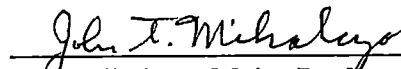
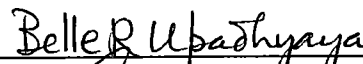


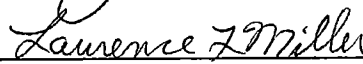
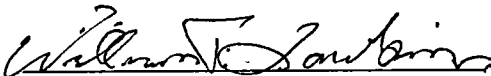
To the Graduate Council:

I am submitting herewith a dissertation written by J. K. Mattingly entitled "High Order Statistical Signatures from Source-Driven Measurements of Subcritical Fissile Systems." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Nuclear Engineering.


J. T. Mihalcz, Major Professor

We have read this dissertation
and recommend its acceptance:




Accepted for the Council:


Associate Vice Chancellor
and Dean of the Graduate School

HIGH ORDER STATISTICAL SIGNATURES
FROM SOURCE-DRIVEN MEASUREMENTS
OF SUBCRITICAL FISSILE SYSTEMS

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

J. K. Mattingly
August 1998

For Laura, for Nakita, and for me.

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developed the theoretical models of the higher order temporal signatures. In particular I thank him for deciphering Gordon E. Hansen's "deductions." Without that contribution the second chapter of this dissertation would be very short indeed.

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this work was finished – I always promised that I'd pay her more attention after I finished.
Now it's too late.

I won't even attempt to describe the unhappiness that my wife Laura must have experienced due to my inattention and general foul mood during the writing of this dissertation. As in everything, only her graceful nature and generous spirit shone through – she really is my better half.

Abstract

This research focuses on the development and application of high order statistical analyses applied to measurements performed with subcritical fissile systems driven by an introduced neutron source. The signatures presented are derived from counting statistics of the introduced source and radiation detectors that observe the response of the fissile system. It is demonstrated that successively higher order counting statistics possess progressively higher sensitivity to reactivity. Consequently, these signatures are more sensitive to changes in the composition, fissile mass, and configuration of the fissile assembly. Furthermore, it is shown that these techniques are capable of distinguishing the response of the fissile system to the introduced source from its response to any internal or inherent sources. This ability combined with the enhanced sensitivity of higher order signatures indicates that these techniques will be of significant utility in a variety of applications. Potential applications include enhanced radiation signature identification of weapons components for nuclear disarmament and safeguards applications and augmented nondestructive analysis of spent nuclear fuel. In general, these techniques expand present capabilities in the analysis of subcritical measurements.

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1 Introduction

Imagine the response of a subcritical fissile system to an injected neutron. If this neutron induces fission, the resultant “progeny” of this reaction, the fission neutrons, may survive to induce more fissions. As this chain reaction proceeds, entire generations of “chain-related” neutrons are born and die. If the system is extremely subcritical, on average only a few generations will pass before the chain reaction is terminated. Consequently, at any given time, usually only a few related neutrons will be present. However, if the system is nearly critical, the chain reaction may survive through potentially thousands, even millions, of generations, so at any given time there is usually an abundance of related neutrons. Therefore, in an extremely subcritical system, neutrons will often exist as singlets, i.e., a single neutron with no living relatives. In contrast, if the system is close to critical, neutrons will often form doublets, triplets, quadruplets, etc., with their surviving relatives. So the number of n -tuplets is an indication of the system’s reactivity, which is affected by the system’s composition, fissile mass, and configuration.

Now recognize that because neutrons from a single fission are all born at the same time, related neutrons are correlated in time. That is, neutrons observed at the same time have a relatively higher probability of being related than do neutrons seen at different times, and the probability of relation declines as the time between observations increases. This is the nature of fission-chain decay in a subcritical fissile system. Consequently, a measurement that correlates two neutrons detected at different times produces a “signature” that indicates the rate of fission-chain decay. This signature is a measure of the distribution of doublets

dispersed in time. A similar measurement can be performed that correlates three neutrons observed at different times to estimate the distribution of triplets dispersed in time. The first measurement, because it is concerned with the distribution of doublets, estimates a second-order neutron counting statistic. A measurement that just counts neutrons without correlating them to others estimates a first-order counting statistic. The measurement that produces a temporal distribution of triplets is a third-order statistic. Recognize that higher order neutron counting statistics can be estimated from measurement.

So what? Recall that fission-chain decay proceeds at a rate dictated by reactivity. Consequently, more reactive systems will sustain longer fission-chains, and hence larger numbers of related neutrons will tend to be produced by a single initiating fission. Because the rate of decay is supralinear with reactivity, the number of doublets will be more sensitive to reactivity than the number of singlets. Likewise, the number of triplets will be more sensitive than the number of doublets, quadruplets will be more sensitive than triplets, and so on. So it is intuitively evident that progressively higher order signatures will possess progressively higher sensitivity to reactivity.

Of course, it's easy to say "it's intuitive." Science is about proof. As evidence of these intuitive deductions, some relevant experimental results are provided in Fig. 1.1. This measurement was performed in 1969 by J. T. Mihalcz and J. J. Lynn at the Oak Ridge critical experiments facility on a set of three 177.8-mm-diameter 93.15%-enriched uranium-metal cylinders each of different height [Mih9]. Neutrons were injected into each cylinder by a ^{252}Cf spontaneous fission source adjacent to the cylinder's axial face. Individual source fissions were observed by an ionization chamber that contained the ^{252}Cf source, and

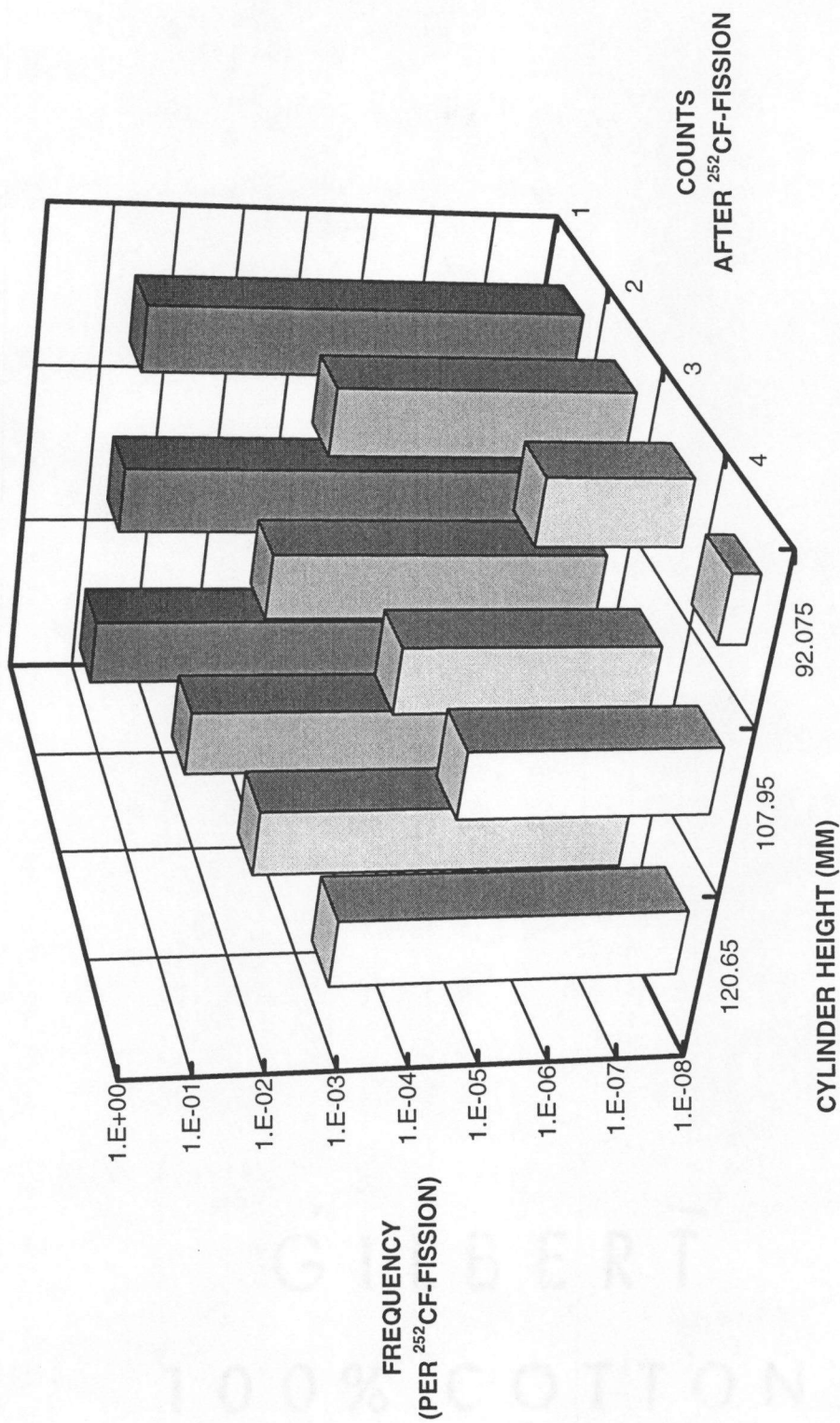


Figure 1.1 Number of counts per ^{252}Cf -fission following ^{252}Cf -fission measured from three 177.8-mm-diameter 93.15%-enriched uranium-metal cylinders (J. T. Mihalcz, Oak Ridge Critical Experiments Facility, 1969, previously unpublished [Mih9])

neutrons escaping each cylinder were counted by a 50-mm-diameter 12.7-mm-thick scintillator adjacent to the cylinder's radial surface. The number of detector counts in 19.025 μ sec following each ^{252}Cf -fission was recorded following each source fission. The chart represents the results obtained after forty million ^{252}Cf fissions, and it depicts the relative frequency of one, two, three, and four counts following each source fission as a function of cylinder height. Of course each of these groupings occurs more frequently as cylinder height increases – the taller cylinders are closer to critical. However, note that the frequency of four counts increases more rapidly than the frequency of three counts, which in turn increases more rapidly than two counts. The frequency of a single count following ^{252}Cf -fission changes the least with cylinder height. So this experiment indicates, though it does not generally prove¹, that quadruplets are more sensitive to reactivity than triplets, triplets are more sensitive than doublets, and doublets are more sensitive than singlets. This experiment led Mihalcz to suggest higher order counting statistics as a dissertation topic. So, in a way, it is the single most important contribution to the present work.

1.1 Review of Previous Developments in Source-Driven Measurements and High Order Statistics

Neither source-driven subcritical measurements nor higher order statistical analysis is a new field of research. Their union was probably an inevitable step in the evolution of

¹ Of course, not every group of four counts is a true quadruplet. Likewise groups of three counts and two counts are not always real triplets and doublets. A means to account for these “accidental” groupings is described in the following chapter.

both techniques.

Present source-driven noise-analysis measurements trace their origins to Rossi- α and pulsed neutron experiments with fissile materials performed at Los Alamos National Laboratory in the early 1940's². In 1943, Bruno Rossi suggested the method that later came to be named after him, but the early theoretical work describing the method was developed by Richard Feynman. The Rossi- α method measures the distribution of counts subsequent to an initial count in another (or the same) detector as a function of the time-delay between the initial and subsequent counts³. Consequently, modern-day auto- and cross-correlation measurements derived from neutron counting are equivalent to measurements of a Rossi- α distribution [Dav, Fey, Orn].

Some of the earliest pulsed neutron measurements with fissile materials were performed at Los Alamos during the 1940's. This method measures the distribution of detector counts subsequent to an initial burst of ($\sim 10^6$ or more) neutrons injected into the fissile system by a pulsed neutron generator. Hence the distribution of detector counts as a function of delay following the initial source pulse measures the cross-correlation between the source and detector. However, the neutron generators employed were accelerators that were typically extremely large [Cor, Kee].

In 1968, John Mihalczko at Oak Ridge National Laboratory developed the randomly

² At the time, it was referred to as the Los Alamos Scientific Laboratory.

³ Single-detector Rossi- α was the measurement first suggested by Bruno Rossi; two-detector Rossi- α measurements were developed later to avoid problems due to the dead-time of a single detector.

pulsed neutron measurement to produce a more robust and portable realization of pulsed neutron measurements. By plating californium-252 on a platinum foil on one electrode of a small (~ 1 "-diameter $\times$ 1"-long) ionization chamber, Mihalcz and collaborators J. Bigelow and F. Gillespie produced a portable, observable source of neutrons. Consequently, detection events could be correlated with random source fissions detected in the ionization chamber to measure a distribution of detection events following ^{252}Cf -fission that was equivalent to the distribution measured by an ordinary pulsed neutron measurement [Mih1,7]. Furthermore, the ^{252}Cf -source-driven measurement evolved to simultaneously correlate detection events between detectors and within a single detector, thereby producing a measurement that also encompassed the Rossi- α method [Mih2].

Unfortunately, the ^{252}Cf -source-driven method was limited by the throughput of available pulse mode electronics, such that extremely large ^{252}Cf sources could not be used. However, the development of noise-analysis techniques allowed the acquisition of equivalent distributions in the frequency domain, which could be done with the source and detectors operating in current mode. Frisch and Littler performed the first measurement of a detector autospectrum, the frequency-domain counterpart to the detector autocorrelation [Fri]. Later Seifritz, et al., and Nomura, et al. performed independent measurements of detector cross-spectra, the frequency-domain counterparts to detector cross-correlations [Sei, Nom]. Subsequently, ^{252}Cf -source-driven measurements were supplemented with noise-analysis techniques, which were later significantly enhanced by the development of modern digital signal processing techniques [Mih3,8].

Currently the method encompasses Rossi- α , randomly pulsed neutron, and noise-

analysis methods to measure cross-correlations and cross-spectra between each pair of detectors and between the source and each detector and to measure autocorrelations and autospectra from the source and each detector. One of the primary advancements made in the ^{252}Cf -source-driven method was the introduction of the spectral ratio by Mihalcz and Paré. This particular signature was observed to be extremely robust – not only was it independent of intrinsic detection efficiency and ^{252}Cf -source strength, but it was also observed to withstand a number of other perturbations without change, including equipment malfunctions that were deliberately induced for demonstrative purposes. The primary utility of the spectral ratio was in its application to the interpretation of subcritical measurements – the signature was observed to vary linearly with reactivity over a broad range of subcritical reactivity for a wide variety of systems [Mih3,4,5].

Physical and statistical models used to interpret these kinds of measurements have also grown more sophisticated in recent years. In particular, the work begun by Pál and Bell to produce a purely stochastic model of neutron transport from first principles was completed by Muñoz-Cobo, Perez, and Verdú and applied to produce the most comprehensive models of detector cross-correlations and cross-spectra to date [Bel, Pál, Muñ1]. Recently Muñoz-Cobo and Verdú have added the representation of delayed neutrons to this theory [Muñ2]. Concurrent with the development of the present research, Perez developed stochastic models of the correlation between three detectors, and the corresponding spectral signature, but he has yet to publish them [Per]. Each of the preceding stochastic models was developed to predict moments (beyond the first) of the neutron distribution in phase space.

Calculational models for these kinds of measurements have also evolved substantially

if more slowly – the primary impediment to the implementation of calculational methods is and has always been the speed and storage capacity of computing resources. Monte Carlo simulations of neutron and gamma transport have been around since the late 1950's [Goe, Kal]. In the late 1980's, Petrie developed a modification of the criticality code KENO-Va to calculate the temporal distribution of counts in a detector following spontaneous fissions in ^{252}Cf [Pet]. In 1991, KENO-Va was modified by Ficaro to produce KENO-NR, a code to simulate ^{252}Cf source-driven neutron noise-analysis measurements [Fic1,2]. To eliminate the necessity of group cross-section representations used in KENO-NR and to add the capability to simulate gamma transport in source-driven measurements, in 1995 Valentine modified MCNPTM-4A to produce MCNP-DSP, a completely analog Monte Carlo simulation of source-driven noise-analysis measurements using point energy cross-section representations and coupled neutron/photon transport⁴ [Val1-3]. Each of these modifications to standard Monte Carlo transport codes was necessary to correctly calculate quantities related to second and higher moments of the neutron and gamma distribution in phase space.

Furthermore, a few years before the randomly pulsed neutron measurement was developed, a handful of researchers, among them Edelmann and Hansen, developed models of third- and higher order correlation signatures, i.e., those that correlate several subsequent detection events to a single initial detection event as a function of delay between events [Ede, Han]. However, probably due to limitations in instrumentation and computing resources, these signatures were never measured, and little else has been done with them until recently.

⁴ MCNP is a trademark of the regents of the University of California, Los Alamos National Laboratory [Brie].

There is one notable exception, however. Recent years have witnessed the application of higher order multiplicities in reactor physics and fissile system identification. This method correlates an arbitrary number (e.g., one, two, three, or more) of subsequent detection events with a single initial event via coincidence counting techniques, but it does not record the temporal distribution of events [Ens, Hag, Rob]. Furthermore, both the early exploration of higher order temporal distributions and the more recent application of higher order multiplicity counting confined their analyses to univariate signatures, i.e., to signatures involving only a single detector. The present work concerns multivariate higher order signatures acquired from source-driven measurements.

Higher order statistical analyses were not originally developed for application in nuclear counting measurements. References indicate that the earliest work in this field began around the turn of the century, though only a little attention was paid to the subject until the 1950's [Bend1, Bri1,2, Loh, Mats, Ros1,2]. The theory of univariate high order statistical analysis is in fact now quite mature [Men]. Most of the recent work in this field has focused upon parametric, as opposed to conventional, estimation of high order spectra (so called polyspectra) [Men, Nik, Noo, Rag], and these developments have been applied to study numerous nonlinear processes including, but not limited to, geophysical phenomena [Hin], turbulence [Lii1,2], and ocean waves [Has]. The commercial software package MATLAB even offers a higher order analysis toolbox now [MATL].

In retrospect, the application of multivariate higher order analyses to nuclear counting statistics seems to be the next logical step in the evolution of source-driven measurements. In fact it was probably inevitable.

1.2 Contributions of the Present Research

This work resulted in some practical, tangible contributions to the field of source-driven subcritical measurements. The present research prescribes an elementary probabilistic methodology that can be extended to develop theoretical models of the temporal distribution of detection events correlated between an arbitrary number of detectors. This methodology is used to develop models of several multivariate higher order temporal covariance signatures. In particular, the “bicovariance” between a neutron source driving a subcritical fissile system and two unique neutron detectors observing the system’s response is modeled. Furthermore, the “tricovariance” between a source and three detectors is modeled, and from this the bicovariance between three unique detectors is derived. The corresponding spectral signatures over the Laplace domain are derived, and these are used to demonstrate the sensitivity of the higher order signatures to changes in reactivity. Each of these developments is new. Finally, a set of new higher order spectral ratios that are independent of the strength of the introduced source, the efficiency of the detectors observing the fissile system, and the strength of the system’s inherent source are derived. A general methodology for constructing similar higher order spectral ratios is prescribed.

Conventional procedures for estimating higher order temporal and spectral signatures from time-discrete processes are described. These procedures are also applicable to discretely sampled time-continuous processes. The detail of the procedures’ description given here is such that they may be implemented with little effort. These procedures are now implemented in the Monte Carlo simulation code, MCNP-DSP, so calculational studies of

the higher order signatures from a variety of subcritical assemblies are now possible. This research presents the first calculational study of the sensitivity of these higher order signatures to reactivity.

A new algorithm to estimate these higher order signatures from binary signals acquired during measurements is described and implemented in a high speed data acquisition system, so the tools to experiment with these methods are now available. The first measurement of multivariate third-order temporal and spectral signatures for a subcritical fissile assembly is presented.

On a less corporeal level, this work represents a unification of several related techniques including coincidence counting, correlation analysis, noise-analysis, and higher order statistical analysis. All these methods are aspects of the same entity.

1.3 Organization of the Dissertation

Chapter 2 develops an elementary probabilistic framework to assist in the interpretation of multivariate high order statistical signatures acquired from source-driven measurements of subcritical fissile systems. This framework is developed from the simplest model of fission-chain propagation through a subcritical assembly – the point reactor model. However, a number of general conclusions that transcend the limitations of the point reactor model are drawn, and these make it possible to interpret some characteristics of high order signatures acquired from systems that do not behave as a point. The description of the theoretical framework begins by developing models for lower order signatures that have been

previously studied. Readers familiar with earlier research in the field of neutron noise-analysis will note that the lower order signatures derived in Chapter 2 are identical to those predicted by earlier models employed in neutron noise-analysis [Benn, Coh1,2, Cou, deH1,2, Dif, Fic1, Ede, Han, Moo, Thi, Uhr]. Subsequently, a number of new multivariate higher order signatures are derived. Some general conclusions are drawn regarding the processes that contribute to these higher order signatures and the relative importance of each contributing process. In particular, these conclusions demonstrate the fundamental reasons for the sensitivity of higher order signatures. The second chapter concludes by developing a set of new spectral ratios that are independent of introduced source strength, detection efficiency, and inherent source strength. A general methodology for developing other similar signatures is prescribed.

Chapter 3 acknowledges the limitations of theoretical models and develops a set of generally applicable numerical methods to estimate multivariate higher order signatures using signals acquired from source-driven measurements or from Monte Carlo simulations of source-driven measurements. The estimation procedures developed are applicable to sequences representing any stationary time-discrete process or discretely sampled time-continuous process, so their specialization to source-driven measurements is deferred until conventional and Fourier-based estimators for first- through fourth-order multivariate temporal and spectral signatures have been described. Then the implementation of these procedures as a modification to the Monte Carlo code MCNP-DSP is outlined. Signatures estimated from an MCNP-DSP calculational model of a 93.15%-enriched uranium-metal sphere are presented, and the features of the higher order signatures are described in detail.

Subsequently, a calculational study of these signatures' sensitivity to reactivity is described. The sensitivity study is based upon statistical signatures estimated from MCNP-DSP calculational models of five uranium-metal spheres of enrichments between 75% and 100%. The results of this study support the general conclusions of the preceding theoretical developments. Chapter 3 concludes by acknowledging that calculational models cannot account for all the factors affecting the behavior of real subcritical systems – the estimation procedures must be implemented in a measurement system and signatures acquired from measurements must be studied.

The fourth chapter presents algorithms optimized for binary signals acquired from source-driven measurements, and the implementation of these algorithms in a high speed data acquisition system designed for source-driven measurements is described. Subsequently, a measurement performed on a 93%-enriched uranium-metal annular casting is described, and the signatures acquired from this measurement are presented. Unlike the preceding theoretical and calculational developments, which only address the neutron response of subcritical fissile systems, the measurement characterizes both the neutron and gamma response of the casting. The features of these signatures allow a number of qualitative observations to be made regarding the response of the casting to the ^{252}Cf source employed in the measurement. In particular, it is noted that the higher order signatures exhibit features that reveal information hidden within the lower order signatures. The measurement primarily serves to demonstrate the practicality and potential utility of these techniques. Further experimentation with higher order signatures is clearly necessary.

The fifth chapter of this dissertation summarizes the most important observations

made regarding the preceding theoretical and calculational models and the measurement of the single uranium casting. It also acknowledges that the present work merely introduces the application of high order statistical analyses to the field of subcritical measurements. Further study of these techniques is recommended so that their utility can be more comprehensively evaluated. General conclusions regarding this research are drawn in the final chapter, and some recommendations regarding future applications of these techniques are made.

2 High Order Statistics of Neutron Counting in Source-Driven Measurements

The fundamental unit of many nuclear measurements is the count. This chapter focuses on neutron counting statistics derived from source-driven measurements performed on subcritical fissile assemblies, or systems. Such measurements use an observable source of neutrons, i.e., a source which is itself a detector capable of registering the time of individual neutron emission events. The response of the subcritical system to the introduced source is observed by an array of neutron-sensitive detectors capable of registering the time of individual detection events. The neutron counting statistics of the system's observed response comprise a set of "signatures" that characterize that fissile assembly. It is prudent to note that this characterization is not necessarily unique, but that in general neutron counting statistical signatures have enjoyed wide success at distinguishing between similar but different fissile assemblies [Mih6]. This study emphasizes the signatures' sensitivity to reactivity, which is of course affected by numerous features of a given fissile system including, but not limited to, its fissile mass, geometry, reflection, moderation, and interaction with other fissile assemblies. There are a number of ways to count neutrons. This study describes a few, some of which are new.

Unfortunately, words are needed to describe these ways of counting neutrons. Since a portion of this study reviews standard neutron counting techniques, some of these words will be familiar. However, because some of the neutron counting statistics are new, so are some of the words used to describe them.

The concept of a count is assumed to be understood at the outset. Figure 2.1 depicts a detection event, a count, as an open circle; the progression of time is indicated by vertical tick marks on a horizontal axis, a “time-line.” For now, the detection event is assumed to be singular, i.e., it exists at only one point in the time continuum, as opposed to being distributed over time. Of course, a real count is measured from an electrical pulse of finite duration, so the actual time of the detection event is somewhat uncertain. However, this is a secondary concern, a detail that is addressed later this chapter. It’s intuitively obvious that, in a static measurement, the number of counts that can be expected during a given period is determined by the average count rate. That is, if $x(t)$ is the measured number of counts per unit time t , then the mean count rate,

$$\bar{x} = E[x(t)] , \quad (2.1)$$

is the expected number of events per unit time. For a stationary process, the mean is independent of time.

Count rates are useful because they give some indication of the activity of a fissile system. However, a count is a marginal quantity; by itself it gives no indication of the processes that caused the count to occur. Presumably it is *correlated* with other counts, since some of these others may have occurred due to related processes.

Figure 2.2 depicts two counts that may or may not be correlated. If pairs of counts consistently appear at the same time, that is, if they are consistently *coincident*, then they are correlated. Of course, perfect correlation between pairs of events is unusual, unless the two are not unique, i.e., if in fact they are always the same detection event twice. So the fraction of pairs that are coincident indicates the degree to which the detection events are correlated.

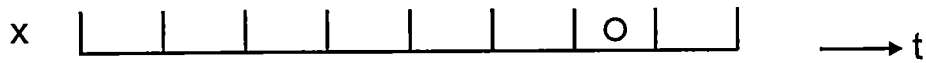


Figure 2.1 A count: the open circle represents a detection event $x(t)$; the passage of time t is indicated by the tick marks on the horizontal axis

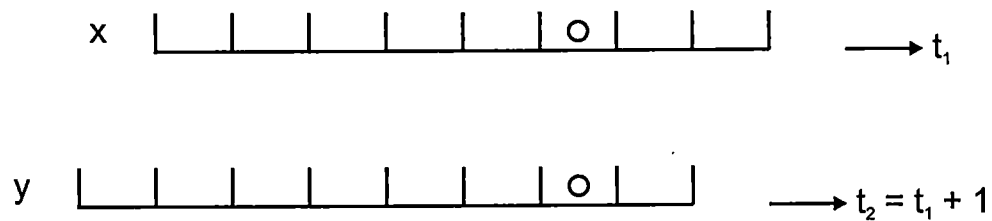
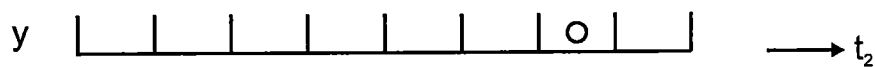
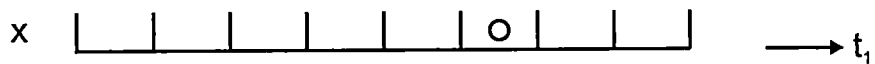


Figure 2.2 Two counts $x(t_1)$ and $y(t_2)$ made coincident by delaying one count

Furthermore, the delay between correlated counts is important. Suppose that the two counts depicted in the figure *are* related. But, as the lower portion of the figure illustrates, they are only coincident when the first event is delayed. So it's important to determine the coincidence rate between events as a function of the delay between them. For stationary processes, this coincidence rate is independent of when the individual counts occurred and only depends upon the delay between them. This quantity,

$$R_{xy}(\tau) = E[x(t_1)y(t_2)] \Big|_{\tau=t_2-t_1}, \quad (2.2)$$

is popularly termed the correlation between x and y , although this is a misnomer. It is in fact the coincidence rate between x and y . Some coincident pairs will be related, and some will not. In general, the frequency of pairs of unrelated counts will be uniformly distributed over the delay between events. In other words, when two events are unrelated, neither event “cares” when the other occurred. This kind of coincidence between unrelated events is termed “accidental” coincidence. The “real” coincidence rate, i.e., the degree of true correlation between x and y , is the covariance

$$C_{xy}(\tau) = R_{xy}(\tau) - \bar{x}\bar{y}, \quad (2.3)$$

where $\bar{x}\bar{y}$ is the accidental coincidence rate. Another expression for the covariance is

$$C_{xy}(\tau) = E\left[\left(x(t_1) - \bar{x}\right)\left(y(t_2) - \bar{y}\right)\right] \Big|_{\tau=t_2-t_1}, \quad (2.4)$$

which accounts for the frequency of uncorrelated coincident events.

Coincidence counting and correlation analysis are one in the same. This technique is the basis for innumerable neutron counting measurements, and it is the original concept around which source-driven neutron noise measurements were developed. These measurements make inferences regarding the state of the system observed by analyzing the

degree of correlation between the source and each detector, between pairs of detectors, and between each detector paired with itself. The basic unit of this measurement is the coincident count rate.

But consider that a count may be related to more than just one other count. Figure 2.3 depicts three counts which may or may not be related to each other. When the counts are shifted as shown in the lower portion of the figure, two counts coincide with the remaining count. That is, the three counts are *bicoincident*. If events are frequently bicoincident this way, then they are probably related. This rate of bicoincidence between x , y , and z ,

$$R_{xyz}(\tau_1, \tau_2) = E[x(t_1)y(t_2)z(t_3)] \Big|_{\substack{\tau_1 = t_2 - t_1 \\ \tau_2 = t_3 - t_1}}, \quad (2.5)$$

depends upon the delays τ_1 and τ_2 between x , y , and z . Consistent with the use of the word “correlation” to denote the coincidence rate between pairs of events, the word “bicorrelation” will be used to denote the bicoincidence rate between triplets of events, even though this perpetuates the misuse of the word correlation. Of course, some bicoincident triplets will be unrelated, i.e., accidental, and others will be truly bicorrelated. Accidentally bicoincident triplets come in two types. The simpler occurs when none of the three events is related to either of the other two. The second type occurs when two of the events are related, but this pair is unrelated to the third event. Hence, the real bicoincidence rate, i.e., the degree of true bicorrelation between x , y , and z is then

$$C_{xyz}(\tau_1, \tau_2) = R_{xyz}(\tau_1, \tau_2) - \bar{x} C_{yz}(\tau_2 - \tau_1) - \bar{y} C_{xz}(\tau_2) - \bar{z} C_{xy}(\tau_1) - \bar{x} \bar{y} \bar{z}, \quad (2.6)$$

which will be called the *bicovariance*. Alternatively,

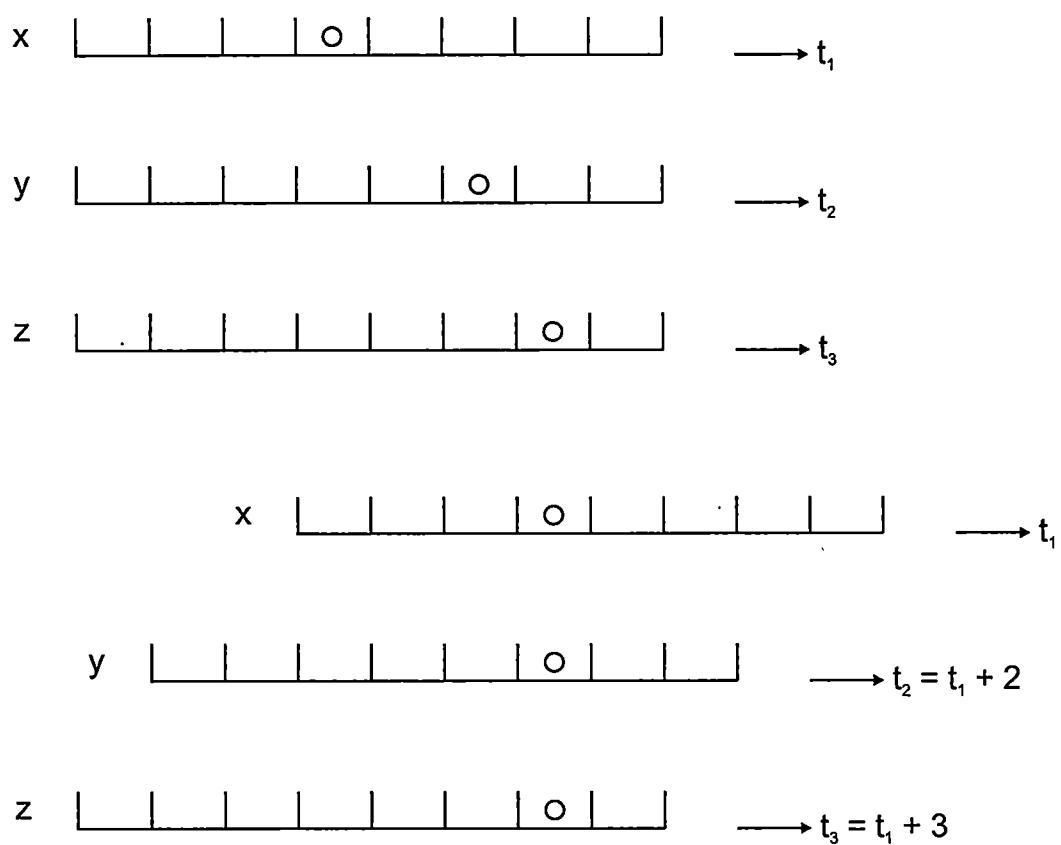


Figure 2.3 Three counts $x(t_1)$, $y(t_2)$, and $z(t_3)$ made bicoincident by delaying two counts

$$C_{xyz}(\tau_1, \tau_2) = E\left[\left(x(t_1) - \bar{x}\right)\left(y(t_2) - \bar{y}\right)\left(z(t_3) - \bar{z}\right)\right] \Bigg|_{\substack{\tau_1 = t_2 - t_1 \\ \tau_2 = t_3 - t_1}} \quad (2.7)$$

accounts for the rate of accidentally bicoincident events.

Furthermore, if four counts are frequently *tricoincident*, as shown in the lower portion of Fig. 2.4, then these quadruplets are *tricorrelated*. The tricoincidence rate

$$R_{wxyz}(\tau_1, \tau_2, \tau_3) = E[w(t_0)x(t_1)y(t_2)z(t_3)] \Bigg|_{\substack{\tau_1 = t_1 - t_0 \\ \tau_2 = t_2 - t_0 \\ \tau_3 = t_3 - t_0}} \quad (2.8)$$

(a.k.a. the “tricorrelation”) is consequently a measure of tricorrelation and accidental tricoincidence. Accidentally tricoincident quadruplets occur when none of the four events is related, when two independent events bicoincide with a correlated doublet, when two correlated doublets accidentally coincide, and when a bicorrelated triplet coincides with an independent event. The remaining real tricoincidence rate is then the tricovariance:

$$\begin{aligned} C_{wxyz}(\tau_1, \tau_2, \tau_3) &= R_{wxyz}(\tau_1, \tau_2, \tau_3) \\ &- \bar{w} C_{xyz}(\tau_2 - \tau_1, \tau_3 - \tau_1) - \bar{x} C_{wyz}(\tau_2, \tau_3) - \bar{y} C_{wxz}(\tau_1, \tau_3) - \bar{z} C_{wxy}(\tau_1, \tau_2) \\ &- C_{wx}(\tau_1) C_{yz}(\tau_3 - \tau_2) - C_{wy}(\tau_2) C_{xz}(\tau_3 - \tau_1) - C_{wz}(\tau_3) C_{xy}(\tau_2 - \tau_1) \\ &- \bar{w} \bar{x} C_{yz}(\tau_3 - \tau_2) - \bar{w} \bar{y} C_{xz}(\tau_3 - \tau_1) - \bar{w} \bar{z} C_{xy}(\tau_2 - \tau_1) \\ &- \bar{x} \bar{y} C_{wz}(\tau_3) - \bar{x} \bar{z} C_{wy}(\tau_2) - \bar{y} \bar{z} C_{wx}(\tau_1) \\ &- \bar{w} \bar{x} \bar{y} \bar{z} . \end{aligned} \quad (2.9)$$

Note that this is a departure from the conventional definition of the word “covariance,” since this ordinarily pertains to a joint central moment; however

$$C_{wxyz}(\tau_1, \tau_2, \tau_3) \neq E\left[\left(w(t_0) - \bar{w}\right)\left(x(t_1) - \bar{x}\right)\left(y(t_2) - \bar{y}\right)\left(z(t_3) - \bar{z}\right)\right] \Bigg|_{\substack{\tau_1 = t_1 - t_0 \\ \tau_2 = t_2 - t_0 \\ \tau_3 = t_3 - t_0}} . \quad (2.10)$$

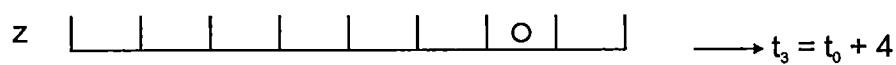
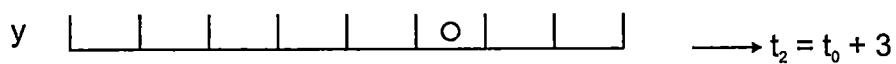
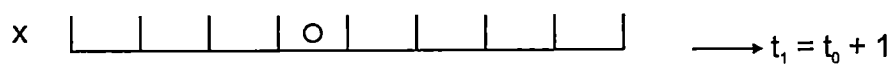
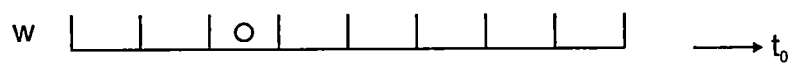
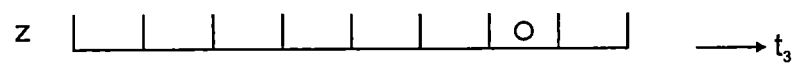
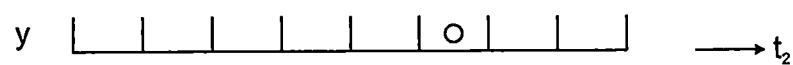
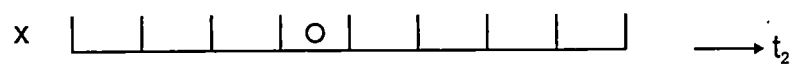
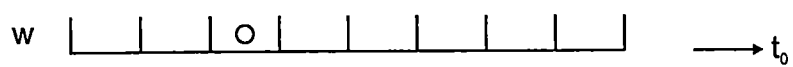


Figure 2.4 Four counts $w(t_0)$, $x(t_1)$, $y(t_2)$, and $z(t_3)$ made tricoincident by delaying three counts

Instead,

$$\begin{aligned}
 C_{wxyz}(\tau_1, \tau_2, \tau_3) &= E \left[\left(w(t_0) - \bar{w} \right) \left(x(t_1) - \bar{x} \right) \left(y(t_2) - \bar{y} \right) \left(z(t_3) - \bar{z} \right) \right] \bigg|_{\substack{\tau_1 = t_1 - t_0 \\ \tau_2 = t_2 - t_0 \\ \tau_3 = t_3 - t_0}} \\
 &- C_{wx}(\tau_1) C_{yz}(\tau_3 - \tau_2) - C_{wy}(\tau_2) C_{xz}(\tau_3 - \tau_1) - C_{wz}(\tau_3) C_{xy}(\tau_2 - \tau_1),
 \end{aligned} \tag{2.11}$$

because the tricovariance indicates the degree of true tricorrelation. The fourth central moment does not account for the frequency of accidentally coincident doublets.

Obviously this progression could continue. However, this discourse will only address statistics up to and including fourth order. This is sufficient to illustrate the behavior and demonstrate the utility of high order signatures acquired from source-driven measurements.

2.1 Statistics of Observable Californium-252 Sources

Californium-252 is a spontaneously fissioning isotope that is commonly employed as the source for neutron noise measurements on fissile systems, although in the past alternative sources such as deuteron-triton (D-T) 14 MeV neutron generators have been applied in related pulsed neutron experiments [Cor, Kee]. In these neutron noise applications, the ^{252}Cf is plated on a platinum foil in a compact parallel plate ionization chamber like that shown in Figure 2.5 [Mih1,3,5,8]. The chamber is typically operated in pulse mode so that individual source fissions are observable.

Californium-252 also decays by alpha emission at a rate roughly 31 times greater than the rate of spontaneous fission as noted in Table 2.1. However, since the average energy deposited in the ionization chamber fill gas by alpha particles is substantially smaller than

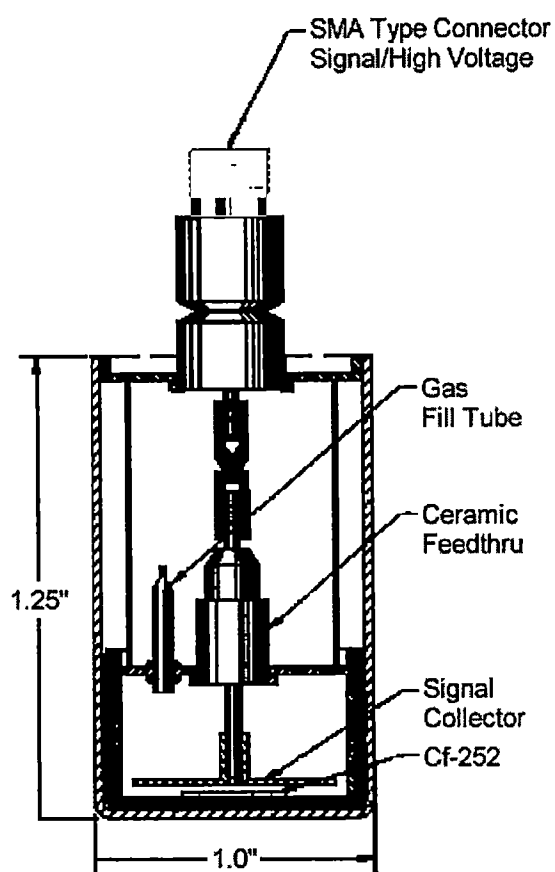


Figure 2.5 Schematic of a typical Californium-252 parallel plate ionization chamber

Table 2.1 Some properties of Californium-252

spontaneous fission specific activity ⁽¹⁾	$6.14 \times 10^5 / \text{s} \cdot \mu\text{g}$
alpha-decay specific activity ⁽¹⁾	$1.92 \times 10^7 / \text{s} \cdot \mu\text{g}$
most probable light fission fragment energy ⁽²⁾	$\sim 105 \text{ MeV}$
most probable heavy fission fragment energy ⁽²⁾	$\sim 80 \text{ MeV}$
average alpha energy ⁽¹⁾	6.11 MeV
average energy of prompt neutrons from spontaneous fission ⁽¹⁾	2.13 MeV
average number of prompt neutrons from spontaneous fission ⁽³⁾ – $\bar{\nu}_0$	$3.77 / \text{fission}$
second reduced moment of spontaneous fission prompt neutron distribution ⁽³⁾ – $\bar{\nu}_0(\bar{\nu}_0 - 1)$	$12.04 / \text{fission}$
third reduced moment of spontaneous fission prompt neutron distribution ⁽³⁾ – $\bar{\nu}_0(\bar{\nu}_0 - 1)(\bar{\nu}_0 - 2)$	$31.96 / \text{fission}$

⁽¹⁾ Taken from [Rei].

⁽²⁾ Estimated from distribution plotted in [Wag].

⁽³⁾ Calculated from Spencer distribution [Spe, Val1,4].

the average energy deposited by fission fragments, it is possible to discriminate between alpha- decay and spontaneous fission on the basis of pulse height. Consequently, ^{252}Cf ionization chamber detection electronics may be set to ignore nearly all pulses corresponding to alpha-decay events while still registering nearly all pulses corresponding to spontaneous fission events.

In typical ^{252}Cf ionization chambers, the plating is sufficiently thin that fission fragments have a high probability of escaping the deposit to ionize the chamber fill gas.

Consequently, fission detection efficiencies approaching 100% have been observed when the ionization chamber detection electronics are properly set. However, it should be noted that degradation in detection efficiency becomes noticeable when the deposit mass exceeds 5 μg ^{252}Cf -mass due to pileup of α -pulses and limitations in the throughput of pulse mode electronics.

Because the ^{252}Cf deposit is so thin, fission neutrons also have a high probability of escaping the plating without interaction, so induced fission events in the deposit are rare enough to be negligible. Therefore, each spontaneous fission event is independent of any other fission event, so the observed ^{252}Cf fission rate may be treated as a white noise process.

Now recognize that the distribution of the number of spontaneous fissions occurring within a given period is binomial:

$$p_n = \binom{N}{n} p^n (1 - p)^{N-n} . \quad (2.12)$$

In this context, N represents the number of trials, i.e., the number of ^{252}Cf nuclei since, conceptually, they are all “trying” to fission. The number of successful trials, i.e., the number of fissions that actually occur, is denoted by n , and p denotes the probability per trial of a success, i.e., the probability that an individual nucleus will fission. However, note that, in a source a few micrograms in size, at any given instant there are on the order of 10^{15} ^{252}Cf nuclei “trying” to fission. Yet only on the order of 10^6 nuclei successfully fission each second. Consequently, although the distribution of fissions occurring in a given period is binomial, the number of trials is very large while the probability of success per trial is very low. Under these conditions, the binomial distribution approaches the Poisson distribution:

$$p_n = \frac{\bar{n}^n}{n!} e^{-\bar{n}}, \quad (2.13)$$

where $\bar{n}$ is the expected number of fissions and n is again the actual number of fissions observed. Hence the ^{252}Cf spontaneous fission rate appears macroscopically as a Poisson-distributed white noise process, which is completely characterized by a single statistic, the mean fission rate.

Recall that the time between sequential nuclear decay events is essentially exponentially distributed:

$$P(t_0 > T) = e^{-T/T_0}. \quad (2.14)$$

Assuming that a nuclear decay event occurs at $t_0 = 0$, the probability that another nuclear decay will occur more than T later is e^{-T/T_0} , where T_0 is the average time between sequential decays. For ^{252}Cf spontaneous fission,

$$T_0 = \frac{1}{F_0}, \quad (2.15)$$

where F_0 is the average fission rate. Note that if spontaneous fission is approximated as a time-continuous process¹, the average fission rate is the expected number of fissions ($\bar{n}$) during the incremental period dt_0 :

$$F_0 = \frac{d\bar{n}}{dt_0} \quad \therefore \quad F_0 dt_0 = d\bar{n}. \quad (2.16)$$

Of course, F_0 is not constant since each fission event leaves one fewer nuclei to fission later.

¹ Of course, nuclear processes like spontaneous fission are time-discrete, not time-continuous.

However, the duration of a typical measurement is so short compared to the spontaneous fission half-life of ^{252}Cf that the decay of the source fission rate during the measurement is usually negligible.

Because the probability that the next spontaneous fission occurs within T seconds is

$$P(t_0 \leq T) = 1 - e^{-F_0 T}, \quad (2.17)$$

the probability that the next fission occurs in dt_0 about t_0 is

$$p(t_0) dt_0 = F_0 e^{-F_0 t_0} dt_0. \quad (2.18)$$

Now observe that

$$e^{-F_0 t_0} = \text{probability that the next fission event will not occur before } t_0, \quad (2.19)$$

in accordance with (2.14), such that

$$F_0 dt_0 = \text{probability that a fission event will occur in } dt_0 \text{ about } t_0. \quad (2.20)$$

Consequently, if $w(t_0)$ denotes a fission event w at time t_0 , without regard to the time of preceding or subsequent fission events, the probability that w occurs in dt_0 about t_0 is

$$p_w(t_0) dt_0 = F_0 dt_0. \quad (2.21)$$

Therefore, the probability of observing a spontaneous fission event within a given period is uniform.

Consequently, the first moment of the observed ^{252}Cf spontaneous fission rate is simply the average source fission rate:

$$\bar{w} = E[w(t_0)] = F_0. \quad (2.22)$$

All the higher moments are determined by this single statistic.

Consider that the joint probability of registering two fissions at t_0 and t'_0 ,

respectively, is

$$p_{ww}(t_0, t'_0) dt_0 dt'_0 = p_w(t_0) dt_0 \delta(t'_0 - t_0) dt'_0 + p_w(t_0) dt_0 p_w(t'_0) dt'_0 . \quad (2.23)$$

Figure 2.6 illustrates this: the first term in the autocorrelation arises when coincident events are identical; the second term arises due to accidental coincidence between fission events that are not identical and thus independent. Consequently, the second moment, or autocorrelation, of the observed ^{252}Cf fission rate is

$$R_{ww}(\tau) = E[w(t_0)w(t'_0)] \Big|_{\tau=t'_0-t_0} = C_{ww}(\tau) + \bar{w}^2 , \quad (2.24)$$

and since no two unique fission events are correlated, the source autocovariance is simply

$$C_{ww}(\tau) = \bar{w} \delta(\tau) . \quad (2.25)$$

Now consider this: as illustrated in Fig. 2.7, the joint probability of registering three bicoincident fission events at t_0 , t'_0 , and t''_0 , respectively, has three components corresponding to the following disjoint occurrences:

- (1) All three events are identical;
- (2) Two events are identical while the third accidentally coincides with this doublet;
- (3) The three events are accidentally bicoincident.

Consequently, the joint probability of registering three fission events at t_0 , t'_0 , and t''_0 is

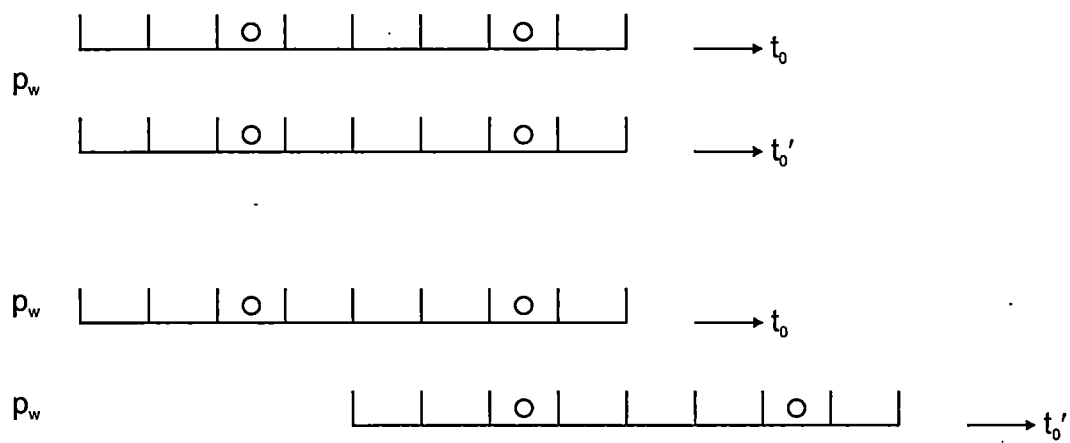


Figure 2.6 Contributions to the source autocorrelation due to real and accidental coincidence

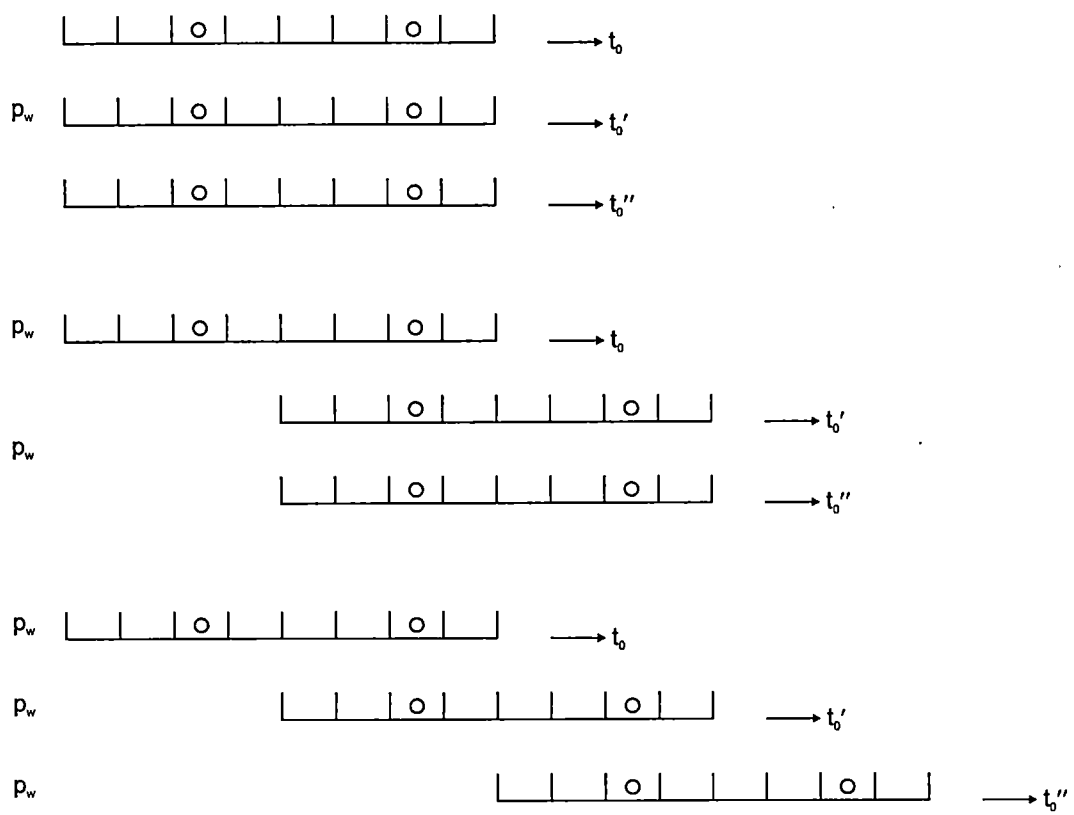


Figure 2.7 Contributions to the source autocorrelation due to real and accidental bicoincidence

$$\begin{aligned}
p_{www}(t_0, t_0', t_0'') dt_0 dt_0' dt_0'' &= p_w(t_0) dt_0 \delta(t_0' - t_0) dt_0' \delta(t_0'' - t_0) dt_0'' + \\
& p_w(t_0) dt_0 p_w(t_0') dt_0' \delta(t_0'' - t_0') dt_0'' + \\
& p_w(t_0') dt_0' p_w(t_0) dt_0 \delta(t_0'' - t_0) dt_0'' + \\
& p_w(t_0'') dt_0'' p_w(t_0) dt_0 \delta(t_0' - t_0) dt_0' + \\
& p_w(t_0) dt_0 p_w(t_0') dt_0' p_w(t_0'') dt_0'' .
\end{aligned} \tag{2.26}$$

Recognize that the first term is the probability of occurrence (1) above, the middle three terms correspond to the three possible ways that (2) may occur, and the last term reflects the probability of occurrence (3). Therefore, the third moment, or autobicorrelation, of the observed ^{252}Cf fission rate is

$$\begin{aligned}
R_{www}(\tau_1, \tau_2) &= E[w(t_0) w(t_0') w(t_0'')] \Bigg|_{\substack{\tau_1 = t_0' - t_0 \\ \tau_2 = t_0'' - t_0}} \\
&= C_{www}(\tau_1, \tau_2) + \bar{w} \left(C_{ww}(\tau_2 - \tau_1) + C_{ww}(\tau_2) + C_{ww}(\tau_1) \right) + \bar{w}^3 .
\end{aligned} \tag{2.27}$$

Each fission event is only bicorrelated with itself, so the source autobicovariance is just

$$C_{www}(\tau_1, \tau_2) = \bar{w} \delta(\tau_1) \delta(\tau_2) . \tag{2.28}$$

Finally, consider that there are five disjoint conditions under which four tricoincident fission events may be respectively registered at t_0 , t_0' , t_0'' , and t_0''' :

- (1) All four events are identical and hence tricorrelated;
- (2) Three events are identical while the fourth accidentally coincides with this triplet;
- (3) Two doublets of identical events coincide accidentally.
- (4) Two events are identical while the other two are accidentally bicoincident

with this doublet;

- (5) The four events are accidentally tricoincident.

These five possibilities are illustrated in Fig. 2.8. Consequently, the joint probability of registering four events at t_0 , t_0' , t_0'' , and t_0''' is simply

$$\begin{aligned}
 p_{www}(t_0, t_0', t_0'', t_0''') &= p_{www}^{(1)}(t_0, t_0', t_0'', t_0''') + \\
 & p_{www}^{(2)}(t_0, t_0', t_0'', t_0''') + \\
 & p_{www}^{(3)}(t_0, t_0', t_0'', t_0''') + \\
 & p_{www}^{(4)}(t_0, t_0', t_0'', t_0''') + \\
 & p_{www}^{(5)}(t_0, t_0', t_0'', t_0''') ,
 \end{aligned} \tag{2.29}$$

where each of the five components directly corresponds to one of the five conditions for tricoincident events. The probability of four identical events is

$$\begin{aligned}
 p_{www}^{(1)}(t_0, t_0', t_0'', t_0''') dt_0 dt_0' dt_0'' dt_0''' &= \\
 p_w(t_0) dt_0 \delta(t_0' - t_0) dt_0' \delta(t_0'' - t_0) dt_0'' \delta(t_0''' - t_0) dt_0''' .
 \end{aligned} \tag{2.30}$$

The probability of three identical events accidentally coincident with a fourth is

$$\begin{aligned}
 p_{www}^{(2)}(t_0, t_0', t_0'', t_0''') dt_0 dt_0' dt_0'' dt_0''' &= \\
 p_w(t_0) dt_0 p_w(t_0') dt_0' \delta(t_0'' - t_0') dt_0'' \delta(t_0''' - t_0') dt_0''' + \\
 p_w(t_0') dt_0' p_w(t_0) dt_0 \delta(t_0'' - t_0) dt_0'' \delta(t_0''' - t_0) dt_0''' + \\
 p_w(t_0'') dt_0'' p_w(t_0) dt_0 \delta(t_0' - t_0) dt_0' \delta(t_0''' - t_0) dt_0''' + \\
 p_w(t_0''') dt_0''' p_w(t_0) dt_0 \delta(t_0' - t_0) dt_0' \delta(t_0'' - t_0) dt_0'' .
 \end{aligned} \tag{2.31}$$

The probability of two doublets of identical events that accidentally coincide is

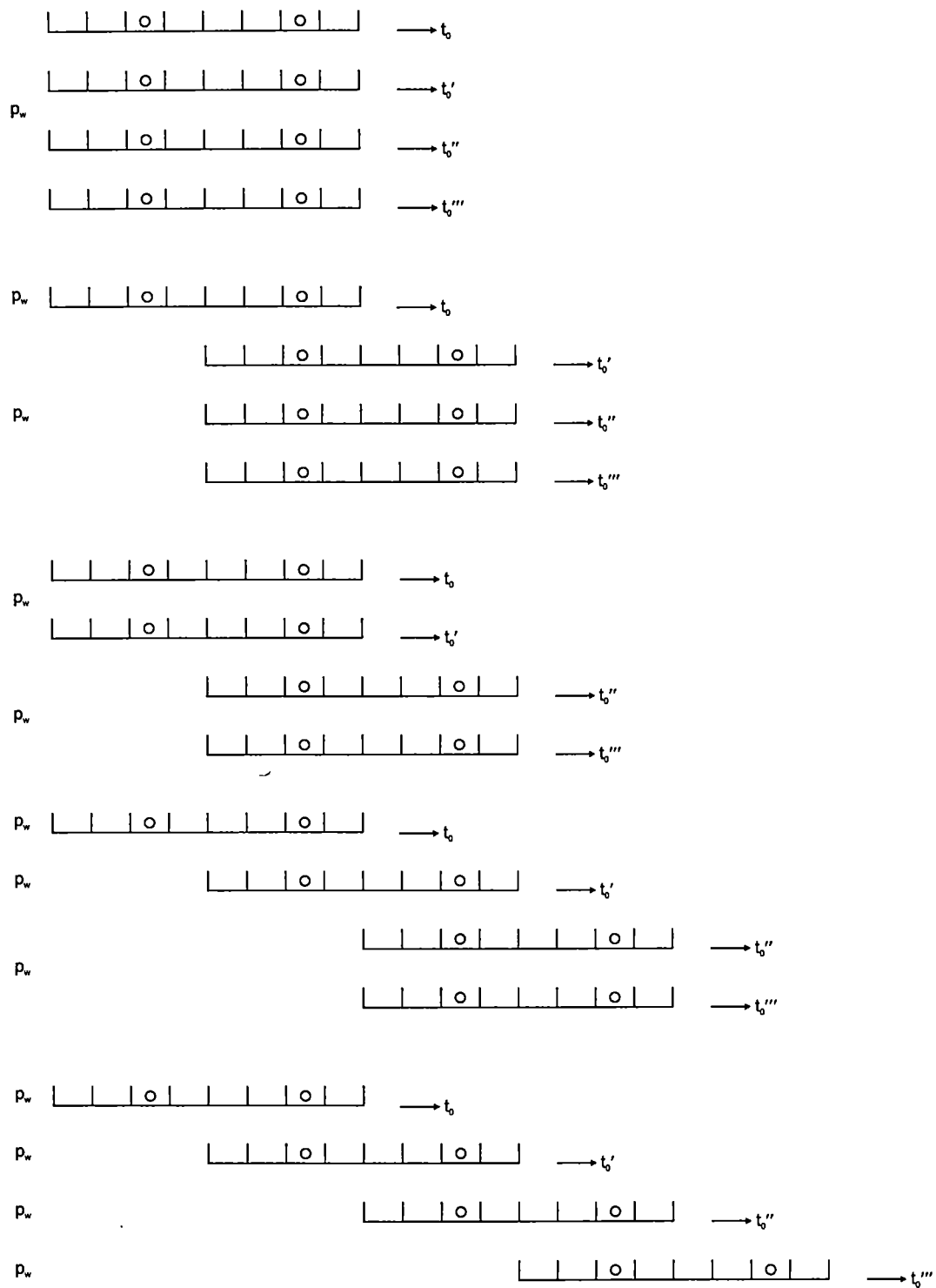


Figure 2.8 Contributions to the source autotricorrelation due to real and accidental tricoincidence

$$\begin{aligned}
P_{wwww}^{(3)}(t_0, t_0', t_0'', t_0''') &= p_w(t_0) dt_0 \delta(t_0' - t_0) dt_0' p_w(t_0'') dt_0'' \delta(t_0''' - t_0'') dt_0''' + \\
& p_w(t_0) dt_0 \delta(t_0'' - t_0) dt_0'' p_w(t_0') dt_0' \delta(t_0''' - t_0') dt_0''' + \quad (2.32) \\
& p_w(t_0) dt_0 \delta(t_0''' - t_0) dt_0''' p_w(t_0') dt_0' \delta(t_0'' - t_0') dt_0'' .
\end{aligned}$$

The probability of two identical events accidentally bicoincident with two other independent events is

$$\begin{aligned}
P_{wwww}^{(4)}(t_0, t_0', t_0'', t_0''') dt_0 dt_0' dt_0'' dt_0''' &= \\
& p_w(t_0) dt_0 p_w(t_0') dt_0' p_w(t_0'') dt_0'' \delta(t_0''' - t_0'') dt_0''' + \\
& p_w(t_0) dt_0 p_w(t_0'') dt_0'' p_w(t_0') dt_0' \delta(t_0''' - t_0') dt_0''' + \\
& p_w(t_0) dt_0 p_w(t_0''') dt_0''' p_w(t_0') dt_0' \delta(t_0'' - t_0') dt_0'' + \quad (2.33) \\
& p_w(t_0') dt_0' p_w(t_0'') dt_0'' p_w(t_0) dt_0 \delta(t_0''' - t_0) dt_0''' + \\
& p_w(t_0') dt_0' p_w(t_0''') dt_0''' p_w(t_0) dt_0 \delta(t_0'' - t_0) dt_0'' + \\
& p_w(t_0'') dt_0'' p_w(t_0''') dt_0''' p_w(t_0) dt_0 \delta(t_0' - t_0) dt_0' ,
\end{aligned}$$

and the probability of accidentally tricoincident events is

$$P_{wwww}^{(5)}(t_0, t_0', t_0'', t_0''') dt_0 dt_0' dt_0'' dt_0''' = p_w(t_0) dt_0 p_w(t_0') dt_0' p_w(t_0'') dt_0'' p_w(t_0''') dt_0''' . \quad (2.34)$$

Consequently, the source fourth moment, or autotricorrelation, is

$$\begin{aligned}
R_{wwww}(\tau_1, \tau_2, \tau_3) &= E[w(t_0) w(t_0') w(t_0'') w(t_0''')] \Bigg|_{\substack{\tau_1 = t_0' - t_0 \\ \tau_2 = t_0'' - t_0 \\ \tau_3 = t_0''' - t_0}} \\
&= C_{wwww}(\tau_1, \tau_2, \tau_3) \\
&+ \bar{w} \left(C_{www}(\tau_3 - \tau_1, \tau_2 - \tau_1) + C_{www}(\tau_2, \tau_3) + C_{www}(\tau_1, \tau_3) + C_{www}(\tau_1, \tau_2) \right) \quad (2.35) \\
&+ C_{ww}(\tau_1) C_{ww}(\tau_3 - \tau_2) + C_{ww}(\tau_2) C_{ww}(\tau_3 - \tau_1) + C_{ww}(\tau_3) C_{ww}(\tau_2 - \tau_1) \\
&+ \bar{w}^2 \left(C_{ww}(\tau_3 - \tau_2) + C_{ww}(\tau_3 - \tau_1) + C_{ww}(\tau_2 - \tau_1) + \right. \\
&\quad \left. C_{ww}(\tau_3) + C_{ww}(\tau_2) + C_{ww}(\tau_1) \right) \\
&+ \bar{w}^4 .
\end{aligned}$$

Note that all terms but the first are accidental since each fission is tricorrelated only with itself. Hence the source autotricovariance is simply

$$C_{www}(\tau_1, \tau_2, \tau_3) = \bar{w} \delta(\tau_1) \delta(\tau_2) \delta(\tau_3) . \quad (2.36)$$

Furthermore, in general the n^{th} -order covariance is just

$$C_{www \dots w}(\tau_1, \tau_2, \dots, \tau_{n-1}) = \bar{w} \cdot \prod_{m=1}^n \delta(\tau_m) \quad (2.37)$$

because each spontaneous fission event is independent of every other.

