee: There's two halves in one of those, and you've got

three of those.

Twice Lee called upon physical referents to explain her answers and once she used them to make a greater-than judgment. Her discussion indicated that she could have taken a procedural approach but found the physical referents easier in this case.

(Interview 4, question 11)

Qstn 11: Which is greater: 2/5 or 5/8

Lee: [Places 2/5 on table; 4/8 on top of 2/5.] 5/8.

Int: Was there another way you could have decided that?

Lee: Yeah, you could have said; you could have found a least common multiple which would be 40, and eight times two, sixteen --.

Int: Okay.

On other occasions Lee used syntactic procedural knowledge, but she did not appear to be tied to any algorithm. Only once did she overgeneralize procedures, on a division problem which she worked in her head, employing common denominators and keeping her denominator as in addition.

Like Sam, Lee's rate of errors was very low; she generated only nine error statements in all four interviews, while her rates of conceptual knowledge and automaticity statements were high.

Jay

Jay communicated his conceptual knowledge of fractions well, generating the highest number of conceptual knowledge statements of any of the twelve participants. He proposed a

fairly complete part-whole definition of a fraction as follows:

Jay: Not a whole number, but part of a whole number, with the whole number on the bottom and what part of it it is on the top.

Jay also diagramed, explained, or gave examples of the fractions 1/2, 1/3, 3/4, 5/8, and 2/3.

Relating his concept of fraction to his own experiences, Jay gave numerous examples of ways fractions are used. He seemed genuinely surprised at how many ways he really used fractions outside of school. Following two prompts about sharing and measuring, Jay responded with wide eyes, "You're bringing up all these things I never thought of!"

Relative nature of fraction values. In discussing an example of telling time, Jay revealed the flexibility of his concept of fraction, noting that 1/4 can be 1/4 of anything, therefore representing different values.

(Interview 1, question 11)

Int: Have you ever heard anyone use fractions in telling time?

Jay: Yeah, like it's a quarter till noon, which is a fourth.

Int: So, a quarter till means a fourth of what?

Jay: Of anything. In time, since there's 60 minutes in an hour a quarter is 15 minutes. In a dollar, a quarter is 25 cents. A quarter moon is 1/4 of the moon.

Int: So it depends on what you're talking about a quarter of.

Jay: Yeah.

Density of fractions. Jay was one of the only two participants who indicated a conceptual understanding of the density of fractional numbers.

Int: How many fractions do you think you could squeeze in between 1/2 and 6/10? (Both 1/2 and 6/10 had been generated by Jay as examples earlier in the episode.)

Jay: [Pause.] Oh, there'd probably be quite a few, 150, or something. Or more than that, because numbers are endless.

Similarly, Jay went well beyond the typical examples of fractions given by other participants when he cited 999/1000, and 999,999/1,000,000 as fractions that are close to one. Other participants cited fractions like 7/8, 3/4, 5/6, 11/12, 15/16, and once 99/100.

Metacognition

Jay's metacognitive skills were remarkable, in that all of the errors he made in the first three interviews he caught and corrected on his own. In one case, the interview had proceeded on to the next question when Jay recognized what he had done on the previous question, and he asked to go back in order to repair his error. He was also aware in many cases of his own use of automaticity in producing answers.

(Interview 2, question 11)

Int: What would 9/3 be the same as?

Jay: Three.

Int: How'd you know that?

Jay: I'm not quite sure. Uh, cause it takes three to equal one, and if you had nine of em, that's three times --; yeah, it would --; I'm not, uh --.

Int: You were explaining that very well.

Jay: I just do it in my head.

Int: What did you do in your head?

Jay: I just see the answer.

Equivalence and Ordering

Jay displayed fairly complete conceptual knowledge of equivalence and ordering of fractions. He seemed equally comfortable dealing with fractions as comparisons of length, subdivision of continuous quantity, and partitioning of discrete sets.

Operations on Fractions

Only in dealing with symbolically presented problems did Jay sometimes use procedures which were apparently syntactic and not interconnected with conceptual knowledge; only then did he produce errors which remained uncorrected.

(Interview 4, question 22)

Qstn 22: Multiply: 1/2 X 2/3.

Jay: [Writes 1/2 beside 2/3.] I don't remember what to do. [Pause; erases 2/3; writes 3/2; pause]

Int: Want to skip that one?

Jay: 3/4. Is that right?

Int: How'd you get that?

Jay: Well, what I did --, I think I remember; it was something like that. I turned the last number over, and then I multiplied those.

During the remainder of the fourth interview, Jay seemed somewhat inflexible in his choice of approaches. He did not utilize the available concrete materials, and when an answer was not immediately apparent, he searched for the appropriate syntactic procedure to follow, sometimes expressing uncertainty as to the correctness of the subprocedures involved.

Pat

Concept of Fraction

Pat's conceptual knowledge of fractions was strong and highly interconnected with experiences and with procedures. She defined a fraction as:

Pat: . . . a way of like dividing things up into portions and telling how much is given or used or something like that.

Asked to explain the fractions 1/3, 4/8, and 3/4, Pat told the most complete stories of any participant, explaining what was going to be divided, why she would want to divide it up, and who was going to get how many pieces, and she often concluded her example with a summarizing statement.

Pat moved easily from considering fractions as comparisons of length to subdivision of continuous quantities, to partitioning of discrete sets. None of Pat's

few errors revealed any deficiency in her concept of fraction. Like Jay, Pat caught most of her own errors and corrected them, suggesting metacognitive monitoring.

Ordering and Equivalence

Pat had a well-integrated conceptual schema for 1/2 which she utilized in ordering. She could easily construct fractions less than or greater than 1/2, a process which she explained as follows:

(Interview 1, question 12)

Pat: . . . You could take, like if I had 16, and made it into 16 pieces, I'd have to pick a number that's less than 8 cause 8's half of 16.

(Interview 1, question 13)

Pat: Cause when you divide your fraction, or your whole number into half, it [the numerator] has to be greater than half.

Pat was equally comfortable with equivalence.

(Interview 2, question 7)

Int: Could you build 1/3 (of a 12-rod unit) any other way?

Pat: Well, you could take 3 of these [places 3 1-rods beside 4-rod]. Maybe 4 of these. [Smiles, places fourth 1-rod beside 4-rod].

Int: What other name is there for that fraction besides 1/3?

Pat: 2/6 or 3/9 or 4/12.

Operations on Fractions

Like Jay, on the fourth interview Pat either displayed automaticity, producing answers with no apparent time for thought, or she talked her way through well-rehearsed algorithms. Even though her procedures were apparently not guided by concepts, Pat exhibited no problems in her use of procedural knowledge. Why-do-you-do-that type questions designed to search for connections between conceptual and procedural knowledge were generally answered by participants with an "I don't know." Pat was the only participant who made any attempt to explain why you multiply denominators but do not add them, or why you invert the divisor and then multiply when solving a division problem with fractions. However she fell short of making clear connections between concept and procedures.

(Interview 4, question 24)

Int: . . . how come you didn't add the denominators there but you did multiply the denominators here?

Pat: Well, cause, see, in multiplying you're getting like a whole new problem; here [adding] you're just getting a larger numerator; here [multiplying] you get the whole new problem.

Int: Does it make sense to you that you do that?

Pat: Yeah, it makes sense, it's just kinda hard to explain.

(Interview 4, question 27)

Int: Do you know, why is it that you flip one of those like that?

Pat: 'Cause I think it's kinda like cross-multiplication. Uh, and so you wouldn't have to go, you wouldn't have to cross multiply, you just go ahead and switch it so it makes it easier.

Pat took a procedural approach to three questions that her cohorts generally approached in a more conceptual manner. Pat had learned that "of" can be changed to a multiplication sign when working with fractions. So when asked, what is 1/2 of 16, 1/3 of 12, and 3/4 of 20, she talked through the subprocesses of putting in the multiplication sign, multiplying, and then dividing.

(Interview 4, question 28)

Qstn 28: What is 3/4 of 20?

Pat: Okay, change that to a multiplication sign, let's see, okay, 3 times 20, that's 60, over 4, and, let's see, four goes into 60. [Divides 60 by 4 using long division.] Okay, it's 15.

Summary

Concept of Fraction

Definition. Lee, Pat and Jay all gave part-whole definitions of what a fraction is. Lee and Jay gave less concrete definitions than other participants, stating that a fraction is a part of a number, rather than a part of something being divided up. Although Sam did not give a verbal definition of fraction, all four explained the meanings of fraction symbols. Three gave examples of food

being shared, while Lee, the only participant to refer to discrete sets, used pennies as examples. However, her subsets were unitary (eg. in 2/3 she described having 3 pennies with only two of them in her pocket), rather than multiples, making her examples cognitively similar to part-whole definitions.

Connections with experiences. In contrast with participants who were less successful in math, Sam, Lee, Pat and Jay all gave numerous examples of the possible uses of fractions, suggesting that their concepts of fraction were, at least to some degree, interconnected with their own life experiences.

Relativity and density of fractions. Two fairly sophisticated concepts -- that of the relative nature of the value of a fraction and that of the density of fractions -- were expressed by two members of this group. Both Sam and Jay explained that 1/4 could mean 15 minutes, 25 cents, or one of many other values. Jay indicated that there were probably more than 150 fractions between 1/2 and 6/10, because "numbers are endless," and Sam stated that there were "millions" of fractions less than 1/2.

Contexts of comparing, subdividing, and partitioning. All four of the people in this group were comfortable with discussing the meaning of fractions, order and equivalence of fractions, and operations on fractions in the settings of comparing lengths, of subdividing continuous quantities, and of partitioning discrete sets. Only one example of whole number interference was noted, in Sam's protocols during one

partitioning episode. Members of this group alone appeared to have integrated fraction and whole number knowledge completely enough to avoid such interference entirely.

Ordering and Equivalence

Pat, Sam, Lee and Jay all utilized their strong concept of 1/2 and related knowledge to produce fractions less than, greater than or equal to 1/2. All used physical referents to discuss equivalent fractions in the subdivision setting, and all produced equivalent fractions in the context of comparing lengths, although Sam, in one episode, did not understand a request to state another name for the equivalent fractions he had created.

Operations on Fractions

Confronted with verbal problems requiring operations on fractions in all three contexts, all members of this group utilized physical referents. When operations problems were presented in symbol form, only Lee sometimes used the available physical referents. She was highly flexible in her approach to solving the symbolically presented problems in interview four, at times using her conceptual knowledge to think through a problem with or without physical referents, in other episodes responding with obvious automaticity, and at times utilizing syntactic procedures. Pat and Sam revealed some flexibility, choosing between syntactic procedures, thinking conceptually, or responding with automaticity, but not utilizing physical referents. Jay was the least flexible, either producing an answer with automaticity or using syntactic procedures.

The four members of this group generated very low levels of error statements, producing only seven to ten errors throughout all episodes of the four interviews, indicating high levels of successful reasoning not only concerning operations, but also discussing meanings and making judgments of ordering and equivalence. In addition, all four members of this group caught and corrected their own errors on more than one occasion. Their high levels of conceptual knowledge and automaticity statements coupled with their flexibility of approach and low levels of errors suggests that students whose procedures are informed by concepts may be more successful and certainly may be more efficient in dealing with fractions than students who must rely on syntactic procedures alone.

CHAPTER 8

Results: Analysis of Data

Participants' Knowledge of Fractions

The first research question asked: What is the content of sixth graders knowledge of fractions related to meaning of fraction and fraction symbols, ordering and equivalence of fractions, and operations on fractions? Each of these areas was explored through several episodes of questioning and task protocols. The results in each of these areas of observation are explicated below in four distinct sections and a summary.

Meaning/concept of Fraction

Verbal discussion of meaning. Asked to explain what a fraction is in general, to explain the meaning of several specific fractions, and to draw a picture of one or more fractions, participants largely described fractions in part-whole terms. There were a total of 57 responses in which students replied to the above questions. In the 57 instances, 49 were part-whole definitions or drawings, an 86.0% portion of the responses. Examples of typical part-whole definitions follow.

(Interview 1, question 4)

Pat: I'd tell him that a fraction is a way of like dividing things up into portions and telling how much is given or used or something like that.

(Interview 1, question 7)

Jay: If you had a pizza that had eight slices in it and you got five and your parents got three; then you would get 5/8 of that pizza.

A few participants explained fractions in terms of discrete items. This occurred in six of the 57 responses, or 10.5% of the time. In the six explanations of fraction meanings that employed discrete sets, participants described sets whose cardinal number was the same as the denominator of the fraction. This situation is cognitively similar to part-whole, especially if the parts of a subdivided whole are seen as wholes in themselves. For example 3/8 is 3 of 8 marbles, a discrete set example, just as it is 3 of 8 pieces of a pizza, a part-whole example. Explanations were cognitively similar in this sense.

Lee: Say you had three pennies; say, if you had one penny of those three in your pocket. Then you have 1/3 of your pennies that you had altogether.

Partitioning of sets of multiples, that is, with the cardinal number being a multiple of the denominator, (eg., 8 out of 20 is 2/5 because it is 2 sets-of-4 out of 5 sets-of-4), was never used by any student to illustrate or explain the meaning of a fraction.

A second departure from the typical part-whole approach was a relational approach explaining a fraction in terms of its relationship with the fraction 1/2 or with the whole number one. This approach was taken by only one participant on two occasions, constituting only 3.5% of the 57 responses

in this area. Ron described 1/3 as requiring "just a little bit more to make it half," and 3/4 as "almost a whole number."

Drawings produced by participants in connection with explanations of the meaning of fractions were typically circles, subdivided by repeated halving. Less often, rectangular strips subdivided with parallel lines were produced. Only one of the diagrams drawn in response to questions of meaning of fractions was other than part-whole, and the verbal discussion that accompanied it was a part-whole description.

Of all the drawings made in response to questions throughout the entire first interview, which included five episodes related to ordering, and one on equivalence, 95.6% were part-whole drawings. Only one participant, Ann, drew two diagrams each consisting of a row of separate circles representing the value of the denominator of the fraction in question, with the number of circles shaded corresponding to the numerator of the fraction.

Meaning of fractions as comparisons of length. In three episodes related to comparison of length, students demonstrated their understanding of fraction as comparison by recognizing paper strips being 1/2, 1/3, and 2/3 as long as a comparison strip. Success rates for recognizing and naming the fractions were 91.7%, 66.7%, and 41.7%, respectively. Both the strength of the concept of 1/2, and the students' greater ease with unit fractions as opposed to non-unit fractions were suggested by these results.

In six episodes with Cuisenaire rods, students similarly were asked to compare lengths. However, they had the advantage of having access to multiple rods of all sizes to aid in making relative size judgments. Students recognized and named 1/2, 1/3, 1/6, and 3/4 with 100%, 83.3%, 81.8%, and 80% rates of success. While 1/2 still emerged as the strongest concept among the participants in this case, the non-unit fraction 3/4 did not exceed the unit fractions in apparent difficulty. All of the students who successfully named the 3/4-rod used three 1/4-rods to make a comparison, or used one 1/4-rod to compete the length of the unit. These students were clearly able to make use of physical referents in the context of comparison of length to make sense of the meaning of the fractions in question and to assist in their judgments of the relative size of the non-unit fraction rod.

The final episode in this section asked students to consider a second unit of comparison and to talk about how it could be possible for one rod to be 1/4 of one thing but 1/6 of something else. Nine of the ten students queried, or 90%, indicated an understanding in this case that different length units could cause the same thing to be named differently in the world of fractions. However, the sense of the relativity of fraction values was only clearly expressed on any other occasion by two of the twelve participants, and, in contrast, two people expressed confusion about the relative nature of fractional values during other episodes.

Meaning of fractions as subdivisions. Questions related to the meaning of fractions as subdivisions of continuous quantities presented participants with the task of recognizing 1/2, 1/3, 1/4, 1/6, and 3/4 as pie-shaped pieces of circles. Rates of successful recognition and naming were 100 %, 91.1%, 100%, 100%, and 75%, respectively. Once more, the non-unit fraction, even though it was the relatively familiar 3/4, was not as easily named. Among participants, one common misnomer applied to 3/4 was 1/3, reflecting a whole-number interpretation of the meaning of fraction symbols, with the numerator representing the one missing piece, and the denominator, the three pieces remaining. Five of the twelve participants, 41.7%, initially answered 1/3 when asked what three of the green pieces (fourths) would be, and two of the five remained convinced that 1/3 was the appropriate name, even after probing questions and discussion.

Meaning of fractions as partitioning. Partitioning of discrete sets was cognitively the most difficult context for considering the meaning of fractions. In this setting, students must ignore the potential interference of the cardinal numbers of the sets and subsets and focus on the numbers of subsets as units. Asked to recognize and name 1/2 of a set of 16 candies and 2/3 of a set of 12, success rates were 91.7% and 58.3% respectively. One-half emerged again as a strong concept even in this difficult setting. Reasons interviewees gave for answering 1/2 included formal explanations such as:

(Interview 3, question 14)

Tad: Well, there's sixteen in all and eight's yellow, and eight over sixteen can go down to 1/2.

The more intuitive explanations were similar to the following:

(Interview 3, question 14)

Bea: 'Cause they're both even.

(Interview 3, question 14)

Tes: 'Cause there's eight on that side and eight over on that side.

Analysis of the errors related to both 1/2 of 16 and 2/3 of a set of 12 did suggest interference from the cardinal numbers of the subsets, coupled with a whole number interpretation of fraction symbols.

(Interview 3, question 14)

Dan: Yeah, I was fixing to say 1/8.

Int: Why were you about to say that?

Dan: Because I had eight right here and I had eight right there and to make a fraction, that would be it.

(Interview 3, question 15)

Int: What part of this bowl of candies is red? (8 of 12)

Bea: [Counts reds]. 1/8.

Int: And how did you decide that?

Bea: Well, there's eight pieces there.

Meaning related to fractions presented in symbol form included recognition of 2/3, 4/8, and 5/12 as diagrams of geometric figures with shaded segments. Success rates among

the twelve participants were 90.9%, 100%, and 90.9%. A whole number interpretation of the meaning of fraction symbols would predict the two errors which occurred: 2/1 for 2/3, and 5/7 for 5/12. In each case the numerator and denominator separately represented whole number counts of the areas shaded and the areas not shaded, or vice versa, rather than a ratio of part to whole.

Statement categories. Within all episodes related to meaning of fraction, the two most commonly occurring statements made by interviewees were in the categories of action statements which recorded an action taken or described, and conceptual statements, indicative of semantic understanding of fractions or fraction symbols. 35.7% of all statements made by interviewees were action statements. The students were clearly able to utilize physical referents and actions on those referents to make sense of the meaning of fractions, and they did so with similar frequencies. All participants generated nearly equal numbers of action statements with nine of twelve generating between 21 and 26 such statements, and the other 3 producing 17, 34, and 39 each.

The second most frequently occurring category in the meaning episodes was that of conceptual knowledge. Of all statements made related to the meaning of fractions, 29.1% were conceptual statements, suggesting that the sixth grade participants did have substantial conceptual knowledge of fractions, at least in the situations of discussing, representing, or interpreting representations of fractions

and fraction symbols. The total statements in each category and corresponding percents of total statements in all episodes related to the meaning of fractions are given in Table 8-1 below.

Ordering

Discussions of fraction size relative to 1/2 or one. Asked to give examples of fractions greater than, less than, and equal to 1/2, six students out of 12, 50%, were able to do so. There was an all or nothing pattern in these responses with those who were successful being able to complete all tasks, and those who were unsuccessful completing none. Although all participants in the study exhibited some concept of 1/2, especially in the contexts of comparing and subdivision, the six who were successful in

Table 8-1: Frequencies and Corresponding Percents of Statements in Meaning Episodes

Category	Number of statements	% of total statements
ER	83	9.5%
CN	254	29.1%
AT	30	3.4%
AC	312	35.7%
PR	10	1.1%
EX	17	1.9%
EV	39	4.5%
AN	129	14.8%
Total	874	100.0%

these tasks were able to extend that concept, using it to create like and unlike fractions. They demonstrated a sense of the relationship of numerator and denominator that the other students did not express.

(Interview 1, question 13)

Int: Could you give me an example of a fraction that's greater than 1/2?

Lee: 3/4 or 12/18. [Laughs.]

Int: How do you know that 12/18 is greater than 1/2?

Lee: Um, 12/24 is a half, so 18, you've only got six left over after you pull the 12 out.

A similar pattern occurred when questions were posed requiring students to construct fractions close to one. The same six people accomplished this task, with four of the six expressing a sense of the relationship of numerator and denominator and the size of the resulting fractions. For example:

(Interview 1, question 15)

Int: Tell me a fraction that's close to 1.

Pat: Uh, l'see, 15 out of 16.

Int: Okay, 15/16. What is it about that that let's you know it's close to one?

Pat: Because if you have maybe just one part less than what would be maybe 16 over 16, cause 16 over 16 equals one whole.

Int: Can you tell me another fraction that's close to one?

Pat: Eleven out of twelve.

Int: And which is closer, 11/12 or 15/16?

Pat: Uh, 15/16.

Int: How come?

Pat: Because the parts are smaller in 15/16, and the one part would take up less space.

Ordering in the context of subdivision. Comparing the relative size of two fractions in a forced choice situation (eg. Which is greater, 2/5 or 5/8?), success rates of correct answers accompanied by appropriate explanations varied across tasks, as noted in Table 8-2.

