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AN EXPLORATORY STUDY OF SECONDARY SCHOOL GEOMETRY STUDENTS: PROBLEM SOLVING RELATED TO CONVERGENT-DIVERGENT PRODUCTIVITY

A Dissertation

Presented for the

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ABSTRACT

This exploratory study was undertaken in order to better understand the problem-solving processes of secondary school students and to generate hypotheses for future experimental study.

A six-problem test, called the CD Test was constructed by the writer and administered to 105 geometry students in Knoxville City School System. A C-score was obtained from three problems, hypothesized to be convergent-type; a D-score was obtained from three problems, hypothesized to be divergent-type. Each student was classified as belonging to one of the following four groups based on these scores: high all-rounder, converger, diverger, or low all-rounder.

Taking samples from each of the above groups, 49 students were then observed individually as each solved a problem, designated the IO Block Problem. Two trials were given to the student. The writer took notes recording the student's problem-solving processes during the first trial, then had the student write a protocol stating his methods of solving the problem. The writer then had the student rework the problem and the processes were coded in sequence. The times for both trials of the IO Block Problem were recorded.

From the data obtained the writer was able to draw some generalizations concerning problem solving as it related to divergent thinking, trial and error methods, and sex differences. Some of the generalizations stated in the study were:

1. Students who scored high on divergent-type problems from the CD Test, made fewer generalizations in their written protocols, used trial and error methods more frequently, and took more time on the second trial of the IO Block Problem than did the students who scored low on the divergent-type problems.

2. Trial and error played a major role initially in the problem-solving task, and a minor role as the solution was approached. Continued use of trial and error increased the time needed to work the problem and seemed to be one of the main characteristics of an ineffective problem solver.

3. Girls made fewer generalizations in their written protocols, used trial and error methods more frequently, and on the average required more time to solve the problem a second time than did the boys.

The study seemed to support the following findings from previous studies: (1) divergent thinking played a minor role in mathematical thinking (Richards and Bolton, 1971); (2) students who were unable to dissect their own operations were also unsuccessful in solving complex problems (Gurova, 1969); (3) boys were better problem solvers than girls at the high school level (Sheehan, 1968). The study did not seem to support Kilpatrick's (1968) conclusion that boys used trial and error more frequently than did girls.

Eleven hypotheses based upon the generalizations drawn from the study are proposed for experimental research. The CD Test, the IO Block Problem, the students' written protocols, and a summary table of data are contained in the Appendix.

TABLE OF CONTENTS

CHAPTER		PAGE
I.	INTRODUCTION	1
	Introduction	1
	Statement of the Problem	2
	Procedures and Sources of Data	3
	Preparation for the study	3
	Procedures for the study	3
	Selection of the population	5
	Selection of the samples	5
	Elements needed for the study	6
	Importance of the Study	6
	Basic Assumptions	9
	Scope and Limitations	
	Definitions	10
	Organization of the Study	11
II.	REVIEW OF RELATED LITERATURE	12
	Introduction	12
	Classical Theories Concerning	
	the Mental Process	13
	Philosophy	13
	Mathematics	18
	Psychology	22

CHAPTER PAGE

Experimental and Exploratory Studies of the Problem-Solving Process 25
Review of experimental studies 25
Review of exploratory studies 26
Review of convergent-divergent studies 32
Methodology of exploratory studies 35
Methodology of convergent-divergent studies 38

III. PROCEDURE AND DESIGN FOR THE STUDY 41
The Study: An Overview 41
The Field Study 43
Purpose 43
Subjects 43
Development of materials and of the design of the study 43
The CD Test 46
Subjects, presentation, and administration 46
The problems 47
Method of scoring 49
Type of data 51
The IO Block Problem 51
Subjects, presentation, and administration 51

CHAPTER PAGE

The problem 53

Methods of obtaining data 54

Coding 55

Types of data 57

IV. PRESENTATION AND INTERPRETATION OF DATA 59

Introduction 59

The CD Test Results 60

Total sample 60

Subsample 61

Comparison of the distributions with Hudson's classification 61

IO Block Results Related to Time on the Second Trial 62

Classification of written protocols 62

Classification of processes 65

Generalizations 71

Variables from IO Block Problem Related to CD Test Results 72

Sex as a Variable 74

V. OBSERVATIONS AND HYPOTHESES 78

Summary of Observations 78

Comparisons with Other Studies 80

Implications for Teaching 82

Hypotheses for Experimental Research 84

CHAPTER PAGE

REFERENCES 86

APPENDIXES 95

A. Contents of CD Test 96

B. IO Block Problem 105

C. Students' Protocols 106

D. Table 14: Summary of Data for Students Working the IO Block Problem: Classifications, Times, Sex, and Scores From the CD Test 120

VITA 122

LIST OF TABLES

TABLE		PAGE
1.	Number of Students Used for IO Block Problem Sessions Classified by Schools and Groups I, II, III, IV	52
2.	Distributions in Hudson's Study, the CD Test, and the IO Block Problem	62
3.	Protocol Classification, Number of Students, and Average Time on the Second Trial	65
4.	A" Classification, Process-sequence, and Time on Second Trial (T_2)	66
5.	B" Classification, Process-sequence, and Time on Second Trial (T_2)	67
6.	C" Classification, Process-sequence, and Time on Second Trial (T_2)	68
7.	NS Classification and Observer's Comments	70
8.	Process Classification, Number of Students, and Average Time on the Second Trial	70
9.	Percent of Students in Certain Combined Classifications from Groups I, II, III, IV	74
10.	Distribution of Sexes in Samples from the CD Test and the IO Block Problem	75

TABLE PAGE

11. Distribution of Sexes from CD Test Results Compared with Hudson's Results 75

12. Males and Females in Groups for IO Block Problem and Average Time for Both Trials 76

13. Males and Females Classified by Protocols and Process-sequence Coding 77

14. Summary of Data for Students Working the IO Block Problem: Classifications, Times, Sex, and Scores from CD Test (Appendix D) 113

CHAPTER I

INTRODUCTION

I. INTRODUCTION

Einstein once said to Wertheimer: "I am not sure whether there can be a way of really understanding the miracle of thinking. Certainly you are right in trying to get at a deeper understanding of what really goes on in a thinking process. . . . (23, p. 182)." Problem solving is a significant aspect of the human condition. Since some of the most exciting and potentially contributory research in education and psychology is being conducted in the area of human development, the researcher in mathematics education should be looking at what is happening to the mathematical student. The need for exploratory studies of problem solving has been voiced by many mathematics educators. Some of these are Polya (79), Kilpatrick (59), Henderson and Pingry (41), and Dessart and Frandsen (18).

Problem solving among students varies in two ways: according to performance and according to approach. Guilford (34) divided performance during problem solving into divergent productivity and convergent productivity. Perhaps these two aspects of performance during problem solving can partly account for the differences in problem-solving approaches among students.

II. STATEMENT OF THE PROBLEM

The problem of this study was to attempt to relate the performance of a geometry student on a set of problems to his approach to another problem.

This exploratory study, a hypotheses generating one, was in two parts. The first part classified the performance of the geometry students by a test consisting of convergent and divergent type problems. The second part observed the approach of the students as they become involved in a new problem.

More specifically, the study attempted to do four things:

1. It classified students into four categories based upon their performance on a test written for this study by the researcher. This test, called the CD Test hereafter, consisted of some problems which were hypothesized to require convergent productivity and some problems which were hypothesized to require divergent productivity.

2. It noted the processes used by some students, selected from the four categories, while they solved a second problem during an individual problem-solving session.

3. It related these processes to the performance of the students on the CD Test.

4. It formulated hypotheses based upon these observations which could be tested by future experimental research.

III. PROCEDURES AND SOURCES OF DATA

Preparation for the Study

For this study the following preparatory steps were taken:

1. The relevant literature was collected and studied.

2. A search for appropriate problems was made.

3. A field study was conducted to aid in the selection of problems for classroom use and individual observations.

4. One set of problems was constructed and designated as the CD Test.

5. One observational type problem was selected based upon a field study which pretested its suitability. It was designated as the IO Block Problem.

6. A process-sequence coding system similar to Kilpatrick's (57) was developed to aid in reporting the problem-solving processes during the individual sessions.

7. Uniform wording of oral and written statements and questions to be used during the administration of the CD Test and the IO Block Problem was decided upon.

Procedures for the Study

The study, which involved seven geometry classes, was conducted in the Knoxville City School System at Young High School and at Bearden Senior High School. The following procedures were used for the study:

1. The purpose of the study was explained to the teachers and the students.

2. The CD Test was administered to these students.

3. The tests were scored and each student was classified into one of four groups.

4. A minimum number of 10 students was selected from each category. These students were used as the sample for the second part of the study.

5. Each student selected for the second part of the study was observed separately as he solved the IO Block Problem.

6. The observable behavior and the time required to solve the problem were recorded by the observer after the first trial.

7. The student wrote out answers to questions about his own mental processes during the solving of the IO Block Problem. This formed the student's written protocol.

8. The student was asked to solve the problem a second time.

9. The second sequence of observable processes in solving the IO Block Problem was recorded according to the coded system developed in the prestudy. This formed the process-sequence strategy used by the student during the second trial. Also, the time required for this second attempt was recorded.

10. The students' protocols and coded-sequence strategies were classified and analyzed.

11. The data from the IO Block Problem were compared to the students' performances on the CD Test.

12. Generalizations and hypotheses for experimental research were formed.

Selection of the Population

Geometry students from the Knoxville City School System were selected as the target population. Geometry students were chosen on the assumption that they had had at least nine years of preparatory mathematics. This would include at least one year of algebra and exposure to the methods of formal proof and inductive and deductive reasoning.

Selection of the Samples

For the field study a total of 94 geometry students from Central High School and South High School of the Knoxville City School System were administered various forms of the CD Test. They were interviewed as to the content, wording, workability and appeal of the problems. Of these students 30 were observed individually while working the IO Block Problem. From this sample the writer derived objective standards for administering and interpreting the IO Block Problem and the CD Test.

For the study 105 students from Young High School and Bearden Senior High School of the Knoxville City School System were contacted. These were administered the CD Test. Separate convergent and divergent scores were tabulated from the CD Test. Based upon this tabulation, the sample was divided into four groups. From these four groups a sample of 49 students was selected for observation while working the IO Block Problem.

Elements Needed for the Study

1. Instruments:

a. An instrument for determining convergent-divergent productivity was required. This was called the CD Test and was developed by the writer.

b. It was necessary to have a problem, mathematical in nature, which caused the subject to exhibit observable behavior while he was solving it. This was the IO Block Problem.

2. Data:

a. The C score (convergent) and the D score (divergent) from all students taking the CD Test were recorded.

b. Written descriptions of the observable activities of the students during their first attempt to solve the IO Block Problem were made.

c. The time required to solve the IO Block Problem in each attempt was recorded.

d. Protocols were obtained from the students who attempted the IO Block Problem.

e. The coded process-sequence pattern of the problem-solving activity during the second trial on the IO Block Problem was recorded.

IV. IMPORTANCE OF THE STUDY

One of the most important tasks of education is the development of good problem solvers. The study of mathematics is a natural area for

developing, improving, and modifying problem-solving processes among students. In order to accomplish such a task it is necessary to understand some of the problem-solving processes which students are capable of using and to distinguish a good problem solver from a bad one. The solution to a problem and the formal steps for solving it are insufficient indices of the actual processes a student uses when engaged in such activity. Polya (79), Kilpatrick (59), and Dessart and Frandsen (18) observed that more clues concerning these processes could be gained by using clinical and empirical observations upon them.

Productive thinking, while solving problems, was classified by Guilford (32) as convergent productivity and divergent productivity. He suggested that educational curriculum and methodology ignored certain aspects of thinking. This was especially the case with the divergent variety (33). Since most mathematics textbooks were written to elicit convergent thinking, one might suppose that the inclusion of more divergent type problems could improve the problem-solving skills and foster creativity among certain students. Thus, more time in the mathematics classroom should be given to looking at alternative methods of solving problems. This study may show what some of those alternative methods are.

Henderson and Pingry (41) have said that teachers of mathematics must understand the psychological processes of problem solving as well as the structure of mathematics. The knowledge of the different processes being used by different students on the same problem should reinforce the need for emphasizing a diversity of teaching approaches

rather than uniformity of presentation. Buswell (10) concluded that either a great variety in the process of problem solving is normal and to be accepted, or schools have been ineffective in teaching a technique of problem solving. The former alternative is more in keeping with today's emphasis on individualized instruction.

A more individualized kind of mathematics education may be obtained from a more complete understanding of the thinking processes of the human mind during a problem-solving activity. This study may help the design of such a program. It is felt that the results from a study of this type may bring about the following:

1. Help mathematics educators better understand the thinking processes of students.

2. Help mathematics teachers provide for individual differences in problem solving.

3. Influence curriculum writers of mathematics to include more problems which would elicit divergent productivity.

4. Aid curriculum writers in producing realistic behavioral objectives in mathematics.

5. Aid writers of computer-assisted instruction to sequence problems in more naturalistic modes which conform to student thinking processes.

6. Help teachers appreciate and encourage diversity in problem-solving methods among students.

7. Provide teachers and researchers with an observational method for charting the sequences of problem-solving processes.

8. Provide hypotheses for scientific testing of problem-solving processes and productive thinking.

V. BASIC ASSUMPTIONS

In this study the following assumptions were made:

1. There were convergent-type mathematical problems and divergent-type mathematical problems. These types required different types of mental productivity in their solutions.

2. The test instrument, the CD Test, provided problems of both types.

3. The test was readable by all subjects.

4. The students in the subsample did not communicate their processes and strategies to one another.

5. The observer was able to detect the outward problem-solving processes during the individual sessions.

6. The students' behavior was not seriously affected by the presence of the observer.

7. The geometry students had similar backgrounds in mathematics and were fairly homogeneous in ability (average and above average).

VI. SCOPE AND LIMITATIONS

The study drew from geometry students in the Knoxville City School System. The group of 105 students were administered the CD Test and 49 of these students were observed during an individual problem-solving

session. From this second set of students observable processes were available.

The study was limited in number due to the time required for individual observations. The CD Test was not validated though it was evaluated by a group of educators. The protocols were given in retrospect and may not have always coincided with the true sequence of thinking. However, the observations of the writer overcame some of the discrepancies.

VII. DEFINITIONS

Convergent Productivity: This is the generation of logical conclusions from given information where the emphasis is upon achieving unique or conventionally best outcomes. It might be categorized as "rigorous thinking" or "zeroing in" on the solution. Here the orderly procedure is so well recognized that it might be considered axiomatic (70, pp. 19-20).

Convergent-type Problem: This is a problem hypothesized to elicit one correct answer.

Divergent Productivity: This is the generation of logical alternatives from given information where the emphasis is upon variety, quantity, and relevance of output from the same source (70, p. 20).

Divergent-type Problem: This is an open-ended problem hypothesized to elicit many responses.

VIII. ORGANIZATION OF THE STUDY

Chapter I contains an introduction to the study of problem solving and productive thinking. It presents the problem, procedures and sources of data, the importance of the study, basic assumptions, scope and limitations of the study, definitions, and the organization of the study.

Chapter II contains a review of the classical literature related to the mental process as well as a review of the experimental and exploratory studies related to the problem-solving process.

Chapter III presents the design and procedures employed in carrying out the study.

Chapter IV is a presentation and interpretation of the data collected in the study.

Chapter V contains a summary of the observations made, comparisons with previous studies, implications for teaching, and hypotheses for experimental research.

CHAPTER II

REVIEW OF RELATED LITERATURE

I. INTRODUCTION

From what source does the solution to a problem flash forth? What are the steps which lead to this resolution? What are the decisive features of such a process? For more than two thousand years scholars have attempted to answer these questions which emerge out of the act of productive thinking. In the earliest stages of this search the answers proposed were largely theoretical. The advent of modern science has moved the inquiry into the view of close observation and experimentation.

Mathematics education has become concerned with many of these mental processes. However, since the viewpoints of scholars affect their research, the approaches have differed among these persons. Generally, there are two categories. The empiricist approaches the subject as an applied science and his work is governed by the rules of good experimentation. He strives to find answers that are reliable and at the same time fit into some theoretical framework. The behavioristic schools of psychology are generally quite influential in efforts of this type. On the other hand, the intuitionist approaches mathematics education as an art. He places more reliance on his own and the judgments of others rather than on data and statistical methods. These studies are exploratory and clinical in nature. The gestalt and cognitive schools of psychology are the major influence here.

The remainder of this chapter will present a representative review of many of the facets of the problem-solving process. The first major portion of the chapter considers the classical theories about mental processes. This is subdivided into philosophy, mathematics, and psychology. The second major part considers the experimental and exploratory studies of problem-solving processes and has five subdivisions. The first subdivision is a brief review of experimental studies of an empirical nature. The second portion covers some of the exploratory and clinical type research which has been done by intuitionists. Thirdly, studies of convergent and divergent thinking patterns are examined. Finally, there is an examination of the methodological techniques used in exploratory studies and in the research into convergent-divergent thinking patterns.