So only the first two moments of the observed ^{252}Cf fission rate need be measured. Because it is a Poisson-distributed process, all moments are determined by the first moment, the mean fission rate. A measurement of the source autocovariance confirms that the source ion chamber signal behaves as a white noise process. There is nothing particularly interesting about the third and higher moments. The numerous ways that source neutrons interact with the fissile system to spawn families of fission chain-related detection events are, in contrast, engaging.

2.2 Point Reactor Model of Fission-Chain Propagation

Granted: models of neutron transport that are more sophisticated than the point reactor model exist. In fact, the stochastic transport models developed by Muñoz-Cobo, Perez, and Verdú, which describe the distribution of neutrons in phase space and their interaction with detectors, are certainly the most complete models yet published and would be well-suited to this study [Muñ1,2]. However, the intent of this discourse is not to capture

the minute details and subtle nuances of high order signatures acquired from source-driven measurements. Quite the opposite, this study aims to describe the salient features of these signatures. The general conclusions derived from the following simple description of these signatures will still hold even when the assumptions supporting the point reactor model no longer hold.

The point model possesses no dependence on neutron position or velocity but instead only describes the temporal evolution of the average neutron density [Hen]:

$$\dot{N} + \alpha N = S . \quad (2.38)$$

Above, N and S respectively represent instantaneous neutron population and source neutron emission rate, and α denotes the prompt neutron fission-chain decay constant (or Rossi- α),

$$\alpha = - \frac{\rho_0 - \beta}{\Lambda_0} = - \frac{\rho}{\Lambda} . \quad (2.39)$$

Above, Λ_0 denotes the mean neutron generation time

$$\Lambda_0 = \frac{\ell}{k_0} , \quad (2.40)$$

where k_0 is the effective multiplication factor and ℓ is the mean neutron lifetime. Total reactivity is represented by

$$\rho_0 = \frac{k_0 - 1}{k_0} , \quad (2.41)$$

and β denotes the effective delayed neutron fraction. Consequently, ρ represents prompt reactivity

$$\rho = \frac{k - 1}{k} = \frac{(1 - \beta)k_0 - 1}{(1 - \beta)k_0}, \quad (2.42)$$

since the multiplication factor due to prompt neutrons alone is

$$k = (1 - \beta)k_0, \quad (2.43)$$

and Λ is the prompt neutron generation time,

$$\Lambda = \frac{\ell}{k} = \frac{\ell}{(1 - \beta)k_0}. \quad (2.44)$$

Observe that this model conspicuously neglects the contribution of delayed neutrons to the temporal evolution of the neutron density. Consider that delayed neutron generations reproduce much more slowly than generations of prompt neutrons and exist in smaller numbers than prompt neutrons. Consequently, in many subcritical systems, fission-chains primarily consist of generations of prompt neutrons. However, delayed neutrons do act as an “inherent” source for fission-chain initiation. Because delayed neutrons are emitted by fission products long after the initial fission event, and not during the fission event itself, they typically act as an independent source of fission-chain initiation and so appear uncorrelated with any introduced source. Furthermore, because delayed neutrons have a spectrum that is much “softer” than that of prompt fission neutrons, their relative importance in fission-chain initiation is often disproportionately greater than their fraction of the entire neutron population; this is particularly true in thermal systems. However, the effect of this inherent source is most significant near delayed critical where fission-chains are extremely long-lived; it can usually be neglected for systems that are several dollars or more subcritical. Therefore, the effect of delayed neutrons upon fission-chain propagation is usually only significant in

the long term behavior of transients or in near delayed-critical systems at steady state. Throughout most of the following discussions, only systems at steady state below delayed critical shall be considered.

Observe then that, if source neutron emission is steady at $\bar{S}$, following the introduction of the source at time $t = 0$, the neutron population evolves as

$$N = \frac{\bar{S}}{\alpha} (1 - e^{-\alpha t}) \quad (2.45)$$

under the assumption of zero initial conditions on the neutron population. First assume that S is a spontaneous fission source (e.g., ^{252}Cf) such that the average source neutron emission rate is

$$\bar{S} = \bar{\nu}_0 F_0, \quad (2.46)$$

where $\bar{\nu}_0$ is the mean number of neutrons emitted per source fission, and F_0 is the mean source fission rate. Further assume that no other neutron sources interact with the system and that the inherent source due to delayed neutrons is negligible by comparison to the introduced spontaneous fission source. Now recall that the average system neutron production rate $\bar{\nu}F$ is related to the average neutron population $\bar{N}$ by

$$\frac{\bar{N}}{\Lambda} = \bar{\nu}F, \quad (2.47)$$

where $\bar{\nu}$ is the expected number of neutrons emergent from induced fission, and F is the mean system fission rate. Hence, the steady system neutron production rate is related to the mean source neutron emission rate by

$$\lim_{t \rightarrow \infty} N = \frac{\bar{S}}{\alpha} \quad \therefore \quad \bar{v}F = \frac{\bar{v}_0 F_0}{\alpha \Lambda} . \quad (2.48)$$

However, the steady neutron production rate is of less interest than the system's response to random fluctuations in source neutron emission. So let

$$\begin{aligned} \delta N &= N - \bar{N} , \\ \delta S &= S - \bar{S} , \end{aligned} \quad (2.49)$$

such that δN and δS represent fluctuations in the neutron density and source neutron emission about their respective averages. Subsequently

$$\delta \dot{N} + \alpha \delta N = \delta S , \quad (2.50)$$

so the system's response is

$$\delta N(t) = \int_{-\infty}^t dt' e^{-\alpha(t-t')} u(t-t') \delta S(t') , \quad (2.51)$$

where u is the causal function (a.k.a., the Heaviside unit step). In terms of system neutron production,

$$\delta(vF(t)) = \int_{-\infty}^t dt' \frac{e^{-\alpha(t-t')}}{\Lambda} u(t-t') \delta S(t') . \quad (2.52)$$

Therefore, the system's impulse response is

$$h(t|t') dt = \frac{e^{-\alpha(t-t')}}{\Lambda} u(t-t') dt , \quad (2.53)$$

which is a simple representation of the system's neutron production following the injection of a neutron. In the following development, the point reactor kernel shall be interpreted in the following manner: per neutron injected into the system at t' , the expected number of

neutrons to be found at time t is $h(t|t')dt$. Further observe that

$$\frac{1}{\bar{v}} h(t|t') dt = \frac{e^{-\alpha(t-t')}}{\bar{v}\Lambda} u(t-t') dt, \quad (2.54)$$

is the expected number of induced fissions following the injection of a neutron – $\bar{v}\Lambda$ is the mean time between fissions.

Note in particular that the total number of neutrons chain-related to a single neutron injected in the remote past is

$$\int_{-\infty}^t dt' h(t|t') = \frac{1}{\alpha\Lambda} = \frac{1}{|\rho|}. \quad (2.55)$$

Consequently, in systems approaching critical, families of chain-related neutrons are abundant. Even in systems that are far subcritical, the expected number of neutrons related to a single injected neutron can be substantial. In the following, this simple kernel will be applied to generate point models of joint moments between the source fission process, the system fission process, and the neutron detection process.

There are a multitude of ways in which source and detection events may be chain-related as Fig. 2.9 illustrates. For example, each detection event depicted is a descendent of the single initiating source fission X_1 . Subgroups of detection events, i.e., doublets, triplets, etc., are related by the various processes that give rise to their common ancestors, which often include fissions induced in the system [Fey]. For example, the doublet (X_2, X_3) , though it is descended from the source fission X_1 , has the induced fission F_2 as its most recent ancestor. The triplet (X_3, X_4, X_5) is of course also descended from the source fission X_1 , but

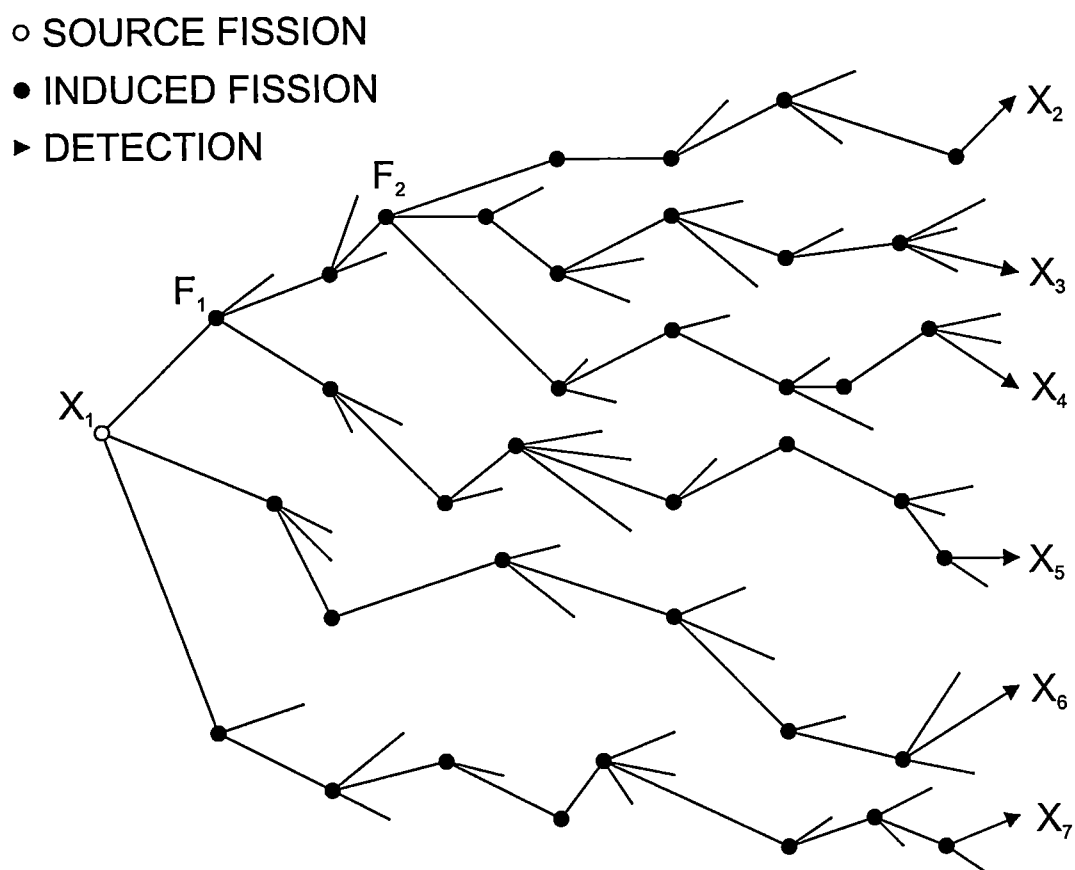


Figure 2.9 Chain-related detection events due to a single source fission

its most recent ancestors are the induced fissions F_1 and F_2 . In the following, a systematic procedure for modeling the processes that give rise to groups of chain-related events is described.

2.3 Statistical Signatures Derived from Neutron Detection

Neutron detection is achieved via a wide variety of physical processes; these are so numerous that even a cursory review of them should not be given here. There are several excellent texts covering this subject; Knoll's book provides a comprehensive description of those neutron detectors most frequently encountered in practice [Kno]. For the remainder of this discourse, it will be tacitly assumed that neutron detection involves the capture of a neutron, which effectively removes it from the fission-chain. Alternative detection mechanisms, like scatter or fission, do not remove the neutron from its fission-chain. In fact, fission detectors often put more neutrons back into the system. The basic processes giving rise to chain-related detection events are fundamentally the same when these alternative detectors are used, but the connections between correlated events undergo subtle changes because these detectors can return (or even supply) neutrons to the fission-chain. There are several publications that address these issues explicitly, e.g., [Dif].

Detection efficiency, in the context of a measurement performed on a real (as opposed to a point) system is a complicated issue. In the following, detection efficiency represents the probability per neutron that a detection event will occur. This sounds simple, but in fact there are a number of factors affecting efficiency. Obviously a detector's "view"

of a real system can have significant effects on the measurement. A detector's field of view is that portion of phase space that the detector can "see", and it includes how clearly the detector can see it. Obviously a detector will respond to neutrons originating from different regions in location, velocity, and time phase space according to the probability that a given neutron will reach the detector and then interact with it. Muñoz-Cobo, Perez, and Verdú, provide a detailed description of "field-of-view" effects in their papers on stochastic neutron transport [Muñ1,2]. Solid angle effects of course do not appear in the point model, because it treats all of space as a point. However, the probability that a neutron will fail to be detected because it contributed to fission-chain propagation is implicit in the following simple treatment of detection efficiency. Furthermore, the intrinsic efficiency of a real detector depends upon the spectrum of neutrons interacting with the detection medium. The following treatment also includes this intrinsic efficiency; however, there is no *explicit* accounting for the fissile system's leakage spectrum in the following treatment. In addition, any real measurement will require a host of electronic devices to translate detection events into observable electrical pulses. A discussion of the effects of detection electronics on measured signatures is deferred to the next section. Finally, note that the point model does not account for the transmission of source neutrons through the system. Hence, according to the point model, all counts result from a system neutron born during an induced fission; but in a real measurement, some counts will always occur due to source neutrons that have entered and then escaped the system. Because the following models are developed from the point model, this effect is not addressed, and efficiency reflects the probability of detecting a system neutron born of induced fission; the probability of counting a source neutron does

not enter into the following models. However, despite these deficiencies, the following descriptions do illustrate the general behavior of these signatures due to the processes that contribute to them.

First consider the response of a neutron capture detector to a subcritical system driven by an observable spontaneous fission source, e.g., a ^{252}Cf ion chamber. The probability of a source fission event w at t_0 resulting in ν_0 neutrons is

$$p_w(t_0, \nu_0) dt_0 = p_w(t_0) dt_0 p_0(\nu_0) = F_0 dt_0 p_0(\nu_0) , \quad (2.56)$$

where $p_0(\nu_0)$ is the source fission neutron multiplicity distribution. Given that the source injected ν_0 neutrons into the system at t_0 , the conditional probability of a subsequent detection event x at t_1 is

$$p_{x|w}(t_1 | t_0, \nu_0) dt_1 = \varepsilon_x \nu_0 \frac{e^{-\alpha(t_1 - t_0)}}{\Lambda} u(t_1 - t_0) dt_1 , \quad (2.57)$$

where ε_x denotes the efficiency of detector x , i.e., the probability per neutron that a detection event (a count) will occur². So the joint probability of a source fission at t_0 and a chain-related count at t_1 is

² Several previous developments expressed detection efficiency ε'_x in terms of counts per fission; obviously the relationship between these efficiencies is $\varepsilon_x = \varepsilon'_x / \bar{\nu}$, where $\bar{\nu}$ denotes the average number of neutrons per induced fission.

$$\begin{aligned}
p_{wx}^{U_1}(t_0, t_1) dt_0 dt_1 &= \sum_{v_0} p_{x|w}(t_1 | t_0, v_0) dt_1 p_w(t_0, v_0) dt_0 \\
&= \varepsilon_x \frac{\bar{v}_0 F_0}{\Lambda} e^{-\alpha(t_1 - t_0)} u(t_1 - t_0) dt_0 dt_1,
\end{aligned} \tag{2.58}$$

where the superscript “ U_1 ” simply denotes that the two events are chain-related, i.e., correlated. This process is graphically depicted in Fig. 2.10, which is a simple representation of any fission-chain connecting a source emission and a single detection event.

However, recognize that sometimes events at t_0 and t_1 may be accidentally coincident (i.e., uncorrelated), in which case

$$p_{wx}^{U'_1}(t_0, t_1) dt_0 dt_1 = p_w(t_0) dt_0 p_x(t_1) dt_1, \tag{2.59}$$

where the marginal probability of a count at any given t_1 is

$$p_x(t_1) dt_1 = \int_{-\infty}^{t_1} dt'_0 p_{wx}^{U'_1}(t'_0, t_1) dt'_0 = \varepsilon_x \frac{\bar{v}_0 F_0}{\alpha \Lambda} = \varepsilon_x \bar{v} F. \tag{2.60}$$

So the joint probability of a source fission w at t_0 and a count x at t_1 is

$$\begin{aligned}
p_{wx}(t_0, t_1) dt_0 dt_1 &= p_{wx}^{U_1}(t_0, t_1) dt_0 dt_1 + p_{wx}^{U'_1}(t_0, t_1) dt_0 dt_1 \\
&= \varepsilon_x \frac{\bar{v}_0 F_0}{\Lambda} e^{-\alpha(t_1 - t_0)} u(t_1 - t_0) dt_0 dt_1 + F_0 dt_0 \varepsilon_x \frac{\bar{v}_0 F_0}{\alpha \Lambda} dt_1.
\end{aligned} \tag{2.61}$$

Consequently, the source-detector correlation is

$$R_{wx}(\tau) = E[w(t_0)x(t_1)] \Big|_{\tau=t_1-t_0} = C_{wx}(\tau) + \bar{w} \bar{x}, \tag{2.62}$$

where C_{wx} is the source-detector covariance,

$$C_{wx}(\tau) = \varepsilon_x \frac{\bar{v}_0 F_0}{\Lambda} e^{-\alpha\tau} u(\tau), \tag{2.63}$$

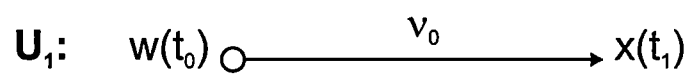


Figure 2.10 A fission-chain relating a source fission and a single count

$\bar{w}$ is the mean source fission rate F_0 , and $\bar{x}$ is the mean detector count rate,

$$\bar{x} = \varepsilon_x \frac{\bar{v}_0 F_0}{\alpha \Lambda} , \quad (2.64)$$

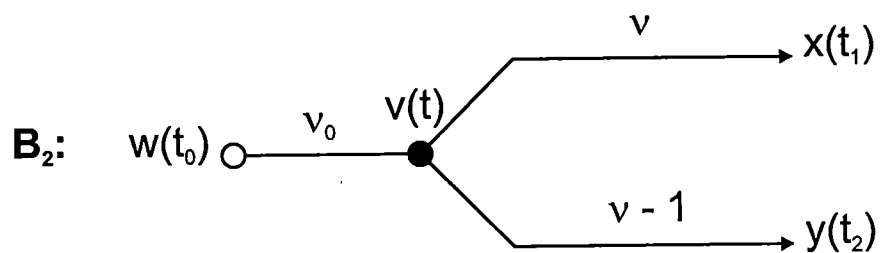
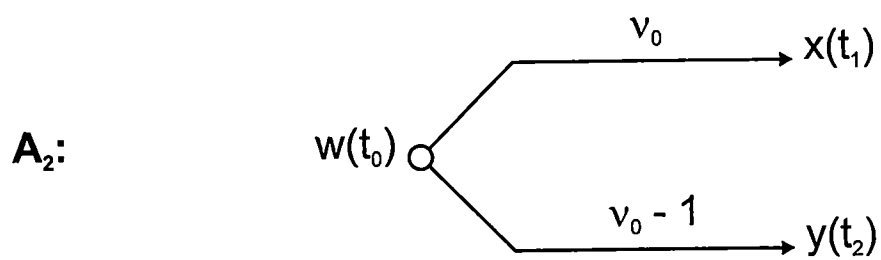
in the absence of any additional neutron sources. Under this condition, all counts are correlated with some initiating source emission as (2.60) asserts. Hence, the source-detector covariance reflects the expected distribution of detector counts following an initiating source fission:

$$\int_0^{\infty} d\tau C_{wx}(\tau) = \bar{x} . \quad (2.65)$$

In other words – and this is intuitively obvious – because each count may be traced to some source emission, each count x is a member of some source-correlated doublet (w, x) . Note that since the source may emit multiple neutrons during a single fission, there may be many such doublets associated with a single source fission.

Next consider the response of two capture detectors, say x and y , simultaneously observing a subcritical system driven by an observable spontaneous fission source w . There are two processes that result in a triplet of bicornelated events (w, x, y) , and these are graphically depicted in Fig. 2.11.

The first process, A_2 , requires that at least two neutrons emerge from the source fission w at t_0 . These neutrons initiate fission-chains that propagate to t_1 and t_2 when their progeny are captured in detection events x and y . Given the occurrence of the initiating source fission at t_0 , the probability of a chain-related count at t_1 is



$$\mathbf{U}_2 = \mathbf{A}_2 \cup \mathbf{B}_2$$

Figure 2.11 Fission-chains relating a source fission and a doublet of counts

$$p_{x|w}(t_1 | t_0, v_0) dt_1 = \varepsilon_x v_0 \frac{e^{-\alpha(t_1 - t_0)}}{\Lambda} u(t_1 - t_0) dt_1, \quad (2.66)$$

and the probability of a subsequent chain-related count at t_2 is

$$p_{y|xw}(t_2 | t_1, t_0, v_0) dt_2 = \varepsilon_y (v_0 - 1) \frac{e^{-\alpha(t_2 - t_0)}}{\Lambda} u(t_2 - t_0) dt_2, \quad t_2 > t_1, \quad (2.67)$$

because only $v_0 - 1$ neutrons were available to sustain a fission-chain surviving to t_2 . So

the probability of a bicornelated triplet (w, x, y) via process A_2 is

$$\begin{aligned} p_{wxy}^{A_2}(t_0, t_1, t_2) dt_0 dt_1 dt_2 &= \sum_{v_0} p_{y|xw}(t_2 | t_1, t_0, v_0) dt_2 p_{x|w}(t_1 | t_0, v_0) dt_1 p_w(t_0) dt_0 p_0(v_0) \\ &= \varepsilon_x \varepsilon_y \frac{\overline{v_0(v_0 - 1)} F_0}{\Lambda^2} e^{-\alpha(t_1 - t_0)} u(t_1 - t_0) e^{-\alpha(t_2 - t_0)} u(t_2 - t_0) dt_0 dt_1 dt_2, \quad t_2 > t_1. \end{aligned} \quad (2.68)$$

The second process, B_2 , can result in a bicornelated triplet even if only one neutron is emitted during the source fission w . This is possible because an intermediate system fission v contributes at least two neutrons to sustain fission-chains whose progeny may be detected by x and y . Given the occurrence of the initial source fission w at t_0 , the conditional probability of an induced fission v at t resulting in v neutrons is

$$p_{v|w}(t, v | t_0, v_0) dt = p_{v|w}(t | t_0, v_0) dt p(v) = v_0 \frac{e^{-\alpha(t - t_0)}}{\bar{v} \Lambda} u(t - t_0) p(v) dt, \quad (2.69)$$

where $p(v)$ is the neutron multiplicity distribution for induced fission. Recall that the mean time between fissions is $\bar{v} \Lambda$. The probability of a following chain-related count x at t_1 is

$$p_{x|vw}(t_1 | t, v, t_0, v_0) dt_1 = \varepsilon_x v \frac{e^{-\alpha(t_1 - t)}}{\Lambda} u(t_1 - t) dt_1, \quad (2.70)$$

and the probability of a second chain-related count y at t_2 is

$$p_{y|v, w}(t_2 | t_1, t, v, t_0, v_0) dt_2 = \varepsilon_y (v - 1) \frac{e^{-\alpha(t_2 - t)}}{\Lambda} u(t_2 - t) dt_2, \quad t_2 > t_1. \quad (2.71)$$

Consequently, the probability of a tricorrelated quadruplet (w, v, x, y) is

$$\begin{aligned} p_{wvxy}(t_0, t, t_1, t_2) dt_0 dt dt_1 dt_2 \\ = \sum_{v_0} \sum_v p_{y|v, w}(t_2 | t_1, t, v, t_0, v_0) dt_2 p_{x|v, w}(t_1 | t, v, t_0, v_0) dt_1 \\ \cdot p_{v|w}(t | t_0, v_0) dt p(v) p_w(t_0) dt_0 p_0(v_0) \\ = \varepsilon_x \varepsilon_y \frac{\overline{v_0 v(v-1)} F_0}{\bar{v} \Lambda^3} e^{-\alpha(t-t_0)} u(t-t_0) e^{-\alpha(t_1-t)} u(t_1-t) e^{-\alpha(t_2-t)} u(t_2-t) dt_0 dt dt_1 dt_2, \quad t_2 > t_1. \end{aligned} \quad (2.72)$$

Of course, such quadruplets are usually unobservable because there are no direct means to detect an individual induced fission event³. However, the probability of an observable bicorrelated triplet (w, x, y) via process B_2 is

$$\begin{aligned} p_{wxy}^{B_2}(t_0, t_1, t_2) dt_0 dt_1 dt_2 &= \int_{t_0}^{t_1} dt p_{wvxy}(t_0, t, t_1, t_2) dt_0 dt_1 dt_2 \\ &= \varepsilon_x \varepsilon_y \frac{\overline{v_0 v(v-1)} F_0}{\bar{v} \alpha \Lambda^3} e^{-\alpha(t_1-t_0)} u(t_1-t_0) \\ &\quad \cdot \left(e^{-\alpha(t_2-t_1)} u(t_2-t_1) - e^{-\alpha(t_2-t_0)} u(t_2-t_0) \right) dt_0 dt_1 dt_2, \quad t_2 > t_1. \end{aligned} \quad (2.73)$$

So the probability of a triplet (w, x, y) is

³ This of course excludes systems specifically designed to observe individual induced fission events, e.g., a system designed to measure $p(v)$ for a particular fissile isotope.

$$\begin{aligned}
p_{wxy}(t_0, t_1, t_2) dt_0 dt_1 dt_2 &= p_{wxy}^{U_2}(t_0, t_1, t_2) dt_0 dt_1 dt_2 \\
&+ p_w(t_0) dt_0 p_{xy}^{U_2}(t_1, t_2) dt_1 dt_2 + p_x(t_1) dt_1 p_{wy}^{U_1}(t_0, t_2) dt_0 dt_2 \\
&+ p_y(t_2) dt_2 p_{wx}^{U_1}(t_0, t_1) dt_0 dt_1 + p_w(t_0) dt_0 p_x(t_1) dt_1 p_y(t_2) dt_2,
\end{aligned} \quad (2.74)$$

where $U_2 = A_2 \cup B_2$, i.e., the two processes giving rise to bicornelated triplets (w, x, y) . The last four terms reflect the probability of triplets of accidentally bicornelated events, which can result if only two of the three events are chain-related or if none of the three events is chain-related.

Consequently, the source-detector bicornelation is

$$\begin{aligned}
R_{wxy}(\tau_1, \tau_2) &= E[w(t_0)x(t_1)y(t_2)] \Big|_{\substack{\tau_1 = t_1 - t_0 \\ \tau_2 = t_2 - t_0}} \\
&= C_{wxy}(\tau_1, \tau_2) + \bar{w} C_{xy}(\tau_2 - \tau_1) + \bar{x} C_{wy}(\tau_2) + \bar{y} C_{wx}(\tau_1) + \bar{w} \bar{x} \bar{y}.
\end{aligned} \quad (2.75)$$

where C_{wxy} is the source-detector bicovariance:

$$\begin{aligned}
C_{wxy}(\tau_1, \tau_2) &= \varepsilon_x \varepsilon_y \left(\frac{\overline{v_0(v_0 - 1)}}{v_0(v_0 - 1)} - \frac{\overline{v_0 v(v - 1)}}{\bar{v} \alpha \Lambda} \right) \frac{F_0}{\Lambda^2} e^{-\alpha \tau_1} e^{-\alpha \tau_2} u(\tau_1) u(\tau_2) \\
&+ \varepsilon_x \varepsilon_y \frac{\overline{v_0 v(v - 1)}}{\bar{v} \alpha \Lambda} \frac{F_0}{\Lambda^2} e^{-\alpha \tau_2} u(\tau_1) u(\tau_2 - \tau_1), \quad \tau_2 > \tau_1.
\end{aligned} \quad (2.76)$$

Of course, such bicornelated triplets (w, x, y) will sometimes (actually, half of the time – when $\tau_2 < \tau_1$) occur in permuted sequence (w, y, x) , so:

$$\begin{aligned}
C_{wxy}(\tau_1, \tau_2) &= \varepsilon_x \varepsilon_y \left(\frac{\overline{v_0(v_0 - 1)}}{v_0(v_0 - 1)} - \frac{\overline{v_0 v(v - 1)}}{\bar{v} \alpha \Lambda} \right) \frac{F_0}{\Lambda^2} e^{-\alpha \tau_1} e^{-\alpha \tau_2} u(\tau_1) u(\tau_2) \\
&+ \varepsilon_x \varepsilon_y \frac{\overline{v_0 v(v - 1)}}{\bar{v} \alpha \Lambda} \frac{F_0}{\Lambda^2} \left(e^{-\alpha \tau_1} u(\tau_1 - \tau_2) u(\tau_2) + e^{-\alpha \tau_2} u(\tau_1) u(\tau_2 - \tau_1) \right)
\end{aligned} \quad (2.77)$$

everywhere over the (τ_1, τ_2) plane.

The accidental contributions to the source-detector bicornelation deserve some

attention. The last term, $\bar{w} \bar{x} \bar{y}$, results from triplets of independent events, and the preceding two terms, $\bar{x} C_{wy}(\tau_2)$ and $\bar{y} C_{wx}(\tau_1)$, result when one count, x or y , is independent of the remaining correlated source-fission/detector-count doublet, (w, y) or (w, x) . The first accidental term results when the two counts comprising a doublet (x, y) are chain-related to each other, but are bicorrelated with a fission event w different from one that occurred (τ_1, τ_2) earlier. The expected rate of correlated detector count doublets (x, y) is the between-detectors covariance C_{xy} .

Recall that all fission-chains in a subcritical system devoid of any internal or inherent sources may be traced to an initiating fission event in the introduced source. Hence under these conditions the probability of a correlated doublet (x, y) is just

$$p_{xy}^{U_2}(t_1, t_2) dt_1 dt_2 = \int_{-\infty}^{\min(t_1, t_2)} dt'_0 p_{wxy}^{U_2}(t'_0, t_1, t_2) dt_1 dt_2, \quad (2.78)$$

where again the superscript “ U_2 ” denotes that the doublet was produced by either process A_2 or B_2 depicted in Fig. 2.11. On the other hand, the probability of an accidentally coincident doublet (x, y) is simply $p_x(t_1) dt_1 p_y(t_2) dt_2$, so

$$p_{xy}(t_1, t_2) dt_1 dt_2 = p_{xy}^{U_2}(t_1, t_2) dt_1 dt_2 + p_x(t_1) dt_1 p_y(t_2) dt_2. \quad (2.79)$$

Consequently the between-detectors correlation is

$$R_{xy}(\tau) = E[x(t_1)y(t_2)] \Big|_{\tau=t_2-t_1} = C_{xy}(\tau) + \bar{x} \bar{y}, \quad (2.80)$$

where the between-detectors covariance C_{xy} is the expected rate of correlated detector counts:

$$C_{xy}(\tau) = \varepsilon_x \varepsilon_y \left(\frac{1}{\bar{v}_0(\bar{v}_0 - 1)} + \frac{\bar{v}_0 \bar{v}(\bar{v} - 1)}{\bar{v} \alpha \Lambda} \right) \frac{F_0}{2 \alpha \Lambda^2} e^{-\alpha |\tau|}, \quad (2.81)$$

and $\bar{x} \bar{y}$ is the accidental coincidence rate.

Now observe that because all pairs of correlated counts may be traced to some initiating source emission, the source-detector bicovariance reflects the expected distribution of correlated doublets following a source fission:

$$\int_0^\infty d\tau_1 C_{wxy}(\tau_1, \tau_1 + \tau) = \int_0^\infty d\tau_2 C_{wxy}(\tau_2 - \tau, \tau_2) = C_{xy}(\tau). \quad (2.82)$$

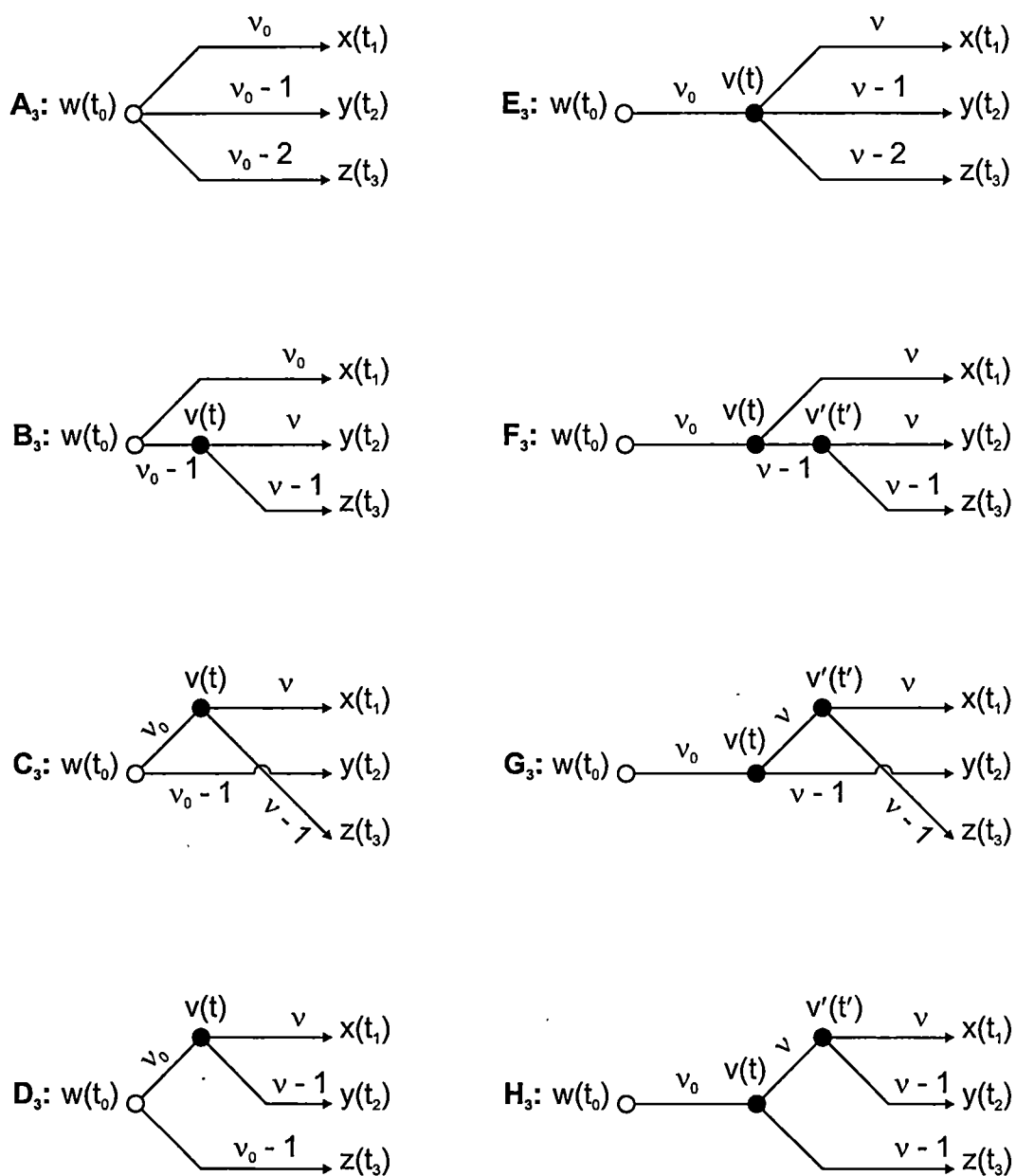
Furthermore, the occurrence of a correlated doublet of counts is as probable as the occurrence of a source-bicorrelated triplet

$$\int_0^\infty d\tau_1 \int_0^\infty d\tau_2 C_{wxy}(\tau_1, \tau_2) = \int_{-\infty}^\infty d\tau C_{xy}(\tau), \quad (2.83)$$

because, as (2.78) asserts, each correlated doublet of counts (x, y) is a member of some source-bicorrelated triplet (w, x, y) .

Now consider that a single source fission w may result in three chain-related counts⁴, say x , y , and z , to form an observable tricorrelated quadruplet (w, x, y, z) . There are eight processes, each depicted in Fig. 2.12, that result in such a tricorrelated quadruplet. Observe that the first process, A_3 , requires that at least three neutrons emerge from the initiating source fission. The next three processes, B_3 , C_3 , and D_3 require that the source fission emit

⁴ In theory, a single source fission may actually result in an unlimited number of chain-related counts due to fission-chain multiplication even in a subcritical system; but three is sufficient for this demonstration.



$$U_3 = A_3 \cup B_3 \cup C_3 \cup D_3 \cup E_3 \cup F_3 \cup G_3 \cup H_3$$

Figure 2.12 Fission-chains relating a source fission and a triplet of counts

at least two neutrons, and the last four processes, E_3 through H_3 , can result in a tricorrelated quadruplet even if only a single neutron emerges from the initial source fission.

The methodology to determine the probability of a quadruplet from each of these processes is the same as previously applied to determine the probability of triplets, doublets, and singlets (i.e., counts) from similar processes. It is essentially the sequential application of Bayes' rule to the chain of events. Begin with the probability of the first event, the source fission. Then use the conditional probability of the second event given the occurrence of the first to determine the joint probability of the first two events. Next use the conditional probability of the third event given the first two to determine the joint probability of the first three events. Continue this procedure until the chain of events is complete. Then integrate (or sum) over the unobservables, e.g., the time of an individual induced fission or the number of neutrons emitted during fission. The result is the probability of an observable chain of events.

To ensure that all processes leading to n chain-related counts are modeled, first begin with the process that requires at least n neutrons to emerge from the initial source fission. Then model all processes that lead to n chain-related counts if only $n - 1$ neutrons are emitted during the initial source fission. Next model all processes that produce n chain-related counts even if only $n - 2$ emerge from the source fission. Continue this procedure until the processes that require only a single neutron to emerge from the initial source fission are modeled.

So recognize that the probability⁵ of a quadruplet due to process A_3 is

$$\begin{aligned}
 p_{wxyz}^{A_3}(t_0, t_1, t_2, t_3) &= \sum_{v_0} \varepsilon_z(v_0-2) \frac{e^{-\alpha(t_3-t_0)}}{\Lambda} u(t_3-t_0) \cdot \varepsilon_y(v_0-1) \frac{e^{-\alpha(t_2-t_0)}}{\Lambda} u(t_2-t_0) \\
 &\quad \cdot \varepsilon_x v_0 \frac{e^{-\alpha(t_1-t_0)}}{\Lambda} u(t_1-t_0) \cdot F_0 p_0(v_0) \\
 &= \varepsilon_x \varepsilon_y \varepsilon_z \frac{\overline{v_0(v_0-1)(v_0-2)} F_0}{\Lambda^3} e^{-\alpha(t_1-t_0)} u(t_1-t_0) e^{-\alpha(t_2-t_0)} u(t_2-t_0) e^{-\alpha(t_3-t_0)} u(t_3-t_0),
 \end{aligned} \tag{2.84}$$

and note that this process requires that at least three neutrons emerge from the initial source fission.

The respective probabilities of a quadruplet arising from processes B_3 , C_3 , and D_3 are

$$\begin{aligned}
 p_{wxyz}^{B_3}(t_0, t_1, t_2, t_3) &= \sum_{v_0} \sum_v \int_{t_0}^{t_2} dt \varepsilon_z(v-1) \frac{e^{-\alpha(t_3-t)}}{\Lambda} u(t_3-t) \cdot \varepsilon_y v \frac{e^{-\alpha(t_2-t)}}{\Lambda} u(t_2-t) \\
 &\quad \cdot (v_0-1) \frac{e^{-\alpha(t-t_0)}}{\bar{v}\Lambda} u(t-t_0) p(v) \cdot \varepsilon_x v_0 \frac{e^{-\alpha(t_1-t_0)}}{\Lambda} u(t_1-t_0) \cdot F_0 p_0(v_0) \\
 &= \varepsilon_x \varepsilon_y \varepsilon_z \frac{\overline{v_0(v_0-1)} \overline{v(v-1)} F_0}{\bar{v}\alpha\Lambda^4} e^{-\alpha(t_1-t_0)} u(t_1-t_0) e^{-\alpha(t_2-t_0)} u(t_2-t_0) \\
 &\quad \cdot \left(e^{-\alpha(t_3-t_2)} u(t_3-t_2) - e^{-\alpha(t_3-t_0)} u(t_3-t_0) \right),
 \end{aligned} \tag{2.85}$$

⁵ In the following, each probability is per incremental element $dt_0 dt_1 dt_2 dt_3$; in order to save space, this increment will not be explicitly shown. In addition, it will temporarily be tacitly assumed that the counts x , y , and z occur in exactly that order, i.e., that $t_3 > t_2 > t_1$; subsequent results will be generalized for permuted sequences of counts.

$$\begin{aligned}
P_{wxyz}^{C_3}(t_0, t_1, t_2, t_3) &= \sum_{v_0} \sum_v \int_{t_0}^{t_1} dt \varepsilon_z(v-1) \frac{e^{-\alpha(t_3-t)}}{\Lambda} u(t_3-t) \cdot \varepsilon_y(v_0-1) \frac{e^{-\alpha(t_2-t_0)}}{\Lambda} u(t_2-t_0) \\
&\quad \cdot \varepsilon_x v \frac{e^{-\alpha(t_1-t)}}{\Lambda} u(t_1-t) \cdot v_0 \frac{e^{-\alpha(t-t_0)}}{\bar{v}\Lambda} u(t-t_0) p(v) \cdot F_0 p_0(v_0) \\
&= \varepsilon_x \varepsilon_y \varepsilon_z \frac{\overline{v_0(v_0-1)} \overline{v(v-1)} F_0}{\bar{v} \alpha \Lambda^4} e^{-\alpha(t_1-t_0)} u(t_1-t_0) e^{-\alpha(t_2-t_0)} u(t_2-t_0) \\
&\quad \cdot \left(e^{-\alpha(t_3-t_1)} u(t_3-t_1) - e^{-\alpha(t_3-t_0)} u(t_3-t_0) \right),
\end{aligned} \tag{2.86}$$

and

$$\begin{aligned}
P_{wxyz}^{D_3}(t_0, t_1, t_2, t_3) &= \sum_{v_0} \sum_v \int_{t_0}^{t_1} dt \varepsilon_z(v_0-1) \frac{e^{-\alpha(t_3-t_0)}}{\Lambda} u(t_3-t_0) \cdot \varepsilon_y(v-1) \frac{e^{-\alpha(t_2-t)}}{\Lambda} u(t_2-t) \\
&\quad \cdot \varepsilon_x v \frac{e^{-\alpha(t_1-t)}}{\Lambda} u(t_1-t) \cdot v_0 \frac{e^{-\alpha(t-t_0)}}{\bar{v}\Lambda} u(t-t_0) p(v) \cdot F_0 p_0(v_0) \\
&= \varepsilon_x \varepsilon_y \varepsilon_z \frac{\overline{v_0(v_0-1)} \overline{v(v-1)} F_0}{\bar{v} \alpha \Lambda^4} e^{-\alpha(t_1-t_0)} u(t_1-t_0) e^{-\alpha(t_2-t_0)} u(t_2-t_0) \\
&\quad \cdot \left(e^{-\alpha(t_3-t_1)} u(t_3-t_1) - e^{-\alpha(t_3-t_0)} u(t_3-t_0) \right).
\end{aligned} \tag{2.87}$$

These processes can occur if two or more neutrons emerge from the initiating source fission.

Process E_3 may take place even if only a single neutron is emitted during the initiating source fission. However, process E_3 is similar to process A_3 , so, naturally,

$$\begin{aligned}
p_{wxyz}^{E_3}(t_0, t_1, t_2, t_3) &= \sum_{v_0} \sum_v \int_{t_0}^{t_1} dt \varepsilon_z(v-2) \frac{e^{-\alpha(t_3-t)}}{\Lambda} u(t_3-t) \cdot \varepsilon_y(v-1) \frac{e^{-\alpha(t_2-t)}}{\Lambda} u(t_2-t) \\
&\quad \cdot \varepsilon_x v \frac{e^{-\alpha(t_1-t)}}{\Lambda} u(t_1-t) \cdot v_0 \frac{e^{-\alpha(t-t_0)}}{\bar{v}\Lambda} u(t-t_0) p(v) \cdot F_0 p_0(v_0) \\
&= \varepsilon_x \varepsilon_y \varepsilon_z \frac{\overline{v_0 v(v-1)(v-2)} F_0}{2\bar{v}\alpha\Lambda^4} e^{-\alpha(t_1-t_0)} u(t_1-t_0) \\
&\quad \cdot \left(e^{-\alpha(t_2-t_1)} u(t_2-t_1) e^{-\alpha(t_3-t_1)} u(t_3-t_1) - e^{-\alpha(t_2-t_0)} u(t_2-t_0) e^{-\alpha(t_3-t_0)} u(t_3-t_0) \right),
\end{aligned} \tag{2.88}$$

Likewise, because processes F_3 , G_3 , and H_3 are similar to B_3 , C_3 , and D_3 ,

$$\begin{aligned}
p_{wxyz}^{F_3}(t_0, t_1, t_2, t_3) &= \sum_{v_0} \sum_v \sum_{v'} \int_{t_0}^{t_1} dt \int_t^{t_2} dt' \varepsilon_z(v'-1) \frac{e^{-\alpha(t_3-t')}}{\Lambda} u(t_3-t') \cdot \varepsilon_y v' \frac{e^{-\alpha(t_2-t')}}{\Lambda} u(t_2-t') \\
&\quad \cdot (v-1) \frac{e^{-\alpha(t'-t)}}{\bar{v}\Lambda} u(t'-t) p(v') \cdot \varepsilon_x v \frac{e^{-\alpha(t_1-t)}}{\Lambda} u(t_1-t) \cdot v_0 \frac{e^{-\alpha(t-t_0)}}{\bar{v}\Lambda} u(t-t_0) p(v) \cdot F_0 p_0(v_0) \Big|_{v'=v} \\
&= \varepsilon_x \varepsilon_y \varepsilon_z \frac{\overline{v_0 v(v-1)^2} F_0}{\bar{v}^2 \alpha \Lambda^5} e^{-\alpha(t_1-t_0)} u(t_1-t_0) \\
&\quad \cdot \left[e^{-\alpha(t_3-t_2)} u(t_3-t_2) \left(e^{-\alpha(t_2-t_1)} u(t_2-t_1) - e^{-\alpha(t_2-t_0)} u(t_2-t_0) \right) \right. \\
&\quad \left. - \frac{1}{2} \left(e^{-\alpha(t_2-t_1)} u(t_2-t_1) e^{-\alpha(t_3-t_1)} u(t_3-t_1) - e^{-\alpha(t_2-t_0)} u(t_2-t_0) e^{-\alpha(t_3-t_0)} u(t_3-t_0) \right) \right],
\end{aligned} \tag{2.89}$$

$$\begin{aligned}
p_{wxyz}^{G_3}(t_0, t_1, t_2, t_3) &= \sum_{v_0} \sum_v \sum_{v'} \int_{t_0}^{t_1} dt \int_t^{t_1} dt' \varepsilon_z(v'-1) \frac{e^{-\alpha(t_3-t')}}{\Lambda} u(t_3-t') \cdot \varepsilon_y(v-1) \frac{e^{-\alpha(t_2-t)}}{\Lambda} u(t_2-t) \\
&\quad \cdot \varepsilon_x v' \frac{e^{-\alpha(t_1-t')}}{\Lambda} u(t_1-t') \cdot v \frac{e^{-\alpha(t'-t)}}{\bar{v}\Lambda} u(t'-t) p(v') \cdot v_0 \frac{e^{-\alpha(t-t_0)}}{\bar{v}\Lambda} u(t-t_0) p(v) \cdot F_0 p_0(v_0) \Bigg|_{v'=v} \\
&= \varepsilon_x \varepsilon_y \varepsilon_z \frac{\bar{v}_0 \bar{v}(v-1)^2 F_0}{\bar{v}^2 \alpha \Lambda^5} e^{-\alpha(t_1-t_0)} u(t_1-t_0) \\
&\quad \cdot \left[e^{-\alpha(t_3-t_1)} u(t_3-t_1) \left(e^{-\alpha(t_2-t_1)} u(t_2-t_1) - e^{-\alpha(t_2-t_0)} u(t_2-t_0) \right) \right. \\
&\quad \left. - \frac{1}{2} \left(e^{-\alpha(t_2-t_1)} u(t_2-t_1) e^{-\alpha(t_3-t_1)} u(t_3-t_1) - e^{-\alpha(t_2-t_0)} u(t_2-t_0) e^{-\alpha(t_3-t_0)} u(t_3-t_0) \right) \right],
\end{aligned} \tag{2.90}$$

and

$$\begin{aligned}
p_{wxyz}^{H_3}(t_0, t_1, t_2, t_3) &= \sum_{v_0} \sum_v \sum_{v'} \int_{t_0}^{t_1} dt \int_t^{t_1} dt' \varepsilon_z(v-1) \frac{e^{-\alpha(t_3-t)}}{\Lambda} u(t_3-t) \cdot \varepsilon_y(v'-1) \frac{e^{-\alpha(t_2-t')}}{\Lambda} u(t_2-t') \\
&\quad \cdot \varepsilon_x v' \frac{e^{-\alpha(t_1-t')}}{\Lambda} u(t_1-t') \cdot v \frac{e^{-\alpha(t'-t)}}{\bar{v}\Lambda} u(t'-t) p(v') \cdot v_0 \frac{e^{-\alpha(t-t_0)}}{\bar{v}\Lambda} u(t-t_0) p(v) \cdot F_0 p_0(v_0) \Bigg|_{v'=v} \\
&= \varepsilon_x \varepsilon_y \varepsilon_z \frac{\bar{v}_0 \bar{v}(v-1)^2 F_0}{\bar{v}^2 \alpha \Lambda^5} e^{-\alpha(t_1-t_0)} u(t_1-t_0) \\
&\quad \cdot \left[e^{-\alpha(t_2-t_1)} u(t_2-t_1) \left(e^{-\alpha(t_3-t_1)} u(t_3-t_1) - e^{-\alpha(t_3-t_0)} u(t_3-t_0) \right) \right. \\
&\quad \left. - \frac{1}{2} \left(e^{-\alpha(t_2-t_1)} u(t_2-t_1) e^{-\alpha(t_3-t_1)} u(t_3-t_1) - e^{-\alpha(t_2-t_0)} u(t_2-t_0) e^{-\alpha(t_3-t_0)} u(t_3-t_0) \right) \right].
\end{aligned} \tag{2.91}$$

Like process E_3 , these last three processes may occur even if only one neutron emerges from the initial source emission.