A whole number concept of fraction, with numerators and denominators seen as two separate comparison pairs, was the most common cause of problems in this segment. For example:

(Interview 2, question 18)

Int: Which is greater 2/5 or 5/8?

Table 8-2: Success rates: Ordering in the Context of Subdivision

Comparison fractions	Correct / Students responses/ asked	Percent of successes
2/5 ; 3/5	10/12	83.3%
1/3 ; 1/2	9/12	75.0%
2/5 ; 5/8	7/10 *	70.0%
7/8 or 5/8 ; 3/4	9/12	75.0%
4/8 ; 2/4	8/11 *	72.7%

* Questions were asked of fewer than 12 students. When two or more items had been missed by a student in an increasingly difficult series of questions, a question was often skipped to avoid frustration. Therefore, percents related to those questions would most likely have been lower, had the questions been posed to all twelve people.

Tad: 5/8.

Int: How come?

Tad: 'Cause five is bigger than two and eight is bigger than five.

(Interview 2, question 19)

Int: Which is greater 5/8 or 3/4?

Dan: 3/4.

Int: How did you decide that?

Dan: 'Cause the 3/4 is just half the size of a 5/8. (Dan has apparently focused on the pair of numbers in the denominators).

Successful responses emerged most often from the use of physical referents to make a visual comparison or to confirm a relationship (eg., each 1/4 = 2/8). In addition to physical referents, other reasons cited or behaviors observed in conjunction with correct answers were automaticity, that is, knowing the answer without having to think about it, and use of the concept of 1/2 as a reference.

Ordering questions posed in symbol form. The same forced choice questions regarding ordering of fractions seen in the second interview were presented in symbol form in the fourth interview. Eleven students participated in the fourth interview. Results were as indicated in Table 8-3 below.

The most often cited reason or observed behavior associated with correct answers in these episodes, even though they were identical in numerical content to those

Table 8-3: Success Rates: Ordering Questions in Symbol Form

Fractions compared	Correct / Students answers / asked	Percent successes
2/5 ; 3/5	10/11	90.9%
1/3 ; 1/2	9/11	81.8%
2/5 ; 5/8	10/11	90.9%
5/8 ; 3/4	9/11	81.8%
4/8 ; 2/4	8/11	72.7%

presented in the context of subdivision, was changing the fractions to a common denominator. Procedural knowledge, therefore, emerged in conjunction with symbolically presented ordering problems that had not been observed in the previous episodes. The second and third most common reasons for correct answers were the use of physical referents and automaticity, as noted in the subdivision context. However, in these episodes, the use of 1/2 as a reference was apparently supplanted by the common denominator procedures.

More similarly to the subdivision episodes, the major reason for errors in this section appeared to be whole number interference, with comparisons made of numerators separately from denominators.

Statement categories. In the combined set of all episodes related to ordering of the rational numbers, the two most common categories of statements produced by

participants were conceptual knowledge, 28.8%, and errors, 15.5%, followed closely by unelaborated answers, 14.8%, and action statements, 13.1%. Table 8-4, displaying numbers of statements in each category and corresponding percents of total statements in the ordering sections, follows.

The most noteworthy difference between this and the previous table of percents, that related to the meaning of fraction, is the lower percent of action statements. Students, within the confines of the interview questions posed, were less likely to use or describe actions in considering questions of ordering of the rational numbers than when contemplating or discussing the meaning of fractions and fraction symbols.

Table 8-4: Frequencies and Corresponding Percents of Statements in Ordering Episodes

Category	Number of statements	% of total statements
ER	70	15.5%
CN	130	28.8%
AT	38	8.4%
AC	59	13.1%
EX	40	8.9%
EV	18	4.0%
PR	29	6.4%
AN	67	14.8%
Total	451	99.9%

Errors, in contrast, were considerably higher in frequency. Ordering was the area of questioning that posed the greatest difficulty for students, indicating that their concept of fraction was somewhat static and, for several participants, did not include conceptual schemas for relating fractions in terms of their relative size.

Equivalence

Verbal discussion of equivalence. The first episode related to equivalence began with a request that participants "tell me a fraction that is the same as 1/2." Approaching this as a pass-fail question, the success rate was 7 of 12, or 58.3%. This episode split the participants precisely into the two groups of high performing and less proficient math students. Those who did create fractions equivalent to 1/2 all produced two, and most produced three in rapid succession.

(Interview 1, question 16)

Int: Can you tell me a fraction that is the same as 1/2?

Jay: Mm-hm, how many of 'em?

Int: Oh, just give me a couple.

Jay: 5/10, 2/4, 3/6.

The indication here was similar to that in the ordering questions in the same interview: those students with a rich enough concept of 1/2 had apparently developed a related schema for easily building fractions equal to (and by extension, greater or less than) 1/2.

(Interview 1, question 16)

Int: Can you tell me a fraction that's the same as 1/2?

Sam: Two--; 2/4 and 3/6 and 6/12.

Int: How are you coming up with those?

Sam: Well, just thinking of like the number 12 and half of that would be 6; so it'd be 6/12.

Equivalence in the context of comparison. While working with Cuisenaire rods, students had the opportunity to construct and then name fractions equivalent to 1/3 and 1/2 of the unit rod. Success rates were 6/9 and 5/10 or 66.7% and 50% of those asked. Because several students who had been having difficulty with prior comparison episodes were not asked these two questions, percents of successes for both questions would probably have been lower for the full group. In the context of comparison of lengths, equivalents of 1/2 did not seem to be any easier for participants to construct and name than were equivalents of 1/3. This finding seems contradictory to findings in other episodes, all of which had suggested that 1/2 was a stronger, more intuitive concept for the students in the study. It appears that the task of producing equivalents utilizing Cuisenaire rods was equally challenging for participants, regardless of the base fraction from which they worked.

Equivalence in the context of subdivision. Five episodes posed several fractions asking what they would be "the same as," or asking for a specific equivalent (eg., how many fourths are the same as 6/8). The fractions, which were presented in verbal form in the context of "pizzas" represented by circles and fractional pieces, are listed in

Table 8-5 below with corresponding frequencies and percents of correct answers. Parallel problems to those in the subdivision and comparison episodes were presented in interview four in symbol form with instructions to "write another way." The problems included and the corresponding results are indicated in Table 8-6.

In both verbal and symbolic presentations, 1/2 and 2/2 elicited the highest percent of successful responses, lending further credence to the notion that the concept of 'half' is unique in its strength and intuitive nature.

Statement categories. Across all episodes related to equivalence, conceptual knowledge and actions were the two categories represented most often by participants' statements, conceptual knowledge comprising 27.5% of all statements, and actions, 23.8%. The results indicate that the students utilized actions on physical referents to make sense of fractions, in this case equivalence, and in the

Table 8-5: Success rates: Equivalence in the Context of Subdivision

Fractions converted	Correct / Students responses/ asked	Percent successes
2/2	11/12	91.7%
9/3	10/11 *	90.9%
6/4	9/12	75.0%
1/2 = ?/6	12/12	100.0%
?/4 = 6/8	6/9 *	66.7%

*Because students not queried were those having difficulties with equivalence questions, percents would likely have been lower for the entire group.

Table 8-6: Success rates: Equivalence Problems in Symbol Form

Fractions converted	Correct / Students answers/ asked	Percent successes
2/2	9/11	81.8%
9/3	7/11	63.6%
6/4	7/11	63.6%
1/3 = ?/6	8/11	72.7%
1/2 = ?/8	9/11	81.8%

process they displayed conceptual knowledge at relatively high frequencies. A table summarizing frequencies and percent frequencies of all categories of statements has been constructed and is displayed in Table 8-7 below.

Table 8-7: Frequencies and Corresponding Percents of Statements in Equivalence Episodes

Category	Number of statements	% of total statements
ER	39	8.0%
CN	134	27.5%
AT	33	6.8%
AC	116	23.8%
EX	16	3.3%
EV	15	3.1%
PR	52	10.7%
AN	82	16.8%
Total	487	100.0%

Operations on Fractions

Operations in the context of subdivision. Questions in the third interview were presented verbally and were related to pizzas represented by cardboard circles and fractional pieces, which interviewees were free to utilize as they wished. In order to explore addition of fractions, verbal questions, as described above, were presented with the numerical content indicated in Table 8-8 below. Success rates have been recorded as ratios and percents.

All students used the physical referents to consider at least one of the addition problems. However, the last problem required students to think in terms of a fractional part not represented by the denominator of either addend. The cognitive leap required was obviously difficult. The common pattern observed in the five students who solved this problem was to place the 1/2 and 1/3 pieces on the circle and then search for a piece which fit the remaining space. Once identified, the 1/6-space indicated that the 1/3 + 1/2 must equal 5/6. Four of the five problem solvers displayed this pattern. The fifth went on to cover the entire array

Table 8-8: Success rates: Operations in the Context of Subdivision

Question content	Correct / Students answers/ asked	Percent successes
1/5 + 2/5	9/12	75.0%
3/8 + 1/4	8/12	66.7%
1/2 + 1/3	5/12	41.7%

with sixths before being satisfied that 5/6 was the solution. Subtraction problems were similarly presented as questions about pizzas represented by circles with fractional parts. Results were as indicated in Table 8-9 below. Success rates for solving verbal subtraction problems were generally higher than those for addition. Only the last problem, which involved subtraction from a whole number, resulted in a lower percent of correct responses than the addition problems.

Multiplication and division questions were framed in similar fashion, asking students to think about pizzas and utilizing circles with fractional pieces. Results were as listed in Table 8-10 below.

Students demonstrated difficulty in translating verbal multiplication problems into actions with referents. For example:

(Interview 3, question 9)

Int: You have 2/3 of a pizza, what would 1/2 of your 2/3 be?

Table 8-9: Success rates: Subtraction in the Context of Subdivision

Question content	Correct /Students responses/ asked	Percent successes
2/3 - 1/3	12/12	100.0%
4/5 - 4/5	11/12	91.7%
1/2 - 3/8	7/11	63.3%
2 - 2/5	4/10	40.0%

Table 8-10: Success Rates: Multiplication and Division in the Context of Subdivision

Question content	Correct / Students responses/ asked	Percent successes
2/3 X 1/2	2/11	18.2%
1/6 X 4	6/12	50.0%
1/2 ÷ 1/8	12/12	100.0%
3 ÷ 1/2	11/12	91.7%
1/4 ÷ 1/8	10/12	83.3%

Ron: [Places 2 thirds on circle; places 1/2 on top of 2/3.] Won't fit. L'see, hmm, would you repeat the question?

Int: You have 2/3 of a pizza, what's 1/2 of your 2/3?

Ron: [Removes 1/2; long pause.] L'see, hmm.

Int: That's a tough one. Want to let that one go?

Ron: Yeah.

Only two people solved the above problem, and their responses appeared to have been based on their sense of 1/2 rather than on any action on the referents. One placed 2/3 on the circle, the other did not. Both answered quite directly that 1/3 would be half of 2/3.

Division, in contrast, was more easily interpreted by the participants. Asked, "How many one-eighths are there in 1/2," all twelve participants decided there were four. While some used their conceptual knowledge of the relationship of four as half of eight, others utilized the

physical referents to determine their solution. Success with the question, "how many one-halves are there in three whole pizzas," may have had as much to do with the students' concept of 1/2 as with their facility with making sense of verbally-presented division questions.

Operations in the context of partitioning. Within the partitioning context, two episodes required multiplication and two were essentially division situations. However, it was clear from students' responses that they took a "common sense" approach to solving the last two problems and did not consider them as requiring division. The multiplication problems involved finding 1/3 of a set of 12 candies, and 2/5 of 10 candies. The division questions asked students to determine how many pieces of candy had been in a full box of candy if the remaining 5 pieces represented 1/3 of the full box, and 9 pieces as 3/5 of the box. There was some variation in numerical content between subjects on three of the four questions, but the form remained constant. Success rates are shown in Table 8-11 below.

Two solution paths were observed in the multiplication problems in the partitioning context. They were solved either by physically dividing the candies into appropriate subsets or by dividing the whole number by the denominator of the fraction (and multiplying by the numerator, in the last case).

The division problems were solved, not by division, but by multiplication or addition. None of the students

Table 8-11: Success rates: Operations in the Context of Partitioning

Question content	Correct responses / Students asked	Percent successes
1/3 X 12	7/12	58.3%
2/5 X 10	6/8 *	75.0%
5 ÷ 1/3	7/12	58.3%
9 ÷ 3/5	6/7 *	85.7%

* Because several students who were having trouble with partitioning were not asked these questions, percents of correct responses would probably have been considerably lower for the entire group.

indicated any sense that division was required. Their interpretation of the situation as either additive or multiplicative appeared to be intuitive. The two solution paths observed in the division problems evolved around the two problems, the first being solved by multiplying five times three, and the second by considering the nine candies as three subsets of three plus two more subsets of three, or nine plus six.

Operations on fractions presented in symbol form. Problems paralleling the verbal operations problems described in earlier sections were presented to students in symbol form in interview four. The same concrete referents were available to students who were encouraged to solve problems using any method they thought best. The addition questions are listed below in Table 8-12 with observed success rates.

Table 8-12: Success rates: Operations Problems Presented in Symbol form

Question posed	Correct / Students responses/ asked	Percent successes
1/5 + 2/5	10/11	90.9%
1/4 + 3/8	8/11	72.7%
1/2 + 1/3	9/11	81.8%

Typical reasoning underlying correct responses was of three types. Most common was automaticity, with students responding with little time for thought and/or indicating that they "just knew" the answer. The second reasoning pattern was to add the numerators only, with operations completed "in the head." Finally students in some cases wrote the problem in column form, changing to common denominators if required, and adding denominators.

The typical error for all three addition problems revealed whole number interference, as two students consistently added numerators and denominators as two separate whole number pairs. The symbolically presented subtraction problems are listed in Table 8-13 below with their rates of success.

Typical solution paths diverged between the first two and the last two of the subtraction problems. The majority of those queried responded to the first two questions with automaticity, while the last two problems were more typically written in column form, and common denominators

Table 8-13: Success rates: Subtraction Problems Presented in Symbol Form

Question posed	Correct / Students responses/ asked	Percent successes
2/3 - 1/3	10/11	90.9%
4/5 - 4/5	7/11	63.6%
1/2 - 3/8	9/11	81.8%
2 - 2/5	7/11	63.6%

found. The final problem was rewritten most often as 1 5/5 - 2/5, but two students worked with 10/5 - 2/5.

Error patterns differed across the problems. The first two problems elicited whole number interference similar to the addition problems. However, a new error pattern arose with the second problem in which subtraction in column form had led to the result 0/5. Two students then decided that the zero meant "nothing" and could be "crossed out," concluding that 0/5 is equal to five. One student, instead, misapplied the subprocedure allowing a one to be placed in the slot of a missing denominator to this result which appeared to her to have a missing numerator.

(Interview 4, question 19)

Ann: Be one over 5 . . . because you have nothing up here [pointing to the numerator].

The symbolically presented multiplication and division problems and success rates are listed in Table 8-14 below.

Table 8-14: Multiplication and Division Problems Presented in Symbol Form

Problem posed	Correct /Students responses/ asked	Percent successes
1/2 X 2/3	5/11	45.5%
4 X 1/5	5/11	45.5%
1/3 of 12	7/11	49.5%
3/4 of 20	6/11	54.5%
1/2 ÷ 1/8	5/11	45.5%
3 ÷ 1/2	5/11	45.5%

Success related to the first two multiplication problems typically resulted from the multiply numerators and multiply denominators algorithm, while the following two problems were seen as divide by the denominator (and on the 3/4 of twenty, multiply by the numerator) problems. Those who solved division problems typically did so using the invert and multiply algorithm, although on one problem one person used the lowest common denominator method, appropriately dropping the denominator. On the second division problem, 3 ÷ 1/2, one person used his knowledge of halves to reason that there would be two halves in one, so there must be six halves in three. Thinking through the meaning of an operation rather than manipulating symbols was not common in response to operations problems on the fourth interview where questions were presented in symbol form.

The most common and consistent error in multiplication and division was the overgeneralization of the addition algorithm. The problem was written in column form, the fractions were converted to common denominators, the numerators were multiplied or divided, and the denominator kept, following, in effect, all of the subprocedures of addition.

Statement categories. Across the combined episodes related to operations on fractions, the most frequently occurring categories of statements were procedures (20.3%), followed by actions (18.7%), and errors (18.1%). The frequency for each category and the percent of total responses in operations episodes each frequency represents are displayed in Table 8-15 below.

Table 8-15: Frequencies and Corresponding Percents of Statements in Operations Episodes

Category	Number of statements	% of total statements
ER	178	18.1%
CN	132	13.4%
AT	75	7.6%
AC	184	18.7%
EX	1	0.1%
EV	41	4.2%
PR	199	20.3%
AN	172	17.5%
Total	982	99.9%

The high levels of procedure statements indicate that students have learned rules and algorithms which they call upon more frequently in the presence of operations on fractions than in making ordering or equivalence judgments, or in considering the meaning of fractions. Lower frequencies of conceptual statements suggest a possible lack of connection between procedures and concepts. The high level of errors reflects both whole-number interference and overgeneralization of algorithms, which would be less likely to occur if concepts and procedures were highly interconnected.

Summary

The sixth grade participants' knowledge of fractions was not limited to rules and algorithms, but included conceptual knowledge supporting their ability to interpret fraction symbols and to utilize physical referents to make sense of ordering, equivalence and operations on fractions.

Students acted on physical referents or described actions more frequently in verbal discussions of the meaning of fractions and fraction symbols than in any of the other areas of questioning: equivalence, ordering, or operations. Action statements comprised 35.7% of all statements in meaning episodes compared with 23.8, 13.1, and 18.7% respectively in the other areas listed above. Across all participants, their sense of meaning rested almost totally on a part-whole definition of fraction. While 1/2 was recognized and named by all participants in the settings of comparison and partitioning, and other fractions were

successfully recognized and named in the subdivision context, cognitively most consonant with a part-whole definition, recognition and naming in the contexts of comparing and partitioning met with lower success frequencies. Non-unit fractions were the most problematic in all settings.

Statements expressing conceptual knowledge remained constant across the first three questioning areas: meaning, ordering, and equivalence (29.1%, 28.8%, and 27.5% respectively). Questions requiring operations on fractions resulted in a lower rate of statements indicative of conceptual knowledge, 13.4%. Frequencies of procedural statements, in contrast, remained low across the first three areas (1.1 6.4, and 10.7%), but were the most frequently represented category (20.3%) in relation to operations.

Conceptual and Procedural Knowledge Related to Operations on Fractions

Research question II asked: How do sixth graders use conceptual and procedural knowledge related to operations on fractions? Contrasting of participants' use of conceptual and procedural knowledge will be presented in two ways. First, to highlight differences that may have been missed in combining episodes across settings, the table of frequencies and corresponding percents of the categories in the area of operations will be broken down to compare verbally presented and symbolically presented problems. Second, the results of parallel problems presented in both formats, verbal and

symbolic, will be analyzed and participants underlying reasoning examined for conceptual and procedural content.

Category frequencies. Summary Table 8-16, which includes frequencies and percents of statement categories for verbally presented problems from interview three, and for corresponding symbolically presented problems from interview four, is displayed in Table 8-16 below.

Frequencies and percents of participants' statements in each category strongly suggests that the students were more inclined to use conceptual knowledge when dealing with verbal operations problems than with similar problems in

Table 8-16: Frequencies and Percents of Statements Generated in Response to Parallel Problems Presented in Verbal and Symbol Form

Category	Verbal Format		Symbol Format	
	Number of statements	% of total statements	Number of statements	% of total statements
ER	93	17.0%	85	19.6%
CN	118	21.5%	14	3.2%
AT	31	5.7%	44	10.1%
AC	176	32.1%	8	1.8%
EX	0	0%	1	0.2%
EV	18	3.3%	23	5.3%
PR	32	5.8%	167	38.5%
AN	80	14.6%	92	21.2%
Total	548	100.0%	434	99.9%

symbol form. Physical referents were utilized more often to represent the content of problems, aiding students in using and expressing their conceptual knowledge, while they worked through solutions to verbal problems. Even though the same physical referents, which students had used successfully in prior interviews, were available when the symbolic problems were presented, and even though the numerical content of the problems was identical or highly similar, students generated procedural statements at a considerably higher rate when problems were confronted in symbol form.

The three categories of statements displaying the largest differences between the conditions of verbal as opposed to symbolic presentation of questions were procedural statements, conceptual knowledge statements, and action statements. The apparent tie between conceptual knowledge and action statements reflects the fact that students regularly used concrete referents to aid in thinking about fraction situations, providing them with a way to demonstrate conceptual knowledge they might otherwise have been unable to verbalize.

There was demonstrated in this small group of people an inverse relationship between conceptual statements and procedural statements. Students were much more likely to make statements indicative of conceptual knowledge when considering verbally presented problems associated with physical referents than when confronted with symbolically presented problems, even though the same referents were available. The reverse was true of procedural statements

which were made much more frequently in responding to problems in symbol form than in verbal form. The eleven students who took part in all four interviews did appear to see symbolically presented problems as different in kind, utilizing their conceptual and procedural knowledge with them in very different ways.

Error statements, in contrast, remained constant across both sets of problems. Students were apparently able to make use of conceptual and procedural knowledge with approximately equal success, even though they were not necessarily used in concert.

