To the Graduate Council:

I am submitting herewith a dissertation written by David Carter Harris entitled "Almost Everywhere Divergence of Two-Dimensional Walsh-Fourier Series." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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ALMOST EVERYWHERE DIVERGENCE OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

A Dissertation

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David Carter Harris

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ABSTRACT

C. Fefferman [1], [2], [3] has shown that the two-dimensional Fourier series of an $f \in L^p$, $p < 2$, may diverge a.e. when summed over expanding circles, but converges a.e. when summed over expanding polygonal arcs. We show this dichotomy does not prevail for two-dimensional Walsh-Fourier series.

To prove our results we prove the unboundedness of a large class of multiplier operators on L^p , $p \neq 2$.

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INTRODUCTION

The Walsh functions $\psi_0, \psi_1, \ldots$ form a complete orthonormal system on $[0, 1)$. In Chapter I we review basic properties of this system.

In Chapter II we obtain an N-dimensional version of the Paley inequalities. We then use this fundamental result to obtain Stein's necessary condition for a.e. convergence in the setting of N-dimensional Walsh analysis.

In Chapter III we investigate the half plane multiplier operator D_σ defined by

$$D_\sigma f \equiv \sum_{n<\sigma m} \hat{f}(m, n)\psi_m\psi_n ,$$

for $\sigma > 0$. We shall prove for $p \neq 2$, that D_σ is not bounded on $L^p([0, 1) \times [0, 1))$.

We also consider partial sums for double Walsh-Fourier series

$$S_\gamma f \equiv \sum_{(m, n) \in \gamma S} \hat{f}(m, n)\psi_m\psi_n ,$$

defined for any region S in the first quadrant of $\mathbb{R}^2$ and any $\gamma > 0$, where γS represents dilation of S by γ. We shall prove for $p < 2$ and S with boundary not always parallel to the coordinate axes, that there exists $f \in L^p([0, 1) \times [0, 1))$, such that $S_\gamma f$ diverges a.e. and in L^p, as $\gamma \to \infty$.

CHAPTER I

FUNDAMENTALS OF WALSH FOURIER ANALYSIS

The material included in Chapter I is well known. A good general reference is Fine's original article on the Walsh functions [4]. For section F (Annihilators) see Rudin [7].

A. Definition of the Walsh Functions.

Every x in $[0, 1)$ has a binary expansion,

$$(1) \qquad x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}$$

in which each x_k is equal to 0 or 1. If we specify that $\lim_{k\to\infty} x_k \neq 1$, then this expansion is unique. Each Walsh function $\psi_n(x)$ is defined in terms of the binary expansion (1) of x. Any time we write $x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}$ or a similar expression we shall mean the above uniquely determined expansion unless otherwise stated.

Walsh functions are conveniently defined in terms of Rademacher functions, a proper subset of the Walsh functions. Thus we begin with a discussion of the Rademacher functions.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. The nth Rademacher function $r_n(x)$ is defined by

$$(2) \qquad r_n(x) = (-1)^{x_n} \qquad n \in \mathbb{N}$$

where x_n is the nth binary digit of x as in (1). We can extend the domain of the Rademacher functions to all of R by periodicity. One then can see that

$$(3) \qquad r_{n+1}(x) = r_n(2x) \qquad n \in \mathbb{N} .$$

Thus

$$(4) \qquad r_0(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ -1 & x \in [\frac{1}{2}, 1) \end{cases}$$

and

$$(5) \qquad r_n(x) = r_0(2^n x) \qquad n \in \mathbb{N} .$$

With the Rademacher functions in hand we can define the Walsh functions to be the possible finite products of Rademacher functions. We specify the nth Walsh function by using the binary expansion of n :

$$(6) \qquad n = \sum_{k=0}^{\infty} n_k 2^k$$

where each n_k is 0 or 1 and, of course, all but a finite number of the n_k's are 0 . The nth Walsh function is

$$(7) \qquad \psi_n = \prod_{k=0}^{\infty} (r_k)^{n_k} .$$

(2) and (7) show that

$$(8) \qquad \psi_n(x) = \prod_{k=0}^{\infty} (-1)^{x_k n_k} .$$

B. The Dyadic Group.

The measure space $([0, 1), dx)$, where dx is Lebesque measure, is equivalent as a measure space to the dyadic group G with its Haar measure. Many of the properties of the Walsh functions arise more naturally when their domain is taken to be G rather than $[0, 1)$. More particularly, the results obtained in this paper depend on the group structure of G. For this reason we now discuss the dyadic group, its basic properties, and its relationship to $[0, 1)$ and the Walsh functions. Furthermore, after this section we take G to be the domain of the Walsh functions. With reluctance, we will hardly mention the interval $[0, 1)$. However, most results obtained can be transferred immediately to $[0, 1)$.

The dyadic group is a set of sequences,

$$G = \{x = (x_k)_{k=0}^{\infty} | x_k = 0 \text{ or } 1\} .$$

Addition is component-wise modulo 2. Thus $x + y = (|x_k - y_k|)_{k=0}^{\infty}$. It is clear that G is an abelian (additive) group. Defining G_n by

$$G_n = \{x \in G | x_k = 0 , k = 0, \ldots , n - 1\} ,$$

we easily see that G_n is a subgroup of G. Notice that the index $[G_{n-1}: G_n]$ of G_n in G_{n-1} is 2. So, in fact, $[G: G_n] = 2^n$. Furthermore $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset \bigcap_{n=0}^{\infty} G_n = \{0\}$. Thus G_n used as a neighborhood base at 0 in G gives G a Hausdorff topology. Each G_n and each coset of G_n is open by definition and thus is also closed since the complement of a coset of G_n is a union of cosets of G_n (which are open). If $x = (x_k)_{k=0}^{\infty}$ is in G with $x_k = 0$ for $k < n$, but $x_n = 1$, then we define $|x| = 2^{-n}$. In other words $|x| = 2^{-n}$ if and only if $x \in G_n \sim G_{n+1}$. We set $|0| = 0$. Notice that $|x + y| \leq |x| + |y|$ and in fact $|x + y| = \max(|x|, |y|)$ if $|x| \neq |y|$. We obtain a metric on G by letting $|x - y|$ be the distance from x to y. Since

$$G_n = \{x| \ |x| < 2^{-n+1}\} = \{x| \ |x| \leq 2^{-n}\} ,$$

we see that this metric agrees with the topology given by the G_n. An elementary argument shows that G is sequentially compact. So in fact, since G is metrizable, we have that G is compact.

An alternate way to see that G is compact is to note that G (initially as a set) is the countable product of copies of the two element group $Z/2Z = \{0, 1\}$. Now note that the topology generated by G_n and their translates is the product topology. Thus Tychonoff's theorem yields that G is compact.

Let μ be the Borel measure on G determined by setting $\mu(x + G_n) = 2^{-n}$ for all $x \in G$, $n \in N$. Then μ is translation-invariant; thus μ is the unique Haar measure on G normalized by $\mu(G) = 1$.

We wish now to show that (G, μ) is equivalent to $([0, 1), dx)$ as a measure space. Define a map $\lambda : [0, 1) \to G$ by $\lambda(x) = (x_0, x_1, x_2, \ldots)$ where $x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}$ (see (1), page 2). Since $\lambda^{-1}(G_n) = [0, 2^{-n})$ and $\lambda^{-1}(x + G_n)$ = dyadic interval of length 2^{-n} in $[0, 1)$ determined by specifying that the first n binary digits are to be the first n coordinates of x, we see that λ is measurable. From these equalities we also see that λ carries Haar (Lebesgue) measure on $[0, 1)$ to Haar measure on G : $\mu(\lambda(E)) = |E|$ where $|E|$ denotes the Lebesgue measure of $E \subset [0, 1)$. It is clear that λ is one-to-one. λ is onto all but a countable subset of G. The exceptional set is

$$G_0^* := \{x \in G \mid \lim_{k \to \infty} x_k = 1\} .$$

Thus $\lambda([0, 2^{-n})) = G_n \sim G_0^*$ and

$$\lambda([k2^{-n}, (k + 1)2^{-n})) = (\lambda(k2^{-n}) + G_n) \sim G_0^* \quad \text{if} \quad k < 2^n .$$

It follows that $\lambda^{-1} : G \sim G_0^* \to [0, 1)$ is measurable. So (G, μ) is indeed equivalent to $([0, 1), dx)$ as a measure space via the map λ.

We can now define the Rademacher and Walsh functions on G. Let $r_n(x) = (-1)^{x_n}$ for all $x \in G$ and let $\psi_n = \prod_{k=0}^{\infty} r_k^{n_k}$. Notice that, for $x \in [0, 1)$, we have $r_n(x) = r_n(\lambda(x))$ and thus $\psi_n(x) = \psi_n(\lambda(x))$. That is to say, the definitions of r_n and ψ_n on G are equivalent to their definition on $[0, 1)$ via composition with the map λ. The L^p-spaces on G are carried in this way to the L^p-spaces on $[0, 1)$ for $1 \leq p \leq \infty$. One should note however that $f(\lambda(x))$ need NOT be continuous on $[0, 1)$ when $f(x)$ is continuous on G. For example any Walsh function (other than ψ_0) is continuous on G but not on $[0, 1)$. For the remainder of this paper we will work with functions defined on G. In particular dx will from now on denote Haar measure on G. If E is measurable in G we let $|E|$ denote the Haar measure of E.

C. <u>The Characters of G</u>.

Each Rademacher function is a character of G, i.e. a continuous group homomorphism from G to the set of complex numbers of modulus one. The finite product of characters is also a character as is the complex conjugate (the multiplicative inverse) of any character. It follows that the set of characters on G forms a group under multiplication. This group is called the dual group of G and is denoted $\hat{G}$. It follows that each Walsh function ψ_n is in $\hat{G}$.

If $\psi \in \hat{G}$, then for any $x \in G$ we have that

$$1 = \psi(0) = \psi(x + x) = \psi(x)\psi(x) .$$

Thus $\psi(x) = \pm 1$. Trivially then $\psi^{-1}(1) = \psi^{-1}\{z|\text{Re}(z) > 0\}$. Continuity of ψ now implies $\psi^{-1}(1) \supset G_n$ for some n . We have shown ψ is identically 1 on some G_n . Any ψ identically 1 on G_n is determined by its values on e_j for $j = 0, 1, 2, \ldots, n-1$; where e_j is defined to be the sequence $(\delta_{kj})_{k=0}^{\infty}$. Thus there are at most 2^n characters identically 1 on G_n . But the Walsh functions ψ_k for $0 \leq k < 2^n$ are identically 1 on G_n and thus ψ must be one of these Walsh functions. We have proven

$$\hat{G} \equiv \{\text{Walsh functions}\} .$$

If m and n are integers, then $\psi_m\psi_n$ is another Walsh function. We use $m \dotplus n$ to denote the subscript of this Walsh function:

$$\psi_m\psi_n = \psi_{m \dotplus n} .$$

If we use (7) we see that

$$\psi_m\psi_n = \prod_{k=0}^{\infty} (r_k)^{m_k} \prod_{k=0}^{\infty} (r_k)^{n_k} = \prod_{k=0}^{\infty} (r_k)^{m_k+n_k} = \prod_{k=0}^{\infty} (r_k)^{|m_k-n_k|}$$

where the last equality is obtained by noting that $(r_k)^{\ell}$ is 1 if ℓ is even and is r_k if ℓ is odd. Since $|m_k - n_k| = 0$ or 1 , $\psi_{m \dotplus n} = \prod_{k=0}^{\infty} (r_k)^{(m \dotplus n)_k}$, and the binary expansion of an integer is unique, we have that $(m \dotplus n)_k = |m_k - n_k|$. Or

$$m \dot{+} n = \sum_{k=0}^{\infty} |m_k - n_k| 2^k .$$

Thus to perform the operation $m \dot{+} n$ we add the binary digits of m and n with no carry.

