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I am submitting herewith a thesis written by David Leland Wolfe entitled "The Asymptotic Behavior of the Convolution Product of Real-Valued Sequences." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts, with a major in Mathematics.

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Thesis
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THE ASYMPTOTIC BEHAVIOR OF THE CONVOLUTION PRODUCT OF REAL-VALUED SEQUENCES

A Thesis

Presented for the

Master of Arts

Degree

The University of Tennessee

David Leland Wolfe

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ABSTRACT

The convolution product of real-valued sequences $\{a_n\}$ and $\{b_n\}$ is a new sequence $\{c_n\}$ defined by

$$c_n = \sum_{j=0}^{n} a_j b_{n-j} \quad (n = 0, 1, 2, \ldots) .$$

The purpose of this thesis is to describe the asymptotic behavior of $\{c_n\}$ under various hypotheses on $\{a_n\}$ and $\{b_n\}$.

It is shown, for example, that if $\sum_{j=0}^{\infty} a_j x^j$ has radius of convergence R and if $\{b_n\}$ is a positive sequence satisfying

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = \lambda ,$$

where $0 < \lambda < R$, then

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \sum_{j=0}^{\infty} a_j \lambda^j .$$

The main result gives an estimate of the rate of convergence of $\{c_n/b_n\}$ when, in addition to the above hypotheses, one has an estimate on the rate of convergence of $\{b_{n-1}/b_n\}$.

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I. INTRODUCTION

Given two real-valued sequences $\{a_n\}$ and $\{b_n\}$, define another sequence $\{c_n\}$ by

$$c_n = \sum_{j=0}^{n} a_j b_{n-j} \qquad (n = 0, 1, 2, \ldots) .$$

This new sequence is called the convolution product of $\{a_n\}$ and $\{b_n\}$. The purpose of this thesis is to describe the asymptotic behavior of $\{c_n\}$ under various hypotheses on $\{a_n\}$ and $\{b_n\}$.

An elementary observation is that if $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series and $\{b_n\}$ is a bounded sequence, then $\{c_n\}$ is also a bounded sequence. Although it is simple, this observation is rather typical in that a property of $\{b_n\}$ is propagated to the convolution product. A somewhat less trivial result is

THEOREM 1. If $\sum_{j=0}^{\infty} a_j$ is absolutely convergent and

$$\lim_{n\to\infty} b_n = b , \tag{1.1}$$

then

$$\lim_{n\to\infty} c_n = b \sum_{j=0}^{\infty} a_j .$$

It follows from Theorem 1 that when $b \neq 0$

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \sum_{j=0}^{\infty} a_j .$$

The next theorem shows that this type of result is valid under much weaker hypotheses on the sequence $\{b_n\}$.

THEOREM 2. Let $\sum_{j=0}^{\infty} a_j x^j$ have radius of convergence R and let $\{b_n\}$ be a positive sequence such that

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = \lambda$$

where $0 < \lambda < R$. Then

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \sum_{j=0}^{\infty} a_j \lambda^j .$$

Clearly, if (1.1) holds with $b \neq 0$, then

(1.2) $$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} \text{ exists.}$$

However, (1.2) may hold even when $\{b_n\}$ diverges or converges to zero; for example, both $\{\log n\}$ and $\{1/\log n\}$ satisfy (1.2). Implications of the property (1.2) will be discussed in Section II.

The main theorem of the thesis gives an estimate of the rate of convergence of $\{c_n/b_n\}$ under an assumption on the rate of convergence of $\{b_{n-1}/b_n\}$. In particular, we have

THEOREM 3. Suppose that $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R > 0$ and that

$$\left| \frac{b_{n-1}}{b_n} - \lambda \right| \leq \delta_{n-1} \quad (n = 1, 2, \ldots) \tag{1.3}$$

where δ_n is a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{\delta_{n-1}}{\delta_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \delta_n = 0 . \tag{1.4}$$

If $0 < \lambda < R$, then

$$\limsup_{n\to\infty} \frac{1}{\delta_n} \left| \frac{c_n}{b_n} - \sum_{k=0}^{\infty} a_k \lambda^k \right| \leq \sum_{k=1}^{\infty} k|a_k|\lambda^{k-1} < \infty . \tag{1.5}$$

Proofs of Theorems 1 and 2 appear in Section III. The proof of Theorem 3 as well as the statement and proof of a useful lemma concerning the hypothesis (1.3) appear in Section IV.