Parallel problems. The numerical content of problems presented verbally in interview three in the context of subdivision or partitioning are listed below. Each was paired with a corresponding symbolic format problem in interview four. Success rates are given for each in Table 8-17.

Underlying reasoning associated with correct answers was strikingly different when students were responding to verbal as compared to symbolic problems. Reasons given or observations of behavior associated with correct answers in the verbal condition were most often connected with physical referents and actions on the referents. Manipulation of physical referents comprised 41.4% of all explanations, other conceptual knowledge explanations, 33.3%, automaticity, 17.6%, visualization of circles, 5.9%, and procedures were cited only once, representing 2.0% of all explanations given related to correct answers to verbally

Table 8-17: Success Rates: Comparison of Verbal and Symbolic Problems

Problem content	Verbal % successes	Symbolic % successes
1/5 + 2/5	75.0%	90.9%
3/8 + 1/4	72.2%	72.7%
1/2 + 1/3	41.7%	81.8%
2/3 - 1/3	100.0%	90.9%
4/5 - 4/5	91.7%	63.3%
1/2 - 3/8	63.3%	81.8%
2 - 2/5	40.0%	63.6%
2/3 X 1/2	18.2%	45.5%
4 X 1/5	50.0%	45.5%
1/3 of 12	58.3%	49.5%
3/4 of 20	50.0%	54.5%
1/2 ÷ 1/8	100.0%	45.5%
3 ÷ 1/2	91.7%	45.5%

presented problems. In contrast, symbolically presented problems evoked 62.5% procedural explanations, 15.2% conceptual knowledge of equivalence (used to facilitate addition or subtraction), 20.8% automaticity, and visualization of "pies" was cited once representing 1.4% of all explanations given in association with correct answers to symbolic problems. Physical referents were never cited in deriving or explaining correct solutions to problems in symbol form.

Common errors also contrasted sharply between the two modes of presentation of problems. Verbal problems resulted in errors associated with the use of physical referents on eighteen occasions, with procedures on six occasions, and a conceptual reasoning error on one occasion. Errors on symbolic problems evoked errors associated with procedures almost exclusively, on 34 occasions. Only three other explanations of errors were given, two associated with diagrams and one with physical referents.

As students worked problems and explained their answers, whether correct or errorful, they revealed the use of conceptual knowledge and procedural knowledge in a dissociated fashion. Procedural knowledge was called upon much more often in conjunction with problems in symbol form. Actions on physical referents, which facilitated expressions of conceptual knowledge, were only common in association with verbal problems. In contrast the use of referents was virtually absent from consideration of problems in symbol form.

Contrast Between More and Less Proficient Math Students

Research question III asked: How do successful sixth grade math students think about fractions and operations on fractions in comparison with less proficient sixth grade math students? Analysis of the data included grouping the responses of those students who had experienced success in mathematics classes and contrasting them with the collected responses of students who had received below-average grades in math class during the current year.

Similarities

Before considering observed differences in the thinking of more and less advanced students, some similarities in their responses will be noted. Two important similarities were that the groups were equally verbal and that they interacted with manipulatives equally.

Verbal rates. Rates of statements averaged across episodes revealed that the members of the lower performing math group (hereafter referred to as Group L), were slightly more verbal than members of the higher performing group (hereafter referred to as Group H) when discussing the meaning of fractions. Group L members produced a mean of 84.8 statements per participant contrasted with 77.2 for Group H. When considering order and equivalence and operations on fractions, rates of statements per participant averaged slightly lower for Group L than for their counterparts, 14.5 to 18.8 and 84.4 to 91.1. (See Group Contrast Data Summaries, Appendix D.)

However, it should be noted that statements coded as comments were not included in data analysis. Based on tabulation of comment statements from eight randomly selected episodes, 2 from each interview, members of Group L generated comments at a considerably higher rate than did Group H students (10 comments per student per episode versus 4.6). The data analysis, which excludes these statements, therefore tends to portray the members of Group L as being comparatively less verbal than they in fact were.

It is safe to conclude that Group L students were at least equally as verbal as were members of Group H, if not slightly more verbal overall. Members of both groups appeared to have been equally willing to discuss mathematical ideas related to fractions. This similarity is important because, along with the selection procedures requiring all participants to have tested average or above in language and/or reading skills, it supports the contention that any differences observed were differences in mathematical thinking, not simply differences in general verbal ability.

Actions on physical referents. The second similarity noted was that of interaction with manipulatives. Responses coded as action statements included actions verbally described by participants as well as physical activities observed by the researcher during the sessions and in transcribing the video tapes. Action statements, therefore, encoded rates of students' involvement with the available manipulative materials. In all areas of questioning, Group

L members exhibited somewhat higher rates of interaction with manipulatives, as reflected by rates of action statements, than did Group H. Overall rates were 72.4 action statements per participant in Group L compared to 53.9 for Group H. Participants in Group L were at least as willing to interact with manipulative materials as were the Group H members. The importance of this observation is that it also supports the notion that differences observed were differences in mathematical thinking, and were not reflective of an unwillingness on the part of Group L members to engage in activities with the manipulative materials made available in the study.

Differences

The two most striking differences evident in the data produced by contrasting Group H and Group L participants' responses were in the statement categories coded error and conceptual knowledge.

Errors. Statements containing errors occurred in 27% of all statements made by participants in Group L. In contrast, error statements constituted only 5.6% of all statements generated by Group H members. Translated into rates, the mean number of error statements made by each participant in Group L across all four interviews was 66.0, contrasted with a rate of 14.9 per student in Group H.

The summary in Table 8-18 below displays the percents and rates of error statements produced in each group in the areas of meaning, order and equivalence, and operations.

Table 8-18: Contrast of Groups H and L: Error Statements

Group	H			L		
	number	%	rate	number	%	rate
Meaning/ concept	14	2.7%	2.1	71	19.0%	16.1
Order & Equivalence	23	3.6%	3.3	86	28.7%	21.5
Operations on fractions	62	9.7%	8.9	119	34.4%	29.0

Breaking the comparison down by topic area, the largest contrast was recorded in discussions of the meaning or concept of fraction where a rate of errors 7.7 times higher was recorded for Group L than for Group H. A close second was observed in ordering and equivalence of fractions with a factor of 6.5 differentiating the rates of errors in Group L over Group H. Only in the episodes dealing with operations on fractions did the rate of errors become somewhat less discrepant, with the difference in the rates for Group L and Group H dropping to a factor of 3.3.

In addition to the discrepancy in numbers of errors, analysis of all errors generated by the members of the two groups revealed a striking difference in the types of errors produced. A great deal of whole number interference was observed in Group L, and very little in Group H. Meanings of fractions were obfuscated by whole number knowledge on many occasions for members of Group L, indicated, for example, by their difficulty in naming embodiments of the fractions 1/3 and 1/6. Most often, an entire array was seen

as representing the unit fraction (eg. 3 thirds on a circle or 6 sixths placed beside a unit rod were referred to as 1/3 and 1/6.) One variation on this theme involved whole number representation of both numerator and denominator (eg, 1/3 was illustrated by placing one candy above three candies).

In equivalence problems, renaming 2/2 and 6/4 also resulted in whole number interference with 2/2 being thought equal to 1/2, and 6/4 equal to 6 or 1/6. Ordering tasks evoked high levels of whole number interference, with numerators and denominators treated as separate pairs of values to be compared (eg. 5/8 > 3/4 because 5 > 3 and 8 > 4). One member of Group H exhibited this same tendency in the case of ordering, and did so consistently in all settings, except in solving symbolically presented problems where he used a common denominator algorithm and made no errors. His was the only evidence of whole number interference observed in Group H, and it occurred only in ordering tasks.

Operations problems presented in symbol form resulted in two people in Group L consistently operating on numerators and denominators as separate whole number values, adding, subtracting, multiplying and dividing them separately. Whole number interference was also scattered throughout responses of Group L members to verbally presented operations problems associated with the use of physical referents. Most commonly, the numbers of pieces present would be placed in the denominator of the answer (eg. 3/4 + 1/8 = 6/8 + 1/8 were correctly represented by

fractional pieces on a circle, with the answer being given, 1/7).

In contrast, the errors that occurred in Group H were more likely to be procedural errors, especially overgeneralization of the addition algorithm to multiplication and division. Further differentiating between the groups was the fact that many of the errors produced by Group H were recognized and corrected in the course of discussion. Their metacognitive critics caught slips especially well. Consistent bugs such as overgeneralization were more likely to remain undetected.

The only error pattern common to both groups reflected an apparent inability to translate verbal multiplication problems into meaningful actions with physical referents, or to make sense of them in any other way. All students in both groups had decided that 2/3 of a pizza with 1/3 eaten would result in 1/3. But only two students in Group H and none in Group L determined that if you had 2/3 of a pizza, 1/2 of the 2/3 would be 1/3. Considering having 1/6 of a pizza four times also resulted in errors in both groups, with the most common error answer, 5/6, appearing in protocols of both groups.

Conceptual knowledge was the second area of notable difference between Group H and Group L participants. Assertions reflecting conceptual knowledge constituted 29% of all statements made by members of Group H, as contrasted with 13.6% for students in Group L. In the form of rates, each Group H sixth grader produced a mean of 77.8 total

statements indicative of conceptual knowledge, while the Group L members each generated an average of 33.3 such statements. Table 8-19 summarizes conceptual knowledge statements, percents and rates for both groups in all three areas of questioning. Differences remained constant across all three major topic areas of meaning/concept of fractions, ordering and equivalence of fractions, and operations on fractions, with Group H generating conceptual knowledge statements at a rate 2.0 to 2.4 times that of Group L.

Expressions of conceptual knowledge at considerably higher rates reflected the relative richness and strength of schemas possessed by members of Group H. Supporting evidence was found in episodes related to the meaning of fractions in which Group H members were consistently more likely to be able to explain and illustrate the meanings of fraction symbols. In ordering and equivalence, they were able to relate other fractions to their concept of 1/2, and to use that same concept to produce fractions greater than,

Table 8-19: Contrast of groups H and L: Conceptual Knowledge Statements

Group	H			L		
	number	%	rate	number	%	rate
Meaning/ concept	191	36.4%	26.9	61	16.4%	2.0
Ordering & equivalence	214	33.5%	31.0	51	17 %	12.8
Operations on fractions	113	17.7%	16.1	28	8.1%	6.8

less than, or equivalent to 1/2. In contrast, possible solutions to these ordering and equivalence tasks eluded all members of Group L.

Additional evidence of the difference between the groups in richness and integration of schemas was provided by responses to questions about their own and others experiences with fractions. Members of groups L, prior to prompting produced no such connections, while group H produced six specific examples of fraction use. After prompts, members of Group H generated on average 2.7 specific examples of fraction use contrasted with 1.4 by each member of group L. Two members of the first group went on to add information about experiences not directly related to the prompts, further supporting the assertion of differences in complexity and connectedness of schemas.

Automaticity. One would expect that well developed schemas and corresponding high levels of conceptual knowledge would have allowed members of Group H greater flexibility than their counterparts in Group L in selecting effective and efficient methods for solving problems. Frequencies of statements of automaticity, which were coded when students produced answers with no pause for thought or when they stated that they "just knew" the answer, indicated that this was the case. These students have begun moving from highly cognitive procedures with fractions, which require conscious thought regarding each subprocedure or relationship, toward greater integration and efficiency, with some procedures and relationships having become

automatically apparent to them. They were able to produce, therefore, answers to some questions with no apparent need for thought. Table 8-20 below indicates the percents and rates of automaticity statements produced by both groups.

Although frequencies of automaticity statements were low overall in comparison to total statements, it seems safe to conclude that rates of automaticity were higher for Group H than for Group L. Their integrated schemas and higher levels of automaticity suggest that Group H members had begun to move toward expertise in dealing with fractions.

The bar graph in Figure 8-1 below summarizes the three statement categories that had indicated the greatest differences between the groups of higher and lower achieving math students (ie. errors, conceptual knowledge, and automaticity).

Table 8-20: Contrast of Groups H and L: Automaticity Statements

Group	H			L		
	number	%	rate	number	%	rate
Meaning/ concept	32	6.1%	4.7	6	1.6%	1.4
Ordering & equivalence	64	10.0%	9.3	7	2.3%	1.8
Operations on fractions	71	11.1%	10.1	4	1.2%	1.0

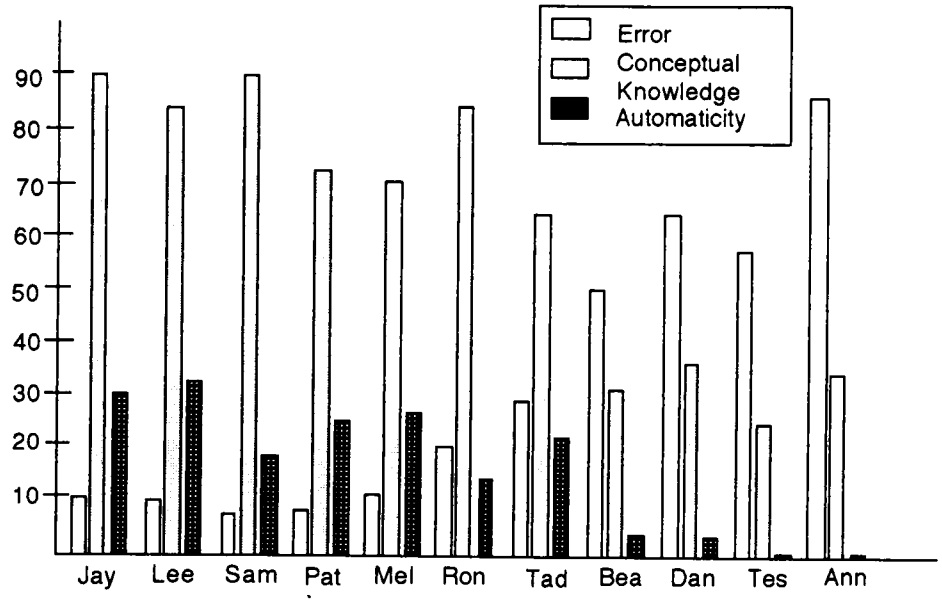

Figure 8-1: Contrast of Conceptual Knowledge, Automaticity, and Errors

Contexts of Comparison, Subdivision and Partitioning

Research question IV asked: How do sixth graders think about fractions when encountered in the contexts of comparisons of objects, subdivision of continuous quantities, and partitioning of discrete sets? The three contexts will be explored in the areas of meaning, ordering and equivalence, and operations on fractions.

Meaning/concept of fraction. Students were able to identify 1/2 as a fraction in comparison of lengths, in subdivision of a circle and in partitioning of a set of candies. Success rates were 100% for both comparison and subdivision, and 91.7% for partitioning. The unit fraction 1/3 was appropriately named by 83.3% and 91.7% of students in comparing and subdividing contexts. Recognition of 1/4

was also 100% in the subdivision context. The more difficult task of identifying a rod 1/4 the length of the unit, in contrast, was achieved by 83.3% of the participants. Non-unit fractions were more difficult in all settings. Identifying a rod 3/4 as long as the unit, or 3 pieces (fourths) of a circle, or 8 of 12 candies as 2/3 of the set, were accomplished by only 80%, 58.3%, and 75% of participants, respectively. Partitioning was affected by whole number interference as the numeral '8' appeared in the denominators of four of five error answers, and the numeral '4' appeared in the denominator of the fifth.

Ordering and equivalence. Ordering and equivalence were considered within the realms of comparison of length and of subdivision of continuous quantities. In comparison of length, students were again asked to produce rather than identify, equivalents to 1/2 and 1/3, with success rates of 50% and 41.7% Dissimilar episodes in the subdivision setting asked students for equivalents of 2/2, 9/3 and 6/4. Corresponding success rates were 91.7%, 83.3%, and 75%. Confronted with the comparison tasks, many students were simply unable to produce and/or to name the requested equivalents, whereas in the subdivision setting, whole number interference produced the greatest number of incorrect responses (eg. the numeral 6 appeared in all error answers of equivalents for 6/4).

Operations on fractions. Operations on fractions were considered in the context of subdivision and briefly in the context of partitioning. Multiplication was the only

operation common to both, since division problems in partitioning were not recognized or treated as such by participants. In the subdivision setting, questions asked students to find 1/2 of 2/3 of a pizza, and to determine the result of having 1/6 of a pizza four times. Partitioning problems were: find 1/3 of 12 candies and 2/5 of 10 candies. Success rates were 18.1% and 50% in the subdivision mode, and 58.3% and 75% in the partitioning mode, although only eight students were asked the second partitioning question (four who were having difficulty with partitioning were not asked), which suggests that the success rate would have been lower had all students been included. Students had trouble making sense of the multiplication problems in subdivision, placing inappropriate pieces or numbers of pieces on the "pizza" circle in attempting to translate the problem onto the physical referents. In contrast, the partitioning problems left students stymied most often by whole number interference (eg. 1/3 should have three pieces in it).

Statement categories. There were considerably greater contrasts to be found between verbal discussions and symbolic encounters with fractions than among the three contexts of comparison, subdivision and partitioning. The chart below summarizes percents of statements across all episodes in each context.

Most categories remained relatively constant across the three contexts. The two areas of greatest difference were action statements and error statements. Action statements were somewhat higher in the comparison setting, indicating

Table 8-21: Percents of Statements in the Three Contexts of Comparison, Subdivision, and Partitioning

Category	Comparison context %	Subdivision context %	Partitioning context %
ER	8.5%	12.7%	17.8%
CN	29.0%	23.1%	26.4%
AT	2.0%	9.0%	4.8%
AC	41.6%	28.8%	30.8%
EX	0.2%	0%	0%
EV	6.3%	2.7%	4.8%
PR	0%	3.9%	5.8%
AN	12.4%	19.8%	9.6%

that students may have been more involved with the physical referents in that setting. However, the differences were not large (41.6%, 30.8%, and 28.8% in comparison, partitioning and subdivision, respectively).

The differences in errors were also relatively small (8.5% in comparison, 12.7% in subdivision, and 17.8% in partitioning). These results do coincide with prior observations that participants had more difficulty with partitioning largely because of the interference of whole number information from the cardinal numbers of sets and subsets.

In addition, small differences in automaticity point to subdivision as the setting in which students were most comfortable. Of all three contexts, the part-whole schemas the students had demonstrated would be most directly related

to the subdivision of wholes into parts. However, the differences in automaticity were small (9.0% in subdivision, 4.8% in partitioning, 2.0% in comparison).

Conceptual knowledge remained constant at relatively high levels (29.0%, 26.4%, 23.1%), while procedural statements remained fairly constant at low frequencies (0%, 3.9%, 5.8%). These results coincide with the scenario of participants using actions on referents to make sense of the mathematical situations at hand, and expressing their conceptual knowledge in the process. It further suggests that students engaged with physical referents in any of the three contexts were not directly connecting their activities with procedures utilized in more formal symbolic settings.

CHAPTER 9

Summary, Conclusions, and Recommendations

Summary

The research questions that prompted this study called for exploration of sixth grade participants' knowledge of fractions in terms of the meaning of fractions and fraction symbols, ordering and equivalence of fractions and operations on fractions; investigation of their use of conceptual and procedural knowledge in operating on fractions; contrast of the cognition of more successful and less successful mathematics students related to fractions and operations on fractions; and examination of students' cognition related to fractions in contexts of comparing, subdividing and partitioning.

Meaning of fraction. Findings related to the meaning or concept of fraction may be summarized as follows:

Participants had unanimously adopted a part-whole concept of fraction which they displayed in their definitions, explanations and illustrations of fractions.

One-half was a relatively strong and intuitive concept for all participants, although it was more integrated and therefore useful for some students than for others.

A whole number interpretation of fraction symbols was observed in some participants protocols, and whole number interference was the major contributor to errors in interpreting or illustrating fraction symbols.

Conceptual and procedural knowledge. Findings related to the use of conceptual and procedural knowledge in operations on fractions were contrary to expectations. Participants were not limited to procedural knowledge nor did they depend on procedural knowledge more than conceptual knowledge. In all areas and across all contexts other than the symbolic presentation of fraction problems, students utilized actions on physical referents and displayed relatively high frequencies of conceptual knowledge statements, in thinking about the meanings of fractions, in ordering and equivalence tasks, and in operating on fractions in response to verbal operations questions. Only in the context of symbolically presented problems did students rely heavily on algorithms and procedures, virtually ignoring the physical referents available, generating conceptual knowledge statements at a considerably lower rate and procedural knowledge statements at considerably higher frequencies.

Contrast of more and less successful mathematics students. Students in the high performing math group (Group H) exhibited greater conceptual knowledge of fractions, accompanied by considerably lower levels of errors. Their concept of fraction appeared to be richer and more highly integrated, and they were more flexible in their approaches to solving problems, calling upon conceptual knowledge when it served them efficiently, or on other occasions utilizing procedures.

Students in Group H displayed higher levels of automaticity which added to their flexibility in approaching problems. The automaticity they displayed indicated less necessity for cognitive control of procedures, and movement toward expertise.

There was a differentiation in types of errors produced by the two groups. Students who were less proficient in math (Group L) tended to produce errors dictated by their whole number interpretation of fraction symbols. In contrast, students in Group H tended to produce procedural errors, with only one member in one situation producing errors consistent with whole number interference.

Members of group H tended to be more metacognitively aware of their own performance, catching and correcting their errors as interviews progressed.