II. CLASSICAL THEORIES CONCERNING THE MENTAL PROCESS

Philosophy

Through the mouth of Socrates, Plato taught that all significant knowledge is placed in the universal (73, pp. 113-114). The universal was what today might be called a general idea of an abstract nature. This knowledge, innate in man, was obtained in a prior existence (75, pp. 226-228). Learning then was an activity of recall, a process of recollection (76, pp. 389-390; 75, p. 228). The activity of man's mind moved between his present existence on earch and his previous condition in a world of ideas. The environment of this world stimulated the mind to recall the universals of the previous world. This view was

illustrated in the Meno (74, pp. 180-182). In that dialogue Socrates presented a problem to an untutored, Greek slave. Given a square whose area is eight square units, the slave calculated the approximate length of the side of the square. Yet, the answer was elicited from the youth rather than taught to him, implying that the knowledge was innate.

Aristotle also considered significant knowledge to be the grasp of the universal (3, pp. 100-101). However, in contrast to Plato, he did not hold the datum to be innate. The mind of man was a tabula rasa which was written on by experience (1, p. 664). In coming to knowledge the mind became congruent with the object perceived and then, by a process of abstraction, the perception was refined to more and more general proportions (1, p. 664). The object in the world was composed of a knowable and unknowable principle (2, pp. 269-270). In abstraction, the mind locked upon the knowable portion, the "form" (2, p. 271). The initial grasp of the form was intuitive (3, p. 136). This implied a process of induction, of experimentation. Given the primary intuitions obtained by induction, the mind could come to other, valid conclusions by the activity of deduction (3, p. 104).

With Plato and Aristotle the parameters of the field of learning were drawn. Idealism or the emphasis upon ideas was paired off against realism or the emphasis upon the thing. Since the writings of Aristotle were the more systematic of the two authors, the logical studies of the first 1500 years after Christ were dominated by his thought (66, pp. 1035-1036). However, the emphasis was placed upon the deductive aspects of Aristotle's studies. Induction and experimentation were not given due

consideration until the birth of modern science in the time of Francis Bacon and Rene Descartes (66, p. 1040).

In the hands of Bacon the world ceased to be an object presented for contemplation and became a matrix of force to be studied, controlled, and manipulated by men (7, p. 107). On the negative side he brought forth a devastating criticism of Aristotelian logic (7, p. 108) and showed the impossibility of obtaining a complete enumeration of observable incidences. Consequently, the universal which was intended to apply to all incidences of the data became an impossibility, being at best a probable assumption. This moved the emphasis from so-called "certain" knowledge to degrees of probability. In a positive vein, Bacon attempted to lay out a method by which inductions and experimentation could be carried on (7, pp. 138-195). Though the attempt appeared decadent today, it has the accolade of being the first systematic attempt.

Descartes attempted to reduce thought to a mathematical model. In his Rules for the Direction of the Mind he stated guidelines which were intended to reduce data to the most simple portions (17, p. 7) upon which the mind performed an intuitive grasp of these simple units and proceeded upon a course of reconstruction (17, p. 7). Thus, truth was slowly but surely established and man moved forward. The criterion for certitude was the clarity of the idea under scrutiny (16, p. 82). Confronted with clarity, the mind intuitively grasped the notion with certitude (17, p. 18). In spite of his acknowledgment of the need for experimentation and observation, Descartes held on to a belief in the

innate nature of certain ideas (16, p. 83). It remained to the British Empiricists to attack and destroy this view.

John Locke and David Hume held firmly to the new-found emphasis upon observation, induction, and experimentation (50, p. 240). In his An Essay Concerning Human Understanding, Locke attacked the notion of innate ideas; knowledge was reduced to what could be traced back to original perceptions (65, pp. 95-122). Hume continued this emphasis in a more coherent and rigorous approach by explaining transitions from one idea to another by association, thus laying the definitive foundation of the empiricist creed (45).

In reaction against Hume, Emmanuel Kant postulated a priori intuitions and categories in the human mind (53, pp. 14-15). In this view the data provided by the world of perception was screened by the mind, was processed into the forms which the consciousness knew. Accordingly, ideas were a blend of materials. A portion was provided by the mind and a part was provided from the outside world. Since man resided as it were on the inside of his consciousness, it followed that the data apart from the perception of it was unknowable. This noumenal world then was a "I know not what" (53, p. 98).

For Fichte it was but a brief step from saying there was no knowledge of the noumenal world to saying it was to be left out of the discussion (26, pp. 54-55). The German Idealists, especially Hegel, by concentrating on the data of consciousness, prepared the way for the phenomenological and existential investigations of the present day (51, pp. 385-388).

At this point philosophy seemed to have reached two clear traditions: the empiricist position derived from Hume and the phenomenological-existential view derived from Kant. The former took consciousness to be transparent: it was not considered to add anything significant to man's knowledge. The latter considered consciousness to be translucent and to radically affect thought. Though the two traditions were distinguished, they continued to meander into each other. John Dewey's work was an example of such a mixture.

Dewey preferred the term, "inquiry," when referring to the concept of intelligence (51, p. 282), thus reflecting his view that the mind was directive and active and had an input into the end products. Yet, the thrust of his studies of mental activity was on what was done with a problem (51, p. 282). The input of the mind was not given the emphasis of an Immanuel Kant. In this last sense Dewey seemed oriented toward an empirical view having held that the limits of any unit of thinking was a confused situation at the beginning and a cleared-up, resolved position at the end (20, pp. 100-107). Between these two extremes Dewey listed five landmarks on the problem-solving road: suggestions, intellectualization of the felt difficulty, formation of an hypothesis, reasoning from the hypothesis, and testing each tentative position (20, pp. 100-107). The entire mental process for Dewey was dynamic and, since no positions were absolute, allowance was made for an evolutionary pattern (19, pp. 321-332).

Like Dewey, Whitehead placed emphasis on the evolutionary thesis. Through his "philosophy of organism," Whitehead attempted to overcome

the subject-object dualism which he attributed to Descartes (104, pp. 9 & 168). Placing the thinker in his thought did away with the dualism (104, pp. 9-10). In striving to construct his view Whitehead distinguished the movement of thought or inference in the mind, the eternal objects thought about, and the possible exemplification of these eternal truths in the physical world (51, p. 311). It was not altogether clear where these eternal truths resided but they were taken to be an objective reality (104, pp. 22-25). Mathematical reasoning therefore traversed a truly real pattern; it was something men found, not a subjective order which they imposed on experience. This brought matters through the full circle. Eternal objects, though they were not placed in a world of ideas, did have a reality outside the mind. The Platonic trend is apparent.

Mathematics

Like the proverbial elephant, mathematics has been described according to the viewpoints which were taken of it. It was considered as a process by mathematicians such as Euclid (24), von Neuman (101), Polya (79), Poincare (77), Hadamard (36), and Einstein (23); as a creation of the mathematician by Kasner and Newman (55) and Wittgenstein (105); as the discovery of an objective reality apart from the thinker by Hardy (38), Whitehead (104) and Russell (86); and finally as being rooted in logic (64). The following pages of this section will consider each of these viewpoints.

Considering mathematics as a process, the question is: What is happening when it is happening? Is the procedure which can be called mathematical a deductive one, an inductive one, or an intuitive one? The classical answer was deduction. Given axioms which were taken as self-evident, the mathematical consequences were deduced by rigorous reasoning. Euclid's geometry was taken as the example par excellence of this view (24). The tradition flowed through Plato and Aristotle up to the blossoming of mathematical science in the sixteenth centyry (50, p. 77). With the work of Gilbert, Bacon, Descartes and others the emphasis shifted to inductive and intuitive processes (50, pp. 67-87). Actually, in part, the new emphasis was not in conflict with the classical one. Part of the time a new question was being asked: What processes are involved in establishing the so-called, self-evident axioms? Given the new question along with the old, the answers were induction and intuition. Von Neuman thought that mathematical ideas originated in empirics and, consequently, he could be placed in the inductive camp (101, p. 2063). Polya, with von Neuman, said that mathematical results were found by induction first and proved later (79, pp. 73-78). Polya made a distinction in the mathematical field: "Mathematics presented with rigor is a systematic deductive science but mathematics in the making is an experimental inductive science (78, p. vii)." He went on to give general methods for problem solving inductively as well as approaching the solution to a problem backwards. This last method he attributed to the Greek mathematician, Pappus (78, pp. 225-232). In his work, Patterns of Plausible Inference, he listed the

inductive process as a particular case of plausible reasoning in mathematical discovery (80, p. 140) and found that the inductive process followed rules which were different from demonstrative reasoning. Further, the conclusion of the former attained only probability. His later efforts looked into case histories with a view toward problem solving and the preparation of teachers of mathematics (79).

Closely related to the inductive view was that of intuition. Poincare regarded the "sudden illumination" as one of the striking aspects of his work in mathematical creation. He ascribed the illumination to unconscious work which he placed between slices of conscious effort (77). Hadamard agreed in part with Poincare. He believed the procedure consisted of preparation, incubation, illumination, and verification but he did not put significance in the subconscious processes which might be involved (36). Hadamard discussed three types of mathematical minds: those who reasoned by common sense where much was supplied by the unconscious processes, the student mathematician who differed from an inventor only in degree, and the mathematicians who combined logic and intuition (36). In discussing the process of thought which led him to his discovery of the theory of relativity, Einstein told Wertheimer that it was not the axioms that came first. What came first was a feeling of direction, of going straight toward something concrete (102, pp. 183-184). Thus, Einstein attributed his creativity to an indefinable intuition.

Mathematical activity has been taken by some as an act of creating rather than of finding. Kasner and Newman contended that non-Euclidean

geometry proved that mathematics was the handiwork of man (55, p. 359). Wittgenstein claimed that "the mathematician creates essence" and was free at every point (105, p. 32) while Poincare held that the axioms of geometry were mere conventions or customs of one's particular, mathematical culture (103, p. 2355).

Contrary to the subjective view of mathematical production is the view that the mathematician discovered an objective reality. Whitehead placed mathematical truths in the real realm of eternal truths (104, pp. 8-9). Hardy felt these truths were outside us to be discovered or observed and that theorems were simply notes of our observations of the realities (38, pp. 63-64). Einstein claimed the necessity and naturalness of mathematical products forced the impression of an objective order (23, p. 33). Russell considered the discoveries of Peano to have definitively refuted the subjectivity of Kantian, a priori intuitions (86, p. 4). Developing this line of thought, Russell with the assistance of Whitehead attempted an ultimate reduction of mathematics to a purely logical foundation (64, p. 5).

Logic became a science in the hands of Aristotle. The treatises of his Organon were the first systematic presentations (66, p. 1036). Symbolic logic, a more perfect system of analyzing thought, began in the seventeenth century with Leibniz (64, p. 5), who foresaw the general character which the proposed logistic was to have. However, the development of a calculus of logic was credited to Sir William Hamilton, DeMorgan, and Boole (64, pp. 36 & 79). Having swept aside the accumulations of nonsymbolic, Aristotelian logic, Boole introduced significant

symbols (64, p. 79). Pierce improved upon Boole and DeMorgan (64, p. 79) and in 1894 Peano and his collaborators began the publication of the Formulaire de Mathematiques in which all branches of mathematics were to be presented in the universal language of logistic (64, p. 5). It remained for Whitehead and Russell in Principia Mathematica to exhibit the perfect union of symbolic logic and logistic method in mathematics (64, p. 5).

Psychology

Modern experimental psychology developed under the influence of the association theories which grew out of the British empirical tradition. The associationist psychologists, interested in the activity of learning, sought to understand how complex ideas were grasped by the mind. Following Hume, they generally agreed that a new idea required contiguity of the component parts of the idea and the repetition of these same parts (46, p. 19).

American psychologists like William James (47, pp. 360-395) and John Dewey (20, pp. 100-107) viewed learning from a new perspective. Seeking to discover the function of the behavioral event of thinking led them to place the emphasis upon the nervous system rather than the mind, thus making action a critical factor in the act of learning (27, pp. 9-10).

A trial and error learning system was developed by Thorndike who performed experiments upon animals and extrapolated an application to

humans (27, p. 10). In more recent times learning theorists such as Guthrie, Hull, and Skinner also used animals in their research (43, p. 297).

The gestalt tradition stood over against the associationist one. The scholars with this view considered learning to be an insight which reorganized the field of experience. This phenomenon was expressed as "having a new idea" or "discovering a solution to a problem." In his observations of chimpanzies, Kohler presented this view (27, p. 15). A similar position was reflected in the writings of Koffka and Wertheimer (27, p. 16).

Gagne attempted a merger of the two traditions by developing a unified theory of learning in a hierarchical model. He listed eight types of learning, each with its own rules. The hierarchy moved from simple to complex; eacher higher order depended upon the mastery of the one below it. The highest type, Type I, called "Problem-Solving Learning," required the internal events of thinking and synthesis. The latter was a combination of previously acquired principles which formed a new, higher-order capability (43, p. 569). Gagne contended that the entire act of problem solving, though requiring some degree of time, found its solution suddenly in a "flash of insight (27, p. 57)."

Placing a great deal of importance on the intuitive process, Brunner said that intuition encompassed "the intellectual technique of arriving at plausible but tentative formulations without going through the analytic steps by which such formulations would be found to be valid or invalid conclusions (8, p. 13)." It was Brunner who was largely

responsible for putting the concept of structure into the discussion, feeling that structure was at the heart of mathematics. Cases in point were the functions of commutation, distribution, and association which were involved in arranging symbols and numbers into knowable equations. The student became familiar with these as a matter of course. Once known, the learner recognized that "new" problems were not new at all but actually variants of old questions (8, pp. 7-8).

Guilford thought that the American psychological theories had been too much dominated by the stimulus-response models of Thorndike, Hull, and Skinner (31, p. 126). He wished to give more weight to insight and the other, intellectual activities which had so often been brushed aside. Rather than to study animals and extrapolate to humans, he preferred to examine the human condition as a self-contained unit. This view resulted in his creation of the Structure of the Intellect (SI) Model. It conceptualized intellectual abilities on a three-dimensional matrix: operations, contents, and products. There were five operations: memory, cognition, evaluation, divergent production, and convergent production. The four types of content were figurial, symbolic, semantic, and behavioral. There were six products: units, classes, relations, systems, transformations, and implications. One hundred and twenty different intellectual abilities were referred to this model (32, pp. 20-26). Problem-solving abilities were placed by Guilford under Productive Thinking (34, pp. 472-474) which he subdivided into convergent and divergent productivity. Convergent production was that generation of information from given information where the emphasis was upon achieving

conventionally accepted best outcomes. The information given determined the response. In divergent production an attempt was made for variety of output from the same source. Here there were no absolute answers and transference of learning and creative potential were brought into full play (70, pp. 19-20). Although divergent production was the SI ability most frequently expected in life, the school most frequently expected convergent production. Guilford felt that the schools should strive to develop divergent tendencies in students and lay less stress upon convergent abilities (33, p. 8).

III. EXPERIMENTAL AND EXPLORATORY STUDIES OF THE PROBLEM-SOLVING PROCESS

Review of Experimental Studies

In November of each year Suydam and Weaver publish "Research on Mathematics Education (K-12) (95, 96, & 97)." However, they do not separate the research into subcategories such as problem solving.

General articles reviewing problem-solving research in mathematics education during the sixties were included in works by Atkin and Romberg (4), Romberg (85), Cronbach and Suppes (12), and Kilpatrick (59). Scandura reviewed his own work on rule-learning experiments (88 & 89). Since 1968 he has continued studies in problem-solving behavior with the aid of his Set-Function-Language Model, which was based on S-R psychology (90).

Much of the research on problem solving in elementary school mathematics can be found in the reviews of Johnson (48 & 49). They

covered the period up to 1944. Riedesel's bibliography updated this collection through 1969 (83).

Dessart and Frandsen noted the need for empirical studies in problem solving on the secondary level (18). They maintained that mathematical problem solving "probably deserves a designation as a 'branch' of mathematics education as much as any other topic . . . (18, p. 1187)."

Stern and Keislar found no significant differences in problem-solving abilities among boys and girls on the elementary school level (94). However, Sheehan (93) reported a difference in favor of boys at the high school level. In 1974 Fennema (25) published a critical review of literature related to sex differences in mathematics achievement.

Finally, Kilpatrick noted the poor quality of methodology used in the research of mathematics education. He suggested that doctoral candidates were best advised to undertake clinical type observations of individual students (58, pp. 531-532).