Now recall the ways accidentally tricoincident quadruplets may occur. These are congruent to the number of ways that the four events may be broken into two, three, and four nonempty subsets. First, one event may be accidentally coincident with a bicorrelated triplet,

$$\begin{aligned}
p_{wxyz|1\cdot3}(t_0, t_1, t_2, t_3) &= p_w(t_0) p_{xyz}^{U_3}(t_1, t_2, t_3) + p_x(t_1) p_{wyz}^{U_2}(t_0, t_2, t_3) \\
&+ p_y(t_2) p_{wxz}^{U_2}(t_0, t_1, t_3) + p_z(t_3) p_{wxy}^{U_2}(t_0, t_1, t_2) ,
\end{aligned} \tag{2.92}$$

or two doublets of correlated events may be accidentally coincident with each other:

$$\begin{aligned}
p_{wxyz|2\cdot2}(t_0, t_1, t_2, t_3) &= p_{wx}^{U_1}(t_0, t_1) p_{yz}^{U_2}(t_2, t_3) \\
&+ p_{wy}^{U_1}(t_0, t_2) p_{xz}^{U_2}(t_1, t_3) \\
&+ p_{wz}^{U_1}(t_0, t_3) p_{xy}^{U_2}(t_1, t_2) .
\end{aligned} \tag{2.93}$$

Next, two independent events may be accidentally bicoincident with a pair of correlated events:

$$\begin{aligned}
p_{wxyz|1\cdot1\cdot2}(t_0, t_1, t_2, t_3) &= p_w(t_0) p_x(t_1) p_{yz}^{U_2}(t_2, t_3) + p_w(t_0) p_y(t_2) p_{xz}^{U_2}(t_1, t_3) \\
&+ p_w(t_0) p_z(t_3) p_{xy}^{U_2}(t_1, t_2) + p_x(t_1) p_y(t_2) p_{wz}^{U_1}(t_0, t_3) \\
&+ p_x(t_1) p_z(t_3) p_{wy}^{U_1}(t_0, t_2) + p_y(t_2) p_z(t_3) p_{wx}^{U_1}(t_0, t_1) .
\end{aligned} \tag{2.94}$$

Finally, none of the four events may be related:

$$p_{wxyz|4}(t_0, t_1, t_2, t_3) = p_w(t_0) p_x(t_1) p_y(t_2) p_z(t_3) . \tag{2.95}$$

So the probability of a quadruplet is

$$\begin{aligned}
p_{wxyz}(t_0, t_1, t_2, t_3) &= p_{wxyz}^{U_3}(t_0, t_1, t_2, t_3) \\
&+ p_{wxyz|1\cdot3}(t_0, t_1, t_2, t_3) + p_{wxyz|2\cdot2}(t_0, t_1, t_2, t_3) \\
&+ p_{wxyz|1\cdot1\cdot2}(t_0, t_1, t_2, t_3) \\
&+ p_{wxyz|4}(t_0, t_1, t_2, t_3) ,
\end{aligned} \tag{2.96}$$

where U_3 is shorthand for the processes leading to tricorrelated quadruplets (w, x, y, z):

$$U_3 = A_3 \cup B_3 \cup C_3 \cup D_3 \cup E_3 \cup F_3 \cup G_3 \cup H_3 . \tag{2.97}$$

Consequently, the source-detector tricorrelation is

$$\begin{aligned}
R_{wxyz}(\tau_1, \tau_2, \tau_3) &= E[w(t_0)x(t_1)y(t_2)z(t_3)] \Bigg| \begin{array}{l} \tau_1 = t_1 - t_0 \\ \tau_2 = t_2 - t_0 \\ \tau_3 = t_3 - t_0 \end{array} \\
&= C_{wxyz}(\tau_1, \tau_2, \tau_3) \\
&+ \bar{w} C_{xyz}(\tau_2 - \tau_1, \tau_3 - \tau_1) + \bar{x} C_{wyz}(\tau_2, \tau_3) + \bar{y} C_{wxz}(\tau_1, \tau_3) + \bar{z} C_{wxy}(\tau_1, \tau_2) \\
&+ C_{wx}(\tau_1) C_{yz}(\tau_3 - \tau_2) + C_{wy}(\tau_2) C_{xz}(\tau_3 - \tau_1) + C_{wz}(\tau_3) C_{xy}(\tau_2 - \tau_1) \\
&+ \bar{w} \bar{x} C_{yz}(\tau_3 - \tau_2) + \bar{w} \bar{y} C_{xz}(\tau_3 - \tau_1) + \bar{w} \bar{z} C_{xy}(\tau_2 - \tau_1) \\
&+ \bar{x} \bar{y} C_{wz}(\tau_3) + \bar{x} \bar{z} C_{wy}(\tau_2) + \bar{y} \bar{z} C_{wx}(\tau_1) \\
&+ \bar{w} \bar{x} \bar{y} \bar{z} ,
\end{aligned} \tag{2.98}$$

where C_{wxyz} is the source-detector trivariance:

$$\begin{aligned}
C_{wxyz}(\tau_1, \tau_2, \tau_3) &= \varepsilon_x \varepsilon_y \varepsilon_z \left[\left(\frac{\overline{v_0(v_0-1)(v_0-2)}}{v_0(v_0-1)(v_0-2)} - 3 \frac{\overline{v_0(v_0-1)} \overline{v(v-1)}}{\bar{v} \alpha \Lambda} \right) \right. \\
&- \frac{\bar{v}_0}{2 \bar{v} \alpha \Lambda} \left(\frac{\overline{v(v-1)(v-2)}}{v(v-1)(v-2)} - 3 \frac{\overline{v(v-1)^2}}{\bar{v} \alpha \Lambda} \right) \left. \right] \frac{F_0}{\Lambda^3} e^{-\alpha \tau_1} e^{-\alpha \tau_2} e^{-\alpha \tau_3} u(\tau_1) u(\tau_2) u(\tau_3) \\
&+ \varepsilon_x \varepsilon_y \varepsilon_z \left(\frac{\overline{v_0(v_0-1)}}{v_0(v_0-1)} - \frac{\overline{v_0} \overline{v(v-1)}}{\bar{v} \alpha \Lambda} \right) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda} \frac{F_0}{\Lambda^3} \left(e^{-\alpha \tau_1} e^{-\alpha \tau_3} u(\tau_1) u(\tau_2) u(\tau_3 - \tau_2) \right. \\
&+ e^{-\alpha \tau_2} e^{-\alpha \tau_3} u(\tau_1) u(\tau_2) u(\tau_3 - \tau_1) + e^{-\alpha \tau_2} e^{-\alpha \tau_3} u(\tau_1) u(\tau_2 - \tau_1) u(\tau_3) \left. \right) \\
&+ \varepsilon_x \varepsilon_y \varepsilon_z \frac{\bar{v}_0}{2 \bar{v} \alpha \Lambda} \left(\frac{\overline{v(v-1)(v-2)}}{v(v-1)(v-2)} + \frac{\overline{v(v-1)^2}}{\bar{v} \alpha \Lambda} \right) \frac{F_0}{\Lambda^3} e^{-\alpha \tau_2} e^{-\alpha(\tau_3 - \tau_1)} u(\tau_1) u(\tau_2 - \tau_1) u(\tau_3 - \tau_1) \\
&+ \varepsilon_x \varepsilon_y \varepsilon_z \frac{\overline{v_0} \overline{v(v-1)^2}}{\bar{v}^2 \alpha^2 \Lambda^2} \frac{F_0}{\Lambda^3} e^{-\alpha \tau_3} u(\tau_1) u(\tau_2 - \tau_1) u(\tau_3 - \tau_2) , \quad \tau_3 > \tau_2 > \tau_1 .
\end{aligned} \tag{2.99}$$

Of course, a tricorrelated quadruplet (w, x, y, z) need not occur in exactly that sequence, although the source fission w must occur first (it starts everything). The other five permuted sequences fill the remainder of the first octant of (τ_1, τ_2, τ_3) space where the trivariance

is non-zero:

$$\begin{aligned}
 (w, x, y, z) &: \tau_3 > \tau_2 > \tau_1 > 0 \\
 (w, x, z, y) &: \tau_2 > \tau_3 > \tau_1 > 0 \\
 (w, y, x, z) &: \tau_3 > \tau_1 > \tau_2 > 0 \\
 (w, y, z, x) &: \tau_1 > \tau_3 > \tau_2 > 0 \\
 (w, z, x, y) &: \tau_2 > \tau_1 > \tau_3 > 0 \\
 (w, z, y, x) &: \tau_1 > \tau_2 > \tau_3 > 0
 \end{aligned} \tag{2.100}$$

So the permutation relations

$$\begin{aligned}
 C_{wxyz}(\tau_1, \tau_2, \tau_3) &= C_{wxzy}(\tau_1, \tau_3, \tau_2) \\
 &= C_{wyxz}(\tau_2, \tau_1, \tau_3) \\
 &= C_{wyzx}(\tau_2, \tau_3, \tau_1) \\
 &= C_{wzxy}(\tau_3, \tau_1, \tau_2) \\
 &= C_{wzyx}(\tau_3, \tau_2, \tau_1)
 \end{aligned} \tag{2.101}$$

complete the tricovariance within all of (τ_1, τ_2, τ_3) space:

$$\begin{aligned}
C_{wxyz}(\tau_1, \tau_2, \tau_3) &= \varepsilon_x \varepsilon_y \varepsilon_z \left[\left(\frac{\overline{v_0(v_0-1)(v_0-2)}}{\bar{v}\alpha\Lambda} - 3 \frac{\overline{v_0(v_0-1)} \overline{v(v-1)}}{\bar{v}\alpha\Lambda} \right) \right. \\
&- \frac{\bar{v}_0}{2\bar{v}\alpha\Lambda} \left(\frac{\overline{v(v-1)(v-2)}}{\bar{v}\alpha\Lambda} - 3 \frac{\overline{v(v-1)}^2}{\bar{v}\alpha\Lambda} \right) \left. \frac{F_0}{\Lambda^3} e^{-\alpha\tau_1} e^{-\alpha\tau_2} e^{-\alpha\tau_3} u(\tau_1) u(\tau_2) u(\tau_3) \right. \\
&+ \varepsilon_x \varepsilon_y \varepsilon_z \left(\frac{\overline{v_0(v_0-1)}}{\bar{v}\alpha\Lambda} - \frac{\overline{v_0} \overline{v(v-1)}}{\bar{v}\alpha\Lambda} \right) \frac{\overline{v(v-1)}}{\bar{v}\alpha\Lambda} \frac{F_0}{\Lambda^3} \left(e^{-\alpha\tau_1} e^{-\alpha\tau_3} u(\tau_1 - \tau_2) u(\tau_2) u(\tau_3) \right. \\
&+ e^{-\alpha\tau_1} e^{-\alpha\tau_2} u(\tau_1 - \tau_3) u(\tau_2) u(\tau_3) + e^{-\alpha\tau_1} e^{-\alpha\tau_2} u(\tau_1) u(\tau_2 - \tau_3) u(\tau_3) \\
&+ e^{-\alpha\tau_2} e^{-\alpha\tau_3} u(\tau_1) u(\tau_2 - \tau_1) u(\tau_3) + e^{-\alpha\tau_2} e^{-\alpha\tau_3} u(\tau_1) u(\tau_2) u(\tau_3 - \tau_1) \\
&\left. \left. + e^{-\alpha\tau_1} e^{-\alpha\tau_3} u(\tau_1) u(\tau_2) u(\tau_3 - \tau_2) \right) \right] \quad (2.102) \\
&+ \varepsilon_x \varepsilon_y \varepsilon_z \frac{\bar{v}_0}{2\bar{v}\alpha\Lambda} \left(\frac{\overline{v(v-1)(v-2)}}{\bar{v}\alpha\Lambda} + \frac{\overline{v(v-1)}^2}{\bar{v}\alpha\Lambda} \right) \frac{F_0}{\Lambda^3} \left(e^{-\alpha\tau_1} e^{-\alpha(\tau_2-\tau_3)} u(\tau_1 - \tau_2) u(\tau_2 - \tau_3) u(\tau_3) \right. \\
&+ e^{-\alpha\tau_1} e^{-\alpha(\tau_3-\tau_2)} u(\tau_1 - \tau_2) u(\tau_2) u(\tau_3 - \tau_2) + e^{-\alpha\tau_2} e^{-\alpha(\tau_3-\tau_1)} u(\tau_1) u(\tau_2 - \tau_1) u(\tau_3 - \tau_1) \left. \right) \\
&+ \varepsilon_x \varepsilon_y \varepsilon_z \frac{\overline{v_0} \overline{v(v-1)}^2}{\bar{v}^2 \alpha^2 \Lambda^2} \frac{F_0}{\Lambda^3} \left(e^{-\alpha\tau_1} u(\tau_3) u(\tau_2 - \tau_3) u(\tau_1 - \tau_2) + e^{-\alpha\tau_1} u(\tau_2) u(\tau_1 - \tau_2) u(\tau_3 - \tau_1) \right. \\
&+ e^{-\alpha\tau_2} u(\tau_3) u(\tau_1 - \tau_3) u(\tau_2 - \tau_1) + e^{-\alpha\tau_2} u(\tau_1) u(\tau_3 - \tau_1) u(\tau_2 - \tau_3) \\
&\left. \left. + e^{-\alpha\tau_3} u(\tau_2) u(\tau_1 - \tau_2) u(\tau_3 - \tau_1) + e^{-\alpha\tau_3} u(\tau_1) u(\tau_2 - \tau_1) u(\tau_3 - \tau_2) \right) \right].
\end{aligned}$$

Further recognize that three bicornelated counts x , y , and z may be observed without attributing them to a particular source fission. That is

$$p_{xyz}^{U_3}(t_1, t_2, t_3) dt_1 dt_2 dt_3 = \int_{-\infty}^{\min(t_1, t_2, t_3)} dt_0' \dot{p}_{wxyz}^{U_3}(t_0', t_1, t_2, t_3) dt_1 dt_2 dt_3, \quad (2.103)$$

where again U_3 is shorthand for the eight ways that a bicornelated triplet (x, y, z) may be produced from a source fission w . Accidentally bicornelated triplets are produced via the processes

$$\begin{aligned}
p_{xyz|1,2}(t_1, t_2, t_3) dt_1 dt_2 dt_3 &= p_x(t_1) dt_1 p_{yz}^{U_2}(t_2, t_3) dt_2 dt_3 \\
&+ p_y(t_1) dt_2 p_{xz}^{U_2}(t_1, t_3) dt_1 dt_3 \\
&+ p_z(t_1) dt_3 p_{xy}^{U_2}(t_1, t_2) dt_1 dt_2 ,
\end{aligned} \tag{2.104}$$

$$p_{xyz|3}(t_1, t_2, t_3) dt_1 dt_2 dt_3 = p_x(t_1) dt_1 p_y(t_2) dt_2 p_z(t_3) dt_3 .$$

such that the between-detectors bicornelation is

$$\begin{aligned}
R_{xyz}(\tau_1, \tau_2) &= E[x(t_1)y(t_2)z(t_3)] \Big|_{\substack{\tau_1 = t_2 - t_1 \\ \tau_2 = t_3 - t_1}} \\
&= C_{xyz}(\tau_1, \tau_2) + \bar{x} C_{yz}(\tau_2 - \tau_1) + \bar{y} C_{xz}(\tau_2) + \bar{z} C_{xy}(\tau_1) + \bar{x} \bar{y} \bar{z} ,
\end{aligned} \tag{2.105}$$

where the between-detectors covariance,

$$\begin{aligned}
C_{xyz}(\tau_1, \tau_2) &= \\
&\varepsilon_x \varepsilon_y \varepsilon_z \left(\frac{1}{v_0(v_0 - 1)(v_0 - 2)} + \frac{\bar{v}_0 \bar{v}(\bar{v} - 1)(\bar{v} - 2)}{\bar{v} \alpha \Lambda} \right) \frac{F_0}{3 \alpha \Lambda^3} \\
&\cdot \left(e^{-\alpha \tau_1} e^{-\alpha \tau_2} u(\tau_1) u(\tau_2) + e^{\alpha \tau_1} e^{-\alpha(\tau_2 - \tau_1)} u(-\tau_1) u(\tau_2 - \tau_1) + e^{\alpha \tau_2} e^{-\alpha(\tau_1 - \tau_2)} u(\tau_1 - \tau_2) u(-\tau_2) \right) \\
&+ \varepsilon_x \varepsilon_y \varepsilon_z \left(\frac{1}{v_0(v_0 - 1)} + \frac{\bar{v}_0 \bar{v}(\bar{v} - 1)}{\bar{v} \alpha \Lambda} \right) \frac{\bar{v}(\bar{v} - 1)}{\bar{v} \alpha \Lambda} \frac{F_0}{2 \alpha \Lambda^3} \\
&\cdot \left(e^{-\alpha \tau_2} u(\tau_1) u(\tau_2 - \tau_1) + e^{-\alpha \tau_1} u(\tau_1 - \tau_2) u(\tau_2) + e^{-\alpha(\tau_2 - \tau_1)} u(-\tau_1) u(\tau_2) \right. \\
&\left. e^{\alpha \tau_1} u(\tau_2 - \tau_1) u(-\tau_2) + e^{-\alpha(\tau_1 - \tau_2)} u(\tau_1) u(-\tau_2) + e^{\alpha \tau_2} u(-\tau_1) u(\tau_1 - \tau_2) \right) ,
\end{aligned} \tag{2.106}$$

is the expected rate of bicornelated triplets of counts.

Now, observe that the source-detector tricovariance reflects the expected distribution of bicornelated triplets following a source fission

$$\begin{aligned}
C_{xyz}(\tau_1, \tau_2) &= \int_0^\infty d\tau'_1 C_{wxyz}(\tau'_1, \tau'_1 + \tau_1, \tau'_1 + \tau_2) \\
&= \int_0^\infty d\tau'_2 C_{wxyz}(\tau'_2 - \tau_1, \tau'_2, \tau'_2 + \tau_2 - \tau_1) \\
&= \int_0^\infty d\tau'_3 C_{wxyz}(\tau'_3 - \tau_2, \tau'_3 - \tau_2 + \tau_1, \tau'_3) .
\end{aligned} \tag{2.107}$$

Consequently,

$$\int_{-\infty}^\infty d\tau_1 \int_{-\infty}^\infty d\tau_2 C_{xyz}(\tau_1, \tau_2) = \int_0^\infty d\tau'_1 \int_0^\infty d\tau'_2 \int_0^\infty d\tau'_3 C_{wxyz}(\tau'_1, \tau'_2, \tau'_3) , \tag{2.108}$$

because each bicorrelated triplet of detector counts (x, y, z) belongs to some source-tricorrelated quadruplet (w, x, y, z) if the system contains no internal sources and the inherent source due to delayed neutrons is negligible.

There are some interesting observations to be made regarding the relationships between singlets, doublets, and triplets. First recall the probability of a singlet (a count) is

$$p_x(t_1) dt_1 = \varepsilon_x \frac{\bar{v}_0}{|\rho|} F_0 dt_1 , \tag{2.109}$$

i.e., the expected count rate follows the rate of neutron injection into the system $(\bar{v}_0 F_0)$; the factor $1/|\rho|$ accounts for prompt fission-chain multiplication of neutrons injected into the system. Hence the number of singlets is inversely proportional to reactivity.

Now recall the probability of a correlated doublet of counts is

$$p_{xy}(t_1, t_2) dt_1 dt_2 = \varepsilon_x \varepsilon_y \frac{1}{2|\rho|} \left(\frac{\bar{v}_0}{v_0(v_0-1)} + \frac{\bar{v}_0 \overline{v(v-1)}}{\bar{v}|\rho|} \right) F_0 h(t_2 | t_1) dt_1 dt_2 , \tag{2.110}$$

where $h(t | t') dt$ denotes the point reactor kernel, i.e., the expected number of neutrons

present at t given that one neutron was injected into the system at t' :

$$p(t|t') dt = \frac{e^{-\alpha(t-t')}}{\Lambda} u(t-t') dt. \quad (2.111)$$

So, given the occurrence of the first count, the probability of a subsequent correlated count is

$$p_{y|x}(t_2|t_1) dt_2 = \epsilon_y \frac{1}{2} \left(\frac{\overline{v_0(v_0-1)}}{\bar{v}_0} + \frac{\overline{v(v-1)}}{\bar{v}|\rho|} \right) h(t_2|t_1) dt_2, \quad (2.112)$$

which is the probability that the second count shares an ancestor with the first. Consequently, $p_{y|x}$ represents the distribution of doublets per singlet. The first term, $\overline{v_0(v_0-1)}/\bar{v}_0$, reflects the expected number of dual-branch fission-chains ($\overline{v_0(v_0-1)}$) originating at the source per source neutron ($1/\bar{v}_0$). The second term, $\overline{v(v-1)}/\bar{v}|\rho|$, indicates the number of dual-branch chains induced in the system per neutron; the factor $1/|\rho|$ accounts for prompt fission-chain multiplication of neutrons injected into the system. In other words, the first term reflects the probability that the two counts are descendants of a single source emission of multiple neutrons, while the second term reflects the probability that the two counts are descended from a single source neutron. Figure 2.13 illustrates these two processes. Clearly, the second process will dominate in systems approaching critical, while the first will dominate in extremely subcritical systems. In fact, the first process only becomes the prevailing source of doublets when

$$\rho < - \frac{\overline{v(v-1)}}{\bar{v}} \bigg/ \frac{\bar{v}_0}{\overline{v_0(v_0-1)}}, \quad (2.113)$$

○ SOURCE FISSION

● INDUCED FISSION



Figure 2.13 Processes that produce correlated doublets

which is typically on the order of a hundred dollars subcritical. Consequently, in more reactive systems, the distribution of doublets per singlet is inversely proportional to reactivity according to $\overline{v(v-1)}/\bar{v}|\rho|$. In systems that are farther subcritical than (2.113), the distribution of doublets per singlet is controlled by the source multiplicity according to $\overline{v_0(v_0-1)}/\bar{v}_0$ and is essentially insensitive to reactivity.

Now recall that the probability of a bicorrelated triplet is

$$\begin{aligned}
 & p_{xyz}(t_1, t_2, t_3) dt_1 dt_2 dt_3 \\
 &= \varepsilon_x \varepsilon_y \varepsilon_z \frac{1}{3|\rho|} \left(\frac{\overline{v_0(v_0-1)(v_0-2)}}{\bar{v}_0} + \frac{\overline{v_0 v(v-1)(v-2)}}{\bar{v}|\rho|} \right) F_0 h(t_2|t_1) h(t_3|t_1) dt_1 dt_2 dt_3 \quad (2.114) \\
 &+ \varepsilon_x \varepsilon_y \varepsilon_z \frac{1}{2|\rho|} \left(\frac{\overline{v_0(v_0-1)}}{\bar{v}_0} + \frac{\overline{v_0 v(v-1)}}{\bar{v}|\rho|} \right) \frac{\overline{v(v-1)}}{\bar{v}|\rho|} F_0 h(t_2|t_1) h(t_3|t_2) dt_1 dt_2 dt_3,
 \end{aligned}$$

such that, given the occurrence of the first count, the probability of the subsequent doublet is

$$\begin{aligned}
 & p_{yz|x}(t_2, t_3|t_1) dt_2 dt_3 \\
 &= \varepsilon_y \varepsilon_z \frac{1}{3} \left(\frac{\overline{v_0(v_0-1)(v_0-2)}}{\bar{v}_0} + \frac{\overline{v(v-1)(v-2)}}{\bar{v}|\rho|} \right) h(t_2|t_1) dt_2 h(t_3|t_1) dt_3 \quad (2.115) \\
 &+ \varepsilon_y \varepsilon_z \frac{1}{2} \left(\frac{\overline{v_0(v_0-1)}}{\bar{v}_0} + \frac{\overline{v(v-1)}}{\bar{v}|\rho|} \right) \frac{\overline{v(v-1)}}{\bar{v}|\rho|} h(t_2|t_1) dt_2 h(t_3|t_2) dt_3,
 \end{aligned}$$

which is the probability that the correlated doublet has an ancestor in common with the initial count. Hence $p_{yz|x}$ represents the distribution of bicorrelated triplets per singlet. The first term, $\overline{v_0(v_0-1)(v_0-2)}/\bar{v}_0$, reflects the expected number of triple-branch fission-chains originating at the source per source neutron. In contrast, the second term, $\overline{v(v-1)(v-2)}/\bar{v}|\rho|$, indicates the number of triple-branch chains induced in the system per

injected neutron. The third term, $\overline{v_0(v_0-1)} / \overline{v_0} \cdot \overline{v(v-1)} / \overline{v} |\rho|$, reflects the expected number of dual-branch chains originating at the source ($\overline{v_0(v_0-1)} / \overline{v_0}$) that survive to spawn a second dual-branch chain in the system ($\overline{v(v-1)} / \overline{v} |\rho|$). Similarly, the last term, $\overline{v(v-1)^2} / \overline{v}^2 |\rho|^2$, indicates the number of dual-branch chains induced in the system that, in turn, initiate a second dual-branch chain. Refer to Fig. 2.14 for illustrations of these processes. The first term reflects the probability of process 1, the second term the probability of process 2. The third term indicates the probability of process 3, and the last term reflects the probability of process 4.

Now refer to Fig. 2.15. Process 1 is clearly only dominant in extremely subcritical systems. Processes 2 and 3 have roughly the same influence on the distribution of doublets per singlet; however, these prevail over process 4 only in extremely subcritical systems, at which point process 1 is most influential. Of course process 4 is preeminent in systems approaching critical, but it is also the primary process contributing to the occurrence of triplets over a broad range of subcritical reactivity. So processes 2 and 3 are only significant over a narrow range of subcritical reactivity; the asymptotic behavior of triplet production follows processes 1 and 4. In fact, process 1 only becomes the prevailing source of triplets when

$$\rho < - \frac{\overline{v(v-1)}}{\overline{v}} \left/ \sqrt{\frac{2}{3} \frac{\overline{v_0(v_0-1)(v_0-2)}}{\overline{v_0}}} \right. , \quad (2.116)$$

○ SOURCE FISSION

● INDUCED FISSION

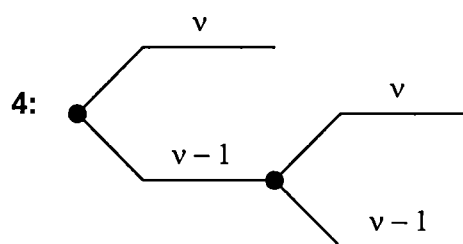
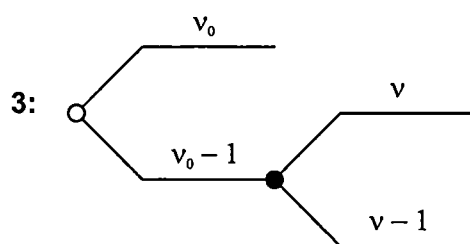
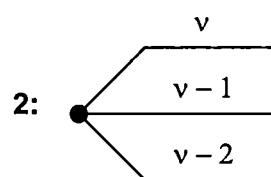
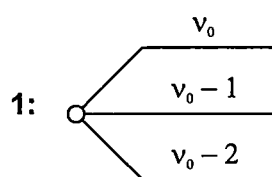


Figure 2.14 Processes that produce bicorrelated triplets of counts

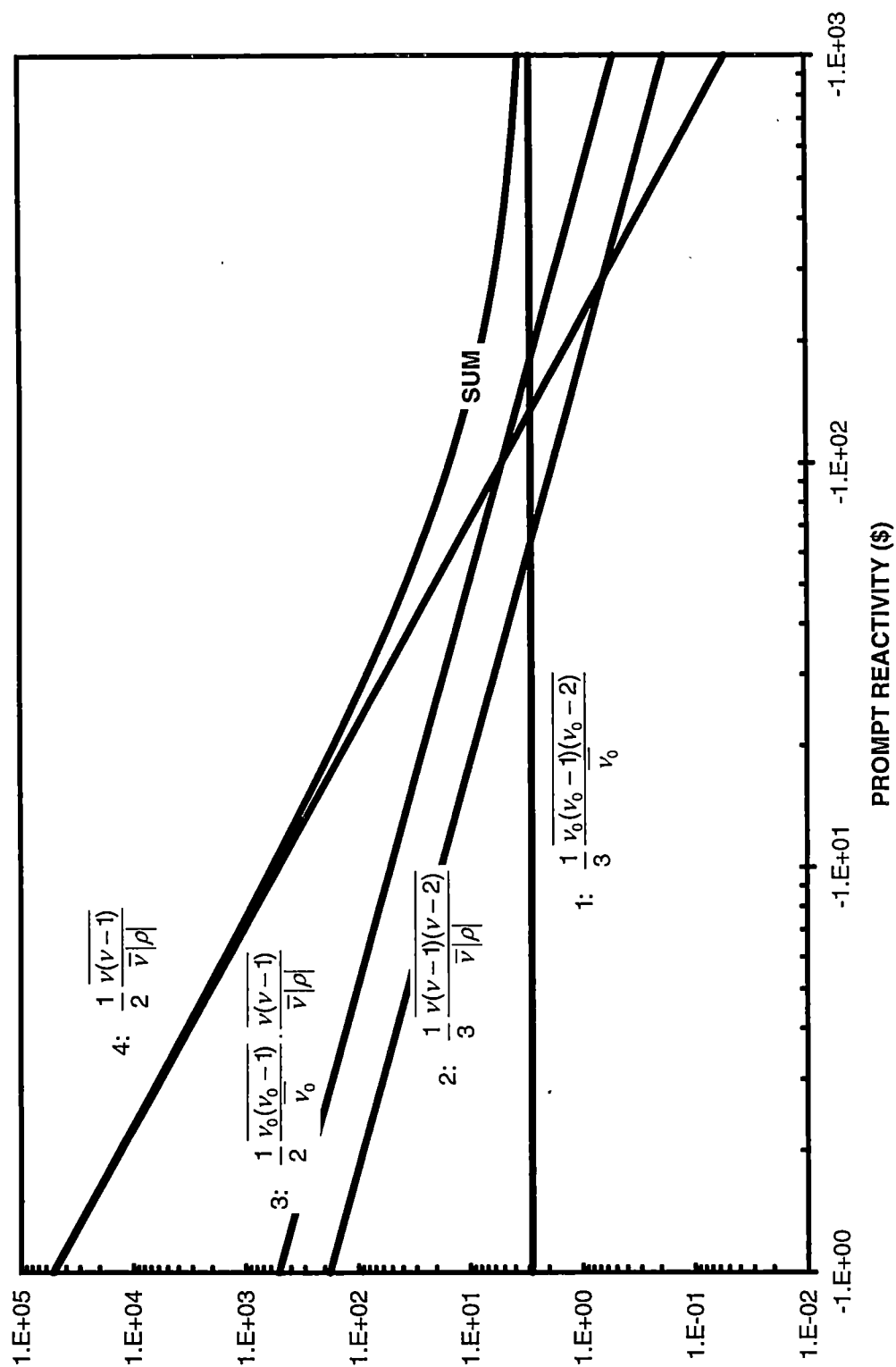


Figure 2.15 Influence of the processes that produce bicorrelated triplets on the distribution of triplets per singlet

which again is typically on the order of one hundred dollars subcritical, as Fig. 2.15 illustrates.⁶ In other words, for most subcritical systems, triplets are produced by series of dual-branch fission-chains initiated by a single source neutron; triple-branch fission-chains originating as the source are only significant in extremely subcritical systems. Consequently, for systems more reactive than (2.116), the distribution of triplets per singlet is inversely proportional to the square of reactivity according to $\overline{v(v-1)}^2 / \bar{v}^2 |\rho|^2$. Hence the number of triplets is more sensitive to reactivity than the number of doublets. Furthermore, in systems less reactive than (2.116), the distribution of triplets per singlet is essentially controlled by $\overline{v_0(v_0-1)(v_0-2)} / \bar{v}_0$, which is insensitive to reactivity.

This behavior is further illustrated by the following. Consider the probability of a third bicorrelated count following a correlated doublet:

$$\begin{aligned}
 p_{z|xy}(t_3|t_1, t_2) dt_3 \\
 = \varepsilon_z \frac{2}{3} \left(\frac{\overline{v_0(v_0-1)(v_0-2)}}{\bar{v}_0} + \frac{\overline{v(v-1)(v-2)}}{\bar{v}|\rho|} \right) \bigg/ \left(\frac{\overline{v_0(v_0-1)}}{\bar{v}_0} + \frac{\overline{v(v-1)}}{\bar{v}|\rho|} \right) h(t_3|t_1) dt_3 \quad (2.117) \\
 + \varepsilon_z \frac{\overline{v(v-1)}}{\bar{v}|\rho|} h(t_3|t_2) dt_3.
 \end{aligned}$$

That is, $p_{z|yx}$ represents the distribution of bicorrelated triplets per correlated doublet. The first term reflects the number of triple-branch chains relative to the number of dual-branch chains and hence the probability that each count in the triplet is descended from a single

⁶ The induced fission multiplicity moments $\bar{v} \approx 2.6$, $\overline{v(v-1)} \approx 5.5$, and $\overline{v(v-1)(v-2)} \approx 9.3$ used to generate Figs. 2.15 and 2.16 were calculated from Zucker-Holden $p(v|E)$ distributions folded with an MCNP-calculated spectrum for neutrons inducing fission in a 150-mm-diameter 93.15%-enriched uranium-metal sphere [Brie, Val1,4, Zuc]. Understand that the trends depicted have essentially the same asymptotic behavior regardless of the induced fission multiplicity moments.

fission event originating either at the source or within the system. In contrast, the second term, $\overline{v(v-1)}/\bar{v}|\rho|$, reflects the probability that the third count is related to the first two by an intermediate system fission. Now refer to Fig 2.16. Of course triple-branch chains originating at the source are dominant in extremely subcritical systems, while triple-branch chains originating in the system become more prevalent as criticality is approached. However, triplet production by the union of these two processes is essentially constant and is negligible compared to triplet production by series of dual-branch chains; this holds until the system is a hundred dollars or more subcritical. Therefore, in extremely subcritical systems, the distribution of triplets per doublet is essentially insensitive to reactivity. However, in more reactive systems, the distribution of triplets per doublet is inversely proportional to reactivity according to $\overline{v(v-1)}/\bar{v}|\rho|$.

Now observe that a pattern emerges. In more reactive systems, the number of singlets, the distribution of doublets per singlet, and the distribution of triplets per doublet are all sensitive to reactivity as $1/|\rho|$. Therefore the distribution of doublets is sensitive to reactivity as $1/|\rho|^2$, and the distribution of triplets is sensitive to reactivity as $1/|\rho|^3$. Consequently, triplets are more sensitive to reactivity than doublets, which are in turn more sensitive to reactivity than singlets. This is true unless the system is extremely subcritical, and then singlets, doublets, triplets, etc., are controlled by the source multiplicity distribution, which is insensitive to reactivity.

From the preceding pattern it may be deduced that, in more reactive systems, the distribution of n -tuplets per $(n-1)$ -tuplets is sensitive to reactivity as $1/|\rho|$, such that the

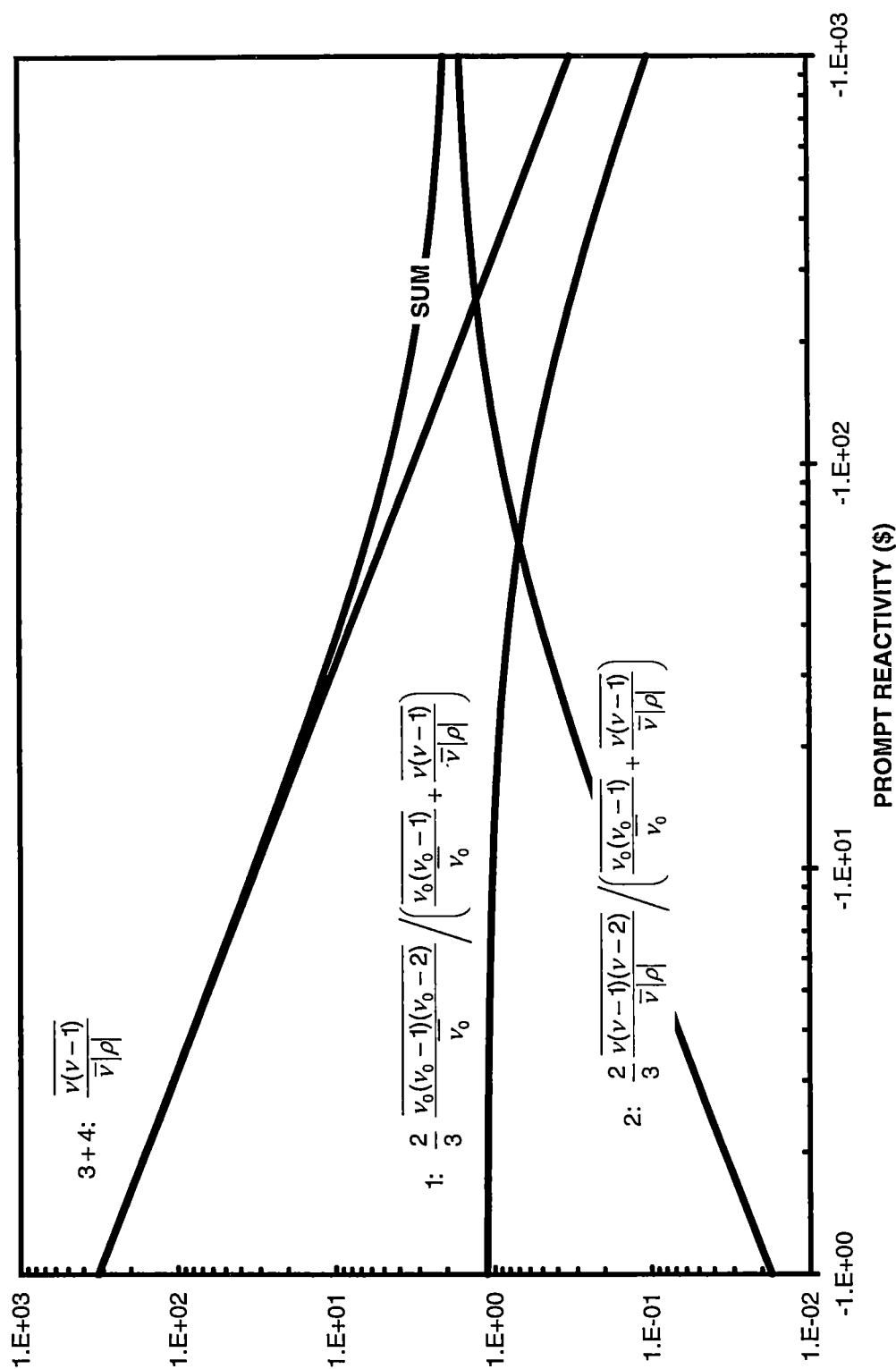


Figure 2.16 Influence of the processes that produce bicorrelated triplets on the distribution of triplets per doublet

number of n -tuplets is always more sensitive to reactivity than the number of $(n - 1)$ -tuplets. This is the fundamental reason that higher order statistics possess enhanced sensitivity to reactivity.

The observations made thus far were limited to a subcritical point reactor driven only by an observable spontaneous fission source. Of course, numerous fissile assemblies contain additional internal neutron sources that cannot be observed directly. These may include other spontaneous fission sources, like ^{240}Pu ; single-neutron emission sources, e.g., isotopes pairs that form (α, n) sources, may also be present. Furthermore, every fissile system contains an inherent source due to delayed neutrons. This source is substantial near delayed critical but less important farther from critical. Finally, most measurements observe some non-negligible background radiation that will be counted in the absence of any introduced source or fissile system. To distinguish it from other neutron sources, it is assumed that background radiation interacts with the fissile system at a negligible rate and so appears only as random counting events from some unidentifiable source (e.g., cosmic radiation, electronic noise, etc.).

The source-detector covariance, bicovariance, and tricovariance signatures C_{wx} , C_{wxy} , and C_{wxyz} of course do not respond to background radiation, internal neutron sources, or the inherent source because these are uncorrelated with the introduced, observable source w . However, the detector count rate $\bar{x}$ will respond to both background and any additional neutron sources that may be present. That is,

$$\bar{x} \rightarrow \varepsilon_x \frac{\bar{v}_0 F_0 + S'}{\alpha \Lambda} + B, \quad (2.118)$$

where S' is the neutron emission rate of internal and inherent neutron sources and B denotes the background count rate. Consequently, the generalization of (2.65) is

$$\int_0^{\infty} d\tau C_{wx}(\tau) \leq \bar{x}. \quad (2.119)$$

Still, the integral of the source-detector covariance will yield that portion of the detector count rate due to the introduced spontaneous fission source.

Assuming that background is not due to some unidentified spontaneous fission source interacting with the detectors, background events will not be correlated between detectors. However, the between-detectors covariance and bicovariance signatures, C_{xy} and C_{xyz} , will respond to any unobservable neutron sources that interact with the system. The response of these signatures to a spontaneous fission source will differ from that to a single-neutron emission source, so it is necessary to make a distinction between the two:

$$S' = \bar{v}_1 F_1 + Q. \quad (2.120)$$

Above, $\bar{v}_1$ and F_1 respectively denote the mean multiplicity and mean fission rate of unobservable spontaneous fission sources interacting with the system. The mean neutron emission rate of single-neutron sources, including the inherent delayed neutron source, is denoted by Q . The between-detectors covariance will respond to these additional sources as

$$C_{xy}(\tau) \rightarrow \varepsilon_x \varepsilon_y \left(\overline{v_0(v_0-1)} F_0 + \overline{v_1(v_1-1)} F_1 + (\bar{v}_0 F_0 + \bar{v}_1 F_1 + Q) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda} \right) \frac{e^{-\alpha \tau}}{2 \alpha \Lambda^2}, \quad \tau > 0. \quad (2.121)$$

In fact situations have been encountered where these additional sources mask this signature's response to the introduced source. However, although in general

$$\int_0^{\infty} d\tau_1 \int_0^{\infty} d\tau_2 C_{wxy}(\tau_1, \tau_2) \leq \int_{-\infty}^{\infty} d\tau C_{xy}(\tau) , \quad (2.122)$$

the integral of the source-detector bicovariance still yields that portion of the between-detectors covariance that is due to the introduced, observable source.

The between-detectors bicovariance will similarly respond to additional spontaneous fission and single-neutron sources according to

$$\begin{aligned} C_{xyz}(\tau_1, \tau_2) \rightarrow & \varepsilon_x \varepsilon_y \varepsilon_z \left(\frac{\overline{v_0(v_0-1)(v_0-2)} F_0 + \overline{v_1(v_1-1)(v_1-2)} F_1 + (\bar{v}_0 F_0 + \bar{v}_1 F_1 + Q) \frac{\overline{v(v-1)(v-2)}}{\bar{v} \alpha \Lambda}}{\bar{v} \alpha \Lambda} \right) \\ & \cdot \frac{e^{-\alpha \tau_1} e^{-\alpha \tau_2}}{3 \alpha \Lambda^3} u(\tau_1) u(\tau_2) \\ & + \varepsilon_x \varepsilon_y \varepsilon_z \left(\frac{\overline{v_0(v_0-1)} F_0 + \overline{v_1(v_1-1)} F_1 + (\bar{v}_0 F_0 + \bar{v}_1 F_1 + Q) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda}}{\bar{v} \alpha \Lambda} \right) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda} \\ & \cdot \frac{e^{-\alpha \tau_2}}{2 \alpha \Lambda^3} u(\tau_1) u(\tau_2 - \tau_1) , \quad \tau_2 > \tau_1 > 0 , \end{aligned} \quad (2.123)$$

and it is easily conceivable that these additional sources could mask a system's bicorrelated response to the introduced source, although enough measurements of this signature have not yet been performed to observe this phenomenon quantitatively. So in general,

$$\int_0^{\infty} d\tau'_1 \int_0^{\infty} d\tau'_2 \int_0^{\infty} d\tau'_3 C_{wxyz}(\tau'_1, \tau'_2, \tau'_3) \leq \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 C_{xyz}(\tau_1, \tau_2) . \quad (2.124)$$

However, that portion of the between-detectors bicovariance that is due to the introduced source can still be obtained by the integral of the source-detector tricovariance.

There are a couple of univariate signatures that, although they are not the focus of this

discourse, still deserve some brief discussion. First is the detector autocovariance,

$$C_{xx}(\tau) = \bar{x} \delta(\tau) + \varepsilon_x^2 \left(\overline{v_0(v_0-1)} F_0 + \overline{v_1(v_1-1)} F_1 + (\bar{v}_0 F_0 + \bar{v}_1 F_1 + Q) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda} \right) \frac{e^{-\alpha \tau}}{2 \alpha \Lambda^2} u(\tau), \quad \tau > 0. \quad (2.125)$$

The first term appears because each count, besides being correlated with other counts, is of course correlated with itself:

$$p_{xx}(t_1, t_2) dt_1 dt_2 = \delta(t_2 - t_1) dt_2 p_x(t_1) dt_1 + p_{x|x}(t_2 | t_1) dt_2 p_x(t_1) dt_1. \quad (2.126)$$

Consequently, the detector autocovariance C_{xx} , like the detector count rate $\bar{x}$, is sensitive to background. In addition, note that because the detector count rate follows ε_x , while the autocovariance follows ε_x^2 , a detector autocovariance measured under conditions of extremely low detection efficiency will essentially yield another measure of the count rate:

$$C_{xx}(\tau) \approx \bar{x} \delta(\tau), \quad \varepsilon_x \ll 1. \quad (2.127)$$

Furthermore, because each count is bicornelated with itself as well as with other counts,

$$p_{xxx}(t_1, t_2) dt_1 dt_2 = \delta(t_3 - t_2) dt_3 \delta(t_2 - t_1) dt_2 p_x(t_1) dt_1 + p_{x|xx}(t_3 | t_1, t_2) dt_3 p_{x|x}(t_2 | t_1) dt_2 p_x(t_1) dt_1, \quad (2.128)$$

the detector autobicovariance exhibits similar behavior:

$$\begin{aligned}
C_{xx}(\tau_1, \tau_2) &= \bar{x} \delta(\tau_1) \delta(\tau_2) \\
&+ \varepsilon_x^3 \left(\overline{v_0(v_0-1)(v_0-2)} F_0 + \overline{v_1(v_1-1)(v_1-2)} F_1 + (\bar{v}_0 F_0 + \bar{v}_1 F_1 + Q) \frac{\overline{v(v-1)(v-2)}}{\bar{v} \alpha \Lambda} \right) \\
&\quad \cdot \frac{e^{-\alpha \tau_1} e^{-\alpha \tau_2}}{3 \alpha \Lambda^3} u(\tau_1) u(\tau_2) \\
&+ \varepsilon_x^3 \left(\overline{v_0(v_0-1)} F_0 + \overline{v_1(v_1-1)} F_1 + (\bar{v}_0 F_0 + \bar{v}_1 F_1 + Q) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda} \right) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda} \\
&\quad \cdot \frac{e^{-\alpha \tau_2}}{2 \alpha \Lambda^3} u(\tau_1) u(\tau_2 - \tau_1), \quad \tau_2 > \tau_1 > 0,
\end{aligned} \tag{2.129}$$

such that, for extremely low detection efficiency

$$C_{xx}(\tau_1, \tau_2) \approx \bar{x} \delta(\tau_1) \delta(\tau_2), \quad \varepsilon_x \ll 1. \tag{2.130}$$

In other words, these signatures may yield little more than another measure of count rate when detection efficiency is low, as is sometimes the case in practice.

Finally, detector responses typically exhibit some characteristic dead-time, i.e., a period after a count is registered during which no further counts can be registered or during which the probability of further counts is diminished. The simplest model is that of a single fixed dead-time, say τ_x , such that following a count no further counts may be registered for a period of τ_x :

$$p_{x|x}(t' | t) dt' \rightarrow u(t' - t - \tau_x) p_{x|x}(t' | t) dt', \quad t' > t. \tag{2.131}$$

This characteristic will appear in both the autocovariance and autocovariance as

$$\begin{aligned}
C_{xx}(\tau) &= \bar{x} \delta(\tau) \\
&+ \varepsilon_x^2 \left(\overline{v_0(v_0-1)} F_0 + \overline{v_1(v_1-1)} F_1 + (\bar{v}_0 F_0 + \bar{v}_1 F_1 + Q) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda} \right) \frac{e^{-\alpha \tau}}{2 \alpha \Lambda^2} u(\tau - \tau_x), \quad (2.132) \\
&\tau > 0,
\end{aligned}$$

and

$$\begin{aligned}
C_{xx}(\tau_1, \tau_2) &= \bar{x} \delta(\tau_1) \delta(\tau_2) \\
&+ \varepsilon_x^3 \left(\overline{v_0(v_0-1)(v_0-2)} F_0 + \overline{v_1(v_1-1)(v_1-2)} F_1 + (\bar{v}_0 F_0 + \bar{v}_1 F_1 + Q) \frac{\overline{v(v-1)(v-2)}}{\bar{v} \alpha \Lambda} \right) \\
&\quad \cdot \frac{e^{-\alpha \tau_1} e^{-\alpha \tau_2}}{3 \alpha \Lambda^3} u(\tau_1 - \tau_x) u(\tau_2 - \tau_x) \\
&\quad + \varepsilon_x^3 \left(\overline{v_0(v_0-1)} F_0 + \overline{v_1(v_1-1)} F_1 + (\bar{v}_0 F_0 + \bar{v}_1 F_1 + Q) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda} \right) \frac{\overline{v(v-1)}}{\bar{v} \alpha \Lambda} \\
&\quad \cdot \frac{e^{-\alpha \tau_2}}{2 \alpha \Lambda^3} u(\tau_1 - \tau_x) u(\tau_2 - \tau_1 - \tau_x), \quad \tau_2 > \tau_1 > 0. \quad (2.133)
\end{aligned}$$

Observe that a sufficiently long dead-time may in fact eliminate the majority of correlated and bicornelated counts, since their probability declines at an exponential rate. The dead-time characteristic of univariate detector signatures, their deficiency at low detection efficiency, and their sensitivity to background led to their infrequent use in favor of multivariate detector signatures in the past. For these same reasons, they will receive only passing mention in subsequent discussions.

2.4 Spectral Signatures

Thus far, the signatures as described do not account for the effects of less-than-ideal detection electronics, which are an integral feature of any measurement. Specifically, measured signatures are the convolution of the actual signatures with the response of detection system electronics. Consider that the measured response of a detector, $a^1(t)$, is the convolution of its actual response, $a(t)$, with the detector's impulse response $h_a(t)$:

$$a^1(t) = \int dt_a h_a(t_a) a(t - t_a) . \quad (2.134)$$

The actual detection events, i.e., the neutron capture events, occur on such a small time-scale that they are essentially impulse events. The detector's response to each such impulse event is typically much slower. For example, in a proportional counter some time is required for the avalanche of electrons to reach the anode following a capture event. The slow rise and even slower decay of the ensuing electrical pulse is characteristic of that detector's response. Hence, although the detector and its associated electronics produce a signal that permits measurement of the time of an individual capture event, they do so imprecisely. The uncertainty in timing "blurs" the actual event. Consequently, the detector's impulse response filters the sequence of impulse events to produce a stream of electrical pulses of finite duration. The effects of this filtering process are more easily examined in the Laplace domain, where time-domain convolution appears as simple multiplication. Each of the following spectral signatures is simply the Laplace domain counterpart of one of the preceding temporal covariance signatures.

Consider then that the measured detector count rate is related to the actual event rate by

$$\bar{a}^1 = \left(\int dt h_a(t) \right) \bar{a} = H_a(0) \bar{a} = \varepsilon_a H_a(0) H(0) \bar{v}_0 F_0 , \quad (2.135)$$

where $H(s)$ is the transfer function of the fissile system,

$$H(s) = \int dt e^{-st} \frac{e^{-\alpha t}}{\Lambda} u(t) = \frac{1}{\Lambda(\alpha + s)} , \quad (2.136)$$

and $H_a(s)$ is the detector's transfer function,

$$H_a(s) = \int dt e^{-st} h_a(t) . \quad (2.137)$$

Specifically, $H_a(0)$ is the gain of the detector electronics:

$$H_a(0) = \frac{\bar{a}^1}{\bar{a}} . \quad (2.138)$$

How may the detection system response be deconvolved from the measured count rate? Let $\bar{a}'$ represent the actual event rate when the detector is presented to a bare source, i.e., when no fissile system is present, such that the count rate measured from the bare source is

$$\bar{a}^0 = H_a(0) \bar{a}' = \varepsilon_a^0 H_a(0) \bar{v}_0 F_0 . \quad (2.139)$$

The efficiency ε_a^0 for counting neutrons directly from the source will necessarily differ from ε_a because the source emission spectrum will differ from the spectrum of neutrons interacting with the detector when the fissile system is present. However, the ratio of the count rate measured in the presence of the fissile system to that measured from the source alone is independent of source strength and the response of the detection electronics

$$\frac{\bar{a}^1}{\bar{a}^0} = \frac{\varepsilon_a}{\varepsilon_a^0} H(0) . \quad (2.140)$$

Of course, this method of deconvolution does not remove the effects of detection efficiency. Furthermore, it obviously cannot be used to correct the measured source count rate, should the source efficiency or the gain of the source electronics happen to deviate from 100%. A discussion of measurements performed under the conditions of degraded source efficiency and response is deferred to later this chapter.

Observe that the measured covariance between a and b is

$$C_{ab}^1(\tau) = \int dt_a \int dt_b h_a(t_a) h_b(t_b) C_{ab}(\tau + t_a - t_b) , \quad (2.141)$$

where C_{ab} is the actual covariance between a and b . Subsequently, observe that the equivalent spectral signature, the measured cross-spectrum between a and b is

$$S_{ab}^1(s) = \int d\tau e^{-s\tau} C_{ab}^1(\tau) = H_a(-s) H_b(s) S_{ab}(s) . \quad (2.142)$$

Consequently the effects of the impulse responses h_a and h_b on this signature may be deconvolved in the Laplace domain using a procedure similar to that previously described for count rates. Consider that, if S_{ab}' represents the actual cross-spectrum between a and b due to a bare source, the cross-spectrum measured from the source alone is

$$S_{ab}^0(s) = H_a(-s) H_b(s) S_{ab}'(s) , \quad (2.143)$$

such that the ratio of the two is independent of the detection electronics response:

$$\frac{S_{ab}^1(s)}{S_{ab}^0(s)} = \frac{S_{ab}(s)}{S_{ab}'(s)} . \quad (2.144)$$

Recognize that this procedure will also remove the effects of degraded source efficiency and response if the one of the signals a or b is acquired from the source.

Consequently, such a measurement will be repeatable for a given fissile system even if some characteristics of the instrumentation change. This ease of deconvolution is the primary reason for using spectral signatures in tandem with temporal covariance signatures. There are of course limits to this technique. Most importantly, detection efficiency in the context of a real measurement is more than just the intrinsic efficiency of the detection material. Detection efficiency is also affected by electronic settings, particularly by discriminator levels that determine, on the basis of pulse height, whether a pulse will be counted or ignored. The use of discriminators is necessary to eliminate spurious counts due to low amplitude electrical noise; the discriminator threshold typically sets a lower bound on pulse heights that will be registered as a count. So clearly discriminator levels can have a significant effect on detection efficiency, particularly if thresholds are set sufficiently high to begin discriminating electrical pulses that correspond to actual detection events. However, if care is taken to recurrently use detectors of the same construction and to ensure that discriminator thresholds are not changed, a measurement can be repeated despite changes in the detection electronics like degraded (or enhanced) timing resolution and gain. Still, major changes in instrumentation will make repeat measurements difficult if not impossible. For example, no one would try to repeat a measurement performed with a ^{252}Cf source and proportional counters later using an D-T source and plastic scintillators. Furthermore, the whole method falls apart if the measurements are performed with detectors whose response is much slower than the response of the fissile system; under such conditions the detectors' timing characteristics, and not the system's, are measured. However, this technique can tolerate significant changes in source strength and moderate changes in detector response.

$$R_{ab}(\tau) = C_{ab}(\tau) + \bar{a} \bar{b} , \quad (2.149)$$

where all the truly correlated information is contained in the covariance C_{ab} . The corresponding spectral signature is

$$\int d\tau e^{-s\tau} R_{ab}(\tau) = S_{ab}(s) + j2\pi \bar{a} \bar{b} \delta(s) , \quad (2.150)$$

where S_{ab} is the cross-spectrum between a and b (j denotes the imaginary number).

Because the accidental coincidence rate $\bar{a} \bar{b}$ only appears at $s = 0$, it may be eliminated by discounting the discontinuity at this point. Similarly, the accidental bicoincidence rate contributes to a bicoincidence signature via

$$R_{abc}(\tau_1, \tau_2) = C_{abc}(\tau_1, \tau_2) + \bar{a} C_{bc}(\tau_2 - \tau_1) + \bar{b} C_{ac}(\tau_2) + \bar{c} C_{ab}(\tau_1) + \bar{a} \bar{b} \bar{c} , \quad (2.151)$$

so the corresponding spectral signature is

$$\begin{aligned} & \int d\tau_1 e^{-s_1 \tau_1} \int d\tau_2 e^{-s_2 \tau_2} R_{abc}(\tau_1, \tau_2) \\ &= S_{abc}(s_1, s_2) \\ &+ j2\pi \left(\bar{a} S_{bc}(s_2) \delta(s_1 + s_2) + \bar{b} S_{ac}(s_2) \delta(s_1) + \bar{c} S_{ab}(s_1) \delta(s_2) \right) \\ &- 4\pi^2 \bar{a} \bar{b} \bar{c} \delta(s_1) \delta(s_2) , \end{aligned} \quad (2.152)$$

where S_{abc} is the cross-bispectrum (bispectrum for short) between a , b , and c . Therefore, the accidental contributions to the spectral signature appear as discontinuities at the point $s_1 = s_2 = 0$ and along the lines $s_1 = 0$, $s_2 = 0$, and $s_1 = -s_2$. Finally, because the accidental tricoincidence rate contributes to a tricoincidence signature via

$$\begin{aligned}
R_{abcd}(\tau_1, \tau_2, \tau_3) &= C_{abcd}(\tau_1, \tau_2, \tau_3) + \\
&\bar{a} C_{bcd}(\tau_2 - \tau_1, \tau_3 - \tau_1) + \bar{b} C_{acd}(\tau_2, \tau_3) + \bar{c} C_{abd}(\tau_1, \tau_3) + \bar{d} C_{abc}(\tau_1, \tau_2) + \\
&C_{ab}(\tau_1) C_{cd}(\tau_3 - \tau_2) + C_{ac}(\tau_2) C_{bd}(\tau_3 - \tau_1) + C_{ad}(\tau_3) C_{bc}(\tau_2 - \tau_1) + \\
&\bar{a} \bar{b} C_{cd}(\tau_3 - \tau_2) + \bar{a} \bar{c} C_{bd}(\tau_3 - \tau_1) + \bar{a} \bar{d} C_{bc}(\tau_2 - \tau_1) + \\
&\bar{b} \bar{c} C_{ad}(\tau_3) + \bar{b} \bar{d} C_{ac}(\tau_2) + \bar{c} \bar{d} C_{ab}(\tau_1) + \\
&\bar{a} \bar{b} \bar{c} \bar{d} ,
\end{aligned} \tag{2.153}$$

the corresponding spectral signature is

$$\begin{aligned}
&\int d\tau_1 e^{-s_1 \tau_1} \int d\tau_2 e^{-s_2 \tau_2} \int d\tau_3 e^{-s_3 \tau_3} R_{abcd}(\tau_1, \tau_2, \tau_3) \\
&= S_{abcd}(s_1, s_2, s_3) \\
&+ j2\pi \left(\bar{a} S_{bcd}(s_2, s_3) \delta(s_1 + s_2 + s_3) + \bar{b} S_{acd}(s_2, s_3) \delta(s_1) + \right. \\
&\quad \bar{c} S_{abd}(s_1, s_3) \delta(s_2) + \bar{d} S_{abc}(s_1, s_2) \delta(s_3) + S_{ab}(s_1) S_{cd}(s_3) \delta(s_2 + s_3) + \\
&\quad \left. S_{ac}(s_2) S_{bd}(s_3) \delta(s_1 + s_3) + S_{ad}(s_3) S_{bc}(s_2) \delta(s_1 + s_2) \right) \\
&- 4\pi^2 \left(\bar{a} \bar{b} S_{cd}(s_3) \delta(s_1) \delta(s_2 + s_3) + \bar{a} \bar{c} S_{bd}(s_3) \delta(s_2) \delta(s_1 + s_3) + \right. \\
&\quad \bar{a} \bar{d} S_{bc}(s_2) \delta(s_3) \delta(s_1 + s_2) + \bar{b} \bar{c} S_{ad}(s_3) \delta(s_1) \delta(s_2) + \\
&\quad \left. \bar{b} \bar{d} S_{ac}(s_2) \delta(s_1) \delta(s_3) + \bar{c} \bar{d} S_{ab}(s_1) \delta(s_2) \delta(s_3) \right) \\
&- j8\pi^3 \bar{a} \bar{b} \bar{c} \bar{d} \delta(s_1) \delta(s_2) \delta(s_3) .
\end{aligned} \tag{2.154}$$

This is of course the trispectrum S_{abcd} between a, b, c , and d with additional discontinuities at the point $s_1 = s_2 = s_3 = 0$, along the lines $s_1 = s_2 = 0$, $s_1 = s_3 = 0$, $s_2 = s_3 = 0$, $s_1 = s_2 + s_3 = 0$, $s_2 = s_1 + s_3 = 0$, and $s_3 = s_1 + s_2 = 0$, and on the planes $s_1 = 0$, $s_2 = 0$, $s_3 = 0$, $s_1 + s_2 + s_3 = 0$, $s_1 = -s_2$, $s_1 = -s_3$, and $s_2 = -s_3$. These discontinuities arise from the various contributions to the accidental tricoincidence rate.

So spectral signatures are really nothing more than another representation of temporal signatures with several convenient features. In particular, because the spectral signatures are integral transformations of the temporal signatures, they tend to more readily indicate small changes in the system under observation.

Now observe that the source-detector cross-spectrum is

$$S_{wx}(s) = \int d\tau e^{-s\tau} C_{wx}(\tau) = \frac{\epsilon_x \bar{v}_0 F_0}{\Lambda(\alpha + s)} \quad (2.155)$$

Recall that for a point reactor, because only the fundamental mode is significant, the source-detector cross-spectrum rolls off at a rate of one decade per decade past its single break-frequency α :

$$S_{wx}(s) \approx \frac{\epsilon_x \bar{v}_0 F_0}{\Lambda s}, \quad s \gg \alpha, \quad (2.156)$$

and its asymptotic behavior at low frequencies is

$$S_{wx}(s) \approx \frac{\epsilon_x \bar{v}_0 F_0}{\alpha \Lambda} = \frac{\epsilon_x \bar{v}_0 F_0}{|\rho|}, \quad s \ll \alpha. \quad (2.157)$$

So the source-detector signature is as sensitive to reactivity as the detector count rate:

$$\bar{x} = \frac{\epsilon_x \bar{v}_0 F_0}{|\rho|}, \quad (2.158)$$

if internal and inherent neutron sources and background contribute to the count rate negligibly. Further consider that, under these conditions, even in a system that does not behave as a point reactor, all counts are correlated with some initiating source fission,

$$p_x(t_1) dt_1 = \int dt_0 p_{x|w}(t_1 | t_0) dt_1 p_w(t_0), \quad (2.159)$$

such that

$$\bar{x} = \int d\tau C_{wx}(\tau) , \quad (2.160)$$

or, in terms of the source-detector cross-spectrum,

$$\bar{x} = S_{wx}(0) . \quad (2.161)$$

However, the sensitivity of the count rate can be masked by background and additional neutron sources, while the source-detector signature will only respond to the fission rate induced by the introduced source. Hence, the source-detector signature can be more useful in applications where contributions to the count rate from background and additional neutron sources are substantial or difficult to evaluate.

Further consider that the count rate and source-detector cross-spectrum may be “calibrated” using a source-in-air measurement to deconvolve the response of detection system electronics. In terms of the system’s transfer function

$$H(s) = \frac{1}{\Lambda(\alpha + s)} , \quad (2.162)$$

the measured detector count rate is

$$\bar{x}^1 = H_x(0) \bar{x} = \epsilon_x H_x(0) H(0) \bar{v}_0 \bar{w} . \quad (2.163)$$

The count rate measured in the absence of the fissile system is

$$\bar{x}^0 = \epsilon_x^0 H_x(0) \bar{v}_0 \bar{w} , \quad (2.164)$$

where ϵ_x^0 is the efficiency for counting neutrons directly from the source. Recall that this is necessarily different from ϵ_x because the source emission spectrum differs from the leakage spectrum seen by the detector when the fissile system is present. At any rate, aside from the ratio of detection efficiencies, the calibrated count rate is essentially a measure of reactivity

$$\frac{\bar{x}^1}{\bar{x}^0} = \frac{\varepsilon_x}{\varepsilon_x^0} H(0) = \frac{\varepsilon_x}{\varepsilon_x^0} \frac{1}{|\rho|} . \quad (2.165)$$

In other words, this signature has first-order sensitivity to reactivity: it rises at a rate of one decade per decade as the system measured approaches critical.

Similarly, the measured source-detector cross-spectrum is

$$S_{wx}^1(s) = H_x(s) S_{wx}(s) = \varepsilon_x H_x(s) H(s) \bar{v}_0 S_{ww}(s) , \quad (2.166)$$

where S_{ww} is the source autospectrum,

$$S_{ww}(s) = \int d\tau e^{-s\tau} C_{ww}(\tau) = F_0 = \bar{w} . \quad (2.167)$$

The source-detector signature may be calibrated using a source-in-air measurement,

$$S_{wx}^0(s) = \varepsilon_x^0 H_x(s) \bar{v}_0 S_{ww}(s) , \quad (2.168)$$

such that the calibrated source-detector cross-spectrum is essentially a measure of the system's transfer function,

$$\frac{S_{wx}^1(s)}{S_{wx}^0(s)} = \frac{\varepsilon_x}{\varepsilon_x^0} H(s) = \frac{\varepsilon_x}{\varepsilon_x^0} \frac{1}{\Lambda(\alpha + s)} , \quad (2.169)$$

so

$$\frac{S_{wx}^1(0)}{S_{wx}^0(0)} = \frac{\varepsilon_x}{\varepsilon_x^0} \frac{1}{|\rho|} = \frac{\bar{x}^1}{\bar{x}^0} . \quad (2.170)$$

Next consider that the source-detector bispectrum is

$$\begin{aligned} S_{wxy}(s_1, s_2) &= \int d\tau_1 e^{-s_1\tau_1} \int d\tau_2 e^{-s_2\tau_2} C_{wxy}(\tau_1, \tau_2) \\ &= \frac{\varepsilon_x \varepsilon_y \bar{v}_0 (\bar{v}_0 - 1) F_0}{\Lambda^2(\alpha + s_1)(\alpha + s_2)} + \frac{\varepsilon_x \varepsilon_y \bar{v}_0 \bar{v} (\bar{v} - 1) F_0}{\bar{v} \Lambda^3(\alpha + s_1 + s_2)(\alpha + s_1)(\alpha + s_2)} . \end{aligned} \quad (2.171)$$

The first term is related to the number of dual-branch fission-chains originating at the source,

and the second term is related to the number of dual-branch fission-chains induced in the system. Recall from preceding discussions that the second process is dominant except in systems that are several hundred dollars or more subcritical.

Observe that the asymptotic behavior of this signature at high frequencies is

$$S_{wxy}(s_1, s_2) \approx \frac{\varepsilon_x \varepsilon_y \overline{v_0(v_0-1)} F_0}{\Lambda^2 s_1 s_2} + \frac{\varepsilon_x \varepsilon_y \overline{v_0} \overline{v(v-1)} F_0}{\bar{v} \Lambda^3 (s_1 + s_2) s_1 s_2}, \quad s_1 \gg \alpha, \quad s_2 \gg \alpha, \quad (2.172)$$

So, if one frequency is fixed, along the second frequency the first term rolls off at a rate of one decade per decade, and the second term rolls off at a rate of two decades per decade.

Along the diagonal $s_1 = s_2$,

$$S_{wxy}(s, s) \approx \frac{\varepsilon_x \varepsilon_y \overline{v_0(v_0-1)} F_0}{\Lambda^2 s^2} + \frac{\varepsilon_x \varepsilon_y \overline{v_0} \overline{v(v-1)} F_0}{2\bar{v} \Lambda^3 s^3}, \quad s \gg \alpha, \quad (2.173)$$

so the first term rolls off at a rate of two decades per decade, and the second term rolls off at a rate of three decades per decade. The asymptotic behavior at low frequencies is

$$S_{wxy}(s_1, s_2) \approx \frac{\varepsilon_x \varepsilon_y \overline{v_0(v_0-1)} F_0}{\alpha^2 \Lambda^2} + \frac{\varepsilon_x \varepsilon_y \overline{v_0} \overline{v(v-1)} F_0}{\bar{v} \alpha^3 \Lambda^3}, \quad s_1 \ll \alpha, \quad s_2 \ll \alpha. \quad (2.174)$$

Hence the first term is sensitive to reactivity as $|\rho|^{-2}$ and the second term as $|\rho|^{-3}$. So, even in far subcritical systems, this signature has second-order sensitivity to reactivity, i.e., it rises at a rate of at least two decades per decade as the measured system approaches critical. Closer to critical, the signature has third-order sensitivity to reactivity.