Comparing, subdividing, and partitioning. A comparison of protocols and statement categories, related to the three contexts explored, did not yield the striking differences observed in other contrasts. However the following findings were based on those differences noted:

Partitioning was cognitively the most difficult context for participants because of the interference of the whole number information inherent in sets and subsets. Partitioning questions resulted in the highest number of errors of any of the three contexts, and the type of errors most common in these episodes did reflect interference by the cardinal numbers of the sets or subsets involved.

Conceptual knowledge was exhibited at a constant and relatively high rate across all three contexts, and procedures at a low and somewhat constant rate.

Conclusions

Meaning and Concept of Fraction

Participants had developed a part-whole definition of fraction and an intuitive sense of 1/2 or half, both of which likely evolved as informal knowledge from everyday experiences, reinforced by school experience. The part-whole definition represents a limited view of fraction, however, and the concept of 1/2 was well-developed and integrated enough to be useful (eg. in ordering and equivalence judgments) only in those students who displayed generally high levels of conceptual knowledge. These students require experiences with fractions in varied contexts, especially in partitioning of discrete sets which proved difficult for most students in this study. In addition, the less-proficient students require opportunities to expand and build upon their concept of 1/2 and to begin to relate it to other fractions.

Conceptual and Procedural Knowledge

Contrary to what was anticipated, it can be concluded from the data in this study that students are neither limited to nor bound by procedural knowledge. Rather the participants in this study utilized physical referents to make sense of mathematical situations, and displayed conceptual knowledge at relatively high levels. Further,

those students who displayed the highest levels of conceptual knowledge also generated the fewest errors, and were more flexible in their approach to solving fraction problems, suggesting that conceptual knowledge supported their ability to problem solve.

However, the data of the study clearly suggest that there is a dissociation between students' use of referents and conceptual knowledge in response to verbal problems, and their procedural responses to symbolically presented problems. Students approached verbal and symbolic problems as two different mathematical animals, seeking solutions to problems with identical numerical content in the one case by utilizing cognitive knowledge and/or by using the available physical referents, and in the other case utilizing procedural knowledge almost exclusively. It seems that if the participants in this study are to wed their procedural and conceptual knowledge, to produce more integrated, useful and flexible schemas related to fractions, there is a need for multiple experiences that would allow for mapping of actions with referents unto symbols.

More Versus Less Proficient Math Students

Whole number interference was a major problem for participants in this study who were less proficient at mathematics. Whole number knowledge intruded upon definitions of fraction symbols most often in the partitioning context and in responding to symbolically presented problems. However, whole number interference caused difficulties, especially with non-unit fractions, in

all settings including comparing and the familiar subdivision setting, which is most consonant with the participant's part-whole definition of fraction. The implication is that these students' understanding of fraction is tentative at best, and that their conceptual knowledge must be bolstered if these students are to deal successfully with fractions.

Because children do attempt to "make sense" in mathematical situations, they are likely to rely on whole number knowledge at inappropriate times until they can make sense of fractions as a set of numbers. Students therefore require experiences in translating from numerous embodiments in a variety of contexts to symbols, and then from symbols to embodiments. Continued use of referents might assist in making sense of fractions and relationships between and among them as ordering and equivalence are explored. Until fractions are experienced as a meaningful set of numbers, including a sense of their ordering and equivalence, whole number knowledge will likely continue to be the students' best base from which to work.

The more proficient students, in contrast, had developed a reasonably useful part whole concept of fraction, and they were generally able to make use of referents to make sense of less familiar contexts of comparing and partitioning. They exhibited automaticity and flexibility that suggested movement toward less cognitive control in carrying out some procedures and recognizing some relationships. These students require expansion of their

knowledge base to include other than part-whole definitions of fraction. They also require experiences that will promote integration of their conceptual knowledge and procedural knowledge of fractions as well as integration of fraction knowledge with that of the systems of whole numbers and decimals.

Contexts of Subdivision, Comparing, and Partitioning

As previously noted, students tended to deal with meaning, ordering and equivalence, and/or operations on fractions more successfully in comparing and subdivision contexts. Partitioning tasks resulted in higher frequencies of errors. The challenging nature of partitioning, with the potential interference of cardinal numbers, requires numerous and varied experiences within the partitioning context.

Recommendations for Future Research

Teaching experiments are needed to determine whether students are able to develop greater conceptual knowledge of fractions and to integrate fraction concepts and procedures through the use of physical referents.

The complex relationship between conceptual and procedural knowledge needs to be further explored, especially in relation to students who are moving toward proceduralization and automaticity.

Finally, it would be helpful to educators if stages were delineated in the development of the fraction concept, from earliest introduction and informal knowledge of

fractions to eventual integration of fractions with the larger world of numbers known to the child.

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APPENDICES

APPENDIX A

Definitions

concept: A concept is defined as a complex mental representation having internal structure, and being highly networked. It is different from a category in that it is not a collection of members, but a relatively stable whole. As instances are encountered in the real world, they are assimilated to the conceptual whole, but are not retained as individuals within the mental representation.(Nelson, 1983).

conceptual knowledge: Conceptual knowledge consists of networks of relationships; it is principled rather than factual; and it is concerned with meaning and intent.

semantic understanding: Semantic understanding denotes meaning-based knowledge representations, especially with reference to the use of meaningful symbols. Although the meanings of conceptual knowledge and semantic understanding differ to some extent, these two terms are often used interchangeably.

procedural knowledge: Procedural knowledge consists of goal-directed knowledge of how to achieve a particular end result. In mathematics it includes mathematical symbols and the algorithms and rules for manipulating those symbols.

syntactic understanding: Syntactic understanding denotes the manipulation of symbols based upon explicit rules. Syntactic understanding and procedural knowledge are at times used synonymously, even though their meanings vary slightly.

comparing: Comparing is one of three experience-based sources of fractions. Fractions "occur" as comparisons are made in the relative sizes of concrete items, resulting in a rational number.

subdividing: Subdividing is the second source of fractions in which a continuous physical quantity is divided into equal parts, with the whole terminating completely into parts.

partitioning: Partitioning is a third source of fractions in which a set of discrete items is divided into equal subsets.

APPENDIX B

Interview Schedules

Interview Schedule #1

PRE-INTERVIEW

1. Would you think back to a time when something you did in math class made you feel really good, and tell me about it?

2. Has anything else in math class ever made you feel really proud of yourself?

3. Can you think back to something you've learned about fractions and tell me about that?

MEANING/CONCEPT OF FRACTION

4. If someone from another planet, like ET, who spoke English but didn't know much about math, asked you, "What is a fraction," what would you say?

5. Can you draw me a picture of what a fraction would look like?

   Tell me about your picture.

6. Suppose that I'm a second grader, and I asked you, "What does 1/3 mean?" How would you explain 1/3 to a second grader?

7. Could you explain a fraction like 3/4; 5/8?

8. Would you draw me a picture of 2/3; of 5/8?

9. Would you tell me a story about this picture (2/3); about this picture (5/8)?

USES OF FRACTIONS

10. If you think about people outside of school, can you think of any time or place when people might use fractions?

11. Tell me about a time when you use fractions, outside of school; another?

    Prompts: Sharing? Measuring? Telling time? Comparing amounts? Cooking?

ORDERING OF THE RATIONAL NUMBERS

12. Can you tell me a fraction that would be less than 1/2? Another?

    What lets you know that _____ is less than 1/2?

13. Could you give me an example of a fraction that's greater than 1/2? Another?

    How do you know that _____ is greater than 1/2?

14. Are there any fractions between 1/2 and _____?

    Can you tell me one of them? Others?

    How many fractions do you think there are between ____ and 1/2?

15. Tell me a fraction that's close to 1.

    Why do you think ____ is close to 1?

EQUIVALENCE

16. Can you tell me a fraction that is the same as 1/2?

    How can _____ be the same as 1/2?

    Others?

FRACTIONS AS COMPARISONS
Manipulatives utilized are paper strips.

16. (Presenting three strips of colored paper representing candy sticks: Jessie's = 4"; Marti's = 8"; Sue's = 12")

    How long is Jessie's candy stick compared to Marti's?

    How long is Jessie's compared to Sue's?

    How long is Marti's compared to Sue's?

Interview Schedule #2

FRACTION AS COMPARISON
Manipulatives utilized are Cuisenaire rods with an added 12-rod.

MEANING/CONCEPT OF FRACTION

1. (Discuss units of length; present 12-rod as new unit; have participant name new unit) Okay, this is one [unit] long. How long would this rod be (6-rod), compared to a [unit] ?

   Explain why you said _____.

2. Same question using 4-rod.

3. Can you show me what rod would be 1/4 as long as a [unit] ?

   Why would you say that is 1/4 as long as a [unit]?

4. How long would this (9-rod) be compared to a [unit]?

   Explain?

5. What length is this (2-rod) compared to a [unit]?

   Explain?

6. Now let's say we have another unit, and it looks like this (8-rod). What do you think we should call this? Now what length is this (2-rod) compared to a [alt. unit]?

   How can this be _____ one time and _____ the next?

EQUIVALENCE

7. Let's go back to our original unit (12-rod). You named this a [unit]. What length was this rod then (4-rod)? Can you think of another way to show the same length as 1/3 of a [unit]?

   Tell me about the fraction you've built.

8. How many ways could you show 1/2 of a [unit]?

   Tell me about the fractions you've built.

FRACTIONS AS SUBDIVISIONS
Manipulatives utilized are circles with fractional parts. Circles are presented as whole pizzas. Participants are encouraged to move pieces around in any way they wish, to help them think about fractions.

MEANING/CONCEPT OF FRACTION

9. Tell me about the orange pieces (thirds).

   The white pieces (sixths).

   The green pieces (fourths).

   Three green pieces (3/4)..

EQUIVALENCE

10. What would the dark blue pieces be?

    What would 2/2 be the same as?

11. What would 9/3 be the same as?

12. What would 6/4 be the same as?

13. How many sixths are the same as 1/2?

14. How many fourths are the same as 6/8?

15. Six pieces of what size would be the same as 3/4?

ORDERING OF THE RATIONAL NUMBERS

16. Which is greater 2/5 or 3/5?

17. Which is greater 1/3 or 1/2?

18. Which is greater 2/5 or 5/8?

19. Which is greater 7/8 or 3/4?

20. Which is greater 4/8 or 2/4?

Interview Schedule #3

FRACTION AS SUBDIVISION
Manipulatives utilized are circles and fractional pieces as in previous interview.

OPERATIONS ON FRACTIONS

1. If you have 1/5 of a pizza and someone gives you 2/5 of theirs, how much do you have?

2. What would you have if you added 3/8 plus 1/4?

3. What is 1/2 plus 1/3?

4. 1/8 plus 3/4?

5. If you start with 2/3 of a pizza, and drop 1/3 of it, what do you have left?

6. What if you had 4/5 of a pizza, and you ate 4/5?

7. If you started with 1/2 and ate 3/8, what would you have?

8. What if you had 2 whole pizzas, and somebody ate 2/5 of a pizza?

9. You have 2/3 of a pizza. What is 1/2 of your 2/3?

10. What if you had 1/6 of a pizza, 4 times?

11. How many 1/8's are there in 1/2 a pizza?

12. How many 1/2's are there in three whole pizzas?

13. How many 1/8's in 1/4 of a pizza?

FRACTION AS PARTITIONING
Manipulatives utilized are candies in red and yellow colored wrappers.

MEANING/CONCEPT OF FRACTION

14. (Presenting 16 candies, 8 red and 8 yellow) What part of this bowl of candies is red?

15. What part of this bowl of candies is yellow (8 of 12)?

OPERATIONS ON FRACTIONS

16. (Presenting 12 candies) What would 1/3 of this bowl of candy look like?

17. (Presenting 10 candies) Here is a box of candies. What would 2/5 of the box of candies be?

18. (Presenting 5 candies) If this is 1/3 of a bag of candies, how many were in the whole bag?

19. (Presenting 9 pieces) If this is 3/5 of a box of candy, what would the full box of candy hold?

Interview Schedule #4

Think-aloud protocols paralleling problems presented in previous interviews. Problems presented individually on 3 X 5 cards. Available manipulatives are Cuisenaire rods, and circles with fractional parts, and candy counters.

MEANING/CONCEPT OF FRACTION

1. What part is shaded (2/3)?

2. What part is shaded (4/8)?

3. What part is shaded (5/12)?

EQUIVALENCE

4. 1/3 = ?/6

5. 1/2 = ?/8

6. Write another way: 2/2 =

7. Write another way: 9/3 =

8. Write another way: 6/4 =

ORDERING OF RATIONAL NUMBERS

9. Which is greater: 2/5 or 3/5

10. Which is greater: 1/3 or 1/2

11. Which is greater: 2/5 or 5/8

12. Which is greater: 7/8 or 3/4

13. Which is greater: 4/8 or 2/4

OPERATIONS ON FRACTIONS

14. Add: 1/5 + 2/5

15. Add: 3/8 + 1/8

16. Add: 1/2 + 1/3

17. Add: 1/8 + 3/4

18. Subtract: 2/3 - 1/3

19. Subtract: 4/5 - 4/5

20. Subtract: 1/2 - 3/8

21. Subtract: 2 - 2/5

22. Multiply: 1/2 X 2/3

23. Multiply: 4 X 1/5

24. Divide: 1/2 + 1/8

25. Divide: 3 + 1/2

26. What is 1/2 of 16?

27. What is 1/3 of 12?

28. What is 3/4 of 20?

APPENDIX C

Data Summary

Interview Schedule #1

MEANING/CONCEPT OF FRACTION

4. If someone from another planet, like ET, who spoke English but didn't know much about math, asked you, "What is a fraction," what would you say?

Answers:

Part-whole definition	5
Example given	3
Dividing things up (discrete?)	1
Can't explain	3

Statements:

CN	7
EX	3

5. Can you draw me a picture of what a fraction would look like?

Fractions illustrated:

1/4	5
2/5	2
1/2; 4/8=1/2	2
Unable to draw and name	3

Statements:

AC	22
CN	15
EX	2
EV	1
ER	5

6. Suppose that I'm a second grader, and I asked you, "What does 1/3 mean?" How would you explain 1/3 to a second grader?

Answers:

Part-whole definition	5
One of 3 discrete items	2
Less than 1/2	1
Don't know	4

Statements:

CN	10
AC	4
EV	3
EX	2
ER	1

7. Could you explain a fraction like a) 3/4?

Answers:

Part-whole	8
3 of 4 discrete items	2
Less than 1	1
Don't know	1

Statements: (totals a+b)

CN	17
AC	8
EX	4

b) 5/8?

Answers:

Part-whole	4
5 of 8 discrete items	1
Not asked	7

8. Would you draw me a picture of a) 2/3?
Illustrations produced:
Part-whole, circle - - - - - 6
Part-whole, other - - - - - 3
Discrete items - - - - - - - 0

b) 5/8?
Illustrations produced:
Part-whole, circle - - - - - 4
Part-whole, other - - - - - 5
Discrete items - - - - - - - 0

Statements:
totals (a+b)
AC - - 20
CN - - 9
EV - - 2
ER - - 1

9. Would you tell me a story a) about this picture (2/3);
Stories:
Life outside school - - - - - 6
School related - - - - - - - 2
Can't - - - - - - - - - - - - 1
N/A - - - - - - - - - - - - - 3

b) about this picture (5/8)?
Stories:
Life outside school - - - - - 3
School related - - - - - - - 0
Can't - - - - - - - - - - - - 2
N/A - - - - - - - - - - - - - 7

Statements:
(totals a+b)
CN - - 6
EX - - 5
AC - - 3
EV - - 1

USES OF FRACTIONS

10. If you think about people outside of school, can you think of any time or place when people might use fractions?
Examples:
Money - - - - - - - - 2
Batting averages - - 1
Time - - - - - - - - 1
% chance - - - - - - 1
Chef cooking - - - - 1

Statements:
EX - - 7
CN - - 6
ER - - 1

11. Tell me about a time when you use fractions, outside of school.
Examples:
Money - - - - - 1
Share food - - 2
Half recipe - - 1

Statements:
EX - - 6
CN - - 1

Prompts: Sharing? Measuring? Telling time? Comparing amounts? Cooking?

Statements:
EX - - 22
CN - - 12
ER - - 3

ORDERING OF THE RATIONAL NUMBERS

12. Can you tell me a fraction that would be less than 1/2?

Fractions named:

1/4 - - - - 4	1/8 - - - - 3	
1/3 - - - - 3	1/100 - - - 1	
12/26 - - - 1	4/10 - - - 1	
2 out of 16 - 1	3/8 - - - - 1	
3 out of 20 - 1	1/10 - - - 1	

Error answers:

0 - - - - - - 2
1/1 or 1 - 2
Don't know - - 4

Statements:

CN - - 14
EX - - 13
EV - - 2
ER - - 6

13. Could you give me an example of a fraction that's greater than 1/2?

Fractions named:

2/3 - - - - 5	3/4 - - - - 5	
6/10 or		
6 out of 10 - 3	5/8 - - - - 1	
4/5 - - - - 1	12/18 - - - 1	
6/8 - - - - 1	5/6 - - - - 1	

Statements:

CN - - 8
EX - - 8
EV - - 2
ER - - 7

14.a. Are there any fractions between 1/2 and ____ (previous named fraction); could you name one?

Answers:

Yes - - - - 6
No - - - - 3
N/A - - - - 3

b) How many fractions do you think there are between ____ and 1/2?

Answers:

Endless - - - - - 1
A lot - - - - - - 1
About 150 - - - - 1
About 100 - - - - 1
3 or 4 (16ths) - 1
Don't know - - - 2

Statements: (totals a+b)

CN - - 10
EX - - 9
EV - - 2
ER - - 3

15.a. Tell me a fraction that's close to 1.

Fractions named:

7/8 - - - - 3	8/9 - - - - - 2	
2/3 - - - - 2	11/12 or	
15/16 - - - 1	11 out of 12 - 2	
5/6 - - - - 1	3/4 - - - - - 1	
9/10 - - - 1	99/100 - - - 1	
999/1000 - 1	999,999/1,000,000 - 1	

Can't - - - 4

Statements:

EX - - 9
CN - - 6
ER - - 4

Error fractions named:
1/2 - - - - 2
0/1 - - - - 1
1/1 - - - - 1
Don't know - - 1

15.b. Which of two (previously named) fractions is closest to 1?

Explanations:
Each 1 piece away from 1; larger denominator means smaller pieces - 2
Diagram - - - - - - - - - - - - - - - - 1
Larger contains more pieces - - - - 1
Cross multiplied - - - - - - - - - - 1

Statements:
CN - - 4
AT - - 1
AC - - 1
PR - - 1
ER - - 1

EQUIVALENCE

16. Can you tell me a fraction that is the same as 1/2?

Fractions named:
2/4 - - - - 5
4/8 - - - - 4
6/12 - - - 2
10/20 - - - 1
3/6 - - - - 5
5/10 - - - 3
8/16 - - - 1
Can't - - - 4

Statements:
PR - - 10
EX - - 9
CN - - 8
AC - - 3
EV - - 1

FRACTIONS AS COMPARISONS

Manipulatives utilized are paper strips.

MEANING/CONCEPT OF FRACTION

17.a. (Presenting three strips of colored paper representing candy sticks: Jessie's = 4"; Marti's = 8"; Sue's = 12")
How long is Jessie's red candy stick (4") compared to Marti's yellow (8")?

Answers:
1/2 - - - - 11

Error answers:
3/5 - - - - 1

Statements:
AC - - 18
AN - - 11
CN - - 9
AT - - 3
ER - - 3

b. How long is Jessie's red candy stick (4") compared to Sue's pink (12")?

Answers:
1/3 - - - - - - 10

Error answers:
1 whole - - - - 1
1/2 - - - - - - 1
2/3 - - - - - - 1

Statements:
AC - - 18
CN - - 9
AN - - 7
EV - - 1
ER - - 9

c. How long is Marti's yellow candy stick (8") compared to Sue's pink (12")?

Answers:

2/3 - - - - - - 5

Error answers:

3/4 - - - - - - 1
1 1/2 - - - - - 1
1/3 - - - - - - 1
1/4 - - - - - - 1
1/2 - - - - - - 1

Statements:

AC - - - 8
CN - - - 8
AN - - - 5
AT - - - 1
ER - - - 6

DATA SUMMARY

Interview Schedule #2

FRACTIONS AS COMPARISONS

MEANING/CONCEPT OF FRACTION

1. (Discuss units of length; present 12-rod as new unit; have participant name new unit) Okay, this is one [unit] long. How long would this rod be (6-rod), compared to a [unit] ?