Review of Exploratory Studies

Cognitive-gestalt psychologists think that a problem by its very nature disturbs the equilibrium of the organism. Psychic balance is restored when the problem is solved (43, p. 260). The solution is a matter of achieving "insight (27, p. 166)." There are several variations which are the equivalent of insight. They are: cognitive restructuring, perceptual reorganization, illumination, or recentering (14, pp. 29-39).

While American psychologists were occupied with experimental studies of a behavioral nature, the European and Russian psychologists were conducting exploratory research. They sought to understand the processes of human thinking. This group was willing to use data from verbal protocols in order to examine the cognitive activities. The methods most often used to understand problem-solving behavior were "think aloud" accounts, written explanations, and on-the-spot observations (15, pp. 79-86). More recently, some significant work has been done with computer simulation (29, pp. 16-17). This section examines a representative selection of these studies.

In Productive Thinking Wertheimer (102) gave an account of children's attempts to solve the problem for finding the area of a rectangle or the Gauss problem of finding the sum of the first six counting numbers. He studied the children's processes and searched for theoretical, law-like classifications. He observed some fine productive processes (A reactions) and blind, stupid activities (B reactions) (102, p. 97). He concluded that even when the process took no more than a minute there was a threefold development. The idea of the solution came in a colloidal manner, was followed by dynamic direction of activity, and terminated with an adequate structuring of the pieces into a whole (102, pp. 189-200). His work gave impetus to many other studies on the quality of the problem-solving process.

In his observational studies of the "genesis" of a solution, Duncker (22, p. 14) noted a configuration like unto a family tree. The

final form of the solution was not generally reached by a single step. Rather, it typically arose as a reformulation of the original problem. By having students think aloud and by questioning them, he identified two areas of importance. First, "poor" mathematicians were those who were unable to restructure the thought material in diverse ways. Secondly, "functional fixedness" (i.e., the inability of a person to break with traditional patterns of thought) interfered with the ability to solve problems (22, p. 110). This functional fixedness of Duncker was the same mental set which Luchins called the "Einstellung effect (68)."

In Luchins' experiments with water jar problems it was the task of the subjects to obtain a required volume of water for a number of specific empty jars. After a demonstration problem the group was given seven problems to solve. After two or three solutions the average testee detected a formula and quickly applied it to the remaining problems. However, the last two problems, though they had easier solutions, had to be solved by a different method. Thus, Luchins showed that even a habit or mental set which was established in minutes can blind a person to a simple answer (67).

Sheerer (91) extended Duncker's (22) and Luchins" (67) investigations. In 1963 he described the damaging effects of fixating upon problem approaches based upon incorrect assumptions (91, p. 120).

More recently, Cunningham (13) studied rigidity in the problem-solving behavior of children. He found that the Einstellung effect affected perception, reasoning, and memory in adverse ways (13, p. 377).

He suggested that Socratic questioning may be a way to overcome the difficulty and to promote flexibility in the mental process (13, p. 386).

In A Study of Thinking, Bruner, Goodnow, and Austin (9) researched human concept learning. Concept learning was thought of as categorizing, i.e., a process of simplifying a complex situation. In order to study this mental process Bruner and his cohorts presented their subjects with a nine-by-nine array of geometric-patterned, stimulus cards which differed one from the other in shape, color, and number of patterns and borders. The problem was to find the stimulus characteristics which defined membership in the concept class. While the subjects studied the cards Bruner and his companions studied the subjects. The result was a classification of the strategies used in the experiment (9, pp. 156-181). These types of strategies are often taken as evidence that the human organism is an active and purposeful organizer in solving its problems.

Zykova (107) attempted to discover how pupils learn to master geometric concepts, the most effective methods for developing this ability, and how it all related to problem solving. Her study showed that, although the pupils knew the definitions of the concepts, many did not relate them to a problem based upon these ideas. Often, if the concept was related to a drawing, the pupils' mastery was limited to figures in standard form and position (107, pp. 140-147).

In the Soviet Union students are taught that a geometry problem requires identification of the principal parts, location of the parts in

a given diagram, and recall of relevant theorems (60). Talyzina (98) held that a mathematics methodology based upon an extensive theory of thinking and aimed at developing the thought processes of the pupils would produce more satisfactory results. However, no such procedure exists (98, p. 51). Accordingly, he has studied the thought processes of students while they are performing their academic tasks. Talyzina concluded three things: subjects execute both complete and partial logical deductions in solving geometry problems, partial logical deductions were made more frequent than complete ones, and when complete deductions did occur the subjects usually only reproduced parts of the general proposition which led to the solution (98, pp. 98-99).

While investigating individual differences, Yakimanshaya (106) classified students by their approach to problems. At one extreme were those who rapidly isolated data from the verbal communication. At the other extreme were those who worked from a diagram which accompanied the verbal communication. The latter group experienced more difficulty (106, pp. 122-123).

Gurova (35) claimed that low problem-solving ability was due to a superficial understanding of the methods involved and no grasp of the underlying intellectual operations. He attempted to get the students to analyze their own intellectual activity. Those who were unable to dissect their mental operations were also unsuccessful in solving complex problems (35, p. 100).

Henderson and Pingry contended that following a fixed sequence was damaging to both fast and slow students. It was a handicap to agile

minds; it confused the patterns of relationship for the slower student (41, p. 266).

More recent writers who have observed students in the process of problem solving did not think the laws of logic were the laws of human understanding (61, p. 129). This view supported Kemeny (56) and Polya (79 & 80) who stressed the use of intuition in the early stages of mathematical instruction and the use of rigor at the latter stages.

The information processing theory grew out of the work of earlier researchers (102, 22, & 15). According to this view a precise set of human mechanisms are postulated to explain behaviour in problem-solving tasks. The postulates can be converted into computer language and the human problem-solving process can thereby be simulated. Davis (14) maintained that a better understanding of the details of sequencing and information-processing skills will come through computer simulation. He felt that gestalt-cognitive explanations were too vague, that stimulus-response ideas were too simple. He claimed that game-like, role-playing methods pointed out important dimensions which had been ignored (14, p. 162).

Paige and Simon's study of algebra students was guided by a computer program (72). They compared the processes of subjects of diverse age and background to those of the computer. Although there were individual differences among the subjects, the processes of the humans and the machine were similar. They were able to identify some of the important patterns a person must acquire in order to do algebra, word problems (72, p. 118).

In 1972 Newell and Simon published Human Problem Solving (71). The result of 17 years of research, the work focused on adults performing short, problem-solving tasks. The research was an attempt to develop further the information processing theory. The conclusion of this book might well serve as the conclusion of this survey of exploratory studies:

> . . . we brought to bear what evidence we could on the question of how the problem solver, in the face of a new task, generates an appropriate problem space and program and on the commonalities and differences among problem solvers. Our answers to these questions were sketchy, for these areas undoubtedly represent the largest and most important terra incognita on the map of the theory of human problem solving today (71, p. 868).

Review of Convergent-Divergent Studies

Guilford (33) constructed a model for problem solving based upon his concept of the structure of the intellect. Convergent and divergent thinking were among the abilities which he listed. He distinguished these two activities factorially (30). The structure of the problem determined whether the results were of a convergent or divergent nature. If the options were limited and the solution was of a determinate nature, the thinking was convergent; if the problem was otherwise, the thinking was divergent. (see the definitions, pp. 10-11). Helmstadter (40) noted the similarity of this division to the older breakdown of deductive and inductive reasoning. He considered Guilford's conception to be much more inclusive.

According to Davis (14) there was not a strict dichotomy between convergent and divergent thinking. It was often the case that the problem

solver first divergently considered the possibilities and then convergently selected the procedure which best met his own perception of the problem. Thus, both types of thinking were likely to appear in the same problem-solving episode (14, 1. 22).

Guilford (33) had claimed that creative thinking and the divergent variety were essentially the same phenomenon. He held that IQ tests measured convergent thinking. Following these guidelines, researchers have classified their subjects according to their relative performance on intelligence and open-ended tests (28 & 44). Classifications have also been made on a high/low basis (81 & 11). Convergent thinkers scored high and divergent thinkers scored low on problems requiring one conventionally accepted answer. The situation was reversed on problems requiring the generation of several equally acceptable answers (11, p. 57).

Open-ended tests were measurements of creativity according to Hudson (44). He identified three types of subjects: convergers (those who possess high IQ ratings), divergers (those who were high in creative tendencies), and all-rounders (those who were more or less equally good or bad on both types of tests (44, p. 52). He identified 30 percent of his sample as convergent, 30 percent as divergent, and 40 percent as all-round. Considering the all-rounders as unimportant, Hudson directed his work toward the other two types (44, p. 55). The results of his study showed that art specialists were on the whole divergers and that mathematics and science majors were convergers. MacKay and Cameron (69) extended the work of Hudson to university students.

Joyce and Hudson (52) employed a self-rated scale of convergence and divergence to introduce a kind of subjective questionnaire to students in a medical school. However, Child and Smithers (11) have shown that the questionnaire was a measure of science and nonscience orientation. The linkage with convergence and divergence was indirect.

In a study of creativity among seventh grade mathematics students Prouse (81) found some correlation with IQ (r = 0.48). This agreed with the correlation found between other creativity-test scores and the Iowa Test of Basic Skills. Prouse also found a stronger correlation between divergent thinking and creativity (r = p.10 to 4 = 0.64) than between convergent thinking and creativity (r = 0.01 to 4 = 0.23) (81, p. 879). This corroborated Guilford's assertion that the more prominent creative abilities were concentrated in the divergent thinking categories.

Most studies of convergent-divergent thinking were related to verbal testing and/or creativity research. An extensive study of the relationships between divergent production, IQ, creativity, and school achievement may be found in works by Guilford (31, 32, & 34) and Ausubel (5 & 6).

There had been speculation about whether there was any natural inclination toward convergent production types rather than divergent production types. Since the two types were not mutually exclusive, it was suggested that the organism simply favors one over the other. However, in terms of demonstrated ability a person might be typed as one or the other even though he was equally proficient in both forms of productivity. Another interpretation related abilities to interests and

motivation. Merrifield and Barron found that people preferred convergent tasks to divergent tasks regardless of their demonstrated proficiency (70, p. 21). Haddon and Lytton (37) demonstrated that informal settings can encourage independent and divergent thinking modes.

The British Journal of Educational Psychology published the bulk of research on convergent-divergent thinking in mathematics. Generally, these studies sprang from Getzel and Jackson's work (28) on the creative child or from Hudson's work (44) with convergers and divergers. That divergent thinking played only a minor role in the mathematical performance of students was demonstrated by Richards and Bolton (82) who contended that general ability was the most important factor in mathematical aptitude. Accordingly, they felt that any emphasis upon divergent thinking development will have only minimal results (82, p. 37). Dodson (21), however, reported that one of the characteristics of successful problem solvers was divergent thinking ability. In measuring mathematical abilities Hiatt (42) developed a test which rated the ability to solve nonroutine problems. He found that ninth graders did as well as twelfth graders on this kind of problem and that the divergent thinking ability of sixth graders did not differ significantly from that of ninth graders.

Methodology of Exploratory Studies

In exploratory studies of problem solving the data are qualitative; the method is descriptive. The experimentalist presses for simplicity, clear relationships, and unencumbered theories. The researcher in

descriptive studies, however, must be satisfied with complex data which are immersed in the complexities of human behavior. One of the difficulties has always been that the researcher can never be sure just what was going on in the mind of the subject. He could ask for an explanation of the mental activity after the problem was solved. This was called retrospection or introspection. Another approach was to have the subject talk about his thought processes while he was working on the problem. This was known as "thinking aloud" (22, p. 2).

For best results retrospective methods were restricted to a simple task of short duration, the subject reporting on his introspections immediately after the task (15, p. 78). This reflects the view of Polya:

> The best time to think about methods may be when the reader has finished solving a problem, or reading its solution, or reading a case history. With his task accomplished and his experience still fresh in mind the reader, in looking back at his effort, can profitably explore the nature of the difficulty he has just overcome. He may ask himself many useful questions: 'What was the decisive point? What was the main difficulty? What could I have done better?'-all these questions are good and there are many others-but the best question is the one that comes spontaneously to mind (79, p. xii).

The advantage of this approach was that the person did not have to think about solving the problem and introspect on his processes at the same time. This method was used extensively before 1930 and has been employed more recently by Karplus (54).

Duncker (22) carried out an investigation of thought processes which involved the "think aloud" method which was considered a novelty in 1935. Although it allowed the investigator to analyze complicated, lengthy processes of thinking, the difficulty of verbalization added an

extra burden to the subject's task. On the one hand thinking aloud influenced the thought process itself and on the other solving the problem was hampered by having to talk about it (15, p. 81). In addition to Duncker (22) and Wertheimer's (102) studies there have been many others who used the think aloud method. Some of these others were deGroot (15), Krutetskii (62), Buswell (10), Kilpatrick (57), Law (63), Paige and Simon (72), and Newell and Simon (71). All but deGroot used tape recordings in their research. Sometimes they encouraged or questioned the subjects if silences or difficulties arose.

In exploratory studies retrospection and think aloud methods were used in a variety of ways. One way was to use the method as the sole source of information as did deGroot (15), Wertheimer (102), and Duncker (22). The other way was to use the method to support other data on the subject's behavior (10). Some of these supportive frameworks have been the subject's manipulation of materials (39, 10, 91, & 87), the trying out of solution proposals (84), the study of intermediate steps (9), and the use of written protocols (63, 54, & 10). The introspective data itself may be given the primary emphasis or it may be used as a commentary upon observable behavior (15, p. 37). Among those who used supportive frameworks were Scheerer (91) and Buswell (10). Scheerer constructed a problem that encouraged a running translation of thought into visible action. He used the think aloud method and the physical manipulation of a boat and mannikins. In this way he was able to follow the progress of the subject in solving the river puzzle (91, p. 125). Along with the think aloud method Buswell (10) used written

questions and the choice of an envelope by the subject to determine the sequencing of the problem solution. He then used the sequencing to detect patterns of problem-solving processes among the individuals being tested.

Data obtained in exploratory studies are often cumbersome. It sometimes becomes necessary to condense the data and develop a coding technique. Kilpatrick (57) developed a code for recording problem-solving protocols by breaking the various steps into stages. He then developed a method for describing the process sequences. He coded the activity by a letter and its result by a number. Of course, much of the problem-solving activity was lost by coding and interpretation. However, the simplicity and clearness which were gained facilitated the writing of computer programs which duplicated human thought. Newell and Simon (71) and Paige and Simon (72) have made progress in this direction.

Rigorous testing of the hypotheses gained from exploratory studies was the natural consequence to be expected. One method possessing such rigor was simulation. An hypothesis was expressed in the form of a computer program. A comparison was then made between the behavior of the program and the behavior of the subjects being tested (72, pp. 124-125). Another equally demanding approach has been the use of the statistical methods of experimental psychology.

Methodology of Convergent-Divergent Studies

In tests of convergent and divergent thinkers the subject must produce new information from the given data. As has been indicated,

tests of convergency require only one right answer; tests of divergency require a number of different answers (33, p. 8). Most tests devised for examining these two processes use verbal examples. Meeker (70, pp. 87-99) listed only three tests for divergent thinking in mathematics: Match Problems, Number Rules, and Waterjar Problems. The situation was similar for convergent tests of mathematical ability. However, it was generally recognized that any IQ or standard test of mathematics probably measured convergent ability (33, p. 4). Meeker (70, p. 114) gave a Decision Flow Chart which identified the mental operation involved. This assisted a test-maker in determining whether he was testing convergent-divergent production. In addition to the lack of appropriate tests there was also a lack of agreement on the scoring of divergent productivity. This difficulty is taken up at the end of this section.

Guilford (33) tested for convergent production in the following manner. He told the subjects to write the number "5" and arrive at "1" using only the steps "x2, ÷7, and -3." The task was to state the order which would make the transition. In searching out divergent productivity he asked the student to write several examples of a quadratic equation. In another approach he gave two simple equations: D + 2 = C - 7 and D + C = 2x. From these the student was asked to generate new, simple equations (33, p. 168).

In 1961 Prouse (81) developed a mathematics test of creativity consisting of 10 items. Three items were hypothesized to be of a convergent nature and the remaining seven were taken as divergent. The following is one of Prouse's tests for divergent productivity: "Using

the operation symbols +, -x, ÷, and the grouping symbols, (), if needed, write as many true equations as you can within the time limit: 7 __ 6 __ 4 = __ (81, p. 877)." In the testing situation 15 lines were presented to the student with the directions that more lines could be added if time permitted.