After completing this thesis the author has discovered that the results of Section IV were previously obtained in [2].

II. IMPLICATIONS OF THE HYPOTHESIS "$\lim_{n\to\infty} \frac{b_{n-1}}{b_n}$ EXISTS"

In this section we prove two basic lemmas concerning positive sequences $\{b_n\}$ which satisfy the hypothesis

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} \text{ exists.}$$

We have already observed that when $\lim_{n\to\infty} b_n = b$, $b \neq 0$, then $\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = 1$. We have also seen a divergent sequence, $\{\log n\}$, and a sequence converging to zero, $\{1/\log n\}$, for which

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = 1 .$$

The following lemma characterizes positive sequences with this property.

LEMMA 1. Let $\{\delta_n\}$ be a sequence of positive numbers. Then

$$\lim_{n\to\infty} \frac{\delta_{n-1}}{\delta_n} = 1 \tag{2.1}$$

if and only if, for any $\rho > 1$, there exists a positive integer $N = N(\rho)$ such that for $n \geq N$

$$\rho^n \delta_n \leq \rho^{n+1} \delta_{n+1} \quad \text{and} \quad \rho^{-n} \delta_n \geq \rho^{-(n+1)} \delta_{n+1} . \tag{2.2}$$

Moreover,

$$\rho^n\delta_n \to \infty \quad (n \to \infty) \quad .$$

We remark that a sequence $\{\rho^n\delta_n\}$ ($\{\rho^{-n}\delta_n\}$) satisfying the first (second) inequality of (2.2) is said to be eventually increasing (decreasing).

PROOF. Condition (2.1) is clearly equivalent to

$$\lim_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} = 1 \; ; \tag{2.3}$$

hence, it will suffice to show the equivalence of (2.2) and (2.3). First, assume (2.2) is true. Then for fixed $\rho > 1$ and $n \geq N(\rho)$, we have

$$\frac{1}{\rho} \leq \frac{\delta_{n+1}}{\delta_n} \leq \rho \; , \tag{2.4}$$

from which it follows that

$$\frac{1}{\rho} \leq \liminf_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} \leq \limsup_{n\to\infty} \frac{\delta_{n+1}}{\delta n} \leq \rho \; .$$

Letting $\rho \to 1+$, we obtain (2.3).

Now assume (2.3) is true. Then for each $\rho > 1$ there exists $N = N(\rho)$ such that (2.4) holds for $n \geq N$. Clearly, (2.2) follows from (2.4) for $n \geq N$.

We need only to show $\rho^n\delta_n \to \infty$ $(n \to \infty)$ for each $\rho > 1$. Fix $\rho > 1$ and choose ρ_0, $\rho > \rho_0 > 1$. There exists $N_0 = N_0(\rho_0)$

such that $\rho_0^n \delta_n$ is nondecreasing for $n \geq N_0$. Thus, for $n \geq N_0$

$$\rho^n \delta_n = (\rho/\rho_0)^n \rho_0^n \delta_n \geq (\rho/\rho_0)^n \rho_0^{N_0} \delta_{N_0} \quad .$$

Since $\rho/\rho_0 > 1$, $\rho^n \delta_n \to \infty$ $(n \to \infty)$, and the proof is complete.

The second lemma establishes an important property of positive sequences $\{b_n\}$ for which $\lim_{n\to\infty} b_{n-1}/b_n = \lambda$ exists, but is not necessarily 1.

LEMMA 2. Let $\{b_n\}$ be a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = \lambda > 0 \quad .$$

Then

$$\lim_{n\to\infty} \sqrt[n]{b_n} = \frac{1}{\lambda} \quad . \tag{2.4}$$

PROOF. Let $0 < \varepsilon < \lambda$ such that there exists an integer N such that for $n \geq N$

$$\left| \frac{b_{n-1}}{b_n} - \lambda \right| < \varepsilon \quad ,$$

or equivalently,

$$-\varepsilon < \frac{b_{n-1}}{b_n} - \lambda < \varepsilon \quad .$$

Hence, for $n \geq N$ we have

$$b_N \frac{1}{(\lambda-\varepsilon)^{n-N}} > \ldots > b_{n-1} \frac{1}{\lambda-\varepsilon} > b_n > \frac{1}{\lambda+\varepsilon} b_{n-1} > \ldots > \frac{1}{(\lambda+\varepsilon)^{n-N}} b_N ,$$

so that taking n^{th} roots yields

$$(2.5) \qquad \sqrt[n]{b_N} \frac{(\lambda-\varepsilon)^{N/n}}{(\lambda-\varepsilon)} > \sqrt[n]{b_n} > \frac{(\lambda+\varepsilon)^{N/n}}{(\lambda+\varepsilon)} \sqrt[n]{b_N} .$$