This sensitivity to reactivity is shared by the between-detectors cross-spectrum

$$S_{xy}(s) = \int d\tau e^{-s\tau} C_{xy}(\tau), \quad (2.175)$$

because

$$\int d\tau C_{xy}(\tau) = \int d\tau_1 \int d\tau_2 C_{wxy}(\tau_1, \tau_2) , \quad (2.176)$$

so

$$S_{xy}(0) = S_{wxy}(0, 0) . \quad (2.177)$$

Observe that this is true even if the fissile system does not behave as a point, because in general each correlated doublet of counts is descended from some initial source emission:

$$p_{xy}(t_1, t_2) dt_1 dt_2 = \int dt_0 p_{xy|w}(t_1, t_2 | t_0) dt_1 dt_2 p_w(t_0) , \quad (2.178)$$

if no additional neutron sources besides the introduced source interact with the system.

Because the between-detectors covariance and the source-detector bicovariance are related by

$$C_{xy}(\tau) = \int d\tau_1 C_{wxy}(\tau_1, \tau_1 + \tau) , \quad (2.179)$$

where

$$\tau = \tau_2 - \tau_1 , \quad (2.180)$$

the between-detectors cross-spectrum is related to the source-detector bispectrum by

$$S_{xy}(s) = S_{wxy}(-s, s) = \varepsilon_x \varepsilon_y \left(\frac{1}{v_0(v_0-1)} + \frac{\bar{v}_0 \bar{v}(v-1)}{\bar{v} \alpha \Lambda} \right) \frac{F_0}{\Lambda^2 (\alpha - s)(\alpha + s)} . \quad (2.181)$$

However, it is important to note that, if additional neutron sources are present, the between-detectors cross-spectrum will respond to these; the source-detector bispectrum will not. So under these more general conditions, which are frequently encountered in practice, the response of the system to the introduced source as measured by the between-detectors cross-spectrum may be masked by the additional sources; this can be a significant problem if these unobservable sources vary unpredictably. But the source-detector bispectrum will measure

only the response of the system to the introduced source, and it possesses the same sensitivity to reactivity as the between-detectors cross-spectrum. Situations in which the source-detector bispectral signature provides a decided advantage are easily conceivable.

Each of these signatures may be calibrated using a source-in-air measurement to adjust for the effects of detection electronics. First consider that the measured between-detectors cross-spectrum is

$$\begin{aligned} S_{xy}^1(s) &= H_x(-s) H_y(s) S_{xy}(s) \\ &= \varepsilon_x H_x(-s) \varepsilon_y H_y(s) H(-s) H(s) \left(\frac{\overline{v_0(v_0-1)}}{v_0(v_0-1)} + \frac{\overline{v(v-1)}}{\bar{v}} H(0) \bar{v}_0 \right) S_{ww}(s) , \end{aligned} \quad (2.182)$$

while this signature's response to a bare source is

$$S_{xy}^0(s) = \varepsilon_x^0 H_x(-s) \varepsilon_y^0 H_y(s) \overline{v_0(v_0-1)} S_{ww}(s) , \quad (2.183)$$

such that the calibrated between-detectors cross-spectrum is independent of each detector's transfer function:

$$\frac{S_{xy}^1(s)}{S_{xy}^0(s)} = \frac{\varepsilon_x \varepsilon_y}{\varepsilon_x^0 \varepsilon_y^0} H(-s) H(s) \left(1 + \frac{\bar{v}_0}{\bar{v}} \frac{\overline{v(v-1)}}{v_0(v_0-1)} H(0) \right) . \quad (2.184)$$

Similarly, the source-detector bispectrum measured in the presence of the fissile system is

$$\begin{aligned} S_{wxy}^1(s_1, s_2) &= H_x(s_1) H_y(s_2) S_{wxy}(s_1, s_2) \\ &= \varepsilon_x H_x(s_1) \varepsilon_y H_y(s_2) H(s_1) H(s_2) \left(\frac{\overline{v_0(v_0-1)}}{v_0(v_0-1)} + \frac{\overline{v(v-1)}}{\bar{v}} H(s_1+s_2) \bar{v}_0 \right) S_{www}(s_1, s_2) , \end{aligned} \quad (2.185)$$

where S_{www} is the source autobispectrum

$$S_{www}(s_1, s_2) = \int d\tau_1 e^{-s_1 \tau_1} \int d\tau_2 e^{-s_2 \tau_2} C_{www}(\tau_1, \tau_2) = F_0 = \bar{w} . \quad (2.186)$$

In the absence of the fissile system,

$$S_{wxy}^0(s_1, s_2) = \varepsilon_x^0 H_x(s_1) \varepsilon_y^0 H_y(s_2) \overline{v_0(v_0-1)} S_{www}(s_1, s_2), \quad (2.187)$$

such that the calibrated signature is

$$\frac{S_{wxy}^1(s_1, s_2)}{S_{wxy}^0(s_1, s_2)} = \frac{\varepsilon_x \varepsilon_y}{\varepsilon_x^0 \varepsilon_y^0} H(s_1) H(s_2) \left(1 + \frac{\bar{v}_0}{\bar{v}} \frac{\overline{v(v-1)}}{v_0(v_0-1)} H(s_1 + s_2) \right). \quad (2.188)$$

Furthermore, the low-frequency asymptotic behavior of both calibrated signatures provides the same measure of reactivity:

$$\frac{S_{xy}^1(0)}{S_{xy}^0(0)} = \frac{S_{wxy}^1(0, 0)}{S_{wxy}^0(0, 0)} = \frac{\varepsilon_x \varepsilon_y}{\varepsilon_x^0 \varepsilon_y^0} \frac{1}{|\rho|^2} \left(1 + \frac{\bar{v}_0}{\bar{v}} \frac{\overline{v(v-1)}}{v_0(v_0-1)} \frac{1}{|\rho|} \right). \quad (2.189)$$

Now observe that the source-detector trispectrum is

$$\begin{aligned} S_{wxyz}(s_1, s_2, s_3) &= \int d\tau_1 e^{-s_1 \tau_1} \int d\tau_2 e^{-s_2 \tau_2} \int d\tau_3 e^{-s_3 \tau_3} C_{wxyz}(\tau_1, \tau_2, \tau_3) \\ &= \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1)(v_0-2)} F_0}{\Lambda^3(\alpha+s_1)(\alpha+s_2)(\alpha+s_3)} + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0 v(v-1)(v-2)} F_0}{\bar{v} \Lambda^4(\alpha+s_1+s_2+s_3)(\alpha+s_1)(\alpha+s_2)(\alpha+s_3)} \\ &\quad + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1) v(v-1)} F_0}{\bar{v} \Lambda^3(\alpha+s_1)(\alpha+s_2)(\alpha+s_3)} \left(\frac{1}{\Lambda(\alpha+s_1+s_2)} + \frac{1}{\Lambda(\alpha+s_1+s_3)} + \frac{1}{\Lambda(\alpha+s_2+s_3)} \right) \\ &\quad + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v(v-1)^2} F_0}{\bar{v}^2 \Lambda^4(\alpha+s_1+s_2+s_3)(\alpha+s_1)(\alpha+s_2)(\alpha+s_3)} \left(\frac{1}{\Lambda(\alpha+s_1+s_2)} + \frac{1}{\Lambda(\alpha+s_1+s_3)} + \frac{1}{\Lambda(\alpha+s_2+s_3)} \right) \end{aligned} \quad (2.190)$$

The first term is related to the number of triple-branch fission-chains originating at the source, and the second term is related to the number of triple-branch fission-chains induced in the system. The third term reflects the number of dual-branch fission-chains originating at the source that spawn dual branch fission-chains in the system, and the last term reflects the number of dual-branch fission-chains induced in the system that in turn initiate a second dual-branch fission-chain. As was noted earlier, this last process is dominant over the others

except in systems several hundred dollars or more subcritical.

As each frequency grows large compared to the break frequency α , this signature asymptotically behaves as

$$\begin{aligned}
 S_{wxyz}(s_1, s_2, s_3) \approx & \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1)(v_0-2)} F_0}{\Lambda^3 s_1 s_2 s_3} + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0} \overline{v(v-1)(v-2)} F_0}{\bar{v} \Lambda^4 (s_1 + s_2 + s_3) s_1 s_2 s_3} \\
 & + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1)} \overline{v(v-1)} F_0}{\bar{v} \Lambda^3 s_1 s_2 s_3} \left(\frac{1}{\Lambda(s_1 + s_2)} + \frac{1}{\Lambda(s_1 + s_3)} + \frac{1}{\Lambda(s_2 + s_3)} \right) \\
 & + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v(v-1)^2} F_0}{\bar{v}^2 \Lambda^4 (s_1 + s_2 + s_3) s_1 s_2 s_3} \left(\frac{1}{\Lambda(s_1 + s_2)} + \frac{1}{\Lambda(s_1 + s_3)} + \frac{1}{\Lambda(s_2 + s_3)} \right), \quad s_1 \gg \alpha, s_2 \gg \alpha, s_3 \gg \alpha.
 \end{aligned} \tag{2.191}$$

So, if two frequencies are fixed, along the third frequency the first term rolls off at a rate of one decade per decade, the second and third terms at two decades per decade, and the fourth term at three decades per decade. Note that along the lines where any two frequencies are identical, the source-detector trispectrum behaves as

$$\begin{aligned}
 S_{wxyz}(s, s, s') &= S_{wxyz}(s, s', s) = S_{wxyz}(s', s, s) \\
 \approx & \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1)(v_0-2)} F_0}{\Lambda^3 s^2 s'} + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0} \overline{v(v-1)(v-2)} F_0}{\bar{v} \Lambda^4 (2s + s') s^2 s'} \\
 & + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1)} \overline{v(v-1)} F_0}{\bar{v} \Lambda^3 s^2 s'} \left(\frac{1}{2\Lambda s} + \frac{2}{\Lambda(s + s')} \right) \\
 & + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v(v-1)^2} F_0}{\bar{v}^2 \Lambda^4 (2s + s') s^2 s'} \left(\frac{1}{2\Lambda s} + \frac{2}{\Lambda(s + s')} \right), \quad s \gg \alpha, s' \gg \alpha,
 \end{aligned} \tag{2.192}$$

so the first term rolls off at a rate of two decades per decade, the second and third terms at three decades per decade, and the fourth at four decades per decade. Finally, along the line $s_1 = s_2 = s_3$, this signature behaves as

$$\begin{aligned}
S_{wxyz}(s, s, s) \approx & \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1)(v_0-2)} F_0}{\Lambda^3 s^3} + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v(v-1)(v-2)} F_0}{3\bar{v}\Lambda^4 s^4} \\
& + \frac{3\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1)} \overline{v(v-1)} F_0}{2\bar{v}\Lambda^4 s^4} + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v(v-1)^2} F_0}{2\bar{v}^2\Lambda^5 s^5}, \quad s \gg \alpha,
\end{aligned} \tag{2.193}$$

so the first term rolls off at a rate of three decades per decade, the second and third terms at four decades per decade, and the fourth term at five decades per decade. On the other hand, the asymptotic behavior of this signature at low frequencies is simply

$$\begin{aligned}
S_{wxyz}(s_1, s_2, s_3) \approx & \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1)(v_0-2)} F_0}{\alpha^3 \Lambda^3} + \frac{\varepsilon_x \varepsilon_y \varepsilon_z \overline{v(v-1)(v-2)} F_0}{\bar{v}\alpha^4 \Lambda^4} \\
& + \frac{3\varepsilon_x \varepsilon_y \varepsilon_z \overline{v_0(v_0-1)} \overline{v(v-1)} F_0}{\bar{v}\alpha^4 \Lambda^4} + \frac{3\varepsilon_x \varepsilon_y \varepsilon_z \overline{v(v-1)^2} F_0}{\bar{v}^2\alpha^5 \Lambda^5}, \quad s_1 \ll \alpha, s_2 \ll \alpha, s_3 \ll \alpha,
\end{aligned} \tag{2.194}$$

so the first term is sensitive to reactivity as $|\rho|^{-3}$, the second and third terms as $|\rho|^{-4}$, and the last term as $|\rho|^{-5}$. So, near critical where the last process is dominant, this signature has fifth-order sensitivity to reactivity. Even for systems that far subcritical, it has at least third-order sensitivity to reactivity.

Of course the between-detectors bispectrum,

$$S_{xyz}(s_1, s_2) = \int d\tau_1 e^{-s_1\tau_1} \int d\tau_2 e^{-s_2\tau_2} C_{xyz}(\tau_1, \tau_2), \tag{2.195}$$

has this same sensitivity because

$$\int d\tau_1 \int d\tau_2 C_{xyz}(\tau_1, \tau_2) = \int d\tau'_1 \int d\tau'_2 \int d\tau'_3 C_{wxyz}(\tau'_1, \tau'_2, \tau'_3), \tag{2.196}$$

such that

$$S_{xyz}(0, 0) = S_{wxyz}(0, 0, 0). \tag{2.197}$$

This is of course true even if the system does not behave as a point, because each bicorrelated

triplet of counts is descended from some initiating source fission:

$$p_{xyz}(t_1, t_2, t_3) dt_1 dt_2 dt_3 = \int dt_0 p_{xyz|w}(t_1, t_2, t_3 | t_0) dt_1 dt_2 dt_3 p_w(t_0) , \quad (2.198)$$

if no sources additional to the introduced source are significant. Now recall that the between-detectors bicovariance may be obtained from the source-detector tricovariance by

$$C_{xyz}(\tau_1, \tau_2) = \int d\tau'_1 C_{wxyz}(\tau'_1, \tau'_1 + \tau_1, \tau'_1 + \tau_2) , \quad (2.199)$$

where

$$\tau_1 = \tau'_2 - \tau'_1 , \quad \tau_2 = \tau'_3 - \tau'_1 . \quad (2.200)$$

Given the preceding relationship, the between-detectors bispectrum is

$$\begin{aligned} S_{xyz}(s_1, s_2) &= \varepsilon_x \varepsilon_y \varepsilon_z \left(\frac{1}{v_0(v_0-1)(v_0-2)} + \frac{\bar{v}_0 v(v-1)(v-2)}{\bar{v} \alpha \Lambda} \right) \frac{F_0}{\Lambda^3(\alpha-s_1-s_2)(\alpha+s_1)(\alpha+s_2)} \\ &+ \varepsilon_x \varepsilon_y \varepsilon_z \left(\frac{1}{v_0(v_0-1)} + \frac{\bar{v}_0 v(v-1)}{\bar{v} \alpha \Lambda} \right) \frac{v(v-1)}{\bar{v}} \frac{F_0}{\Lambda^3(\alpha-s_1-s_2)(\alpha+s_1)(\alpha+s_2)} \\ &\cdot \left(\frac{1}{\Lambda(\alpha+s_1+s_2)} + \frac{1}{\Lambda(\alpha-s_1)} + \frac{1}{\Lambda(\alpha-s_2)} \right) = S_{wxyz}(-s_1-s_2, s_1, s_2) . \end{aligned} \quad (2.201)$$

However, it must be noted that, because the between-detectors bispectrum is sensitive to additional neutron sources, whereas the source-detector trispectrum is not, the bicorrelated response of the system to the introduced source measured by the between-detectors bispectrum may be masked by additional sources. In contrast, the source-detector trispectrum will respond only to those fission-chains initiated by the introduced source. Consequently, the trispectrum will possess the same sensitivity to reactivity but will not suffer from the bispectrum's sensitivity to changes in the nature or intensity of any unobservable sources.

As has been previously illustrated for other signatures, both the between-detectors

bispectrum and the source-detector trispectrum may be adjusted by a source-in-air measurement to deconvolve the response of detection electronics. Recognize that the measured between-detectors bispectrum is

$$\begin{aligned}
 S_{xyz}^1(s_1, s_2) &= H_x(-s_1-s_2) H_y(s_1) H_z(s_2) S_{xyz}(s_1, s_2) \\
 &= \varepsilon_x H_x(-s_1-s_2) \varepsilon_y H_y(s_1) \varepsilon_z H_z(s_2) H(-s_1-s_2) H(s_1) H(s_2) \\
 &\cdot \left[\frac{1}{v_0(v_0-1)(v_0-2)} + \frac{v(v-1)(v-2)}{\bar{v}} H(0) \bar{v}_0 \right. \\
 &\quad \left. + \frac{v(v-1)}{\bar{v}} (H(s_1+s_2) + H(s_1) + H(s_2)) \left(\frac{1}{v_0(v_0-1)} + \frac{v(v-1)}{\bar{v}} H(0) \bar{v}_0 \right) \right] S_{www}(s_1, s_2)
 \end{aligned} \tag{2.202}$$

and the bispectrum measured by exposing the detectors to a bare source is

$$S_{xyz}^0(s_1, s_2) = \varepsilon_x^0 H_x(-s_1-s_2) \varepsilon_y^0 H_y(s_1) \varepsilon_z^0 H_z(s_2) \frac{1}{v_0(v_0-1)(v_0-2)} S_{www}(s_1, s_2) . \tag{2.203}$$

Consequently, the calibrated bispectrum will be independent of the response of the detection electronics

$$\begin{aligned}
 \frac{S_{xyz}^1(s_1, s_2)}{S_{xyz}^0(s_1, s_2)} &= \frac{\varepsilon_x \varepsilon_y \varepsilon_z}{\varepsilon_x^0 \varepsilon_y^0 \varepsilon_z^0} H(-s_1-s_2) H(s_1) H(s_2) \left[1 + \frac{\bar{v}_0}{\bar{v}} \frac{v(v-1)(v-2)}{v_0(v_0-1)(v_0-2)} H(0) + \right. \\
 &\quad \left. \frac{v(v-1)}{\bar{v} v_0(v_0-1)(v_0-2)} (H(s_1+s_2) + H(s_1) + H(s_2)) \left(\frac{1}{v_0(v_0-1)} + \frac{v(v-1)}{\bar{v}} H(0) \bar{v}_0 \right) \right] S_{www}(s_1, s_2) .
 \end{aligned} \tag{2.204}$$

Similarly, the measured source-detector trispectrum is

$$\begin{aligned}
 S_{wxyz}^1(s_1, s_2, s_3) &= H_x(s_1) H_y(s_2) H_z(s_3) S_{wxyz}(s_1, s_2, s_3) \\
 &= \varepsilon_x H_x(s_1) \varepsilon_y H_y(s_2) \varepsilon_z H_z(s_3) H(s_1) H(s_2) H(s_3) \cdot \left[\frac{1}{v_0(v_0-1)(v_0-2)} \right. \\
 &\quad \left. + \frac{v(v-1)(v-2)}{\bar{v}} H(s_1+s_2+s_3) \bar{v}_0 + \frac{v(v-1)}{\bar{v}} (H(s_1+s_2) + H(s_1+s_3) + H(s_2+s_3)) \right. \\
 &\quad \left. \cdot \left(\frac{1}{v_0(v_0-1)} + \frac{v(v-1)}{\bar{v}} H(s_1+s_2+s_3) \bar{v}_0 \right) \right] S_{www}(s_1, s_2, s_3) ,
 \end{aligned} \tag{2.205}$$

where S_{www} denotes the source autotrispectrum

$$S_{www}(s_1, s_2, s_3) = \int d\tau_1 e^{-s_1\tau_1} \int d\tau_2 e^{-s_2\tau_2} \int d\tau_3 e^{-s_3\tau_3} C_{www}(\tau_1, \tau_2, \tau_3) = F_0 = \bar{w}. \quad (2.206)$$

The trispectrum obtained from a source-in-air measurement is

$$S_{wxyz}^0(s_1, s_2, s_3) = \varepsilon_x^0 H_x(s_1) \varepsilon_y^0 H_y(s_2) \varepsilon_z^0 H_z(s_3) \overline{v_0(v_0-1)(v_0-2)} S_{www}(s_1, s_2, s_3), \quad (2.207)$$

such that the calibrated trispectrum is

$$\begin{aligned} \frac{S_{wxyz}^1(s_1, s_2, s_3)}{S_{wxyz}^0(s_1, s_2, s_3)} &= \frac{\varepsilon_x \varepsilon_y \varepsilon_z}{\varepsilon_x^0 \varepsilon_y^0 \varepsilon_z^0} H(s_1) H(s_2) H(s_3) \left[1 + \frac{\bar{v}_0}{\bar{v}} \frac{\overline{v(v-1)(v-2)}}{v_0(v_0-1)(v_0-2)} H(s_1+s_2+s_3) \right. \\ &\quad + \frac{\overline{v(v-1)}}{\bar{v} v_0(v_0-1)(v_0-2)} (H(s_1+s_2) + H(s_1+s_3) + H(s_2+s_3)) \\ &\quad \left. \cdot \left(\frac{\overline{v(v-1)}}{v_0(v_0-1)} + \frac{\overline{v(v-1)}}{\bar{v}} H(s_1+s_2+s_3) \bar{v}_0 \right) \right] S_{www}(s_1, s_2, s_3). \end{aligned} \quad (2.208)$$

Of course, these two calibrated signatures possess the same sensitivity to reactivity:

$$\begin{aligned} \frac{S_{xyz}^1(0, 0)}{S_{xyz}^0(0, 0)} &= \frac{S_{wxyz}^1(0, 0, 0)}{S_{wxyz}^0(0, 0, 0)} = \frac{\varepsilon_x \varepsilon_y \varepsilon_z}{\varepsilon_x^0 \varepsilon_y^0 \varepsilon_z^0} \frac{1}{|\rho|^3} \\ &\cdot \left[1 + \frac{\bar{v}_0}{\bar{v}} \frac{\overline{v(v-1)(v-2)}}{v_0(v_0-1)(v_0-2)} \frac{1}{|\rho|} + \frac{3 \overline{v(v-1)}}{\bar{v} v_0(v_0-1)(v_0-2)} \frac{1}{|\rho|} \left(\frac{\overline{v(v-1)}}{v_0(v_0-1)} + \frac{\overline{v(v-1)}}{\bar{v}} \frac{\bar{v}_0}{|\rho|} \right) \right], \end{aligned} \quad (2.209)$$

but they will be insensitive to moderate changes in the response of detection electronics and hence more readily repeatable than the uncalibrated signatures.

2.5 Spectral Ratios

Even the most carefully designed measurement can suffer from equipment malfunctions. Occasionally things break. Suitable – even seemingly identical –

replacements can usually be found, but unless great care is taken to reconstruct the measurement system, including the precise settings of all associated electronics, measurements can sometimes prove difficult to repeat. The purpose of the preceding calibrated signatures was to mitigate some of these concerns by eliminating the intrinsic response of the detection electronics from the measured signatures. However, even in that the calibrated signatures are not entirely successful. Recall that, though none of the calibrated signatures depends upon the response of the detection electronics, each one did depend upon the ratio of the efficiency for detecting neutrons from the system to that for detecting neutrons directly from the source. Of course, if these were simply the intrinsic efficiencies of the detection material, i.e., the probability per neutron of an interaction in the detector, then these ratios would be invariant as long as the detection material remained constant, e.g., if one detector were replaced with another of identical construction. Unfortunately, this is not the case. These efficiencies can still be affected by electronic settings, particularly by discriminator levels which determine, on the basis of pulse height, whether a pulse will be counted or ignored. For detectors whose output pulse heights are proportional to the energy deposited in the detection medium, the discriminator threshold sets a lower (and/or upper) bound on the deposition energies that will be registered as an event. So clearly discriminator levels can have a significant effect on detection efficiency. Consequently, a repeated measurement may yield results significantly different from the original simply if discriminator thresholds change.

So signatures that are independent of electronics responses and detection efficiency really are more robust for application in field measurements. The spectral ratio introduced

by Mihalczko and Paré attempts to achieve this degree of robustness. For the ideal case, where the source is 100% efficient and all electronics have instantaneous and perfect response, the spectral ratio is

$$\Gamma_{wxy}(s) = \frac{S_{wx}(-s) S_{wy}(s)}{S_{wx}(-s) S_{xy}(s)} = \frac{\overline{v_0}}{\frac{v_0(v_0-1)}{\overline{v_0}} + \frac{v(v-1)}{\overline{v}|\rho|}}. \quad (2.210)$$

The first equality is essentially the definition of the spectral ratio as introduced by Mihalczko and Paré. The rightmost equality is the interpretation of the spectral ratio using the point models described during this chapter. It's important to note that the point models of second-order signatures described here are not quite as sophisticated as recent point models used in the interpretation of the spectral ratio. Consequently, (2.210) does not account for every factor affecting the spectral ratio [Mih3,5]. In particular, it does not yet account for the effects of degraded source efficiency or internal and inherent sources.

There has already been some discussion of the behavior of detectors with imperfect responses. However, it has been so far assumed that the source ionization chamber detects all fission fragments and that the response of electronics associated with the source is, to a good approximation, instantaneous and perfect. Although this is often a reasonable approximation, sometimes, particularly for high intensity (i.e., large mass) ^{252}Cf sources, these assumptions are invalid. Occasionally, a threshold setting to discriminate all α -particle pulses but keep all fission fragment pulses cannot be clearly determined from a pulse height analysis, so the threshold is set relatively high to ensure that no α -particles are counted. Unfortunately, this usually discriminates some lower energy fission fragments. Furthermore,

for very high intensity sources, the fission rate may exceed the throughput of pulse mode electronics, so the source must be operated in current mode, where the current is proportional to the sum of the spontaneous fission and α -decay rates. Current mode electronics essentially lowpass filter the source signal, aside from changing the stream of discrete electrical pulses to a continuous current signal.

So in general, the mean measured source fission rate is

$$\bar{w}^1 = \varepsilon_w H_w(0) \bar{w} , \quad (2.211)$$

where ε_w denotes the source “efficiency”, and $H_w(s)$ represents the transfer function of electronics associated with the source. In particular $H_w(0)$ is the gain of the electronics associated with the source:

$$H_w(0) = \int dt h_w(t) = \frac{\bar{w}^1}{\varepsilon_w \bar{w}} , \quad (2.212)$$

where $h_w(t)$ represents the impulse response of the source electronics. In pulse mode, the source “efficiency” ε_w folds the intrinsic efficiency of the ionization chamber together with the effects of a variable discriminator threshold. If the ionization chamber bias voltage is set sufficiently high, the intrinsic efficiency is essentially unity, i.e., the chamber will produce an electrical pulse for every source fission. However, some pulses will also result from the pileup of α -particles, and these must be discriminated. Unfortunately, some pulses corresponding to fission fragments may be discriminated as well. Hence, for pulse mode operation, the portion of the source efficiency due to the discriminator threshold accounts for that fraction of fission fragment pulses that are discounted. If the source is operated in current mode, the source “efficiency” accounts for the fraction f_α of current that results from

α -particles in addition to the current that results from fission fragments:

$$\text{current mode: } \varepsilon_w = 1 + f_\alpha . \quad (2.213)$$

So, in current mode, this “efficiency” is greater than one because both the α -decay rate and the spontaneous fission rate are measured. The current mode transfer function converts the α -decay and spontaneous fission rates to a continuous current signal.

Subsequently, the measured source autospectrum is

$$S_{ww}^1(s) = \varepsilon_w H_w(-s) H_w(s) \bar{w} . \quad (2.214)$$

Similarly, the measured source-detector cross-spectrum is

$$S_{wx}^1(s) = \varepsilon_w H_w(-s) \varepsilon_x H_x(s) H(s) \bar{v}_0 \bar{w} , \quad (2.215)$$

while the measured between-detectors cross-spectrum is “unaware” of the source efficiency or response:

$$S_{xy}^1(s) = \varepsilon_x H_x(-s) \varepsilon_y H_y(s) H(-s) H(s) \left(\frac{1}{\bar{v}_0(\bar{v}_0-1)} + \frac{\bar{v}(\bar{v}-1)}{\bar{v}} H(0) \bar{v}_0 \right) \bar{w} , \quad (2.216)$$

i.e., it responds to the actual number of source fissions, not merely to those source fissions that were counted. So, the measured spectral ratio, Γ_{wxy}^1 , is not identical to the actual spectral

ratio:

$$\Gamma_{wxy}^1(s) = \frac{S_{wx}^1(-s) S_{wy}^1(s)}{S_{ww}^1(-s) S_{xy}^1(s)} = \varepsilon_w \Gamma_{wxy}(s) . \quad (2.217)$$

A simple change in the source discriminator threshold can change this signature.

Furthermore, note that if unobservable neutron sources interact with the system, the spectral ratio will respond to them:

$$\begin{aligned}
S_{xy}(s) &\rightarrow \\
\varepsilon_x \varepsilon_y H(-s) H(s) &\left(\frac{\overline{v_0(v_0-1)} F_0 + \overline{v_1(v_1-1)} F_1 + \frac{\overline{v(v-1)}}{\overline{v}} H(0) (\overline{v_0} F_0 + \overline{v_1} F_1 + Q)}{\overline{v_0}} \right), \\
\therefore \Gamma_{wxy}(s) &\rightarrow \frac{\overline{v_0}}{\frac{\overline{v_0(v_0-1)}}{\overline{v_0}} \left(1 + \frac{\overline{v_1(v_1-1)} F_1}{\overline{v_0(v_0-1)} F_0} \right) + \frac{\overline{v(v-1)}}{\overline{v} |\rho|} \left(1 + \frac{\overline{v_1} F_1 + Q}{\overline{v_0} F_0} \right)}
\end{aligned} \tag{2.218}$$

Observe that if the unobservable sources (F_1 and Q) are sufficiently intense, the spectral ratio may not respond to the introduced observable source (F_0) at all. So, a measurement of two systems that are identical except for their content of internal spontaneous fission and/or single-neutron emission sources may yield significantly different spectral ratio signatures. Although this is a useful property if the purpose of the measurement is to recognize the presence of varying neutron-emitting isotopes, it is a deficiency if the purpose is to classify identical systems as such. The latter application is the more usual.

More importantly, because every fissile system contains an inherent source Q_0 due to delayed neutrons, the spectral ratio responds to the strength of this source even if no other internal sources are present:

$$\Gamma_{wxy}(s) = \frac{\overline{v_0}}{\frac{\overline{v_0(v_0-1)}}{\overline{v_0}} + \frac{\overline{v(v-1)}}{\overline{v} |\rho|} (1 + Q_0)} \tag{2.219}$$

Particularly for thermal-fissioning systems, because delayed neutrons have a relatively higher probability of inducing fission than do prompt neutrons, the importance of this source can be greater than the rate of delayed neutron production. In particular, at delayed critical, fission-chains initiated by delayed neutrons produce a substantial number of the correlated

doublets contributing to the between-detectors cross-spectrum S_{xy} . Relatively fewer counts and correlated doublets are traceable to the introduced source, so the spectral ratio tends to vanish at delayed critical because the source-detector cross-spectra become negligible compared to the between-detectors cross-spectrum. On the other hand, if no fissile material is present

$$\lim_{\rho \rightarrow -\infty} \Gamma_{wxy}(s) = \frac{1}{\chi_0}, \quad (2.220)$$

where χ_0 denotes the source Diven factor [Div],

$$\chi_0 = \frac{\overline{v_0(v_0-1)}}{\overline{v_0}^2}. \quad (2.221)$$

However, the contribution of the inherent delayed neutron source to the spectral ratio can be difficult to evaluate since it requires knowledge of, among other things, the effective delayed neutron fraction particular to the fissile assembly being measured. Consequently, in measurements performed near delayed critical, the spectral ratio can be difficult to interpret.

Before proceeding, reconsider the implied meaning of the spectral ratio:

$$\Gamma_{wxy}(s) = \frac{\overline{v_0}}{\frac{\overline{v_0(v_0-1)}}{\overline{v_0}} + \frac{\overline{v(v-1)}}{\overline{v}|\rho|}}. \quad (2.222)$$

The denominator reflects the probability of a pair of correlated counts per source neutron. The numerator reflects the probability of two independent counts per source neutron – $\overline{v_0}^2/\overline{v_0}$. So the spectral ratio is a measure of the ratio of doublets of independent counts to

doublers of correlated counts. Another measure of a similar quantity, for lack of a better name, is the bispectral ratio

$$\Psi_{x \cdot y|w} = \frac{S_{wx}(0) S_{wy}(0)}{\bar{w} S_{wxy}(0, 0)} = \frac{\overline{v_0}}{\frac{v_0(v_0-1)}{\overline{v_0}} + \frac{v(v-1)}{\overline{v}|\rho|}} . \quad (2.223)$$

Here's why it works. Consider that the bispectral ratio is

$$\Psi_{x \cdot y|w} = \frac{\frac{1}{\bar{w}} \int d\tau C_{wx}(\tau) \cdot \frac{1}{\bar{w}} \int d\tau' C_{wy}(\tau')}{\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 C_{wxy}(\tau_1, \tau_2)} . \quad (2.224)$$

Now consider that

$$\frac{C_{wx}(\tau)}{\bar{w}} \sim \frac{p_{wx}(t_0, t_1)}{p_w(t_0)} = p_{x|w}(t_1 | t_0) , \quad (2.225)$$

such that

$$\frac{1}{\bar{w}} \int d\tau C_{wx}(\tau) \sim \int d\tau C_{x|w}(\tau) \quad (2.226)$$

is the expected rate of counts originating from the introduced source. Hence the product in the numerator is the expected rate of doublers of independent counts originating from the observable source. In contrast,

$$\frac{C_{wxy}(\tau_1, \tau_2)}{\bar{w}} \sim \frac{p_{wxy}(t_0, t_1, t_2)}{p_w(t_0)} = p_{xy|w}(t_1, t_2 | t_0) , \quad (2.227)$$

such that

$$\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 C_{wxy}(\tau_1, \tau_2) \sim \int d\tau_1 \int d\tau_2 C_{xy|w}(\tau_1, \tau_2) \quad (2.228)$$

is the expected rate of correlated doublers originating from the introduced source. So the bispectral ratio is a measure of the ratio of doublers of independent counts originating at the

source to doublets of correlated counts originating at the source. Furthermore, it is unaffected by the introduction of unobservable sources. More importantly, the bispectral ratio is insensitive to the strength of the inherent source.

Finally, the measured bispectral ratio

$$\Psi_{x,y|w}^I = \frac{S_{wx}^I(0) S_{wy}^I(0)}{\bar{w}^I S_{wxy}^I(0,0)}, \quad (2.229)$$

where

$$\begin{aligned} S_{wxy}^I(s_1, s_2) &= \epsilon_w H_w(-s_1 - s_2) \epsilon_x H_x(s_1) \epsilon_y H_y(s_2) \\ &\cdot H(s_1) H(s_2) \left(\frac{1}{\bar{v}_0(\bar{v}_0 - 1)} + \frac{\bar{v}(\bar{v} - 1)}{\bar{v}} H(s_1 + s_2) \bar{v}_0 \right) \bar{w}, \end{aligned} \quad (2.230)$$

is the measured source-detector bispectrum, is invariant over changes in source and detector efficiency and response. Simply put, the measured bispectral ratio is identical to the actual bispectral ratio:

$$\Psi_{x,y|w}^I = \Psi_{x,y|w}. \quad (2.231)$$

The bispectral ratio is plotted as a function of prompt reactivity in Fig. 2.17.⁷ Observe that as the system approaches prompt critical, the bispectral ratio vanishes,

$$\lim_{\rho \rightarrow 0} \Psi_{x,y|w} = 0. \quad (2.232)$$

The reason for this should be obvious. First, because prompt fission-chains can potentially endure forever at prompt critical, the number of correlated doublets increases more rapidly

⁷ The subsequent figures depicting various spectral ratios use the same induce fission multiplicity moments calculated by folding Zucker-Holden $p(v|E)$ distributions with an MCNP-calculated spectrum for neutrons inducing fission in a 150-mm-diameter 93.15%-enriched uranium-metal sphere. The asymptotic behaviors depicted are the same regardless of the multiplicity moments.

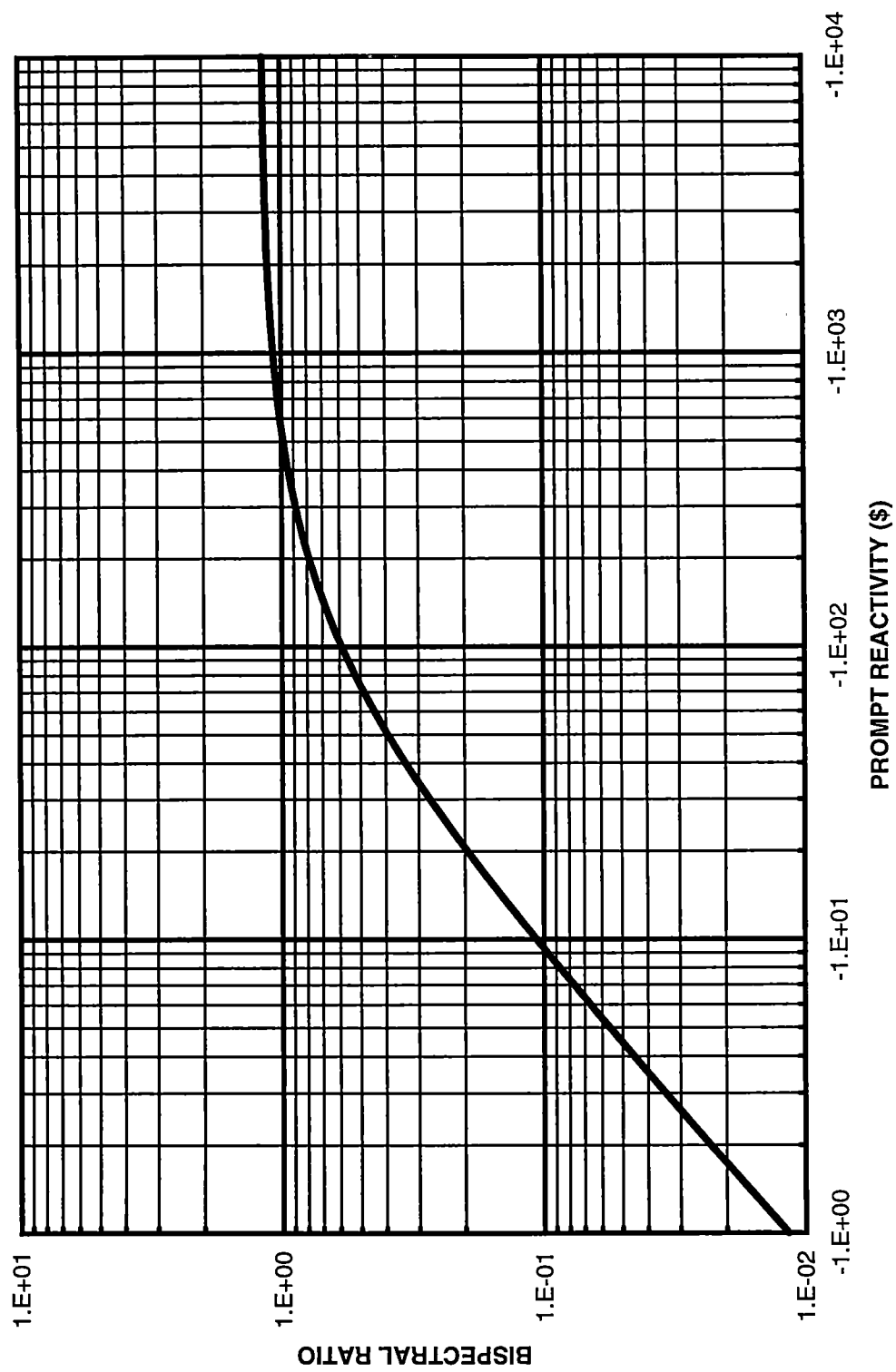


Figure 2.17 The bispectral ratio as a function of prompt reactivity

than the number of singlets as prompt criticality is approached. Hence the bispectral ratio vanishes at prompt critical. This does not occur at delayed critical because the bispectral ratio responds only to fission-chains initiated by the source; these are necessarily prompt neutron fission-chains. For obvious reasons, in the absence of a multiplicative system the bispectral ratio yields the inverse of the source Diven factor:

$$\lim_{\rho \rightarrow -\infty} \Psi_{x,y|w} = \frac{1}{\chi_0} . \quad (2.233)$$

The bispectral ratio rises at a rate of one decade per decade (i.e., linearly) until subcritical reactivity is in the neighborhood of a hundred dollars. At this point, it rolls over to approach its upper asymptote in a sublinear manner. So, over a broad range of subcritical reactivity, the bispectral ratio is directly proportional to subcritical reactivity. Furthermore, because the bispectral ratio does not respond to the inherent delayed neutron source, it may prove easier to interpret than the spectral ratio in some applications. For example, the bispectral ratio measured from a spent UO_2 fuel assembly would be easier to interpret than the spectral ratio measured from the same assembly. Interpretation of the spectral ratio would require some knowledge of, among other things, the concentration of ^{242}Cm and ^{244}Cm , two spontaneously fissioning isotopes found in spent UO_2 nuclear fuel. The bispectral ratio would not respond to these neutron sources.

But wait, there's more. The bispectral ratio is a measure of the ratio of doublets of independent counts to doublets of correlated counts. Consider that the tools are also available to measure the ratio of triplets of independent counts to triplets of bicornelated counts or the ratio of doublets of correlated counts to triplets of bicornelated counts. The

former quantity, for lack of a better name, shall be called a type I trispectral ratio:

$$\Psi_{x \cdot y \cdot z | w} = \frac{S_{wx}(0)S_{wy}(0)S_{wz}(0)}{\bar{w}^2 S_{wxyz}(0, 0, 0)} = \frac{\frac{1}{\bar{w}} \int d\tau C_{wx}(\tau) \cdot \frac{1}{\bar{w}} \int d\tau' C_{wy}(\tau') \cdot \frac{1}{\bar{w}} \int d\tau'' C_{wz}(\tau'')}{\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 \int d\tau_3 C_{wxyz}(\tau_1, \tau_2, \tau_3)} \quad (2.234)$$

In terms of conditional moments,

$$\Psi_{x \cdot y \cdot z | w} \sim \frac{\int d\tau C_{x|w}(\tau) \cdot \int d\tau' C_{y|w}(\tau') \cdot \int d\tau'' C_{z|w}(\tau'')}{\int d\tau_1 \int d\tau_2 \int d\tau_3 C_{xyz|w}(\tau_1, \tau_2, \tau_3)} \quad (2.235)$$

since

$$\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 \int d\tau_3 C_{wxyz}(\tau_1, \tau_2, \tau_3) \sim \int d\tau_1 \int d\tau_2 \int d\tau_3 C_{xyz|w}(\tau_1, \tau_2, \tau_3) \quad (2.236)$$

because

$$\frac{C_{wxyz}(\tau_1, \tau_2, \tau_3)}{\bar{w}} \sim \frac{p_{wxyz}(t_0, t_1, t_2, t_3)}{p_w(t_0)} = p_{xyz|w}(t_1, t_2, t_3 | t_0) \quad (2.237)$$

So clearly the numerator of the type I trispectral ratio reflects the probability per source neutron of three independent counts originating from the source. The denominator reflects the probability per source neutron of three bicorrelated counts originating from the source. Hence,

$$\Psi_{x \cdot y \cdot z | w} = \frac{\bar{v}_0^2}{\frac{v_0(v_0-1)(v_0-2)}{\bar{v}_0} + \frac{v(v-1)(v-2)}{\bar{v}|\rho|} + 3 \left(\frac{v_0(v_0-1)}{\bar{v}_0} + \frac{v(v-1)}{\bar{v}|\rho|} \right) \cdot \frac{v(v-1)}{\bar{v}|\rho|}} \quad (2.238)$$

such that for a prompt critical system, the type I trispectral ratio also vanishes:

$$\lim_{\rho \rightarrow 0} \Psi_{x \cdot y \cdot z | w} = 0 \quad (2.239)$$

On the other hand, if the system measured contains no fissile material,

$$\lim_{\rho \rightarrow -\infty} \Psi_{x \cdot y \cdot z | w} = \left(\frac{v_0(v_0-1)(v_0-2)}{\bar{v}_0^3} \right)^{-1}. \quad (2.240)$$

As was true for the measured bispectral ratio, the measured type I trispectral ratio is identical to the actual type I trispectral ratio

$$\Psi_{x \cdot y \cdot z | w}^1 = \frac{S_{wx}^1(0) S_{wy}^1(0) S_{wz}^1(0)}{(\bar{w}^1)^2 S_{wxyz}^1(0, 0, 0)} = \Psi_{x \cdot y \cdot z | w}, \quad (2.241)$$

because the measured source-detector trispectrum S_{wxyz}^1 compensates for changes in source efficiency and response:

$$\begin{aligned} S_{wxyz}^1(s_1, s_2, s_3) &= \varepsilon_w H_w(-s_1-s_2-s_3) \varepsilon_x H_x(s_1) \varepsilon_y H_y(s_2) \varepsilon_z H_z(s_3) \\ &\cdot H(s_1) H(s_2) H(s_3) \left[\frac{v_0(v_0-1)(v_0-2)}{\bar{v}_0} + \frac{v(v-1)(v-2)}{\bar{v}} H(s_1+s_2+s_3) \bar{v}_0 + \frac{v(v-1)}{\bar{v}} \right. \\ &\cdot (H(s_1+s_2) + H(s_1+s_3) + H(s_2+s_3)) \left. \left(\frac{v_0(v_0-1)}{\bar{v}_0} + \frac{v(v-1)}{\bar{v}} H(s_1+s_2+s_3) \bar{v}_0 \right) \right] \bar{w}. \end{aligned} \quad (2.242)$$

Furthermore, it is clear that this signature is also unaffected by the internal or inherent sources. The type I trispectral ratio is plotted versus prompt reactivity in Fig. 2.18. Observe that over a broad range of subcritical reactivity, this signature rises at a rate of two decades per decade, until it rolls over to approach its upper asymptote in the neighborhood of a hundred dollars subcritical. So, over a wide range of subcritical reactivity, the type I trispectral ratio is proportional to the square of subcritical reactivity.

Now consider a “type II” trispectral ratio

$$\Psi_{xy \cdot z | w} = \frac{S_{wxy}(0, 0) S_{wz}(0)}{\bar{w} S_{wxyz}(0, 0, 0)} = \frac{\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 C_{wxy}(\tau_1, \tau_2) \cdot \frac{1}{\bar{w}} \int d\tau C_{wz}(\tau)}{\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 \int d\tau_3 C_{wxyz}(\tau_1, \tau_2, \tau_3)}, \quad (2.243)$$

or, in terms of conditional moments,

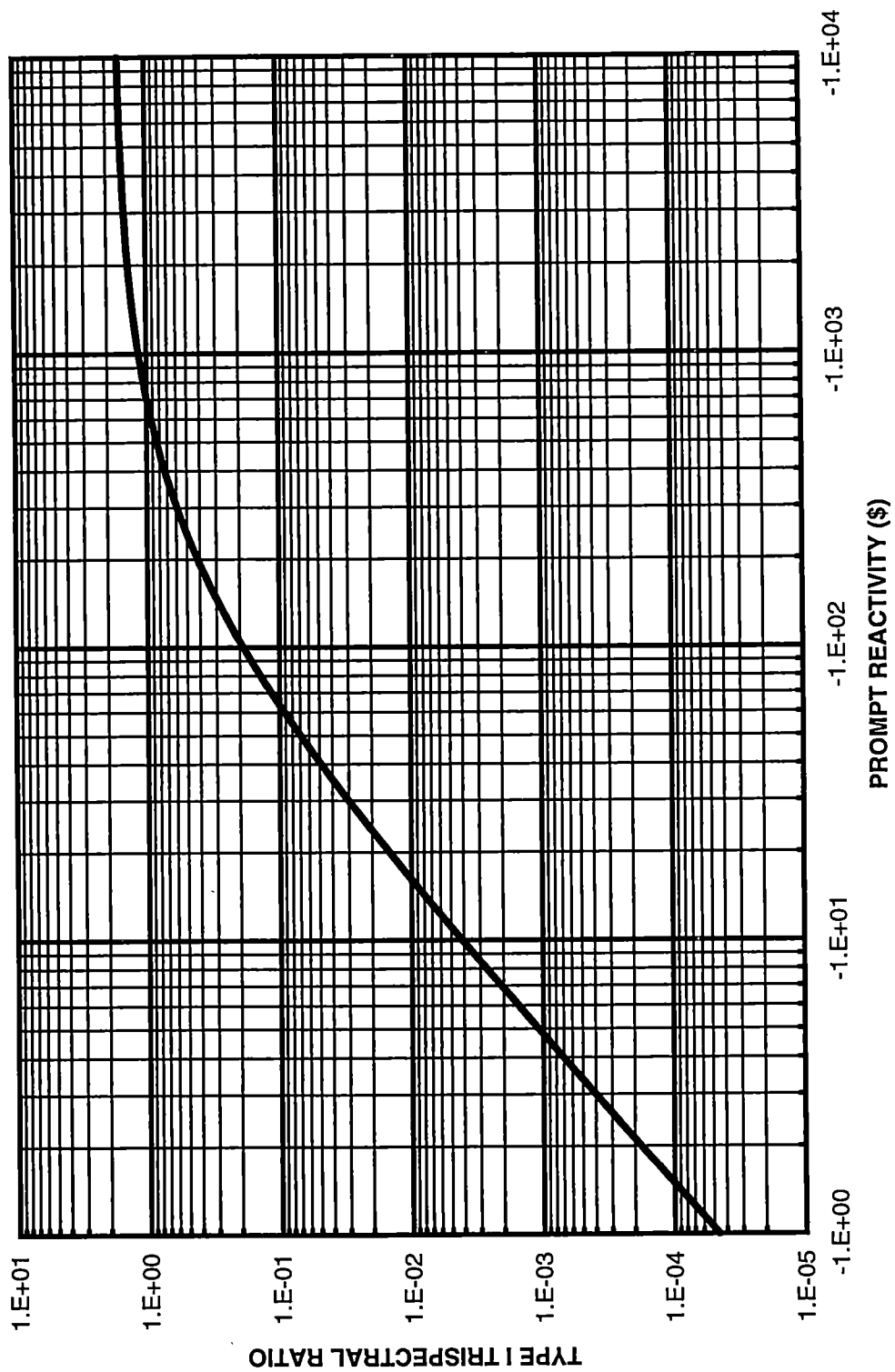


Figure 2.18 The type I trispectral ratio as a function of prompt reactivity

$$\Psi_{xy \cdot z|w} \sim \frac{\int d\tau_1 \int d\tau_2 C_{xy|w}(\tau_1, \tau_2) \cdot \int d\tau C_{z|w}(\tau)}{\int d\tau_1 \int d\tau_2 \int d\tau_3 C_{xyz|w}(\tau_1, \tau_2, \tau_3)} . \quad (2.244)$$

Observe that the denominator again reflects the probability per source neutron of a triplet of bicorrelated counts originating from the source. The numerator of the type II trispectral ratio reflects the probability per source neutron of a doublet of correlated counts (paired with a third unrelated count). Consequently,

$$\Psi_{xy \cdot z|w} = \frac{\bar{v}_0 \left(\frac{v_0(v_0-1)}{\bar{v}_0} + \frac{v(v-1)}{\bar{v}|\rho|} \right)}{\frac{v_0(v_0-1)(v_0-2)}{\bar{v}_0} + \frac{v(v-1)(v-2)}{\bar{v}|\rho|} + 3 \left(\frac{v_0(v_0-1)}{\bar{v}_0} + \frac{v(v-1)}{\bar{v}|\rho|} \right) \frac{v(v-1)}{\bar{v}|\rho|}} . \quad (2.245)$$

This signature is plotted in Fig. 2.19 versus prompt reactivity. Note that the type II trispectral ratio as well vanishes when the system reaches prompt critical,

$$\lim_{\rho \rightarrow 0} \Psi_{xy \cdot z|w} = 0 . \quad (2.246)$$

In the absence of a multiplicative medium, this signature yields

$$\lim_{\rho \rightarrow -\infty} \Psi_{xy \cdot z|w} = \left(\frac{v_0(v_0-1)(v_0-2)}{\bar{v}_0 v_0(v_0-1)} \right)^{-1} . \quad (2.247)$$

This trispectral ratio begins to slowly approach this upper asymptote in the neighborhood of a hundred dollars subcritical. Prior to this, the signature rises at roughly one decade per decade over a broad range of subcritical reactivity, as the figure illustrates.

Obviously the type II trispectral ratio does not respond to unobservable sources since each quantity in the ratio responds only to events correlated with the introduced observable source. However, it has the following added advantage. Each quantity in the ratio

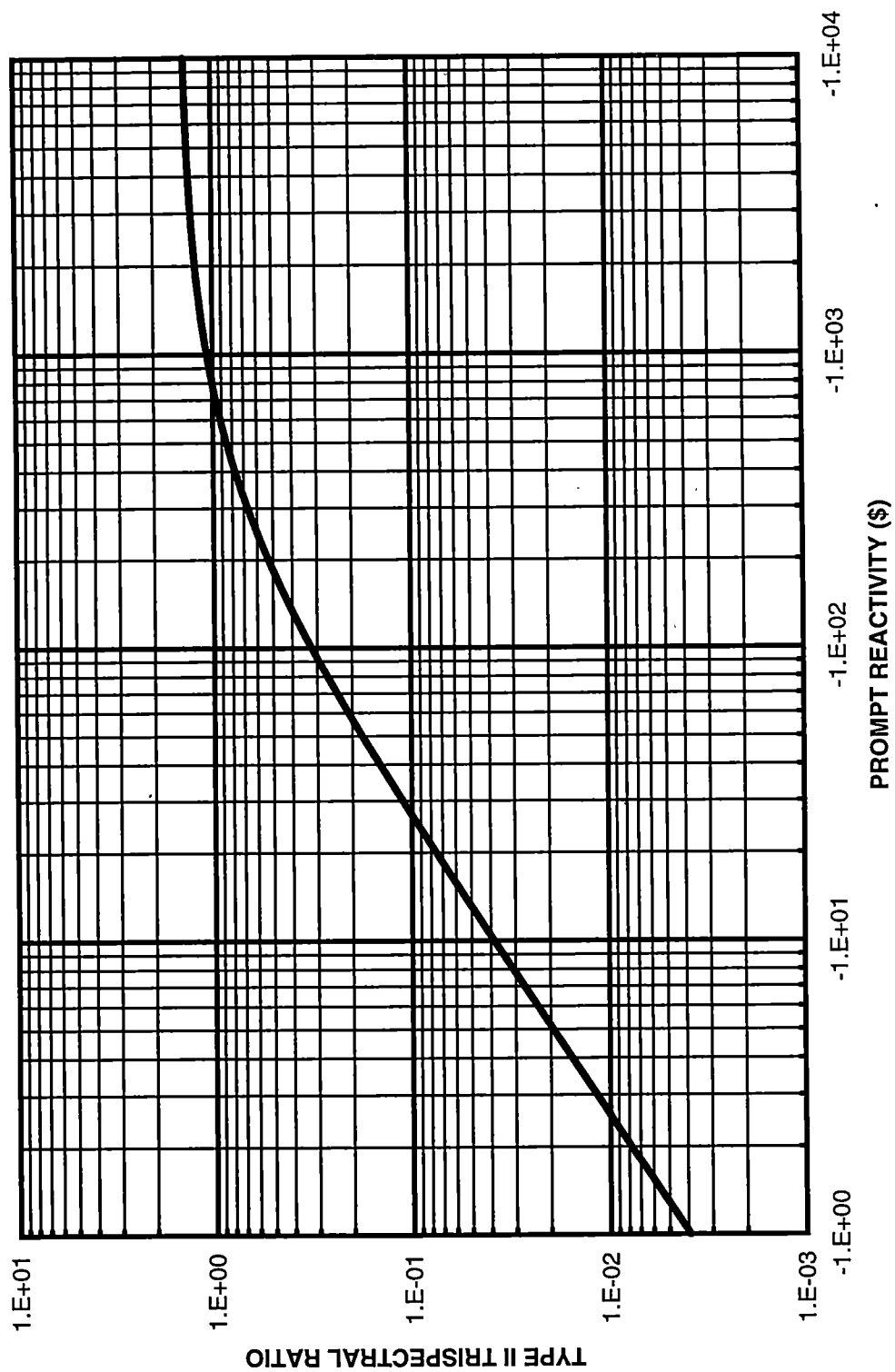


Figure 2.19 The type II trispectral ratio as a function of prompt reactivity

compensates the others for changes in source and detector efficiency and response so that the measured signature is identical to the actual signature:

$$\Psi_{xy \cdot z|w}^1 = \frac{S_{wxy}^1(0, 0) S_{wz}^1(0)}{\bar{w}^1 S_{wxyz}^1(0, 0, 0)} = \Psi_{xy \cdot z|w} . \quad (2.248)$$

Furthermore, although it was not explicitly noted for the preceding ratios, like the preceding ratios the type II trispectral ratio is invariant over permutations in the order of detection events:

$$\Psi_{zy \cdot x|w} = \Psi_{yz \cdot x|w} = \Psi_{zx \cdot y|w} = \Psi_{xz \cdot y|w} = \Psi_{yx \cdot z|w} = \Psi_{xy \cdot z|w} . \quad (2.249)$$

Finally, it should be clear that there are other ratio signatures that may be defined. One could, for example, define a ratio that reflects the probability of quadruplets of independent counts relative to a quadruplet of tricorrelated counts:

$$\Psi_{x_1 \cdot x_2 \cdot x_3 \cdot x_4|w} = \frac{S_{wx_1}(0) S_{wx_2}(0) S_{wx_3}(0) S_{wx_4}(0)}{\bar{w}^3 S_{wx_1 x_2 x_3 x_4}(0, 0, 0, 0)} . \quad (2.250)$$

The behavior of this signature would depend on the processes that give rise to quadruplets of tricorrelated counts, i.e., those that contribute to the source-detector quad-spectral signature $S_{wx_1 x_2 x_3 x_4}(s_1, s_2, s_3, s_4)$. Intuitively, this signature is expected to vanish for a prompt critical system,

$$\lim_{\rho \rightarrow 0} \Psi_{x_1 \cdot x_2 \cdot x_3 \cdot x_4|w} = 0 , \quad (2.251)$$

and to yield the quantity

$$\lim_{\rho \rightarrow -\infty} \Psi_{x_1 \cdot x_2 \cdot x_3 \cdot x_4|w} = \left(\frac{v_0(v_0-1)(v_0-2)(v_0-3)}{\bar{v}_0^4} \right)^{-1} \quad (2.252)$$

in the absence of a multiplicative medium. Finally, the expectation that the signature

approaches its lower asymptote (zero) at a rate of three decades per decade stems from intuition. However, these suspicions will be left for someone else to prove (or disprove). The necessary methodology has been sufficiently described.

3 Estimating High Order Statistical Signatures

None of the process statistics described in the previous chapter are realizable. Even the measured signatures as modeled are idealized. They reflect the behavior of the entire ensemble of all possible fission-chains, which are infinite in number [Coo, Pap]. Real measured signatures are derived from a finite number of observations. Hence they are only estimates of the process statistics. This chapter outlines procedures to estimate the statistics modeled in the previous chapter.

First, the numerical procedures to estimate first- through fourth-order multivariate temporal and spectral signatures are described in general, i.e., outside the context of neutron counting measurements. They are presented in this manner because they are of general value in multivariate process analysis applications, and it is undesirable to taint their general applicability with details specific to neutron counting experiments. These specifics are discussed in the subsequent section.

Next, the implementation of these procedures as part of a Monte Carlo calculational code, specifically MCNP-DSP, is outlined. This section primarily serves to familiarize readers with the physics simulated by MCNP-DSP. Because the additional details of the estimation procedures' implementation regard rather dull book-keeping considerations, this section is brief. It directly segues to the following section, where statistical signatures estimated from a Monte Carlo calculational model of an enriched uranium-metal sphere are presented. Some discussion of the uncertainty in these estimates follows. Finally, results

from a set of calculations of spheres of varying enrichment are presented to further illustrate the enhanced sensitivity of successively higher order signatures.

3.1 Numerical Estimators of High Order Signatures

Let $\{x(n)\}$ represent a stationary time-discrete process, or signal, where time is indexed by n . The mean, or expected value,

$$\bar{x} = E[x(n)] . \quad (3.1)$$

It's assumed that the sequence $\{x(n)\}$ is infinite in length. However, only a finite record of the signal is acquired. Typically this record is partitioned into M segments (or "blocks") each containing N observations (or "samples"),

$$\{x(n)\} \supset \bigcup_{m=0}^{M-1} \bigcup_{n=0}^{N-1} \{x^{(m)}(n)\} , \quad (3.2)$$

and an estimate of the mean is obtained from

$$\hat{\bar{x}} = \frac{1}{M \cdot N} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x^{(m)}(n) . \quad (3.3)$$

In other words, the estimator is accumulated over N observations in each segment and then averaged over M segments. Note that $\hat{\bar{x}}$ is an unbiased estimator, i.e., its expected value is the true mean:

$$E[\hat{\bar{x}}] = \bar{x} . \quad (3.4)$$

The joint second moment sequence, or cross-correlation, between signals x and y is

$$R_{xy}(k) = E[x(n)y(n+k)] , \quad (3.5)$$

and it may be estimated from $M \cdot N$ observations of x and y . However, before an explicit description of the estimator is given, it's necessary to first take a brief detour to describe conventions observed in computer implementations of the estimator. As in the implementation of the estimator $\hat{x}$, estimates of the correlation R_{xy} are obtained from N observations of x and y in each segment m , and the final estimate is obtained by averaging over M segments. The estimator accumulated over each segment does not use observations from previous or subsequent segments, so $\{x^{(m)}(n)\}$ and $\{y^{(m)}(n)\}$ are assumed to vanish for $n < 0$ and $n \geq N$. So the estimate of $R_{xy}(k)$ is obtained at N positive values of the delay k (including zero) and at $N - 1$ negative values of k , and the estimate of $R_{xy}(\pm N)$ is always zero. Now refer to Fig. 3.1, which illustrates the convention for storing the estimator in an array with $2N$ elements. Logically, estimates of the correlation at the N positive delays (including zero) $k \in [0, N - 1]$ are stored in the first half of the array. The estimate of $R_{xy}(\pm N)$, which is always zero, is stored in array element N . However, because it's inconvenient to index the array using negative numbers, estimates of the correlation at the N negative delays $-k \in [1, N - 1]$ are stored in reverse order beginning at the end of the array. This is sometimes called a “wrap-around” structure. It may seem awkward at first, but it is the accepted convention, and it is also necessary for compatibility with standard implementations of the fast Fourier transform (FFT), which are an integral part of conventional spectral analysis [Ell, Opp, Pre].