The complexity of divergent tests led to a lack of agreement on scoring. Prouse (81) assigned two scores to his divergent tests: a fluency score and an originality score. Fluency was tabulated by counting the number of acceptable responses made by the student. Originality was based upon the frequency of response. A weight was then assigned to each response according to the frequency of its use (81, p. 878). Torrance (99) scores items on fluency, flexibility, originality, and elaboration. However, the intercorrelation of these four scores was so high that Vernon thought it was a waste of time to distinguish them (100, p. 251). Further, he remarked that it was difficult to rely upon a frequency criterion. Subjective evaluations of cleverness and originality infiltrate the scoring. He felt it was reasonable to allow such subjective judgments provided they were applied to and recorded for each testee. Accordingly, Vernon chose 100 pages at random and labeled as unusual responses given by not more than five subjects. In addition he identified frequent and intermediate answers. Unusual responses were weighted at two points, intermediate at one point, and frequent ones at a value of zero. These values were then added to the fluency scores. Finally, there have been some tests which were scored only on the basis of the unusual response (100, 252).

CHAPTER III

PROCEDURE AND DESIGN FOR THE STUDY

I. THE STUDY: AN OVERVIEW

This study of problem-solving processes utilized geometry students in the secondary schools of the Knoxville City System. In the field study various problems were presented to 94 students at Central High School and South High School. During this phase the methodology of the final study was developed. In the final study the CD Test, written to elicit convergent and divergent productivity, was administered to 105 students at Young High School and Bearden Senior High School. On the basis of the CD Test the students were classified as belonging to one of four groups. Of these students 49, with at least 10 from each of the four groups, were then observed individually as they worked upon the IO Block Problem. The data on problem-solving behavior, student thinking processes, and strategies were obtained by the writer during these observation sessions.

In a study of this nature Kilpatrick (57, p. 32) warned about two extremes. If the group was too homogeneous, the results would be atypical. If the population was too heterogeneous, it would be difficult to detect common patterns of thinking. Accordingly, the geometry students in this study were chosen because of the similarity of their mathematical backgrounds and the dissimilarity of the other aspects of

their life. The backgrounds of the students were alike in these respects:

1. All had had a minimum of nine years of mathematics, including the first year of algebra.

2. All were currently enrolled in the second semester of a high school geometry class.

3. All had been exposed to the method of formal proof, to the processes of inductive and deductive reasoning, and to the concept of congruency.

4. All were using Goodwin's Geometry published in 1970 by Merrill Publishing Company. All had covered a minimum of nine chapters in this text which included a study of quadrilaterals. The backgrounds of the students were unlike in these respects:

1. The students differed among themselves in ability, age, achievement, and sex.

2. The total sample involved different schools and different teachers.

3. The students differed in their performances upon the CD Test. (This last variable was the criterion for selecting the students that were involved in the individual problem-solving sessions.)

II. THE FIELD STUDY

Purpose

The purpose of this field study was twofold:

1. It was conducted to enable the writer to test out and select problems suitable for use in the final study.

2. It was carried out to give this researcher actual practice in recording problem-solving behavior.

The CD Test, the selection of the IO Block Problem, and the methods used in the individual sessions were all developed from this field study.

Subjects

The 94 subjects in the field study were members of four geometry classes in the Knoxville City System. A total of 45 students in two classes from Central High School had the same teacher as did the 49 students in two classes from South High School. These students participated in the pretesting which gave form to the CD Test. Some of these same students were used in the observations which led to the selection of the IO Block Problem and to the formulation of the methods used in the final study.

Development of Materials and of the Design of the Study

The writer visited Central High School daily for a period of two weeks. After a two-week interval of evaluation and rewriting of materials she visited South High School for two weeks. At each of the schools

she was allowed the use of two regular class periods and was permitted to see students individually in a room set aside for that purpose. The students were observed either during their regular geometry class or during a free period.

During the visits to Central High School, 12 different problems were tried out with 45 students in classroom situations. This brought into perspective the difficulty, appeal, and approximate time-consumption element of each problem. The problems were coded as C-1, D-1, C-2, D-2, etc. The C problems were thought to be convergent types. Each had parts requiring only single answers. The D problems were thought to be divergent types. These problems were of an open-ended nature and called for a variety of answers. After each class period the students were asked to choose the problems they preferred, to rank them, and to tell why they were so ranked. The students were encouraged to ask questions and to criticize the problems. By means of the questions and criticisms the writer was able to identify areas of ambiguity in the verbalization of the problems.

Each student at Central High School was asked if he would be willing to work with the writer on an individual basis. It was explained that this would allow his individual strategy to be observed while he solved a puzzle-type problem. There were 23 students who indicated a willingness to participate in this aspect of the prestudy. Schedules were arranged so that the geometry periods and the free periods could be utilized. The IO Block Problem was presented to some of these students. Other puzzles and some written problems were given to others. After

observation of these variations all but the IO Block Problem proved unsuitable. In receiving directions for working the IO Block Problem many students did not understand the meaning of the words "vertically" and "horizontally." Later, these words were changed to "columns" and "rows" respectively. Although some of the students understood "diagonally" as referring only to the main diagonals, this word was not changed. Instead the writer decided to answer questions about it whenever they were asked. These sessions led to the development of a coding form similar to Kilpatrick's (57), which made it possible to keep a record of the observable, problem-solving processes of each student. But since the student often took 40 to 50 minutes to solve the IO Block Problem on the first trial making the coded sequences very long, it was finally used only on the second trial. Written observations were found to be more informative on the first trial.

The CD Test took on a better form at South High School. Problems which were too easy or too difficult were dropped and modifications were made in the acceptable ones. The test now consisted of five convergent-type and five divergent-type problems. At this time a method for scoring the divergent types was decided upon. This new form of the CD Test was administered to 49 students. From this group subjects for the IO Block Problem were selected based upon their willingness to undergo further testing and upon their performance upon the CD Test.

The students who took the IO Block Problem at South High School were required to do three things:

1. They were to solve the problem.

2. They were required to answer questions about their thinking processes during the first working of the problem.

3. They were asked to work the problem a second time. It was this procedure which became the final method for collecting data in the final study. The coding for recording behavior during the IO Block Problem was simplified at this time. It was decided that the student's behavior would be coded only during the second trial. Notable behavior during the first trial would be described in writing by the observer. Later, the descriptions would be condensed and used in the interpretation of the subject's behavior. There was still some question about judging "trial and error" activity. Kilpatrick (57) had distinguished between "trial and error" and "exploratory" activity. This writer, however, failed to detect the difference and coded both activities under the same symbol.

III. THE CD TEST

Subjects, Presentation, and Administration

In the Knoxville City System 105 students from two high schools were administered the CD Test. The 67 students from Young High School were taken from three geometry classes. Two teachers were involved with this group. The CD Test was administered to each geometry class during its regular period. The teachers were not present during the administration of the tests. The 38 students from Bearden Senior High School volunteered for the study. They were drawn from five geometry classes. The four teachers involved with these classes were not present when the CD

Test was administered in the auditorium of the school at a period designated by the principal.

In each case the writer had spoken to the classes on previous occasions. At those times the purposes and procedures of the study were explained. Every attempt was made on these occasions and during the administration of the test to put the students at ease and to make the situation friendly and nonthreatening. This approach was particularly necessary in order to establish rapport with the students who would be dealt with individually in the second part of the study.

The following information was obtained from the students before the test: name, grade, geometry class, and free period. A code number was written on the form. The student was informed that his name would not be used in the final version of the study. It was only necessary to know the student's name so that he could be identified when called in for the individual, problem-solving session.

The Problems

During the second half of the field study the preliminary version of the CD Test had consisted of 10 problems. It required two class periods to administer it. At the end of the field study two more problems were dropped and several were shortened. This refined version of the pre-CD Test was submitted to three mathematicians and one educational psychologist at the University of Tennessee. These were to give an evaluation of the questions.

Four of the problems were hypothesized to be convergent and four were considered to be divergent. The four evaluators were given definitions, a decision flow chart, and a rating page. They were asked to identify the problem on a continuum scale.

Consistent agreement among the evaluators was found for problems 1, 2, and 6. All but one rated problem 3 and problem 5 divergent. A split on problems 4, 7, and 8 developed. Problems 4 and 7 were discarded. Problem 8 was rewritten according to the suggestions of two of the raters. This revised version of problem 8 was retained.

The final version of the CD Test consisted of three convergent-type problems and three divergent-type problems. The problems were not numbered for testing purposes. With the exception of two related problems, Problem A and Problem B, the sequencing of the problems was varied. Since the instructions for Problem A were also used in Problem B, this pair was stapled together.

Students could work the problems in any order with the exception of the stapled pair. Problem A was to be worked before Problem B. (See Appendix A.)

Although the problems were not named or numbered during the testing, they have been so designated for the posttesting period. This is to facilitate analysis, discussion, and interpretation. The convergent problems are identified as follows:

1. The Cube Problem is 1-C.
2. The Archaeologist Problem is 2-C.

3. The Flip Problem (Problem A) is 3-C.

The divergent problems are identified as follows:

1. The Generate Paths Problem (Problem B) is 4-D.

2. The Quadrilateral Problem is 5-D.

3. The Equations Problem is 6-D.

These designations will be used in future discussions of the problems.

Method of Scoring

The convergent problems were scored in general by giving points for each correct part. The maximum possible convergent score was 26 points. The divergent problems were scored in general by giving points for fluency and originality. There was no maximum divergent score.

The scoring for the individual problems follows:

1. Problem 1-C, the Cube Problem, was scored by giving one point for each correctly circled pattern and by subtracting one point for each incorrectly circled pattern. There was a maximum of nine points possible for this problem.

2. Problem 2-C, the Archaeologist Problem, had two parts. On Tablet A one point was given if only the second row or column was filled in. An additional point was given if the third row or column was filled in. Six points were given if the entire tablet was correctly filled in. No points were deducted if no more than two computational errors occurred. Six points were given if the scorer detected that the subject recognized the pattern and was applying the correct rule even if the entire table was not completed. On Tablet B one point was given for the correct

placement of a "3" with only one being counted. One point was given for the correct placement of a "15" with only one being counted. The same scoring applied to the placement of a "30." If the entire bottom row was correctly placed, the problem was worth six points.

3. Problem 3-C, the Flip Problem (Problem A), was scored by giving one point for each correct path on the first three figures. Two points were given for the fourth which called for two paths. If arrows were left out but the points were placed correctly, the scorer considered the problem to be correct. There was a maximum of five points in the scoring of this problem.

4. Problem 4-D, the Generate Paths Problem (Problem B), was scored by giving one point for each path sketched. This was considered to be the "fluency" score. To this was added an "originality" score. This was calculated by adding one point for each new figure used after the first one.

5. Problem 5-D, the Quadrilateral-Problem, was scored by allowing one point for each pair of congruent quadrilaterals identified. This was a fluency score. If four quadrilaterals were written as congruent to each other, a fluency score of six was given. This was because it involved the identification of six different pairs.

6. Problem 6-D, the Equations Problem, was scored by allowing the first correct equation in each set to be counted one point each on the convergent score. Every correct equation generated after the first one was considered as one point of a fluency score and was added to the divergent score.

Type of Data

The data were obtained in the form of two scores. The C score was the convergent measurement. It was obtained from problems 1-C, 2-C, 3-C, and the first of each set in 6-D. The D score was the divergent measurement. It was obtained from the fluency scores of problems 4-D, 5-D, 6-D, and the originality score of 4-D.

The mean C score and the mean D score were calculated from the total group of 105 students. A student was classified as High C if his C score was above or equal to the mean C score. Otherwise, he was classified as Low C. The classification of High D or Low D was done in the same way. Accordingly, four groups were formed:

1. Group I was High C and High D. These were the High All-Rounders.

2. Group II was High C and Low D. These were the Convergers.

3. Group III was Low C and High D. These were the Divergers.

4. Group IV was Low C and Low D. These were the Low All-Rounders.

The sample for the IO Block Problem was drawn in part from each of these four groups.

IV. THE IO BLOCK PROBLEM

Subjects, Presentation, and Administration

Following the administration and scoring of the CD Test, the students were designated as belonging to one of the groups mentioned above. The student code numbers were randomized within each group to determine the order in which the individuals would be taken. Schedules

were arranged to fit the convenience of the student and the writer. If one student was not available on a particular date, the next number in line was chosen. Sometimes the "missing" student was observed at a later date; sometimes he was skipped. A minimum of 10 students per classification was set as a goal for the IO Block Problem observations. Ultimately, 49 students were observed. There were 21 taken from Young High School and 28 taken from Bearden Senior High School. The actual breakdown is shown in Table 1.

TABLE 1

NUMBER OF STUDENTS USED FOR IO BLOCK PROBLEM SESSIONS CLASSIFIED BY SCHOOLS AND GROUPS I, II, III, IV

Groups	Number of Students		
	Young High	Bearden Senior	Total
I	8	5	13
II	4	7	11
III	7	3	10
IV	2	13	15

The problem-solving sessions took place in library conference rooms set aside by the respective school librarians. The student worked at a table, facing a wall. The observer sat slightly to the back and to the right of the student.

The student was told not to discuss the problem with others after he left. He was told to try to block out the presence of the observer from his mind. It was suggested that he pretend that he was alone and that he should not be self-conscious about his work or the time that he took. It was stressed again that the observer was looking for the different ways that students might follow in solving the problem. There was no one way that the problem should be solved. Then the student was told to read the directions on the paper and to begin. (See Appendix B.) The colored blocks were in an envelope on the table. The 4 x 4 checkerboard was printed on the problem sheet.

The Problem

The IO Block Problem is a variation of the "15 Puzzle" invented by Sam Loyd (55, p. 2429). The "15 Puzzle" consists of a square, shallow box which holds 15 little, square blocks. The blocks are numbered 1 to 15. Since there is room for 16 blocks in the box, the 15 blocks can be moved about and their places interchanged. The problem consists of bringing about a specified arrangement of the blocks from a given initial position (55, pp. 2430-2431).

The variation used in this study employs 16 blocks. It assigns colors to the blocks and allows free movement of them on a 4 x 4 checkerboard. There are four blue blocks, and three each of red, green, yellow, and white. The objective is to arrange the blocks on the board so that no two of the same color are in a horizontal, vertical, or diagonal line.

In analyzing this problem Schwartz (92) remarked that the most stringent requirement was imposed by the blue blocks. Their location was equivalent to the problem of placing four nonattacking queens on a 4 x 4 chessboard making the solution for blue block placement unique except for a reflection of this pattern. A blue block in a corner formed an impasse for the correct solution as did a blue block on one of the center cells. Another restriction involved the center four cells. There could not be any two of the same color there since these cells are adjacent. Thus, placing a red, a yellow, a green, and a white block in the center cells determined the position of the two other blocks of each of these colors. Schwartz said that the probability that a random pattern will be a solution is about 10^{-7} (92).

Methods of Obtaining Data

When the student started to work the problem, the observer noted the time and began to take notes on the subject's behavior during the first attempt. The behavior observed included such things as speed in working, strategies, emotional reactions, and questions or remarks. These notes were written in phrase form and were later used as supportive material in the interpretation of data.

After the student had completed the problem (if he did) the time was noted. Then he was presented a blank piece of paper and a pencil. He was asked to write an answer to some questions. The first question was: "How did you solve the problem?" If this was not clear, the writer added: "I mean tell me how you figured it out?" or "What were

you thinking about?" When the answer was written down, the second question was asked: "How would you tell another person to work it?" The third and last question was: "Do you think you could do it again?"

If the student answered "Yes" to the third question, he was asked to try it again. When the student had cleared the board and began to work, the time was noted. The observer then began recording the student's behavior in coded form. When the student finished, the time was noted. Then, the student was thanked for his cooperation and dismissed.

The students' answers to the three questions form the data for the written protocols of the problem-solving processes of the subjects. The writer felt that it was during the student's formulation of these answers that the protocols became structured and strategies were formed. These protocols are found in Appendix C.

Coding

In his coding method Kilpatrick (57, p. 50) had used a tripartite division: preparation, production, and evaluation. Under production he had placed three activities: deduction from condition (D), setting up equations (E), and trial and error (T). In analyzing his data he combined D and E into a single variable which he called the "deductive process" and compared this variable with T, the "trial and error" one. He distinguished trial and error from exploratory manipulation in this manner: "The latter are always performed on the data of the problem, whereas the former involved values the subject has chosen to try (57,

p. 162)." Kilpatrick always followed the production symbols with a number from 1 to 5. These represented the outcome of the problem-solving effort (57, p. 50).