D. Fourier Analysis on G.

If $f \in L^1(G)$ we define the nth Walsh Fourier coefficient of f by

$$\hat{f}(n) = \int_G f(x)\psi_n(x)dx \qquad n = 0, 1, 2, \dots .$$

More generally if μ is a finite Borel measure on G we let $\hat{\mu}(n)$ be defined by

$$\hat{\mu}(n) = \int_G \psi_n(x)d\mu(x) \qquad n = 0, 1, 2, \dots .$$

The Walsh-Fourier series of $f \in L^1(G)$ (or of $\mu \in B(G)$) is a formal series given by

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)\psi_n(x) .$$

If f is in some Banach space of functions on G, then an important question is whether the Fourier series of f converges to $f(x)$ in the norm of the Banach space. Another important but more difficult question is whether the Fourier series of f converges to $f(x)$ pointwise for all x or pointwise almost everywhere. Before we can address these questions we need to establish some of the basic facts of Walsh functions.

In a standard way one can easily prove that ψ_n are orthogonal. Indeed

$$\int_G \psi_k(x)\overline{\psi_\ell(x)}dx = \int_G \psi_k(x)\psi_\ell(x)dx = \int_G \psi_{k\dot{+}\ell}(x)dx \ .$$

So it suffices to show $\int_G \psi_n(x)dx = 0$ if $n \neq 0$. Toward this end, note there is x_0 such that $\psi_n(x_0) \neq 1$ (since $n \neq 0$). So $\int_G \psi_n(x)dx = \int_G \psi_n(x + x_0)dx$ (by translation invariance of dx) $=$ $\int_G \psi_n(x)\psi_n(x_0)dx = \psi_n(x_0)\int_G \psi_n(x)dx$. We must therefore have that $\int_G \psi_n(x)dx = 0$ or we obtain the contradiction that $\psi_n(x_0) = 1$.

The first 2^n Walsh functions are identically 1 on G_n and are easily seen to form a subgroup of $\hat{G}$. Obviously $\sum_{k=0}^{2^n-1} \psi_k(x) = 2^n$ on G_n. We claim this sum is 0 off G_n. Indeed suppose $x \in G$ is not in G_n. Choose ψ_ℓ such that $\psi_\ell(x) \neq 1$ with $\ell < 2^n$. Then

$$\sum_{k=0}^{2^n-1} \psi_k(x) = \sum_{k=0}^{2^n-1} \psi_{k+\ell}(x) = \sum_{k=0}^{2^n-1} \psi_k(x)\psi_\ell(x) = \psi_\ell(x)\sum_{k=0}^{2^n-1} \psi_k(x)$$

which forces $\sum_k \psi_k(x)$ to be 0 as desired.

For an alternate proof we note that

$$\int_{G_n} (\sum_{k=0}^{2^n-1} \psi_k(x))^2 dx = \int_{G_n} (2^n)^2 dx = |G_n| 2^{2n} = 2^{-n}2^{2n} = 2^n \ .$$

But by orthogonality we have

$$\int_G \Big(\sum_{k=0}^{2^n-1} \psi_k(x)\Big)^2 dx = \sum_{k=0}^{2^n-1} \int_G (\psi_k(x))^2 dx = \sum_{k=0}^{2^n-1} \int_G 1dx = \sum_{k=0}^{2^n-1} 1 = 2^n .$$

Thus we must have $\sum_{k=0}^{2^n-1} \psi_k(x) = 0$ off G_n finishing the proof.

We have shown, therefore, that

$$\sum_{k=0}^{2^n-1} \psi_k(x_0)\psi_k(x) = \sum_{k=0}^{2^n-1} \psi_k(x + x_0) = \begin{cases} 2^n & \text{on } G_n + x_0 \\ 0 & \text{off } G_n + x_0 \end{cases}$$

Thus the span of the first 2^n Walsh functions includes the characteristic functions of the cosets of G_n. From this fact we see that the Walsh polynomials (i.e. finite linear combinations of Walsh functions) are dense in $L^1(G)$. The Riemann-Lebesque Lemma can be proved from this last statement. Indeed let $f \in L^1(G)$ be given. Choose a finite sum $\sum a_k\psi_k$ which approximates f in L^1-norm by ϵ. For any ψ_ℓ not appearing in the sum we have $\left|\int_G f\psi_\ell\right| = \left|\int_G (f - \sum a_k\psi_k)\psi_\ell\right|$ (since ψ_ℓ is orthogonal to each ψ_k in the sum) $\leq \int_G |f - \sum a_k\psi_k| \leq \epsilon$.

Actually it is also immediate that the Walsh polynomials are dense in $C(G)$. From this we can prove the completeness of ψ_n in $\mathcal{B}(G)$. Suppose $\mu \in \mathcal{B}(G)$ and $\int_G \psi_n d\mu = 0$ for all n. Then $\int_G Pd\mu = 0$ for any Walsh polynomial P. For an arbitrary $f \in C(G)$ we may choose Walsh polynomials $P_j \to f$ uniformly. Thus $\int_G fd\mu = \int_G \lim_{j\to\infty} P_j d\mu = \lim_{j\to\infty} \int_G P_j d\mu = 0$. (We have used Lebesque dominated convergence.) It follows that $\mu = 0$. Now let us note that the completeness of ψ_n in $\mathcal{B}(G)$ implies the completeness in $L^2(G)$. So we have proven ψ_n is a complete orthonormal system in $L^2(G)$.

E. Higher Dimensions.

Let G^N be the product of N copies of G. If $x \in G^N$ we write $x = (x^{(1)}, x^{(2)}, \ldots, x^{(N)})$ reserving the subscript notation for binary expansions. We have $(G^N)\hat{} = (G\hat{})^N$, where $\psi_n(x) = \psi_{n^{(1)}}(x) \cdot \ldots \cdot \psi_{n^{(N)}}(x)$, with $n = (n^{(1)}, \ldots, n^{(N)})$. The basic definitions we made above for G extend to G^N. Thus we have

$$\hat{f}(n) = \int_{G^N} f(x)\psi_n(x)dx \qquad \text{for} \quad f \in L^1(G^N)$$

$$\hat{\mu}(n) = \int_{G^N}\psi_n(x)d\mu(x) \qquad \text{for} \quad \mu \in \mathcal{B}(G^N)$$

we write

$$f(x) \sim \sum_{n \in \mathbb{N}^N} \hat{f}(n)\psi_n(x) \qquad \text{or}$$

$$f(x) \sim \sum_{n \in (G^N)\hat{}} \hat{f}(n)\psi_n(x) ,$$

identifying $\mathbb{N}^N$ with $(G^N)\hat{}$. Several of the theorems in the previous section can be proven in much the same way for G^N. For example, $\{\psi_n\}_{n \in \mathbb{N}^N}$ is complete in $\mathcal{B}(G^N)$ and forms a complete orthonormal system in $L^2(G^N)$.

Let us at this time define and discuss conductors. In section C we showed any character of G is identically 1 on some G_n. The same argument shows any character ψ of G^N is identically 1 on some G_n^N. The largest such G_n^N is called the conductor of ψ, denoted cond (ψ). If $G_n^N = \text{cond}(\psi)$ we will also call n the conductor of ψ. No confusion results. Let an integer $n \geq 1$ be given. Notice that $\text{cond}(\psi_k) \leq n$ if and only if k satisfies

$$k^{(j)} < 2^n \qquad \text{for } 1 \leq j \leq N .$$

Also $\text{cond}(\psi_0) = 0$. Thus there are $2^{Nn} - 2^{N(n-1)}$ characters of conductor n for $n \geq 1$ and one character of conductor 0.

If $H \subset G^N$ is a closed subgroup and $\psi \in \hat{H}$ we can define the conductor of ψ (with respect to G^N) to be the largest G_n^N such that $\psi^{-1}(1) \supset G_n^N \cap H$. That such G_n^N exists is shown in the same way as for G or G^N. An important application of this fact is the following:

<u>Lemma 1</u>. Any character ψ of a closed subgroup $H \subset G^N$ can be extended to a character of G^N.

<u>Proof</u>: Let $n = \text{cond}(\psi)$. Define $\psi(G_n^N) \equiv 1$ and $\psi(h + G_n^N) \equiv \psi(h)$ for any $h \in H$. Note that ψ is well defined on $G_n^N + H$. One can easily verify now that $\psi \in (G_n^N + H)\hat{}$. Since $G_n^N + H$ is open in G^N it has finite index in G^N. Thus to finish our proof we need only show we can extend ψ to any group properly containing $G_n^N + H$. Toward this end, simply choose $x \notin G_n^N + H$ and define ψ on $(G_n^N + H) \cup (x + G_n^N + H)$ by setting $\psi(x + h) = \psi(h)$ for $h \in G_n^N + H$.

One can easily verify this definition extends ψ appropriately and the lemma is finished.

F. Annihilators.

We let N be fixed for this discussion. Let H be a closed subgroup of G^N. We define the annihilator of H by

$$\operatorname{ann}(H) = \{\psi \in \widehat{G^N} \mid \psi|_H \equiv 1\} .$$

Ann(H) is easily verified to be a subgroup of $(G^N)\hat{}$. Furthermore any $\psi \in \operatorname{ann}(H)$ gives a character of G^N/H via $\psi(x + H) = \psi(x)$. Conversely any element ψ of $(G^N/H)\hat{}$ gives an element of $\operatorname{ann}(H) \subset (G^N)\hat{}$ via the same equation. Thus we may identify ann(H) with $(G^N/H)\hat{}$.

In a similar way, for subgroups Γ of $\widehat{G^N}$, we define

$$\operatorname{ann}(\Gamma) = \{x \in G^N \mid \psi(x) = 1 \text{ for all } \psi \in \Gamma\} .$$

Ann(Γ) is a closed subgroup of G^N. Note that any element of $(G^N)\hat{}/\Gamma$ gives a well-defined character of ann(Γ) by letting $(\psi \cdot \Gamma)(x) = \psi(x)$ for any ψ in $(G^N)\hat{}$ and any x in ann(Γ) . Furthermore $\psi|_{\operatorname{ann}(\Gamma)} \equiv \tilde{\psi}|_{\operatorname{ann}(\Gamma)}$ implies $\psi \cdot \Gamma = \tilde{\psi} \cdot \Gamma$ as elements of $(\operatorname{ann}(\Gamma))\hat{}$. Thus $(G^N)\hat{}/\Gamma \subset (\operatorname{ann}(\Gamma))\hat{}$. By Lemma E.1 any character of $(\operatorname{ann}(\Gamma))\hat{}$ is of the form $\psi \cdot \Gamma$ (since $\Gamma|_{\operatorname{ann}(\Gamma)} \equiv 1$) . So we have proven $(G^N)\hat{}/\Gamma \cong (\operatorname{ann}(\Gamma))\hat{}$. Let us summarize the results obtained thus far in this section in the following theorem.

Theorem 1. $\mathrm{ann}(H) \cong (G^N/H)^\wedge$ and

$$(\mathrm{ann}(\Gamma))^\wedge \cong (G^N)^\wedge/\Gamma .$$

To obtain more detailed information about annihilators we introduce a bit more notation. Let L be the collection of closed subgroups of G^N. Let $\hat{L}$ be the collection of subgroups of $(G^N)^\wedge$. Both L and $\hat{L}$ form lattices with set inclusion providing the partial ordering. Our preceding discussion shows $\mathrm{ann} : L \to \hat{L}$ and $\mathrm{ann} : \hat{L} \to L$. It is easy to see that

$$H_1 \subset H_2 \text{ implies } \mathrm{ann}(H_1) \supset \mathrm{ann}(H_2)$$

for all H_1, H_2 in L and

$$\Gamma_1 \subset \Gamma_2 \text{ implies } \mathrm{ann}(\Gamma_1) \supset \mathrm{ann}(\Gamma_2)$$

for all Γ_1, Γ_2 in $\hat{L}$. That is to say, ann reverses the partial ordering of L and $\hat{L}$. One can also easily verify that

$$\mathrm{ann}(\mathrm{ann}(H)) \supset H \quad \text{and}$$

$$\mathrm{ann}(\mathrm{ann}(\Gamma)) \supset \Gamma .$$

In fact equality holds as indicated in the following.