Thus,

$$\frac{1}{\lambda-\varepsilon} \geq \limsup_{n\to\infty} \sqrt[n]{b_n} \geq \liminf_{n\to\infty} \sqrt[n]{b_n} \geq \frac{1}{\lambda+\varepsilon}$$

and (2.4) follows upon letting $\varepsilon \to 0$.

III. PROOFS OF THEOREMS 1 AND 2

The proofs of Theorems 1 and 2 are somewhat similar since each theorem asserts the existence of a limit. Although estimates needed to prove Theorem 2 are more difficult than those needed to prove Theorem 1, the underlying ideas of the two proofs are basically the same. We assume that the reader is familiar with the elementary theory of series as found, for example, in [1].

PROOF OF THEOREM 1. Let $\varepsilon > 0$ and put $A = \sum_{j=0}^{\infty} |a_j|$. First, fix an integer $N_0 \geq 1$ such that for $n \geq N_0$

$$\sum_{j=N_0}^{\infty} |a_j| < \frac{\varepsilon}{3(|b|+1)} , \tag{3.1}$$

and

$$|b_n - b| < \frac{\varepsilon}{3A} . \tag{3.2}$$

Next, define $B = \sum_{j=0}^{N_0-1} |b_j - b| + 1$ and determine an integer $N_1 > N_0$ such that

$$\sum_{j=N_1}^{\infty} |a_j| < \frac{\varepsilon}{3B} . \tag{3.3}$$

Now for $n \geq N_1 + N_0$, we have

$$|c_n - (\sum_{j=0}^{\infty} a_j)b| \le (\sum_{j=0}^{n-N_0} + \sum_{j=n-N_0+1}^{n})|a_j|\left|b_{n-j}-b\right| + \sum_{j=n+1}^{\infty}|a_j|\ |b|$$

(3.4)

$$= I_n + J_n + K_n \ .$$

Applying (3.2) to I_n and (3.1) to K_n yields

(3.5) $$I_n + K_n < \frac{2\varepsilon}{3} \quad (n \ge N_1) \ .$$

We next estimate J_n by making a change of index and using (3.3) to obtain

(3.6) $$J_n < \sum_{k=0}^{N_0-1} |b_k-b| \ \frac{\varepsilon}{3B} < \frac{\varepsilon}{3} \ .$$

Combining (3.5) and (3.6) with (3.4) completes the proof.

PROOF OF THEOREM 2. Let $\varepsilon > 0$. Choose $c > 0$ such that $(1+c)\lambda < R$ and put $A = \sum_{j=0}^{\infty} |a_j|\lambda^j$, $\eta = c/(1+c)$, and $b_n = \lambda^{-n}q_n$ $(n = 0, 1, 2, \ldots)$. Then

$$\lim_{n\to\infty} \frac{q_{n-1}}{q_n} = 1$$

and, in fact, since

(3.7) $$\frac{q_{n-j}}{q_n} = \frac{q_{n-j}}{q_{n-j+1}} \frac{q_{n-j+1}}{q_{n-j+2}} \cdots \frac{q_{n-1}}{q_n} \qquad (n > j,\ j = 1, 2, 3, \ldots) \ ,$$

we have

$$\lim_{n\to\infty} \frac{q_{n-j}}{q_n} = 1 \quad (j = 1, 2, 3, \ldots) .$$

Thus, we may first fix a positive integer $N_0 \geq 1$ so large that

$$\sum_{j=N_0+1}^{\infty} |a_j|[(1+c)^j + 1]\lambda^j < \varepsilon/5 \tag{3.8}$$

and then determine $N_1 > 2N_0$ such that for $n \geq N_1$ and $j = 0, 1, \ldots, N_0$,

$$\left|\frac{q_{n-j}}{q_n} - 1\right| < \min \{\varepsilon/(5A) , c\} . \tag{3.9}$$

Finally, if $B = \max \{q_0, q_1, \ldots, q_{N_1-1}\}$, then there exists an integer $N_2 > N_1 + N_0$ such that for $n \geq N_2$

$$\sum_{j=n-N_1+1}^{\infty} |a_j| (1+c)^j \lambda^j < \frac{\varepsilon q_{N_1} (1+c)}{5B} . \tag{3.10}$$