The method used in this study was a modification of that of Kilpatrick. The second trial at solving the IO Block Problem was recorded in code. Since the preparatory activity in this study was trivial, it was not coded. In its place, however, a record was kept of activities involving a change of method. In place of Kilpatrick's "D" and "E" variables under production this study placed "DS" and "DA" under process. The former referred to a deliberate use of all the blocks of the same color in any one sequence. The latter referred to a deliberate use of different colored blocks in any one sequence. Kilpatrick's "T" is replaced by "AT." However, in this study no attempt was made to distinguish "trial and error" methods from "exploratory manipulation." "AT" was characterized by the behavior of the subject as he shifted pieces from one cell to another or rearranged them after their initial placement on the board. The process symbols were followed by a number from 1 to 5. This represented the outcome of the problem-solving effort. Underlining a process and outcome symbol indicated that the subject was having difficulty and persisted in the same activity for at least one minute. With regard to evaluation it was decided to use the single letter, "C," to designate completion of the problem. This ordinarily involved checking by the student or the observer. It did not mean the solution was correct but rather that the student felt the

solution was correct. The following is a summary of the symbols and their meaning as they were used in the study:

Activity Symbols (showing a reformulation of method):

R: removes blocks of one color from the board

B: cleans the board to begin anew

Process Symbols:

DS: places blocks of the same color in some arrangement: deliberation-sameness

DA: places blocks of alternate colors in some arrangement: deliberation-alternation

AT: shifts blocks around looking for some new arrangement: alternation-trial, error, exploration

Outcome Symbols:

1: incomplete result

2. impasse

3. intermediary result

4. incorrect result

5. correct result

Difficulty Symbol:

This is indicated by a line (____) under a process and outcome symbol, e.g., DS-1.

Termination Symbol:

C: completion of the problem from the viewpoint of the problem solver.

Types of Data

During the problem-solving sessions five types of data were collected by the writer:

1. The time required for the subject to solve the problem the first time was recorded.

2. A written record of observable behavior during the first problem-solving trial was made.

3. The student's written protocol of his own processes (written in response to the questions of the observer) was obtained.

4. The time required for the subject to solve the problem the second time was recorded.

5. The coded protocol of the student's processes during the second problem-solving trial was made.

These data along with that obtained from the CD Test were used to obtain the results found in Chapter IV and to formulate the proposed hypotheses for future research in Chapter V.

CHAPTER IV

PRESENTATION AND INTERPRETATION OF DATA

I. INTRODUCTION

The data presented in this study were obtained from three sources:

1. The CD Test results.
2. IO Block Problem sessions.
3. Certain variables obtained from the IO Block Problem sessions related to the CD Test results.

The first section presents the mean score from the CD Tests. It shows the resulting classification of all of the students in one of four groups and a similar classification of the students selected for the IO Block Problem. The distributions from these samples are compared with Hudson's results. The second section classifies the students who solved the IO Block Problem according to their written protocols and according to the coded form of their problem-solving processes. Each of these classifications is related to the time taken for the second trial (T_2). From these categories are drawn the generalizations of effective and ineffective problem solvers. The third section relates some variables from the IO Block Problem to the students' performances on the CD Test. The last section is a post hoc look at sex as a variable in the problem-solving process.

The time (T_1) taken for the student to solve the problem on the first trial was also recorded. However, this datum was no indicator of the time which would be required on the second trial. Also, it did not predict the effectiveness of the strategies which grew out of the first trial. Solving the problem in a short length of time on the first trial did not mean that the student could do so again, nor did taking a longer time on the first trial indicate that the student had formed a good understanding of the problem. These times (T_1) are found in Table 14 in Appendix D. They were not used by the writer to draw generalizations. They are given only for the sake of completeness.

The writer's observations form the other source of data obtained during the IO Block Problem session. Most of this was found to be unrelated to the student's final performance. On occasion these observations were used as supportive evidence for drawing generalizations from other sources. Sometimes they were used to strengthen incidental remarks which may be the basis for a future hypothesis.

II. THE CD TEST RESULTS

Total Sample

The CD Test was administered to 105 students from two schools. The student's solutions to three problems, hypothesized to be convergent type problems, resulted in a C score for each student. The solutions to three problems, hypothesized to be divergent type problems, resulted in a D score for each student.

The mean C score was 14.8 and the mean D score was 16.4. Using these mean scores the 105 students were classified as follows:

30 students: Group I (high all-rounders).

22 students: Group II (convergers).

14 students: Group III (divergers).

39 students: Group IV (low all-rounders).

Subsample

A total of 49 students was observed individually while solving the IO Block Problem. The distribution of this sample is as follows:

13 students: Group I (high all-rounders).

11 students: Group II (convergers).

10 students: Group III (divergers).

15 students: Group IV (low all-rounders).

Comparison of the Distributions with Hudson's Classification

Hudson (44, p. 55) defined 30 percent of his sample as convergers, 30 percent as divergers, and 40 percent as all-rounders. All of his sample were boys. A comparison of Hudson's findings with the CD Test findings and the IO Block sample used in this study is found in Table 2.

TABLE 2

DISTRIBUTIONS IN HUDSON'S STUDY, THE CD TEST, AND THE IO BLOCK PROBLEM

Group	Hudson	CD Test	IO Block
Convergers	30%	20.9%	22.4%
All-Rounders	40%	65.6%	67.1%
Divergers	30%	13.3%	20.4%

III. IO BLOCK RESULTS RELATED TO TIME ON THE SECOND TRIAL

Classification of Written Protocols

After each student had completed the IO Block Problem the first time, he was asked questions. This elicited written material which gave the researcher some idea of the mental processes of the student while he was solving the problem. It also helped to formulate strategies that would aid another in solving the problem. The answers to these questions form the written protocols for the IO Block Problem.

Using these answers, it was possible to determine in most cases whether the student had seen patterns or made generalizations that would enable him to effectively solve the problem a second time. A classification was made as follows:

A' :students whose written protocols indicated that they had formed more than one generalization for solving the problem.

C' :students whose written protocols indicated that they had formed only one generalization for solving the problem.

T :students whose written protocols indicated that they had worked the problem by "trial and error" and would tell another to work the problem the same way. This classification did not account for all of the students. Seven students did not complete the problem on the first trial and did not submit a written protocol. These seven cases are discussed in the next section.

The A' group consisted of 16 students: 5-I, 7-I, 17-II, 21-I, 31-II, 38-IV, 54-III, 66-I, 68-I, 69-IV, 76-IV, 79-IV, 80-IV, 91-II, 97-IV, 102-II. Their complete written protocols are found in Appendix C. That the blue blocks should be placed first and that the other colors should be placed on cells in the same pattern as the knight moves in chess were suggested by 5-I, 7-I, and 97-IV. Two students, 80-IV and 17-II, indicated that the corners should be filled each with a different color after placing the four blues correctly. The directions given by 21-I, 31-II, and 102-II were to place the four blues correctly and then to fill the center cells with different colors. In addition to these directions, 68-I mentioned the pattern of the "L" for the colors other than blue. The most specific and unique directions of all were given by 69-IV, 79-IV, and 91-II. They said to place the blues correctly, fill the center cells with different colors, then place the remaining colors one at a time, rotating the board 90° after the placement of each block. It was suggested by 38-IV and 54-III that the placement of the blues should be followed by the placement of two blocks of the same color. A lengthy protocol of 66-I described placement of each color in a pattern that related one color to the other.

The C' group consisted of 19 students: 6-I, 9-II, 15-III, 19-II, 22-III, 23-II, 27-III, 29-III, 50-I, 67-IV, 73-IV, 75-IV, 83-I, 86-IV, 92-III, 93-IV, 96-II, 98-III, 99-I. All of these students gave essentially the same rule, i.e., one should start with the correct placement of the four blue blocks and figure it out from there.

The T group consisted of seven students: 11-I, 24-III, 25-III, 30-I, 89-IV, 100-II, 74-II. Typical of the protocols for this group is the one written by 74-II: "I just put them down there and tried different ways. I would look in a row and see what I didn't have that I could use." The student did not know how to tell another to work the problem. Two students, 74-II and 100-II, could not solve the problem the second time. They were aided by the observer whose questions led them to form the solution.

The times for the second trial (T_2) are found in Table 14, in Appendix D. The average times were calculated for each protocol classification. The average time for students in the A' classification was less on the second trial than that of either of the other classifications. While the average time for students having made no generalizations in their protocols was the greatest, more than 83 percent of this sample made at least one generalization in their protocol. Table 3 shows the average times for the respective groups.

TABLE 3

PROTOCOL CLASSIFICATION, NUMBER OF STUDENTS, AND AVERAGE TIME ON THE SECOND TRIAL

Protocol	Number of Students	T_2 (Min.)
A'	16	2.4
C'	19	8.2
T	7	21.6

Classification of Processes

During the second trial of solving the IO Block Problem the students' moves were coded by using the symbols developed for this study. (See Chapter III, p. 57.) The classification is as follows:

A" :Students who completed the problem, showed only DA or DS process symbols, and had no incomplete, impasse or incorrect results.

B" :Students who completed the problem, had at most one AT symbol, and/or had incomplete, impasse or incorrect results.

C" :Students who completed the problem and had more than one AT symbol.

NS :Students who failed to complete the problem the first or second time or who received some type of help from the observer. These were the nonsolvers.

The classifications of these students, their coded strategies, and their time consumption on the second trial (T_2) are found in Tables 4, 5, 6,

and 7. There are also brief discussions of certain of the students (indicated by an asterisk) in these tables.

In Table 4 the protocols of some of these students (5-I, 7-I, 17-II, 102-II, 69-IV, 79-IV) indicated that they began to search for patterns after trial and error arrangements did not work. No trial and error was used in their second trials. One student (69-IV) said that he discovered the patterns by trial and error. One student (73-IV) solved the problem very quickly but failed to see that it was correct. He studied the board for some time thereafter.

TABLE 4

A'' CLASSIFICATION, PROCESS-SEQUENCE, AND TIME ON SECOND TRIAL (T_2)

Student	Process-Sequence	T_2(Min.)
5-I*	DA-3, DS-3, DA-5, C	1
7-I*	DS-3, DS-5, C	1
21-I	DS-3, DA-5, C	2
9-II	DA-5, C	1
17-II*	DS-3, DS-5, C	1
96-II	DS-3, DA-5, C	1
102-II*	DA-3, DA-3, DA-3, DA-5, C	1
69-IV*	DS-3, DA-5, C	1
38-IV	DS-3, DA-5, C	1
76-IV	DS-3, DS-5, C	1
67-IV	DS-3, DS-5, C	1
79-IV*	DS-3, DS-5, C	1
97-IV	DS-3, DS-5, C	2
73-IV*	DS-3, DS-5, C	5

In Table 5 one student (91-II) had developed an elaborate written protocol (which was not now before him) and seemed to adhere to the

repetition of an unfruitful sequence for a long time. The strategy he had developed involved turning the board 90° after the placement of two blocks of the same color (with the exception of the blues which he placed first). He finally perfected his method on the fifth attempt and finished the problem. This took almost as long as his first trial.

TABLE 5

B" CLASSIFICATION, PROCESS-SEQUENCE, AND TIME ON SECOND TRIAL (T_2)

Student	Process-Sequence	T_2 (Min.)
6-I	DS-3, DS-2, DS-2, B, DS-3, DS-2, DA-2, AT-2, B, DS-3, DS-2, DS-2, DA-2, B, DS-3, DS-2, DA-5, C	20
11-I	DS-3, DA-2, DS-3, AT-2, DA-5, C	10
66-I	DS-3, DS-1, R, DS-5, C	2
68-I	DS-3, DA-2, AT-5, C	1
83-I	DS-3, DA-2, AT-5, C	7
99-I	DS-3, DS-1, R, AT-2, B, DS-3, DS-5, C	5
19-II	DS-3, DA-2, B, DS-3, DS-5, C	4
23-II	DS-3, DS-3, DS-3, DS-3, DS-2, DA-2, B, DS-3, DS-3, DS-2, DA-2, AT-5, C	7
31-II	DS-3, DA-3, DA-2, B, DS-3, DA-5, C	5
91-II*	DS-3, DS-1, DS-3, DS-2, B, DS-3, DS-1 DS-2, B, DS-3, DS-1, DS-2, B, DS-3, DA-3, DA-2, AT-2, B, DS-3, DS-5, C	15
15-III	DS-2, DA-2, B, DS-3, DA-2, AT-5, C	10
22-III	DS-3, DA-2, DA-5, C	1
27-III	DS-3, DA-2, B, DS-3, DA-2, B, DS-3, DA-2, B, DS-3, B, DS-3, DA-2, B, DS-3, DS-2, B, DS-3, DA-5, C	15
29-III	DS-3, DS-2, DA-2, AT-2, B, DS-3, DA-2, B, DS-3, DA-2, B, DS-3, DS-2, AT-2, B, DS-3, DS-2, DA-2, B, DS-3, DS-2, DA-2, AT-5, C	35
54-III	DS-3, DS-2, DA-5, C	1
92-III	DS-3, DS-2, AT-5, C	7
98-III	DS-3, DS-2, AT-5, C	4
80-IV	DA-3, DS-3, DS-2, DA-2, R, DS-5, C	2
93-IV	DS-3, DS-1, DA-5, C	5

For most of the students in Table 6 it was as difficult to work the problem the second time as it was the first time. One student (89-IV) admitted that he saw no patterns even after the second trial. She said it would probably take as long the third time.

TABLE 6

C" CLASSIFICATION, PROCESS-SEQUENCE, AND TIME ON SECOND TRIAL (T_2)

Student	Process-Sequence	T_2 (Min.)
30-I	DA-2, AT-2, R, DS-3, DA-2, AT-2, DA-2, AT-2, AT-5, C	35
50-I	DS-3, DS-2, AT-2, R, AT-2, B, DS-3, DS-2, DA-2, AT-5, C	10
24-III	DA-2, AT-3, AT-5, C	15
25-III	DA-3, DA-1, C, AT-2, DA-3, AT-3, DA-3, AT-2, DS-3, DS-2, DS-3, AT-2, DS-3, AT-2, DS-3, AT-2, DS-3, AT-5, C	45
75-IV	DS-3, DS-2, AT-2, R, DS-2, B, DS-3, DA-1, AT-5, C	3
86-IV	DS-3, DS-2, AT-2, R, DA-2, DA-2, B, DS-3, DA-1, AT-5, C	15
89-IV	DA-3, DS-1, DA-2, AT-2, DS-3, DA-2, AT-3, DA-5, C	18

The students in Table 7 are nonsolvers. They present a challenge to any educator and may defy analysis. There were nine students in this category. Six of them worked the problem during the last period of the day; three made their attempt during the first period of the day. (Others, of course, had been successful during these periods but perhaps for these students the time of day was significant.) One student (16-I)

was called back the following day. His protocol included a reference to the previous day's experience: "Yesterday, since it was the sixth period, it had been a long and extremely slow day. . . . All I really had on my mind was sleep and rest. . . ." This student was able to work the problem the second day in one minute. The observer gave help to some of the students (100-II, 28-IV, 26-IV) who had shown no productive thinking for 20 minutes or more. This help was in the form of questioning (Socratic method):

O. "What is giving you the most trouble?"

S. "The blues."

O. "Try to place them." (S. does.)

O. "Now look back at your own written strategy about the center blocks." (S. does and completes the problem.)

Of the nine nonsolvers, six were girls and three were boys.

The times for the second trial (T_2) are found in Table 14 in Appendix D. The average times for each process classification were calculated. It was found that these average times were directly proportional to the use of the AT activity for each classification. The greater use of the AT activity produced the higher average times for the second trial. Table 8 shows the average times for the respective groups.

TABLE 7

NS CLASSIFICATION AND OBSERVER'S COMMENTS

Student	Observer's Comments
16-I*	Could not work the problem on the first trial; last period of the day.
101-I	Could not work the problem on the first trial; last period of the day.
62-II	Stopped during the first trial; declared she could not work the problem; last period of the day.
74-II	Student could not complete the second trial; the following day she succeeded on the third trial.
100-II*	Completed the first trial; could not work the problem the second time without help from the observer.
10-III	Completed the first trial; took the full hour; was not called back; last period of the day.
26-IV*	Did not complete the second trial; appeared to be in a hurry to leave and stopped trying.
28-IV*	Stopped working on the first trial; declared that that was the best he could do.
94-IV	Did not complete the problem in one hour; was not called back.

TABLE 8

PROCESS CLASSIFICATION, NUMBER OF STUDENTS, AND AVERAGE TIME ON THE SECOND TRIAL

Process	Number of Students	T_2 (Min.)
A''	14	1.4
B''	19	7.2
C''	7	20.1
NS	9	----

Generalizations

By looking at the data obtained during the IO Block Problem session and classifying students first by protocols and then by processes, the writer was able to make some general statements about the problem-solving activity of students involved in this study.

1. All students used trial and error in the first solving of the IO Block Problem.

2. Many students began to search for patterns after several impasse situations or after reaching a frustration level.

3. During the first trial session most solutions occurred immediately after periods of frustrations or after several sequences of impasse situations.

4. Students who made more generalizations in their protocols required less time to solve the problem on the second trial than those who had made fewer generalizations or who used trial and error.

5. There was a marked difference in the operations employed in the processes that led to effective problem solving (A" and B") as compared to those which led to ineffective problem solving (C" and T).