Hence, the accumulator for positive values of the delay k is

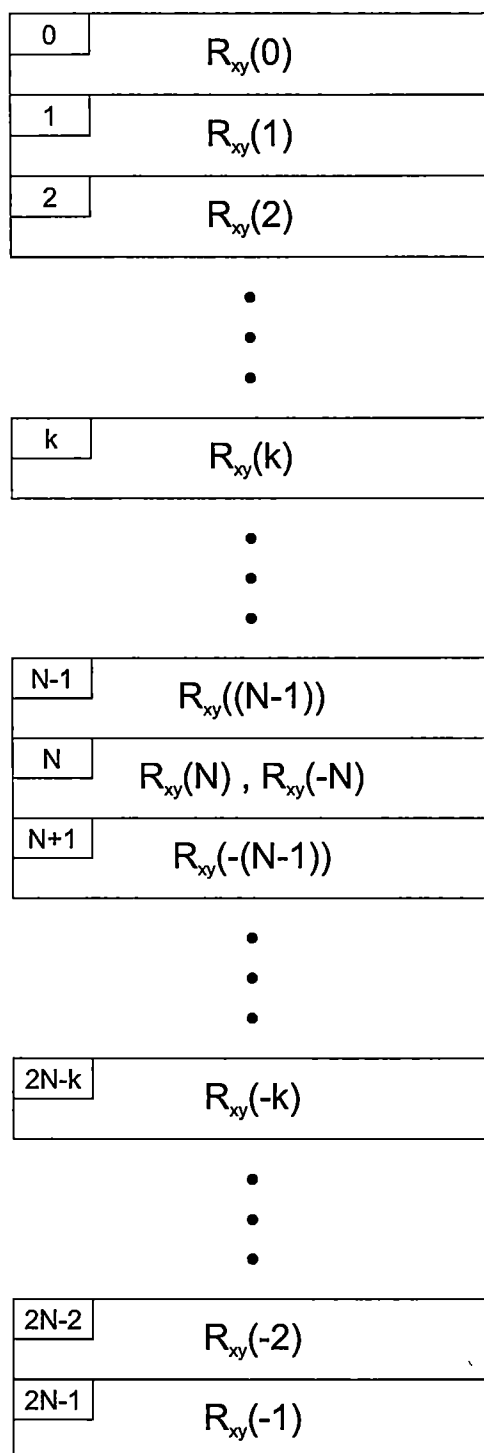


Figure 3.1 Wrap-around structure of correlation accumulator and estimator arrays

$$\hat{R}'_{xy}(k) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1-k} x^{(m)}(n) y^{(m)}(n+k) , \quad k \in [0, N-1] . \quad (3.6)$$

The accumulator for negative values of the delay k is stored in reverse order beginning at the end of the array:

$$\hat{R}'_{xy}(2N-k) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1-k} y^{(m)}(n) x^{(m)}(n+k) , \quad k \in [1, N-1] . \quad (3.7)$$

Note that this makes use of the identity $R_{xy}(-k) = E[y(n)x(n+k)]$ to avoid indexing the sequence $y^{(m)}$ with negative numbers. The accumulator is averaged over M segments to obtain the estimate

$$\begin{aligned} \hat{R}_{xy}(k) &= \frac{\hat{R}'_{xy}(k)}{M \cdot (N-k)} , \quad k \in [0, N-1] ; \\ \hat{R}_{xy}(2N-k) &= \frac{\hat{R}'_{xy}(2N-k)}{M \cdot (N-k)} , \quad k \in [1, N-1] , \end{aligned} \quad (3.8)$$

where the factor $(N-k)^{-1}$ accounts for the fact that only $N-k$ observations of x and y were used each segment to estimate the correlation at delay $\pm k$. Consequently, $\hat{R}_{xy}$ is an unbiased estimator of the correlation R_{xy} :

$$\begin{aligned} E[\hat{R}_{xy}(k)] &= R_{xy}(k) , \quad k \in [0, N-1] ; \\ E[\hat{R}_{xy}(2N-k)] &= R_{yx}(k) = R_{xy}(-k) , \quad k \in [1, N-1] . \end{aligned} \quad (3.9)$$

The covariance between x and y is then estimated from

$$\hat{C}_{xy}(k) = \hat{R}_{xy}(k) - \hat{\bar{x}} \hat{\bar{y}} , \quad (3.10)$$

and the estimated cross-spectrum between x and y is obtained from the discrete Fourier transform (DFT) of the covariance estimate:

$$\hat{S}_{xy}(n) = \sum_{k=0}^{2N-1} e^{-j2\pi nk/2N} \hat{C}_{xy}(k) = \mathbf{F}_{2N}(n|k) \hat{C}_{xy}(k), \quad n \in [0, 2N-1]. \quad (3.11)$$

Note that the array containing the spectrum estimate $\hat{S}_{xy}$ has structure similar to that depicted in Fig. 3.1. The first N elements will correspond to positive frequencies $n \in [0, N-1]$ including zero. The N -th element holds $\hat{S}_{xy}(\pm N)$. Negative frequencies $-n \in [1, N-1]$ progress in reverse order beginning at the end of the array.

Above, $\mathbf{F}_{2N}(n|k)$ is shorthand notation for the DFT operator that transforms a $2N$ -element sequence indexed by k (time) to one indexed by n (frequency). Recall that $\mathbf{F}_N$ is an $N \times N$ array operator, such that

$$\mathbf{F}_N(n|k) \bullet(k) = \sum_{k=0}^{N-1} e^{-j2\pi nk/N} \bullet(k), \quad n \in [0, N-1], \quad (3.12)$$

where $\bullet(k)$ represents the operand, an $N \times 1$ column array indexed by k . The inverse DFT matrix is simply

$$\mathbf{F}_N^{-1} = \frac{1}{N} \mathbf{F}_N^{T*}, \quad (3.13)$$

such that $\mathbf{F}_N^{-1}$ transforms a frequency-domain sequence back to a time-domain sequence:

$$\mathbf{F}_N^{-1}(k|n) \bullet(n) = \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi kn/N} \bullet(n), \quad k \in [0, N-1] \quad (3.14)$$

Two important properties of these matrix operators are (1) its orthonormality,

$$F_N^{-1}(\ell | n) F_N(n | k) \bullet(k) = \bullet(\ell) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} e^{-j2\pi(k-\ell)n/N} \bullet(k)$$

$$\therefore \frac{1}{N} \sum_{n=0}^{N-1} e^{-j2\pi(k-\ell)n/N} = \delta_k^\ell,$$

where

$$\delta_k^\ell = \begin{cases} 1, & \ell = k \\ 0, & \ell \neq k \end{cases} \quad (3.16)$$

is the Kronecker delta, and (2) its circularity:

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{-j2\pi((k+p)-\ell)n/N} = \delta_{k \oplus_N p}^\ell, \quad (3.17)$$

where $\oplus_N$ denotes addition modulo N . This circularity is an artifact of the phasor $e^{j2\pi kn/N}$

that rotates through the complex plane returning to the same position every N steps:

$$e^{j2\pi(k+N)n/N} = e^{j2\pi kn/N}. \quad (3.18)$$

These two properties of the DFT can be used to develop a procedure for estimating the covariance and cross-spectrum that is computationally more efficient than the one just described.

Recall that the correlation accumulators (3.6) and (3.7) each require on the order of $N \times N$, or $O(N^2)$, operations per segment. However, this is quite inefficient. Further recall that a fast implementation of the DFT, like the radix-2 FFT, requires $O(N \log_2 N)$ operations per segment, which is substantially fewer for most N . Conventional spectral analyses take advantage of the FFT's efficiency to estimate the cross-spectrum and covariance [Ell, Opp, Pre].

First, let

$$X^{(m)}(n) = F_{2N}(n|k) x^{(m)'}(k) , \quad (3.19)$$

where $x^{(m)'}$ is the N -element sequence $x^{(m)}$ augmented by N zeros:

$$x^{(m)'}(k) = \begin{cases} x^{(m)}(k) & , \quad k \in [0, N-1] \\ 0 & , \quad k \in [N, 2N-1] \end{cases} . \quad (3.20)$$

This is sometimes referred to as a zero-padded sequence [Ell, Pre]. Let $Y^{(m)}$ be defined similar to $X^{(m)}$. Next, define the spectral accumulator

$$\hat{Q}_{xy}(n) = \sum_{m=0}^{M-1} X^{(m)}(2N \ominus_{2N} n) Y^{(m)}(n) , \quad (3.21)$$

where $\ominus_{2N}$ denotes “subtraction” modulo $2N$. In other words,

$$\begin{aligned} \hat{Q}_{xy}(0) &= \sum_{m=0}^{M-1} X^{(m)}(0) Y^{(m)}(0) , \\ \hat{Q}_{xy}(n) &= \sum_{m=0}^{M-1} X^{(m)}(2N - n) Y^{(m)}(n) , \quad n \in [1, N-1] , \\ \hat{Q}_{xy}(2N - n) &= \sum_{m=0}^{M-1} Y^{(m)}(2N - n) X^{(m)}(n) , \quad n \in [1, N-1] . \end{aligned} \quad (3.22)$$

Now observe that the inverse DFT of the spectral accumulator is just the correlation accumulator:

$$F_{2N}^{-1}(k|n) \hat{Q}_{xy}(n) = \hat{R}'_{xy}(k) . \quad (3.23)$$

This works because, due to the DFT's orthonormality and circularity properties,

$$F_{2N}^{-1}(k|n) \hat{Q}_{xy}(n) = \sum_{m=0}^{M-1} \sum_{n=0}^{2N-1} x^{(m)'}(n) y^{(m)'}(n \oplus_{2N} k) , \quad (3.24)$$

which, due to the nature of modulo addition and the use of zero-padded sequences, yields (3.6) for $k \in [0, N - 1]$, zero for $k = N$, and (3.7) for $k \in [N + 1, 2N - 1]$. Try it!

Subsequently, (3.8) is applied to the resulting correlation accumulator $\hat{R}'_{xy}$ to obtain an unbiased estimate $\hat{R}_{xy}$ of the correlation. A covariance estimate $\hat{C}_{xy}$ is then obtained from the correlation estimate $\hat{R}_{xy}$ according to (3.10). The final spectral estimate $\hat{S}_{xy}$ is then simply the forward DFT of the resultant covariance estimate $\hat{C}_{xy}$. By using the FFT, all this is efficiently accomplished in $O(M \cdot N \log_2 N)$ operations, whereas the first procedure requires $O(M \cdot N^2)$ operations. The latter method is the one implemented to obtain the calculational results presented later this chapter. In the following chapter, a different technique is described that, under some conditions, is even faster.

Next consider that the joint third moment sequence, or bicornelation, between x , y , and z is

$$R_{xyz}(k_1, k_2) = E[x(n)y(n + k_1)z(n + k_2)] . \quad (3.25)$$

Currently, there is no consensus regarding conventions that should be observed in computer implementations of bicornelation estimators. However, since heavy reliance on two-dimensional (2-D) FFT algorithms is a necessity for conventional bispectral analysis, compatibility with existing 2-D FFT codes requires that the bicornelation matrix take the form depicted in Fig 3.2. The $2N \times 2N$ bicornelation array follows the same convention as the correlation array in that it has a “wrap-around” structure. The estimator is stored in the

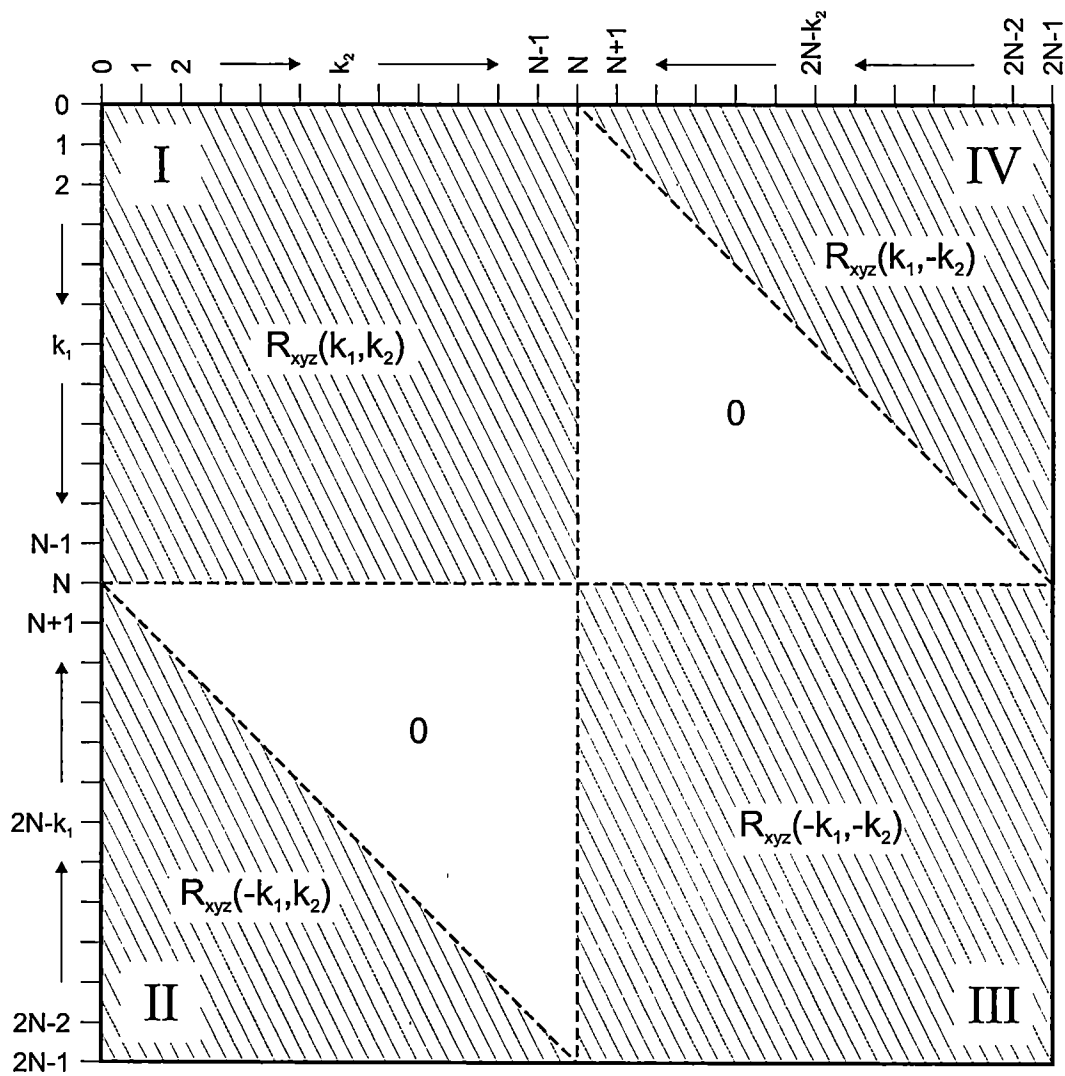


Figure 3.2 Wrap-around structure of bicornelation accumulator and estimator arrays

first quadrant of the array when both delays are positive, and when the two delays are negative, the estimator is stored in reverse order beginning at the end of the third quadrant. When one delay is positive and the other negative, the estimator is stored in the second and fourth quadrants of the array as shown in Fig. 3.2. The “unwrapped” arrays maps to the hexagonal region shown in Fig. 3.3. For each segment m , the sequences $\{x^{(m)}(n)\}$, $\{y^{(m)}(n)\}$, and $\{z^{(m)}(n)\}$ are again assumed to vanish when $n < 0$ or $n \geq N$. Hence when the two delays have the same sign, neither delay may exceed $N - 1$; otherwise, at least one of the sequences will fail to coincide with the other two. When the two delays are of opposite sign, $|k_1| + |k_2|$ must be less than N to ensure that the three sequences coincide.

Subsequently, for both delays positive, the bicorrelation accumulator is

$$\hat{R}'_{xyz}(k_1, k_2) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1-\max(k_1, k_2)} x^{(m)}(n) y^{(m)}(n+k_1) z^{(m)}(n+k_2), \quad (3.26)$$

$$(k_1, k_2) \in [0, N-1] \times [0, N-1].$$

When the first delay is negative and the second positive,

$$\hat{R}'_{xyz}(2N-k_1, k_2) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1-(k_1+k_2)} y^{(m)}(n) x^{(m)}(n+k_1) z^{(m)}(n+k_1+k_2), \quad (3.27)$$

$$k_1 \in [1, N-1], \quad k_2 \in [0, N-1-k_1].$$

Note that this makes use of the identity $R_{xyz}(-k_1, k_2) = E[y(n)x(n+k_1)z(n+k_1+k_2)]$; similar identities are employed by the other accumulators. Also note that the upper limit on k_2 is necessary to ensure that $k_1 + k_2 < N$. Similarly, when the first delay is positive and the

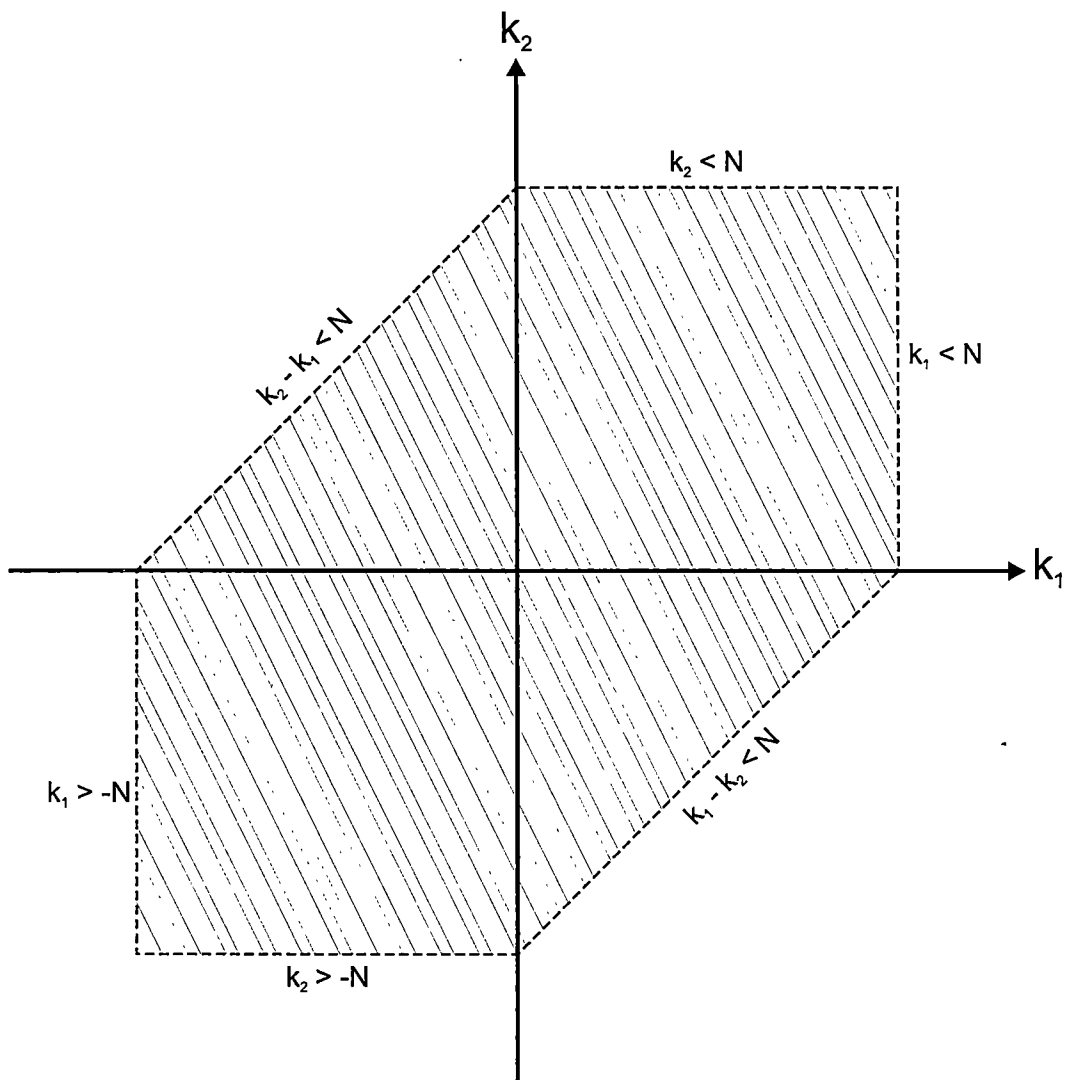


Figure 3.3 A bicorrelation estimator array maps to a hexagonal region in the (k_1, k_2) plane

second negative,

$$\hat{R}'_{xyz}(k_1, 2N - k_2) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1-(k_1+k_2)} z^{(m)}(n) x^{(m)}(n+k_2) y^{(m)}(n+k_1+k_2), \quad (3.28)$$

$$k_2 \in [1, N-1], \quad k_1 \in [0, N-1-k_2].$$

Finally, when both delays are negative,

$$\hat{R}'_{xyz}(2N - k_1, 2N - k_2) = \sum_{m=0}^{M-1} \begin{cases} \sum_{n=0}^{N-1-k_1} y^{(m)}(n) x^{(m)}(n+k_1) z^{(m)}(n+k_1-k_2), & k_2 < k_1 \\ \sum_{n=0}^{N-1-k_2} z^{(m)}(n) x^{(m)}(n+k_2) y^{(m)}(n+k_2-k_1), & k_2 \geq k_1 \end{cases}, \quad (3.29)$$

$$(k_1, k_2) \in [1, N-1] \times [1, N-1].$$

The accumulator is averaged over M segments and adjusted for the number of observations used to estimate R_{xyz} at each pair of delays,

$$\hat{R}_{xyz}(k_1, k_2) = \frac{\hat{R}'_{xyz}(k_1, k_2)}{M \cdot (N - \max(k_1, k_2))}, \quad (k_1, k_2) \in [0, N-1] \times [0, N-1];$$

$$\hat{R}_{xyz}(2N - k_1, k_2) = \frac{\hat{R}'_{xyz}(2N - k_1, k_2)}{M \cdot (N - (k_1 + k_2))}, \quad k_1 \in [1, N-1], \quad k_2 \in [0, N-1-k_1];$$

$$\hat{R}_{xyz}(k_1, 2N - k_2) = \frac{\hat{R}'_{xyz}(k_1, 2N - k_2)}{M \cdot (N - (k_1 + k_2))}, \quad k_2 \in [1, N-1], \quad k_1 \in [1, N-1-k_2];$$

$$\hat{R}_{xyz}(2N - k_1, 2N - k_2) = \frac{\hat{R}'_{xyz}(2N - k_1, 2N - k_2)}{M \cdot (N - \max(k_1, k_2))}, \quad (k_1, k_2) \in [1, N-1] \times [1, N-1], \quad (3.30)$$

to obtain the unbiased estimator $\hat{R}_{xyz}$:

$$\begin{aligned}
E\left[\hat{R}_{xyz}(k_1, k_2)\right] &= R_{xyz}(k_1, k_2) , \\
(k_1, k_2) &\in [0, N-1] \times [0, N-1] ; \\
E\left[\hat{R}_{xyz}(2N-k_1, k_2)\right] &= R_{yxz}(k_1, k_1+k_2) = R_{xyz}(-k_1, k_2) , \\
k_1 &\in [1, N-1] , \quad k_2 \in [0, N-1-k_1] ; \\
E\left[\hat{R}_{xyz}(k_1, 2N-k_2)\right] &= R_{zyx}(k_2, k_1+k_2) = R_{xyz}(k_1, -k_2) , \\
k_2 &\in [1, N-1] , \quad k_1 \in [0, N-1-k_2] ; \\
E\left[\hat{R}_{xyz}(2N-k_1, 2N-k_2)\right] &= \begin{cases} R_{yxz}(k_1, k_1-k_2) , & k_2 < k_1 \\ R_{zyx}(k_2, k_2-k_1) , & k_2 \geq k_1 \end{cases} = R_{xyz}(-k_1, -k_2) , \\
(k_1, k_2) &\in [1, N-1] \times [1, N-1] .
\end{aligned} \tag{3.31}$$

The bicovariance is then estimated from

$$\hat{C}_{xyz}(k_1, k_2) = \hat{R}_{xyz}(k_1, k_2) - \hat{x} \hat{C}_{yz}(k_2 - k_1) - \hat{y} \hat{C}_{xz}(k_2) - \hat{z} \hat{C}_{xy}(k_1) - \hat{x} \hat{y} \hat{z} . \tag{3.32}$$

Note that the substitution

$$k_2 - k_1 \rightarrow 2N - (k_1 - k_2) , \quad k_2 < k_1 \tag{3.33}$$

is used to account for the wrap-around structure of the covariance array $\hat{C}_{yz}$. Finally, the

bispectrum estimate is obtained from the double FFT of the bicovariance:

$$\hat{S}_{xyz}(n_1, n_2) = \mathbf{F}_{2N}(n_1 | k_1) \mathbf{F}_{2N}(n_2 | k_2) \hat{C}_{xyz}(k_1, k_2) . \tag{3.34}$$

Observe that the bispectrum array has wrap-around structure similar to the bicorrelation and

bicovariance arrays. Also note that, in practice, the bispectrum estimate $\hat{S}_{xyz}$ is not obtained

by transforming first the columns of $\hat{C}_{xyz}$ and then its rows, as (3.34) appears to imply. This

is too inefficient. Instead, standard 2-D FFT algorithms that accomplish this task in

$O(2N \log_2 N)$ operations are employed [Ell, Pre].

A more practical implementation takes advantage of the FFT's speed to estimate the bispectrum and bicovariance. The procedure is similar to that employed to estimate the cross-spectrum and covariance. Define the bispectral accumulator

$$\hat{Q}_{xyz}(n_1, n_2) = \sum_{m=0}^{M-1} X^{(m)}(2N \ominus_{2N} (n_1 \oplus_{2N} n_2)) Y^{(m)}(n_1) Z^{(m)}(n_2), \quad (3.35)$$

where $X^{(m)}$, $Y^{(m)}$, and $Z^{(m)}$ are respectively obtained by FFT'ing the zero-padded sequences $x^{(m)'}$, $y^{(m)'}$, and $z^{(m)'}$. The inverse transform of the bispectral accumulator yields the bicorrelation accumulator,

$$F_{2N}^{-1}(k_1 | n_1) F_{2N}^{-1}(k_2 | n_2) \hat{Q}_{xyz}(n_1, n_2) = \sum_{m=0}^{M-1} \sum_{n=0}^{2N-1} x^{(m)'}(n) y^{(m)'}(n \oplus_{2N} k_1) z^{(m)'}(n \oplus_{2N} k_2) = \hat{R}'_{xyz}(k_1, k_2), \quad (3.36)$$

due to the orthonormality and circularity of the DFT. Subsequently, application of (3.30) to the bicorrelation accumulator yields an unbiased estimate of the bicorrelation. Then (3.32) may be applied to estimate the bicovariance. The double FFT of the covariance estimate yields the estimate of the bispectrum. This is the procedure employed to obtain the bispectral and bicovariance estimates presented later this chapter.

Finally, the tricorrelation is the joint fourth moment sequence

$$R_{wxyz}(k_1, k_2, k_3) = E[w(n)x(n+k_1)y(n+k_2)z(n+k_3)]. \quad (3.37)$$

The same conventions observed in the implementation of the correlation and bicorrelation estimator are applied to the following implementation of a tricorrelation estimator. The

estimator for zero and positive delays less than N is stored in forward order, and for negative delays the estimator is stored in reverse order beginning at the last index, $2N - 1$. The resultant $2N \times 2N \times 2N$ array is compatible with standard implementations of three-dimensional (3-D) FFT algorithms¹. At this point it's important to note that even for relatively small N , the tricorrelation estimator array requires a substantial amount of memory because it has $8N^3$ elements. For example, in an application requiring 128 samples/segment and using 32-bit floating point arithmetic to estimate the tricorrelation, a single estimator array will occupy 64 megabytes (Mb) in memory. The corresponding trispectrum will require twice that space, 128 Mb, since it has both real and imaginary parts. If 64-bit precision is required, the necessary storage increases by a factor of eight. Furthermore, the mere act of indexing this large number of memory addresses requires substantially more computational effort than observed in implementations of lower order signatures. However, recently the price of memory for personal computers has fallen substantially (\$50 - \$100 per 64 Mb), and memory access times are as well decreasing rapidly, though they are still on the order of 50 ns - 100 ns if the address is outside the cache. Still, the capacity of cache memory has been increasing at a steady pace. So, although when this work began three years ago tricorrelation estimation was only somewhat impractical, it is now marginally practical, so long as only one or two fourth-order signatures are estimated at a time.

Of course, since by convention the sequences $w^{(m)}$, $x^{(m)}$, $y^{(m)}$, and $z^{(m)}$ are assumed

¹ Three- and two-dimensional (even one-dimensional) FFT algorithms are actually implemented in a single multidimensional FFT code that can operate on rectangular arrays of any dimensionality [Pre].

to vanish beyond the boundaries of the m -th segment, each octant of the tricorrelation array is not completely filled. In fact, only the first and seventh octants (where all three delays are of the same sign) are completely full. The requirement that all four sequences coincide at one or more elements leaves the other six octants only half-full, the remainder of each being populated by zeros. However, a cubic array is required for compatibility with current 3-D FFT codes. Although it is conceivable that multidimensional FFT algorithms that can operate on slightly irregular arrays might some day be developed, the development of such a tool is quite beyond the scope of this work.

The bounds of the tricorrelation accumulator follow a regular pattern depending on the number of negative delays. So it's more convenient, and illustrative, to present the tricorrelation accumulator in order of (1) all delays positive, (2) one delay negative, (3) two delays negative, and (4) all delays negative, rather than simply progressing through the octants. The octant will be indicated for each accumulator. When all three delays are positive, the accumulator over the first octant is

$$\hat{R}'_{wxyz}(k_1, k_2, k_3) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1-\max(k_1, k_2, k_3)} w^{(m)}(n) x^{(m)}(n+k_1) y^{(m)}(n+k_2) z^{(m)}(n+k_3) , \quad (3.38)$$

$$(k_1, k_2, k_3) \in [0, N-1] \times [0, N-1] \times [0, N-1] .$$

Note that it completely fills the first octant. When one delay is negative, the accumulator partially fills the (1) second octant:

$$\begin{aligned}
\hat{R}'_{wxyz}(2N - k_1, k_2, k_3) &= \\
\sum_{m=0}^{M-1} \sum_{n=0}^{N-1-(k_1+\max(k_2, k_3))} x^{(m)}(n) w^{(m)}(n+k_1) y^{(m)}(n+k_1+k_2) z^{(m)}(n+k_1+k_3) , & \quad (3.39) \\
k_1 \in [1, N-1] , \quad k_2 \in [0, N-1-k_1] , \quad k_3 \in [0, N-1-(k_1+k_2)] ; &
\end{aligned}$$

(2) fourth octant:

$$\begin{aligned}
\hat{R}'_{wxyz}(k_1, 2N - k_2, k_3) &= \\
\sum_{m=0}^{M-1} \sum_{n=0}^{N-1-(k_2+\max(k_1, k_3))} y^{(m)}(n) w^{(m)}(n+k_2) x^{(m)}(n+k_1+k_2) z^{(m)}(n+k_2+k_3) , & \quad (3.40) \\
k_2 \in [1, N-1] , \quad k_3 \in [0, N-1-k_2] , \quad k_1 \in [0, N-1-(k_2+k_3)] ; &
\end{aligned}$$

and (3) fifth octant:

$$\begin{aligned}
\hat{R}'_{wxyz}(k_1, k_2, 2N - k_3) &= \\
\sum_{m=0}^{M-1} \sum_{n=0}^{N-1-(k_3+\max(k_1, k_2))} z^{(m)}(n) w^{(m)}(n+k_3) x^{(m)}(n+k_1+k_3) y^{(m)}(n+k_2+k_3) , & \quad (3.41) \\
k_3 \in [1, N-1] , \quad k_1 \in [0, N-1-k_3] , \quad k_2 \in [0, N-1-(k_1+k_3)] . &
\end{aligned}$$

Similarly, when two delays are negative, the accumulator partially fills the (1) third octant:

$$\begin{aligned}
\hat{R}'_{wxyz}(2N - k_1, 2N - k_2, k_3) &= \\
\sum_{m=0}^{M-1} \left\{ \begin{array}{l} \sum_{n=0}^{N-1-(k_1+k_3)} x^{(m)}(n) w^{(m)}(n+k_1) y^{(m)}(n+k_1-k_2) z^{(m)}(n+k_1+k_3) , \quad k_2 < k_1 \\ \sum_{n=0}^{N-1-(k_2+k_3)} y^{(m)}(n) w^{(m)}(n+k_2) x^{(m)}(n+k_2-k_1) z^{(m)}(n+k_2+k_3) , \quad k_2 \geq k_1 \end{array} \right. , & \quad (3.42) \\
k_1 \in [1, N-1] , \quad k_3 \in [0, N-1-k_1] , \quad k_2 \in [1, N-1-(k_1+k_3)] ; &
\end{aligned}$$

(2) sixth octant:

$$\hat{R}'_{wxyz}(2N - k_1, k_2, 2N - k_3) = \sum_{m=0}^{M-1} \begin{cases} \sum_{n=0}^{N-1-(k_1+k_2)} x^{(m)}(n) w^{(m)}(n+k_1) y^{(m)}(n+k_1+k_2) z^{(m)}(n+k_1-k_3), & k_3 < k_1 \\ \sum_{n=0}^{N-1-(k_2+k_3)} z^{(m)}(n) w^{(m)}(n+k_3) x^{(m)}(n+k_3-k_1) y^{(m)}(n+k_2+k_3), & k_3 \geq k_1 \end{cases}, \quad (3.43)$$

$$k_3 \in [1, N-1], \quad k_2 \in [0, N-1-k_3], \quad k_1 \in [1, N-1-(k_2+k_3)];$$

and (3) eighth octant:

$$\hat{R}'_{wxyz}(k_1, 2N - k_2, 2N - k_3) = \sum_{m=0}^{M-1} \begin{cases} \sum_{n=0}^{N-1-(k_1+k_2)} y^{(m)}(n) w^{(m)}(n+k_2) x^{(m)}(n+k_1+k_2) z^{(m)}(n+k_2-k_3), & k_3 < k_2 \\ \sum_{n=0}^{N-1-(k_1+k_3)} z^{(m)}(n) w^{(m)}(n+k_3) x^{(m)}(n+k_1+k_3) y^{(m)}(n+k_3-k_2), & k_3 \geq k_2 \end{cases}, \quad (3.44)$$

$$k_2 \in [1, N-1], \quad k_1 \in [0, N-1-k_2], \quad k_3 \in [1, N-1-(k_1+k_2)].$$

Finally, when all three delays are negative, the accumulator

$$\hat{R}'_{wxyz}(2N - k_1, k_2, 2N - k_3) = \sum_{m=0}^{M-1} \begin{cases} \sum_{n=0}^{N-1-k_1} x^{(m)}(n) w^{(m)}(n+k_1) y^{(m)}(n+k_1-k_2) z^{(m)}(n+k_1-k_3), & k_2 < k_1 \cap k_3 < k_1 \\ \sum_{n=0}^{N-1-k_2} y^{(m)}(n) w^{(m)}(n+k_2) x^{(m)}(n+k_2-k_1) z^{(m)}(n+k_2-k_3), & k_1 < k_2 \cap k_3 < k_2 \\ \sum_{n=0}^{N-1-k_3} z^{(m)}(n) w^{(m)}(n+k_3) x^{(m)}(n+k_3-k_1) y^{(m)}(n+k_3-k_2), & k_1 < k_3 \cap k_2 < k_3 \end{cases}, \quad (3.45)$$

$$(k_1, k_2, k_3) \in [1, N-1] \times [1, N-1] \times [1, N-1]$$

completely fills the seventh octant. The accumulator is then averaged over M segments and

adjusted for the number of observations used to estimate the tricorrelation at each triplet of delays:

$$\begin{aligned}
 \hat{R}_{wxyz}(k_1, k_2, k_3) &= \frac{\hat{R}'_{wxyz}(k_1, k_2, k_3)}{M \cdot (N - \max(k_1, k_2, k_3))} , \\
 (k_1, k_2, k_3) &\in [0, N-1] \times [0, N-1] \times [0, N-1] ; \\
 \hat{R}_{wxyz}(2N - k_1, k_2, k_3) &= \frac{\hat{R}'_{wxyz}(2N - k_1, k_2, k_3)}{M \cdot (N - (k_1 + \max(k_2, k_3)))} , \\
 k_1 &\in [1, N-1] , \quad k_2 \in [0, N-1 - k_1] , \quad k_3 \in [0, N-1 - (k_1 + k_2)] ; \\
 \hat{R}_{wxyz}(k_1, 2N - k_2, k_3) &= \frac{\hat{R}'_{wxyz}(k_1, 2N - k_2, k_3)}{M \cdot (N - (k_2 + \max(k_1, k_3)))} , \\
 k_2 &\in [1, N-1] , \quad k_3 \in [0, N-1 - k_2] , \quad k_1 \in [0, N-1 - (k_2 + k_3)] ; \\
 \hat{R}_{wxyz}(k_1, k_2, 2N - k_3) &= \frac{\hat{R}'_{wxyz}(k_1, k_2, 2N - k_3)}{M \cdot (N - (k_3 + \max(k_1, k_2)))} , \\
 k_3 &\in [1, N-1] , \quad k_1 \in [0, N-1 - k_3] , \quad k_2 \in [0, N-1 - (k_1 + k_3)] ; \\
 \hat{R}_{wxyz}(2N - k_1, 2N - k_2, k_3) &= \frac{\hat{R}'_{wxyz}(2N - k_1, 2N - k_2, k_3)}{M \cdot (N - (k_3 + \max(k_1, k_2)))} , \\
 k_1 &\in [1, N-1] , \quad k_3 \in [0, N-1 - k_1] , \quad k_2 \in [1, N-1 - (k_1 + k_3)] ; \\
 \hat{R}_{wxyz}(2N - k_1, k_2, 2N - k_3) &= \frac{\hat{R}'_{wxyz}(2N - k_1, k_2, 2N - k_3)}{M \cdot (N - (k_2 + \max(k_1, k_3)))} , \\
 k_3 &\in [1, N-1] , \quad k_2 \in [0, N-1 - k_3] , \quad k_1 \in [1, N-1 - (k_2 + k_3)] ; \\
 \hat{R}_{wxyz}(k_1, 2N - k_2, 2N - k_3) &= \frac{\hat{R}'_{wxyz}(k_1, 2N - k_2, 2N - k_3)}{M \cdot (N - (k_1 + \max(k_2, k_3)))} , \\
 k_2 &\in [1, N-1] , \quad k_1 \in [0, N-1 - k_2] , \quad k_3 \in [1, N-1 - (k_1 + k_2)] ; \\
 \hat{R}_{wxyz}(2N - k_1, 2N - k_2, 2N - k_3) &= \frac{\hat{R}'_{wxyz}(2N - k_1, 2N - k_2, 2N - k_3)}{M \cdot (N - \max(k_1, k_2, k_3))} , \\
 (k_1, k_2, k_3) &\in [1, N-1] \times [1, N-1] \times [1, N-1] ;
 \end{aligned} \tag{3.46}$$

to obtain an unbiased estimate of the tricorrelation:

$$\begin{aligned}
E[\hat{R}_{wxyz}(k_1, k_2, k_3)] &= R_{wxyz}(k_1, k_2, k_3), \\
(k_1, k_2, k_3) &\in [0, N-1] \times [0, N-1] \times [0, N-1]; \\
E[\hat{R}_{wxyz}(2N-k_1, k_2, k_3)] &= R_{xwyz}(k_1, k_1+k_2, k_1+k_3) = R_{wxyz}(-k_1, k_2, k_3), \\
k_1 &\in [1, N-1], \quad k_2 \in [0, N-1-k_1], \quad k_3 \in [0, N-1-(k_1+k_2)]; \\
E[\hat{R}_{wxyz}(k_1, 2N-k_2, k_3)] &= R_{ywzx}(k_2, k_1+k_2, k_2+k_3) = R_{wxyz}(k_1, -k_2, k_3), \\
k_2 &\in [1, N-1], \quad k_3 \in [0, N-1-k_2], \quad k_1 \in [0, N-1-(k_2+k_3)]; \\
E[\hat{R}_{wxyz}(k_1, k_2, 2N-k_3)] &= R_{zwx}(k_3, k_1+k_3, k_2+k_3) = R_{wxyz}(k_1, k_2, -k_3), \\
k_3 &\in [1, N-1], \quad k_1 \in [0, N-1-k_3], \quad k_2 \in [0, N-1-(k_1+k_3)]; \\
E[\hat{R}_{wxyz}(2N-k_1, 2N-k_2, k_3)] &= \begin{cases} R_{xwyz}(k_1, k_1-k_2, k_1+k_3), & k_2 < k_1 \\ R_{ywzx}(k_2, k_2-k_1, k_2+k_3), & k_2 \geq k_1 \end{cases} = R_{wxyz}(-k_1, -k_2, k_3), \\
k_1 &\in [1, N-1], \quad k_3 \in [0, N-1-k_1], \quad k_2 \in [1, N-1-(k_1+k_3)]; \quad (3.47) \\
E[\hat{R}_{wxyz}(2N-k_1, k_2, 2N-k_3)] &= \begin{cases} R_{xwyz}(k_1, k_1+k_2, k_1-k_3), & k_3 < k_1 \\ R_{zwx}(k_3, k_3-k_1, k_2+k_3), & k_3 \geq k_1 \end{cases} = R_{wxyz}(k_1, -k_2, -k_3), \\
k_3 &\in [1, N-1], \quad k_2 \in [0, N-1-k_3], \quad k_1 \in [1, N-1-(k_2+k_3)]; \\
E[\hat{R}_{wxyz}(k_1, 2N-k_2, 2N-k_3)] &= \begin{cases} R_{ywzx}(k_2, k_1+k_2, k_2-k_3), & k_3 < k_2 \\ R_{zwx}(k_3, k_1+k_3, k_3-k_2), & k_3 \geq k_2 \end{cases} = R_{wxyz}(k_1, -k_2, -k_3), \\
k_2 &\in [1, N-1], \quad k_1 \in [0, N-1-k_2], \quad k_3 \in [1, N-1-(k_1+k_2)]; \\
E[\hat{R}_{wxyz}(2N-k_1, 2N-k_2, 2N-k_3)] &= \begin{cases} R_{xwyz}(k_1, k_1-k_2, k_1-k_3), & k_2 < k_1 \cap k_3 < k_1 \\ R_{ywzx}(k_2, k_2-k_1, k_2-k_3), & k_1 < k_2 \cap k_3 < k_2 \\ R_{zwx}(k_3, k_3-k_1, k_3-k_2), & k_1 < k_3 \cap k_2 < k_3 \end{cases} \\
&= R_{wxyz}(-k_1, -k_2, -k_3), \quad (k_1, k_2, k_3) \in [1, N-1] \times [1, N-1] \times [1, N-1].
\end{aligned}$$

Subsequently, an estimate of the tricovariance is obtained from

$$\begin{aligned}
\hat{C}_{wxyz}(k_1, k_2, k_3) &= \hat{R}_{wxyz}(k_1, k_2, k_3) \\
&- \hat{w} \hat{C}_{xyz}(k_2 - k_1, k_3 - k_1) - \hat{x} \hat{C}_{wyz}(k_2, k_3) - \hat{y} \hat{C}_{wxz}(k_1, k_3) - \hat{z} \hat{C}_{wxy}(k_1, k_2) \\
&- \hat{C}_{wx}(k_1) \hat{C}_{yz}(k_3 - k_2) - \hat{C}_{wy}(k_2) \hat{C}_{xz}(k_3 - k_1) - \hat{C}_{wz}(k_3) \hat{C}_{xy}(k_2 - k_1) \\
&- \hat{w} \hat{x} \hat{C}_{yz}(k_3 - k_2) - \hat{w} \hat{y} \hat{C}_{xz}(k_3 - k_1) - \hat{w} \hat{z} \hat{C}_{xy}(k_2 - k_1) \\
&- \hat{x} \hat{y} \hat{C}_{wz}(k_3) - \hat{x} \hat{z} \hat{C}_{wy}(k_2) - \hat{y} \hat{z} \hat{C}_{wx}(k_1) \\
&- \hat{w} \hat{x} \hat{y} \hat{z} .
\end{aligned} \tag{3.48}$$

Observe that the following substitutions,

$$\begin{aligned}
k_2 - k_1 &\rightarrow 2N - (k_1 - k_2) , \quad k_2 < k_1 , \\
k_3 - k_1 &\rightarrow 2N - (k_1 - k_3) , \quad k_3 < k_1 , \\
k_3 - k_2 &\rightarrow 2N - (k_2 - k_3) , \quad k_3 < k_2 ,
\end{aligned} \tag{3.49}$$

are necessary to account for the wrap-around structure of the covariance arrays $\hat{C}_{xy}$, $\hat{C}_{xz}$, and $\hat{C}_{yz}$

and the bicovariance array $\hat{C}_{xyz}$. Finally, an estimate of the trispectrum is obtained from the

3-D FFT of the tricovariance:

$$\hat{S}_{wxyz}(n_1, n_2, n_3) = \mathbf{F}_{2N}(n_1 | k_1) \mathbf{F}_{2N}(n_2 | k_2) \mathbf{F}_{2N}(n_3 | k_3) \hat{C}_{wxyz}(k_1, k_2, k_3) . \tag{3.50}$$

Furthermore, an alternative, and more efficient, procedure to estimate the trispectrum and tricovariance takes advantage of the speed of the FFT. The procedure, in its formulation, is identical to that described to estimate the lower order signatures using the FFT. First, define a trispectral accumulator

$$\begin{aligned}
\hat{Q}_{wxyz}(n_1, n_2, n_3) &= \\
&\sum_{m=0}^{M-1} W^{(m)}(2N \ominus_{2N} (n_1 \oplus_{2N} n_2 \oplus_{2N} n_3)) X^{(m)}(n_1) Y^{(m)}(n_2) Z^{(m)}(n_3) ,
\end{aligned} \tag{3.51}$$

where the preceding Fourier-sequences are obtained from the FFT of the zero-padded time-sequences. The tricorrelation accumulator is obtained from the inverse transform of the trispectral accumulator,

$$\begin{aligned}
 F_{2N}^{-1}(k_1 | n_1) F_{2N}^{-1}(k_2 | n_2) F_{2N}^{-1}(k_3 | n_3) \hat{Q}_{wxyz}(n_1, n_2, n_3) &= \\
 \sum_{m=0}^{M-1} \sum_{n=0}^{2N-1} w^{(m)'}(n) x^{(m)'}(n \oplus_{2N} k_1) y^{(m)'}(n \oplus_{2N} k_2) z^{(m)'}(n \oplus_{2N} k_3) & \quad (3.52) \\
 = \hat{R}'_{wxyz}(k_1, k_2, k_3) , &
 \end{aligned}$$

by virtue of the DFT's orthonormality and circularity. Subsequently, the accumulator is “unbiased” via application of (3.46) to obtain the tricorrelation estimator $\hat{R}_{wxyz}$. Application of (3.48) to $\hat{R}_{wxyz}$ yields the tricovariance $\hat{C}_{wxyz}$, and the forward transform of the tricovariance yields the final trispectrum estimate $\hat{S}_{wxyz}$. This technique was applied to acquire the tricovariance and trispectrum estimates presented later this chapter.

There are some further refinements to the preceding procedures that deserve brief mention, though they are not implemented in this work. Data windowing and segment overlap are two procedures developed to reduce the uncertainty in a given spectral estimate [Ell, Opp, Pre]. Data windowing reduces uncertainty in spectral estimates by applying a window function that typically vanishes at both boundaries of a segment. The intent is to mitigate uncertainty introduced by edge discontinuities at the segment boundaries imposed by partitioning a sampled time-continuous signal. However, the essentially time-discrete nature of nuclear counting signals acquired from pulse-mode electronics suggests that data windowing is unnecessary in this context. Segment overlap advances the segment

boundaries a fraction of the segment width, rather than a full segment width, per partition thereby increasing the effective number of segments and introducing some correlation between segments. This of course tends to reduce uncertainty in spectral estimates by increasing the number of observations and essentially treating adjacent segments as correlated observations rather than independent ones. However, it does require more sophisticated “book-keeping” procedures than those represented by the accumulators presented previously: a portion of each segment must be retained as the partition advances through the time-series. These two techniques have been frequently applied in second-order analyses and have enjoyed varying degrees of success [Ell, Pre]. Furthermore, extensions of these procedures to third-order spectral analyses have been outlined and implemented in several publications [Nik, Piñ, Rag]. Interested readers are encouraged to examine these references.

Finally, the procedures described belong to the larger family of conventional statistical analyses. Conventional analyses essentially take the data’s behavior at face value. The statistical estimates are computed based upon the behavior of a large number of observations, and no explicit attempt is made to characterize any cause and effect relationships between variables. In contrast, parametric estimation procedures², which should probably be categorized with the larger family of experimental design techniques, essentially attempt to characterize the process by developing a few-parameter model of cause and effect relationships between variables [Kay, Mar]. Specifically, the process from which

² Parametric estimation procedures are sometimes referred to as time-series models, predictive models, or autoregressive moving average (ARMA) models [Upa1, Upa2].

the signals originate is modeled as a time-discrete process with a finite number of poles and zeros. The locations of these poles and zeros in the complex (Laplace) plane are estimated from the behavior of the collection of signals over a fairly small number of observations (relative to conventional estimators). Frequently the model is applied to predict the future response of the process to changes in input. However, the present work is primarily concerned with changes in the state of the system itself, not with changes in the input to the process.

Parametric estimation procedures have been successfully applied in a number of second-order multivariate spectral analyses, and extensions of these procedures to third-order univariate bispectral analyses have been described in the literature [Men, Nik, Rag]. They appear to be particularly useful in the analysis of wide-band processes, a family to which source-driven fission-chain proliferation belongs. Their advantages may in the future make them competitive with or superior to conventional analyses in source-driven measurement applications. However, they do have the following disadvantage: some potentially dubious assumptions regarding the behavior of the process must be made a-priori. Specifically, a model order, i.e., the number of poles and zeros, must be determined. Procedures to select a near-optimal model order have been developed for second-order analyses. However, higher order parametric analysis has not yet reached this level of sophistication. Furthermore, nuclear assemblies that exhibit a continuum of eigenvalues have been observed [Cor]. For these, the finite number of discrete poles estimated by the parametric model are fictitious – they do not reflect the actual physical behavior of the process as they are sometimes touted to do. So, being a bit old-fashioned this work adheres to the underlying philosophy of

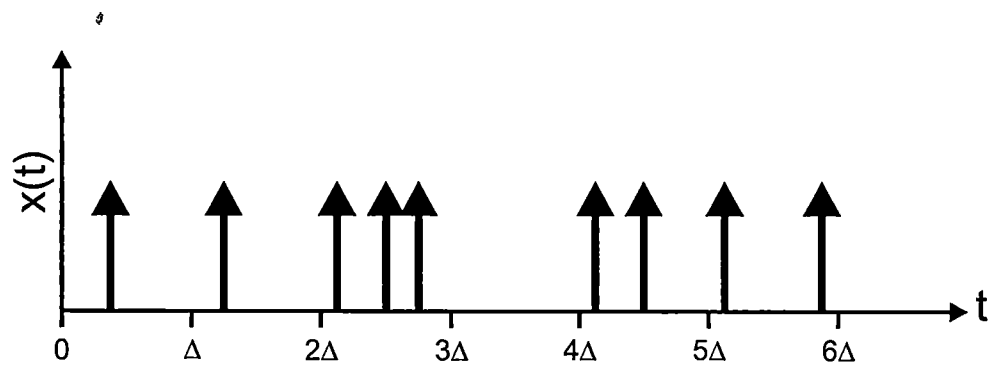
conventional process analysis: the behavior of the process is that which is observed.

3.2 Interpretation as Neutron Counting Statistics

The numerical procedures described in the previous section are generally applicable; they may be used to estimate first- through fourth-order signatures from any process provided that the signals are acquired as or adequately represented by sequences of discrete observations. Many processes are, or appear to be, time-continuous, so for these the sequences represent digitally sampled signals. Source-driven neutron counting measurements acquired in pulse mode are, on the other hand, essentially time-discrete.

However, the discrete counting events do not appear at regular intervals, as depicted in Fig. 3.4. The vertical arrows represent discrete counting events, which are typically measured from the leading edge of square pulses output by a single channel analyzer or constant fraction discriminator. A detector signal acquired in current mode is quite different; the continuous current signal essentially represents an integral of the recent count rate, and it must be digitally sampled to submit to numerical analysis. Furthermore, the blurring effect of current mode operation makes it difficult to express estimated signatures in terms of fundamental counting units. So for the most part, this work continues to focus on signals acquired in pulse mode, which are already “quasi-digital” in nature. Pulse mode signals do not have to be digitally sampled; they can be sampled via simple counting.

This simple process illustrated by Fig. 3.4. During sequential intervals of width Δ , the number of events that occurred during that particular counting interval is recorded.



$$x(n) : \{ 1, 1, 3, 0, 2, 2, \dots \}$$

Figure 3.4 Illustration of a counting event sequence and a record of counts per discrete counting interval

Hence a sequence of counts per interval is generated which may be analyzed in conjunction with similar sequences acquired from other detector signals to estimate process statistics.

In this context, each element of the sequences $\{x^{(m)}(n)\}$, $\{y^{(m)}(n)\}$, etc., can be interpreted as the observed number of events per interval n in segment m :

$$x^{(m)}(n) = \int_{m \cdot N\Delta + n\Delta}^{m \cdot N\Delta + (n+1)\Delta} dt x(t) . \quad (3.53)$$

Hence the estimated mean $\hat{\bar{x}}$ defined by (3.2) is the average number of events per interval.

However, observe that for a fixed count rate this estimate can vary depending on the width of the counting interval Δ . Hence, the estimate should be normalized to the counting interval

width, such that $\hat{\bar{x}} / \Delta$ is an estimate of the mean count rate in channel x . Note that, if w

denotes the source channel, $\hat{\bar{w}} / \Delta$ estimates the mean source fission rate. This normalization

is equivalent to redefining the sequences $\{x^{(m)}(n)\}$, etc., such that they represent the

observed number of events per unit time in segment m . Now recall that in the absence of any

unobservable sources, the count rate (and every other signature as well) is directly

proportional to the source fission rate. Consequently, in source-driven measurements the

measured detector count rate, and every other signature, is frequently normalized to the

measured source fission rate. Therefore, $\hat{\bar{x}} / \hat{\bar{w}}$ estimates the mean number of counts per

²⁵²Cf-fission. The distribution of these following ²⁵²Cf-fission is the normalized source-

detector covariance, $\hat{C}_{wx} / \hat{\bar{w}} \cdot \Delta$. Table 3.1 summarizes the units of each normalized

signature.

Table 3.1 Units of normalized estimators

Estimator	Normalization	Units ⁽¹⁾
detector count rate	$\hat{x} / \hat{w}$	counts/ ²⁵² Cf-fission
source-detector covariance	$\hat{C}_{wx} / \hat{w} \cdot \Delta$	counts/ ²⁵² Cf-fission/nsec coincident with ²⁵² Cf-fission
between-detectors covariance	$\hat{C}_{xy} / \hat{w} \cdot \Delta$	coincident counts/ ²⁵² Cf- fission/nsec
source-detector bicovariance	$\hat{C}_{wxy} / \hat{w} \cdot \Delta^2$	coincident counts/ ²⁵² Cf- fission/nsec ² bicoincident with ²⁵² Cf-fission
between-detector bicovariance	$\hat{C}_{xyz} / \hat{w} \cdot \Delta^2$	bicoincident counts/ ²⁵² Cf- fission/nsec ²
source-detector tricovariance	$\hat{C}_{wxyz} / \hat{w} \cdot \Delta^3$	bicoincident counts/ ²⁵² Cf- fission/nsec ³ tricoincident with ²⁵² Cf-fission
source-detector spectrum	$\hat{S}_{wx} / \hat{w}$	counts/ ²⁵² Cf-fission coincident with ²⁵² Cf-fission
between-detectors spectrum	$\hat{S}_{xy} / \hat{w}$	coincident counts/ ²⁵² Cf-fission
source-detector bispectrum	$\hat{S}_{wxy} / \hat{w}$	coincident counts/ ²⁵² Cf-fission bicoincident with ²⁵² Cf-fission
between-detectors bispectrum	$\hat{S}_{xyz} / \hat{w}$	bicoincident counts/ ²⁵² Cf- fission
source-detector trispectrum	$\hat{S}_{wxyz} / \hat{w}$	bicoincident counts/ ²⁵² Cf- fission tricoincident with ²⁵² Cf-fission

⁽¹⁾ Counting interval width Δ expressed in units of nanoseconds (nsec).

Recall that each spectral signature is an integral of the corresponding temporal distribution. It should be obvious that the temporal distribution is over delay(s)

$$\tau_i = k_i \Delta . \quad (3.54)$$

A negative delay, which simply corresponds to a prior event, rather than a subsequent event, maps from

$$-\tau_i = (2N - k_i) \Delta , \quad (3.55)$$

due to the wrap-around structure of the estimators. The corresponding spectral signature represents the distribution over frequency. For example, the units of the source-detector spectrum could be expressed as counts/²⁵²Cf-fission/nsec/GHz. Natural frequency maps from

$$f_i = \frac{n_i}{2N \Delta} , \quad (3.56)$$

and negative frequencies are mapped from

$$-f_i = \frac{2N - n_i}{2N \Delta} . \quad (3.57)$$

However, these are non-physical. Given the symmetrical nature of each spectral signature³, negative frequencies are usually ignored. The distribution over positive frequency is essentially the same.

Of course, given the equivalence between real coincidence and correlation, the estimated signatures have the interpretations listed in Table. 3.2. These are essentially the same interpretations derived in the previous chapter, except that the signatures are

³ The symmetries possessed by the spectral signatures are implicit in the symmetries of the temporal signatures. Explicit descriptions of some spectral symmetry properties are given later this chapter.

Table 3.2 Interpretation of normalized signatures

Normalized Signature	Interpretation ⁽¹⁾
detector count rate	counts/ ²⁵² Cf-fission
source-detector covariance	counts/ ²⁵² Cf-fission/nsec correlated with ²⁵² Cf fission
between-detectors covariance	correlated pairs/ ²⁵² Cf-fission/nsec
source-detector bicovariance	correlated pairs/ ²⁵² Cf-fission/nsec ² bicorrelated with ²⁵² Cf fission
between-detector bicovariance	bicorrelated triplets/ ²⁵² Cf-fission/nsec ²
source-detector tricovariance	bicorrelated triplets/ ²⁵² Cf-fission/nsec ³ tricorrelated with ²⁵² Cf fission
source-detector spectrum	counts/ ²⁵² Cf-fission correlated with ²⁵² Cf-fission
between-detectors spectrum	correlated pairs/ ²⁵² Cf-fission
source-detector bispectrum	correlated pairs/ ²⁵² Cf-fission bicorrelated with ²⁵² Cf-fission
between-detectors bispectrum	bicorrelated triplets/ ²⁵² Cf-fission
source-detector trispectrum	bicorrelated triplets/ ²⁵² Cf-fission tricorrelated with ²⁵² Cf-fission

⁽¹⁾ Counting interval width Δ expressed in units of nanoseconds (nsec).

normalized to the measured ^{252}Cf -fission rate. This renders the source-detector signatures independent of source strength. If no additional unobservable source is present, the normalized between-detectors signatures are as well independent of source strength. This is also true of the detector count rates if background is also negligible.

Furthermore, recall that “correlated” implies “chain-related.” Hence the signatures measure relation between singlets, doublets, and triplets of detection events and source fissions.

3.3 Implementation of Estimation Procedures in Monte Carlo Calculational Models

The theoretical models developed in the previous chapter are “nice” in that they readily demonstrate the processes contributing to the various signatures and illustrate the signatures’ sensitivity to reactivity. However, they were developed from a point model of a fissile system, which is not generally applicable. The point model simply does not account for numerous factors affecting a real measurement. These include, but are not limited to:

- (1) the spectrum of source neutrons inducing fission – this affects the distribution of neutrons from induced fission;
- (2) the spectrum of system neutrons inducing fission – this also affects the distribution of neutrons from fission;
- (3) the importance of source neutrons in initiating fission-chains relative to that of system neutrons in sustaining fission-chains – this weights effects (1) and (2) to determine the distribution of neutrons from fission;
- (4) the spatial distribution of source and system neutrons;
- (5) the spectrum of neutrons transmitted through the system and the probability

that these will interact with the detectors – this is a “field-of-view” effect that depends on source and detector positions relative to the system;

- (6) the leakage spectrum of system neutrons and the probability that these will interact with the detectors – this is also a field-of-view effect;
- (7) the relative importance of effects (5) and (6);
- (8) reflection of neutrons back into the system or into the detectors;
- (9) interaction with other fissile systems;
- (10) the behavior of extremely subcritical systems, which exhibit long-lived spatial modes above the fundamental.

A change in any of these factors can have a significant effect on the outcome of a measurement. So planning or analyzing a real measurement using the point models is at best a dubious endeavor.

However, a properly constructed Monte Carlo calculational model will account for all of these factors. Furthermore, because the distribution of the stochastic neutron field is implicit in Monte Carlo calculations, it is possible to calculate the response of detectors observing a fissile system driven by a ^{252}Cf source. Consequently, by recording the calculated response over a large number of neutron histories, the process statistics specific to the system modeled may be estimated. MCNP-DSP was developed to perform this very task [Val1,2,3]. Specifically, it was designed to plan and analyze source-driven measurements.