Now, for all $n \geq N_3 = N_0 + 2N_1 + N_2$ we have

$$\left|\frac{c_n}{b_n} - \sum_{j=0}^{\infty} a_j\lambda^j\right| \leq \left(\sum_{j=0}^{N_0} + \sum_{j=N_0+1}^{n-N_1} + \sum_{j=n-N_1+1}^{n}\right)\left(|a_j|\left|\frac{q_{n-j}}{q_n} - 1\right|\lambda^j\right) + \sum_{j=n+1}^{\infty} |a_j|\lambda^j$$

$$= I_n + J_n + K_n + L_n . \tag{3.11}$$

It then follows immediately from (3.8), (3.9) and the definition of A that

$$I_n + L_n \leq 2\varepsilon/5 \quad (n \geq N_3) \ . \tag{3.12}$$

In order to estimate J_n we first observe that $j \leq n - N_1$ implies $n - j \geq N_1$, so that we may apply the case $j = 1$ of (3.9) to each factor on the right side of equation (3.7) to obtain

$$\left|\frac{q_{n-j}}{q_n} - 1\right| \leq \frac{q_{n-j}}{q_n} + 1 \leq (1 + c)^j + 1 \quad (N_0 + 1 \leq j \leq n - N_1) \ .$$

Thus, by (3.8) and (3.11)

$$J_n \leq \sum_{j=N_0+1}^{n-N_1} |a_j| \ [(1+c)^j + 1]\lambda^j < \varepsilon/5 \ . \tag{3.13}$$

The estimate of K_n is somewhat more difficult even though K_n contains N_1 terms for all n. Clearly, by the definition of B,

$$\left|\frac{q_{n-j}}{q_n} - 1\right| \leq \frac{B}{q_n} + 1 \quad (n - N_1 + 1 \leq j \leq n) \ .$$

In addition, (3.9) with $j = 1$ and $n = k$ yields

$$q_{k-1}/q_k < 1 + c \quad (k \geq N_1)$$

or, equivalently,

$$q_k > q_{k-1} \ (1 - \frac{c}{1+c}) = q_{k-1} \ (1 - \eta) \quad (k \geq N_1) \ .$$

Since $n \geq N_3 \geq N_1$, we therefore have

$$q_n > q_{n-1}(1-\eta) > q_{n-2}(1-\eta)^2 > \ldots > q_{N_1}(1-\eta)^{n-N_1} \quad .$$

Thus

$$\begin{aligned} K_n &\leq \sum_{j=n-N_1+1}^{n} |a_j| \left(\frac{B}{q_n} + 1\right)\lambda^j \\ &\leq \frac{B}{q_{N_1}} \sum_{j=n-N_1+1}^{n} |a_j| (1-\eta)^{N_1-n}\lambda^j + \sum_{j=n-N_1+1}^{n} |a_j|\lambda^j \\ &\leq \frac{B(1-\eta)^{N_1}}{q_{N_1}} \sum_{j=n-N_1+1}^{n} |a_j| (1+c)^n\lambda^j + \sum_{j=n-N_1+1}^{n} |a_j|\lambda^j \\ &\leq \frac{B(1-\eta)^{N_1}(1+c)^{N_1-1}}{q_{N_1}} \sum_{j=n-N_1+1}^{n} |a_j| (1+c)^j\lambda^j + \sum_{j=n-N_1+1}^{n} |a_j|\lambda^j \quad . \end{aligned}$$

Since $(1-\eta)^{N_1}(1+c)^{N_1-1} = 1/(1+c)$, it follows from (3.8) and (3.10) that

$$K_n \leq 2\varepsilon/5 \quad . \tag{3.14}$$

Combining (3.12), (3.13), and (3.14) with (3.11) now yields

$$\left|\frac{c_n}{b_n} - \sum_{j=0}^{n} a_j\lambda^j\right| < \varepsilon$$

and completes the proof.

IV. PROOF OF THEOREM 3

The proof of Theorem 3 depends on the following lemma.