Theorem 2. $\mathrm{ann} : L \to \hat{L}$ is an (anti) isomorphism of lattices with inverse $\mathrm{ann} : \hat{L} \to L$. To prove the theorem we will need two lemmas.

Lemma 3. If $H \in L$ and $x \notin H$, then there is $\psi \in \text{ann}(H)$ such that $\psi(x) \neq 1$.

Proof: Since H is closed and $x \notin H$, there is n such that $(x + G_n^N) \cap H \neq \emptyset$. Note that $H + G_n^N$ is open since G_n^N is open. Thus $H + G_n^N \in L$. Also $x \notin H + G_n^N$. Thus, be replacing H by $H + G_n^N$ if necessary, we may assume H is open. Now we may define ψ by $\psi(x + H) = -1$ and $\psi(y) = 1$ for $y \notin x + H$. Since H is open (and closed) we easily see that ψ is continuous. Since $x + x = 0$ we have furthermore that ψ is in $(G^N)\hat{}$ and in fact $\psi \in \text{ann}(H)$. Lemma 3 is finished.

Lemma 4. If $\Gamma \in \hat{L}$ and $\psi \notin \Gamma$, then there is $x \in \text{ann}(\Gamma)$ such that $\psi(x) \neq 1$.

Proof: From Theorem 1 we see that distinct elements of $(G^N)\hat{}/\Gamma$ give distinct characters of $\text{ann}(\Gamma)$. Since $\psi \cdot \Gamma \neq \Gamma$ it follows that there is x in $\text{ann}(\Gamma)$ with $(\psi \cdot \Gamma)(x) \neq \Gamma(x)$, i.e. $\psi(x) \neq 1$. Thus Lemma 4 is finished.

Proof of Theorem 2: These two lemmas give us the theorem immediately. Indeed Lemma 3 implies $\text{ann}(\text{ann}(H)) \subset H$ for all $H \in L$ while Lemma 4 implies

$$\text{ann}(\text{ann}(\Gamma)) \subset \Gamma \qquad \text{for all } \Gamma \in \hat{L}$$

The opposite inclusions were noted above. Thus equality holds and the theorem is proven. We wish to extend Theorem 2 by showing that ann preserves index:

Theorem 5. Suppose H_1 , $H_2 \in L$ and Γ_1 , $\Gamma_2 \in \hat{L}$. Suppose $H_1 \supset H_2$ and $\Gamma_1 \supset \Gamma_2$. Then

(1) $\quad [\mathrm{ann}(H_2) : \mathrm{ann}(H_1)] = [H_1 : H_2]$ and

(2) $\quad [\mathrm{ann}(\Gamma_2) : \mathrm{ann}(\Gamma_1)] = [\Gamma_1 : \Gamma_2]$

where we allow possibility of infinite index.

Proof: If $x \in H_1$, $x \notin H_2$, then $(x + H_2) \cup H_2 \in L$ and $H_1 \supset (x + H_2) \cup H_2 \supset H_2$. Furthermore the index of H_2 in $(x + H_2) \cup H_2$ is 2 . We can successively obtain larger groups of index 2 to get

$$H_2 \equiv \tilde{H}_0 \subset \tilde{H}_1 \subset \dots .$$

If at some point we have $\tilde{H}_N = H_1$, then $[H_1 : H_2] = 2^N$ and Theorem 2 shows $[\mathrm{ann}\ (H_2) : \mathrm{ann}(H_1)] = 2^N$. If $\tilde{H}_0 \subset \tilde{H}_1 \subset \dots$ is infinite, then $[H_1 : H_2] = \infty$ and Theorem 2 shows $[\mathrm{ann}\ (H_2) : \mathrm{ann}(H_1)] = \infty$. (2) can be proved in the same way as (1) or one can simply apply Theorem 2 to (1) to get (2). Theorem 5 is proved.

As a first application of annihilators we generalize the result that $\sum_{k=0}^{2^n-1} \psi_k = 2^n$ on G_n and is 0 off G_n . Notice that $\{\psi_k\}_{k=0}^{2^n-1}$ is a subgroup of $\hat{G}$ and G_n is the annihilator of this subgroup. These facts suggest the following generalization.

Theorem 6. If Γ is a finite subgroup of order 2^n in $(G^N)\hat{}$, then

$$\sum_{\psi\in\Gamma} \psi = 2^n \cdot \chi_{\mathrm{ann}(\Gamma)}$$

where χ denotes characteristic function.

Proof 1: Clearly $\sum_{\psi\in\Gamma} \psi = 2^n$ on $\mathrm{ann}(\Gamma)$. Let $x \notin \mathrm{ann}(\Gamma)$ and choose $\tilde{\psi} \in \Gamma$ such that $\tilde{\psi}(x) \neq 1$. Then

$$\sum_{\psi\in\Gamma} \psi(x) = \sum_{\psi\in\Gamma} \tilde{\psi}(x)\psi(x) = \tilde{\psi}(x) \sum_{\psi\in\Gamma} \psi(x) . \quad \text{Since } \tilde{\psi}(x) \neq 1 ,$$

we must have $\sum_{\psi\in\Gamma} \psi(x) = 0$. Proof 1 is finished.

Proof 2: Again note that $\sum_{\psi\in\Gamma} \psi = 2^n$ on $\mathrm{ann}(\Gamma)$. Now $[G^N : \mathrm{ann}(\Gamma)] = [\Gamma : \{0\}] = 2^n$. Thus $|\mathrm{ann}(\Gamma)| = 2^{-n}$. Thus

$$\int_{\mathrm{ann}(\Gamma)} \Big(\sum_{\psi\in\Gamma} \psi\Big)^2 = \int_{\mathrm{ann}(\Gamma)} (2^n)^2 = |\mathrm{ann}(\Gamma)| \cdot 2^{2n} = 2^{-n} \cdot 2^{2n} = 2^n .$$

But by orthogonality we have $\int_{G^N} \Big(\sum_{\psi\in\Gamma} \psi\Big)^2 = \sum_{\psi\in\Gamma} 1 = 2^n$. So $\sum_{\psi\in\Gamma} \psi$ must be 0 off $\mathrm{ann}(\Gamma)$. Proof 2 is finished.

Theorem 6 is well known, but never-the-less it is a fundamental element of our proof of Theorem III.B.1.

Theorem III.B.1 also requires the following result.

Proposition 7. Let H be a compact abelian group. Let μ be Haar measure normalized so $\mu(H) = 1$. Let $X_1, \ldots, X_M$ be characters of H taking on the values ± 1 and only these values. (In particular no X_j is identically 1.) Then the following are equivalent.

i) $X_1, \ldots, X_M$ are independent random variables.

ii) For $j = 1, \ldots, M$ there is x_j such that $X_j(x_j) = -1$, and $X_k(x_j) = 1$ for $k \neq j$.

iii) The group of functions Γ generated by $X_1, \ldots, X_M$ under pointwise multiplication has order 2^M.

Proof: First note that for any j, $X_j^{-1}(1)$ is a proper subgroup of H with index 2. So $\mu(X_j^{-1}(1)) = \frac{1}{2} = \mu(X_j^{-1}(-1))$.

i) $\Rightarrow$ ii) $\mu\{X_j = -1 , X_k = 1 \text{ for } k \neq j\}$

$$= \mu\{X_j = -1\} \cdot \prod_{k \neq j} \mu\{X_k = 1\} = 2^{-M} .$$

In particular $\{X_j = -1 , X_k = 1 \text{ for } k \neq j\} \neq \emptyset$. This first implication is proven.

ii) $\Rightarrow$ iii) Since each X_j has order 2 we must have order $\Gamma \leq 2^M$. But ii) easily implies order $\Gamma \geq 2^M$. So indeed ii) $\Rightarrow$ iii)

iii) $\Rightarrow$ i). The 2^M characters in Γ are orthogonal. (This fact can be proved just as for G.) Thus they are linearly independent. Since these characters are constant on cosets of $\text{ann}(\Gamma)$ there must be at least 2^M cosets of $\text{ann}(\Gamma)$. That is to say $[H : \text{ann}(\Gamma)] \geq 2^M$. Now suppose $X_j(x + \text{ann}(\Gamma)) = X_j(y + \text{ann}(\Gamma))$ for all j.

Then $\qquad X_j(x) = X_j(y) \quad$ for all j,

$$X(x) = X(y) \qquad \text{for all } X \in \Gamma ,$$

$$X(x - y) = 1 \qquad \text{for all } X \in \Gamma \text{ so that}$$

$$x + \text{ann}(\Gamma) = y + \text{ann}(\Gamma) .$$

Thus $\{X_j\}_{j=1}^M$ separates cosets of $\text{ann}(\Gamma)$. Since $\{X_j\}_{j=1}^M$ can separate at most 2^M cosets we have $[H : \text{ann}(\Gamma)] \leq 2^M$. Combining with the reverse inequality above, we have

$$[H : \text{ann}(\Gamma)] = 2^M .$$

Since there are 2^M cosets, the X_j separate these cosets, and there are only 2^M possible sequences $(a_j)_{j=1}^M$ with $a_j = \pm 1$; we must have for each such sequence an $x \in H$ such that $X_j(x) = a_j$. Furthermore $\{X_j = a_j$ for $j = 1, \ldots, M\} = x + \text{ann}(\Gamma)$ which has measure $= \mu(\text{ann}(\Gamma)) = 2^{-M} = \prod_{j=1}^{M} \mu\{X_j = a_j\}$. iii) $\Rightarrow$ i) is proved.

CHAPTER II

PALEY INEQUALITIES AND A.E. CONVERGENCE

A. The Paley Inequalities.

Some of the deepest results in Walsh analysis are contained in the Paley inequalities [6]. The purpose of this section is to give a more or less self-contained proof of these inequalities. Little that we present here is new. However we felt it wise to include the material because of its overall significance in Walsh analysis, and because our approach is in some ways more modern and more direct than other proofs in print.

We shall use the Calderon-Zygmund decomposition in the dyadic setting and therefore begin with a proof of it.

Let $G =$ the dyadic group. Let N be a positive integer. We shall refer to cosets of G_n^N (for any n) as dyadic cubes. Let H be a separable Hilbert space. If $y \in H$, we let $|y|$ denote the H-norm of y.

The following is a dyadic version of the well-known Calderon-Zygmund decomposition (see [9]).

Theorem 1. Given $f \in L^1(G^N, H)$ and $\alpha > 0$, G^N may be decomposed as follows.

i) $G^N = F \cup \Omega$, $F \cap \Omega = \emptyset$.

ii) $f(x) \leq \alpha$ almost everywhere on F.

iii) Ω is the disjoint union of dyadic cubes Q such that

a) $\alpha < \frac{1}{|Q|} \int_Q |f| \leq 2^N \alpha$ and

b) $|\Omega| \leq \frac{1}{\alpha} \|f\|_1$.

Proof: Let $\alpha > 0$ be given. Let A be the collection of dyadic cubes Q such that

1) $\frac{1}{|Q|} \int_Q |f| > \alpha$ and yet for any Q' containing Q we have

2) $\frac{1}{|Q'|} \int_{Q'} |f| \leq \alpha$.