An MCNP-DSP calculation follows the natural chain of events on a per-segment basis. The number of source fissions that occur during a particular segment are randomly selected from a Poisson distribution given the expected number of source fissions per

segment. Subsequently the time of an individual source fission is randomly selected from a uniform distribution over the segment duration. The individual source fissions are simulated at a point in space specified by the model geometry. The number of neutrons from an individual source fission is randomly selected from the Spencer multiplicity distribution for ^{252}Cf [Spe,Val1], and the energy of each source neutron is randomly selected from a modified Maxwellian spectrum for ^{252}Cf [Man, Val1]. Then the direction of each source neutron relative to an isotropically distributed light fission fragment is selected from a distribution measured by Budtz-Jorgensen and Knitter, if the user chooses [Bud, Val1]. Otherwise each neutron's direction is selected from an isotropic distribution. Thus each source neutron is "born" into position, speed, direction, and time phase space.

If a particular source neutron does not immediately escape by exiting the bounds of the modeled geometry, it is tracked through its history of interactions until it escapes or is lost in an absorption interaction. From its starting point in space, the site of a source neutron's first interaction is randomly selected from the probability of an interaction for that particular neutron's energy along its direction of travel. The time of the interaction is simply the distance traveled divided by the neutron's speed. The type of interaction, absorption or scatter, is selected congruent to the cross-sections for these reactions at the neutron's incident energy. If the neutron scatters, the neutron's emergent direction and energy are selected from the differential scatter cross-section, and it is transported to its next interaction site. This continues until the neutron escapes or is absorbed.

If the neutron is absorbed, the type of absorption, capture or fission, is selected congruent to the cross-section for these reactions at the neutron's incident energy. If a fission

that spawns one or more fission neutrons occurs, or if a capture interaction that results in one or more ejected neutrons occurs (e.g. an (n, xn) reaction), the incident source neutron “dies” and the newly born progeny are tracked. Each descendant neutron is born into phase space with direction and energy selected from the respective distributions for neutrons emergent from that particular reaction that spawned the neutron. In particular, if a ^{235}U , ^{238}U , or ^{239}Pu fission occurs, the number of neutrons emergent from this fission is selected from a Zucker- Holden multiplicity distribution particular to that isotope [Zuc, Val1]⁴. These distributions account for the energy of the neutron inducing fission. The direction of an individual fission neutron is selected from the Budtz-Jorgensen and Knitter distribution relative to the direction of an isotropically distributed light fission fragment if the user makes this specification; otherwise it is selected from an isotropic distribution [Bud, Val1].

All descendant neutrons are tracked through their history of interactions until they escape or are lost in an absorption. This process is repeated for each neutron born of a single source fission until it and all its progeny have been tracked to extinction through escape or absorption. Then the process is repeated for each source fission in that particular segment until all source neutrons and progeny have been tracked to extinction. This is repeated for a specified, presumably large, number of segments.

If a neutron interaction occurs within a detector modeled in the geometry, then a count is recorded at the time if the interaction is appropriate to that type of detector. For example, a neutron capture event in a capture detector results in a count. If the detector operates via neutron scatter, the code even registers multiple interactions of a single neutron

⁴ This is a feature particular to MCNP-DSP.

as a single count if these interactions occur within a user-specified pulse generation time and if the total energy deposition is sufficient to cross the user-specified threshold. The counts are “binned” into counting intervals of a user-specified width. Consequently, for each segment a sequence of counting events as described by (3.52) is recorded for each channel [Val1]. At this point, the estimation procedures described in §3.1 and §3.2 come into play. These were appended to MCNP-DSP as a static library of C-functions that are interfaced to the main code through a single function call. At the end of each segment, the interface passes the detection sequence for each simulated channel to the estimation procedures.

The procedures accumulate the number of events in each channel including the source channel. In addition, the second-, third-, and fourth-order univariate signatures are collected for each channel. The source-detector cross-correlation is accumulated for each detector, and the between-detectors cross-correlation is accumulated for each pair of detectors. For a simulation of q channels including the source, the number of second-order multivariate signatures is $qC2$, where

$$qC_r = \binom{q}{r} \quad (3.58)$$

is the number of combinations of r selected from a set of q . Similarly, the source-detector bicorrelation is accumulated for each pair of detectors, and the between-detectors bicorrelation is accumulated for each triplet of detectors, resulting in $qC3$ multivariate third-order signatures. Finally, the source-detector tricorrelation is accumulated for each triplet of detectors such that $qC3$ additional fourth-order signatures are accumulated. If four

detectors are modeled⁵, the procedures will also accumulate the between-detectors tricorrelation for each quadruplet of detectors resulting in one more fourth-order signature. Because the memory requirements of such a calculation could potentially exceed the available memory, the user may specify the maximum order (≤ 4) to be calculated. The code will calculate all signatures up to and including this order.

Each signature is collected using the spectral, bispectral, and trispectral accumulators (3.20), (3.34), and (3.50). At the end of the calculation, each of these accumulators is inverse transformed to obtain the corresponding correlation, bicorrelation, or tricorrelation accumulator. These accumulators are used to obtain unbiased estimates of the correlation, bicorrelation, and tricorrelation via (3.7), (3.29), and (3.45), respectively. Then each unbiased signature is written to permanent storage as a binary file. The corresponding covariances, bicovariances, and tricovariances, as well as the cross-spectra, bispectra, and trispectra, are calculated by a post-processing code that writes these signatures to text files suitable for plotting or analysis. Each signature is normalized to the counting interval and mean source fission rate as described in Table 3.1. Consequently, the calculated signatures are interpreted as summarized by Table 3.2.

3.4 Monte Carlo Calculation of High Order Signatures from an Enriched Uranium-Metal Sphere

In order to illustrate the properties of high order signatures under conditions more general than those encompassed by the theoretical point model, an MCNP-DSP calculational

⁵ Currently MCNP-DSP will simulate a maximum of five channels including the source.

model of a 93.15%-enriched uranium-metal sphere was constructed. Some basic properties of the sphere are listed in Table 3.3. The enrichment was chosen because numerous measurements of this material have been made. These measurements include, but are not limited to, annular castings and isolated and coupled metal cylinders at subcritical and delayed critical. A sphere was chosen for its simplicity.

The diameter was chosen such that the system was strongly multiplicative but still far below delayed critical, as shown in the table [Pax]. The effective multiplication factor of the unreflected sphere was estimated using an MCNP-DSP analog k_{eff} -calculation [Val1]. The prompt reactivity,

$$\rho = \frac{(1 - \beta)k_{eff} - 1}{(1 - \beta)k_{eff}}, \quad (3.59)$$

Table 3.3 Some basic properties of the 93.15%-enriched uranium-metal sphere

diameter	150 mm
density	18.76 kg/l
composition	93.15 wt% ^{235}U 5.64 wt% ^{238}U 0.97 wt% ^{234}U 0.24 wt% ^{236}U
effective multiplication factor ⁽¹⁾	0.8736 ± 0.0003
prompt reactivity ⁽²⁾ (\$ subcritical)	23.08 ± 0.05
prompt neutron fission-chain decay constant (μsec^{-1})	26.47 ± 0.01
prompt neutron generation time (nsec)	5.75 ± 0.01

⁽¹⁾ Uncertainties represent one standard deviation.

⁽²⁾ The value 0.0066 for the effective delayed neutron fraction was assumed.

was estimated assuming that the value $\beta = 0.0066$ for the effective delayed neutron fraction was appropriate [Kee]. The prompt reactivity is relevant to the MCNP-DSP calculation because currently the code does not track delayed neutrons during simulations of source-driven measurements. In addition, it's probably prudent to note that this calculation and all subsequent calculations used the ENDF-V cross-section set.

The prompt neutron fission-chain decay constant, α , was estimated from an MCNP-DSP simulation of a pulsed neutron measurement. A ^{252}Cf point source was modeled at the center of an unreflected sphere, and a distribution of the time between ^{252}Cf -fission events and subsequent induced fission events was estimated. This was done by treating the sphere as a fission detector. This distribution is plotted in Fig. 3.5. Observe that the sphere essentially behaves as a single-mode system, although what appears to be a short-lived higher mode (or modes) is evident between 0 and 10 nsec. An exponential fit to the distribution between 50 and 250 nsec yielded an estimate of α . The uncertainty in this estimate was obtained from the standard deviation of the fitted "slope." Subsequently, the prompt neutron generation time Λ was estimated via $\Lambda = -\rho / \alpha$; the uncertainty in Λ was estimated using propagation of error.

To simulate a measurement of the process statistics, the sphere was surrounded by a 75-mm-thick shell of three "black" detectors as shown in Fig. 3.6. These detectors are a special feature of MCNP-DSP: any interaction in the detection material results in a count and the neutron involved in the interaction is immediately terminated. The detector material and density was chosen such that each detector is $> 10^{30}$ mean-free-paths thick. A ^{252}Cf point source was modeled at the center of the sphere as before. The detectors were simulated as

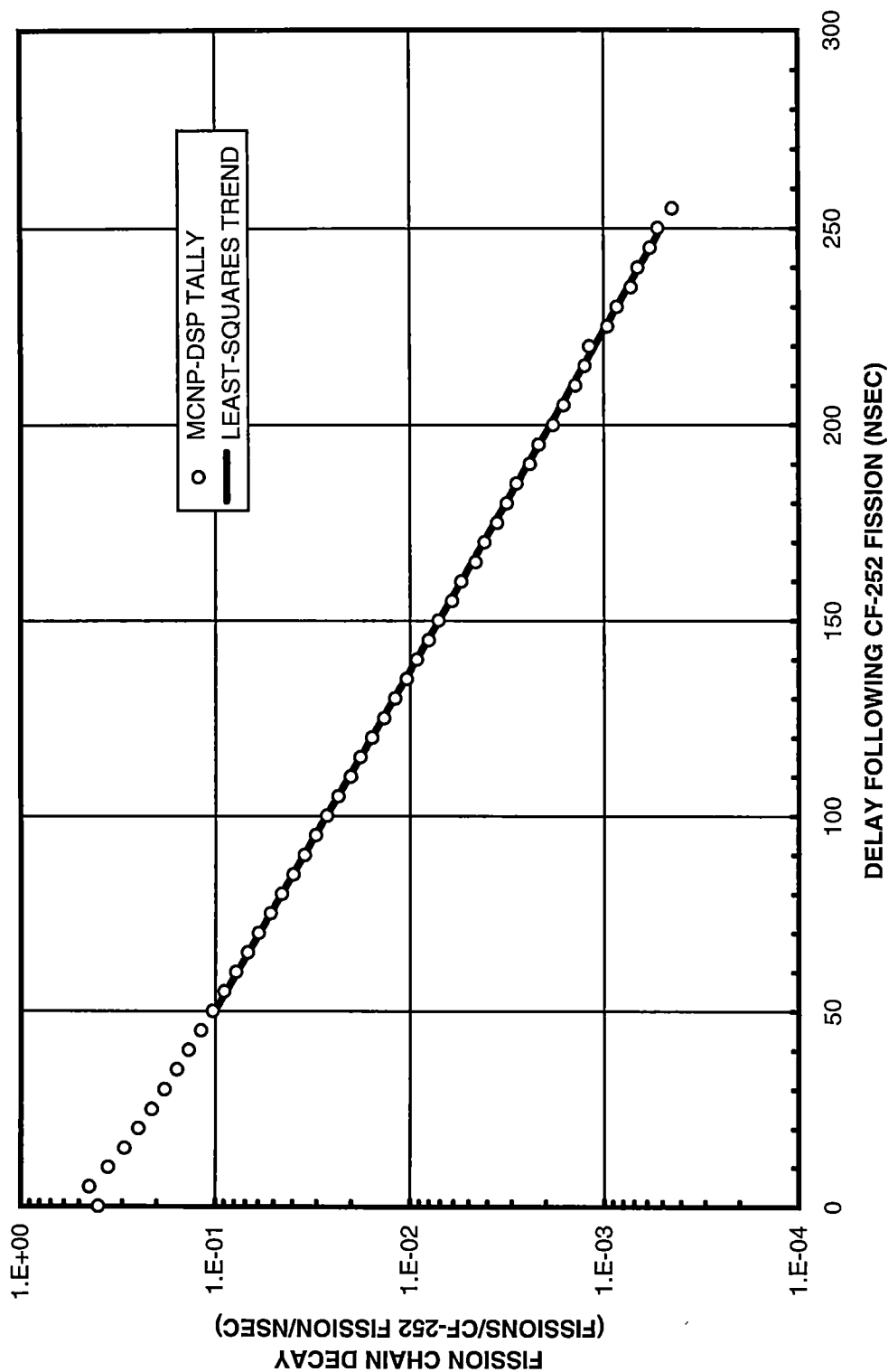


Figure 3.5 Distribution of time between ^{252}Cf fission and subsequent induced fission events estimated by MCNP-DSP for an unreflected 150-mm-diameter 93.15%-enriched uranium-metal sphere

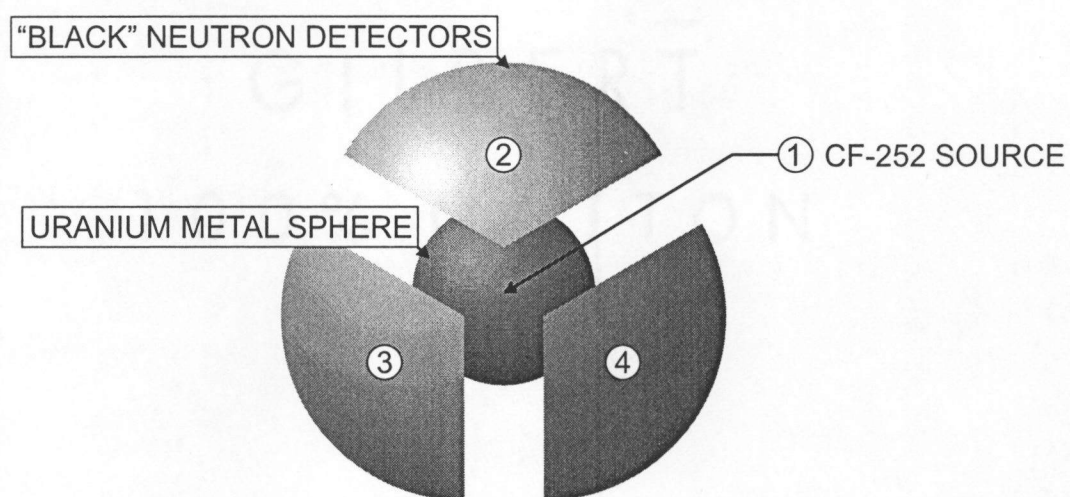


Figure 3.6 Illustration of MCNP-DSP calculational model used to estimate first- through fourth-order statistical signatures

black for two reasons – (1) simulations of source-driven measurements can take an exceptionally long time if the detection efficiency is low, and (2) if the detectors did not capture every neutron that entered them, they would act as reflectors. Reflected systems often exhibit some long-lived modes above the fundamental. Although this last effect might make the following signatures a bit more interesting, the purpose here is to illustrate a simple example.

Detection event sequences were sampled for segments containing 128 intervals each 20 nsec wide, i.e., the sample rate was 50 MHz. The signatures were calculated from a total of 500,000 segments. Although an equivalent measurement would require only 1.28 seconds of real-time, the calculation took about two-and-a-half days to complete on a Sun SPARCStation Ultra. Californium-252 fissions were simulated at an average of one per segment; this is the fission rate of a 0.64 μg ^{252}Cf source, which is a mass similar to that encountered in many measurements⁶.

Its interesting to note that on average 16 induced fissions resulted from each ^{252}Cf -fission, such that

$$\frac{\text{induced fissions}}{\text{source fissions}} \neq \frac{\bar{\nu}_0}{\bar{\nu}} \frac{1}{|\rho|} \quad (3.60)$$

using the average induced fission neutron multiplicity $\bar{\nu} = 2.60$ and source fission neutron

⁶ Due to memory limitations, fourth-order signatures were calculated using only 32 intervals per segment. The sample rate and the number of source fissions per segment were the same as used to calculate the third- and lower-order signatures.

multiplicity $\bar{\nu}_0 = 3.77$ tallied by MCNP-DSP⁷. This is due to the inequity between the importance of ^{252}Cf neutrons in initiating fission-chains and the importance of prompt neutrons in sustaining fission-chains. Put more simply, relative to the number of ^{252}Cf neutrons that succeed in inducing fission, a disproportionately larger number of prompt neutrons survive to induce subsequent fissions. This implies that the majority of the ^{252}Cf neutrons induce fission deep within the sphere, whereas the subsequent fission-chains proliferate further towards the sphere's perimeter. This deduction is evidenced by the small fraction of detection events resulting from capture of a source neutron, $\sim 7\%$. Hence the uranium sphere does not behave as a point.

The first statistic is the average count rate – 9.25 counts per ^{252}Cf -fission. Due to the symmetry of the calculational model, this same count rate was observed in all three detectors to fewer than 0.01 counts per ^{252}Cf -fission absolute deviation. For this reason, all of the following higher order signatures appear the same regardless of which detectors were involved in their estimation.

The covariance between the source (channel 1) and detector 2 is plotted in Fig. 3.7 as a function of the time following ^{252}Cf -fission, i.e, the delay between an event in channel 1 and an event in channel 2. Of course the signature vanishes when the delay is negative –

⁷ MCNP-DSP tallies these quantities by recording the number of neutrons sampled from each source and induced fission [Val4].

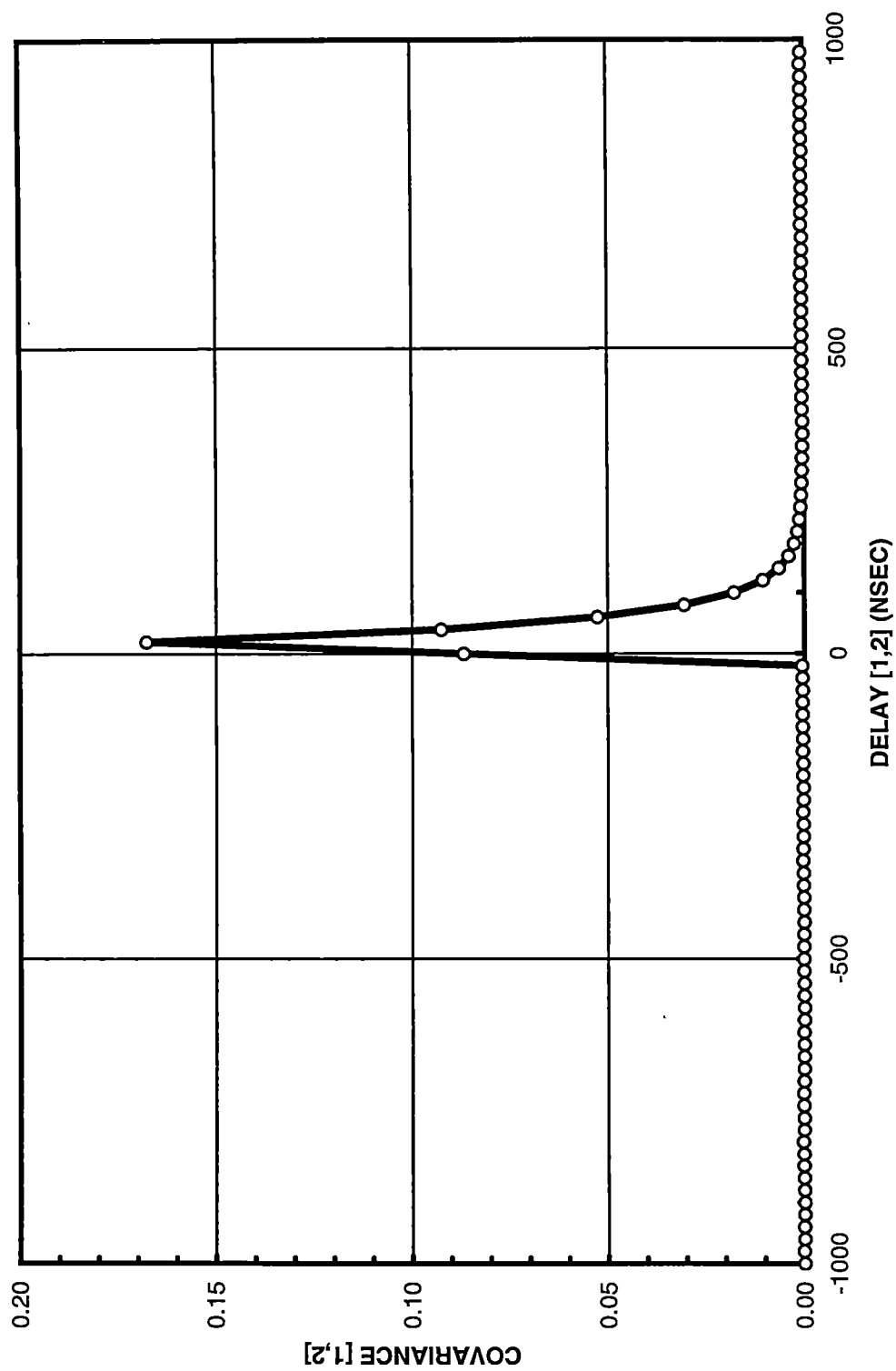


Figure 3.7 Source-detector covariance [1,2] (in counts/²⁵²Cf-fission/nsec) calculated for the 93.15%-enriched uranium-metal sphere

a detection event cannot be correlated with a subsequent source fission⁸. More noteworthy is the fact that the signature decays according to α ; an exponential fit to the signature confirms this. Consequently, the magnitude of the corresponding source-detector spectral signature, cross-spectrum [1,2], rolls off past the break frequency⁹ $\alpha / 2\pi$, as shown in Fig 3.8. Next, the phase of this signature,

$$\angle \hat{S}_{12} = \tan^{-1} \frac{\text{Im } \hat{S}_{12}}{\text{Re } \hat{S}_{12}}, \quad (3.61)$$

is plotted in Fig. 3.9. Note that since every source-correlated detection event must succeed the initial source fission, the phase shift is always negative. The subsequent figure, 3.10, plots the real and imaginary parts of the calculated source-detector cross-spectrum, which are respectively symmetric and antisymmetric since

$$S_{xy}(-f) = S_{yx}^*(f). \quad (3.62)$$

These of course are all well-known properties of the second-order source-detector signature.

The covariance between detector 2 and detector 3 is plotted in Fig. 3.11 as a function of the delay between a detection event in 2 and a detection event in 3. Note that the signature decays at an exponential rate according to α . Also recall that negative delay simply implies that a count occurred in detector 3 first, so the between-detectors covariance is symmetric about zero delay since $C_{xy}(-\tau) = C_{yx}(\tau)$. Due to this symmetry, the corresponding spectral

⁸ Note that due to the quantization of delays, the covariance estimate represents a histogram distribution of counts following ^{252}Cf -fission; hence the covariance estimate at delay zero represents the integral of the covariance from 0 to 20 nsec.

⁹ The decay constant α has units of inverse time. The corresponding break frequency in units of Hz is $\alpha / 2\pi$.

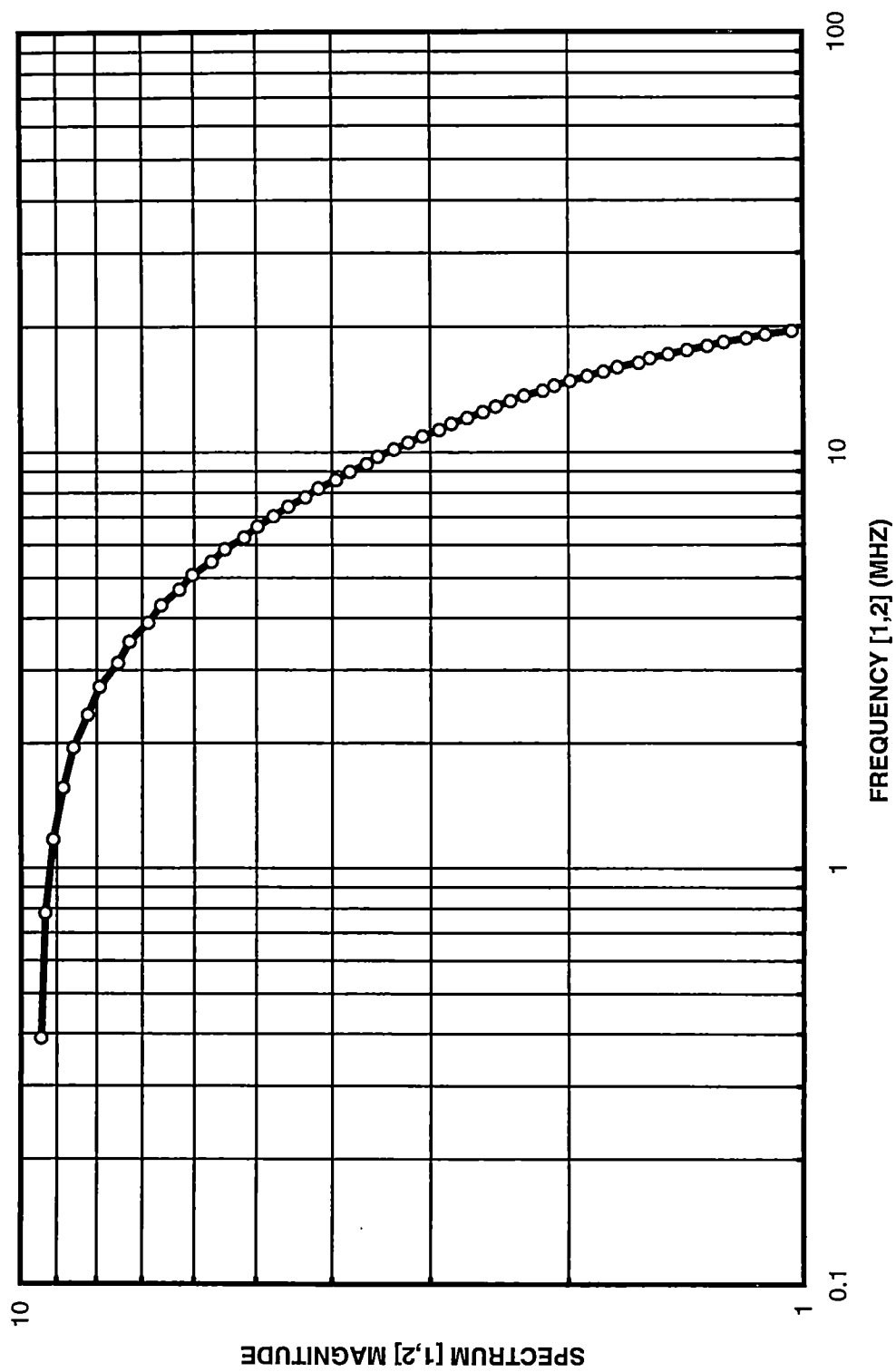


Figure 3.8 Magnitude of source-detector spectrum [1,2] (in counts/ ^{252}Cf -fission) calculated for the 93.15%-enriched uranium-metal sphere

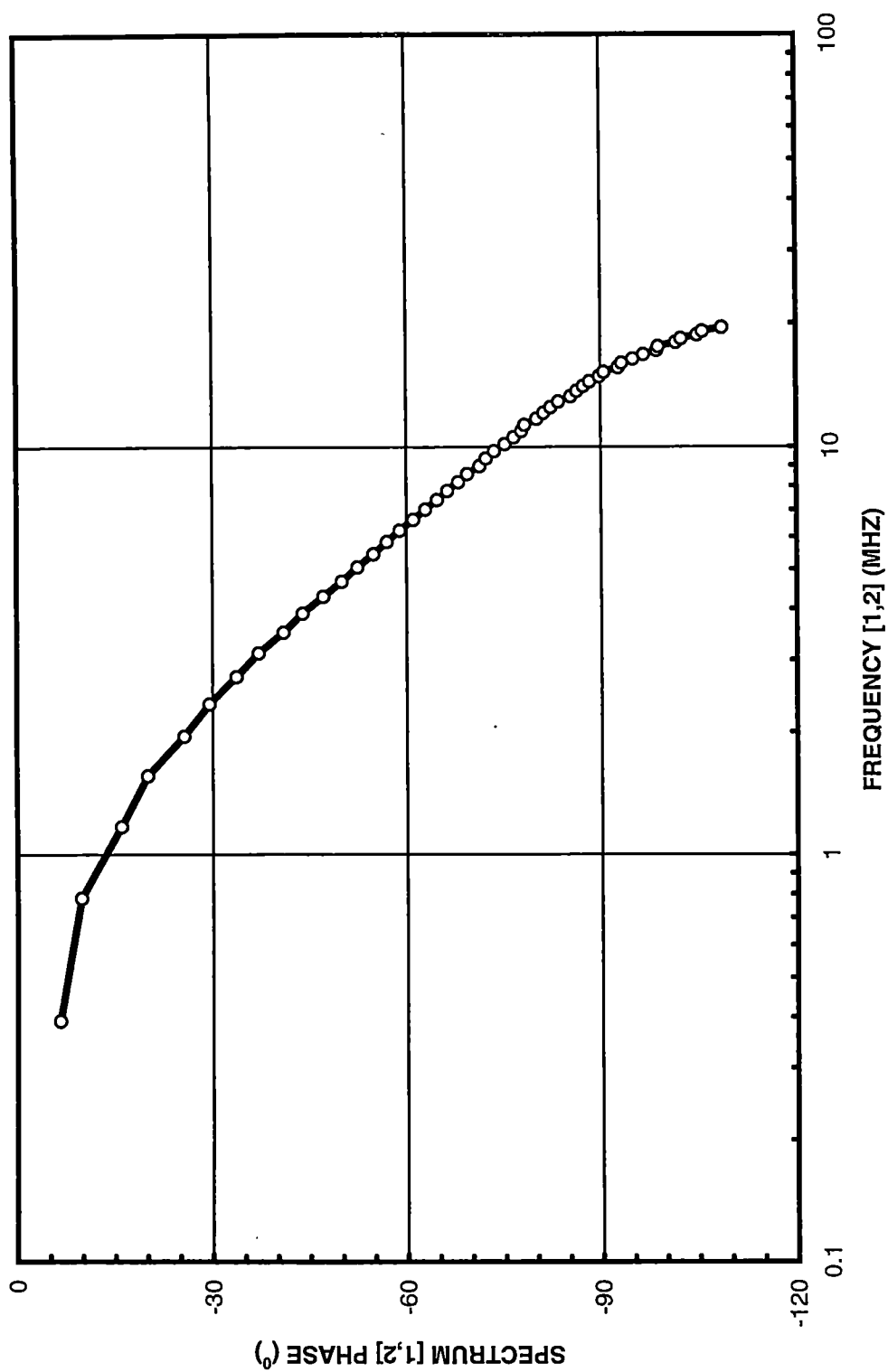


Figure 3.9 Phase of source-detector spectrum [1,2] calculated for the 93.15%-enriched uranium-metal sphere

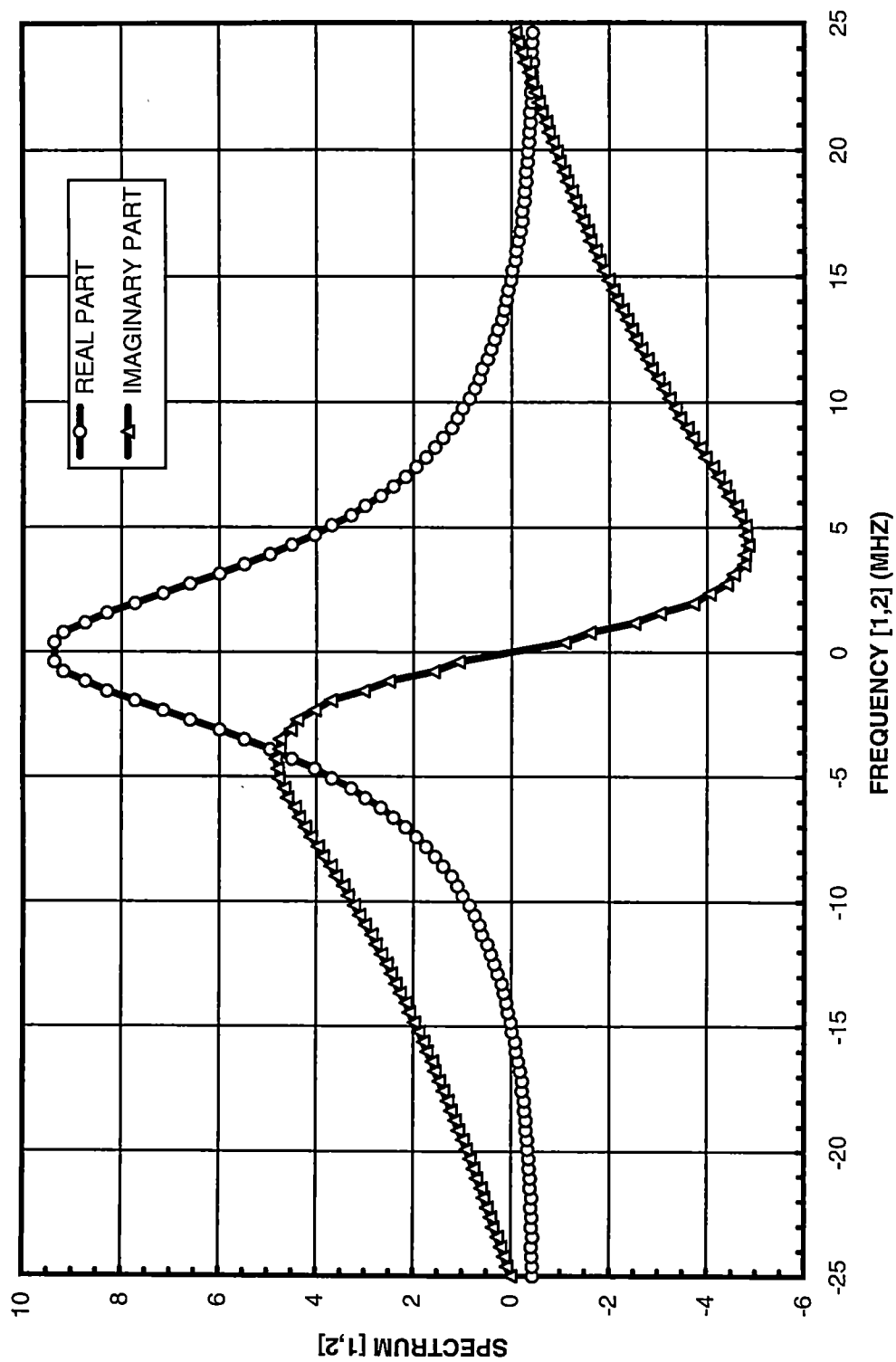


Figure 3.10 Real and imaginary parts of source-detector spectrum [1,2] (in counts/ ^{232}Cf -fission) calculated for the 93.15%-enriched uranium-metal sphere

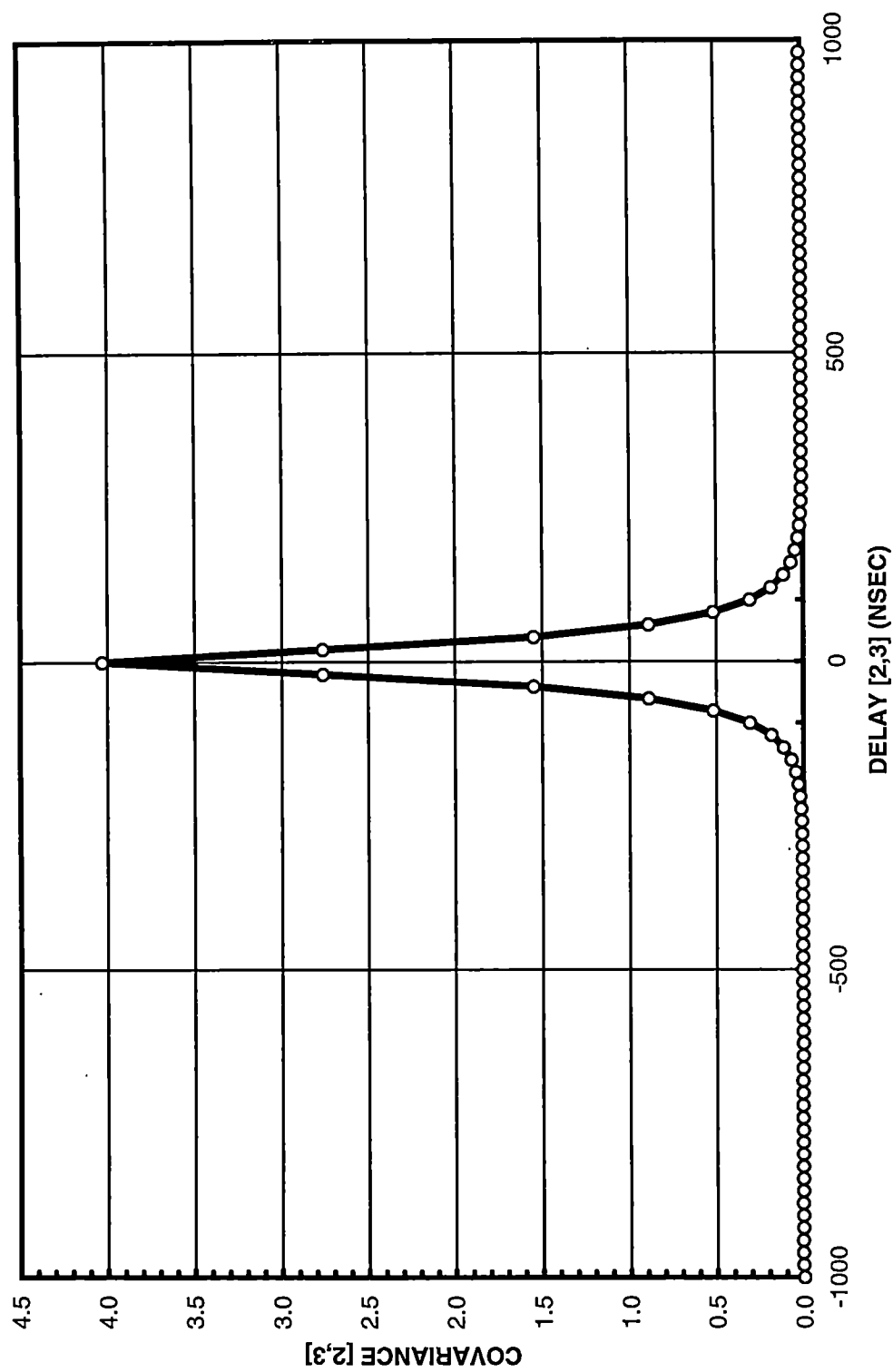


Figure 3.11 Between-detectors covariance [2,3] (in correlated pairs/ ^{252}Cf -fission/nsec) calculated for the 93.15%-enriched uranium-metal sphere

signature, spectrum [2,3], is real-valued. Also observe that as Fig 3.12 depicts, past the frequency $\alpha / 2\pi$ the between-detectors spectrum rolls off more rapidly than does the source-detector spectrum. Again, these properties have been frequently encountered in the past.

The bicovariance between the source, detector 2, and detector 3, i.e., bicovariance [1,2,3] is plotted in Fig. 3.13 versus the delay between source fission and detector 2 count, delay [1,2], and versus the delay between source fission and detector 3 count, delay [1,3]. Of course the signature vanishes when either of these delays are less than zero, and it is symmetric about the diagonal where the two delays are equal because

$$C_{xyz}(\tau_2, \tau_1) = C_{xzy}(\tau_1, \tau_2) . \quad (3.63)$$

Also observe that when one delay is zero, along the other delay the signature decays exponentially according to α . Furthermore, for one delay fixed, as the other delay moves further from the diagonal $\tau_{[1,2]} = \tau_{[1,3]}$ the signature decays at an exponential rate of α . This “ridge” along the diagonal simply reflects that, given the two counts did not occur immediately after the source fission, they are most likely to appear immediately one after the other.

The magnitude of the corresponding spectral signature, bispectrum [1,2,3], is plotted as a surface over the two frequencies [1,2] and [1,3] in Fig. 3.14. Observe that the transition between the “flat” low-frequency behavior and the roll-off at high frequency occurs at $\alpha / 2\pi$. Also note that the roll-off is more rapid along the diagonal $f_{[1,2]} = f_{[1,3]}$. The phase of the source-detector bispectrum is plotted in the subsequent figure, 3.15. Observe that when one frequency is zero, along the other frequency the phase behaves exactly as the phase of the

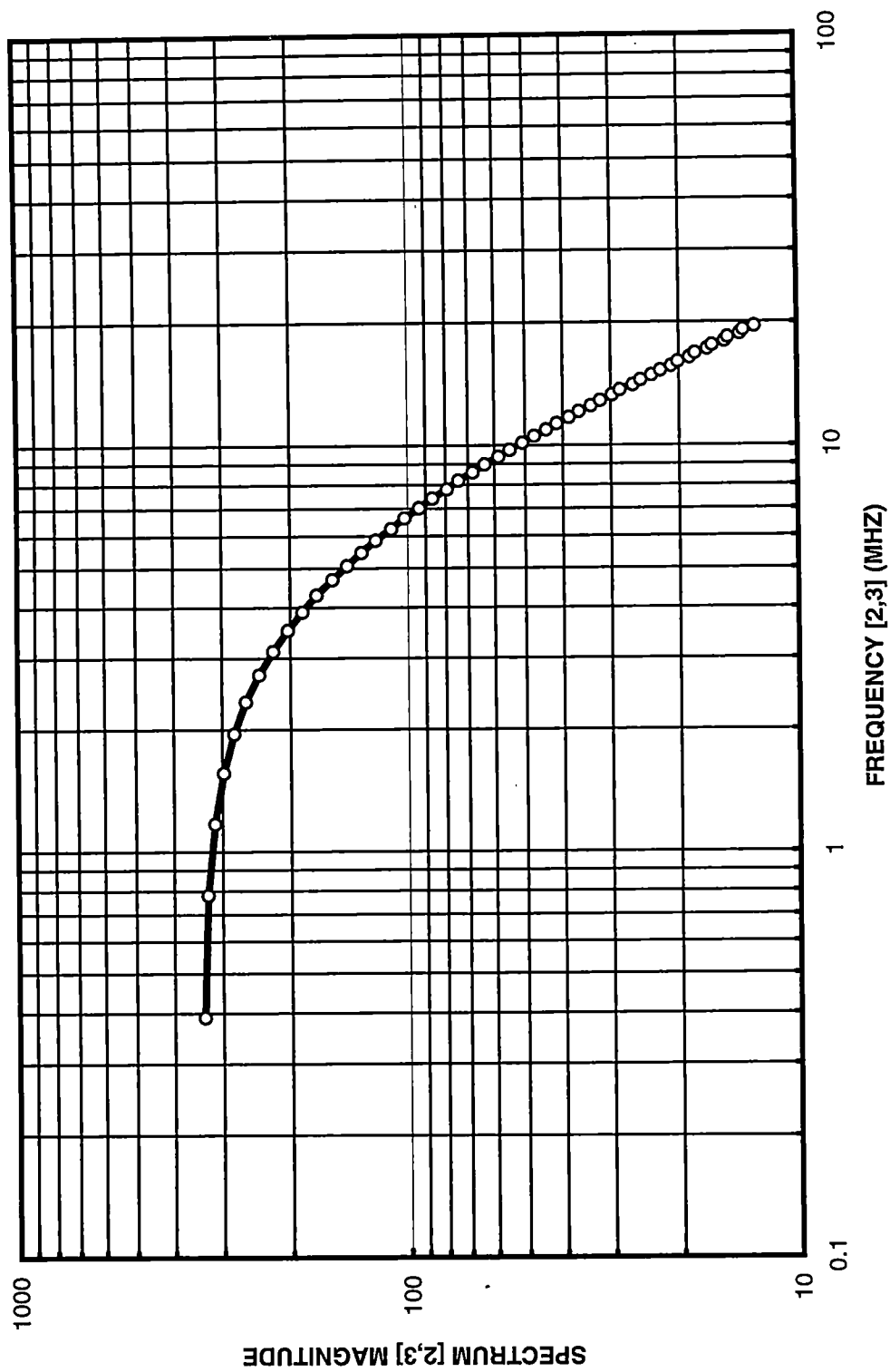


Figure 3.12 Magnitude of between-detectors spectrum [2,3] (in correlated pairs/ ^{252}Cf -fission) calculated for the 93.15%-enriched uranium-metal sphere

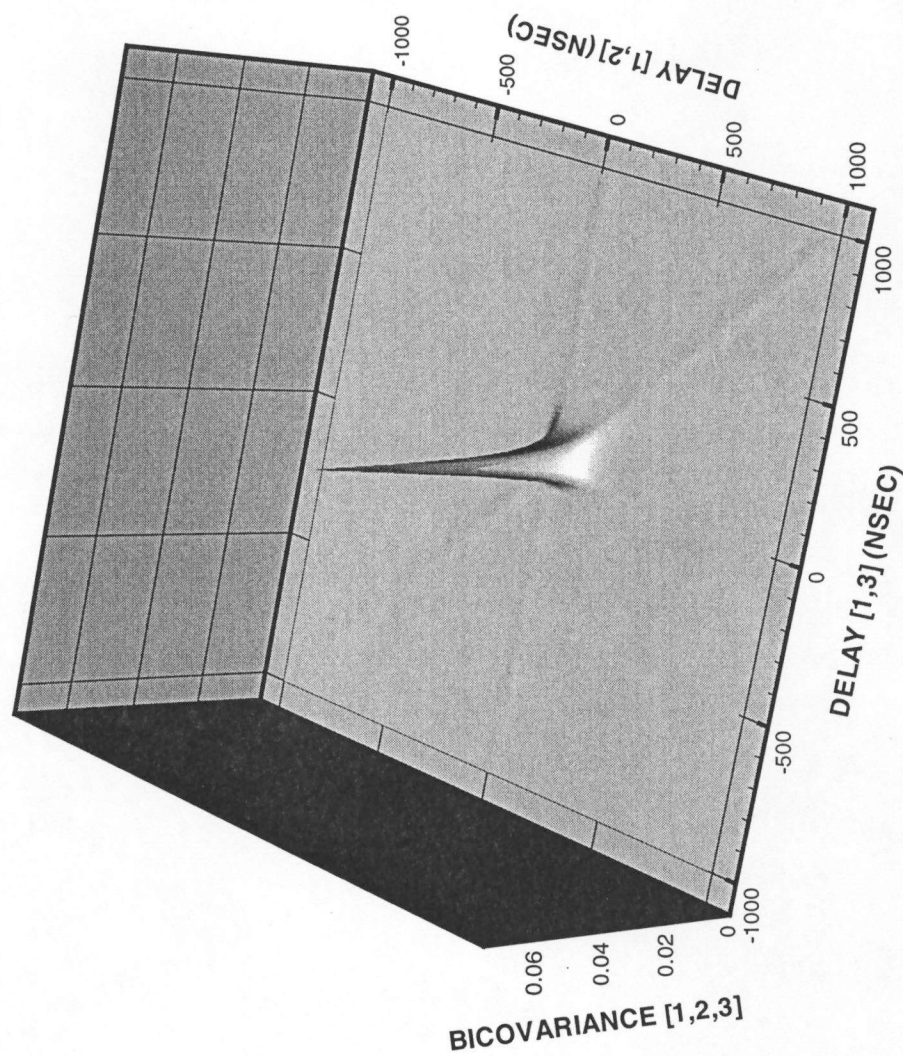


Figure 3.13 Source-detector bicovariance [1,2,3] (in correlated pairs/ ^{252}Cf -fission/nsec 2) calculated for the 93.15%-enriched uranium-metal sphere

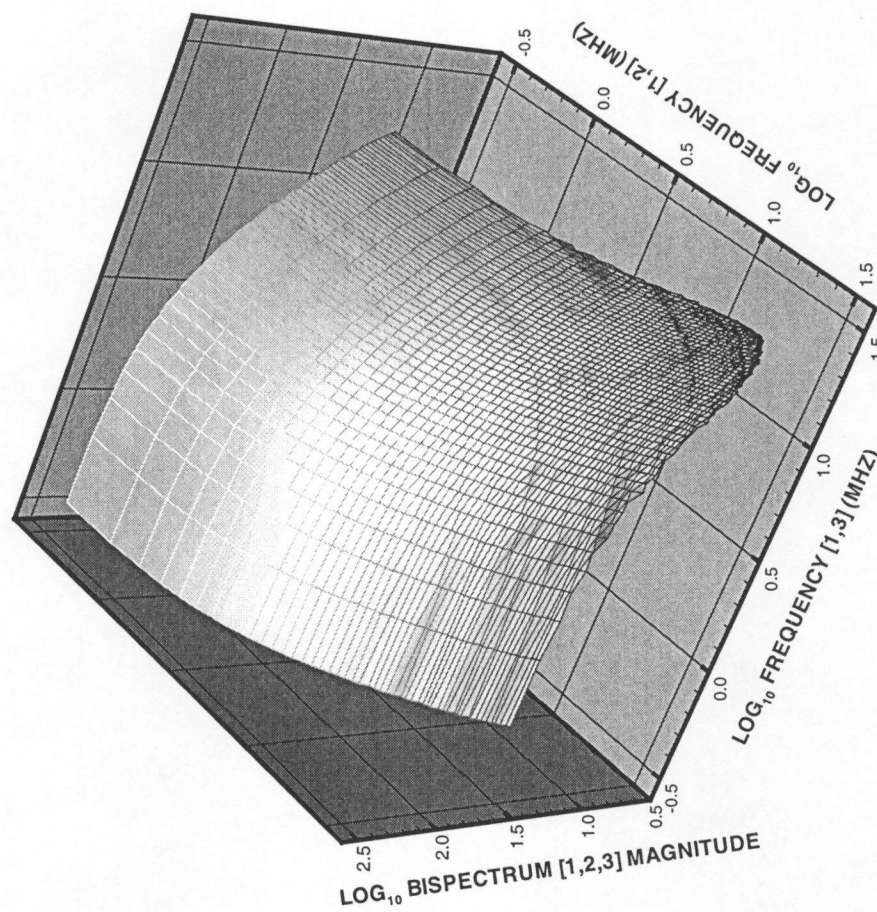


Figure 3.14 Magnitude of source-detector bispectrum [1,2,3] (in correlated pairs/ ^{252}Cf -fission) calculated for the 93.15%-enriched uranium-metal sphere

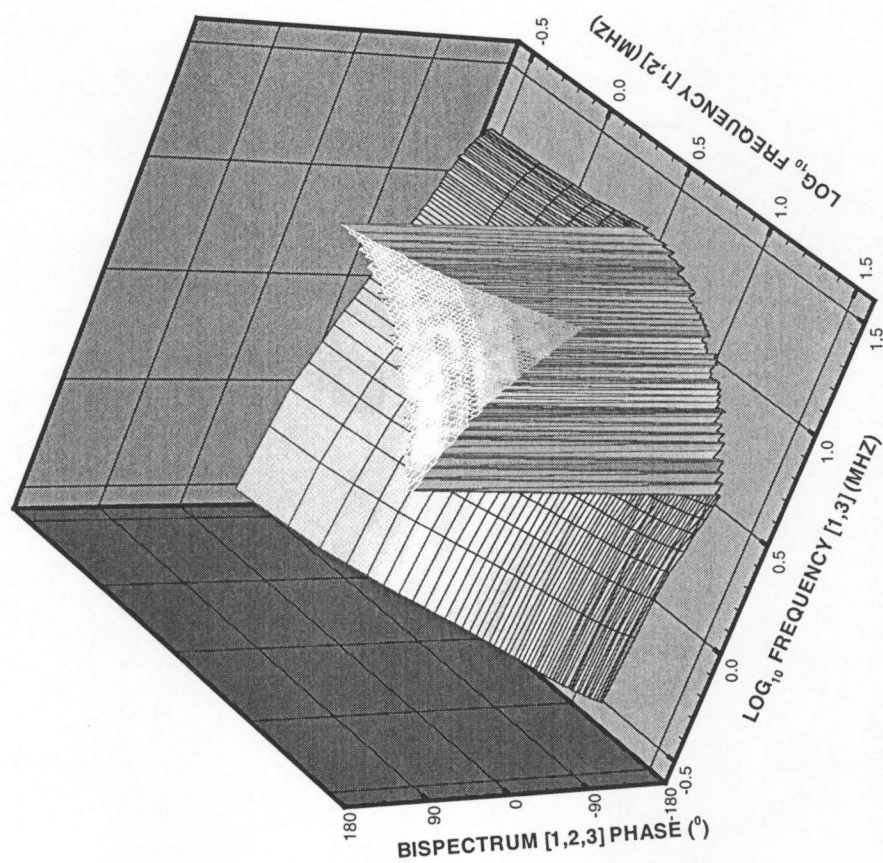


Figure 3.15 Phase of source-detector bispectrum [1,2,3] calculated for the 93.15%-enriched uranium-metal sphere

source-detector spectrum. Along the diagonal, the phase shift decreases more rapidly and passes through -180° . The apparent discontinuity at this transition is due to the cyclic nature of the arctangent. In fact the phase shift is always negative, because pairs of counts bicornelated with a source fission must always occur subsequent to the source fission.

The real and imaginary parts of bispectrum [1,2,3] possess a number of symmetry/antisymmetry properties according to

$$\begin{aligned} S_{xyz}(-f_1, f_2) &= S_{xyz}^*(f_1, -f_2) , \\ S_{xyz}(-f_1, -f_2) &= S_{xyz}^*(f_1, f_2) , \end{aligned} \quad (3.64)$$

and

$$\begin{aligned} S_{xyz}(f_2, f_1) &= S_{xzy}(f_1, f_2) , \\ S_{xyz}(-f_1 - f_2, f_2) &= S_{yxz}(f_1, f_2) , \\ S_{xyz}(-f_1 - f_2, f_1) &= S_{yzx}(f_1, f_2) , \\ S_{xyz}(f_2, f_1 - f_2) &= S_{zxy}(f_1, f_2) , \\ S_{xyz}(f_1, f_2 - f_1) &= S_{zyx}(f_1, f_2) . \end{aligned} \quad (3.65)$$

These impose the lines of symmetry on the plane depicted in Fig 3.16. Observe that the resulting symmetries are evident in the plots of the real and imaginary parts shown in Fig 3.17. First, it's immediately apparent that the real part is symmetric about $f_{[1,2]} = -f_{[1,3]}$ and $f_{[1,2]} = f_{[1,3]}$, while the imaginary part is antisymmetric across $f_{[1,2]} = -f_{[1,3]}$ and symmetric across $f_{[1,2]} = f_{[1,3]}$. Less apparent is the fact that both the real and imaginary part exhibit "skewed" symmetries about $f_{[1,3]} = -f_{[1,2]} - f_{[1,3]}$, $f_{[1,2]} = -f_{[1,2]} - f_{[1,3]}$, $f_{[1,3]} = f_{[1,2]} - f_{[1,3]}$, and $f_{[1,2]} = f_{[1,3]} - f_{[1,2]}$ in accordance with (3.64) and the last four parts of (3.65). This is a

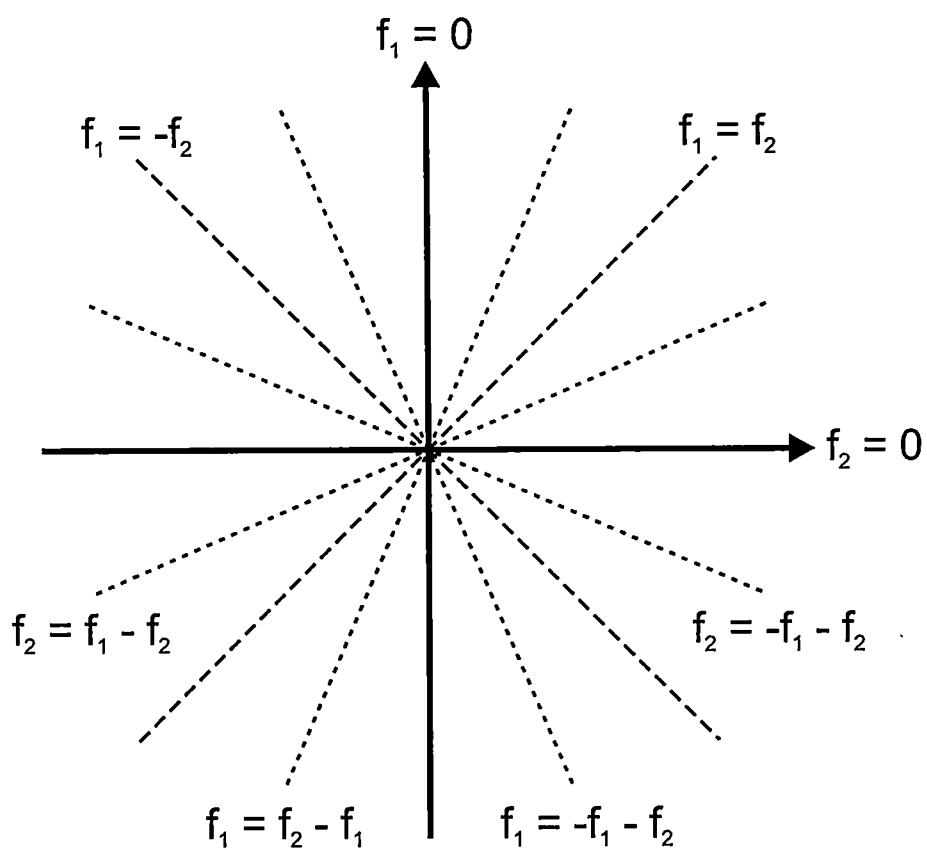


Figure 3.16 Lines of bispectrum symmetry/antisymmetry in (f_1, f_2) plane

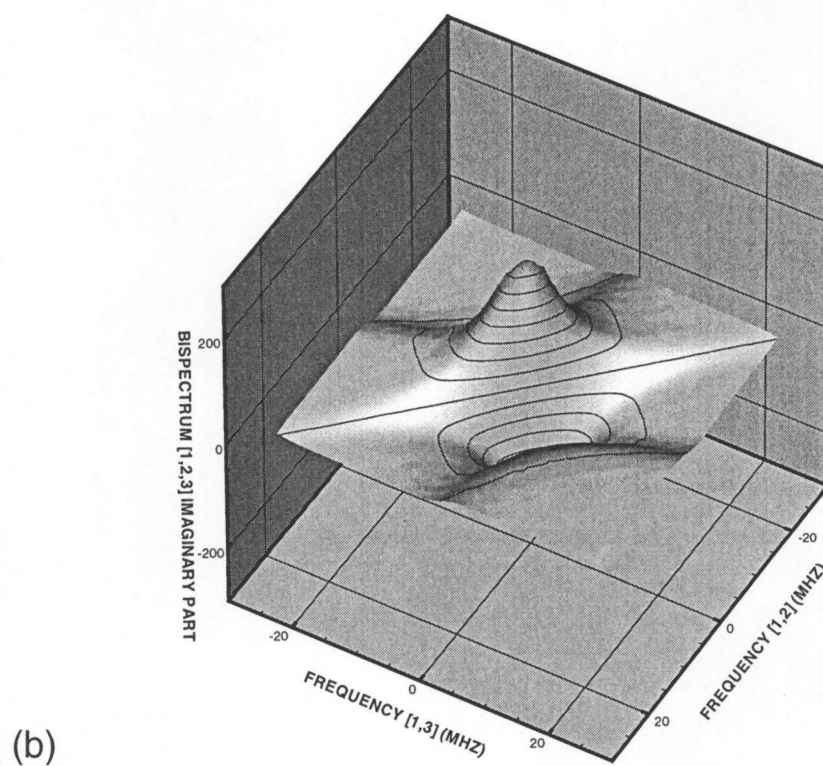
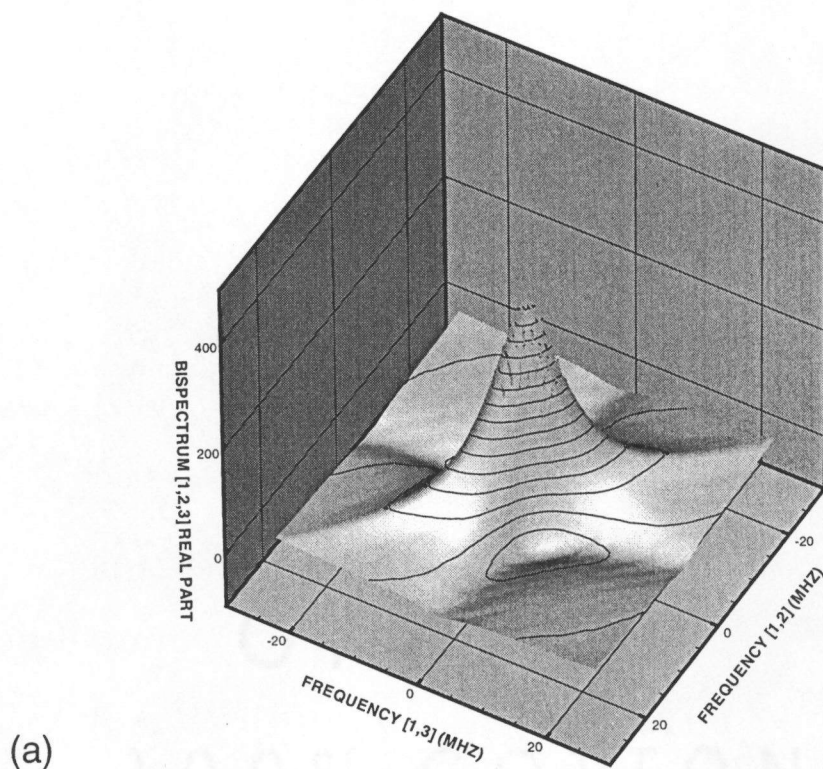


Figure 3.17 Source-detector bispectrum [1,2,3] (a) real and (b) imaginary parts (in correlated pairs/ ^{252}Cf -fission) calculated for the 93.15%-enriched uranium-metal sphere

bit more evident in the contours of the real and imaginary parts plotted in Fig. 3.18. Try folding the contour plots across these lines to see the symmetries. For example, fold the contour of the imaginary part across $f_{[1,3]} = -2f_{[1,2]}$, then across $f_{[1,2]} = 0$ and $f_{[1,3]} = 0$ to see a skewed antisymmetry. These properties are of course entertaining. However, a more practical conclusion from (3.64) is that the bispectrum need only be plotted in the first quadrant. This is a general conclusion that is also true for the between-detectors bispectrum.