6. The use of trial and error in the problem-solving process during the second trial increased the time required to solve the problem.

7. Over two-thirds of the students used no more than one trial and error procedure on the second solving.

8. About one-fourth of the students used no trial and error procedure on the second solving.

9. Most nonsolvers were girls.

10. Most nonsolvers were working during the last period of the day.

IV. VARIABLES FROM IO BLOCK PROBLEM RELATED TO CD TEST RESULTS

From the previous sections it was found that the time taken on the second trial was a good predictor of the effectiveness of the problem-solving strategies used during the second solving of the IO Block Problem. This time (T_2) was inversely proportional to both the number of trial and error moves in the process-sequence coding and to the number of generalizations made in the protocols.

The average times on the second trial for each group were calculated from Table 14 in Appendix D. These times are as follows:

Group I: 8.5 minutes

Group II: 4.4 minutes

Group III: 11.4 minutes

Group IV: 4.6 minutes

Certain combinations of protocol classifications and process-sequence classifications were made. A'A" and A'B" were indicated as effective problem solvers. There was a total of 16 students in these categories. Combined classifications of C'B", TB", TC", or C'C" were indicated as less effective problem solvers. There were 21 students who fell into this category. These were students who made one or no generalizations in their protocols and who used AT at least once or reached an impasse

while solving the problem a second time. The percentage of students in each combined classification in relation to Groups I, II, III, and IV is shown in Table 9.

It can be seem from the average taken on the second trial for each group and from Table 9 that Group III (divergers) had:

1. the highest average time on the second trial;

2. the highest percentage of less efficient problem solvers (students who had made fewer generalizations and used more AT activity);

3. the lowest percentage of effective problem solvers.

Group III was characterized by a low convergent score and a high divergent score on the CD Test.

Group I had the second highest average time as well as the second highest percentage of less effective problem solvers. Group I was characterized by a high convergent score coupled with a high divergent score. The common element between the students in Group I and Group III was the high divertent score on the CD Test.

On the other hand, Group II and Group IV had low average times on the second trial and the lowest percentage of less effective problem solvers. The common characteristic of these two groups was their low divergent scores on the CD Test.

The generalizations which result from this analysis seem to be related to divergent thinking patterns. They are:

1. Those who score high on divergent test items tend to be less effective as problem solvers.

2. Those who score lower on divergent test items tend to be more effective as problem solvers.

TABLE 9

PERCENT OF STUDENTS IN CERTAIN COMBINED CLASSIFICATIONS FROM GROUPS I, II, III, IV

Combined Classifications	Group I N=13)	Group II (N=11)	Group III (N=10)	Group IV (N=15)
A'A"	23 %	18.2%	0%	20 %
A'B"	7.2%	18.2%	10%	13.3%
C'B"	23 %	18.2%	60%	6.7%
T B"	7.8%	0 %	0%	0 %
T C"	7.8%	9.1%	20%	6.7%
C'C"	7.8%	0 %	0%	13.3%

V. SEX AS A VARIABLE IN THIS STUDY

To look at sex as a variable in the problem-solving process was not one of the objectives of this study. However, some data obtained seemed to lend itself to interpretation with regard to the sex differences of the problem solvers.

The distribution ratio between boys and girls was nearly the same for the CD Test sample and the IO Block sample. (See Table 10.) It was again possible to compare the distribution of the CD Test sample with Hudson's (44) results. (See p. 62.) This was done by dividing each group into its male and female components. Table 11 gives the distribution by sex of Group II (convergers), Groups I and IV (all-rounders),

and Group III (divergers). Hudson used all male samples. The male sample from this study more closely approximates the distribution of Hudson than does the female sample.

TABLE 10

DISTRIBUTION OF SEXES IN SAMPLES FROM THE CD TEST AND THE IO BLOCK PROBLEM

Source of Sample	Number		Percent	
	Boys	Girls	Boys	Girls
CD Test	57	49	53.8%	46.2%
IO Block	26	23	53.1%	46.9%

TABLE 11

DISTRIBUTION OF SEXES FROM CD TEST RESULTS COMPARED WITH HUDSON'S RESULTS

Group	Hudson	CD Test (Male)	CD Test (Female)
Convergers	30%	24.6%	16.3%
All-rounders	40%	63.1%	69.4%
Divergers	30%	12.3%	14.3%

Using Table 14 in Appendix D, the average times on the first trial and on the second trial were calculated for boys and for girls who solved the IO Block Problem. The average times on the first trial were nearly the same for both groups but there was a marked difference on the

averages for the second time. The boys took much less time than the girls did. Table 12 shows this time difference and the distribution of males and females by groups.

Another classification was made by using the difference in sex and the number of students in each category formed from the protocols and the process-sequence coding. Table 13 shows a number of results:

1. Four times as many boys as girls made two or more generalizations in their protocols.

2. Seven times as many girls as boys indicated the use of only trial and error in their protocols.

TABLE 12

MALES AND FEMALES IN GROUPS FOR IO BLOCK PROBLEM AND AVERAGE TIMES FOR BOTH TRIALS

Sex	Groups				Average Time (Min.)	
	I	II	III	IV	T_1	T_2
M	7	8	4	7	16	5
F	6	3	6	8	18	12

3. The process-sequence coding showed twice as many boys as girls used no AT activity in solving the problem a second time.

4. Twice as many girls as boys used more than one AT activity in the second trial.

TABLE 13

MALES AND FEMALES CLASSIFIED BY PROTOCOLS AND PROCESS-SEQUENCE CODING

Sex	Protocols			Process-sequence			
	A'	C'	T	A''	B''	C''	NS
M	13	10	1	10	11	2	3
F	3	9	7	4	8	5	6

5. There were twice as many girls as boys who were classified as nonsolvers.

From Tables 10, 11, 12, and 13, it was possible to form the following generalizations regarding the activities of boys and girls during problem-solving sessions:

1. On the average girls required more time than boys to solve the problem a second time.

2. There were more girls than boys in Group III, the divergent group discussed in the previous section.

3. Girls used trial and error more often than boys during the second solving of the problem.

4. The protocols of the girls revealed fewer generalities than did those of the boys.

CHAPTER V

OBSERVATIONS AND HYPOTHESES

I. SUMMARY OF OBSERVATIONS

This exploratory study was concerned with processes used by students while solving a problem. It related this activity with students' performances on a test constructed by the writer for this study. The study was not experimental; it was not a comparison of controlled groups, and it did not assume that the sample used was normally distributed. Rather, it was an observational type study with secondary school geometry students which was undertaken in order to better understand the problem-solving processes and to generate hypotheses for future experimental study.

This last chapter presents a summary of the observations obtained from the study and compares these observations with previous studies. From these observations some implications are drawn for secondary school teachers on mathematics. The chapter ends with a list of hypotheses for experimental research in problem solving.

As a result of this study, the writer was able to make certain general observations. They are presented as follows:

1. Trial and error plays a major role initially in a problem-solving task and a minor role as the solution is approached. (See students' protocols in Appendix C and the references to the writer's observations on pp. 54, 58, and 60.)

2. The solution to the problem followed most often after several impasse situations or during the time of high frustration. (See Table 5, p. 67 and Table 6, p. 68.)

3. Students begin to search for patterns after an initial period of trial and error methods has failed. (See students' protocols in Appendix C.)

4. The discovery of generalizations and patterns during problem-solving activity decreases the time needed to work the problem and seems to be one of the main characteristics of an effective problem solver. (See Table 3, p. 65.)

5. Continued use of trial and error increases the time needed to work the problem and seems to be one of the main characteristics of an ineffective problem solver. (See Table 8, p. 70.)

6. Over-dependence on rules or generalizations interferes with effective problem solving. Some flexibility must be maintained. (See Table 5, p. 67.)

7. Some students seem to operate on intuition. These students solve the problem effectively but see no logical steps leading to the discovery of the pattern. (See Appendix C, protocols from students 9, 11, 17, 19. 23, 24, 25, 26, 30. 89.)

8. Evidently something in addition to the performance on the CD Test accounts for the success of some students in problem solving and the failure of others. (See CD Test scores in Appendix D.)

9. The results from the CD Test do not agree with Hudson's (44) distribution. (See Table 2, p. 62.)

10. Divergers seem to be less effective problem solvers. (See pp. 72 and 73.)

11. Sex may be a factor in the problem-solving process. (See Table 12, p. 76 and Table 13, p. 77.)
(There were more girls than boys who could not solve the problem. There were more girls than boys who used trial and error methods. There were more girls than boys who reported only one or no generalizations in the problem. There were more girls than boys who were divergers.)

12. The time of day may be a factor for those who cannot solve the problem. (See Table 7, p. 70.)

13. Lack of determination and/or motivation may be factors for those who cannot solve the problem. (See Table 7, p. 70.)

14. The use of retrospection as a method of obtaining data should always be accompanied by an observer's written account as is done in this study. (Observations from writer's notes.)

II. COMPARISONS WITH OTHER STUDIES

This study examined again the age-old problem of trying to understand the processes of human thinking. Many theorists and researchers have opinions and findings which this study seems to support. However, there are others whose theories and findings are somewhat contrary to the observations reported in this study. Both sides will be compared to this new attempt.

That trial and error plays an initial role in the problem-solving task but is later replaced by a search or patterns or generalizations

would be supported by such theorists as Bruner (8), von Neuman (101), Polya (78), Hadamard (36), and Einstein (23). In various ways these have said that knowledge results can be gained inductively and proven later. Wertheimer (102) concluded that the idea of the solution came in a threefold development. The last stage was a structuring of the pieces into a whole.

That the solution to a problem usually occurs shortly after periods of frustration may indicate that the entire mental process is dynamic and evolutionary as Dewey (20) and Whitehead (104) proposed. However, this observation is contrary to the findings of Scandura (88) whose research indicated that all problem solving is dependent upon previously learned rules. Indications by students in this study that they looked for patterns are a strong argument in favor of the gestalt traditionalists and Bruner (9) who described the human problem solver as an active and purposeful organizer.

It was found that over-dependence on rules and generalizations interfered with effective problem solving. Duncker (22) found that the inability of a person to break with traditional patterns of thought interfered with his ability to solve problems. Luchins (67), Scheerer (91), and Cunningham (13) also investigated this "functional fixedness." They found similar results. Henderson and Pingry (41) found that following a fixed sequence was a handicap to agile minds.

The results from the CD Test were contrary to Hudson's findings (44). This study indicated that divergent thinkers were poor problem solvers just as had Richards and Bolton's (82) findings in 1971.

Seemingly in contradiction to these findings Dodson (21) had reported that one of the characteristics of successful problem solvers was divergent ability.

Gurova (35) found that students who were unable to dissect their own mental operations were also unsuccessful in solving complex problems. Talyzina (98) observed that partial deductions are more frequent than complete ones. This study showed essentially the same results through the protocols of the students.

Finally, the writer is in agreement with Newell and Simon (71) who believe that the area of trying to understand the human problem-solving activity is "undoubtedly . . . the largest and most important terra incognita on the map. . . . (71, p. 868)."

III. IMPLICATIONS FOR TEACHING

Since the mathematics classroom is a natural area for developing, improving, and modifying problem-solving processes among students, it would seem that this study would have certain implications for the teacher of mathematics.

Students vary widely in their ability in, and their methods of, problem solving. The teacher should learn to observe the more effective problem solvers in order to determine ways to aid the less effective ones. These latter students always present a challenge. They should be given less difficult problems at first so that they may gain confidence and have success in simple problem-solving tasks. Later, they can be required to solve more complex problems.

A good practice would be to frequently introduce nonroutine, content-free problems in the classroom. This would give students more problem-solving experience without the threat of grading. As far as possible, the teacher should provide for individual differences. This can be done by giving problems of varying difficulties, and by not putting strict time limits on a student's thinking processes.

The extent to which divergent or creative thinking should be encouraged in mathematics is still a matter of debate to this writer. This study showed, as did Richards and Bolton (82), that divergent thinking plays a minor role in mathematical thinking. This is contrary to Dodson (21) who reported that one of the characteristics of successful problem solvers was divergent thinking. If divergent thinking is encouraged, it should not be done at the sacrifice of the ability to search for patterns and to make generalizations. It is the opinion of this writer that a teacher should seek to maintain a happy balance between generating ideas and generalizing ideas.

Teachers often assume that a student who has worked a problem has a clear understanding of it. It may be that the student operated only by trial and error or partially generalized a solution. In order to check understanding it might be a good classroom practice to have students formulate these generalizations verbally after some of the problem-solving sessions.

However, students who have too rigid a method of thinking or who cannot get out of unfruitful repetitive patterns need to be helped also. The teacher should have a thorough understanding of the structure of the

problem so that questions could be formulated in such a way that the student is led to discover the solution.

Professor Polya believed that the main principle of active learning, a principle already known to Socrates, is to let the pupil himself discover as large a part of the material as possible (78). A good teacher should be like a midwife, should help bring forth the idea conceived in the pupil's mind (78, pp. 20-22).

IV. HYPOTHESES FOR EXPERIMENTAL RESEARCH

A primary aim of this study was to generate hypotheses which could be tested by future experimental study. These hypotheses, based upon observations which were made in this study, are presented here:

1. Students who score high on divergent-type tests will be less effective problem solvers than students who score low on such tests.

2. Students who score high on divergent-type tests will use trial and error more frequently in problem-solving tasks than will students who score low on such tests.

3. Students who are trained to look for patterns and to draw generalizations become more effective problem solvers.

4. Students who are required to formulate statements concerning generalizations formed during problem-solving tasks will perform better on other problem-solving tasks than those who are not required to formulate the statements.

5. The time taken on the second trial of a complex problem is the best single predictor of the number of generalizations formed by the problem solver.

6. The distribution of convergers, all-rounders, and divergers from secondary school mathematics classes consisting of both sexes will differ from Hudson't (44) distributions of 30 percent convergers, 40 percent all-rounders, and 30 percent divergers.

7. Female samples will differ significantly from male samples with regard to the distribution of convergers, all-rounders, and divergers.

8. Secondary school girls make use of trial and error methods more often during problem solving than do secondary school boys.

9. Secondary school boys make more generalizations and detect more patterns during problem-solving activity than do secondary school girls.

10. The solution to a problem on a first trial is not a criterion for judging the understanding of the problem.

11. Effectiveness in mathematics problem solving changes during the school day.

This study also pointed up the need for developing and validating an instrument similar to the CD Test which could be used to measure convergent and divergent thinking in secondary school mathematics students. It is felt that verbal tests of divergent thinking do not adequately measure the tendency for divergent thinking in mathematics.

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APPENDIXES

APPENDIX A

CONTENTS OF CD TEST

NAME_____

Code_____

Geometry Period_____

Free Period_____

READ ALL THE PROBLEMS IN THE FOLDER.

YOU MAY WORK THE PROBLEMS IN ANY ORDER WITH THE EXCEPTION OF THE TWO WHICH ARE FASTENED TOGETHER: PROBLEM A SHOULD BE WORKED BEFORE PROBLEM B.

DO AS MUCH OF EACH PROBLEM AS YOU CAN AND DO AS MANY PROBLEMS AS YOU CAN DURING THIS PERIOD.

IF YOU HAVE ANY QUESTIONS RAISE YOUR HAND AND YOU WILL RECEIVE FURTHER INSTRUCTIONS.

YOU MAY BEGIN NOW.