Since two cubes Q, Q' have intersection that is either empty or equal to Q or Q', we have that the Q's in A are pairwise disjoint. Let $\Omega = \bigcup_{Q \in A} Q$. Thus by 1) and 2)

$$\alpha < \frac{1}{|Q|} \int_Q |f| \leq 2^N \frac{1}{|Q^d|} \int_{Q^d} |f| \leq 2^N \alpha ,$$

where Q^d = the N-fold double of Q . Also we have

$|\Omega| = \sum_Q |Q| \leq \sum_Q \frac{1}{\alpha} \int_Q |f| = \frac{1}{\alpha} \int_{\cup Q} |f| \leq \frac{1}{\alpha} \|f\|_1$. So iii) is proved.

Let $F = \Omega^c$. Then F is closed and by Lebesque's theorem on differentiation of integrals (see [9, pp. 4 and 5]) we have $|f| \leq \alpha$ a.e. on F . The proof of the Calderon-Zygmund decomposition is finished.

Now let

$$h(x) = \begin{cases} f(x) & \text{for } x \in F \\ \frac{1}{|Q|} \int_Q f & \text{for } x \in Q \in A \end{cases}$$

So
$$|h(x)| \le \begin{cases} \alpha & \text{a.e. in } F \\ 2^N\alpha & \text{for } x \text{ in } F^c = \Omega \end{cases} .$$

We have that $h(x) \in L^2(G^N, H)$ since in fact

$$\|h\|_2^2 = \int_F |h|^2 + \int_\Omega |h|^2$$
$$\le \int_F \alpha|h| + 2^{2N}\alpha^2|\Omega|$$
$$\le \alpha\|f\|_1 + 2^{2N}\alpha\|f\|_1 = (1+2^{2N})\alpha\|f\|_1 .$$

We let $b(x) = f(x) - h(x)$. Thus $b(x) = 0$ on F . Also $\int_Q b(x) = 0$ for all $Q \in A$.

In our proof of the Paley inequalities we shall apply the Calderon-Zygmund decomposition to $L^1(G^N, H)$ for $H = \mathbb{C}$ and for $H = \ell^2 = \{(\beta_j)_{j=-1}^\infty | \beta \in \mathbb{C}\}$. Notice we begin our index j for elements of ℓ^2 at $j = -1$. Of course, if $H = \mathbb{C}$, then $L^1(G^N, H) = L^1(G^N)$.

<u>Definitions:</u> For $f \in L^1(G^N)$, let

$$\Delta_j f(x) = \frac{1}{|G_{j+1}^N|}\int_{G_{j+1}^N(x)} f - \frac{1}{|G_j^N|}\int_{G_j^N(x)} f$$

where $G_j^N(x) = G_j^N + x$. Let $\Delta_{-1}f(x) \equiv \int_{G^N} f$. Define an operator $U : L^2(G^N) \to L^2(G^N, \ell^2)$ by

$$f \longrightarrow (\Delta_j f)_{j=-1}^{\infty} .$$

Note that U is an isometry (but is NOT onto). The dyadic Littlewood-Paley g-function is defined by $g(f)(x) = |Uf|(x)$.

Theorem 2: $U : L^1(G^N) \longrightarrow$ weak $L^1(G^N, \ell^2)$ boundedly, i.e. U , originally defined on $L^2(G^N)$, extends to an operator U on $L^1(G^N)$ which is of weak type $(1, 1)$. (This extension is unique since $L^2(G^N)$ is dense in $L^1(G^N)$.)

Proof: Let $f \in L^1(G^N)$ and let $\alpha > 0$ be given. Let $f = b + h$ according to the discussion immediately following the Calderon-Zygmund decomposition. Since $\int_Q b = 0$ for any Q in A and $b = 0$ a.e. on F we have that $Ub = 0$ on F . (See the definition of U and of Δ_j .) Of course $Uf = Ub + Uh$. Hence $\{|Uf| > \alpha\} \subset \{|Uh| > \alpha\} \cup \Omega$. So $|\{|Uf| > \alpha\}| \leq |\{|Uh| > \alpha\}| + |\Omega|$ $\leq |\{|Uh| > \alpha\}| + \frac{1}{\alpha}\|f\|_1$. But $\|Uh\|_2 = \|h\|_2$ so that

$$|\{|Uh| > \alpha\}| \leq \frac{1}{\alpha^2}\|Uh\|_2^2 = \frac{1}{\alpha^2}\|h\|_2^2 \leq \frac{(1 + 2^{2N})}{\alpha}\|f\|_1 .$$

So $|\{|Uf| > \alpha\}| \leq (2 + 2^{2N})\frac{1}{\alpha}\|f\|_1$. The proof of Theorem 2 is complete.

In a similar manner let us define

$$V : L^2(G^N, \ell^2) \to L^2(G^N) \text{ by}$$

$$f = (f_j)_{j=-1}^{\infty} \to \sum_{j=-1}^{\infty} \Delta_j f_j .$$

Note that V is onto with norm 1 , but is not one-to-one.

Theorem 3. $V : L^1(G^N, \ell^2) \to \text{weak } L^1(G^N)$ boundedly.

Proof: Let $f \in L^2(G^N, \ell^2)$ (A dense subset of $L^1(G^N, \ell^2)$.) Let $f = b + h$ as in the discussion following the Calderon-Zygmund decomposition. As for U we have $Vb = 0$ on F. Also $Vf = Vb + Vh$. Hence

$$\{|Vf| > \alpha\} \subset \{|Vh| > \alpha\} \cup \Omega \quad \text{and}$$

$$|\{|Vf| > \alpha\}| \leq |\{|Vh| > \alpha\}| + |\Omega|$$

$$\leq |\{|Vh| > \alpha\}| + \frac{1}{\alpha} \|f\|_1 .$$

But $\|Vh\|_2 \leq \|h\|_2$ so that

$$|\{|Vh| > \alpha\}| \leq \frac{1}{\alpha^2} \|Vh\|_2^2 \leq \frac{1}{\alpha^2} \|h\|_2^2 \leq \frac{(1 + 2^{2N})}{\alpha} \|f\|_1 .$$

So $|\{|Vf| > \alpha\}| \leq (2 + 2^{2N}) \frac{1}{\alpha} \|f\|_1$. Now extend V to all of $L^1(G^N, \ell^2)$ by continuity. Theorem 3 is proven.

The Marcinkiewicz interpolation theorem [see 9 p. 272, for example] gives that

$$U : L^p(G^N) \to L^p(G^N, \ell^2) \quad \text{boundedly} \quad 1 < p \leq 2 .$$

and $$V : L^p(G^N, \ell^2) \to L^p(G^N) \quad \text{boundedly} \quad 1 < p \leq 2 .$$

We will now show that U and V are adjoints. Let $f \in L^2(G^N)$ and $g \in L^2(G^N, \ell^2)$. Then

$$\langle Uf, g\rangle_{L^2(G^N, \ell^2)} = \int_{G^N} \sum_{j=-1}^{\infty} (Uf)_j(x) g_j(x) dx$$

$$= \int_{G^N} \sum_{j=-1}^{\infty} (\Delta_j f)(x) g_j(x) dx = \int_{G^N} \sum_{j=-1}^{\infty} (\Delta_j f)(x)(\Delta_j g_j)(x) dx$$

$$= \int_{G^N} \sum_{j=-1}^{\infty} f(x)(\Delta_j g_j)(x) dx = \int_{G^N} f(x) Vg(x) dx$$

$$= \langle f, Vg\rangle_{L^2(G^N)} .$$

So indeed U and V are adjoints.

For $2 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ let $f \in L^q(G^N) \subset L^2(G^N)$. We have

$$\| Uf \|_q = \sup_{\substack{g \in L^2(G^N, \ell^2) \\ \| g \|_p \leq 1}} \left| \int_{G^N} \langle Uf(x), g(x)\rangle_{\ell^2} dx \right|$$

$$= \sup_g \left| \int_{G^N} f(x) Vg(x) dx \right| \leq \sup_g \| f \|_q \, \| V_g \|_p$$

$$\leq C_q \| f \|_q .$$

So $U : L^p(G^N) \to L^p(G^N, \ell^2)$ boundedly for $1 < p < \infty$. Thus we have shown $f \to |Uf| = g(f)$ is bounded from $L^p(G^N)$ to $L^p(G^N)$ for $1 < p < \infty$. Similarly, for $2 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, let $f \in L^q(G^N, \ell^2) \subset L^2(G^N, \ell^2)$. We have

$$\| Vf\|_q = \sup_{\substack{g \in L^2(G^N) \\ \| g\|_p = 1}} \left|\int_{G^N} Vf(x)g(x)dx\right|$$

$$= \sup_g \left|\int_{G^N} \langle f(x), U_g(x)\rangle_{\ell^2} dx\right|$$

$$\leq \sup_g \| f\|_{L^q(G^N, \ell^2)} \cdot \| Ug\|_{L^p(G^N, \ell^2)} \leq C_q \| f\|_q .$$

So, $V : L^p(G^N, \ell^2) \to L^p(G^N)$ boundedly for $1 < p < \infty$. Thus, if f_j is a sequence of Walsh polynomials on G^N (with values in $\mathbb{C}$) such that $\Delta_j f_j = f_j$ and such that

$$(\sum |\Delta_j f_j|^2)^{1/2} \in L^p(G^N) \qquad 1 < p < \infty$$

then there is (unique) $f \in L^p(G^N)$ such that $\Delta_j f = f_j$.

We now summarize the main results obtained thus far:

Theorem 4. (The Paley-inequalities) The map $f \to g(f)$ is weak (1, 1) and the L^p-norm of $g(f)$ is equivalent to the L^p-norm of f for $1 < p < \infty$, i.e. there are positive constants A_p and B_p such that

$$A_p \| f\|_p \leq \| g(f)\|_p \leq B_p \| f\|_p \quad \text{for } 1 < p < \infty .$$

As an application of the Paley-inequalities, let us show that Rademacher series have all L^p-norms equivalent.

Theorem 5. If $F \equiv \sum_{j=0}^{\infty} a_j r_j$ is a (formal or non-formal) Rademacher series, then for $1 \leq p < \infty$ there is A_p (independent of the Rademacher series) such that

$$\frac{1}{A_p}\|F\|_p \le \|F\|_2 \le A_p\|F\|_p .$$

In particular, if $\sum|a_j|^2 = \infty$, then $\sum a_j r_j$ is not a Walsh-Fourier series for any function in $L^1(G)$.

Proof: $g(F) \equiv \|F\|_2$. Thus $\|g(F)\|_p \equiv \|F\|_2$ is independent of p . The theorem now follows immediately from the Paley-inequalities.

An alternate proof for Theorem 5 is possible based simply on the fact that the Rademacher functions are independent random variables. See [9, pp. 276-278].

B. Stein's Necessary Condition for Almost Everywhere Convergence.

Let G = dyadic group. All integrals below are with respect to Haar measure on G^N .

Stein's theorem in [8] applies to a general compact abelian group and a general sequence of operators on L^p which commute with translations. We restrict our attention to G^N and special "partial sum" operators sufficient for our purposes. This allows us a shorter proof. (See Theorem 1 below.)

Remember that $\widehat{G^N} = \{\psi_n\}$ with n an integer lattice point of the first "octant" of $\mathbb{R}^N$.

Definition. Let S be a bounded region in the first octant of R^n containing 0 . Let $\gamma \ge 0$ be given. Then the partial sum operator S_γ defined and bounded on $L^1(G^N)$ is given by $S_\gamma f = \sum_{n\in\gamma S} \hat{f}(n)\psi_n$. The maximal operator S^*f is defined to be $\sup_{\gamma\ge 0}|S_\gamma f|$. Notice in the definition of S^* that the supremum can be taken over a countable set since $(G^N)^\wedge$ is discrete. Thus S^*f is measurable.

Of course the operators S_γ (and S^*) commute with translations.

Throughout the rest of this section S will refer to a bounded region in the first octant of $\mathbb{R}^n$ containing 0. Of course $\lim_{\gamma\to\infty} S_\gamma f = y$ is equivalent to $\lim_{m\to\infty} S_{\gamma_m} f(x) = y$ for any sequence of positive reals γ_m tending to ∞ as m approaches ∞.