LEMMA 3. Suppose

$$\left|\frac{b_{n-1}}{b_n} - \lambda\right| \leq \delta_{n-1} \quad (n = 1, 2, \ldots) \tag{4.1}$$

where δ_n is a sequence of positive numbers such that (1.4) holds. Then for every $\rho > 1$ and $\varepsilon > 0$ there exists a positive integer N such that for $n \geq N + \ell$ $(\ell = 1, 2, \ldots)$ we have

$$\left|\frac{b_{n-k}}{b_n} - \lambda^k\right| \leq k\delta_{n-k}\rho^{k-1}(\lambda+\varepsilon)^{k-1} \quad (k = 1, 2, \ldots, \ell) . \tag{4.2}$$

PROOF. Fix $\rho > 1$ and $\varepsilon > 0$. Use (1.4) and Lemma 1 to choose the positive integer N such that for $n \geq N$

$$\rho^{-n}\delta_n \geq \rho^{-(n+1)}\delta_{n+1} \text{ and } \delta_n < \varepsilon . \tag{4.3}$$

We shall prove the lemma by induction on ℓ. Clearly, the inequality (4.2) holds for $\ell = 1$. Assume the lemma is true for $\ell = r$. If $\ell = r + 1$, $1 \leq k \leq r$, and $n \geq N + r + 1$, then $n - 1 \geq N + r$ and (4.2) follows from the inductive hypothesis. Thus, we need only show that

$$\left|\frac{b_{n-r-1}}{b_n} - \lambda^{r+1}\right| \leq (r+1)\delta_{n-r-1}\rho^r(\lambda+\varepsilon)^r . \tag{4.4}$$

For $n \geq N + (r+1)$, we have

$$\left|\frac{b_{n-r-1}}{b_n} - \lambda^{r+1}\right| = \left|\left(\frac{b_{n-r-1}}{b_{n-1}} - \lambda^r\right)\left[\left(\frac{b_{n-1}}{b_n} - \lambda\right)+\lambda\right] + \lambda^r\left(\frac{b_{n-1}}{b_n} - \lambda\right)\right|$$

$$\leq \left|\frac{b_{n-r-1}}{b_{n-1}} - \lambda^r\right|\left(\left|\frac{b_{n-1}}{b_n} - \lambda\right| + \lambda\right) + \lambda^r\left|\frac{b_{n-1}}{b_n} - \lambda\right| .$$

Thus, by the inductive assumption and inequalities (4.1) and (4.3),

$$\left|\frac{b_{n-r-1}}{b_n} - \lambda^{r+1}\right| \leq r\delta_{n-r-1}\rho^{r-1}(\lambda+\varepsilon)^{r-1}(\delta_{n-1}+\lambda) + \lambda^r\delta_{n-1}$$

$$\leq r\delta_{n-r-1}\rho^{r-1}(\lambda+\varepsilon)^r + \lambda^r\delta_{n-1} (n-1 \geq N+r) .$$

Since $\{\rho^{-n}\delta_n\}$ is an eventually decreasing sequence, we have for $n \geq N + r + 1$

$$\rho^{-n+(r+1)}\delta_{n-(r+1)} \geq \rho^{-n+1}\delta_{n-1} ,$$

so that

$$\rho^r\delta_{n-(r+1)} \geq \delta_{n-1} .$$

Hence,

$$\left|\frac{b_{n-r-1}}{b_n} - \lambda^{r+1}\right| \leq r\delta_{n-r-1}\rho^r(\lambda+\varepsilon)^r + \lambda^r\rho^r\delta_{n-r-1}$$

$$\leq (r+1)\delta_{n-r-1}\rho^r(\lambda+\varepsilon)^r ,$$

which completes the proof.

PROOF OF THEOREM 3. Fix $\rho > 1$ and $\varepsilon > 0$ such that $\lambda + \varepsilon < R$. By Lemma 3 choose a positive integer N such that for $n \geq N + \ell$ and $\ell = 1, 2, 3, \ldots$ inequality (4.2) holds. If $n \geq N$, we have

$$\left|\frac{c_n}{b_n} - \sum_{k=0}^{\infty} a_k \lambda^k\right|$$

$$(4.5) \qquad \leq \sum_{k=0}^{n-N} |a_k| \left|\frac{b_{n-k}}{b_n} - \lambda^k\right| + \sum_{k=n-N+1}^{n} |a_k| \frac{b_{n-k}}{b_n} + \sum_{k=n-N+1}^{\infty} |a_k| \lambda^k$$

$$= I_n + J_n + K_n .$$

We shall bound I_n/δ_n and then show J_n/δ_n and K_n/δ_n tend to zero as $n \to \infty$.