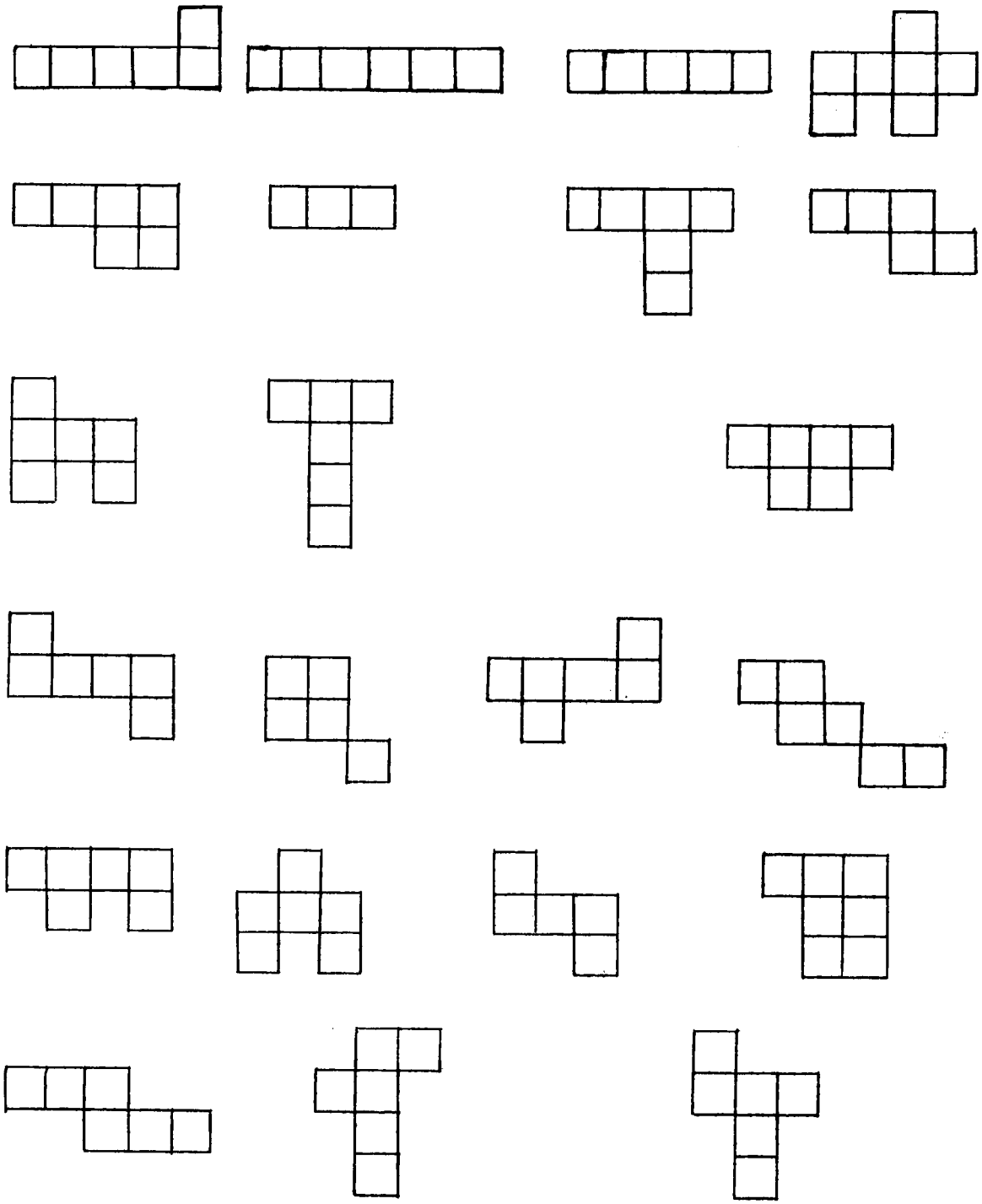
Students were given the following directions at the end of the period:

1. Remove the staple from the A and B problem sheet.
2. Place your code number at the top of each sheet.

3. Arrange the sheets in this order:

Name sheet
Cube problem
Archaeology problem
Problem A
Problem B
Quadrilateral problem
Equations problem

C-1



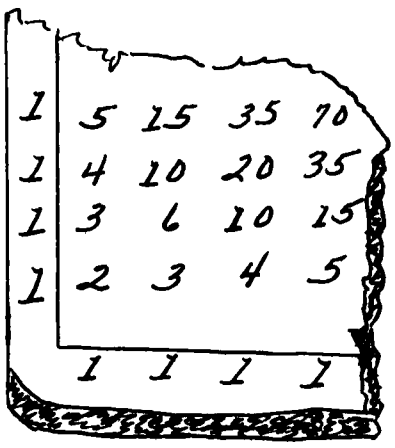
The patterns above were made by using six square pieces of cardboard, all the same size, and placing them side by side on a flat surface so that some edges were lined up. The common edges were taped together. A cube* is to be made by folding the pattern and taping it but not cutting it.

Encircle all the patterns above which could be used to form cubes.*

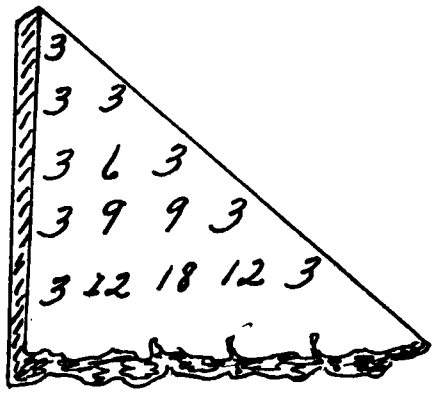
Cube: a six-sided solid with square sides. A cube may be represented by a closed box with its sides shaped like squares.

C-2

An archaeologist, while excavating in an ancient land, found two broken stone tablets in the treasurer's palace. See sketches of each tablet below.



Tablet A

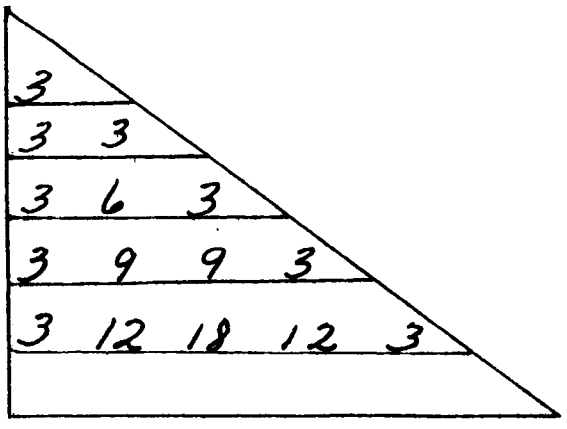


Tablet B

Find the numbers which would have appeared in the next two rows and the next two columns of Tablet A if the tablet had not been broken. Fill in the numbers in the tablet below.

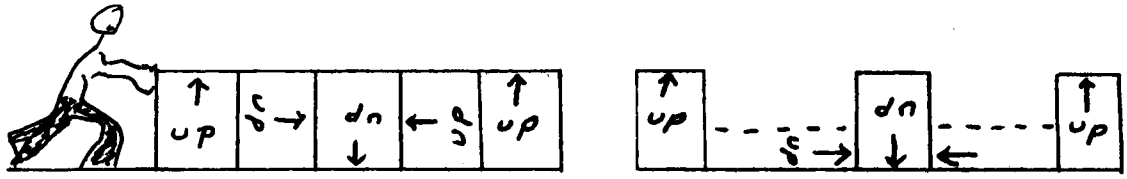
1	5	15	35	70		
1	4	10	20	35		
1	3	6	10	15		
1	2	3	4	5		
	1	1	1	1		

Find the numbers which would have appeared along the lower edge of Tablet B if the tablet had not been broken. Fill in the numbers in the tablet below.

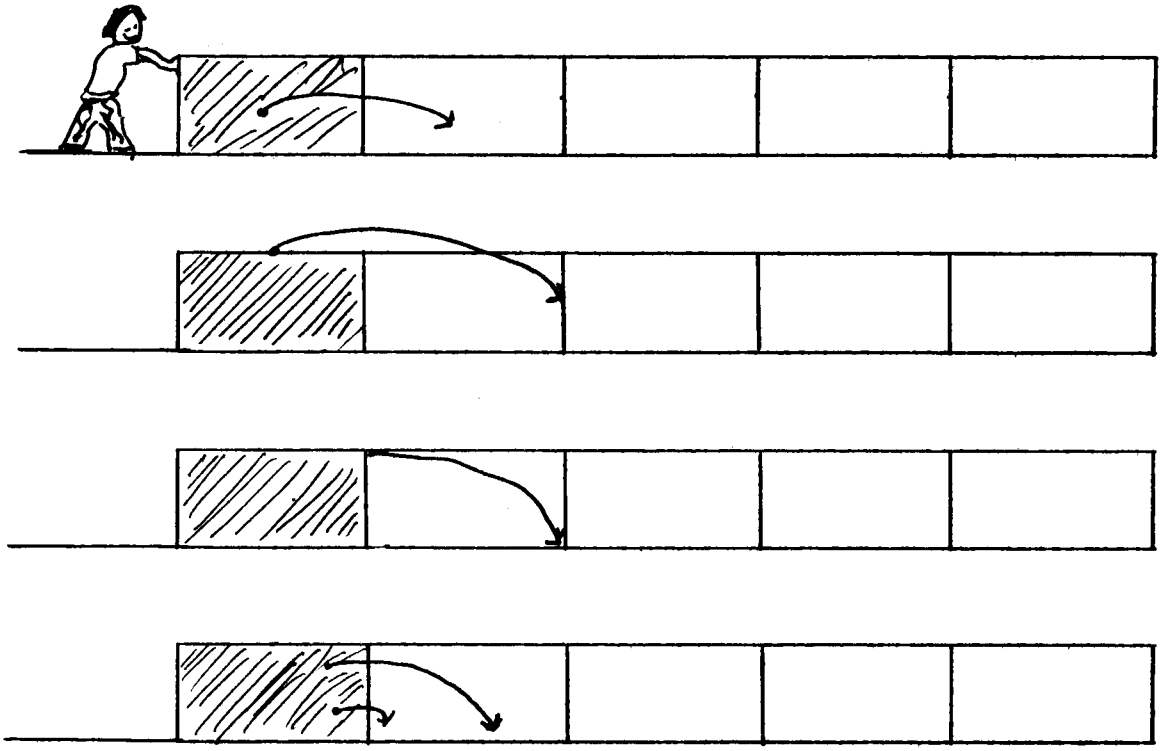


C-3 PROBLEM A

A crate may be moved across the floor by flipping it end over end as shown below.



A crate with square sides has a point (or points) marked on the side facing you and it is flipped end over end across the floor as described above. Complete the paths for each marked point(s) in the figures below. In each case the path for the first flip is given.



D-4

PROBLEM B

Make different paths of points by flipping various figures (not a square or a rectangle) end over end along some line. Show these paths by marking a point(s) on the edge or within the figure and indicating by a dark line the movement of the point until the pattern begins to repeat. (See Problem A as an illustration.)

You may vary the figure and/or the position of the point(s) and/or the line along which the figure is moved.

Make as many different paths as you can.

(Problems A and B, C-3 and D-4, are adaptations of problems presented by Evan M. Maletsky in "Fun with Flips." The Mathematics Teacher 66: 531-535, 1973.)

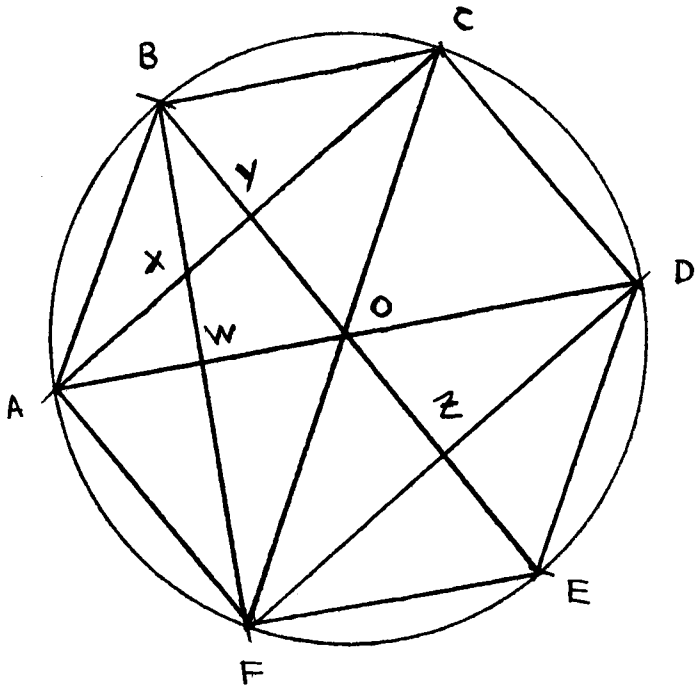
D-5

$\overline{ABCDEF}$ IS A REGULAR HEXAGON INSCRIBED IN CIRCLE O.

$\overline{AB} = \overline{BC} = \overline{CD} = \overline{DE} = \overline{EF} = \overline{FA}$

$\overline{AOD} = \overline{BOE} = \overline{COF}$ because each is a diameter* of the circle.

(*a chord of the circle which passes through the center O).



List as many <u>pairs</u> of congruent quadrilaterals* as you can find. (*four-sided closed figures which would match in every respect if you placed one on the other.)

D-6

Using the following set of numbers [3, 21, 2, 10] and these operations +, -, x, ÷, write the left member of the equation in as many different ways as you can so that it will equal to the number given in the right member. You do not have to use all the numbers each time or all the operations, and if you want to you may use one number or one operation more than once in any equation.

a)_____= 17

b)_____ =1

c)_____= 11-5/6

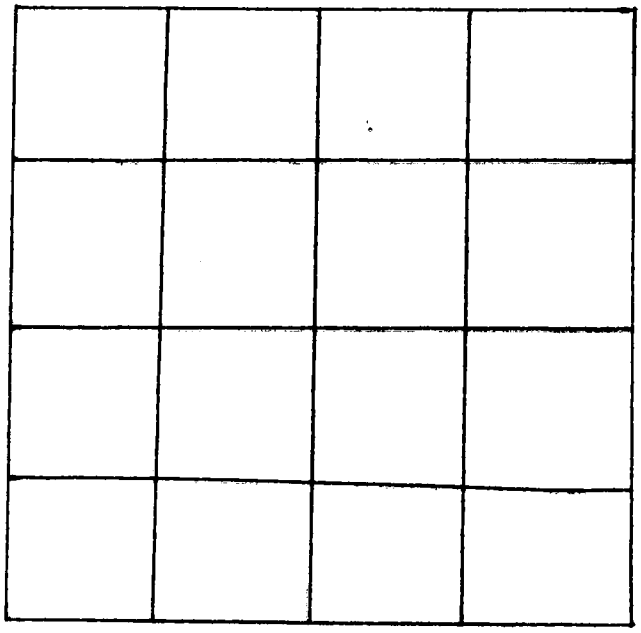
APPENDIX B

10 BLOCK PROBLEM

YOU ARE GIVEN 16 BLOCKS. SOME BLOCKS ARE BLUE, SOME ARE RED, SOME ARE GREEN, SOME ARE YELLOW, AND SOME ARE WHITE.

YOU ARE ASKED TO ARRANGE THESE BLOCKS ON A FOUR-BY-FOUR CHECKERBOARD SO THAT NO TWO OF THE SAME COLOR ARE IN THE SAME ROW OR THE SAME COLUMN OR ON THE SAME DIAGONAL.

(Students were given an envelope containing 16 colored blocks of paper: four blue, three each of red, green, yellow, and white.)



Source: Schwartz: Benjamin L. "A New Sliding Block Puzzle." The Mathematics Teacher: 66: 277-280, 1973.

APPENDIX C

STUDENTS' PROTOCOLS

The students' protocols are the results of their answering in writing the following questions;

1. How did you work the problem?

2. How would you tell another to work the problem?

These answers were copied just as the student wrote them. No attempt was made by this writer to correct spelling or grammar.

5-I Student's protocol: uncorrected.

1) First, I tried trial arrangements to see if one of them would work. Then, when it seemed almost impossible, I decided to look for patterns. I placed one of each color in the 4 corners and tried to fill the center. When it didn't work I started wondering about the odd numbers of the colors: 3 of 4 colors and 4 blues. I thought that maybe the patterns were symmetrical, which meant that the blue ones could not be in the center. I filled in the center, saw how the fianl pattern would be, and filled it in.

2) I'd tell him to visualize the board as a chessboard and think of the markers as knights. Their locations on the board correspond to knight moves for each color.

6-I Student's protocol: uncorrected.

1) I knew that the four blue squares had to be arranged first since the other squares were in colors of three.

2) Arrange the four blue squares first and it is much easier.

7-I Student's protocol: uncorrected.

1) I just stuck them down at first then I tried sticking each color down in a pattern when I got most of them in a pattern I just fixed the others where they would work.

2) Place the blue ones down first where there wouldn't be any two in the same row. Then set the others in where they would be in the same pattern as every other color except blue. They could find the place to start then place one there move two squares over and one down, place one there and then go two squares directly above and one square to the right and place one there. Follow the same pattern for the other two colors with only three blocks of each one.

9-II Student's protocol: uncorrected.

1) Trial and Error by first putting them in horz. and vertical rows and then refitting them for diagonal rows and kept moving until they were in place.

2) Look for diagonal rows for they are more deceiving.

10-III (No protocol was written.)

11-I Student's protocol: uncorrected.

1) I filled up some of the squares with blocks then I put the rest in where they would work and I had to change around a lot.

2) I would tell them that that there is a certain pattern of each color of blocks.

15-III Student's protocol: uncorrected.
(This student was given four red blocks instead of the usual four blue blocks.)

1) I read the directions then I put the different colored blocks into different piles. I counted to see how many of each color I had. I worked the puzzle by just putting the blocks on the paper. After playing around with the blocks I realized tha the four red blocks were what had been giving me trouble. The I reworked the puzzle by first

putting one the red blocks. One on each different part of the puzzle so that they were not diagonally. I then worked in the other blocks around the red ones until thye were right.

2) To first put the red blocks on different lines and work around them.

16-I Student's protocol: uncorrected.
(This was the second day. The student had failed to work the problem the first day.)