<u>Theorem 1</u>. Let S be given. If for every $f \in L^p(G^N)$, $1 \leq p \leq 2$, $\lim_{\gamma\to\infty} S_\gamma f$ exists for almost every $x \in G^N$, <u>then</u> S^* is weak (p, p).

<u>Corollary</u>. Let S be given. If the operators S_γ are not uniformly bounded on some L^{p_0}, $1 < p_0 < 2$, then for any p such that $1 \leq p < p_0$, there is $f \in L^p$ at $\lim_{\gamma\to\infty} S_\gamma f(x)$ does <u>NOT</u> exist for all x in a set of positive measure.

<u>Proof of Corollary</u>: Otherwise $\lim_{\gamma\to\infty} S_\gamma f(x)$ exists almost everywhere for every $f \in L^p \supset L^{p_0} \supset L^2$. Thus by Theorem 1, S^* is weak (p, p) and weak $(2, 2)$. The Marcinkiewicz interpolation theorem implies that S^* is bounded on L^q for q such that $p < q < 2$. In particular S^* is bounded on L^{p_0}. But then the operators S_γ are uniformly bounded on L^{p_0} giving a contradiction. The proof is thus finished.

Our proof of the theorem follows closely the proof of the case $N = 1$, $p = 2$, G = circle group in Zygmund [13], Vol. II, p. 165. Before we begin the proof let us introduce the following notation: For any $f \in L^1$ and $\alpha > 0$, let $E(\alpha) = E(\alpha, f) = \{S^*f > \alpha\}$.

<u>Proof of the Theorem</u>: Suppose by way of contradiction that S^* is not weak (p, p). Then for each $n = 1, 2, \ldots$ there is a function $f_n(x)$ and numbers $w(n)$ and α_n such that $\| f_n \|_p = 1$, $w(n) \to \infty$,

$|E(\alpha_n, f_n)| > w(n)\alpha_n^{-p}$. According to Lemma 2 below we see that we may assume the f_n's are polynomials. Now we note that $w(n) \to \infty$ implies $\alpha_n \to \infty$. Thus upon discarding a finite number of the f_n's we may assume that $\alpha_n > 1$. We may now subtract $\hat{f}_n(0)\psi_0$ from f_n . Then $\| f_n - \hat{f}_n(0)\psi_0\|_p \le 2$ and $E(\alpha_n - 1, f_n - \hat{f}_n(0)\psi_0) \supset E(\alpha_n, f_n)$. Replacing f_n by $(f_n - \hat{f}_n(0)\psi_0)/\| f_n - \hat{f}_n(0)\|_p$ we see that we may assume with no loss of generality that $\hat{f}_n(0) = 0$.

Using the fact $\alpha_n \to \infty$ and applying Proposition 3 below, we can find a non-decreasing sequence $\{n_k\}_{k=1}^{\infty}$ (with possible repetition) such that $\sum\alpha_{n_k}^{-p} < \infty$ and $\sum w(n_k)\alpha_{n_k}^{-p} = \infty$. The last equation implies that $\sum|E(\alpha_{n_k}, f_{n_k})|$ diverges. We relabel to get

$$(1) \qquad \sum\alpha_n^{-p} < \infty , \quad \sum w(n)\alpha_n^{-p} = \infty , \quad \text{and} \quad \sum|E(\alpha_n, f_n)| = \infty .$$

Let δ denote dilation by 2 . Thus $\delta^k f(x) = f(2^k x)$ (where we assume f is "extended by periodicity" as a function on $[0, 1)$). Since $\hat{f}_n(0) = 0$, we may inductively choose integers k_n which increase with n so rapidly that the polynomial $\delta^{k_n} f_n$ does not overlap with $\delta^{k_j} f_j$, $j < n$ and also does not overlap with $\bigcup_{\gamma\in[0,L_n]} S_\gamma$ where L_n is chosen so that $S^*(\delta^{k_j} f_j) = \sup_{\gamma\in[0,\, L_n]} S_\gamma(\delta^{k_j} f_j)$ when $j < n$. Thus it surely follows that degree $(\delta^{k_{n-1}} f_{n-1}) \le 2^M \le$ first non-zero Fourier coefficient of $\delta^{k_n} f_n$ for some integer M . Replacing $\delta^{k_n} f_n$ by f_n we may assume $\{f_n\}$ satisfy this last condition as well as (1). (We are using the fact that $S_{2^k\cdot\gamma}(\delta^k \cdot f) = \delta^k(S_\gamma f)$ to get that $|E(\alpha_n, f_n)| = E(\alpha_n, \delta^k f_n)|$ for any integer k .)

Now we apply Lemma 4 to obtain a sequence $\{x_n\}$ in G^N such that almost every x in G^N belongs to infinitely many of the sets $x_n + E(\alpha_n, f_n)$. Now let $P_n(x) = f_n(x - x_n)\alpha_n^{-1}$. Notice that $\| P_n \|_p = \alpha_n^{-1}$, so that $\sum_{n=1}^{\infty} \| P_n \|_p^p < \infty$. Thus we obtain that

$$\int \left(\sum_{n=1}^{\infty} \sum_j |\Delta_j P_n|^2 \right)^{p/2} \leq \int \sum_{n=1}^{\infty} \left(\sum_j |\Delta_j P_n|^2 \right)^{p/2}$$

$$= \sum_{n=1}^{\infty} \int \left(\sum_j |\Delta_j P_n|^2 \right)^{p/2} \leq \sum_{n=1}^{\infty} A_p^p \| P_n \|_p^p < \infty ,$$

where we have used the Paley inequalities for the second inequality. We have that $(\Delta_j P_{n_1})(\Delta_j P_{n_2}) = 0$ for any $n_1 \neq n_2$. Thus again applying the Paley inequalities gives that $f \equiv \sum_{n=1}^{\infty} P_n \in L^p(G^N)$.

Finally, if $x - x_n \in E(\alpha_n, f_n)$, then there are γ_1 and γ_2 such that $S_{\gamma_1} f(x) - S_{\gamma_2} f(x)$ exceeds $\alpha_n \alpha_n^{-1} = 1$. Thus $S_\gamma f$ diverges at each point belonging to infinitely many $x_n + E(\alpha_n, f_n)$, i.e. diverges almost everywhere. This completes the proof of the theorem.

<u>Lemma 2</u>. Let S be given. Let $\alpha > 0$ and $f \in L^p$ be given. If $|E(\alpha, f)| > C$ for some number C, then there is 2^n-square partial sum f_M of f such that $|E(\alpha, f_M)| > C$ (and of course $\| f_M \|_p \leq \| f \|_p$).

<u>Proof</u>: For each $\gamma \geq 0$ let $E_\gamma = \{ |S_\gamma f| > \alpha \}$. Then $E(\alpha, f) = \bigcup_{\gamma \geq 0} E_\gamma$. Since $|E(\alpha, f)| > C$, there is L such that $\left| \bigcup_{\gamma \in [0,L]} E_\gamma \right| > C$. Choose M so large that $S_\gamma f_M = S_\gamma f$ for all $0 \leq \gamma \leq L$. The desired conclusion follows. The proof of Lemma 2 is finished.

Proposition 3. If $0 < a_n \to 0$ and $w(n) \to \infty$ then there is a non-decreasing sequence n_k (with possible repetition) such that $\sum_k a_{n_k} < \infty$ while $\sum w(n_k)a_{n_k} = \infty$.

Proof: Set $A_1 = \alpha_{n_1}$ where n_1 is chosen so that $a_{n_1} < \frac{1}{2}$ and $w(n_1) > 2$. Set $B_1 = A_1 w(n_1) = a_{n_1} \cdot w(n_1)$. Choose $R = R_1$ repetitions of A_1 so that $(R - 1)A_1 < \frac{1}{2} < R \cdot A_1$. Note $R \geq 2$ and $R \cdot B_1 > R \cdot 2 \cdot A_1 > 1$. In general (by induction on k) choose $n_k \geq n_{k-1}$ so that $A_k \equiv a_{n_k} < 2^{-k}$ and $w(n_k) > 2^k$. Set $B_k = A_k w(n_k)$. Take R_k repetitions of A_k so that $(R_k - 1)A_k < 2^{-k} < R_k A_k$. (So certainly $R_k \geq 2$.) We have $R_k \cdot B_k = R_k A_k w(n_k) > 2^{-k} \cdot 2^k = 1$. Thus $\sum R_k B_k = \infty$. Meanwhile, $\sum R_k A_k = \sum A_k + \sum(R_k - 1)A_k \leq 2 \sum(R_k - 1)A_k \leq 2 \sum 2^{-k} < \infty$. Proposition 3 is proven.

Lemma 4. If $E_n \subset G^N$ satisfies $\sum|E_n| = \infty$, then $\exists\{x_n\}$ such that almost every $x \in G^N$ belongs to infinitely many $x_n + E_n$.

Our proof comes from Zygmund [13, Vol. II, p. 166].

Proof: We will write $C(E)$ for the complement of E in G. For any given $\{x_n\}$, the set of points belonging to only a finite number the sets $x_n + E_n$ is

$$\bigcup_{p=1}^{\infty} \bigcap_{n=1}^{\infty} C(x_n + E_{p+n})$$

and thus is contained in

$$\bigcup_{\ell=0}^{\infty} \bigcap_{n=p_\ell+1}^{p_{\ell+1}} C(x_n + E_n) \quad \text{for any choice of } 0 = p_0 < p_1 < p_2 < \dots$$

Let $\chi_n(t)$ be the characteristic function of $C(E_n)$. Then the characteristic function of $\bigcap_{n=1}^{p_1} C(x_n + E_n)$ is

$$\chi_1(t + x_1) \cdot \chi_2(t + x_2) \cdot \ldots \cdot \chi_{p_1}(t + x_{p_1}) .$$

Now

$$\int_{G^N} \cdots \int_{G^N} \{\int_{G^N} \chi_1(t + x_1) \cdot \ \ldots \cdot \chi_{p_1}(t + x_{p_1}) dt\} dx_1 \ldots dx_{p_1}$$

$$= \prod_{n=1}^{p_1} \int_{G^N} \chi_n(t) dt = \prod_{n=1}^{p_1} (1 - |E_n|) .$$

The hypothesis that $\sum |E_n| = \infty$ implies that $\prod_{n=1}^{p_1} (1 - |E_1|) < \frac{1}{2}$ if p_1 is chosen large enough. (This is a standard fact about products.) It follows that $\exists x_1, \ldots, x_{p_1}$ such that $|\bigcap_{n=1}^{p_1} C(x_n + E_n)| < \frac{1}{2}$. Similarly, we can select p_2 and $x_{p_1+1}, \ldots x_{p_2}$ so that $\bigcap_{n=p_1+1}^{p_2} C(x_n + E_n)$ has measure less than $(\frac{1}{2})^2$. Continuing, we conclude that

$$\sum_{\ell=0}^{\infty} |\bigcap_{n=p_\ell+1}^{p_{\ell+1}} C(x_n + E_n)| < \infty$$

which easily implies that $|\bigcup_{p=1}^{\infty} \bigcap_{n=1}^{\infty} C(x_n + E_{p+n})| = 0$.

Lemma 4 is proven.

CHAPTER III

A.E. DIVERGENCE OF TWO-DIMENSIONAL WALSH SERIES

A. Subgroups of $\hat{G} \times \hat{G}$ Lying Near Straight Lines.

In this section we describe certain subgroups of $\hat{G}$. Two such subgroups can be used to designate a subgroup of $\hat{G} \times \hat{G}$ which is particularly useful for our purposes.