Answers:	Reasons:	Statements:
1/2 - - 12.	2 6-rods fit in unit - 8	AC - - 17
	It just is (AT) - - - 3	CN - - 14
	Measuring - - - - - - - 1	AN - - 8
		AT - - 3
		EV - - 1

2. How long is this (4-rod) compared to a (12-rod) [unit]?

Answers:	Reasons:	Statements:
1/3 - - - 9	3 4-rods fit in unit - 9	AC - - 17
Thirds - - 1	N/A - - - - - - - - - - 1	CN - - 14
Error answers:		AN - - 6
One whole - 1		EV - - 2
Half of a		AT - - 1
half - - 1		ER - - 2

3. Can you show me what rod would be 1/4 as long as a [unit] ?

Initial estimates:	Final selections:	Statements:
2-rod - - - - 4	3-rod - - - - 9	AC - - 29
3-rod - - - - 3	don't know - 1	EV - - 11
5-rod - - - - 2		CN - - 7
8-rod (space		AN - - 3
remaining) - 1		ER - - 11
N/A - - - - - 2		

4. How long would this (9-rod) be compared to a [unit]?

Answers:	Reasons:	Statements:
3/4 - - - - 7	equals 3 3-rods - - - 5	AC - - 23
2/3 -> 3/4 - 1	1 3-rod fills	CN - - 16
Error answers:	remaining space - - 4	AN - - 6
1/3 - - - - 1	Error reasons:	AT - - 2
1 1/3 - - - - 1	unit represents 1;	EV - - 2
N/A - - - - - - 2	3 3-rods = third - - 1	ER - - 5

5. What length is this (2-rod) compared to a [unit]?

Initial est:
- 1/6 - - - - 4
- 1/8 - - - - 2
- 1/5 - - - - 2
- 1/7 - - - - 1
- < 1/4 - - - 1

Answers:
- 1/6 - - - - 8
- sixths - - 1

Error answers
- 1/1 - - - - 1

Don't know - - 1
N/A - - - - - 1

Reasons:
- Six 2-rods = unit - 8
- not room for 8 - - 1

Statements:
- AC - - 21
- CN - - 12
- AN - - 9
- AT - - 1
- EV - - 1
- ER - - 2

6. Now let's say we have another unit, and it looks like this (8-rod). What do you think we should call this? a) Now what length is this (2-rod) compared to an [alt. unit]?

Initial estimates:
- 1/4 - - - - 4
- 1/3 - - - - 2

Answers:
- 1/4 - - - - 9
- N/A - - - - 3

b) How can this be _____ one time and _____ another time?

Reasons:
- Units are different or depends on length of unit - - - 9
- Don't know - - - - - - - - - - - - - 1
- N/A - - - - - - - - - - - - - - - - - 2

Statements: (totals a+b)
- CN - - 18
- AC - - 11
- AN - - 6
- EV - - 3
- EX - - 1
- ER - - 1

EQUIVALENCE

7. Let's go back to our original unit (12-rod). You named this a [unit]. What length was this rod then (4-rod)? Can you think of another way to show the same length as 1/3 of a [unit]?

Fractions produced:
- 2/6 - - - 5
- 4/12 - - - 3

Don't know - 3
N/A - - - - 3

Fractions produced and named:
- 2/6 - - - 5
- 4/12 - - 3

Statements:
- AC - - 23
- CN - - 19
- EV - - 7
- AN - - 3
- ER - - 1

8. How many ways could you show 1/2 of a [unit]?

Fractions produced:
- 2/4 - - - - 7
- 3/6 - - - - 6
- 6/12 - - - 3

Don't know - 2
N/A - - - - - 2

Fractions produced and named:
- 3/6 - - - - - 5
- 2/4 - - - - - 4
- 6/12 - - - - 2
- Don't know - 5

Statements:
- AC - - 26
- CN - - 21
- EV - - 5
- AN - - 2
- ER - - 2

FRACTIONS AS SUBDIVISIONS

Manipulatives utilized are circles and with fractional parts. Circles are presented as whole pizzas. Participants are encouraged to move pieces around in any way they wish, to help them think about fractions.

MEANING/CONCEPT OF FRACTION

9.a. Tell me about the orange pieces (thirds).

Answers:	Reasons:	Statements:
1/3 - - - - - 10	3/3 = 1 whole - 10	AC - - 18
1/4 --> 1/3 - 1	recognize shape - 2	AN - - 11
Error answers:		CN - - 5
1/2 or 3/2 - 1		AT - - 2
		ER - - 5

9.b. What size are the white pieces (sixths)?

Answers:	Reasons:	Statements:
1/6 - - - - - - 8	6 make 1 whole - - 6	CN - - 16
1/8 --> 1/6 - 2	3 make 1/2 - - - - 4	AC - - 15
N/A - - - - - - - 2		AN - - 10
		EV - - 2
		AT - - 1

9.c. What size are the green pieces (fourths)?

Answers:	Statements:
1/4 - - - - 8	AC - - 14
fourths - - 4	CN - - 8
	AN - - 8
	AT - - 7
	EV - - 2
	ER - - 1

9.d. What would you have if you had three green pieces (3/4)?

Answers:	Statements:
3/4 - - - - - - 7	AN - - 10
1/3 --> 3/4 - 2	AT - - 6
Error answers:	AC - - 6
1/3 - - - - - - 2	CN - - 3
1/3 --> 1/2 - 1	ER - - 8

EQUIVALENCE

10.a. What would the dark blue pieces be (1/2)?

Answers:	Statements:
1/2 - - - - - - 9	AN - - 11
halves - - - 3	AT - - 7
	AC - - 4
	CN - - 2

10.b. What would 2/2 be the same as?

Answers:
- 1 whole or a whole - - - 11

Error answers:
- 1/2 - - - - - - - 1

Statements:
- CN - - 12
- AN - - 11
- AT - - 11
- AC - - 1
- ER - - 1

11. What would 9/3 be the same as?

Answers:
- 3 or 3 wholes or 3 pizzas - 10
- Don't know - - - - 1
- N/A - - - - - - - - 1

Reasons:
- 9 3 = 3 - - - - 4
- 3/3 = 1 whole - 5
- (AT) just know - 1

Statements:
- AC - - 17
- CN - - 14
- AN - - 9
- PR - - 7
- AT - - 3
- ER - - 5

12. What would 6/4 be the same as?

Answers:
- 1 1/2 or 1 1/2 pizzas - 7
- 2 -> 1 1/2 - - 1
- 6/6 -> 1 1/2 - 1

Error answers:
- 1/6 - - - - - - 1
- 6 - - - - - - - 2

Reasons:
- 4 = whole; 2 = 1/2 - 6
- divide & reduce - - 2
- take D from N; see what's left - 1

Statements:
- AC - - 15
- CN - - 14
- AN - - 10
- AT - - 3
- PR - - 3
- EV - - 1
- ER - - 9

13. How many sixths are the same as 1/2?

Answers:
- 3 - - - - - - 12

Reasons:
- 3 is half of 6 - - 8
- (AT just knew it - 2

Statements:
- CN - - 16
- AC - - 10
- AN - - 7
- AT - - 4

14. How many fourths are the same as 6/8?

Answers:
- 3 - - - - - - 5
- 5 --> 3 - - 1

Error answers:
- 4 - - - - - - 2
- 1/3 - - - - 1
- N/A - - - - - - 3

Reasons:
- visual inspection - 5
- 4/4 = 1 - - - - - - - 1
- 2/8 = 1/4 - - - - - - 1
- reduce 6/8 to 3/4 - 1

Statements:
- AC - - 19
- CN - - 10
- AN - - 5
- EV - - 3
- PR - - 1
- ER - - 4

ORDERING OF THE RATIONAL NUMBERS

16. Which is greater 2/5 or 3/5?

Answers:
- 3/5 - - - - 8
- 2/5 -> 3/5 - 2

Error answers:
- 2/5 - - - - 1

Reasons:
- visual inspection - 5
- 3 of 5 is more - - 2
- 3 > 2 - - - - - - - - 3
- >1/2 vs <1/2 - - - 1

Error reason:
- larger number = smaller fraction - 1

Statements:
- AN - - 11
- CN - - 10
- AC - - 7
- AT - - 5
- EX - - 1
- PR - - 1
- ER - - 5

17. Which is greater 1/3 or 1/2?

Answers:	Reasons:	Statements:
1/2 - - - - - 8	AT - - - - - - - - - 8	AT - - 13
1/3 --> 1/2 - 1	visual inspection - 1	AN - - 10
Error answers:	Error reasons:	AC - - 8
1/3 - - - - - 3	3 > 2 - - - - - - - - 2	CN - - 2
	visualized - - - - 1	PR - - 2
		ER - - 10

18. Which is greater 2/5 or 5/8?

Answers:	Reasons:	Statements:
5/8 - - - - 8	visual inspection - - 3	AC - - 8
Error answers:	2/5 < 1/2; 5/8 > 1/2 - 2	AN - - 5
the same - 1	AT - - - - - - - - - - 2	CN - - 6
Don't know - 1	AT - - - - - - - - - - 1	AT - - 3
N/A - - - - - 2	Error reasons:	EV - - 1
	5 > 2; 8 > 5 - - - - 2	PR - - 1
		ER - - 5

19.a. Which is greater 7/8 or 3/4?

Answers:	Reasons:
7/8 - - - - 8	1/8 left vs 1/4 left - - - 2
3/4 -> 7/8 - 1	change 3/4 to 6/8 - - - - 1
	7/8 closer to 1 - - - - - 1
	relate 1/2, 1/4, 1/8 - - - 1
	shade more pieces in 7/8 - 1

b. Which is greater 5/8 or 3/4?

Answers:	Reasons:	Statements: (total a+b)
3/4 - - - - - 3	Change 3/4 to 6/8 - - - - 1	AN - - 11
	Don't know - - - - - - - - 1	CN - - 11
	Error reason:	AC - - 8
	3/4 is half of 7/8 - - - 1	PR - - 3
		EV - - 2
		AT - - 1
		ER - - 3

20. Which is greater 4/8 or 2/4?

Answers:	Reasons:	Statements:
= or same - 8	both = 1/2 - - - - - - 4	CN - - 19
Error answers:	1/4 = 2/8 - - - - - - 2	AC - - 5
4/8 - - - - 1	reduce 4/8 to 2/4 - - 1	EV - - 2
4/8 -> 2/4 - 1	2 is half of 4 and	AN - - 1
Don't know - - 1	4 is half of 8 - 1	PR - - 1
N/A - - - - - 1	Error reasons:	ER - - 4
	4 > 2; 8 > 4 - - - - 1	
	visualized shading circles - 1	

DATA SUMMARY

Interview Schedule #3

FRACTIONS AS SUBDIVISIONS

Manipulatives utilized are circles and fractional pieces as in previous interview.

OPERATIONS ON FRACTIONS

1. If you have 1/5 of a pizza and someone gives you 2/5 of theirs, how much do you have?

Answers:
- 3/5 - - - - 9

Error answers:
- 3/3 - - - - 1
- 1 2/3 - - - 1
- 1 2/5 - - - 1

Statements:
- AN - - 9
- AC - - 8
- AT - - 6
- PR - - 3
- CN - - 2
- ER - - 7

2. What would you have if you added 3/8 of a pizza plus 1/4 more?

Answers:
- 5/8 - - - - - 7
- 3 1/2 pcs. --> 5/8 - 1

Error answers:
- 4/8 --> 1/5 - 1
- 4/12 - - - - - 1
- 1/7 - - - - - - 1

N/A - - - - - - - 1

Reasons:
- change 1/4 to 2/8 - - 8

Error reasons:
- visual inspection - - 1
- add N and add D - - - 1
- 3 + 4 = 7 - - - - - - - 1

Statements:
- AC - - 12
- CN - - 11
- PR - - 7
- AN - - 6
- ER - - 8

3. What if you started with 1/2 of a pizza, and someone gave you 1/3 more?

Answers:
- 5/6 - - - - - 4
- 4/6 --> 5/6 - 1

Error answers:
- 4/5 - - - - - 1
- 8/10 or 9/10 - 1
- 6/6 - - - - - 1
- 3/5 - - - - - 1
- 2/3 - - - - - 1
- 7 pieces - - - 1

Don't know - - - 1

Reasons:
- takes 1/6 more to = 1 - 3
- 1/2, 1/3 replaced with sixths - 1

Error reasons
- visual inspection - - - 3
- misnamed 1/6 as 1/5 - - 1
- misnamed 1/5 as 1/3 - - 1

Statements:
- AC - - 26
- CN - - 10
- EV - - 5
- PR - - 3
- ER - - 11

4. What if you had 1/8 of a pizza plus 3/4 more?

Answers:	Reasons:	Statements:
7/8 - - - - - 7	change 3/4 to 6/8 - - - 4	AC - - 12
Error answers:	each 1/4 = 2/8 - - - - 2	CN - - 9
4/5 --> 1/7 - 1	whole minus 1/8 - - - - 1	AN - - 3
1/3 - - - - - 1	Error reasons:	PR - - 3
N/A - - - - - - 3	visual inspection - - - 2	ER - - 4

5. If you start with 2/3 of a pizza, and drop 1/3 of it, what do you have left?

Answers:	Statements:
1/3 - - - - - 12	AN - - 10
	AC - - 7
	CN - - 4
	AT - - 2
	ER - - 1

6. What would be 4/5 - 4/5?

Answers:	Statements:
0, none or nothing - 11	AN - - 10
	AT - - 9
Error answers:	AC - - 2
1/5 - - - - - 1	CN - - 1
	EV - - 1
	ER - - 2

7. What if you started with 1/2 of a pizza and ate 3/8?

Answers:	Reasons:	Statements:
1/8 - - - - - 6	visual inspection - - 5	AC - - 10
5/8 --> 1/8 - 1	AT - - - - - - - - - 1	CN - - 8
Error answers:	1/2 = 4/8 - - - - - - 1	AN - - 3
3/8 --> 1/2 - 1	Error reasons:	EV - - 2
1/6 - - - - - 1	visual inspection - - 2	AT - - 1
1/5 - - - - - 1	8 - 3 = 5 --> 1/5 - - 1	ER - - 9
Don't know - - 1		
N/A - - - - - - 1		

8. What if you had 2 whole pizzas, and someone ate 2/5 of a pizza?

Answers:	Reasons:	Statements:
1 3/5 - - - - 4	AT - - - - - - - - - - 3	AC - - 10
Error answers:	visual inspection - - 2	AN - - 4
1 5/8 - - - - 1	Error reasons:	CN - - 2
8/10 = 4/5 - 1	visual inspection - - 3	PR - - 2
1 1/2 - - - - 1	two things out of 3	AT - - 1
1/8 --> 1/2 - 1		ER - - 9
3/5 - - - - - 1		
1/3 - - - - - 1		
N/A - - - - - - 2		

9. You have 2/3 of a pizza. What is 1/2 of your 2/3?

Answers:	Reasons:	Statements:
1/3 - - - - 2	1 is 1/2 of 2 - - - - 2	AC - - 15
Error answers:	Error reasons:	AN - - 2
3/6 - - - - 2	visual inspection - - 4	CN - - 2
2/5 - - - - 1	have to convert it	EV - - 2
1/5 - - - - 1	to sixths - 1	PR - - 1
1/8 - - - - 1		ER - - 6
Don't know - 4		
N/A - - - - - 1		

10. What if you had 1/6 of a pizza, four times?

Answers:	Reasons:	Statements:
4/6 - - - - 6	visual inspection - - 1	AN - - 8
Error answers:	Error reasons:	AC - - 3
5/6 - - - - 2	divide by 10 or 12 - - 1	CN - - 2
12 - - - - - 1	visual inspection - - 1	AT - - 1
1/24 - - - - 1	6 X 4 = 24 - - - - - - 1	ER - - 8
4 wholes - - 1	3 pcs = 1 whole X 4 - 1	
2 or 2 2/3		
or 2 4/10 - 1		

11. How many 1/8's are there in 1/2 a pizza?

Answers:	Reasons:	Statements:
4 or 4/8 - - 12	Half of 8 is 4 - - - - 4	CN - - 14
	visual inspection - - 2	AC - - 9
	AT - - - - - - - - - - 3	AN - - 5
	estimated - - - - - - 1	AT - - 4

12. How many 1/2's are there in three whole pizzas?

Answers:	Reasons:	Statements:
6 - - - - - 10	2 halves in each; 2X3 - - 6	CN - - 11
2 --> 6 - - 1	visual inspection - - - - 3	AN - - 8
Error answers:	Error answers:	AC - - 7
3 - - - - - 1	visual inspection - - - - 1	PR - - 3
		AT - - 1
		ER - - 2

13. How many 1/8's are there in 1/4?

Answers:	Statements:
2 or 2/8 - - 8	AC - - 9
3 --> 2 - - 1	CN - - 9
1/8 --> 2 - 1	AN - - 5
Error answers:	AT - - 2
1/8 - - - - 1	ER - - 6
Don't know - 1	

PARTITIONING

Manipulatives utilized are candies in red and yellow colored wrappers.

MEANING/CONCEPT OF FRACTION

14. (Presenting 16 candies, 8 red and 8 yellow) What part of this bowl of candies is red?

Answers:	Reasons:	Statements:
Statements:		
1/2 or half - 9	8/16 = 1/2 - - - - - - - - 3	AC - - 11
8/16 - - - - - 1	2 equal parts - - - - - - - 4	CN - - 11
8/8 -> 1/8 -> 1/2 - 1	Error reasons:	AN - - 7
Error answers:	counted 8 of each color - 1	EV - - 1
1/8 - - - - - 1		ER - - 5

15. What part of this bowl of candies is yellow (8 of 12)?

Answers:	Reasons:	Statements:
2/3 - - - - - 4	2 out of 3 groups - - - - 4	CN - - 10
8/12 - - - - - 1	8/12 = 2/3 - - - - - - - - 1	AC - - 7
3/4 --> 2/3 - 1	8 out of 12 - - - - - - - - 1	AN - - 5
1/3 --> 2/3 - 1	4 pcs = 1/3 - - - - - - - - 1	EV - - 2
Error answers:	Error reasons:	PR - - 2
3/8 --> 2/8 - 1	counted 8 pieces - - - - 3	AT - - 1
1/8 or 1/2 - 1	dividing - - - - - - - - - 1	ER - - 11
1/8 - - - - - 2		
1/4 - - - - - 1		

OPERATIONS ON FRACTIONS

16. (Presenting 12 candies) What would 1/3 of this bowl of candy look like?

Answers:	Reasons:	Statements:
4 - - - - - 7	3 groups of 4 - - - - - - - 3	AC - - 20
Error answers:	12 3 = 4 - - - - - - - - 3	CN - - 9
3 - - - - - 1	AT - - - - - - - - - - - - 1	EV - - 3
1 pc over	Error reasons:	PR - - 2
3 pcs - - 1	3 pcs represent the third;	AN - - 1
Don't know - 3	1 pc represents the 1 - 1	AT - - 1
	4 groups of 3 pcs - - - - 1	ER - - 7

17. (Presenting 10 candies) Here is a box of candies. What would 2/5 of the box of candies be?

Answers:	Reasons:	Statements:
4 - - - - 6	2/5 of 5 = 2; of 10 = 4 - - 2	AC - - 16
Error answers:	divide into 5 groups of 2 - 2	CN - - 7
2 - - - - 1	divide by D; multiply by N - 2	EV - - 1
Don't know - 1	Error reasons:	PR - - 1
N/A - - - - 4	5/X 2 = 10 - - - - - - - - - 1	ER - - 3

18. (Presenting 5 candies) If this is 1/3 of a bag of candies, how many were in the whole bag?

Answers:	Reasons:
15 - - - - - 4	multiply 3 X 5 - - - - - - - 4
Error answers:	**Error reasons:**
5 - - - - - 2	counted 5 - - - - - - - - - 2
3 - - - - - 1	

Question 18 variations:

b. (6 candies) this is 1/3 of how many?

Answers:	Reasons:
18 - - - - - 1	multiply 3 X 6 - - - - - - - 1

c. (5 candies) this is 1/5 of how many?

Answers:	Reasons:
25 - - - - - 1	multiply 5 X 5 - - - - - - - 1
Error answers:	**Error reasons:**
10 - - - - - 1	1 pc = 1/5 so that's 1/2 - 1
Don't know - - 1	

d. (5 candies) this is 1/4 of how many?

Answers:	Reasons:
20 - - - - - 1	multiply 4 X 5 - - - - - 1

Statements: (totals a-d)

CN - - 6
PR - - 6
AN - - 4
AT - - 3
AC - - 1
ER - - 5

19. (Presenting 9 pieces) If this is 3/5 of a box of candy, what would the full box of candy hold?

Answers:	Reasons:
15 - - - - - 3	9 + 2 more fifths - - - - 5
14 --> 15 - 1	5 X 3 - - - - - - - - - - - 1
23 --> 15 - 1	
12 --> 15 - 1	

Question 19 variation:

b. (8 pieces) this is 2/5 of how many?

Error answers:	Error reasons:
80 - - - - - 1	8 X 5 X 2 - - - - - - - - - 1

Statements: (totals a+b)

CN - - 11
AC - - 9
EV - - 3
AN - - 2
PR - - 1
ER - - 5

DATA SUMMARY

Interview Schedule #4

Think-aloud protocols based on problems paralleling those presented in previous interviews. Problems presented individually on 3 X 5 cards. Available manipulatives are, Cuisenaire rods, circles with fractional parts, and candy counters, as in previous interviews.