Bicovariance [2,3,4], i.e., the between-detectors bicovariance, is plotted in Fig. 3.19 as a surface over the two delays, [2,3] and [2,4]. Due to the symmetry of the calculational model, and

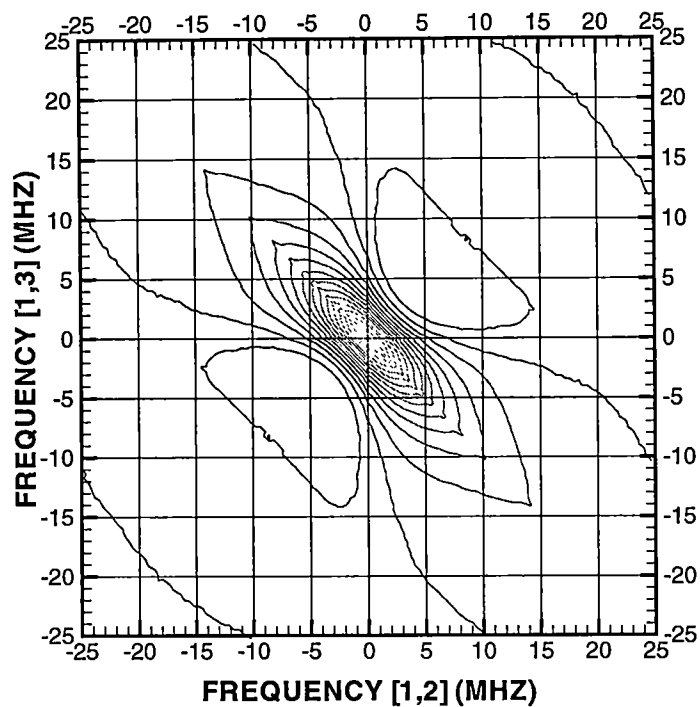
$$\begin{aligned}
 C_{xyz}(\tau_2, \tau_1) &= C_{xzy}(\tau_1, \tau_2) , \\
 C_{xyz}(-\tau_1, \tau_2 - \tau_1) &= C_{yxz}(\tau_1, \tau_2) , \\
 C_{xyz}(\tau_2 - \tau_1, -\tau_1) &= C_{yzx}(\tau_1, \tau_2) , \\
 C_{xyz}(-\tau_2, \tau_1 - \tau_2) &= C_{zxy}(\tau_1, \tau_2) , \\
 C_{xyz}(\tau_1 - \tau_2, -\tau_2) &= C_{zyx}(\tau_1, \tau_2) ,
 \end{aligned} \tag{3.66}$$

such that

$$\begin{aligned}
 C_{xyz}(-\tau_1, \tau_2) &= C_{yxz}(\tau_1, \tau_1 + \tau_2) , \\
 C_{xyz}(\tau_1, -\tau_2) &= C_{zxy}(\tau_2, \tau_1 + \tau_2) , \\
 C_{xyz}(-\tau_1, -\tau_2) &= C_{yxz}(\tau_1, \tau_1 - \tau_2) = C_{zyx}(\tau_2, \tau_2 - \tau_1) ,
 \end{aligned} \tag{3.67}$$

this signature is symmetric about $\tau_{[2,3]} = \tau_{[2,4]}$ and $\tau_{[2,3]} = -\tau_{[2,4]}$; there are also a number of skewed symmetries similar to those found in the bispectrum. These are evident in the contour plot of the between-detectors bicovariance in Fig 3.20. Now observe that, similar to the source-detector bicovariance, the between-detectors bicovariance decays at an exponential

(a)



(b)

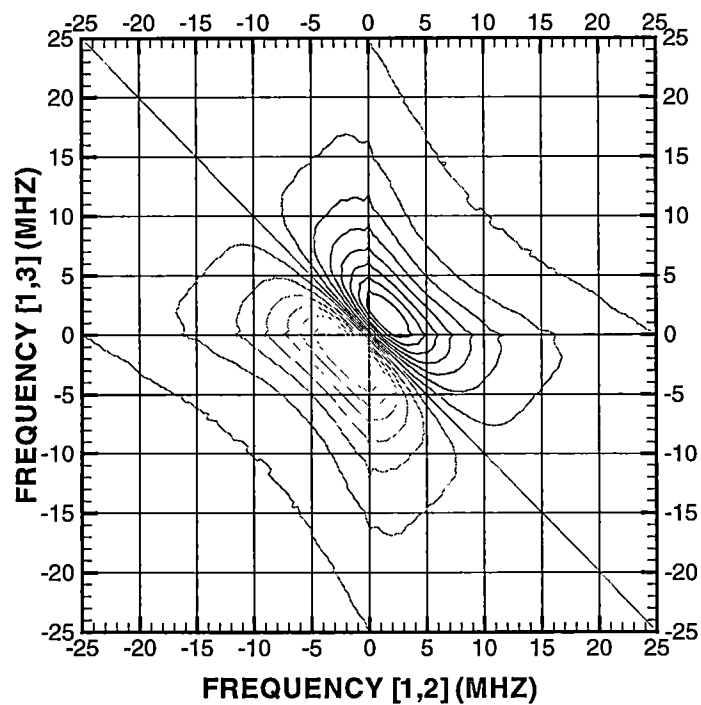


Figure 3.18 Contours of source-detector bispectrum [1,2,3] (a) real and (b) imaginary parts calculated for the 93.15%-enriched uranium-metal sphere

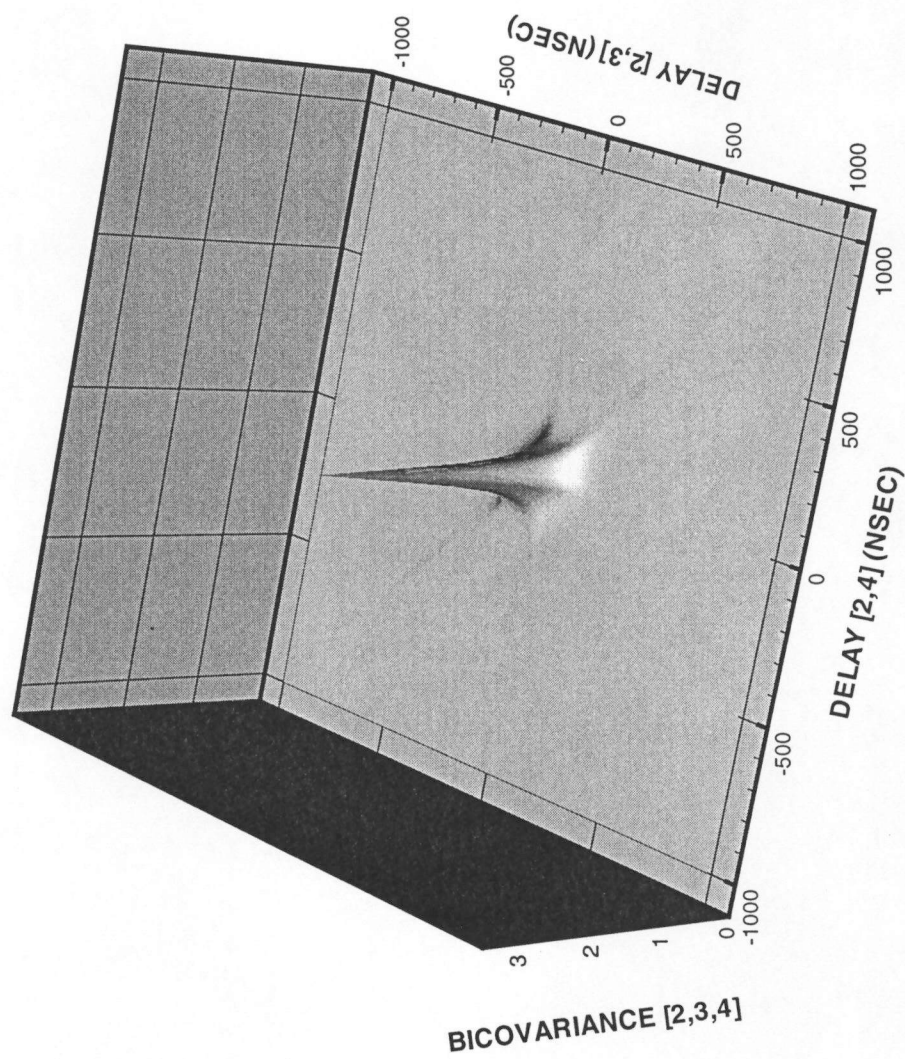


Figure 3.19 Between-detectors bicovariance [2,3,4] (in bico-related triplets/ ^{252}Cf -fission/nsec²) calculated for the 93.15%-enriched uranium-metal sphere

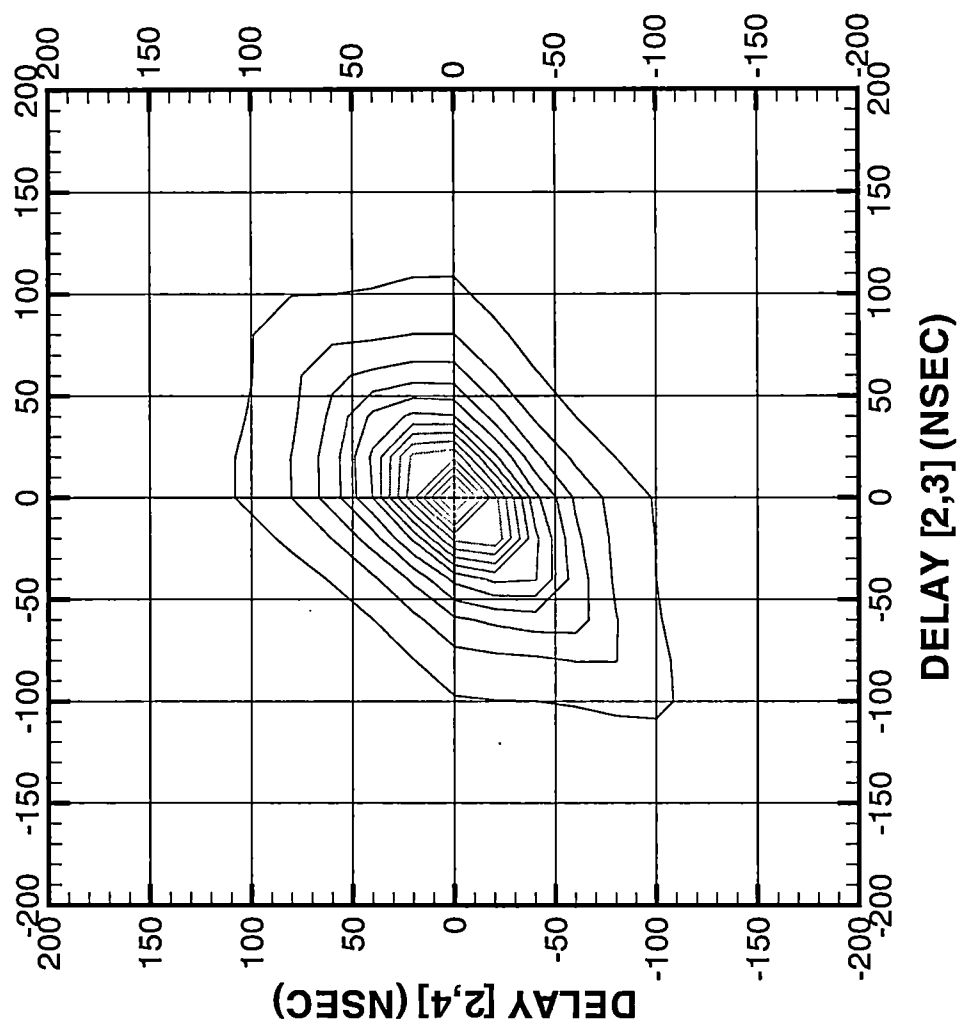


Figure 3.20 Contours of between-detectors bico-variance [2,3,4] calculated for the 93.15%-enriched uranium-metal sphere

rate of α along one delay when the other delay is zero. Further note that, as one delay moves away from the diagonal $\tau_{[2,3]} = \tau_{[2,4]}$ while the other delay is fixed, the signature decays exponentially according to α . Again, this ridge along the diagonal simply reflects that the last two correlated counts are most likely to occur immediately after one another, even if the first bicornelated count occurred some time ago.

The magnitude of the between-detectors bispectrum plotted in Fig 3.21 rolls off more rapidly past the break frequency $\alpha/2\pi$ than does the source-detector bispectrum, as expected. Perhaps unexpected is the fact that the between-detectors bispectrum possesses a non-zero phase, as Fig 3.22 depicts. This is due to the ridge along the diagonal of the between-detectors bicovariance that results from the elevated probability of two counts in rapid sequence following an initial count that occurred in the more remote past. Consequently the phase shift only departs significantly from zero at high frequencies along the diagonal $f_{[2,3]} = f_{[2,4]}$. This is further evidenced by the real and imaginary parts plotted in the subsequent figure, 3.23. Observe that at low frequency, the imaginary part is negligible compared to the real part. The two only become comparable at high frequency. Finally, note that the real and imaginary parts of the between-detectors bispectrum exhibit symmetry/antisymmetry properties similar to those observed in the source-detector bispectrum, as the contours in Fig. 3.24 show. The symmetries (3.64) and (3.65) are general properties of any bispectrum between real-valued signals.

Finally, visualization of the source-detector tricovariance, i.e., tricovariance $[1,2,3,4]$ poses a bit of a problem, since it has three independent delays, $[1,2]$, $[1,3]$, and $[1,4]$.

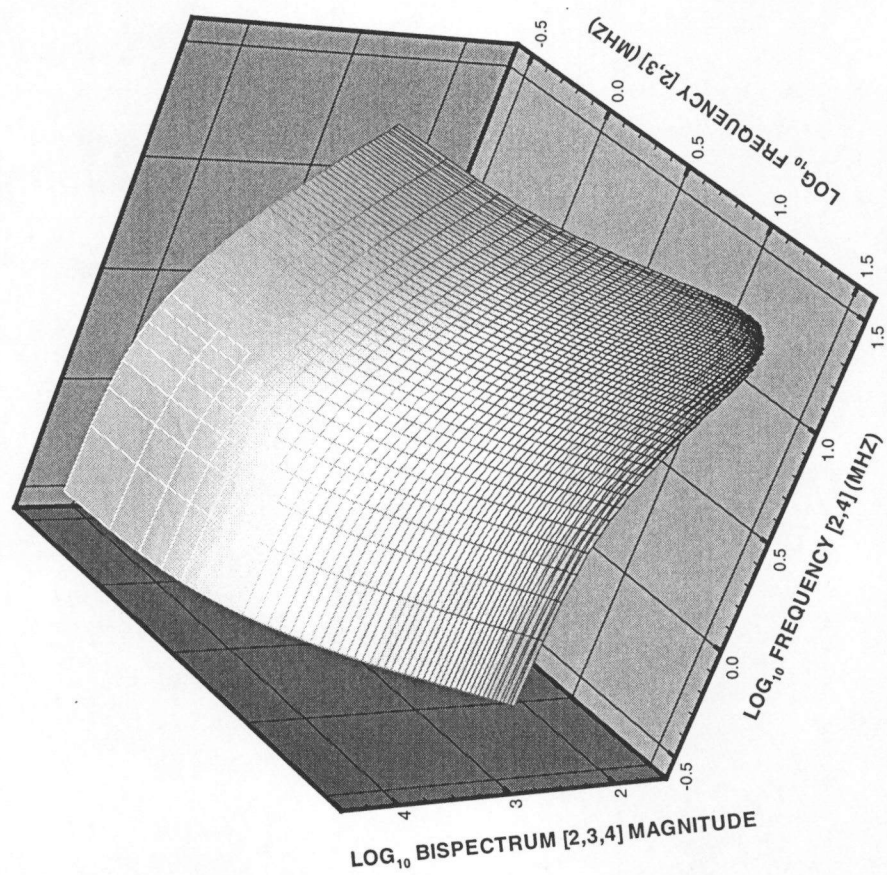


Figure 3.21 Magnitude of between-detectors bispectrum [2,3,4] (in bicornelated triplets/ ^{252}Cf -fission) calculated for the 93.15%-enriched uranium-metal sphere

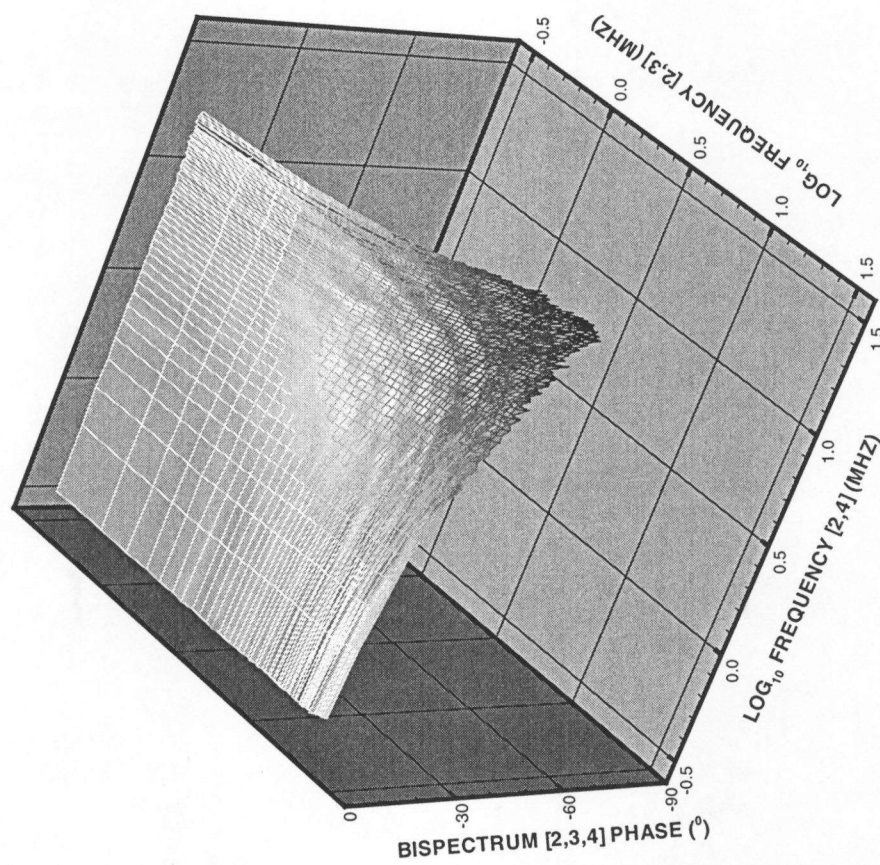
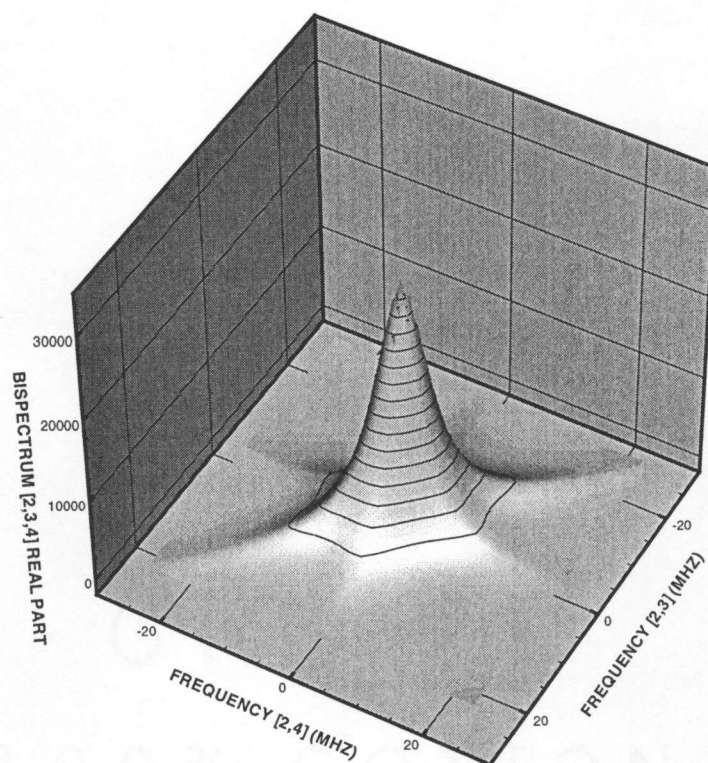
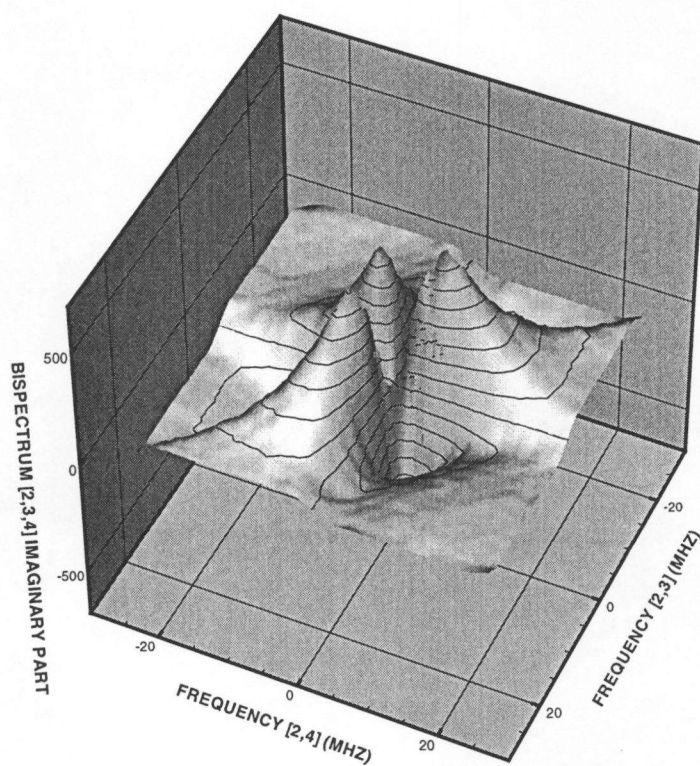


Figure 3.22 Phase of between-detectors bispectrum [2,3,4] calculated for the 93.15%-enriched uranium-metal sphere



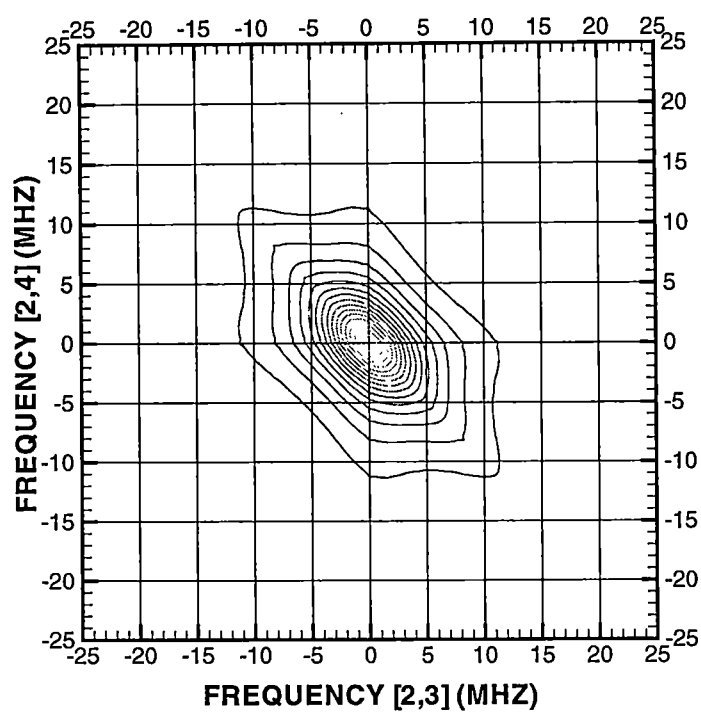
(a)



(b)

Figure 3.23 Between-detectors bispectrum [2,3,4] (a) real and (b) imaginary parts (in bicorrelated triplets/ ^{252}Cf -fission) calculated for the 93.15%-enriched uranium-metal sphere

(a)



(b)

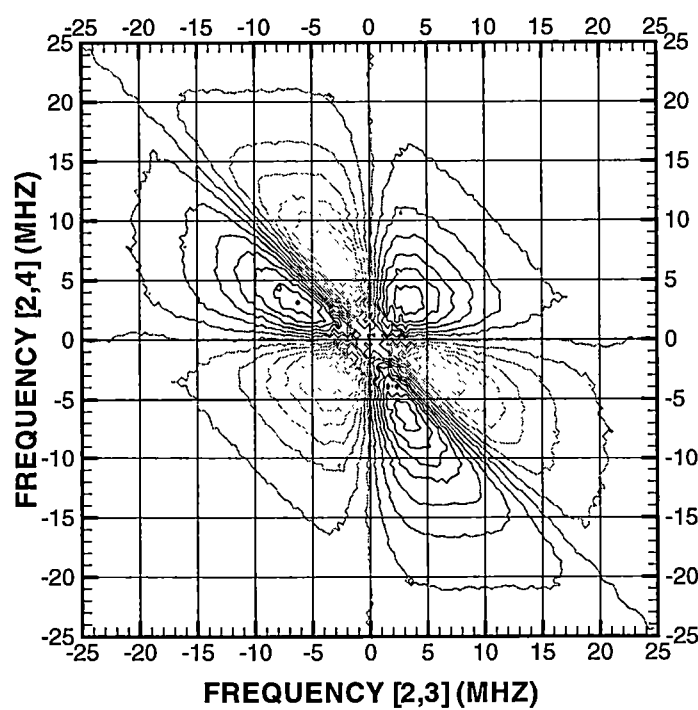


Figure 3.24 Contours of between-detectors bispectrum [2,3,4] (a) real and (b) imaginary parts calculated for the 93.15%-enriched uranium-metal sphere

However, the signature of course vanishes if any of these delays are negative, and it possesses symmetries according to

$$\begin{aligned}
 C_{wxyz}(\tau_1, \tau_2, \tau_3) &= C_{wxzy}(\tau_1, \tau_3, \tau_2) \\
 &= C_{wyxz}(\tau_2, \tau_1, \tau_3) \\
 &= C_{wyzx}(\tau_2, \tau_3, \tau_1) \\
 &= C_{wzxy}(\tau_3, \tau_1, \tau_2) \\
 &= C_{wzyx}(\tau_3, \tau_2, \tau_1) .
 \end{aligned} \tag{3.68}$$

Consequently, most of the source-detector tricovariance's properties are visible if it is plotted as a function of two delays in a "movie" that steps through the third delay. Unfortunately, dissertations are not yet accepted in multimedia format, so Fig. 3.25 shows several frames in such a movie. The full movie, in raster metafile format, is attached to the text on a compact disc (CD). Observe that the source-detector tricovariance shares many of the features possessed by the source-detector bicovariance. First, if two of the delays are zero, along the third delay the signature decays at an exponential rate of α . Furthermore, along any diagonal where two delays are equal, the signature exhibits a ridge due to the heightened probability of two counts in rapid succession following a source fission and count that occurred further back in the past. In addition, the probability of three counts in rapid succession is likely even if the initial source fission occurred in the more remote past. Consequently, a "ridge" (invisible in less than four dimensions) exists along any line where all three delays are equal. Still, its apparent that the fourth-order source-detector tricovariance behaves more or less like the third-order source-detector bicovariance. Consequently, the magnitude and phase of the corresponding source-detector trispectrum behave like that of the source-detector bispectrum. Several frames from movies of the

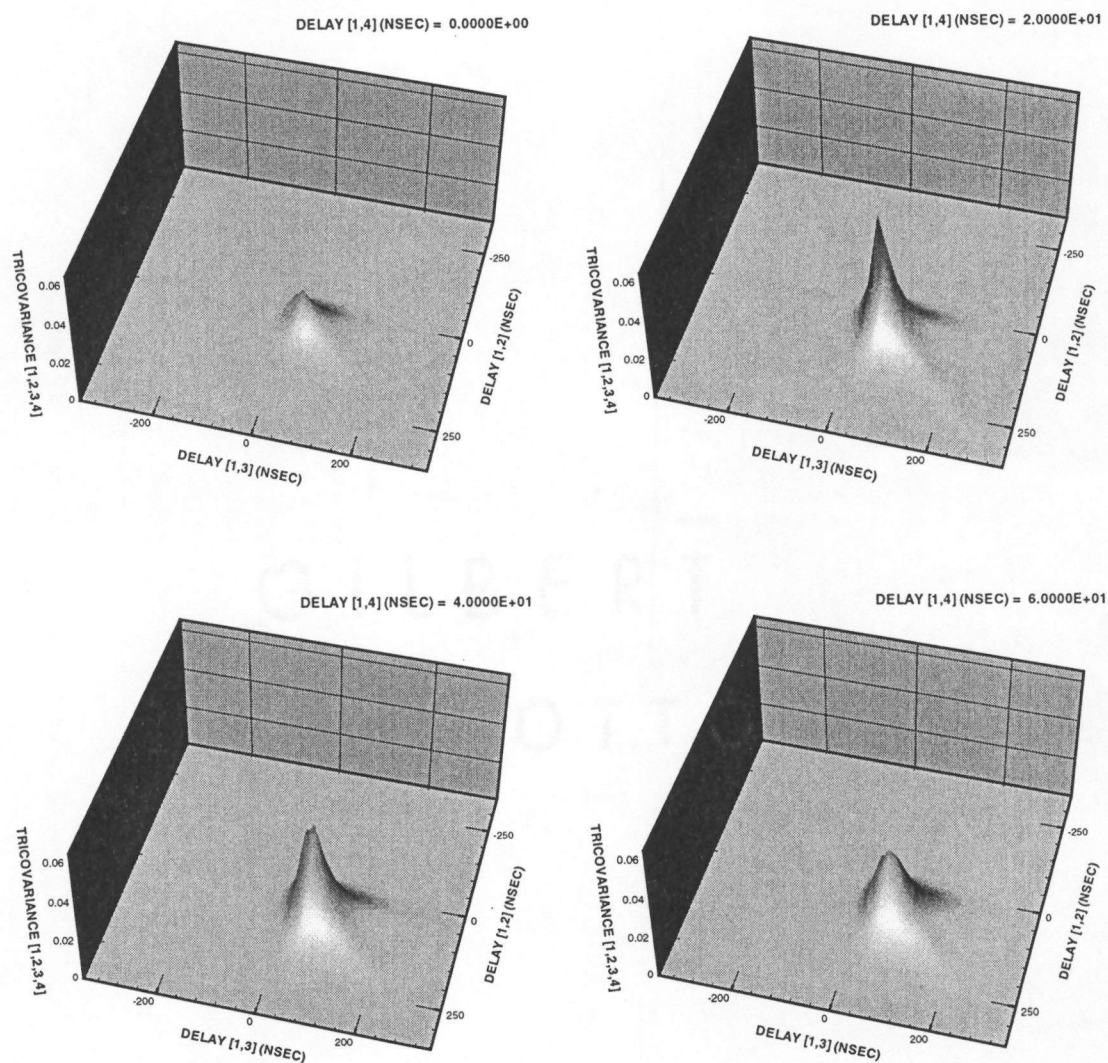


Figure 3.25 Source-detector tricovariance [1,2,3,4] (in bicorrelated triplets/ ^{252}Cf -fission/nsec³) calculated for the 93.15%-enriched uranium-metal sphere

trispectrum's magnitude and phase are shown in Figs. 3.26 and 3.27. The full movies are available in raster metafile format on the enclosed CD.

These are all very interesting properties of high-order signatures, and they serve to depict the behavior of the signatures under conditions more general than those addressed in the point model. However, their value is still largely academic. The evaluation of the signatures' sensitivity to reactivity that appears later this chapter is of more practical value. However, before illustrating this sensitivity, it's prudent to first estimate the uncertainty in these calculated signatures.

3.5 Uncertainty in Estimated Signatures

Of course, the preceding signatures were estimated from a finite number of observations, so each has some uncertainty associated with it. However, estimating this uncertainty is no trivial task. Consider, for example, the variance in the estimate $\hat{\bar{x}}$ of the mean count rate,

$$\sigma_{\hat{\bar{x}}}^2 = E[(\hat{\bar{x}} - \bar{x})^2], \quad (3.69)$$

where $\bar{x}$ denotes the true, unrealizable mean count rate. It can be shown¹⁰ that

$$\sigma_{\hat{\bar{x}}}^2 = \frac{1}{N} \sum_{k=-N}^N \left(\frac{N-k}{N} \right) C_{xx}(k), \quad (3.70)$$

where N denotes the number of observations used to estimate the mean each segment, and C_{xx}

¹⁰ This requires a very subtle "trick." Please refer to Bendat and Piersol's book [Bend2].

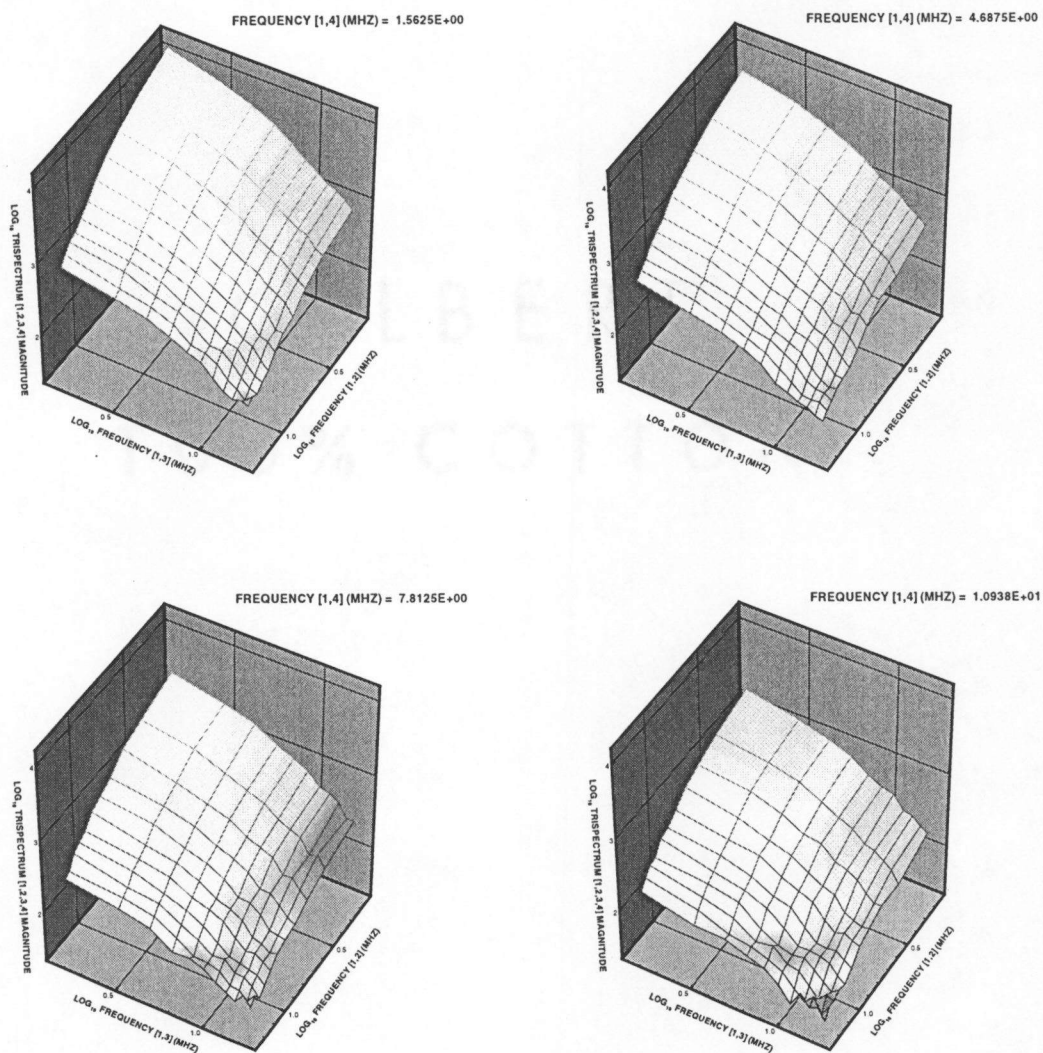


Figure 3.26 Magnitude of source-detector trispectrum [1,2,3,4] (in biccurred triplets/ ^{252}Cf -fission) calculated for the 93.15%-enriched uranium-metal sphere

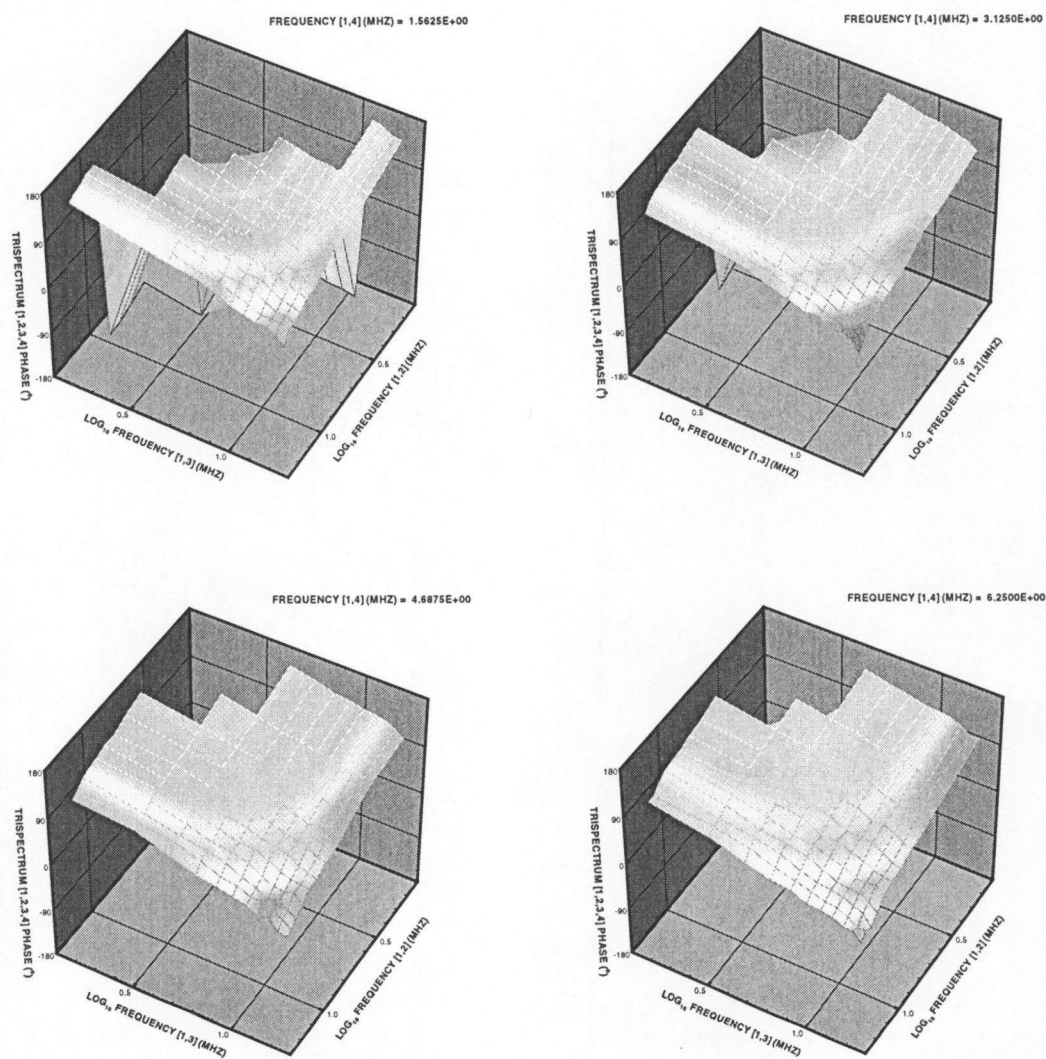


Figure 3.27 Phase of source-detector trispectrum [1,2,3,4] calculated for the 93.15%-enriched uranium-metal sphere

denotes the true autocovariance, which is also unknowable. Hence the true variance of the estimate of the mean is unrealizable. However, (3.70) implies that the variance may be estimated using the autocovariance estimate

$$\hat{\sigma}_{\bar{x}}^2 = \frac{1}{N} \sum_{k=-N}^N \left(\frac{N-k}{N} \right) \hat{C}_{xx}(k) . \quad (3.71)$$

Note that since the autocovariance estimate $\hat{C}_{xx}$ does not account for correlation between segments, the resulting variance estimate $\hat{\sigma}_{\bar{x}}^2$ should be at least conservative. Also note that, since $\hat{\sigma}_{\bar{x}}^2$ is an estimate, it also is uncertain. How big is the variance of the variance?

Now, if this seems like a lot of work, that's because it is. Furthermore, it can be inferred from (3.70) that the variance of the covariance estimate $\hat{C}_{xy}$ is related to the fourth-order trivariance C_{xyxy} [Bend2], that the variance of the bicovariance estimate $\hat{C}_{xyz}$ is related to the sixth-order moment C_{xyzxyz} , etc. So in order to estimate the uncertainty in each high-order signature, an estimate of a signature of twice that order must as well be collected. Forget for a moment that this requires estimation of signatures that are otherwise of no interest; it simply isn't practical. Finally, for second- and higher order signatures, the variance is a function of the delay(s), which only further complicates matters.

However, recall that the sensitivity of each signature is basically reflected by the integral (sum) over that signature. For example, the sensitivity of the covariance C_{xy} to

changes in reactivity is most readily seen in

$$\overline{C_{xy}} = \int d\tau C_{xy}(\tau) , \quad (3.72)$$

which is of course the zero-frequency asymptote of the spectrum S_{xy} . Still, a rigorous theoretical evaluation of the uncertainty in this signature requires a lot of work. An empirical evaluation of its uncertainty would be nearly as illuminating and much easier to achieve.

Recall that each signature is estimated for a given segment independent of the other segments. In particular, MCNP-DSP does not track particles past the end of a segment, so the signature estimates really are independent observations on a segment-by-segment basis. However, the distribution of these independent observations is unknown, so the uncertainty of a segment-wise mean estimate cannot be evaluated. But recall that the average of a large number of independent observations is normally distributed in accordance with the central limit theorem [Net, Pap]. Further recall that the variance of a mean $\bar{\sigma}$ of M observations drawn from a normally distributed population is

$$\sigma_{\bar{\sigma}}^2 = \frac{\sigma_{\sigma}^2}{M} , \quad (3.73)$$

where σ_{σ}^2 is the population variance. The variance of a sample mean $\hat{\sigma}$ estimated from M observations drawn from a normally distribution population is similarly

$$\sigma_{\hat{\sigma}}^2 = \frac{\sigma_{\sigma}^2}{M} , \quad (3.74)$$

where σ_{σ}^2 denotes the sample variance of the M observations.

Consequently, if integrated signatures like (3.72) are estimated each segment, e.g.,

$$\hat{C}_{xy} = \sum_k \hat{C}_{xy}(k) , \quad (3.75)$$

a sample mean and variance for each integrated signature may be accumulated. If the sample means are acquired from an average over a large number M of segments, say $M = 1000$, then the variance of this mean is related to the accumulated sample variance by (3.74). Therefore, the uncertainty in each integrated signature can be evaluated empirically.

A special version of the code described in §3.3 was written to perform this task. Instead of accumulating correlation signatures, the code accumulates integrated covariance signatures and retains the square of each integrated signature. This way an estimate of the mean and mean-square of each signature is accumulated. Every M segments, where M is specified by the user, the code averages the M independent samples of each integrated signature and calculates a mean-square for each integrated signature. The number of segments, the number of source fissions, and the mean and mean-square of each integrated signature are written to permanent storage for subsequent evaluation. If $\hat{\sigma}$ represents an integrated signature estimate obtained from an average over M segments, then the uncertainty of the estimate is

$$\sigma_{\hat{\sigma}} = \sqrt{\frac{\hat{\sigma}^2 - \hat{\sigma}^2}{M}} , \quad (3.76)$$

where $\hat{\sigma}^2$ and $\hat{\sigma}^2$ respectively denote the sample squared-mean and sample mean-square.

The relative uncertainty is simply $\hat{\sigma}_z / \hat{z}$.

Figure 3.28 plots the estimated relative uncertainty in the first- through fourth-order signatures calculated for the 93.15%-enriched uranium-metal sphere as a function of the number of source fissions. These estimates were obtained at 1000 segment intervals over a total of 100,000 segments. Where multiple estimates of the relative error were available, the different estimates were averaged. For example, the relative error in the source-detector covariance represents the average of the relative error in covariance [1,2], [1,3], and [1,4]. The relative error of the source fission rate is plotted as a benchmark. Since the source fission rate is a Poisson-distributed white noise process, the relative error in the mean fission rate should behave as

$$\frac{\hat{\sigma}_{\bar{x}_1}}{\bar{x}_1} = \frac{1}{\sqrt{N_1}}, \quad (3.77)$$

where N_1 denotes the number of source fissions. Observe that it does. As a matter of fact, the relative error in each of the signatures more or less decreases as $1 / \sqrt{N_1}$, or, alternatively, as $1 / \sqrt{M}$ since $N_1 \propto M$, the number of segments. This implies that the sample error is essentially constant regardless of the number of segments acquired.

Consequently, a prescription for determining the number of segments necessary to achieve a desired error is possible. A short run of a problem will yield the sample error σ_z , which is essentially constant. The number of segments M to achieve a desired error σ_z in the

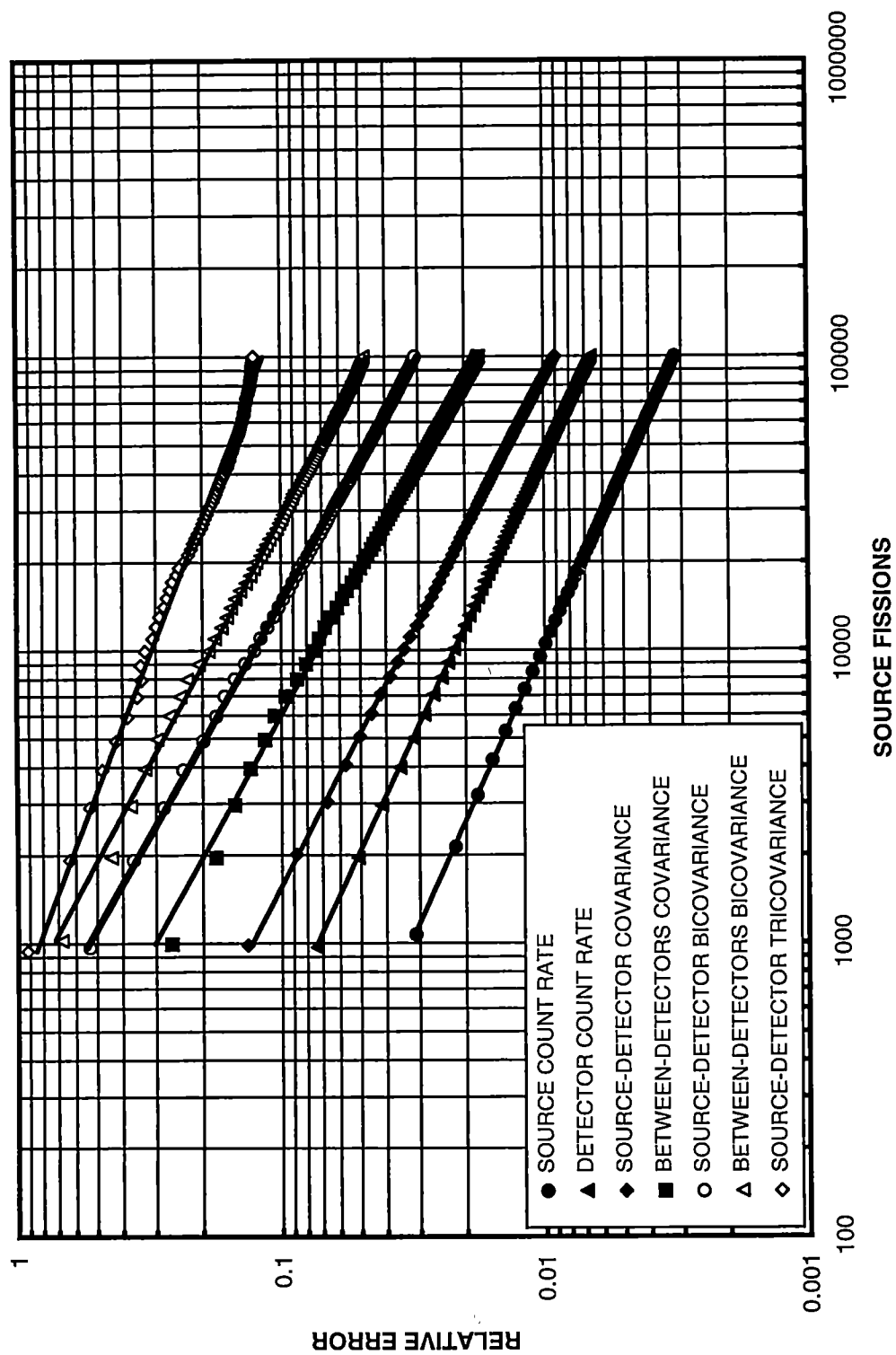


Figure 3.28 Relative uncertainty in signatures calculated for the 93.15%-enriched uranium-metal sphere

mean of a particular signature is simply

$$M = \frac{\sigma_s^2}{\sigma_d^2} . \quad (3.78)$$

Of course, this is not necessarily true in general. That's the problem with an empirical evaluation: only limited generalizations can be inferred. It's expected that this generalization is at least applicable to systems similar to the one modeled, that is, systems that are relatively reactive and are observed by high efficiency detectors.

Note that as the number of detectors involved in the signature increases, the relative error also increases. The reason for this trend is obvious: accumulation of a signature involving q detectors requires that one of at least q correlated events occur in each detector. This becomes less probable as q increases, so the signature essentially involves fewer observations as q increases. For this same reason, relative uncertainty tends to be larger for less reactive systems and for systems driven by weaker sources and/or observed by lower efficiency detectors.

Finally, the trends depicted in Fig. 3.28 indicate that after roughly 500,000 source fissions, the relative error in each integrated signature is less than 10%. Rough estimates of the uncertainty in each integrated signature are listed in Table 3.4. Of course, the uncertainty in any given signature at a particular delay (or set of delays) is somewhat higher. Nonetheless, it can be concluded that the signatures calculated for the 93.15%-enriched sphere are sufficiently certain after 500,000 source fissions. This should at least hold for other fairly high-enriched uranium spheres, particularly those described in the following section.

Table 3.4 Relative uncertainty in integrated signatures calculated for the 93.15%-enriched uranium-metal sphere

Signature	Estimated Relative Uncertainty after 500,000 Source Fissions
source count rate	= 0.14%
detector count rate	≤ 0.3%
source-detector covariance	≤ 0.4%
between-detectors covariance	≤ 0.8%
source-detector bicovariance	≤ 1.3%
between-detectors bicovariance	≤ 2.0%
source-detector tricovariance	≤ 7.0%

3.6 Sensitivity of High Order Signatures Calculated for Uranium-Metal Spheres of Varying Enrichment

To further illustrate the enhanced sensitivity to reactivity of successively higher order signatures, and to do so under conditions more general than those encompassed by the point reactor model, a series of calculations was performed on uranium-metal spheres of enrichments between 75% and 100%. Some of the parameters affecting the sensitivity of the signatures are summarized in Table 3.5.

As noted in the table caption, each of the five spheres was 150 mm in diameter. Furthermore, the modeled density of each sphere was the same – 18.76 kg/ℓ, which is the

Table 3.5 Some properties of 150-mm-diameter unreflected uranium-metal spheres

Parameter	Enrichment				
	75%	80%	85%	93%	100%
^{235}U mass (kg)	24.86	26.52	28.18	30.88	33.15
induced fissions/source fission	7.79	9.21	10.99	15.57	20.76
fraction of fissions in ^{238}U	5.35%	4.09%	2.95%	1.04%	—
mean energy of neutrons inducing fission (MeV)	1.53	1.53	1.53	1.54	1.55
prompt reactivity ^(1,2) (\$ subcritical)	45.17 ± 0.06	38.53 ± 0.06	32.56 ± 0.06	23.08 ± 0.05	17.52 ± 0.05
prompt neutron fission-chain decay constant (μsec^{-1})	35.50 ± 0.06	34.03 ± 0.04	31.92 ± 0.03	26.47 ± 0.01	22.19 ± 0.01
prompt neutron generation time (nsec)	8.40 ± 0.02	7.47 ± 0.02	6.73 ± 0.01	5.75 ± 0.01	5.21 ± 0.02
$\bar{\nu}$ (per induced fission)	2.60	2.60	2.60	2.60	2.61
$\bar{\nu}(\bar{\nu} - 1)$ (per induced fission)	5.52	5.52	5.52	5.52	5.52
$\bar{\nu}(\bar{\nu} - 1)(\bar{\nu} - 2)$ (per induced fission)	9.38	9.36	9.35	9.33	9.33
average detection efficiency (counts/neutron)	20.8%	20.6%	20.3%	19.9%	19.6%
fraction of counts due to source neutron capture	13.3%	11.3%	9.6%	6.7%	5.1%

⁽¹⁾ The value 0.0066 for the effective delayed neutron fraction was assumed.

⁽²⁾ Uncertainties represent one standard deviation.

density of the 93.15%-enriched sphere described in previous sections. These two features were fixed such that the fissile mass was independent of volume and density and only depended upon enrichment. At any rate, spheres of the four enrichments different from 93.15% have never been fabricated, so no information regarding their actual density, which would deviate from the theoretical, is available.

The ratio of induced fissions to source fissions was calculated by MCNP-DSP as before. A standard MCNP model was constructed of each unreflected sphere with a point ^{252}Cf source at its center, and the number of fissions occurring in uranium isotopes 235 and 238 was tallied to estimate the fraction of all fissions occurring in the 238 isotope. The model of the 93.15%-enriched sphere also produced tallies of the number of fissions occurring in isotopes 234 and 236, which turned out to be negligible. Observe that in each of the spheres, fission of ^{235}U accounts for the vast majority of fission-chain proliferation. The same standard MCNP models were used to estimate the mean energy of neutrons inducing fission. This was done by tallying the incident energy distribution of neutrons entering fission reactions with ^{235}U and ^{238}U . The average energy of neutrons inducing fission was subsequently estimated via simple trapezoidal-rule integration. A more sophisticated integration was unnecessary because the distributions were tallied on a fairly fine energy grid. Unfortunately, all this work only revealed that the average energy of neutrons inducing fission is essentially the same for all enrichments – about 1.5 MeV.

These same energy distributions were then applied to estimate the first three reduced moments of the induced fission neutron multiplicity distribution, i.e., $\bar{v}$, $\overline{v(v-1)}$, and $\overline{v(v-1)(v-2)}$. This was done by folding the energy distributions for neutrons inducing

fission in ^{235}U and ^{238}U with the respective $p(v|E)$ distributions as tabulated by Zucker and Holden [Zuc, Val1,4]. Again the integration was performed using the trapezoidal rule. Estimates of the reduced moments later became available from MCNP-DSP as a new modification by Valentine [Val4]. The MCNP-DSP estimates of these quantities yielded the same values to the first decimal place. Observe that only the third reduced moment appears to change between the maximum and minimum enrichments, and even so the change is too small to significantly effect the rate of triplet production. Furthermore, recall that at these reactivities the primary process responsible for triplet production results from sequential dual-branch fissions, which follows $\overline{v(v-1)}$.

The prompt reactivity was estimated for each unreflected sphere using an MCNP-DSP analog k_{eff} -calculation as before. The prompt reactivity was estimated assuming that $\beta = 0.0066$ applies for each enrichment – measurements of this quantity are simply unavailable for the four enrichments different from 93.15%. The prompt neutron fission-chain decay constant α and prompt neutron generation time Λ were estimated as before using an MCNP-DSP pulsed neutron simulation that modeled a ^{252}Cf point source at each unreflected sphere's center and treated each sphere as a fission detector. Finally each sphere was surrounded with three black neutron detectors that formed a 75-mm-thick spherical shell. The average detection efficiency listed in the table represents the ratio of counts to the total number of induced fission neutrons. The portion of these counts that resulted from a source neutron captured in a detector is also indicated. Observe that very few source neutrons survive to the periphery of the sphere. Also note that roughly two-thirds of the induced

fission neutrons survive to escape the sphere – the maximum possible detection efficiency is one-third, and an average efficiency of roughly 20% was observed.

In this calculational study, the most influential parameter by far is of course reactivity. It determines the number of neutrons born per source neutron injected into the sphere. But do the neutron counting signatures adequately reflect the change in reactivity? Apparently so. For example, the source-detector covariances [1,2] plotted in Fig. 3.29 show a clear increase in amplitude as a function of enrichment. Note also that the decline of α is evident in the length of the covariances' "tails" due to the increasing length of prompt neutron fission-chains. This sensitivity is more starkly apparent in the low-frequency asymptote of the source-detector spectra plotted in Fig. 3.30. Recall that a change in the low-frequency magnitude reflects an integral over changes in the covariance:

$$S_{12}(0) - S'_{12}(0) = \int d\tau \left(C_{12}(\tau) - C'_{12}(\tau) \right). \quad (3.79)$$

The relatively higher sensitivity of the second-order between-detectors signature is readily apparent in the plots of covariance [2,3] and spectrum [2,3] shown in Figs. 3.31 and 3.32. Observe that the source-detector bicovariance and bispectrum [1,2,3] appear to follow the same trend as Figs 3.33 and 3.34 show. Of course, this is to be expected. Recall that, because all neutrons in these calculations are chain-related to some initial source-fission, bicovariance [1,2,3] simply reflects the distribution of [2,3] correlated doublets following source fission events recorded in channel [1]. Covariance [2,3] is just the integral of this distribution.