Let M be a positive integer. Let increasing powers of 2 be given:

$$A_0 = 2^{\ell_0}, A_1 = 2^{\ell_1}, \ldots, A_{M-1} = 2^{\ell_{M-1}} \quad \text{with} \quad 0 \leq \ell_0 < \ldots < \ell_{M-1} .$$

Let positive integers s_j be given for $j = 0, \ldots, M-1$. Suppose that

$$(1) \qquad s_j A_j < A_{j+1} \quad \text{for} \quad j = 0, \ldots, M-2 .$$

Example 1. If $A_j = 2^j$ for all j, then our condition on s_j forces $s_j = 1$ for all j.

Example 2. Suppose we specify $s_j = s =$ some fixed positive integer for all j. Then we may satisfy (1) by setting $A_j = A^j$ where $A = 2^{\ell}$ is some fixed power of $2 > s$.

Recall that if a is a nonnegative integer $< 2^M$, then a_j for $j = 0, \ldots, M-1$ is defined by $a = \sum_{j=0}^{2^M-1} a_j 2^j$, $a_j = 0$ or 1. We shall have occasion to denote the finite sequence $s_1, \ldots, s_{M-1}$ by s.

Define Γ (a subgroup of $\hat{G}$, as we will verify) by $\Gamma = \Gamma(M, A_j, s_j) = \{\sum_{j=0}^{M-1} a_j s_j A_j \mid a = 0, 1, \ldots, 2^M-1\}$. (For example, if $A_j = 2^j$ for all j , then $\Gamma = \{0, \ldots, 2^M-1\}$.) Obviously Γ has order 2^M and $0 \in \Gamma$. To verify Γ is a subgroup we need only verify that Γ is closed under the $\dot{+}$ operation. However our condition $s_j A_j < A_{j+1}$ shows that

$$s_j A_j + s_{j+1} A_{j+1} = s_j A_j \dot{+} s_{j+1} A_{j+1} \ .$$

Now by an easy induction on k we obtain

$$\sum_{j=0}^{k} s_j A_j = \sum_{j=0}^{k-1} s_j A_j + s_k A_k = \sum_{j=0}^{k-1} (\dot{+})\ s_j A_j + s_k A_k = \sum_{j=0}^{k} (\dot{+})\ s_j A_j \ .$$

Since each a_j is 0 or 1 it follows that $\sum a_j s_j A_j = \sum(\dot{+})\ a_j s_j A_j$. Thus

$$(2) \quad (\sum a_j s_j A_j) \dot{+} (\sum b_j s_j A_j) = \sum(\dot{+})[(a_j s_j A_j) \dot{+} (b_j s_j A_j)]$$

$$= \sum(a_j \dot{+} b_j) s_j A_j = \sum(a \dot{+} b)_j s_j A_j \ .$$

So Γ is indeed a subgroup. In fact for $a = 0, \ldots, 2^M - 1$; define $I(a) = \sum a_j s_j A_j$. Then (2) actually shows $I(a) \dot{+} I(b) = I(a \dot{+} b)$, i.e. $I : (G/G_M)\hat{} \longrightarrow \Gamma$ is a $\dot{+}$ isomorphism onto Γ . One sees immediatley that I is order preserving, i.e. $a < b$ implies $I(a) < I(b)$.

To obtain a subgroup of $\hat{G} \times \hat{G}$ we need the following data: the positive integer M ; $A_0 = 2^{\ell_0}$, $A_1 = 2^{\ell_1}$, ... , $A_{M-1} = 2^{\ell_{M-1}}$ with $0 \leq \ell_0 < \ell_1 < \dots < \ell_{M-1}$; and positive integers s_j, t_j for $j = 0, \dots, M-1$. We suppose that

$$s_j A_j < A_{j+1} \quad \text{and}$$

$$t_j A_j < A_{j+1} \quad \text{for } j = 0, \dots, M-2 .$$

Let Γ be defined by

$$\Gamma = \Gamma(M, A_j, s_j, t_j) = \{\sum_{j=0}^{M-1} a_j(s_j, t_j)A_j \mid a = 0, 1, \dots, 2^{M-1}\}$$

where $a_j \cdot (s_j, t_j) \cdot A_j = (a_j s_j A_j, a_j t_j A_j)$. If we let $I_s(a) = \sum_{j=0}^{M-1} a_j s_j A_j$ and $I_t(a) = \sum_{j=0}^{M-1} a_j t_j A_j$, then Γ is the image of $(G/G_M)^\wedge$ under the group homomorphism $I_s \times I_t$. ($I_s \times I_t(a)$ is defined to be $(I_s(a), I_t(a))$. Thus Γ is a subroup of $\hat{G} \times \hat{G}$ and $I_s \times I_t : (G/G_M)^\wedge \to \Gamma$ is an isomorphism onto.

Example 1. If $A_j = 2^j$, $s_j = t_j = 1$, then Γ is the diagonal subgroup: $\Gamma = \{(a, a) \mid a = 0, 1, \dots, 2^M - 1\}$.

Example 2. If $A_j = A^j = (2^\ell)^j$, $s_j \equiv s$, and $t_j \equiv t$, then $\Gamma = \{\sum_{j=0}^{M-1} a_j(s, t)A_j\}$ and every element (m, n) of Γ lies on the line $y = \frac{t}{s}x$.

For Theorem 1 below we need the following classical result of Kronecker:

Theorem (Kronecker) [12, Vol. II, p. 20] Given any real x there are infinitely many fractions p/q such that $|x - p/q| \le q^{-2}$.

Theorem 1. Let σ be a positive real number. Let $\epsilon > 0$ be given and let M be a positive integer. Then there is a subgroup Γ of $\hat{G} \times \hat{G}$, $\Gamma = \Gamma(M, A_j, s_j, t_j)$ for some A_j, s_j, t_j ; such that any (m, n) in Γ has vertical distance $< \epsilon$ from the line $y = \sigma x$:

$$|n - \sigma m| < \epsilon \qquad (m, n) \in \Gamma .$$

Proof: Use induction on M . Let $M > 0$ be given. If $M = 1$, let $\Gamma_0 = \{0\}$. If $M > 1$, then assume by induction that $\Gamma_0 = \Gamma_0(M-1, A_j, s_j, t_j)$ is given satisfying $|n - \sigma m| < \frac{\varepsilon}{2}$ for all $(m, n) \in \Gamma_0$. Choose A_{M-1} , a power of 2 , so large that

$$s_{M-2} \cdot A_{M-2} < A_{M-1} \qquad \text{and}$$

$$t_{M-2} \cdot A_{M-2} < A_{M-1} .$$

If $M = 1$, we let $A_0 = 1$ or let A_0 be arbitrary. By Kronecker's theorem, there are integers s_{M-1} and t_{M-1} such that

$$|t_{M-1} - \sigma s_{M-1}| < \epsilon/2 \, A_{M-1} .$$

Thus $|t_{M-1} \cdot A_{M-1} - \sigma s_{M-1} \cdot A_{M-1}| < \epsilon/2$. We claim that $\Gamma = \Gamma(M, A_j, s_j, t_j)$ satisfies the theorem. Indeed

$$\Gamma = \Gamma_0 \cup (\Gamma_0 \dot{+} (s_{M-1} \cdot A_{M-1}, t_{M-1} \cdot A_{M-1})) .$$

Furthermore since A_{M-1} is a power of 2 and $A_{M-1} > m, n$ for any $(m, n) \in \Gamma_0$ we have

$$\Gamma_0 \dot{+} (s_{M-1} \cdot A_{M-1}, t_{M-1} \cdot A_{M-1}) = \Gamma_0 + (s_{M-1} \cdot A_{M-1}, t_{M-1} \cdot A_{M-1}) .$$

That is to say

$$(m, n) \dot{+} (s_{M-1} \cdot A_{M-1}, t_{M-1} \cdot A_{M-1}) = (m, n) + (s_{M-1} \cdot A_{M-1}, t_{M-1} \cdot A_{M-1})$$

for any $(m, n) \in \Gamma_0$. We now obtain:

$$|(n + t_{M-1} \cdot A_{M-1}) - \sigma(m + s_{M-1} \cdot A_{M-1})| \leq$$

$$|n - \sigma m| + |t_{M-1} \cdot A_{M-1} - \sigma s_{M-1} \cdot A_{M-1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon ,$$

for any $(m, n) \in \Gamma_0$. The case $M = 1$ is proved. Also we have proved the requisite induction step. Thus the entire theorem is proved by induction.

<u>Remark</u>. If σ is rational, say $\sigma = t/s$, then we may take $s_j \equiv s$, $t_j \equiv t$, and $A_j = A^j$ where A is some fixed power of 2 greater than s and t. (See Example 2 above.) Γ then lies on the line $y = \sigma x$ and we therefore get the theorem with $\epsilon \equiv 0$ without the use of Kronecker's theorem.

B. Unboundedness of Half-plane Multipliers.

Let a positive real number σ be given. Let $\epsilon = \frac{\sigma}{2}$. Let M be a positive integer. Choose $\Gamma = \Gamma(M, A_j, s_j, t_j)$ according to Theorem 1. Let I_s, I_t be given as before Theorem 1 so that $I_s \times I_t : (G/G_M)\hat{} \longrightarrow \Gamma$ is an isomorphism onto. To shorten notation we let $J(a) = I_s \times I_t(a)$. For $j = 0, \ldots, M-1$; let

$$\Gamma_j = \{J(a) |\ a_j = 0\} = J(\{a \in (G/G_M)\hat{} \ |\ a_j = 0\}) .$$

Now $\{a \in (G/G_M)\hat{}\ |\ a_j = 0\}$ is a subgroup of $(G/G_M)\hat{}$ and furthermore it has index 2 in $(G/G_M)\hat{}$. Since J is an isomorphism it follows that Γ_j has index 2 in Γ for $j = 0, \ldots, M-1$.

Let $H = \mathrm{ann}(\Gamma)$ and let $H_j = \mathrm{ann}(\Gamma_j)$ for $j = 0, \ldots, M-1$. The general theory of annihilators in Chapter I shows that $H_j \neq H_k$ for $j \neq k$ and $[H_j : H] = 2$. It follows that $H_j \cap H_k = H$ if $j \neq k$. We have (see Theorem I.F.6) that

$$\sum_{\psi \in \Gamma} \psi = \left\{ \begin{array}{ll} |\Gamma| = 2^M & \text{on } H \\ 0 & \text{otherwise} \end{array} \right\} = \sum_{k=0}^{2^M-1} \psi_{I_s(k),\, I_t(k)}$$

Using our identification of $\hat{G}$ with the nonnegative integers, we define a function $R_j(x)$ as follows: $R_j(x) = I_S(2^j) = \psi_{I_S(2^j)}(x) =$ image of the jth Rademacher function $r_j(x) = \psi_{2^j}(x)$ under I_S. To avoid writing the arguments (x, y) of numerous functions below, it is convenient to view R_j as a function on $G \times G$ independent of the second variable y. We now define a function $f_j = f_j^M$

by $f_j = R_j \sum_{\psi \in \Gamma} \psi$ and note that

$$f_j = \begin{cases} 2^M \cdot R_j & \text{on } H \\ 0 & \text{otherwise} \end{cases} .$$

From the definition of Γ and R_j in terms of I_S and I_t we see that we also have

$$f_j = \sum_{k=0}^{2^M-1} \psi_{I_S(k \dot{+} 2^j),\, I_t(k)} .$$

Note that $k \dot{+} 2^j > k$ if $k_j = 0$, i.e. if $J(k) \in \Gamma_j$ and

$k \dot{+} 2^j < k$ if $k_j = 1$, i.e. if $J(k) \in \Gamma \sim \Gamma_j$.