MEANING/CONCEPT OF FRACTION

1. What part is shaded (2/3)?

Answers:
  2/3 - - - - 10
Error answers:
  2/1 - - - - 1

Statements:
  AN - - 9
  AT - - 3
  CN - - 2
  ER - - 2

2. What part is shaded (4/8)?

Answers:
  1/2 - - - - - 6
  4/8 - - - - - 4
  1/4 --> 1/2 - 1

Statements:
  CN - - 11
  AC - - 10
  AN - - 5
  PR - - 2
  EV - - 1
  ER - - 1

3. What part is shaded (5/12)?

Answers:
  5/12 - - - - - 9
  5/9 --> 5/12 - 1
Error answer:
  5/7 - - - - - 1

Statements:
  AC - - 10
  CN - - 10
  AN - - 3
  PR - - 2
  ER - - 3

EQUIVALENCE

4. 1/3 = ?/6

Answers:
  2 or 2/6 - - 8
Error answers:
  1/6 --> 3/6 - 1
Don't know - - 2

Reasons:
  multiply N & D by 2 - - - 4
  3 doubled, so doubled 1 - 1
  AT - - - - - - - - - - - - 1
  visualized - - - - - - - - 1
Error reasons:
  3 can't go into 1 - - - - 1

Statements:
  AN - - 7
  PR - - 6
  AT - - 4
  CN - - 2
  AC - - 1
  ER - - 3

5. 1/2 = ?/8

Answers:
  4 or 4/8 - - 9
Error answers:
  8 - - - - - - 1
Don't know - - 1

Reasons:
  multiply N & D by 4 - - - 5
  visual inspection - - - - 1
  4 is half of 8 - - - - - 1

Statements:
  AN - - 6
  PR - - 6
  CN - - 2
  AC - - 1
  AT - - 1
  ER - - 3

6. Write another way: 2/2 =

Answers:
  1, 1/1, or
    a whole - - - 9
Error answers:
  1/2 --> 1 2/2 - 1
  1/2 - - - - - - 1

Statements:
  AN - - 8
  CN - - 6
  AT - - 4
  EV - - 3
  ER - - 3

7. Write another way: 9/3 =

Answers:
  3 - - - - - - 5
  3/1 - - - - - 1
  6/1 --> 3/1 - 1
Error answers:
  3/9 - - - - - 1
  3/1 --> 1/1 - 1
  3/3 - - - - - 1
Don't know - - 1

Reasons:
  Divide 9 by 3 = 3 - - - 6
Error reasons:
  Reduce to 3/1, then
    divide (slip) - - - - 1
  3 into 9 = 3; put 3
    on the bottom - - - - 1

Statements:
  PR - - 9
  AN - - 8
  CN - - 4
  ER - - 5

8. Write another way: 6/4 =

Answers:
  1 1/2 - - - - 6
  4/2 --> 3/2 - 1
Error answers:
  2/3 - - - - - 2
Don't know - - 2

Reasons:
  divide 6 by 4; reduce - - 5
  reduce, then divide - - - 1
  AT - - - - - - - - - - - - 1
  invert, reduce, invert - 1
  6 is larger than 4 by 2 - 1

Statements:
  PR - - 10
  AN - - 6
  CN - - 6
  AT - - 3
  EV - - 2
  ER - - 3

ORDERING OF RATIONAL NUMBERS

9. Which is greater: 2/5 or 3/5

Answers:
  3/5 - - - - 10
Error answers:
  2/5 - - - - 1

Reasons:
  3 > 2; D's same - - - - 5
  visualized - - - - - - - 1
  AT - - - - - - - - - - - 2
Error reasons:
  remember - - - - - - - - 1

Statements:
  AN - - 9
  AT - - 6
  CN - - 6
  AC - - 2
  ER - - 1

10. Which is greater: 1/3 or 1/2

Answers:	Reasons:	Statements:
1/2 - - - - - 8	demonstrates 1/2 > 1/3 - 5	CN - - 8
1/3 --> 1/2 - 1	more pcs = smaller pcs - 1	AN - - 7
Error answers:	AT - - - - - - - - - - - - 1	AC - - 5
1/3 - - - - - 1	change to sixths - - - - 1	AT - - 5
1/3 -> same - 1	cross-multiplication - - 1	PR - - 3
	Error reasons:	EV - - 3
	3 > 2 - - - - - - - - - - 1	ER - - 6
	shade the same number of parts - 1	

11. Which is greater: 2/5 or 5/8

Answers:	Reasons:	Statements:
5/8 - - - - - 10	visual inspection - - - 3	PR - - 7
Error answers:	change to 40ths - - - - 2	AC - - 6
2/5 - - - - - 1	visualized - - - - - - 2	AN - - 5
	cross-multiply - - - - 1	CN - - 3
	Error reasons:	ER - - 4
	compare N's; compare D's - 1	
	compare D's only - - - - - 1	
	identify 1/3 of circle as 2/5; 1/5 as 5/8 - - - 1	

12. Which is greater: 5/8 or 3/4

Answers:	Reasons:	Statements:
3/4 - - - - - 8	change 3/4 to 6/8 - - - 4	AC - - 9
5/8 --> 3/4 - 1	3/4 seems bigger - - - 2	AN - - 8
Error answers:	N close to D - - - - - 1	CN - - 8
5/8 - - - - - 1	5/8 needs 1/8 more to = 3/4 - - 1	PR - - 5
3/4 --> 5/8 - 1	visual inspection - - - 1	EV - - 3
	Error reasons:	AT - - 1
	compare N's only - - - 1	ER - - 6
	shade more pcs in 5/8 - 1	

13. Which is greater: 4/8 or 2/4

Answers:	Reasons:	Statements:
equal or same - 8	both = 1/2 - - - - - - - 4	CN - - 15
Error answers:	8 divided by 4 = 2 - - 1	PR - - 5
4/8 - - - - - - 2	change 2/4 to 4/8 - - 3	AT - - 3
2/4 - - - - - - 1	Error reasons:	EV - - 1
	4 > 2; 8 > 4 - - - - - 1	ER - - 5
	2/4 = 1/2 but 4/8 is less - - 1	

OPERATIONS ON FRACTIONS

14. Add: 1/5 + 2/5

Answers:	Reasons:	Statements:
3/5 - - - - - 10	Add N's only - - - - - - - 4	PR - - 11
Error answers:	AT - - - - - - - - - - - - - 3	AN - - 10
3/10 - - - - 1	Column form; add N's - - 2	AT - - 6
	Error reasons:	CN - - 1
	Add N's & add D's - - - 1	EV - - 1
		EX - - 1
		ER - - 2

15. Add: 3/8 + 1/8

Answers:	Reasons:	Statements:
4/8 or 1/2 - - 10	add and reduce - - - - 3	AN - - 8
Error answers:	Error reasons:	CN - - 8
4/16 - - - - - 1	add N's & D's - - - - - 1	PR - - 7
		AT - - 6
		ER - - 2

16. Add: 1/2 + 1/3

Answers:	Reasons:	Statements:
5/6 - - - - - 9	column form; common D - - 4	PR - - 21
Error answers:	change to sixths - - - - 3	AC - - 1
2/5 - - - - - 2	Error reasons:	AN - - 1
	add N's & D's - - - - - - 1	CN - - 1
		ER - - 4

17. Add: 1/8 + 3/4

Answers:	Reasons:	Statements:
7/8 - - - - - 8	column form; common D - - 5	PR - - 15
Error answers:	change 3/4 to 6/8 - - - - 3	AN - - 3
4/12 - - - - 1	Error reasons:	EV - - 1
14/8 = 7/4 - 1	Add N's & D's - - - - - - 2	ER - - 3
4/12 = 1/3 - 1	change to 8/8 + 6/8 - - - 1	

18. Subtract: 2/3 - 1/3

Answers:	Reasons:	Statements:
1/3 - - - - 10	AT - - - - - - - - - - - - - 6	AN - - 9
Error answers:	Column form; subtract N's - 2	AT - - 7
1/0 - - - - 1	subtract N's - - - - - - - 1	PR - - 4
	Error reasons:	EV - - 1
	subtract N's & D's - - - - 1	ER - - 2

19. Subtract: 4/5 - 4/5

Answers:		
0 or nothing - 6		
0/5 --> 0 - - 1	Error reasons:	Statements:
Error answers:	0/5; cross out the 0 - - 1	AN - - 10
0/0 - - - - - 1	0/5 = 5 - - - - - - - - 1	AT - - 5
5 - - - - - - 1	nothing in N; put a 1 - 1	PR - - 3
0/5 --> 5 - - 1		EV - - 1
1/5 - - - - - 1		ER - - 5

20. Subtract: 1/2 - 3/8

Answers:	Reasons:	Statements:
1/8 - - - - - - 8	column form; common D - - 6	PR - - 16
7/8 --> 1/8 - 1	change 1/2 to 4/8 - - - - 1	AN - - 5
Don't know - - 2		AT - - 1
		CN - - 1
		EV - - 1
		ER - - 2

21. Subtract: 2 - 2/5

Answers:	Reasons:	Statements:
1 3/5 - - - - 7	column form: 10/5 - 2/5 - 3	PR - - 25
Error answers:	column form; borrow - - - 3	EV - - 6
1 7/5 - - - - 1	visualized - - - - - - - - 1	AN - - 4
1 1/5 - - - - 1	Error reasons:	CN - - 4
Don't know - - 2	column form: 1 9/9 - 2/5;	AC - - 2
	common D = 45 - - - - - 1	ER - - 5

22. Multiply: 1/2 X 2/3

Answers:	Reasons:	Statements:
1/3 - - - - - 3	multiply N's & D's - - - 3	PR - - 15
1/2 --> 1/3 - 1	Error reasons:	EV - - 6
Error answers:	column form; common D;	AN - - 3
2 - - - - - - 2	multiply N's only - - - 5	CN - - 1
3/4 - - - - - 1	invert the 2/3 - - - - - 1	ER - - 17
1 1/2 - - - - 1		
54/12 - - - - 1		
6/3 - - - - - 1		

23. Multiply: 4 X 1/5

Answers:	Reasons:	Statements:
4/5 - - - - - 5	multiply N's & D's - - - 4	PR - - 9
Error answers:	Error reasons:	AN - - 5
10 - - - - - 1	invert 1/5; multiply - - 1	EV - - 4
4 1/5 - - - - 1	added - - - - - - - - - 1	AT - - 1
3 4/5 - - - - 1	3 9/9 X 9/45 - - - - - - 1	ER - - 11
1/5 - - - - - 1	multiply 4 X 1; 4 X 5 - 1	
Don't know - - 2		

24. Divide: 1/2 ÷ 1/8

Answers:		Statements:
4 - - - - - - 5		PR - - 18
Error answers:	Error reasons:	AN - - 4
3/8 - - - - - 1	change 1/2 to 4/8;	EV - - 2
4/8 - - - - - 1	divide N's only - - - - - 3	ER - - 11
1/2 - - - - - 1	divide N's and D's - - - - 1	
1/4 - - - - - 1		
Don't know - - 1		

25. Divide: 3 ÷ 1/2

Answers:	Reasons:	Statements:
6 - - - - - 5	3/1; invert 1/2; multiply - - 3	PR - - 5
Error answers:	2/2 in each one of three - - 1	AN - - 3
2 1/2 - - - 1	Error reasons:	CN - - 2
3 - - - - - 1	change to 2 18/18 9/18 - - 1	ER - - 5
Don't know - 4	turn 3 into halves; 6/2 = 3 - 1	

26. What is 1/2 of 16?

Answers:	Reasons:	Statements:
8 - - - - - 8	AT - - - - - - - - - - - - - - - 5	AT - - 11
Error answers:	16 divided by 2 - - - - - - 2	AN - - 7
1/8 - - - - 2	16/1 times 1/2 - - - - - - - 1	PR - - 4
15/2 - - - 1	Error reasons:	AC - - 2
	16 counters separated into	CN - - 1
	into 2 sets of 8 -> 1/8 - 1	ER - - 5
	2 into 16 = 8; keep 1 in N - 1	
	took away 16 (from N?) - - - 1	

27. What is 1/3 of 12?

Answers:	Reasons:	Statements:
4 - - - - - 7	12 divided by 3 - - - - - - 5	AN - - 6
Error answers:	1/3 X 12/1 - - - - - - - - 1	AT - - 6
3/9 - - - - 1	AT - - - - - - - - - - - - - 1	PR - - 5
1/4 - - - - 1	Error reasons:	AC - - 3
Don't know - 2	draws circle; subdivides into	CN - - 2
	12 pcs; shades 3 -> N;	ER - - 6
	unshaded pcs = 9 -> D - - 1	
	3 into 12; keep 1 in N - - 1	

28. What is 3/4 of 20?

Answers:	Reasons:	Statements:
15 - - - - - 1	divide by D; multiply by N - 3	PR - - 9
Error answers:	divide 20 into 4 parts;	AN - - 6
5 - - - - - 1	5 in each; X 3 - - - - - - 2	AT - - 1
3/17 - - - - 1	3/4 X 20/1 - - - - - - - - 1	CN - - 1
4 - - - - - 1	Error reasons:	ER - - 5
3/5 - - - - 1	20 4 - - - - - - - - - - - 1	
	draws 20 circles; shades 3	
	-> N; unshaded = 17 -> D - 1	
	20 4; keep 3 in N - - - - 1	

APPENDIX D

GROUP CONTRAST DATA SUMMARY

Interview Schedule #1

MEANING/CONCEPT OF FRACTION

4. If someone from another planet, like ET, who spoke English but didn't know much about math, asked you, "What is a fraction," what would you say?

Answers:	H	L
Part-whole definition	4	1
Example given	2	1
Discrete set	0	1
Can't explain	1	2

Statements:	H	L
CN	5	2
EX	2	1

5. Can you draw me a picture of what a fraction would look like?

Fractions illustrated:	H	L
1/4	4	1
2/5	1	1
1/2; 4/8=1/2	2	0
Unable to draw and name	0	3

Statements:	H	L
AC	13	9
CN	12	3
EX	1	2
EV	1	3
ER	0	5

6. Suppose that I'm a second grader, and I asked you, "What does 1/3 mean?" How would you explain 1/3 to a second grader?

Answers:	H	L
Part-whole definition	4	1
One of 3 discrete items	2	0
Less than 1/2	1	0
Can't explain	0	4

Statements:	H	L
CN	7	3
AC	0	4
EV	1	3
EX	2	2
ER	0	1

7. Could you explain a fraction like a) 3/4 and/or b) 5/8?

Answers:	(3/4) H	L	(5/8) H	L
Part-whole	5	3	3	1
Discrete set	1	1	1	0
Less than 1	1	0		
Don't know	0	1		
N/A	0	0	3	4

Statements:	H	L
CN	11	6
AC	1	7
EX	4	0

8. Would you draw me a picture of a) 2/3 and/or b) 5/8?

Illustrations:	(2/3) H	L	(5/8) H	L
Part-whole, circle	5	0	4	0
Part-whole, other	2	1	2	2
Can't	0	1	0	0
N/A	0	3	1	3

Statements:	H	L
AC	16	4
CN	9	0
EV	2	0
ER	0	1

9. Would you tell me a story a) about your picture (of 5/8 or 2/3)?

Stories:	H	L
Life outside school	5	0
School related	0	1
Can't	0	2
N/A	2	2

Statements:	H	L
CN	7	0
EX	5	0
AC	3	0
EV	1	0

USES OF FRACTIONS

10. If you think about people outside of school, can you think of any time or place when people might use fractions?

Examples:	H	L
Money	2	0
Batting averages	1	0
Time	1	0
% chance	1	0
Chef, cooking	1	0

Statements:	H	L
EX	7	0
CN	6	0
ER	1	0

11. Tell me about a time when you use fractions, outside of school.
(part a -- before prompts).

Examples:	H	L
Money	1	0
Cut/share food	2	0
Cut recipe in half	1	0

Statements:	H	L
EX	6	0
CN	1	0

(part b -- after prompts: sharing? measuring? telling time? comparing amounts? cooking?)

Examples:	H	L
sharing	3	3
measuring	1	0
telling time	3	0
comparing amounts	2	0
cooking	4	4
money	1	0
musical notes	1	0

Statements:	H	L
EX	15	7
CN	12	0
ER	3	1

ORDERING OF THE RATIONAL NUMBERS

12. Can you tell me a fraction that would be less than 1/2?

Fractions named:	H	L
1/4	4	0
1/8	3	0
1/3	2	1
1/10	1	0
3/8	1	0
4/10	1	0
1/100	1	0
12/26	1	0
2 out of 16	1	0
3 out of 20	1	0

Statements:	H	L
CN	12	2
EX	11	2
EV	1	1
ER	3	3

Error answers:	H	L
0	2	0
1/1 or 1	1	1
Don't know	0	4

13. Could you give me an example of a fraction that's greater than 1/2?

Fractions named:	H	L
2/3	5	0
3/4	5	0
6/10 or 6 out of 10	3	0
4/5	1	0
5/8	1	0
5/6	1	0
6/8	1	0
12/18	1	0

Statements:	H	L
EX	8	0
CN	7	1
EV	2	1
ER	2	5

Error answers:	H	L
1/4	1	0
1/5 --> 2/5	0	1
1/3	0	1
1 whole	0	1
Don't know	0	1

14.a. Are there any fractions between 1/2 and ____ (previously named fraction); could you name one?

Answers:	H	L
Yes	6	0
No	1	2
N/A	0	3

b. How many fractions do you think there are between ____ and 1/2?

Answers:	H	L
Endless	1	N/A
A lot	1	
About 150	1	
About 100	1	
3 or 4 (16ths)	1	
Don't know	1	

Statements: (a & b)	H	L
CN	10	0
EX	9	0
EV	3	0
ER	2	1
ER	2	1

15.a. Tell me a fraction that's close to 1.

Fractions named:	H	L
7/8	3	0
8/9	2	0
2/3	1	1
11/12 or 11 out of 12	1	0
15/16	1	0
5/6	1	0
3/4	1	0
9/10	1	0
99/100	1	0
999/1000	1	0
999,999/1,000,000	1	0
Error fractions named:		
1/2	0	2
0/1	1	0
1/1	1	0
Don't know	0	1

Statements:	H	L
EX	8	1
CN	6	0
ER	2	2

15.b. Which of two (previously named) fractions is closest to 1?

Explanations:	H	L
Each 1 piece away from 1; larger denominator means smaller pieces	2	0
Diagram	1	0
Larger contains more pieces	1	0
Cross multiplied	1	0
N/A	0	5

Statements:	H
CN	4
AT	1
AC	1
PR	1
ER	1

EQUIVALENCE

16. Can you tell me a fraction that is the same as 1/2?

Fractions named:	H	L
2/4	4	1
3/6	5	0
4/8	4	0
5/10	4	0
6/12	2	0
8/16	1	0
10/20	1	0
Don't know	0	4

Statements:	H	L
PR	10	0
EX	9	0
CN	7	1
AC	2	1
EV	1	0

FRACTIONS AS COMPARISONS
Manipulatives utilized are paper strips.

MEANING/CONCEPT OF FRACTION

17. (Presenting three strips of colored paper representing candy sticks: Jessie's = 4"; Marti's = 8"; Sue's = 12")

How long is Jessie's red candy stick (4") compared to Marti's yellow (8")?

Answers:	H	L
1/2 - - - - - -	7	4
Error answers:		
3/5 - - - - - -	0	1

Statements:	H	L
AC - -	10	8
AN - -	6	5
CN - -	7	2
AT - -	3	0
ER - -	1	2

How long is Jessie's red candy stick (4") compared to Sue's pink (12")?

Answers:	H	L
1/3 - - - - - -	7	2
Error answers:		
2/3 - - - - - -	0	1
1 whole - - -	0	1
1/2 - - - - - -	0	1

Statements:	H	L
AC - -	11	7
CN - -	8	1
AN - -	5	1
EV - -	0	1
ER - -	0	10

How long is Marti's yellow candy stick (8") compared to Sue's pink (12")?

Answers:	H	L
2/3 - - - - - -	5	0
Error answers:		
3/4 - - - - - -	1	0
1 1/2 - - - -	1	0
1/3 - - - - - -	0	1
1/4 - - - - - -	0	1
1/2 - - - - - -	0	1

Statements:	H	L
CN - -	10	0
AC - -	7	3
AN - -	5	0
AT - -	1	0
ER - -	2	5

GROUP CONTRAST DATA SUMMARY

Interview Schedule #2

FRACTIONS AS COMPARISONS

MEANING/CONCEPT OF FRACTION
Manipulatives utilized are Cuisenaire rods with added 12-rod.

1. (Discuss units of length; present 12-rod as new unit; have participant name new unit) Okay, this is one [unit] long. How long would this rod be (6-rod), compared to a [unit] ?