By Lemma 3 and the inequality $\rho^{n-k}\delta_{n-k} \leq \rho^n \delta_n$,

$$I_n \leq \sum_{k=0}^{n-N} |a_k| k \delta_{n-k} \rho^{k-1} (\lambda+\varepsilon)^{k-1}$$

$$\leq \delta_n \sum_{k=1}^{n-N} |a_k| k \rho^{2k-1} (\lambda+\varepsilon)^{k-1} .$$

Hence, we have the following estimate:

$$(4.6) \qquad \limsup_{n\to\infty} \frac{I_n}{\delta_n} \leq \sum_{k=0}^{\infty} |a_k| k \rho^{2k-1} (\lambda+\varepsilon)^{k-1} .$$

We next estimate J_n . If $B = \max (b_0, \ldots, b_{N-1})$, then clearly

$$J_n \leq B \frac{1}{b_n} \sum_{k=n-N+1}^{n} |a_k| \quad .$$

Since $\lim_{n\to\infty} \sqrt[n]{b_n} = 1/\lambda$ by Lemma 2 and $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1/R$, as is well known, we may choose θ such that $0 < \theta < \frac{R-\lambda}{R+\lambda}$ and then determine a positive integer N_1 such that for $n \geq N_1$

$$(4.7) \qquad b_n > \left(\frac{1-\theta}{\lambda}\right)^n > \left(\frac{1+\theta}{R}\right)^n \quad \text{and} \quad |a_n| < \left(\frac{1 + \theta/2}{R}\right)^n .$$

It follows that for $n \geq N + N_1$

$$J_n \leq B\left(\frac{R}{1+\theta}\right)^n \sum_{k=n-N+1}^{n} \left(\frac{1 + \theta/2}{R}\right)^k$$

$$\leq B\left(\frac{1+ \theta/2}{1+\theta}\right)^n \sum_{j=0}^{N-1} R^j \quad .$$

Hence,

$$\frac{J_n}{\delta_n} \leq \left(B \sum_{j=0}^{N-1} R^j\right)\left(\frac{1 + \theta/2}{1+\theta}\right)^n \frac{1}{\delta_n} \quad .$$

Since $\frac{1+\theta}{1 + \theta/2} > 1$, $\left(\frac{1+\theta}{1 + \theta/2}\right)^n \delta_n \to \infty$ as $n \to \infty$ by Lemma 1 in Section II, and, therefore

$$(4.8) \qquad \lim_{n\to\infty} J_n/\delta_n = 0 \quad .$$

Finally we consider K_n . By the choice of θ

$$\frac{1 + \theta/2}{R} < \frac{1+\theta}{R} < \frac{1-\theta}{\lambda} ,$$

so that

$$(\frac{1+\theta}{R})\lambda < 1 - \theta < 1 ;$$

thus,

$$K_n \leq \sum_{k=n-N+1}^{\infty} (\frac{1 + \theta/2}{R})^k \lambda^k$$

$$\leq \sum_{k=n-N+1}^{\infty} (\frac{1+\theta}{R})^k \lambda^k$$

$$= ((\frac{1+\theta}{R})\lambda)^{n-N+1} \frac{1}{1-(\frac{1+\theta}{R})\lambda} .$$

Since $(\frac{1+\theta}{R})\lambda < 1$, $(\frac{R\lambda^{-1}}{1+\theta})^n \delta_n \to \infty$ as $n \to \infty$, and, therefore,

$$\lim_{n\to\infty} K_n/\delta_n = 0 . \tag{4.9}$$

Combining (4.6), (4.8) and (4.9) with (4.5) yields

$$\limsup_{n\to\infty} \frac{1}{\delta_n} \left| \frac{c_n}{b_n} - \sum_{k=0}^{\infty} a_k \lambda^k \right| \leq \sum_{k=1}^{\infty} k|a_k| \rho^{2k-1} (\lambda+\varepsilon)^{k-1} ,$$

from which (1.5) follows upon letting $\rho \to 1+$ and $\varepsilon \to 0+$.

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VITA

David Leland Wolfe was born in Sheldon, Iowa, on July 23, 1950. He attended elementary schools in that city until moving to Wheaton, Illinois his sophomore year in high school. Graduating from Wheaton North Community High School in 1968, he first spent a year overseas before attending Bryan College, Dayton, Tennessee. In May, 1973, he received a Bachelor of Arts degree in Mathematics, graduating with honors and chosen for membership in Who's Who in American Colleges and Universities.

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