1) I figured out which colored squares I had the most of and place them on the paper so that they could be arranged according to the directions. Then I just placed the other squares at random where they would fit in.

Yesterday, since it was the sixth period, it had been a long and extremely slow day which caused me to think mostly about when I could get home and get some rest from this tiresome day. All I really had on my mind was sleep and rest, I suppose that was what mostly kept my thoughts astray from the puzzle.

2) To work this puzzle, first read the directions given on the page. Then look over the sixteen squares and discover how many of each colored squares you have or which you think might be the most difficult ones to place on the paper.

17-II Student's protocol: uncorrected.

1) first I placed two rows so that they were In the order ask by the Instructions then I tried filling in the rest of the squares untill I saw a pattern then all of a sudden it fell in place

2) Start in the corners with the yellow first then the green, then red, and then white. place each of these so that the follow the Instructions then try to fill in with the blue.

19-II Student's protocol: uncorrected.

1) Since there were four blue blocks I used those first as my outline or sort of pattern. next I just used the trial and error method until I arrived at the conclusion.

2) I would say that there is probably 5 or 6 different ways to do it and to just keep trying different combinations till they recognize a pattern.

21-I Student's protocol: uncorrected.

1) Set the blue only one way, put 4 different colors in the center then by trial and error

2) Fix the blue and leave them alone and fix the center and finish.

22-III Student's protocol: uncorrected.

1) I had to get the blue blocks placed so that there would be only one blue per colum then just fit the other squares in around the blue ones. I tried to just make sure the greens and reds wer closeer and then fit the whites in later.

2) I would first say that here are 3 of each square except blue so you have to make sure that each blue is on a different numbered square in a different numbered row. Then wich them good luck.

23-I Student's protocol: uncorrected.

1) First I divided them into their color groups, I then started with the color that had the most and arranged them where they would work, then I arranged the rest.

2) Arrange the blocks of colors to where no two of the same colors were in a same row.

24-III Student's protocol: uncorrected.

1) I first took different colors of each row and put them in order. Then I try to figure out if the same colors were not in the same colum or in the same diagonal section. I came up with two colors the same, and try to look over the board to see where I culd place the colors so thaey would not be in the same row, colum or diagonal section.

2) I don't know.

25-III Student's protocol: uncorrected.

1) I put a different color on each block on the first row and then the first column. I then did the same to the last row and the last column and worked from there.

2) By starting from the top and working down, then check it crossways.

26-IV Student's protocol: uncorrected.

1) first I tried to just put them down and figure it out. Then i put all the colors together and one by one took the color that wasn't used in the row, diagonal and put it down.

2) tell them to think about all of the colors first and then put all the same colors together and work it form there.

27-III Student's protocol: uncorrected. (Four red blocks.)

1) More red blocks then the others. There couldn't be the same colors at the ends of the diagonals. I just kept working at the puzzle and found ways that would and wouldn't work.

2) Place the red blocks first, so that they aren't in the same columns and place the other blocks around them.

28-IV (No protocol was written.)

29-III Student's protocol: uncorrected. (Four green blocks.)

1) Because of the number of green boxes I began with that color. After I had located all the green the rest was somewhat of a chance game, using the trail aby error until finished.

2) It would problably be the same way that I had found the answer. Start wtih the green and then just work with it until completed.

30-I Student's protocol: uncorrected.

1) First I went by rows and put all different colored squares in each row. Then I saw if the columns were all different colors. If they weren't I just kept changing around until it came out right.

2) I'd just ell them to start off by rows but sort of check the columns as you're doing it and to concentrate.

31-II Student's protocol: uncorrected. (Four red blocks.)

1) At first I just went at it haphazardly but after a while I concentrated on putting the red ones in place because they are the most numerous and then built from the middle.

2) I would tell them to put the red ones in place first and then build from the middle

---

38-IV Student's protocol: uncorrected.

1) I tried to put them together like a purzle. To spread out the colors so that they would fit. To separate the colors and deceid which color went where.

2) To sereate the colors before you started and put about 2 of one color then start on another. Place the blue one's first there are more of them.

50-I Student's protocol: uncorrected.

1) Trial and error

2) Blue is the key

54-III Student's protocol: uncorrected. (Four red blocks.)

1) I 1st grouped the colors into the same color group 2nd used the red because it had the largest number of blocks. I experimented with the red blocks until none were diagonal or on the same row. I figured that the position of these would not change, so I never moved them when I placed the other colors. Next I fit the dark green close to the red, and next the yellow without any problem. The white and blue I shuffled around not knowing they'd fit as perfectly as they did.

2) I'd tell the person to start with the red and then keep the green under the red and to think the rest out for himself.

62-II (No protocol was written.)

66-I Student's protocol: uncorrected. (Four green blocks.)

1) first I started putting blocks in any where-sort of a trial and error sort of thing. Then I noticed that there were 4 dark greens and they would only fit in one pattern. After I found the pattern, I tried the trial and error method again. I would take one color and fit it in until a color wouldn't work. Then I'd start all over again leaving the greens where they were. I put the blues on top of the greens once and they fit so I thought there must be a pattern and tried to find it. This is what it is: wherever the dark green is put a blue on top except at the one at the top row. On the column that I didn't put a blue I put a white at the bottom. On top of white I put red except on the white on the top row. On the column that I didn't put white I put red at the bottom. On top of red I put yellow except on red at top row. On the column where there was no red I yellow at the bottom.

2) The first dk. green is in the 3rd space on the top row. The second dark green is in the 1st space in second row. The third dk green is in the fourth space in the third row. The fourth dk. green is in the 2nd space in the 4th row. See above and follow pattern.

67-IV Student's protocol: uncorrected. (Four red blocks.)

1) rearangeing the reds so that they fit then check the others to make sure that they didn't break the rules and kept exchanging colors till I got it.

2) tell him to get the reds in place first the arrange the other colors according to the rules till you get it.

68-I Student's protocol: uncorrected.

1) had to put each of one color in the middle, there was only one way in which the blues could go. Made a L so that each would fit then when I had just one left over I asked myself what could go there.

2) find the way the blues would go together then work from the center outward.

69-III Student's protocol: uncorrected.

1) Trial and error- looking for patterns- discovered pattern by trial and error

2) get the color with four squares in first- different color in each corner-four colors in middle same as corners but rotated 1/4 turn- fill in sides from there.

73-IV Student's protocol: uncorrected. (Four red blocks.)

1) First, I saw that there were more reds, so I tried to put all those down first. After that I saw there were 3 of all the other colors so I laid those down making sure they were not in the same row. After finishing 3 colors, the others seemed to fit in.

2) I would tell them that it's not hard and to figure out the red blocks first and after that the order of colors wouldn't matter since there are three of all the other colors.

74-II Student's protocol: uncorrected.

1) I just put them down there and tried different ways. I would look in a row and see what I didn't have that I could use.

2) I don't know.

75-IV Student's protocol: uncorrected.

1) I started by making a row of 1 of each of the different sets of colors in a line and since there were 4 blues, I placed them all around the square so they would be in the right position. Then I tried to reverse the first row I made altering it when I had two of the same color next to each other or in the same row.

2) Start out by placing the color you have the most of around the block and continue from there.

76-IV Student's protocol: uncorrected. (Four red blocks.)

1) After I had so much trouble I figured out that there was more of one color- red than the others. I knew that I must place them first, but I didn't exactly know how. I finnally decided to use the red blocks alone and then place the other blocks around them. After I did this the puzzle seem to work out very easily.

2) I would tell her to count the number of blocks each color has. Then I wuold tell her to place the ones with more first and then place the others around them.

79-IV Student's protocol: uncorrected. (This student made drawings to illustrate each direction given. These have been left out.)

1) At first I tried by trial and error. I figured later that would take to long. So I looked for a pattern the first few did not work. All colors had to be separate. A. I took all the blues 1st because since there were more of them than the others. I place them in an order in which they would not be in the same colume, row, or diagonal. (Here was place an illustration.) Next I arranged the greens in the same way. after more trials in looking for a pattern I found a pattern. Starting in the left hand top corner you put 1 color, green, next put a green color in the block 2 right and 1 down from there put another green color 2 down 1 to left. (Here was placed an illustration.) If you

keep turning the puzzle around and following this pattern for the rest of the colors, red, yellow, white, the problem will be finished. (Three more illustrations were included.)

2) Where the sentence starts at line (A.) on the other side. There are step by step directions on how to work it.

80-IV Student's protocol: uncorrected.

1) First off I just tried the different colors all around but then I saw that there had to be a blue square in each row and that there must be a pattern to it. When I set up the blue pattern, I then took each color separately and fit into the remaining spaces. The last 2 colors needed rearranging a couple times. About 1/2 way through, after the blue pattern was set I tried putting each of the other colors into a pattern which worked out.

2) put each of the colors except blue into a corner. starting with top left-hand corner, put a blue underneath that (left-hand) spuare, and one blue one on the other side of each corner after it. from there you should be able to figure out a pattern fairly easily.

83-I Student's protocol: uncorrected. (Four red blocks.)

1) I guessed

2) Reds stationary Just move them around.

86-IV Student's protocol: uncorrected. (Four green blocks.)

1) There were four green, so I knew that they had to go in the four columns but couldn't be next to each other. Then I just placed the different squares around the green.

2) Tell him to place the green in different places then work around them.

89-IV Student's protocol. uncorrected.

1) I first put different colors in the top row. As I went on I was trying to figure ways in which to put them so that I didn't have all of one color left to fit in. Then I had only one left I thought of ways to switch only a few so I could fit the last one in.

2) Make sure to space out your colors and as you are going along check to make sure its right so far.

91-II Student's protocol. uncorrected.

1) At first I tried just puting them in randonly with no order.
A. Tried taking one of each color and puting it in a row and seeing if it would come out right.
2. Tried a different order using same method
B. as above.
C. Tried aranging one color at a time untill I noticed that there were more blues than any other color.
D. Used blues first using above method. Then placed the other color according to the way a knight moves.

2) You are to arrange all of the different squars so that no two colors are in the same row, colam, or diagonal.

92-III Student's protocol. uncorrected. (Four green blocks.)

1) There are 4 greens and whenever I put them in any certain shape the green messed the whole thing up. So I figured that if I could figure some way to place the greens, the hardest part would be over. After trial and error I figured that I go diagonally and cover one in a certain direction and then the problem would be solved.

2) 1. to look at the colors and see if there were any obvious differences.
2. in some way, figure out how to work the greens
3. look very long for a pattern and follow it for the other colors

---

93-IV Student's protocol: uncorrected. (Four red blocks.)

1) First, I just put it anyway and then checked through to see what was wrong and tried to correct it. Then I kept ending up with a red square with no place, so I put them in order first and went back to my original procedure.

2) get rid of the red ones first because there are more of them.

94-IV (No protocol was written.)

96-II Student's protocol: uncorrected.

1) First I just started putting down the squares and then realizing that since I had four blue ones I whould place them down first and the other squares would fall in place. It took a while to find where to place the blue squares but when I did I just started placeing the other squares down.
I found the place for the blue squares by just trying them in different coums and ways until I came up the right one.

2) My first and only clue to the problem would be to have them place the blue squares first.

97-IV Student's protocol: uncorrected. (Four red blocks.)

1) Since there was four red blocks I knew that they had to go on the outside. The rest of the block I did as trial and error.

2) If I was going to tell someone how to do it I would tell them how the knight moves in a chess game and tell them to use that as to where you would put your blocks. I would also tell them not to get discouraged.

98-III Student's protocol: uncorrected. (Four green blocks.)

1) By process of elimination I first placed the three green squares into nondiagonal slots and worked from there to the blue from the blue to the white and red and finally ending up with one yellow to finish.

2) Place the green pieces into the slots since they are four of them then proceed to the blue which at this stage is not as hard but be careful not to violate the directions. Proceed from the blue to the yellow and from there to the red and white together it shouldn't be hard to tell there proper places at this point.

99-I Student's protocol: uncorrected.

1) Off the table I set up the blocks so I could see how many of each color- the first part I was putting them in place at random but wasn't getting anywhere. After a few minutes I cleared the board and started over by first putting the BLUE squares into a place where they fit since there were more of them. hence: harder. I then just took the blocks a color at at time and placed them so that they fit. Then I just had to make a few small moves.

2) Start with blue squares and fit them in first and then just take a group of colors at a time.

100-II Student's protocol: uncorrected.

1) I strated with making a row at the top and working from there to find the combination, but I could not, so I made a square in the middle and worked the blocks around that.

2) When you start with the square just start making combinations around it. It seems that the red and blues were together more than any color, also the green and yellows did the same for me.

101-I (No protocol was written.)

102-II Student's protocol: uncorrected.

1) First of all I was attempting to find a definite pattern concerning the squares. Second I tried to put the four of the five colors in the center for squares on the chart. By doing this noticed that each of these squares was in triplicate. This at that moment told me that the blue square had to go in each of the four columns. The rest of them worked in a pattern of a sort.

2) To tell some one else how to work this problem I would first tell them that there were four blue blocks and three red, green and yellow blocks. First put one of each color block in each corner ecept for a blue block. Second put the other blocks of the same color in the center four squares. Place each of these blocks to the right of the original square up two squares and over one. Then just place the remaining blocks in the remaining squares without the blue blocks directly across from each other.

APPENDIX D

TABLE 14

SUMMARY OF DATA FOR STUDENTS WORKING THE IO BLOCK PROBLEM: CLASSIFICATIONS, TIMES, SEX, AND SCORES FROM THE CD TEST

Student	Classifications Protocol/Process		Times (Min.) T_1	T_2	Sex	Scores C	D
5-I	A'	A''	5	1	M	28	61
6-I	C'	B''	30	20	F	15	44
7-I	A'	A''	5	1	M	25	37
9-I	C'	A''	7	1	M	20	32
10-III	-	NS	60	-	F	13	18
11-I	T	B''	10	10	F	15	32
15-III	C'	B''	30	10	F	14	25
16-I	-	NS	-	-	F	17	17
17-II	A'	A''	15	1	M	15	13
19-II	C'	B''	20	4	M	18	15
21-I	A'	A''	25	2	F	22	20
22-III	C'	B''	15	1	M	8	24
23-II	C'	B''	3	7	M	20	25
24-III	T	C''	15	15	F	10	22
25-III	T	C''	10	45	F	14	26
26-IV	-	NS	-	-	F	12	11
27-III	C'	B''	45	15	F	12	25
28-IV	-	NS	-	-	M	10	1
29-III	C'	B''	7	35	M	10	57
30-I	T	C''	5	35	F	22	25
31-II	A'	B''	25	5	M	16	14
38-IV	A'	A''	35	1	M	14	1
50-I	C'	C''	15	10	M	16	40
54-III	A'	B''	5	1	M	11	17
62-II	-	NS	-	-	F	19	15
66-IV	A'	B''	13	2	F	19	23
67-IV	C'	A''	15	1	M	12	5
68-I	A'	B''	20	1	M	15	17
69-IV	A'	A''	18	1	M	14	16
73-IV	C'	A''	10	5	F	13	5
74-II	T	C''	20	-	F	19	9
75-IV	C'	C''	20	3	F	13	16
76-IV	A'	A''	20	1	F	10	15

TABLE 14 (continued)

Student	Classifications Protocol/Process		Times (Min.) T_1	T_2	Sex	Scores C	D
79-IV	A'	A''	25	1	M	13	13
80-IV	C'	B''	25	7	M	12	13
83-I	C'	B''	25	7	M	17	19
86-IV	C'	C''	18	15	F	8	10
89-IV	T	C''	10	18	F	8	7
91-II	A'	B''	16	15	M	17	13
92-III	C'	B''	30	7	F	13	17
93-IV	C'	B''	20	5	F	13	12
94-IV	-	NS	-	-	F	14	9
96-II	C'	A''	10	1	F	17	9
97-IV	A'	A''	20	2	M	10	13
98-III	C'	B''	20	4	M	12	23
99-III	C'	B''	8	5	M	21	20
100-II	T	NS	10	-	M	19	9
101-I	-	NS	-	-	M	15	17
102-II	A'	A''	25	1	M	15	16

VITA

Ann Alsobrook Maxwell was born in Chattanooga, Tennessee on February 14, 1929. Her parents were James E. and Catherine Weidner Alsobrook. She attended Notre Dame Elementary and High School, graduating in 1947. Having been awarded a scholarship to Siena College, Memphis, Tennessee, she received a B.S. degree with a major in mathematics and a minor in biology, graduating cum laude. In 1963, the M.T.S. degree with a major in mathematics and a minor in chemistry was obtained from Catholic University in Washington, D. C.

Ann Maxwell has taught in Tennessee in public, private, and parochial schools. She was an instructor in mathematics at Aquinas Junior College in Nashville for three years and worked with the adult education program at St. Louis University in 1968.

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