Answers:	H	L
1/2 - - -	7	5

Reasons:	H	L
(M) 2 6-rods fit in unit -	5	1
2 6-rods would = 1 -	2	0
It just is (AT) - - -	0	3
Measuring - - - - - - -	0	1

Statements:	H	L
AC - -	7	10
CN - -	10	4
AN - -	3	5
AT - -	1	2
EV - -	0	1

2. How long is this (4-rod) compared to a (12-rod) [unit]?

Answers:	H	L
1/3 or thirds - -	7	3
Error answers:		
1 whole - - - - - -	0	1
half of a half - -	0	1

Reasons:	H	L
(M) 3 4-rods fit in unit -	7	2
measurement - - -	0	1
(M) 3 4-rods placed beside unit - - -	0	2

Statements:	H	L
AC - -	11	6
CN - -	9	5
AN - -	3	3
EV - -	2	0
AT - -	1	0
ER - -	0	2

3. Can you show me what rod would be 1/4 as long as a [unit]?

Initial est:	H	L
3-rod - - -	4	1
2-rod - - -	3	1
5-rod - - -	0	2
8-rod (space remaining)-	0	1

Answers:	H	L
3-rod - - -	7	2
don't know -	0	2
N/A - - - - -	0	1

Statements:	H	L
AC - -	9	2
EV - -	2	9
CN - -	6	1
AN - -	0	3
ER - -	1	10

4. How long would this (9-rod) be compared to a [unit]?

Answers:	H	L
3/4	6	1
2/3 --> 3/4	1	0
Error answers:		
1/3	0	1
1 1/3	0	1
N/A	0	2

Reasons:	H	L
(M) 1/4 (3-rod) fills remaining space	4	0
(M) equals 3 3-rods	3	1
(M) 3-rod fills remaining space	0	1
(M) unit rod = one; 3 3-rods = third	0	1

Statements:	H	L
AC	17	6
CN	14	2
AN	6	0
AT	2	0
EV	1	1
ER	1	4

5. What length is this (2-rod) compared to a [unit]?

Answers:	H	L
1/6 or sixths	7	2
Error answers		
1/1	0	1
Don't know	0	1
N/A	0	1

Reasons:	H	L
(M) 6 2-rods fit in unit	7	1
(M) not room for 8 2-rods	0	1
(M) whole array (6/6) is 1/6; each 2-rod is 1/1	0	1

Statements:	H	L
AC	10	11
CN	8	4
AN	5	4
AT	1	0
EV	1	0
ER	0	2

6.a. Now let's say we have another unit, and it looks like this (8-rod). What do you think we should call this? What length is this (2-rod) compared to a [alt. unit]?

Answers:	H	L
1/4	5	1
1/3 --> 1/4	2	1
N/A	0	3

Statements: (parts a + b)	H	L
CN	14	4
AC	8	3
AN	3	3
EV	2	1
EX	1	0
ER	1	0

b. How can this be _____ one time and _____ another time?

Reasons:	H	L
Units are different; or depends on length of unit	7	2
Don't know	0	1
N/A	0	2

EQUIVALENCE

7. Let's go back to our original unit (12-rod). You named this a [unit]. What length was this rod then (4-rod)? Can you think of another way to show the same length as 1/3 of a [unit]?

Fractions produced and named:	H	L
2/6	5	0
4/12	3	0
Don't know	0	3
N/A	1	2

Statements:	H	L
AC	18	5
CN	17	2
EV	6	1
AN	3	0
ER	1	0

8. How many ways could you show 1/2 of a [unit]?

Fractions produced and named:	H	L
3/6	5	0
2/4	4	0
6/12	2	0
Don't know	1	4
N/A	1	1

Statements:	H	L
AC	20	6
CN	19	2
EV	2	3
AN	1	1
ER	1	1

FRACTIONS AS SUBDIVISIONS

Manipulatives utilized are circles and with fractional parts. Circles are presented as whole pizzas. Participants are encouraged to move pieces around in any way they wish, to help them think about fractions.

MEANING/CONCEPT OF FRACTION

9.a. Tell me about the orange pieces (thirds).

Answers:	H	L
1/3	6	3
1/4 --> 1/3	1	0
1 whole -> 1/3	0	1
Error answers:		
2/2 --> 1/2	0	1

Reasons:	H	L
(M) 3/3 equal 1 whole	6	0
recognize shape	1	0
Error reasons:		
(M) 3/3 placed on circle	0	1

Statements:	H	L
AC	8	11
AN	6	6
CN	4	1
AT	2	0
ER	1	4

9.b. What size are the white pieces (sixths)?

Answers:	H	L
1/6 or sixths	3	5
1/8 --> 1/6	2	0
N/A	2	0

Reasons:	H	L
(M) 6 make 1 whole	2	4
(M) 3 make 1/2	3	1

Statements:	H	L
CN - -	10	6
AC - -	6	9
AN - -	5	5
EV - -	1	1
AT - -	0	1

9.c. What size are the green pieces (fourths)?

Answers:	H	L
1/4 or fourths -	7	5

Statements:	H	L
AC - -	7	7
CN - -	5	3
AN - -	5	3
AT - -	5	2
EV - -	1	1
ER - -	0	1

9.d. What would you have if you had three green pieces (3/4)?

Answers:	H	L
3/4 - - - - - -	7	0
1/3 --> 3/4 -	0	2
Error answers:		
1/3 - - - - - -	0	2
1/3 --> 1/2 -	0	1

Statements:	H	L
AN - -	7	3
AT - -	6	0
AC - -	2	4
CN - -	1	2
ER - -	0	8

EQUIVALENCE

10.a. What would the dark blue pieces be (1/2)?

Answers:	H	L
1/2 or halves -	7	5

Statements:	H	L
AN - -	7	4
AT - -	6	1
AC - -	2	2
CN - -	0	2

10.b. What would 2/2 be the same as?

Answers:	H	L
1 whole or a whole - - -	7	4
Error answers:		
1/2 - - - - - - -	0	1

Statements:	H	L
CN - - -	7	5
AN - - -	7	4
AT - - -	7	4
AC - - -	0	1
ER - - -	0	1

11. What would 9/3 be the same as?

Answers:	H	L
3 or 3 wholes - - - - - -	6	2
2 --> 3 wholes - - - - - -	1	0
1/3 -> 6/6 -> 3 wholes -	0	1
Don't know - - - - - - - - -	0	1
N/A - - - - - - - - - - - -	0	1

Reasons:	H	L
9 divided by 3 - -	4	0
3/3 make 1 whole -	2	0
just know it (AT)-	1	0
(M) 3/3 make 1 whole -	0	3

Statements:	H	L
AC	6	11
CN	8	7
AN	7	2
PR	7	0
AT	3	0
ER	1	4

12. What would 6/4 be the same as?

Answers:	H	L
1 1/2 or 1 1/2 pizzas	7	0
a whole & a half	0	1
1/2 --> 1 1/2	0	1
Error answers:		
6	0	1
6 & 4 pcs left	0	1
1/6	0	1

Reasons:	H	L
(M) 4 = whole; 2 = 1/2	4	2
6 divided by 4	2	0
take D from N; see what's left	1	0
Error reasons:		
(M) 4/4 placed on circle; 2/4 beside	0	3

Statements:	H	L
AC	6	9
CN	11	3
AN	7	3
AT	3	0
PR	3	0
EV	0	1
ER	0	9

13. How many sixths are the same as 1/2?

Answers:	H	L
3	7	5

Statements:	H	L
CN	9	7
AC	2	8
AN	5	2
AT	4	0

Reasons:	H	L
(M) 2/6 or 3/6 placed on circle	2	3
(M) 6/6 placed on circle	0	2
3 is half of 6	2	0
using sixths, make whole number [6/6]; take half-	1	0
just knew (AT)	2	0

14. How many fourths are the same as 6/8?

Answers:	H	L
3 or 3/4-	5	0
5 --> 3	1	0
Error answers:		
1/3	0	1
Don't know	0	2
N/A	1	2

Reasons:	H	L
(M) 3/4 placed on top of 6/8	3	0
(M) each 1/4 takes up 2/8	2	0
2/8 = 1/4; 6/8 = 3/4	1	0
Error reasons:		
fourths take up 2/8	0	1

Statements:	H	L
AC	11	8
CN	9	1
AN	2	3
EV	2	1
PR	1	0
ER	1	3

ORDERING OF THE RATIONAL NUMBERS

16. Which is greater 2/5 or 3/5?

Answers:	H	L
3/5	7	1
2/5 --> 3/5	0	2

Reasons:	H	L
3 > 2; D's same	2	1
3 of 5 is more	1	0
3/5 is bigger	0	1
visualizing	1	1
(M) 2/5 < 1/2; 3/5 > 1/2	1	0
(M) physical comparison	2	1

Error answers:		
3/5 --> 2/5	0	1
3/5 --> 2/5	0	1

Error reasons:	H	L
larger number = littler fraction	0	1

Statements:	H	L
AN	6	5
CN	7	3
AC	4	3
AT	4	1
EX	1	0
PR	1	0
ER	0	5

17. Which is greater 1/3 or 1/2?

Answers:	H	L
1/2	6	2
1/3 -> 1/2	0	1

Reasons:	H	L
1/2 is more (AT)	6	2
(M) physical comparison	0	1

Error answers:		
1/3	1	2

Error reasons:	H	L
2 < 3	1	0
visualized shading circles	0	1
1/3 is bigger	0	1

Statements:	H	L
AT	12	1
AN	6	4
AC	4	4
CN	3	0
PR	2	0
ER	2	8

18. Which is greater 2/5 or 5/8?

Answers:	H	L
5/8	7	1
Error answers:		
same	0	1
Don't know		
N/A	0	2

Reasons:	H	L
(M) physical comparison	3	0
2 < 1/2 of 5; 5 > 1/2 of 8	2	0
just know it (AT)	2	0
Error reasons:		
5 > 2; 8 > 5	1	1
N/A	0	1

Statements:	H	L
AC	7	1
AN	4	1
CN	6	0
AT	3	0
EV	1	0
PR	1	0
ER	2	3

19.a. Which is greater 7/8 or 3/4?

Answers:	H	L
7/8	6	2
3/4 --> 7/8	0	1

Reasons:	H	L
1/8 left vs 1/4 left	2	0
change 3/4 to 6/8	1	1
7/8 closer to 1	1	0
(M) 4/8 = 1/2; 2/8 = 1/4; difference is 1/8	1	0
(M) physical comparison	0	1
shade more pieces in 7/8	0	1
Error reasons:		
compare N's and D's	1	0

b. Which is greater 5/8 or 3/4?

Answers:	H	L
3/4	1	2

Reasons:	H	L
Change 3/4 to 6/8	1	0
Don't know why	0	1
Error reasons:		
3/4 is half of 7/8	0	1

Statements:	H	L
CN	8	3
AN	5	6
AC	5	3
PR	2	1
EV	0	2
AT	1	0
ER	1	2

20. Which is greater 4/8 or 2/4?

Answers:	H	L
=; same	7	1

Reasons:	H	L
both = 1/2	3	1
(M) physical comparison	1	0
(M) each 1/4 = 2/8	1	0
2 = half of 4; 4 = half of 8	1	0
reduce 4/8 to 2/4	1	0

Error answers:

4/8	0	1
4/8 -> 2/4	0	1
Don't know	0	1
N/A	0	1

Error reasons:

4 > 2; 8 > 4	0	1
visualized shading circles	0	1

Statements:

	H	L
CN	14	5
AC	3	2
EV	1	1
AN	0	1
PR	1	0
ER	0	4

GROUP CONTRAST DATA SUMMARY

Interview Schedule #3

FRACTIONS AS SUBDIVISIONS

Manipulatives utilized are circles and fractional pieces as in previous interview.

OPERATIONS ON FRACTIONS

1. If you have 1/5 of a pizza and someone gives you 2/5 of theirs,how much do you have?

Answers:	H	L
3/5	7	2
Error answers:		
3/3	0	1
1 2/3	0	1
1 2/5	0	1

Statements:	H	L
AN	8	1
AC	0	8
AT	6	0
PR	3	0
CN	0	2
ER	0	7

2. What would you have if you added 3/8 of a pizza plus 1/4 more?

Answers:	H	L
5/8	6	1
3 1/2 pcs. -> 5/8	1	0
Error answers:		
4/8 -> 1/5	0	1
4/12	0	1
1/7	0	1
N/A	0	1

Reasons:	H	L
change 1/4 to 2/8	5	0
(M) 3/8 + 1/4 -> 2/8 on circle	2	1
Error reasons:		
(M) 3/8 + 1/4 on circle	0	1
add N and add D	0	1
3 + 4 = 7	0	1

Statements:	H	L
AC	3	9
CN	9	2
PR	1	6
AN	5	1
ER	1	7

3. What if you started with 1/2 of a pizza, and someone gave you 1/3 more?

Answers:	H	L
5/6	4	0
4/6 --> 5/6	1	0

Reasons:	H	L
(M) takes 1/6 more to = 1	4	0
(M) 1/2 + 1/3 covered by sixths	1	0

Error answers:

	H	L
4/5	1	0
8/10 or 9/10	1	0
6/6	0	1
7 pcs -> 7/8	0	1
3/5	0	1
2/3	0	1
Don't know	0	1

Error reasons:

	H	L
(M) 1/6 called 1/5	1	0
(M) 1/2 + 1/3 on circle	1	2
(M) 1/2 + 1/5; replaces with 3/5	0	1
just added	0	1

Statements:

	H	L
AC	19	7
CN	10	0
EV	3	2
PR	2	1
ER	5	6

4. 1/8 plus 3/4?

Answers:

	H	L
7/8	6	0
9/10 --> 7/8	1	0

Reasons:

	H	L
change 3/4 to 6/8	4	0
(M) each 1/4 = 2/8	2	0
(M) takes 1/8 more to = 1	1	0

Error answers:

	H	L
4/5 --> 1/7	0	1
1/3	0	1
N/A	0	3

Error reasons:

	H	L
(M) 3/4 + 1/8; 3/4 --> 6/8	0	1
(M) 3/4 + 1/8 on circle	0	1

Statements:

	H	L
AC	7	5
CN	9	0
AN	3	0
PR	2	1
ER	1	3

5. If you start with 2/3 of a pizza, and drop 1/3 of it, what do you have left?

Answers:

	H	L
1/3	7	5

Statements:

	H	L
AN	6	4
AC	1	6
CN	1	3
AT	2	0
ER	0	1

6. What would be 4/5 - 4/5?

Answers:

	H	L
0, none or nothing	7	4

Error answers:

	H	L
1/5	0	1

Statements:

	H	L
AN	7	3
AT	7	2
AC	0	2
CN	0	1
EV	0	1
ER	2	0

7. 1/2 - 3/8

Answers:	H	L
1/8 - - - - -	5	1
5/8 --> 1/8 -	1	0
Error answers:		
3/8 --> 1/2 -	0	1
1/6 - - - - -	0	1
1/5 - - - - -	0	1
Don't know - -	0	1
N/A - - - - - -	1	0

Reasons:	H	L
(M) 3/8 + 1/2 on circle - - - -	4	1
just know it (AT) - - - - -	1	0
4/8 = 1/2 - - - - - - - - -	1	0
Error reasons:		
(M) 3/8 + 1/2 on circle - - - -	0	1
(M) 1/3 + 1/2 on circle - - - -	0	1
subtract 3 off of 8 - - - -	0	1

Statements:	H	L
AC - - -	6	4
CN - - -	7	1
AN - - -	2	1
EV - - -	2	0
AT - - -	1	0
ER - - -	1	8

8. What if you started with 2 whole pizzas and then you ate 2/5 of a pizza?

Answers:	H	L
1 3/5 - - -	5	0
Error answers:		
1 5/8 - - -	1	0
8/10 = 4/5 -	1	0
1 1/2 - - -	0	1
1/8 -> 1/2 -	0	1
3/5 - - - -	0	1
1/3 - - - -	0	1

Reasons:	H	L
just know it (AT) - - - - -	3	0
(M) 2/5 on one circle - - - - -	2	0
Error reasons:		
(M) 2/5 on circle; 2/8 on top -	1	0
(M) 5/5 on 2 circles; removes 2/5 - - - - -	1	0
(M) 2 circles; 2/4 on one - - -	0	1
(M) 2/5 + 5/8 on circle - - - -	0	1
(M) 2 circles; 1/5 on one - - -	0	1
2 things out of 3; 1 left -	0	1

Statements:	H	L
AC - -	6	4
AN - -	3	1
CN - -	2	0
PR - -	2	0
AT - -	1	0
ER - -	2	7

9. You have 2/3 of a pizza. What is 1/2 of your 2/3?

Answers:	H	L
1/3 - - -	2	0

Reasons:	H	L
1 is half of 2 - - - - - - - - -	1	0
(M) 2/3 on circle - - - - - - - - -	1	0

Error answers:		
3/6 - - -	2	0
2/5 - - -	1	0
1/5 - - -	0	1
1/8 - - -	0	1
Don't know -	1	3
N/A - - - -	1	0

Error reasons:		
have to convert it to sixths -	1	0
(M) 1/2 + 1/6 on 2/3 - - - - - - -	1	0
(M) 2/3 on circle - - - - - - - - -	1	0
(M) 2/3 + 3/8 on circle - - - - - -	0	1
(M) 2/3 + 1/8 on circle - - - - - -	0	1

Statements:	H	L
AC - - -	6	9
CN - - -	2	0
AN - - -	1	1
EV - - -	1	1
PR - - -	1	0
ER - - -	3	3

10. What if you had 1/6 of a pizza, four times?

Answers:	H	L
4/6 - - - - -	5	1

Reasons:	H	L
(M) 4/6 on circle - - - - - - -	0	1
N/A - - - - - - - - - - - -	5	0

Error answers:		
2 or 2 2/3 or 2 4/10 -	1	0
5/6 - - - - -	1	1
12 - - - - - -	0	1
1/24 - - - - -	0	1
4 wholes - -	0	1

Error reasons:		
(M) 5/6 on circle - - - - - - -	1	0
dividing by 10 or 12 - -	1	0
6 X 4 = 24 - - - - - - - -	0	1
(M) 3 pcs = 1 whole X 4 - - -	0	1
N/A - - - - - - - - - - - -	0	2

Statements:	H	L
AN - - -	7	1
AC - - -	1	2
CN - - -	2	0
AT - - -	1	0
ER - - -	3	5

11. How many 1/8's are there in 1/2 a pizza?

Answers:	H	L
4 or 4/8 - -	7	5

Reasons:	H	L
Half of 8 is 4 - - - -	3	1
1/2 = 4/8 (AT) - - - -	1	0
(M) 4/8 on circle - - - -	1	2
estimated - - - - - - -	0	1
N/A - - - - - - - - - -	2	1

Statements:	H	L
CN - - -	8	6
AC - - -	2	7
AN - - -	5	0
AT - - -	4	0

12. How many 1/2's are there in three whole pizzas?

Answers:	H	L
6	7	3
2 --> 6	0	1

Reasons:	H	L
2 halves in 1 whole; 2 X 3	5	1
(M) 3 circles; 2 + 2 + 2	1	0
visualized	0	3
N/A	1	0

Error answers:		
3	0	1

Error answers:		
(M) 3 circles; 1/2 on each	0	1

Statements:

CN	9	3
AN	5	3
AC	3	4
PR	3	0
AT	1	0
ER	0	2

13. How many 1/8's are there in 1/4?

Answers:	H	L
2 or 2/8	5	3
3 --> 2	1	0
8 --> 2	0	1

Reasons:	H	L
8/8 = 1 whole; divided by 4	1	0
2/8 would fit into 1/4	1	0
(M) 2/8 on 1/4	1	1
(M) 2/8 on circle	0	1
(M) 4/4 on circle; remove 3/4	0	1
N/A	4	0

Error answers:		
1/8	1	0
Don't know	0	1

Error reasons:		
dividing	0	1

Statements:

	H	L
AC	2	7
CN	5	4
AN	4	1
AT	2	0
EV	1	0
ER	5	1

PARTITIONING

Manipulatives utilized are red and yellow wrapped candies.

MEANING/CONCEPT OF FRACTION

14. (Presenting 16 candies, 8 red and 8 yellow) What part of this bowl of candies is red?

Answers:	H	L
1/2 or half	7	2
8/16	0	1
8/8 -> 1/8 -> 1/2	0	1

Reasons:	H	L
reduce 8/16 = 1/2	3	0
(M) counted 16; 8 red	0	1
(M) 8 red;8 yellow;half of each	1	2
looks like half	0	1
N/A	3	0

Error answers:			Error reasons:		
1/8 - - - - - -	0	1	(M) 8 red; 8 yellow; 1/8 - - -	0	1

Statements:	H	L
AC - - -	7	4
CN - - -	8	3
AN - - -	3	4
EV - - -	0	1
ER - - -	0	5

15. What part of this bowl of candies is yellow (8 of 12)?

Answers:	H	L	Reasons:	H	L
2/3 - - - - - -	5	0	(M) 2 out of 3 groups - - - -	4	0
			8/12; reduce - - - - - -	1	0
8/12 - - - - -	0	1	8 out of 12 - - - - - - - -	0	1
1/3 --> 2/3 -	1	0	4 pcs = 1/3 - - - - - - - -	1	0
Error answers:			**Error reasons:**		
3/8 --> 2/8 -	1	0	4 pcs = 1/8 - - - - - - - -	1	0
1/8 or 1/2 -	0	1	8 pcs out of 12 - - - - - -	0	3
1/8 - - - - - -	0	2	don't know why - - - - - -	0	1
1/4 - - - - - -	0	1			

Statements:	H	L
CN - - -	9	1
AC - - -	4	3
AN - - -	3	2
EV - - -	0	2
PR - - -	1	1
AT - - -	1	0
ER - - -	7	5

OPERATIONS ON FRACTIONS

16. (Presenting 12 candies) What would 1/3 of this bowl of candy look like?

Answers:	H	L	Reasons:	H	L
4 - - - - - -	7	0	(M) 3 groups of 4 - - - - - - - -	3	0
			12 divided by 3 - - - - - - -	2	0
			1/3 times 12/1 - - - - - - -	1	0
			just knew it (AT) - - - - - -	1	0
Error answers:			**Error reasons:**		
3 - - - - - -	0	1	1/3 = 3 pcs in each group - -	0	1
1 pc over			1 pc represents the one;		
3 pcs - -	0	1	3 pcs represent the third - -	0	1
Don't know -	0	3			

Statements:	H	L
AC - -	13	7
CN - - -	7	2
EV - - -	1	2
PR - - -	2	0
AN - - -	1	0
AT - - -	1	0
ER - - -	0	7

17. (Presenting 10 candies) Here is a box of candies. What would 2/5 of the box of candies be?