Thus, since I_s is order preserving, we have

$$I_s(k \dot{+} 2^j) > I_s(k) \quad \text{if} \quad J(k) \in \Gamma_j$$

and $$I_s(k \dot{+} 2^j) < I_s(k) \quad \text{if} \quad J(k) \in \Gamma \sim \Gamma_j .$$

Since 2 distinct integers differ by at least one we obtain

$$I_s(k \dot{+} 2^j) \geq I_s(k) + 1 \quad \text{if} \quad J(k) \in \Gamma_j$$

and $$I_s(k \dot{+} 2^j) \leq I_s(k) - 1 \quad \text{if} \quad J(k) \in \Gamma \sim \Gamma_j .$$

Thus $$\sigma I_s(k \dot{+} 2^j) \geq \sigma I_s(k) + \sigma \geq I_t(k) - \frac{\sigma}{2} + \sigma \geq I_t(k)$$

if $J(k) \in \Gamma_j$ and

$$\sigma I_S(k \dotplus 2^j) \le \sigma I_s(k) - \sigma \le I_t(k) + \frac{\sigma}{2} - \sigma \le I_t(k)$$

if $J(k) \in \Gamma \sim \Gamma_j$. If we define an operator $D = D(\sigma)$ on a Walsh polynomial P on $G \times G$ by $D(P) = \sum_{\ell<\sigma k} \hat{P}(k, \ell)\psi_{k,\ell}$, then the above computations show

$$D(f_j) = \sum_{\substack{k=0 \\ J(k)\in\Gamma_j}}^{2^M-1} \psi_{I_S(k \dotplus 2^j),\, I_t(k)} = R_j \sum_{\psi\in\Gamma_j} \psi$$

$$= |\Gamma_j| \cdot R_j \cdot X_{H_j} = 2^{M-1}R_j \cdot X_{H_j}$$

(X_{H_j} = characteristic function of H_j). Now define f by $f = f^M = \sum_{j=0}^{M-1} f_j$. Then

$$D(f) = \sum_{j=0}^{M-1} D(f_j) = \begin{cases} ? \text{ on } H & \text{(not needed in the sequel)} \\ R_j \cdot 2^{M-1} & \text{on } H_j \sim H \\ 0 & \text{otherwise} \end{cases}.$$

So $\| D(f)\|_p \ge (\sum_{j=0}^{M-1} (2^{M-1})^p \cdot |H_j \sim H|)^{1/p} = (M \cdot 2^{p(M-1)} \cdot 2^{-M})^{1/p}$

$$= M^{1/p} \cdot 2^{M-1} \cdot 2^{-M/p} = \frac{1}{2} M^{1/p} \cdot 2^{M(1-\frac{1}{p})} = \frac{1}{2} M^{1/p} \cdot 2^{M/q}$$

where $q = 1 - \frac{1}{p}$.

We need to compute $\| f \|_p$. Toward this we note that f_j and hence f itself is supported on H . We shall use that $(G^N)^{\wedge}/\Gamma \cong \widehat{\mathrm{ann}(\Gamma)} = \hat{H}$ (see Theorem I.F.1) to "pass to an analysis of f on H ." Actually we shall use more than $(G^N)^{\wedge}/\Gamma = \hat{H}$ by making use of the function $R_j(x)$ and Proposition I.F.7. Let us proceed. Recall that we view $R_j(x) = R_j(x, y)$ with R_j being constant for each fixed x , i.e. we take $G \times G$ as the domain of each R_j .

Let us define a map $\tilde{J} : (G/G_M)^{\wedge} \times (G/G_M)^{\wedge} \to \hat{G} \times \hat{G}$ by $(a, b) \xrightarrow{\tilde{J}} (I_S(a) \dot{+} I_S(b), I_t(b))$. Let Γ_0 be the image of $\tilde{J}$. Since I_s and I_t are one-to-one, it follows immediately that $\tilde{J}$ is one-to-one. Thus $|\Gamma_0| = |(G/G_M)^{\wedge} \times (G/G_M)^{\wedge}| = 2^{2M}$. Note that $(0, b) \to (I_S(b), I_t(b)) = J(b)$. So $\Gamma \subset \Gamma_0$. Also note that $(a, 0) \to (I_S(a), 0)$. In particular $(2^j, 0) \to (I_S(2^j), 0) = R_j(x)$ under our convention that $R_j(x)$ is a function of x and y constant for fixed x . So $\{R_j\} \subset \Gamma_0$. Let Γ' be given by

$$\Gamma' = \text{group generated by } \{R_j\} \equiv \{(I_S(a), 0) \mid a = 0, \dots, 2^M-1\} .$$

Since an element in Γ_0 can be written uniquely as

$$(I_S(a), 0) \dot{+} (I_S(b), I_t(b))$$

it follows that Γ_0 is the direct sum of Γ' and Γ . Now we obtain:

$$\Gamma' \cong \Gamma_0/\Gamma \cong (\mathrm{ann}(\Gamma)/\mathrm{ann}(\Gamma_0))\hat{} = (H/\mathrm{ann}(\Gamma_0))\hat{} \subset \hat{H}_j$$

$$(I_S(a),\ 0) \to (I_S(a),\ 0) + \Gamma \to (I_S(a),\ 0)|_{\mathrm{ann}(\Gamma)} = (I_S(a),\ 0)|_H \ .$$

In particular the elements of Γ' remain distinct when restricted to H. Thus $\mathrm{order}(\Gamma'|_H) = \mathrm{order}(\Gamma') = 2^M$. Remembering that $\{R_j|_H\}$ generate $\Gamma'|_H$, we may use Proposition I.F.7 to obtain that $\{R_j|_H\}$ are linearly independent. The above isomorphism acting on R_j is simply $R_j \longrightarrow R_j|_H$. We may combine the above isomorphism with I_S to see that

$$(G/G_M)\hat{} \cong (H/\mathrm{ann}(\Gamma_0))\hat{} \quad \text{via}$$

$$a \longrightarrow (I_S(a),\ 0)|_H \quad \text{or for} \quad a = 2^j \quad \text{we have}$$

$$r_j \longrightarrow R_j|_H \ .$$

This isomorphism induces an isomorphism from G/G_M to $H/\mathrm{ann}(\Gamma_0)$ which takes Haar measure on G/G_M to Haar measure on $H/\mathrm{ann}(\Gamma')$. We thus in turn obtain an isometry of $L^2\text{-span}(G/G_M)\hat{}$ with $L^2\text{-span}(\Gamma')$ which are, of course, finite dimensional vector spaces contained in $L^2(G)$ and $L^2(H)$ respectively. Since r_j corresponds to $R_j|_H$ under this isometry we obtain that $2^M \sum_{j=0}^{M-1} r_j$ corresponds to $2^M \sum_{j=0}^{M-1} R_j|_H \equiv f|_H$. Note that $|G \times G/H| = 2^M$ so that normalized Haar measure on H eaquals 2^M times normalized Haar measure on

$G \times G$ restricted to H. Thus, finally, we have

$$\| f \|_{L^p(G \times G)} = 2^{-M/p} \| f \|_{L^p(H)} = 2^{-M/p} \| 2^M \sum_{j=0}^{M-1} r_j \|_{L^p(G)} .$$

At this time we invoke Theorem II.A.5 which says Rademacher series have all L^p-norms equivalent. We obtain that

$$\| f \|_{L^p(G\times G)} \leq 2^M \cdot 2^{-M/p} \cdot C_p \| \sum_{j=0}^{M-1} r_j \|_{L^2(G)}$$

$$= C_p \cdot 2^{M(1-1/p)} \cdot M^{1/2} = C_p \cdot 2^{M/q} \cdot M^{1/2} ,$$

where C_p is independent of M and $\frac{1}{q} = 1 - \frac{1}{p}$. Thus for all M we have

$$\| D(\sigma) \|_{p,p} \geq \| D(f^M) \|_p / \| f^M \|_p \geq (\tfrac{1}{2} M^{1/p} \cdot 2^{M/q}) / (C_p \cdot 2^{M/q} \cdot M^{1/2})$$

$$= \frac{1}{2C_p} \cdot M^{(1/p - 1/2)} .$$

If $p < 2$, then this last quantity approaches ∞ as $M \to \infty$. Duality shows $D(\sigma)$ is unbounded on all L^p, $p \neq 2$.

We now summarize this result in the following theorem. The case $\sigma = 1$ is proved in [12].

<u>Theorem 1</u>. Let $\sigma > 0$ be given. Define the operator $D = D(\sigma)$ by

$$D(f) = \sum_{\ell < \sigma k} \hat{f}(k, \ell) \psi_{k,\ell} \qquad \text{for } f \in L^2(G^N) .$$

Then D is not bounded on any L^p other than L^2.

C. Partial Sum Operators and A.E. Divergence.

We wish to use the preceding result to show that certain partial sum operators are not uniformly bounded on $L^p(G \times G)$, $p \neq 2$. Let an integer M be given. We consider integers $k < M$. Note that $(2^M - 1) \dot{+} k = 2^M - k - 1$, since $2^M - 1$ has binary expansion $\sum_{j=0}^{M-1} 2^j$. If we let $m = 2^M - k - 1$, then (k, ℓ) satisfies $\ell < \sigma k$ if and only if (m, ℓ) satisfies $\sigma m + \ell < \sigma(2^M - 1)$. Let us define $\overline{D}_M = \overline{D}_M(\sigma)$ by

$$\overline{D}_M(f) = \sum_{\sigma m+\ell<\sigma(2^M-1)} \hat{f}(m, \ell)\psi_{m,\ell} \qquad \text{for } f \in L^2(\frac{G \times G}{G_M \times G_M}) .$$

Let $D = D(\sigma)$ be as above, but restrict D to $L^2(\frac{G \times G}{G_M \times G_M})$, i.e.

$$D(f) = \sum_{\ell<\sigma k} \hat{f}(k, \ell)\psi_{k,\ell} \qquad \text{for } f \in L^2(\frac{G \times G}{G_M \times G_M}) .$$

Then the following relationship holds between D and $\overline{D}_M$:

$$\psi_{2^M-1}(x)D(f)(x, y) = \psi_{2^M-1}(x) \sum_{\ell<\sigma k} \hat{f}(k, \ell)\psi_{k,\ell}(x, y)$$

$$= \sum_{\ell<\sigma k} \hat{f}(k, \ell) \cdot \psi_{2^M-k-1,\ell}(x, y) = \sum_{\sigma m+\ell<\sigma(2^M-1)} \hat{f}(2^M - m - 1, \ell)\psi_{m,\ell}(x, y)$$

$$= \overline{D}_M [\sum_{m,\ell=0}^{2^M-1} \hat{f}(2^M - m - 1, \ell)\psi_{m,\ell}(x, y)] = \overline{D}_M(\psi_{2^M-1}(x) \cdot f(x, y)) .$$

In short, $\psi_{2^M-1}(x)D(f) = \overline{D}_M(\psi_{2^M-1}(x)f)$. In words: Multiplying by $\psi_{2^M-1}(x)$ intertwines D and $\overline{D}_M$ on $L^2(\frac{G \times G}{G_M \times G_M})$ and, of course, multiplication by $\psi_{2^M-1}(x)$ preserves L^p-norms for

$1 \leq p \leq \infty$. It follows that $\| \overline{D}_M \|_{p,p} = \| D \|_{p,p}$ (D restricted to $L^2(\frac{G \times G}{G_M \times G_M})$). More explicitly, let $f \in L^2(\frac{G \times G}{G_M \times G_M})$ be given, let g be defined as $g(x, y) = \psi_{2^M-1}(x)\, f(x, y)$. Then certainly $\| g \|_p = \| f \|_p$, $1 \leq p \leq \infty$. And our computation above shows

$$\| \overline{D}_M(g) \|_p = \| D(f) \|_p \ .$$

In any case (for fixed σ) it follows from our previous theorem that $\{\overline{D}_M(\sigma)\}_{M=0}^{\infty}$ is not a uniformly bounded collection of operators on $L^p(G \times G)$.

Let σ be fixed. Let S = triangle on $\hat{G} \times \hat{G}$ plane determined by the line $y = -\sigma x + 1$ and the two coordinate axes. Thus S_γ is the triangle determined by the two coordinate axes and the line $y = -\sigma x + \gamma$. So $\{\overline{D}_M(\sigma)\}_{M=0}^{\infty} \subset \{S_\gamma\}_{\gamma \geq 0}$. In fact, $\overline{D}_M = S_\gamma$ with $\gamma = \sigma(2^M - 1)$. The preceding discussion proves the following theorem:

Theorem 1. $\{S_\gamma\}_{\gamma \geq 0}$ are not uniformly bounded on $L^p(G \times G)$, $p \neq 2$.

Corollaries. The uniform boundedness principle implies there is some $f \in L^p(G \times G)$ such that $S_\gamma f$ does not converge to f in L^p-norm $(p \neq 2)$. Finally, Stein's theorem given above, implies, for $p < 2$, that there is $f \in L^p(G \times G)$ such that $\lim_{\gamma \to \infty} S_\gamma f(x, y)$ does not converge for a set of (x, y) having positive measure in $G \times G$.

Now let $h(x)$ be continuous and nonnegative on some interval $[a, b] \subset [0, \infty)$. Suppose also that $h'(x)$ exists on (a, b) and is continuous and negative on (a, b) . Thus $h(x)$ is strictly decreasing on $[a, b]$. Suppose the graph of h is a part of the boundary of a bounded region $\tilde{S}$ containing 0 in the first quadrant of $\mathbb{R}^2$. So $\tilde{S}$ gives rise to the partial sum operators $\tilde{S}_\gamma$, $\gamma \geq 0$.

For simplicity, we suppose that the interior of the angle with vertex $(0, 0)$ and sides passing through, respectively, $(a, f(a))$ and $(b, f(b))$ intersect the boundary of $\tilde{S}$ only in the graph of $h(x)$.

It follows there is line $y = \frac{e}{d} x$ through the origin with rational slope $\frac{e}{d}$ which intersects the graph of h say at the point $(c, h(c))$ with $a < c < b$. Thus $\frac{h(c)}{c} = \frac{e}{d}$. Let d , e , and c be fixed. Let $-\sigma = h'(c)$, $\sigma > 0$.

For any $\gamma > 0$, we let $h_\gamma(x) = \gamma \cdot h(\frac{x}{\gamma})$. Thus $h_\gamma'(x) = h'(\frac{x}{\gamma})$ and $h_{\gamma \cdot \tilde{\gamma}}(x) = \gamma h_{\tilde{\gamma}}(\frac{x}{\gamma})$. The graph of $h_\gamma(x)$ forms a part of the boundary of $\tilde{S}_\gamma$.

Now, as in the preceding theorem, let S be the triangle in the $\hat{G} \times \hat{G}$ plane with sides equal to the x-axis, the y-axis, and the line $y = -\sigma x + 1$. According to our previous theorem $\{S_\gamma\}_{\gamma \geq 0}$ are not uniformly bounded on L^p , $p \neq 2$. We shall show for certain relevant functions and certain γ , $\beta > 0$ that the action of S_γ and $\tilde{S}_\beta$ are the same. We thus obtain the non-uniform boundedness of $\{\tilde{S}_\gamma\}_{\gamma \geq 0}$. Indeed let M be a positive integer and suppose $f = f^M \in L^2(\frac{G \times G}{G_M \times G_M})$ satisfies $\| \bar{D}_M(f) \|_p / \| f \|_p > C(M)$ where $\bar{D}_M = \bar{D}_M(\sigma)$ and $C(M)$ is large for large M . Such f^M exist by the preceding theorem. We shall show that $\tilde{S}_\beta(\psi f) = S_\gamma(\psi f)$ for appropriate β , $\gamma > 0$ and appropriate ψ $\hat{G} \times \hat{G}$. We let such functions f^M be hereby fixed until the end of this proof.

Notice for $\gamma_0 = \sigma(2^M - 1)$ that we have $S_{\gamma_0} = \bar{D}_M$. In fact, since $\hat{G} \times \hat{G}$ is discrete in the first quadrant, we have $S_{\gamma_0 - w} = \bar{D}_M$ for all w satisfying $0 < w < w_0$ for some $w_0 > 0$. We can assume $\gamma_0 - w_0 > 0$. Let this $w_0 > 0$ be fixed.

Set $\epsilon = 2^{-M} \dfrac{w_0}{2}$. Choose $\delta > 0$ so that $a < c - \delta < c + \delta < b$ and so that

$|h'(x) + \sigma| < \epsilon$ when $|x - c| < \delta$. Thus

(1) $|h_\gamma'(x) + \sigma| < \epsilon$ when $|x - \gamma c| < \gamma\delta$ for any $\gamma > 0$.

Set $\dfrac{1}{\gamma_1} = h(c) + \sigma c > 0$. We obtain successively that

$$h(c) = -\sigma c + \frac{1}{\gamma_1}$$

$$\gamma_1 h(c) = -\sigma(\gamma_1 c) + 1$$

$$h_{\gamma_1}(\gamma_1 c) = -\sigma(\gamma_1 c) + 1$$

Choose γ_2 so that $\gamma_1 \cdot \gamma_2 > 2^M/\delta$. Notice that

$$h_\gamma(\gamma c) = \frac{\gamma}{\gamma_1} h_{\gamma_1}(\gamma_1 c) = \frac{\gamma}{\gamma_1} [-\sigma(\gamma_1 c) + 1] = -\sigma(\gamma c) + \frac{\gamma}{\gamma_1} .$$

In short

(2) $$h_\gamma(\gamma c) = -\sigma(\gamma c) + \frac{\gamma}{\gamma_1}.$$

Now let $\gamma > \gamma_1 \cdot \gamma_2$. If x is such that $|x - \gamma c| < 2^M < \gamma\delta$, then the mean value theorem together with (1) implies that

(3) $$|h_\gamma(x) + \sigma x - \frac{\gamma}{\gamma_1}| < \epsilon \cdot 2^M = \frac{w_0}{2} \qquad \text{when}$$

$$\gamma > \gamma_1\gamma_2 \qquad \text{and} \qquad |x - \gamma c| < 2^M .$$

We have shown the geometrically intuitive result that for γ large, $h_\gamma(x)$ has graph very close to the line $y = -\sigma x + \frac{\gamma}{\gamma_1}$ for all x in a wide interval (of length 2^M) .

For any positive integer L , let $\gamma = \gamma(L)$ be given by $\gamma = \gamma_1[\gamma_0 - \frac{w_0}{2} + \sigma Ld2^M + Le2^M]$. With this choice of γ we obtain:

(4) The line $y = -\sigma x + \frac{\gamma}{\gamma_1}$ passes through $((Ld + 1) \cdot 2^M - 1, L \cdot e \cdot 2^M - \frac{w_0}{2})$.

To verify (4) we need to show

$$L e 2^M - \frac{w_0}{2} = -\sigma[(Ld + 1) \cdot 2^M - 1] + \frac{\gamma}{\gamma_1}.$$

In view of the definition of γ this becomes

$$L e 2^M - \frac{w_0}{2} = -\sigma Ld2^M - \sigma(2^M - 1) + [\gamma_0 - \frac{w_0}{2} + \sigma Ld2^M + Le2^M] ,$$

which is equivalent successively to

$$0 = -\sigma Ld2^M - \sigma(2^M - 1) + \gamma_0 + \sigma Ld2^M$$

$$0 = \qquad - \sigma(2^M - 1) + \gamma_0$$

$$\gamma_0 = -\sigma(2^M - 1) .$$

This last equation is simply the definition of γ_0 and thus (4) is verified.

Choose L so large that $\gamma > \gamma_1\gamma_2$. So (3) holds when $|x - \gamma c| < 2^M$.

(5) The line $y = -\sigma x + \frac{\gamma}{\gamma_1}$ passes also through $(\gamma c, h_\gamma(\gamma c))$ by (2) .

(6) The line $y = \frac{e}{d} x$ passes through $(Ld \cdot 2^M, Le2^M)$ and $(\gamma c, h_\gamma(\gamma c))$ since $\frac{e}{d} = \frac{h(c)}{c} = \frac{h_\gamma(\gamma c)}{c}$.

From (4), (5), and (6) it follows that

$$(7) \qquad Ld2^M \leq \gamma c \leq (Ld + 1)2^M - 1 .$$

Thus (3) holds when x is in $[Ld2^M, (Ld + 1)2^M - 1]$.

The partial sum operator S_{γ/γ_1} is determined by the line $y = -\sigma x + \frac{\gamma}{\gamma_1}$ while $\tilde{S}_\gamma$ is determined by h_γ . But (3) implies, for x in $[L \cdot d \cdot 2^M, (Ld + 1)2^M - 1]$, that $h_\gamma(x)$ differs from

$-\sigma x + \frac{\gamma}{\gamma_1}$ by less than $\frac{w_0}{2}$. The choice of w_0 and the fact that $y = -\sigma x + \frac{\gamma}{\gamma_1}$ passes through $((Ld + 1)2^M - 1, Le2^M - \frac{w_0}{2})$ imply that

$$(8) \qquad S_\gamma = S_{\gamma/\gamma_1}$$

as operators acting on functions with Fourier transform support having first coordinate in $[Ld2^M, (Ld + 1)2^M - 1]$. Let $\Psi = \Psi_{L \cdot d \cdot 2^M, Le \cdot 2^M}$. Recall $f = f^M$ satisfies $\hat{f}(m, n) = 0$ if m or $n \geq 2^M$. Thus

$$(9) \qquad \widehat{\Psi f}(m + L \cdot d \cdot 2^M, n + L \cdot e \cdot 2^M) = \hat{f}(m, n) .$$

Thus Ψf has Fourier transform support with first coordinates between $Ld \cdot 2^M$ and $(Ld + 1) \cdot 2^M$. By (8) we have

$$(10) \qquad \tilde{S}_\gamma(\Psi f) = S_{\gamma/\gamma_1}(\Psi f) .$$

Finally we get the following succession of equalities:

$\Psi\bar{D}_M(f) = \Psi S_{\gamma_0}(f)$ by definition of γ_0 .

$\Psi S_{\gamma_0}(f) = \Psi S_{\gamma_0 - w_0/2}(f)$ by choice of w_0 .

$\Psi S_{\gamma_0 - w_0/2}(f) = S_{\gamma_0 - w_0/2 + \sigma Ld2^M + Le2^M}(\Psi f)$ by (9) .

This last quantity equals $S_{\gamma/\gamma_1}(\Psi f)$ by definition of γ . $S_{\gamma/\gamma_1}(\Psi f) = \tilde{S}_\gamma(\Psi f)$ by (10). In summary we have $\Psi\bar{D}_M(f) = \tilde{S}_\gamma(\Psi f)$.

Thus $\| \tilde{S}_\gamma(\Psi f) \|_p / \| \Psi f \|_p = \| \bar{D}_M(f) \|_p / \| f \|_p$. The non-uniform boundedness of $\{\tilde{S}_\gamma\}_{\gamma \geq 0}$ follows from that of $\{\bar{D}_M\}_{M=0}^{\infty}$. We have proved the following theorem:

Theorem 2. $\{\tilde{S}_\gamma\}_{\gamma \geq 0}$ are not uniformly bounded on $L^p(G \times G)$, $p \neq 2$.

As before we get corollaries:

Corollaries. The uniform boundedness principle implies there is some $f \in L^p(G \times G)$ such that $\tilde{S}_\gamma f$ does not converge to f in L^p-norm $(p \neq 2)$. Stein's Theorem II.B.1 above implies, for $p < 2$, that there is $f \in L^p(G \times G)$ such that $\lim_{\gamma \to \infty} \tilde{S}_\gamma f(x, y)$ does not converge for a set of (x, y) having positive measure in $G \times G$.

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