Answers:	H	L
4 - - - - - -	6	0
Error answers:		
2 - - - - - -	1	0
Don't know -	0	1
N/A - - - - - -	0	4

Reasons:	H	L
2 of 5 groups of 2 - - - - - - -	2	0
divide by D; multiply by N - -	1	0
2 out of each 5 - - - - - - - -	1	0
1/5 of 10 = 2; 2/5 = 4 - - - -	1	0
N/A - - - - - - - - - - - - - -	1	0
Error reasons:		
5 X 2 = 10 - - - - - - - - - -	1	0

Statements:	H	L
AC - -	15	1
CN - - -	7	0
EV - - -	1	0
PR - - -	1	0
ER - - -	0	3

18. (Presenting 5 candies) If this is 1/3 of a bag of candies, how many were in the whole bag?

Answers:	H	L
15 - - - - - -	4	0
Error answers:		
5 - - - - - -	0	2
3 - - - - - -	0	1

Reasons:	H	L
multiply 3 X 5 - - - - - - -	4	0
Error reasons:		
counted 5 - - - - - - - - -	0	2
don't know why - - - - - - -	0	1

Question 18 variations:

(5 candies) this is 1/5 of how many?

Answers:	H	L
25 - - - - - -	1	0
Error answers:		
10 - - - - - -	0	1
Don't know - -	0	1

Reasons:	H	L
multiply 5 X 5 - - - - - - -	1	0
Error reasons:		
1 pc = 1/5 so that's 1/2 -	0	1

(6 candies) this is 1/3 of how many?

Answers:	H	L
18 - - - - - -	1	0

Reasons:	H	L
multiply 3 X 6 - - - - - - -	1	0

(5 candies) this is 1/4 of how many?

Answers:	H	L
20 - - - - - -	1	0

Reasons:	H	L
multiply 4 X 5 - - - - - - -	1	0

Statements:

(totals)	H	L
CN	6	0
PR	6	0
AN	4	0
AT	3	0
AC	1	0
ER	0	5

19. (Presenting 9 pieces) If this is 3/5 of a box of candy, what would the full box of candy hold?

Answers:	H	Reasons:	H
15	3	9 + 2 more fifths	5
14 --> 15	1	5 X 3	1
23 --> 15	1		
12 --> 15	1		

Question 19 variation:
(8 pieces) this is 2/5 of how many?

Error answers:	H	Error reasons:	H
80	1	8 X 5 X 2	1

Statements:

(totals)	H
CN	11
AC	9
EV	3
AN	2
PR	1
ER	5

GROUP CONTRAST DATA SUMMARY

Interview Schedule #4

Think-aloud protocols based on problems paralleling those presented in previous interviews. Problems presented individually on 3 X 5 cards. Available manipulatives are, Cuisenaire rods, circles with fractional parts, and candy counters, as in previous interviews.

MEANING/CONCEPT OF FRACTION

1. What part is shaded (2/3)?

Answers:	H	L
2/3	7	3
Error answers:		
2/1	0	1

Statements:	H	L
AN	6	3
AT	3	0
CN	1	1
ER	0	2

2. What part is shaded (4/8)?

Answers:	H	L
1/2	6	0
4/8	1	3
1/4 --> 1/2	0	1

Reasons:	H	L
4 of 8; 4/8 = 1/2	6	0
4 of 8	1	3
N/A	0	1

Statements:	H	L
CN	9	2
AC	8	2
AN	1	4
EV	0	1
PR	1	1
ER	0	1

3. What part is shaded (5/12)?

Answers:	H	L
5/12	7	2
5/9 --> 5/12	0	1
Error answer:		
5/7	0	1

Statements:	H	L
AC	7	3
CN	7	3
AN	1	2
PR	2	0
ER	0	3

EQUIVALENCE

4. 1/3 = ?/6

Answers:	H	L
2 or 2/6	7	1
Error answers:		
1/6 --> 3/6	0	1
Don't know	0	2

Reasons:	H	L
multiply N & D by 2	4	1
D doubled, so doubled N	1	0
knew it (AT)	1	0
N/A	1	0
Error reasons:		
visualized shading circles	0	1

Statements:	H	L
AN	6	1
PR	6	0
AT	4	0
AC	0	1
CN	1	0
ER	0	3

5. 1/2 = ?/8

Answers:	H	L
4 or 4/8	7	2

Error answers:		
8	0	1
Don't know	0	1

Reasons:	H	L
multiply N & D by 4	5	0
(M) 3/8 on top of 1/2	0	1
4 is half of 8	0	1

Error reasons:		
N/A	2	1

Statements:	H	L
AN	5	1
PR	6	0
CN	0	2
AC	0	1
AT	1	0
ER	0	3

6. Write another way: 2/2 =

Answers:	H	L
1, 1/1, or a whole	7	2

Error answers:		
1/2 --> 1 2/2	0	1
1/2	0	1

Statements:	H	L
AN	6	2
CN	5	1
AT	4	0
EV	0	3
ER	0	3

7. Write another way: 9/3 =

Answers:	H	L
3	5	0
3/1	1	0
6/1 --> 3/1	1	0

Error answers:		
3/9	0	1
3/1 --> 1/1	0	1
3/3	0	1
Don't know	0	1

Reasons:	H	L
Divide N by D	6	0
N/A	1	0

Error answers:		
Don't know	0	1
Reduce; divide N by D	0	1
Divide N by D; keep D	0	1

Statements:	H	L
PR	6	3
AN	7	1
CN	4	0
ER	1	4

8. Write another way: 6/4 =

Answers:	H	L
1 1/2	5	1
4/2 --> 3/2	1	0

Reasons:	H	L
divide N by D	4	0
invert;reduce;invert	1	0
6 is larger than 4,by 2	1	0
reduce; divide N by D	0	1

Error answers:		
2/3	1	1
Don't know	0	2

Error reasons:		
Don't know why	0	1
N/A	0	1

Statements:	H	L
PR	5	5
AN	4	2
CN	6	0
AT	3	0
EV	2	0
ER	2	1

ORDERING OF RATIONAL NUMBERS

9. Which is greater: 2/5 or 3/5

Answers:	H	L
3/5	7	3

Reasons:	H	L
3 > 2; D's same	3	2
AT	1	0
(M) demonstrates with fifths	1	0
visualized shading circles	0	1

Error answers:		
2/5	0	1

Error reasons:		
remember it	0	1

Statements:	H	L
AN	6	3
AT	6	0
CN	4	2
AC	1	1
ER	0	1

10. Which is greater: 1/3 or 1/2

Answers:	H	L
1/2	7	1
1/3 --> 1/2	0	1

Reasons:	H	L
(M) demonstrates with pcs	3	2
more pcs = smaller pcs	1	0
AT	1	0
change both to sixths	1	0
cross-multiplication	1	0

Error answers:		
1/3	0	1
1/3 -> same	0	1

Error reasons:		
3 > 2	0	1
visualizes; shade the same number of parts	0	1

Statements:

	H	L
CN	7	1
AN	5	2
AC	2	3
AT	4	1
PR	3	0
EV	0	1
ER	0	6

11. Which is greater: 2/5 or 5/8

Answers:

	H	L
5/8	7	3

Reasons:

		H	L
(M)	demonstrates; pcs on circle	3	0
	change to 40ths	2	0
	cross-multiply	1	0
	visualized circles	1	1

Error answers:

	H	L
2/5	0	1

Error reasons:

		H	L
	2 < 5; 5 < 8	0	2
(M)	identified 1/3 of circle as 2/5; 1/5 as 5/8	0	1

Statements:

	H	L
PR	7	0
AC	5	1
AN	3	2
CN	3	0
ER	0	4

12. Which is greater: 5/8 or 3/4

Answers:

	H	L
3/4	6	2
5/8 --> 3/4	1	0

Reasons:

		H	L
	change 3/4 to 6/8	4	0
	3/4 seems bigger	0	2
(M)	5/8 needs 1/8 more to = 3/4	1	0
	N close to D	1	0

Error answers:

	H	L
5/8	0	1
3/4 --> 5/8	0	1

Error reasons:

		H	L
	shade 5 pcs vs 3 pcs	0	1
	5 > 3	0	1

Statements:

	H	L
AC	3	6
CN	7	1
AN	4	4
PR	5	0
EV	1	2
AT	1	0
ER	1	5

13. Which is greater: 4/8 or 2/4

Answers:

	H	L
equal or same	7	1

Reasons:

	H	L
both = 1/2	4	0
change 2/4 to 4/8	3	0
8 + 4 = 2; 4 2 = 2	0	1

Error answers:

	H	L
4/8	0	2
2/4	0	1

Error reasons:

	H	L
4 > 2; 8 > 4	0	1
2/4 = 1/2; 4/8 is less	0	1
N/A	0	1

Statements:

	H	L
CN	13	2
PR	4	1
AT	3	0
EV	0	1
ER	0	5

OPERATIONS ON FRACTIONS

14. Add: 1/5 + 2/5

Answers:

	H	L
3/5	7	3

Reasons:

	H	L
Add N's only	4	0
AT	3	0
Column form; add N's	0	2
N/A	0	1

Error answers:

	H	L
3/10	0	1

Error reasons:

	H	L
Add N's; add D's	1	0

Statements:

	H	L
PR	7	4
AN	7	3
AT	6	0
CN	0	1
EV	0	1
EX	0	1
ER	0	2

15. Add: 3/8 + 1/8

Answers:

	H	L
4/8 or 1/2	7	3

Reasons:

	H	L
AT	6	0
column form; add N's	0	3
N/A	1	0

Error answers:

	H	L
4/16	0	1

Error reasons:

	H	L
add N's & D's	0	1

Statements:

	H	L
AN	6	2
CN	5	3
AT	6	0
PR	1	6
ER	0	2

16. Add: 1/2 + 1/3

Answers:

	H	L
5/6	7	2

Reasons:

	H	L
column form;writes 3/6 + 2/6	4	1
change to 3/6 + 2/6	3	0
column form; 6/12 + 4/12	0	1

Error answers:

	H	L
2/5	0	2

Error reasons:

	H	L
add N's & D's	0	2

Statements:

	H	L
PR	16	5
AC	1	0
AN	1	0
CN	1	0
ER	0	4

17. Add: 1/8 + 3/4

Answers:

	H	L
7/8	7	1

Reasons:

	H	L
column form; writes 1/8 + 6/8	4	1
change 3/4 to 6/8	3	0

Error answers:

	H	L
4/12	0	1
1/3	0	1
7/4	0	1

Error reasons:

	H	L
Add N's & D's	0	2
column form; writes 8/8 +6/8	0	1

Statements:

	H	L
PR	12	3
AN	3	0
EV	0	1
ER	0	3

18. Subtract: 2/3 - 1/3

Answers:

	H	L
1/3	7	3

Reasons:

	H	L
AT	6	0
Subtract N's	1	0
Column form	0	2
N/A	0	1

Error answers:

	H	L
1/0	0	1

Error reasons:

	H	L
subtract N's; subtract D's	0	1

Statements:

	H	L
AN	7	2
AT	7	0
PR	1	3
EV	0	1
ER	0	2

19. Subtract: 4/5 - 4/5

Answers:

	H	L
0 or nothing	7	0

Reasons:

	H	L
AT	5	0
N/A	2	0

Error answers:

	H	L
0/0	0	1
5	0	1
0/5 --> 5	0	1
1/5	0	1

Error reasons:

	H	L
subtract N's; subtract D's	0	1
0/5; cross out the 0	0	1
0/5; same as 5	0	1
nothing in N; put in a 1	0	1

Statements:	H	L
AN - -	8	2
AT - -	5	0
PR - -	0	3
EV - -	0	1
ER - -	0	5

20. Subtract: 1/2 - 3/8

Answers:	H	L
1/8 - - - - -	6	2
7/8 --> 1/8 -	1	0
Don't know - -	0	2

Reasons:	H	L
column form;writes 4/8 - 3/8 -	4	2
change 1/2 to 4/8 - - - - - -	3	0

Statements:	H	L
PR - -	11	5
AN - -	5	0
AT - -	1	0
CN - -	1	0
EV - -	0	1
ER - -	1	1

21. Subtract: 2 - 2/5

Answers:	H	L
1 3/5 - - -	6	1

Reasons:	H	L
column form; 1 5/5 - 2/5 - - -	2	0
column form; 10/5 - 2/5 - - -	1	1
change 2 to 10/5 - - - - - - -	1	0
change 2 to 1 5/5 - - - - - -	1	0
visualize 2 pies; - - - - - -	1	0

Error answers:	H	L
1 1/5 - - -	1	0
1 7/5 - - -	0	1
Don't know -	0	2

Error reasons:	H	L
column form; 1 45/45 - 18/45 -	1	0
borrow; change 2/5 to 12/5 - -	0	1

Statements:	H	L
PR - -	20	5
EV - -	4	2
AN - -	3	1
CN - -	4	0
AC - -	2	0
ER - -	1	4

22. Multiply: 1/2 X 2/3

Answers:	H	L
2/6 or 1/3 -	3	2
1/2 --> 1/3 -	1	0

Reasons:	H	L
multiply N's; multiply D's -	3	1
N/A - - - - - - - - - - - - - -	1	0

Error answers:	H	L
6/3 or 2 - -	2	1
1 1/2 - - - -	1	0
3/4 - - - - -	1	0
54/12 - - - -	0	1

Error reasons	H	L
find LCD; multiply N's only -	3	1
invert 2nd factor; multiply -	1	0
invert 2nd factor; multiply -	1	0
column form; 6/12 X 8/12; multiply N's - -	0	1

Statements:	H	L
PR - -	11	4
EV - -	3	3
AN - -	1	2
CN - -	1	0
ER - -	11	6

23. Multiply: 4 X 1/5

Answers:	H	L
4/5 - - - -	4	1
Error answers:		
10 - - - -	1	0
4 1/5 - - -	1	0
3 4/5 - - -	1	0
1/5 - - - -	0	1
Don't know -	0	2

Reasons:	H	L
whole number over 1;multiply -	3	0
cross-multiply - - - - - - - - -	0	1
N/A - - - - - - - - - - - - - - -	1	0
Error reasons:		
invert 2nd factor; multiply - -	1	0
added fraction to whole number -	1	0
column form; 3 9/9 X 9/45 - - -	1	0
whole number X N & D; reduce - -	0	1

Statements:	H	L
PR - -	4	5
AN - -	3	2
EV - -	1	3
AT - -	1	0
ER - -	8	3

24. Divide: 1/2 ÷ 1/8

Answers:	H	L
4 - - - - - -	4	1
Error answers:		
4/8 - - - -	1	0
1/2 - - - -	1	0
3/8 - - - -	1	0
1/4 - - - -	0	1
1/16 - - -	0	1
Don't know -	0	1

Reasons:	H	L
invert divisor;multiply;reduce -	4	0
find LCD; divide N's; drop D - -	0	1
Error reasons:		
find LCD; divide N's; keep D - -	3	0
divide N's; divide D's - - - - -	0	1
N/A - - - - - - - - - - - - - - -	0	1

Statements:	H	L
PR - -	13	3
AN - -	4	0
EV - -	2	0
ER - -	8	6

25. Divide: 3 ÷ 1/2

Answers:	H	L
6 - - - - - -	5	0

Reasons:	H	L
whole number over 1; invert divisor; multiply -	3	0
2/2 in each one of three - - -	1	0
N/A - - - - - - - - - - - - - - -	1	0

Error answers:

	H	L
2 1/2	1	0
3	1	0
Don't know	0	4

Error reasons:

	H	L
column form; 2 18/18 9/18	1	0
3 would be 6/2 = 3	1	0

Statements:

	H	L
PR	5	0
AN	3	0
CN	2	0
ER	4	1

26. What is 1/2 of 16?

Answers:

	H	L
8	7	1

Reasons:

	H	L
AT	4	1
16 divided by 2	2	0
16/1 X 1/2	1	0

Error answers:

	H	L
1/8	0	2
15/2	0	1

Error reasons:

	H	L
16 + 2; keep 1 in N	0	1
(M) sorts 16 items into 2 sets	0	1
took away 16 (from N?)	0	1

Statements:

	H	L
AT	9	2
AN	6	1
PR	3	1
AC	0	2
CN	1	0
ER	0	5

27. What is 1/3 of 12?

Answers:

	H	L
4	7	0

Reasons:

	H	L
12 divided by 3	5	0
12/1 X 1/3	1	0
AT	1	0

Error answers:

	H	L
3/9	0	1
1/4	0	1
Don't know	0	2

Error reasons:

	H	L
subdivides circle into 12 pcs; shades 3 pcs = N; counts unshaded = D	0	1
12 divided by 3; keep 1 in N	0	1

Statements:

	H	L
AN	6	0
AT	6	0
PR	4	1
AC	1	2
CN	2	0
ER	0	6

28. What is 3/4 of 20?

Answers:	H	L
15	6	0
Error answers:		
5	1	0
3/17	0	1
3/5	0	1
4	0	1
Don't know	0	1

Reasons:	H	L
divide by D; multiply by N	3	0
divide 20 into 4 parts; 5 in each; X 3	2	0
3/4 X 20/1	1	0
Error reasons:		
divide by D	1	0
draws 20 circles; shades 3 = N; counts unshaded 17 = D	0	1
20 ÷ 4; keep 3 in N	0	1
don't know	0	1

Statements:	H	L
PR	8	1
AN	6	0
AT	1	0
CN	1	0
ER	1	4

APPENDIX F

Parent Letter and Consent Form

Cumberland Gap, TN
March 8, 1993

Dear Parent:

I would like to ask permission to interview your child as a part of a study that I'm working on. I am trying to learn more about how sixth graders understand fractions. This study may be helpful to teachers, because fractions are difficult to teach and seem to be difficult for children to learn in any meaningful way. As a participant in my study, your child would take part in four short interviews (approximately 20 minutes each). We will talk about experiences with fractions; we'll use number rods and other hands-on materials to explore the concept of fractions; and we'll work some fraction problems and talk about the processes involved.

Interviews will be videotaped so that I can carefully analyze what the children have to say about fractions. Interviews will be conducted during the school day, at times that will not interfere with regular class work, sometime between March 22 and May 28.

Although this study will result in a written dissertation report, the children in the study will be anonymous. Children's real names will not be used in the report, nor will the school be identified. Your child and the school will be given pseudonyms (false names) in the dissertation report. You, your child, and I are the only people who will have access to the video tapes, and I will keep them in a locked storage cabinet in my office for a period of three years, after which they will be erased.

There will be no tests given to your child as a participant in this study, and I will be careful not to ask any child to work beyond frustration level. Your child will not be ranked or rated as to ability in math. I am purely interested in how each child thinks about fractions.

In closing, let me tell you just a little about myself. I have taught for several years at Lincoln Memorial University, with teaching experience in both math and psychology. I'm completing my doctorate at U.T. in Educational Psychology. In the past I have taught

elementary school, grades 1-3. I live in the Cumberland Gap area, and have three children.

I would very much appreciate having your child participate in my study. If you are willing to let your child do so, please fill out the attached consent form and have your child return it to his/her math teacher. It would be helpful to have the forms returned on or before Friday, March 19. If you have any questions about the study or your child's participation in it, please call me at 869-6249 on Tuesday or Thursday between 11 am - 1 pm. Thank you very much.

Sincerely

Arlene Sprague
Asst. Professor of Human Development

CONSENT FORM

The purpose of this study is to learn more about how children understand fractions. Your child will participate in four informal 20-30 minute interviews, which will be video-taped. Your child's responses to interview questions will be kept completely confidential, and the tapes will be kept in a locked cabinet in my office. You, your child, and I are the only people who will have access to the tapes. Although the results of this study will be used in my dissertation, the identity of participants will not be revealed in the written report. Your child and his/her school will be given a pseudonym (a false name) so that they will remain anonymous.

There are no foreseeable risks involved in your child's participation in this study, nor are there any direct benefits. However, final results of the study will be made available to you upon request.

Your child's participation in this project is entirely voluntary and he/she may refuse to answer any question or may withdraw at any time without penalty. If you have any further questions about this project, please contact me at the following address or telephone number:

Arlene Sprague
Department of Human Development
Lincoln Memorial University
Harrogate, TN 37752

(615) 869-6249

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PARENTAL CONSENT

I fully understand the explanation of the research project, and I give my permission for my child,

_____ to participate.
(child's full name; please print)

_____
(parent's name; please print)

_____ _____
(parent's signature) (date)

(Please return this form to your child's math teacher. Thank you.)

VITA

Arlene Kay Erlandson Sprague was born in Anoka, Minnesota on November 18, 1946. In June of 1964 she graduated from Anoka High School. She received her Bachelor of Arts degree from the University of Minnesota in the summer of 1971. She directed a Head Start Program in North Carolina, and having subsequently completed teacher certification at the University of North Carolina at Asheville, she taught first and second grades in the North Carolina public schools. She pursued a Masters degree in Early Childhood and Elementary Education which was awarded by Lincoln Memorial University in May, 1985. At present, Arlene Sprague is an assistant professor in the Department of Human Development at Lincoln Memorial University in Harrogate, Tennessee.