The trend of each signature versus reactivity is illustrated most clearly in Fig. 3.35. The quantities plotted in this figure are the low-frequency asymptotes of each spectral

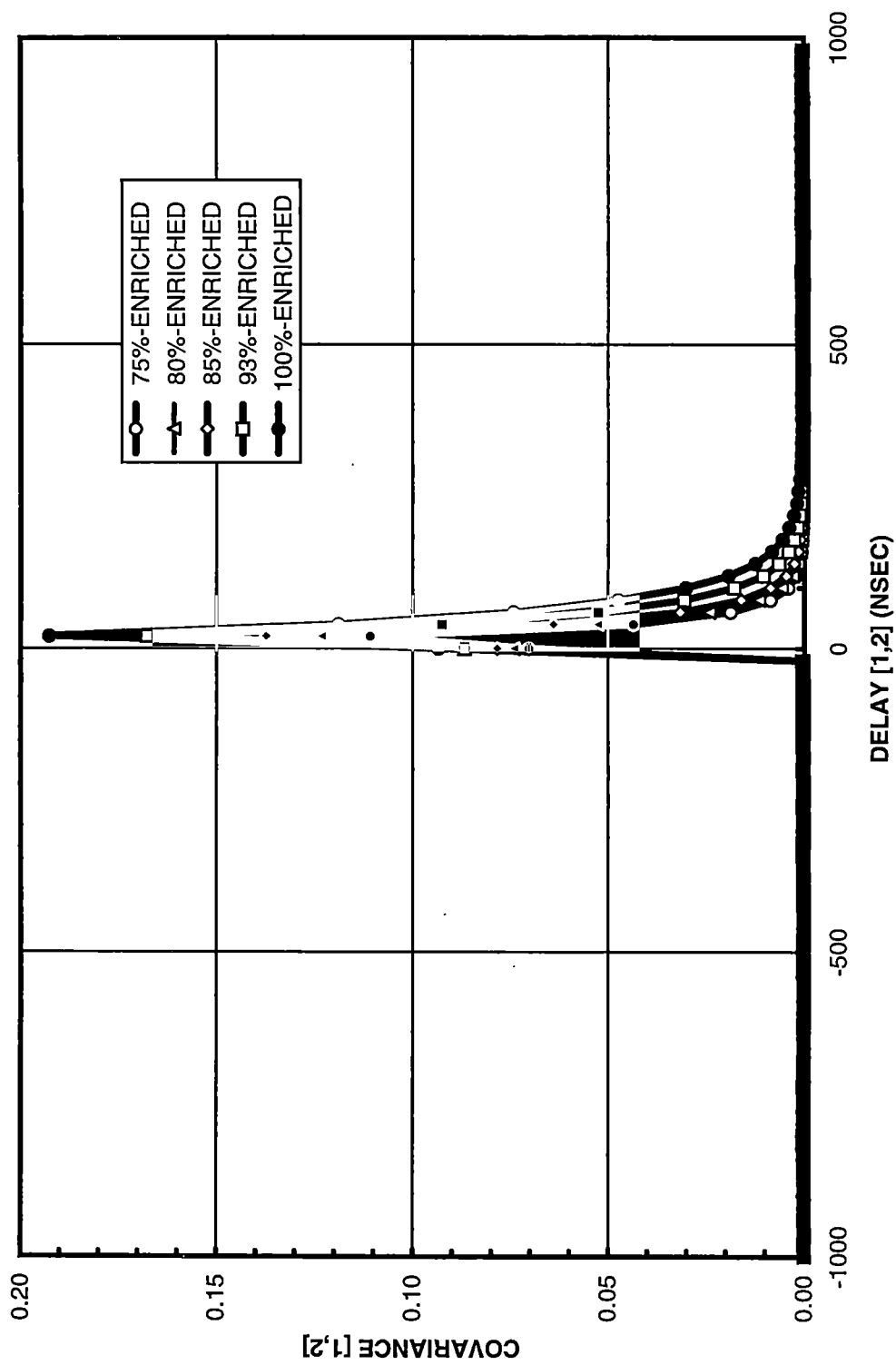


Figure 3.29 Source-detector covariances calculated for uranium-metal spheres of varying enrichment

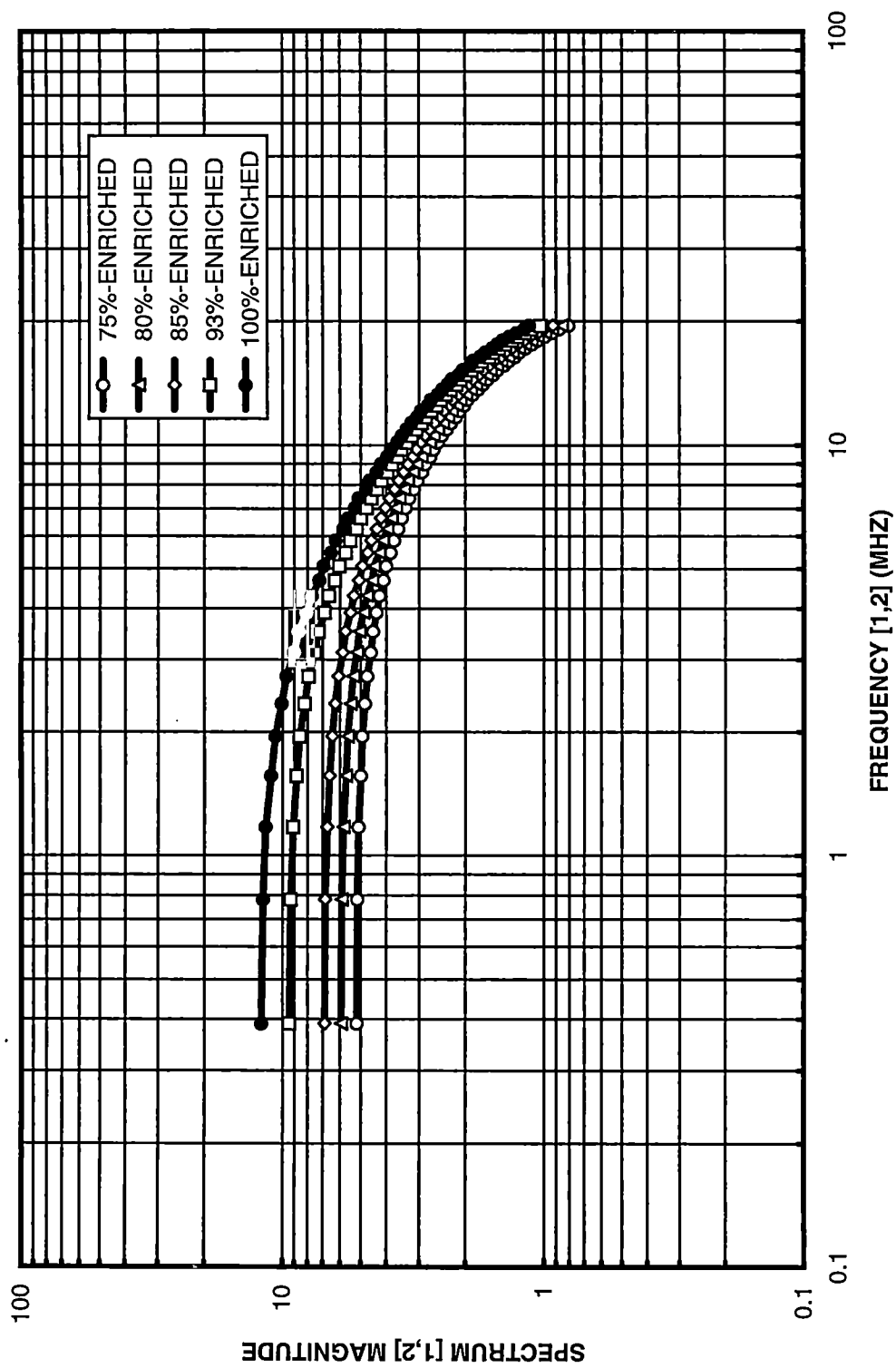


Figure 3.30 Source-detector cross-spectra calculated for uranium-metal spheres of varying enrichment

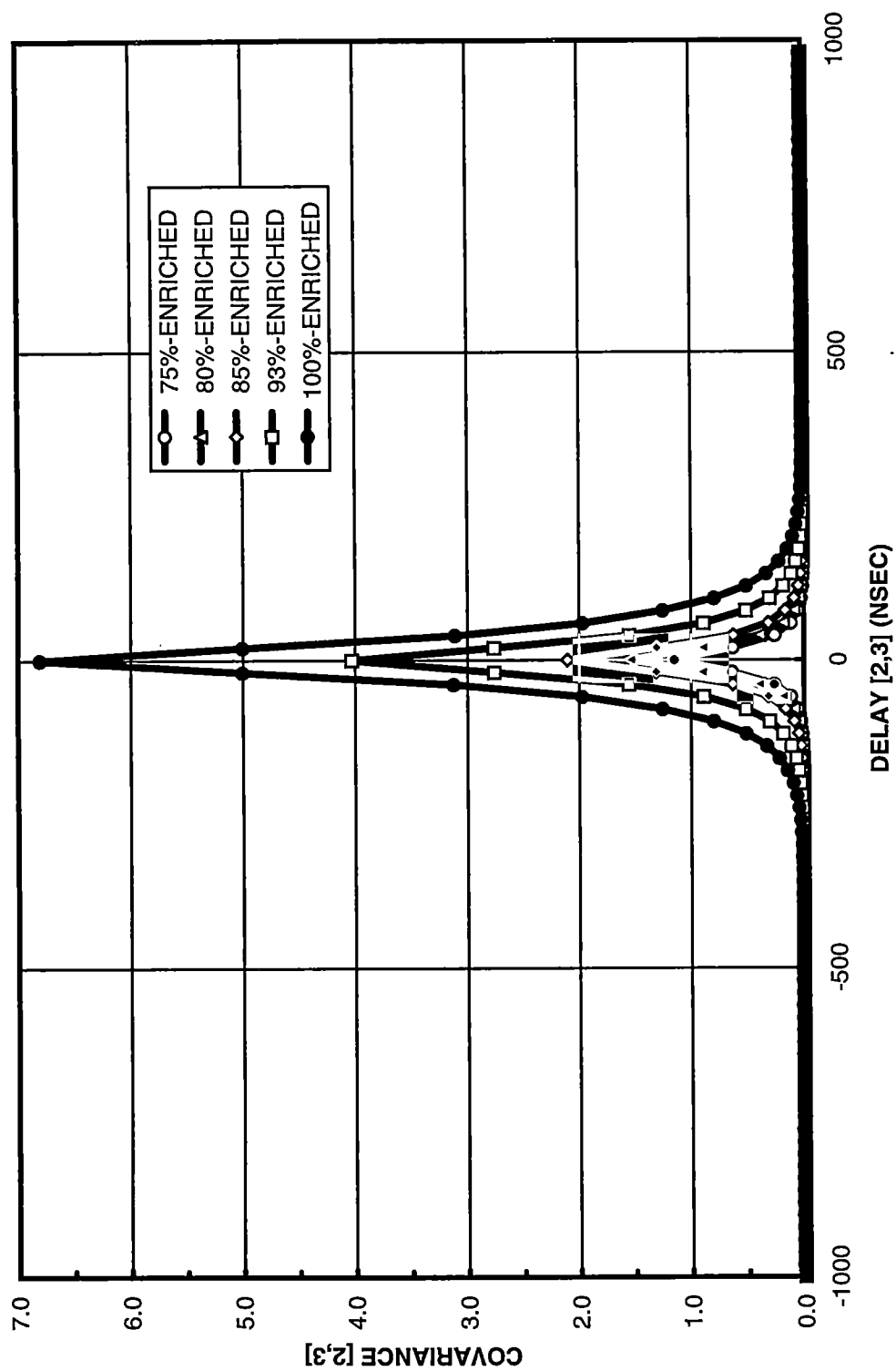


Figure 3.31 Between-detectors covariances calculated for uranium-metal spheres of varying enrichment

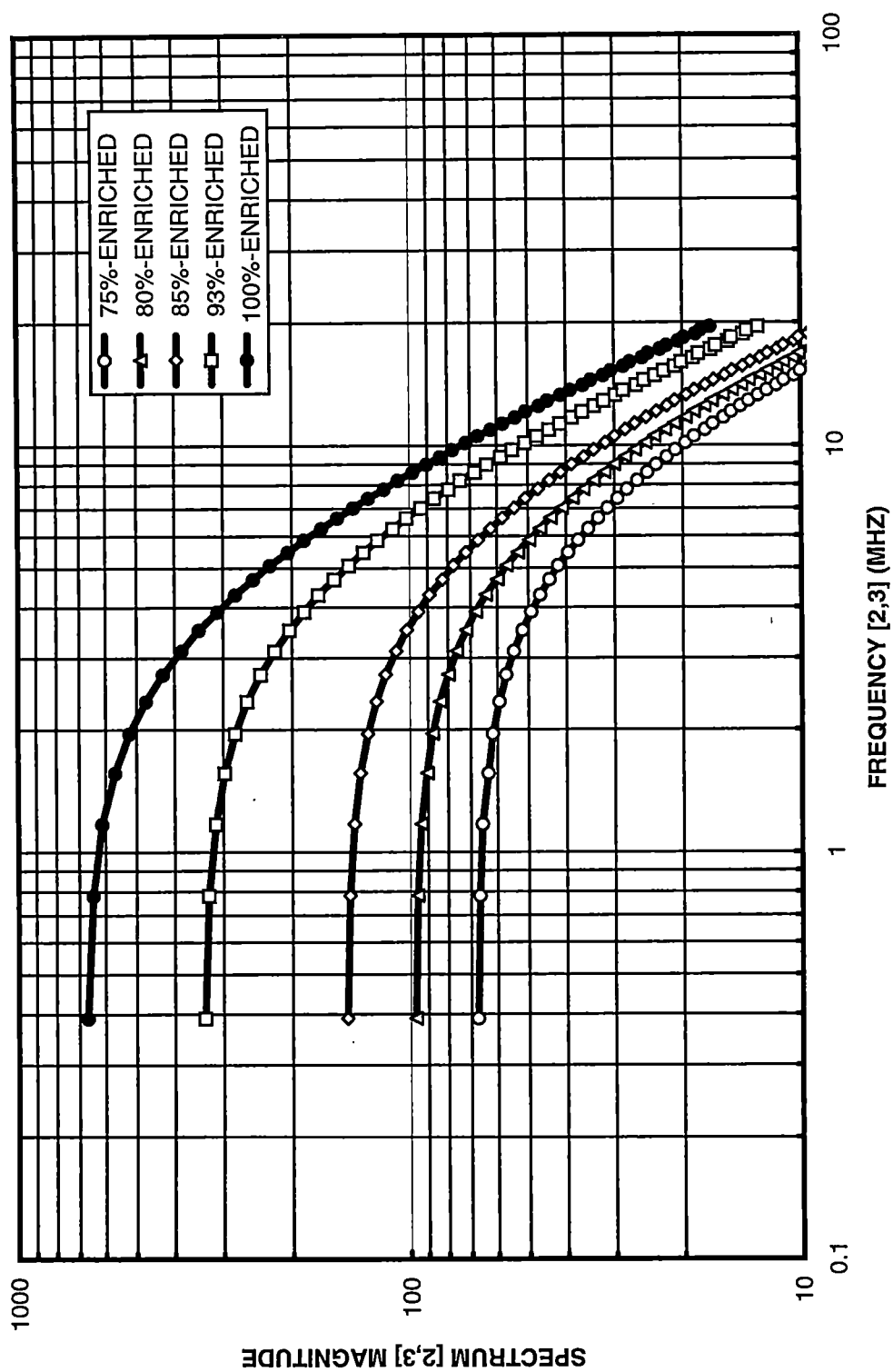


Figure 3.32 Between-detectors cross-spectra calculated for uranium-metal spheres of varying enrichment

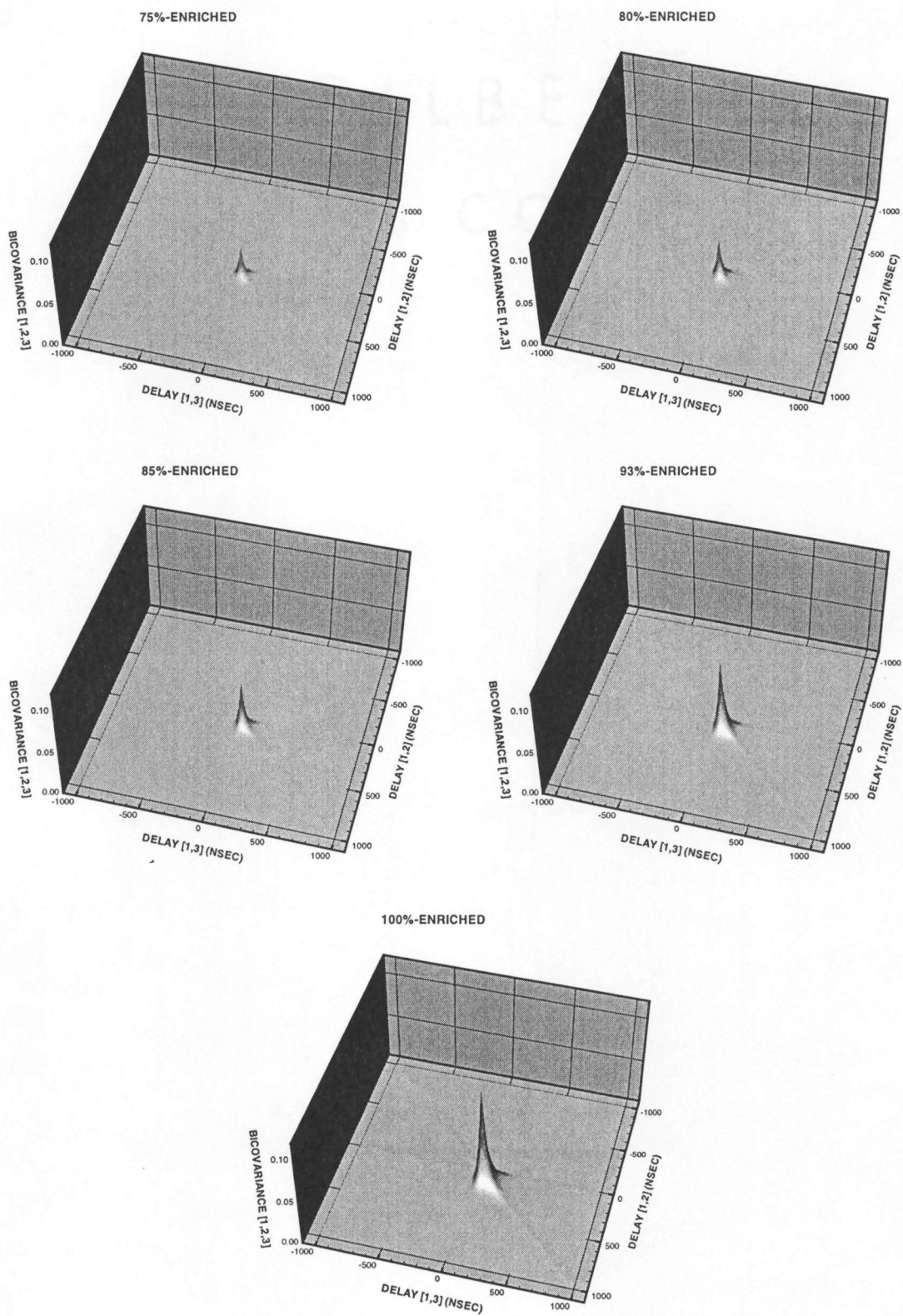


Figure 3.33 Source-detector bicovariances calculated for uranium-metal spheres of varying enrichment

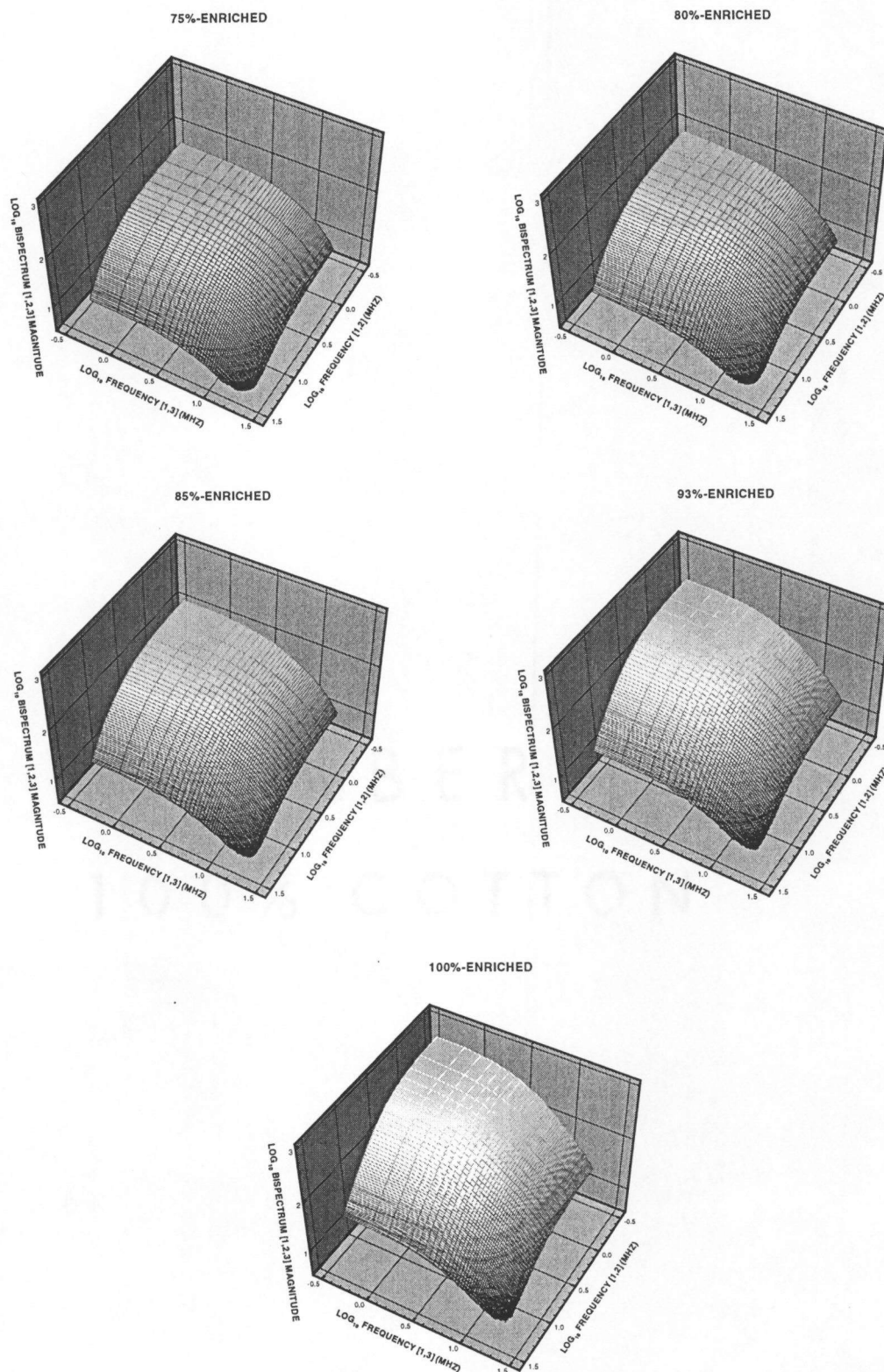


Figure 3.34 Source-detector bispectra calculated for uranium-metal spheres of varying enrichment

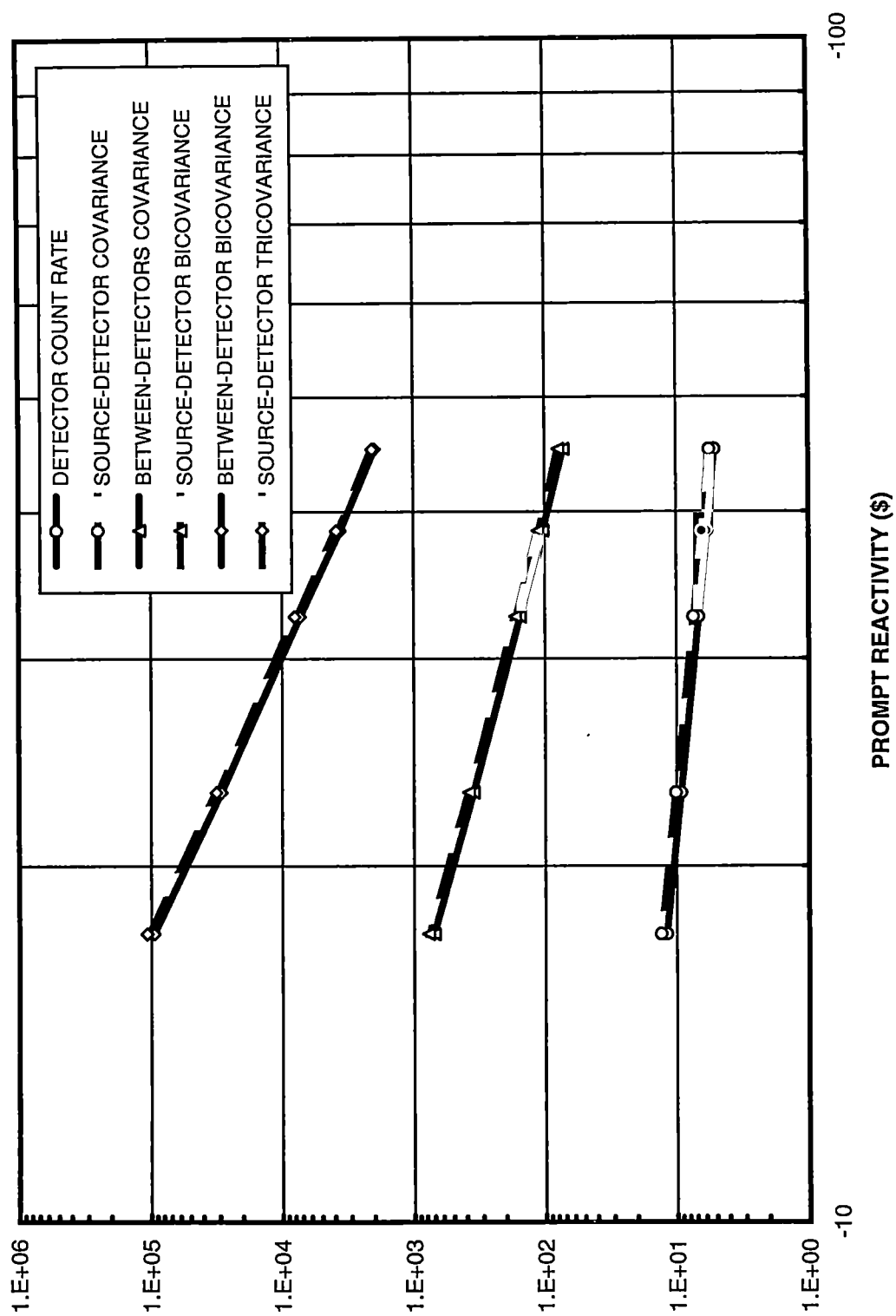


Figure 3.35 Sensitivity of calculated signatures to reactivity

signature, or, equivalently, the integral over delay of each temporal signature. Observe that, as expected, the integral of the source-detector covariance is essentially identical to the detector count rate, and so its sensitivity to reactivity is the same. Similarly, the integral of the source-detector bicovariance is identical to that of the between-detectors covariance. Thus they have the same sensitivity to reactivity, and this sensitivity is relatively higher than that of the source-detector covariance and detector count rate. Finally, the highest sensitivity to reactivity is possessed by the between-detectors bicovariance and source-detector tricovariance.

In the subsequent figure, 3.36, the three spectral ratios defined in the previous chapter are plotted versus reactivity. Observe that these also follow the expected trend. The bispectral and type II trispectral ratio exhibit the same sensitivity to reactivity. The sensitivity of the type I trispectral ratio is relatively higher. This is because the ratio of uncorrelated pairs to correlated pairs has the same sensitivity to reactivity as does the ratio of correlated pairs to bicorrelated triplets. The ratio of uncorrelated triplets to bicorrelated triplets is more sensitive to changes in reactivity.

Even though the calculated spheres do not behave exactly as simple theory predicts, the sensitivity to reactivity of successively higher order signatures does follow a trend remarkably similar to that predicted in the previous chapter. Again, this behavior was expected, and should be observed under a variety of conditions. However, it is advisable to note that these results are after all only calculations and that they do not cover even a full decade of reactivity. Most likely, in accordance with predictions made in the previous chapter, these signatures will exhibit a less dramatic trend in extremely subcritical systems.

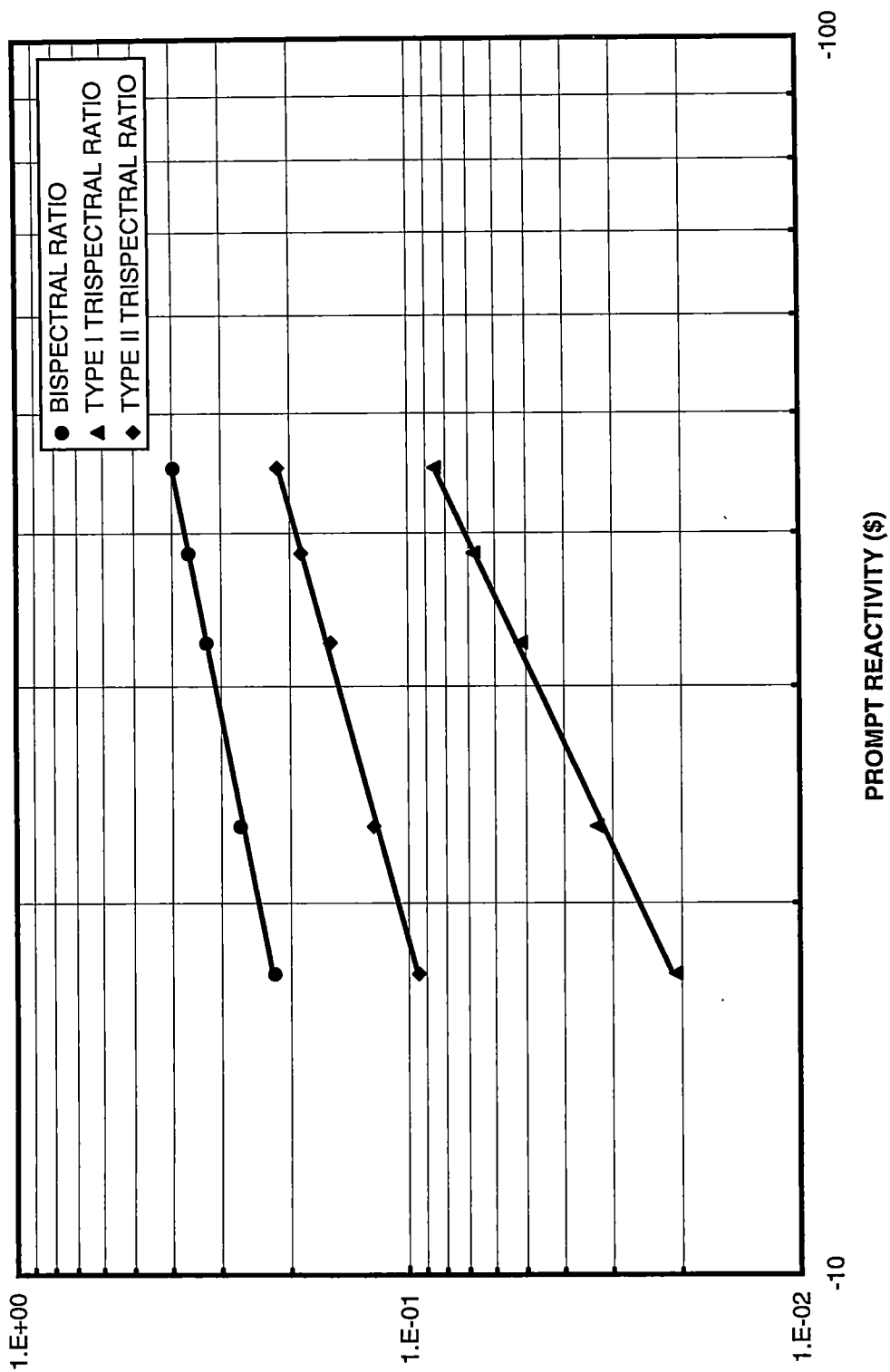


Figure 3.36 Sensitivity of calculated spectral ratios to reactivity

In such systems, fission-chain proliferation is weak, and the signatures are primarily driven by short fission-chains initiated by source neutrons. So, although the calculational results presented here definitely demonstrate the utility of high order signatures, they still only address conditions that are relatively ideal. The next chapter describes measured signatures acquired under decidedly less ideal conditions.

4 Measurement of High Order Nuclear Counting Statistics: One Example

The estimation procedures described in the previous chapter are generally applicable. They are also efficient enough to be practical for the majority of calculational and measurement applications. However, an alternative procedure particularly well-suited to nuclear counting measurements exists. Under the right conditions, this procedure is in fact far more efficient than those previously described. This algorithm and the conditions under which it is practical are described in this chapter, and a brief account of the algorithm's implementation in an advanced measurement system follows. Then signatures acquired by this system from an annular casting of high-enriched uranium-metal are presented for examination. The features of these measured signatures unaddressed by the preceding theoretical and calculational studies are emphasized.

4.1 The Time-Difference Correlation Algorithm

First recall the definition of the coincidence rate distribution, or cross-correlation:

$$R_{xy}(\tau) = E[x(t_1)y(t_2)] \Big|_{\tau = t_2 - t_1} . \quad (4.1)$$

The estimator for this distribution described in the previous chapter was applied to discrete sequences of events that were acquired by digitally sampling a time-continuous signal from each detector or by recording the number of events that occurred in each detector during sequential counting intervals. That particular algorithm became prevalent after the advent

of digital signal processing techniques; the development of the fast Fourier transform (FFT) during the 1960's was the primary advancement that made the algorithm practical. Prior to that, a number of clever techniques were developed to implement correlation and spectral analyses in analog and digital hardware.

Consider that the cross-correlation $R_{xy}(\tau)$, less the accidental coincidence rate $\bar{x}\bar{y}$, essentially measures the distribution of delays (τ) between initial counting events in detector x and subsequent correlated counting events in detector y . The algorithm described in the previous chapter estimates this distribution by accumulating the number of counting events that are coincident between x and y when one is delayed by τ relative to the other. As its name implies, the time-difference algorithm accumulates correlation estimates using the time-difference, or delay, between detection events. Furthermore, the time-difference algorithm is optimized for application to *binary* signals.

Suppose that detector signals $x(t)$ and $y(t)$, each of which is essentially a series of electrical pulses corresponding to counting events, are sampled at an extremely high rate – so high in fact that during any given counting interval the probability of collecting more than one count is either zero or vanishingly small. Under these conditions, the discrete representation of each signal is just a sequence of ones and zeros. In other words, the sequence is a binary number.

The time-difference algorithm is further optimized by using a compressed representation of the binary sequence of events in each channel. First, the signal in each channel is acquired for a large segment, or block, of counting intervals, typically 512. Next, rather than recording the signals as sequences of ones and zeros instead only the *positions*

of the ones are recorded. If the sample rate is sufficiently high, this compressed representation will usually have only a few entries per block. Consequently, each channel may be represented by a sequence containing a small number of event positions.

Now let N_x and N_y respectively denote the number of events in channels x and y during a particular block. Next let $P_x(i)$ and $P_y(i)$ respectively denote the position of the i -th event in channels x and y . This little bit of pseudo-code illustrates the algorithm:

$$\begin{aligned} & \text{for } i = 1 \text{ to } N_x \\ & \quad \text{for } j = 1 \text{ to } N_y \\ & \quad \quad \text{Increment Accumulator}_{xy}(P_y(j) - P_x(i)) \text{ by } 1. \end{aligned} \quad (4.2)$$

Consequently, the number of operations executed during a particular block is $N_x \cdot N_y$, which due to the sparsity of the typical block is much fewer than N^2 , where N is the number of sampling intervals per block. The number of operations¹ is even fewer than $2N \log_2 N$, if the sample rate is sufficiently high that the average number of events per channel per block is fewer than $\sqrt{2N \log_2 N}$. Note that, for 512 samples per block, the average number of events per channel per block can be as high as 96 for the time-difference algorithm to continue to perform fewer operations than a Fourier-based algorithm. Further note that each time-difference operation is a simple integer subtraction followed by an increment of an integer accumulator. Fourier-based algorithms usually multiply and add floating-point operands and typically use floating-point accumulators. Consequently, the time-difference algorithm is still more efficient even when the average number of events per block slightly exceeds $\sqrt{2N \log_2 N}$.

¹ Note that a Fourier-based correlation algorithm requires one FFT per block for each of the two channels.

This algorithm can be extended to accumulate higher order statistics. In fact, this is rather easy. The definitions of the bicorrelation,

$$R_{xyz}(\tau_1, \tau_2) = E[x(t_1)y(t_2)z(t_3)] \left| \begin{array}{l} \tau_1 = t_2 - t_1, \\ \tau_2 = t_3 - t_1 \end{array} \right. \quad (4.3)$$

and tricorrelation,

$$R_{wxyz}(\tau_1, \tau_2, \tau_3) = E[w(t_0)x(t_1)y(t_2)z(t_3)] \left| \begin{array}{l} \tau_1 = t_1 - t_0, \\ \tau_2 = t_2 - t_0, \\ \tau_3 = t_3 - t_0 \end{array} \right. \quad (4.4)$$

practically define the form of each corresponding time-difference accumulator. For example, the pseudo-code to accumulate the bicorrelation is

$$\begin{aligned} & \text{for } i = 1 \text{ to } N_x \\ & \quad \text{for } j = 1 \text{ to } N_y \\ & \quad \quad \text{for } k = 1 \text{ to } N_z \\ & \quad \quad \quad \text{Increment Accumulator}_{xyz}(P_y(j) - P_x(i), P_z(k) - P_x(i)) \text{ by } 1. \end{aligned} \quad (4.5)$$

Note that this requires $N_x \cdot N_y \cdot N_z$ operations per block. Consequently, to maintain efficiency superior to a Fourier-based accumulator, the average number of events per block must be on the order of $\sqrt[3]{3N \log_2 N}$ or fewer. Still, for 512 samples/block, this critical number is still as high as 24. It is significantly fewer for a time-difference tricorrelation accumulator:

$$\begin{aligned} & \text{for } i = 1 \text{ to } N_w \\ & \quad \text{for } j = 1 \text{ to } N_x \\ & \quad \quad \text{for } k = 1 \text{ to } N_y \\ & \quad \quad \quad \text{for } \ell = 1 \text{ to } N_z \\ & \quad \quad \quad \quad \text{Increment Accumulator}_{wxyz}(P_x(j) - P_w(i), P_y(k) - P_w(i), P_z(\ell) - P_w(i)) \text{ by } 1, \end{aligned} \quad (4.6)$$

since $N_w \cdot N_x \cdot N_y \cdot N_z$ operations are required per block. Obviously the average number of counts per block must be on the order of $\sqrt[4]{4N \log_2 N}$ or fewer.

The critical number of events per block at which the time-difference algorithms begin

to perform more operations than Fourier-based algorithms is plotted in Fig. 4.1 versus the number of samples per block. Bear in mind that this is only a rough analysis of the relative efficiency of time-difference and Fourier algorithms. Recall that the time-difference algorithms are still more efficient than the Fourier methods even when the count rate is slightly above the critical limit because the former performs simple operations on integers. Furthermore, recall that for the time-difference algorithm to be applicable the sample rate must be sufficiently high that it is either impossible or highly unlikely that more than one count will be collected per sample. In other words, the detector signals must be adequately represented by binary sequences.

Recent advancements in hardware that made it possible to acquire radiation detector signals at extremely high rates led to the development of the Nuclear Weapons Identification System (NWIS) [Mih8]. The NWIS is a measurement system that performs correlation and spectral analysis on signals acquired from up to five radiation detectors at rates up to and including one gigahertz. As its name implies, NWIS was designed to use these signatures to identify nuclear weapons components by performing source-driven and passive measurements on these items. Clearly this specific application has no place in a Ph.D. dissertation. However, NWIS is capable of performing similar measurements on a variety of nuclear and non-nuclear assemblies, and it is particularly well-suited to fast fissile systems, e.g., those composed of fissile metal. The following sections describe some key features of NWIS, the implementation of higher order signature estimation procedures, and their application to the measurement of a highly enriched uranium-metal annular casting.

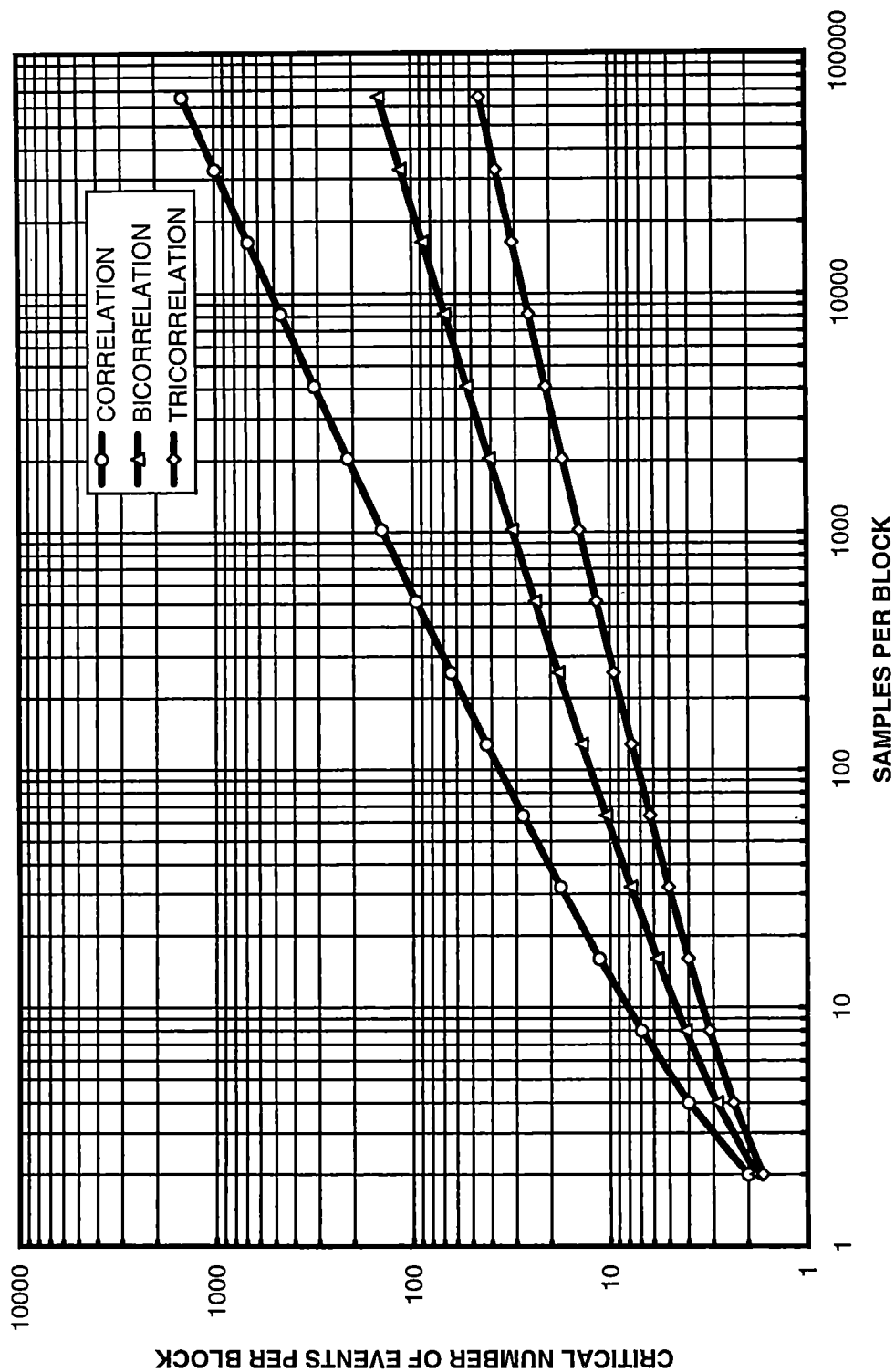


Figure 4.1 Critical number of events per block at which time-difference algorithms begin to perform more operations than Fourier-based algorithms

4.2 Implementation in a High Speed Nuclear Measurements System

Most of the functionality of NWIS is derived from its data capture and compression (DCC) module where the detector signals are acquired. Within the DCC, input detector signals are converted to compressed streams containing the position of events in each channel relative to the beginning of the current block. This function is illustrated in Fig 4.2. The DCC front-end is designed to accept standard nuclear instrumentation module (NIM) pulses. These have negative polarity, nominal 800 mV amplitude, and are typically at least a few nanoseconds in duration. However, the DCC can accept pulses from other instrumentation providing that they have negative polarity, nominal amplitude greater than 400 mV, and are matched to the DCC input impedance of 50 Ω . Following each “tick” of the DCC front-end clock, the DCC detects the input pulse if its leading edge crosses the 400 mV threshold, at which time a captured event is recorded for that channel. No other events will be recorded in this channel until the DCC detects that the input pulse’s trailing edge has fallen back below the threshold. The binary stream results from collecting 512 sequential records each representing one clock interval. This stream is compressed such that only the positions of the captured events are retained, and this compressed stream is passed to data acquisition/processing software running on a personal computer, where the event positions are extracted from the compressed stream.

The software’s data acquisition function allows the user to control, among other parameters, the number of active channels, the sample rate, the block size, and the total

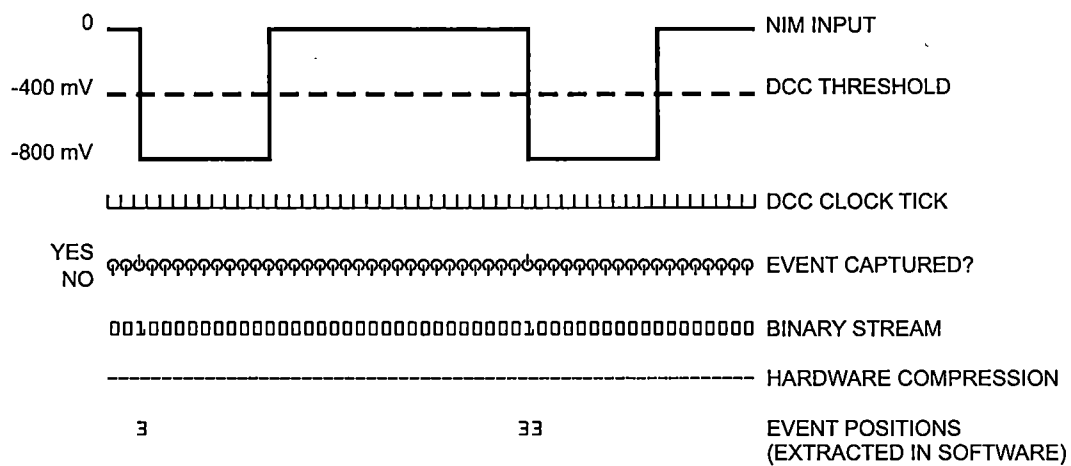
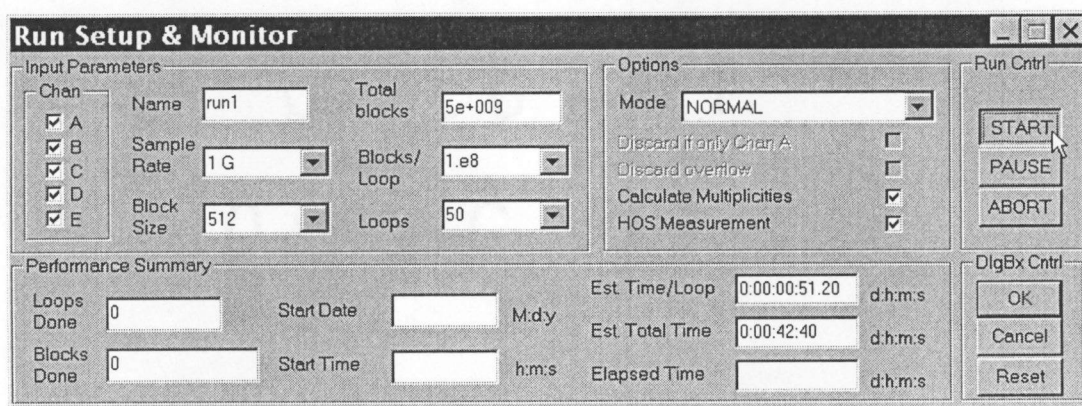


Figure 4.2 Illustration of the function of the data capture and compression (DCC) module in the Nuclear Weapons Identification System (NWIS)

number of blocks to be collected, as Fig. 4.3 illustrates. Currently NWIS can acquire up to five channels. The sample rate is adjustable between 15.625 MHz and 1 GHz; this control adjusts the speed of the DCC front-end clock. The block size is adjustable between 512 and 4096 samples per block; currently block sizes greater than 512 are generated in software by concatenating neighboring blocks of 512 samples. The number of blocks collected can be adjusted between 1 and 10^{14} . The processing function of the software accumulates second-order autocorrelations for all active channels and cross-correlations for each pair of channels using the time-difference algorithm described earlier. During and at the end of a measurement, these signatures are written to permanent storage, and an offline analysis code provides a graphical user interface that computes and displays the temporal signatures, spectral signatures, and a variety of other measured statistics.

A set of C functions implementing the higher order signature (HOS) estimation were developed and appended to the software's data acquisition routines. During each block, the main data acquisition/processing function passes the event position arrays for each channel to the HOS procedures where the signatures are accumulated using the time-difference algorithms previously described. Higher order analyses are activated from the dialog shown in Fig. 4.3, and the signatures to be collected are subsequently selected from the dialog shown in Fig. 4.4². The HOS procedures will only collect those signatures requested by the user. The user may also elect to collect fourth-order signatures. At the end of a

² Jim Mullens, the primary author of the NWIS data acquisition user interface, deserves special thanks for both attending to the details of appending the HOS procedures to the main code and for creating the dialog that allows the user to control which higher order signatures are collected.



The dialog box is titled "Run Setup & Monitor" and contains several sections for configuring a data acquisition run.

Input Parameters:

- Chan:** A list of channels A through E, each with a checked checkbox.
- Name:** A text field containing "run1".
- Sample Rate:** A dropdown menu set to "1 G".
- Block Size:** A dropdown menu set to "512".
- Total blocks:** A text field containing "5e+009".
- Blocks/Loop:** A dropdown menu set to "1.e8".
- Loops:** A dropdown menu set to "50".

Options:

- Mode:** A dropdown menu set to "NORMAL".
- Discard if only Chan A:** An unchecked checkbox.
- Discard overflow:** An unchecked checkbox.
- Calculate Multiplicities:** A checked checkbox.
- HOS Measurement:** A checked checkbox.

Run Cntrl:

- Buttons for "START", "PAUSE", and "ABORT". A mouse cursor is pointing at the "START" button.

Performance Summary:

- Loops Done:** A text field containing "0".
- Start Date:** A text field.
- Start Time:** A text field.
- Est Time/Loop:** A text field containing "0:00:00:51.20".
- Est Total Time:** A text field containing "0:00:42:40".
- Elapsed Time:** A text field.

DlgBx Cntrl:

- Buttons for "OK", "Cancel", and "Reset".

Figure 4.3 Dialog in NWIS data acquisition user interface that allows the user to control measurement parameters

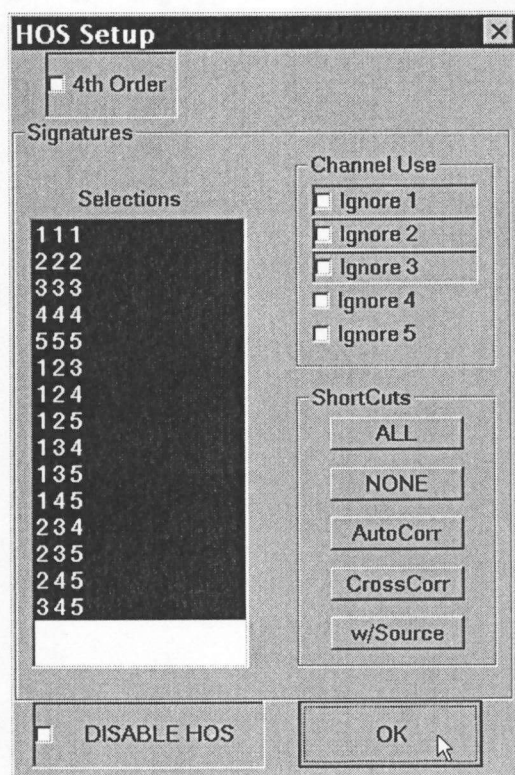


Figure 4.4 Dialog in NWIS data acquisition user interface that allows user to select higher order signatures to be collected

measurement, the accumulated signatures are recorded to permanent storage. A set of codes to process and analyze these signatures offline was developed. These were used to produce the plots of the higher order signatures presented later this chapter.

4.3 Measurement Performed on a High-Enriched Uranium-Metal Annular Casting

In order to demonstrate the implementation of HOS estimation procedures in NWIS, a measurement was performed on a highly enriched uranium-metal annular casting at the Oak Ridge Y-12 Plant. The photographs in Figs. 4.5 and 4.6 depict the arrangement of the ^{252}Cf source, the annular casting (inside a steel can), and the array of four fast plastic scintillation detectors used in the measurement. A similar casting, with the can open, is shown in Fig 4.7. The measured casting itself was fabricated of 93.15%-enriched uranium-metal at a density of approximately 18.74 kg/l and was cast as a 19.05-mm-thick 152.4-mm-tall right annulus with an outside diameter (OD) of 127 mm. Consequently, the mass of the casting is approximately 18.6 kg. The specific shape of the casting is designed to be inherently safe at this mass, i.e., a critical chain-reaction cannot be sustained in the casting even if it were optimally moderated and reflected. The casting was contained inside a 152.4-mm-OD 177.8-mm-tall stainless steel can with a wall thickness of 0.254 mm. A plastic bag surrounded the casting within the can.

Figure 4.8 shows elevation and plan views of the measurement arrangement. The measurement was performed atop a 6.34-mm-thick carbon-steel table approximately one

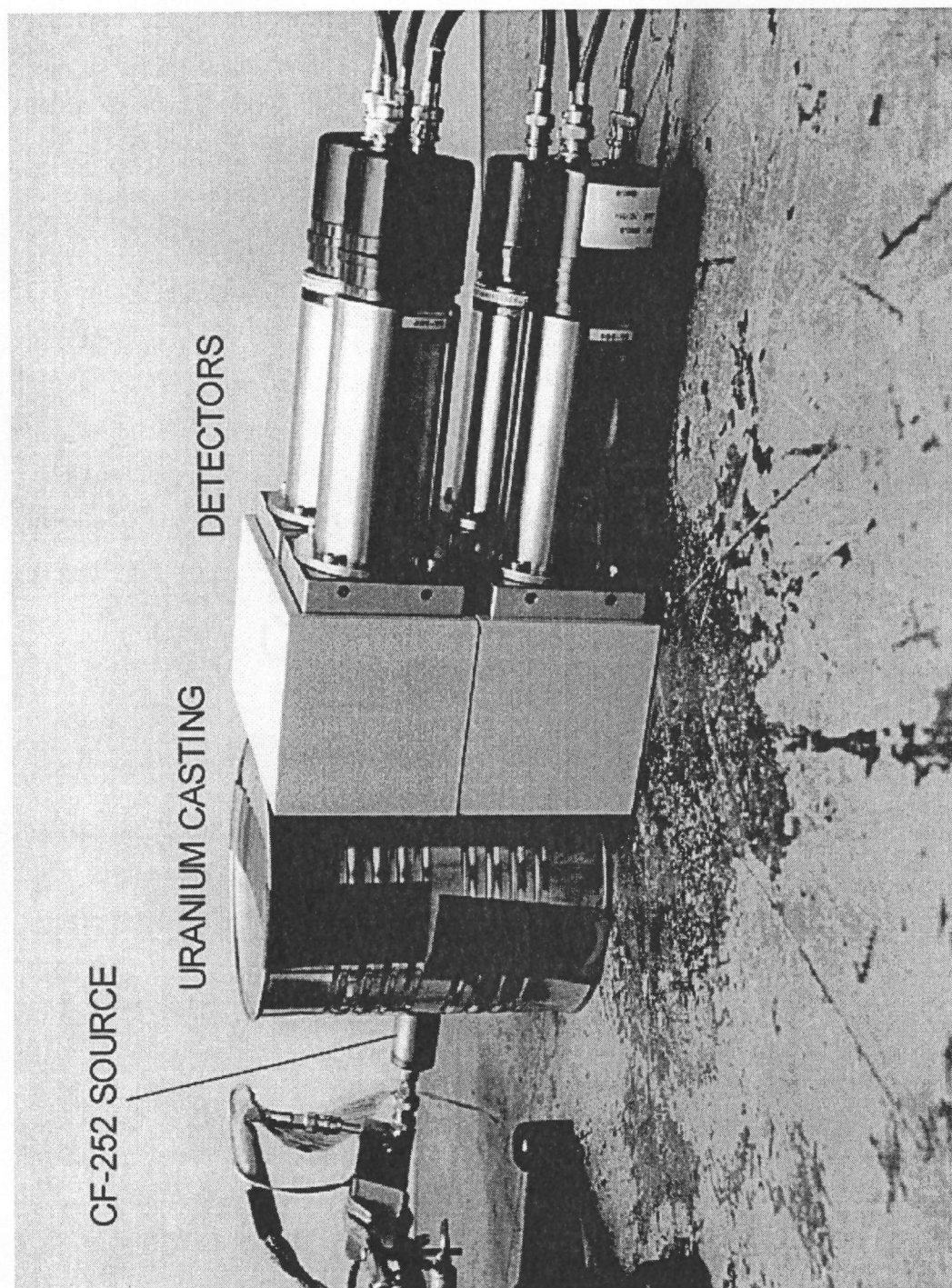


Figure 4.5 Casting measurement arrangement viewed from one side

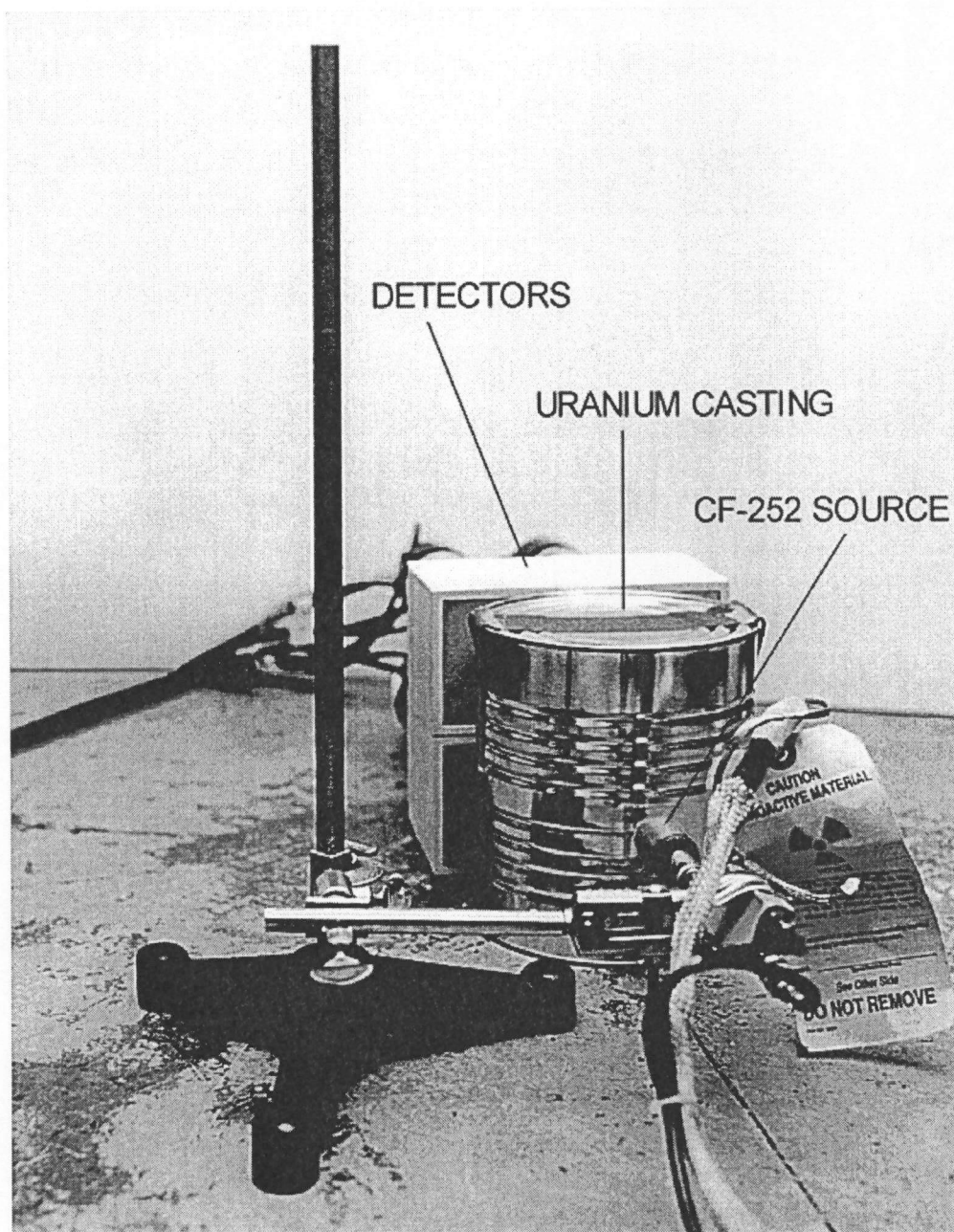


Figure 4.6 Alternative view of casting measurement arrangement

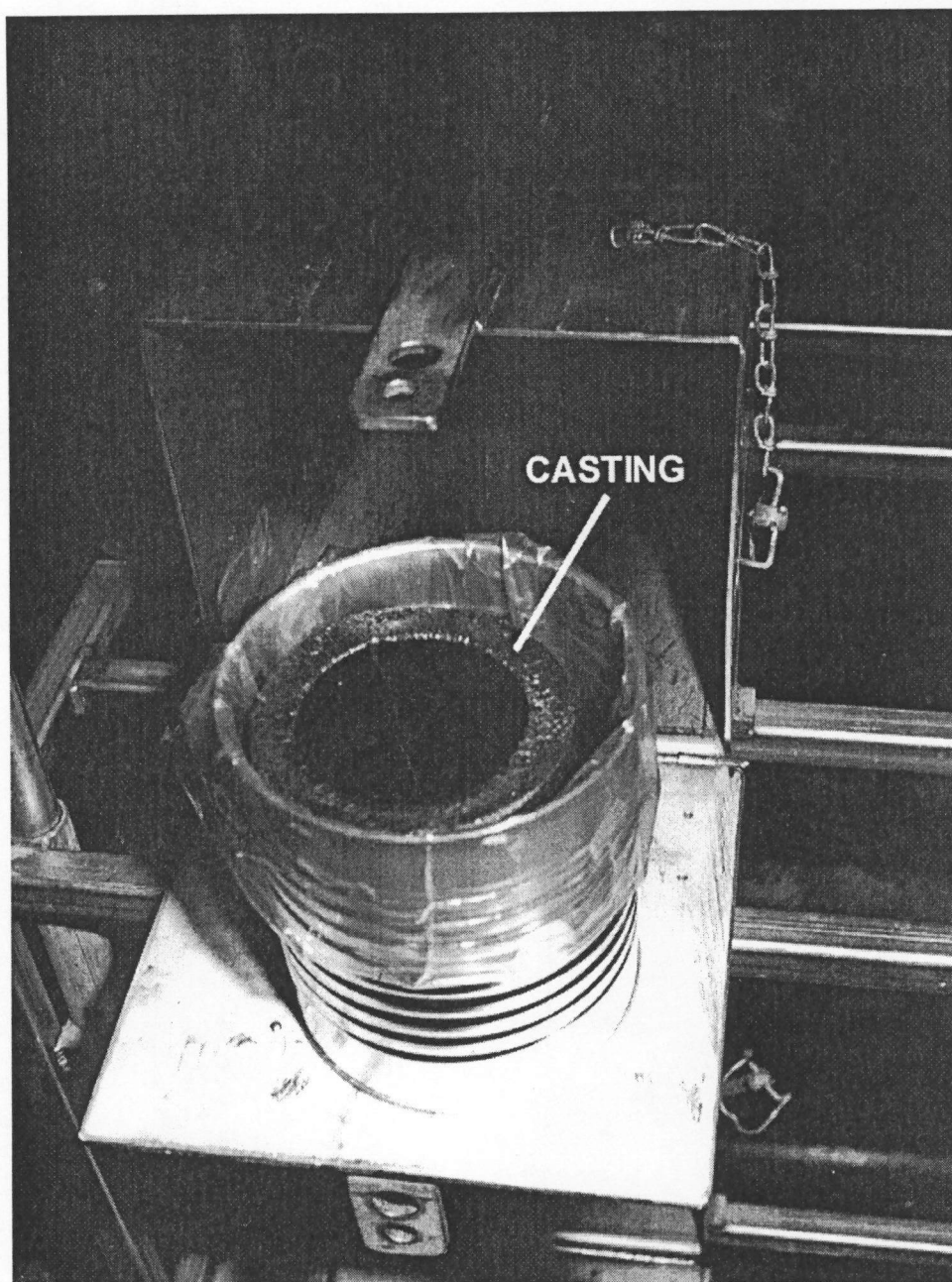


Figure 4.7 View of similar uranium-metal casting with can open

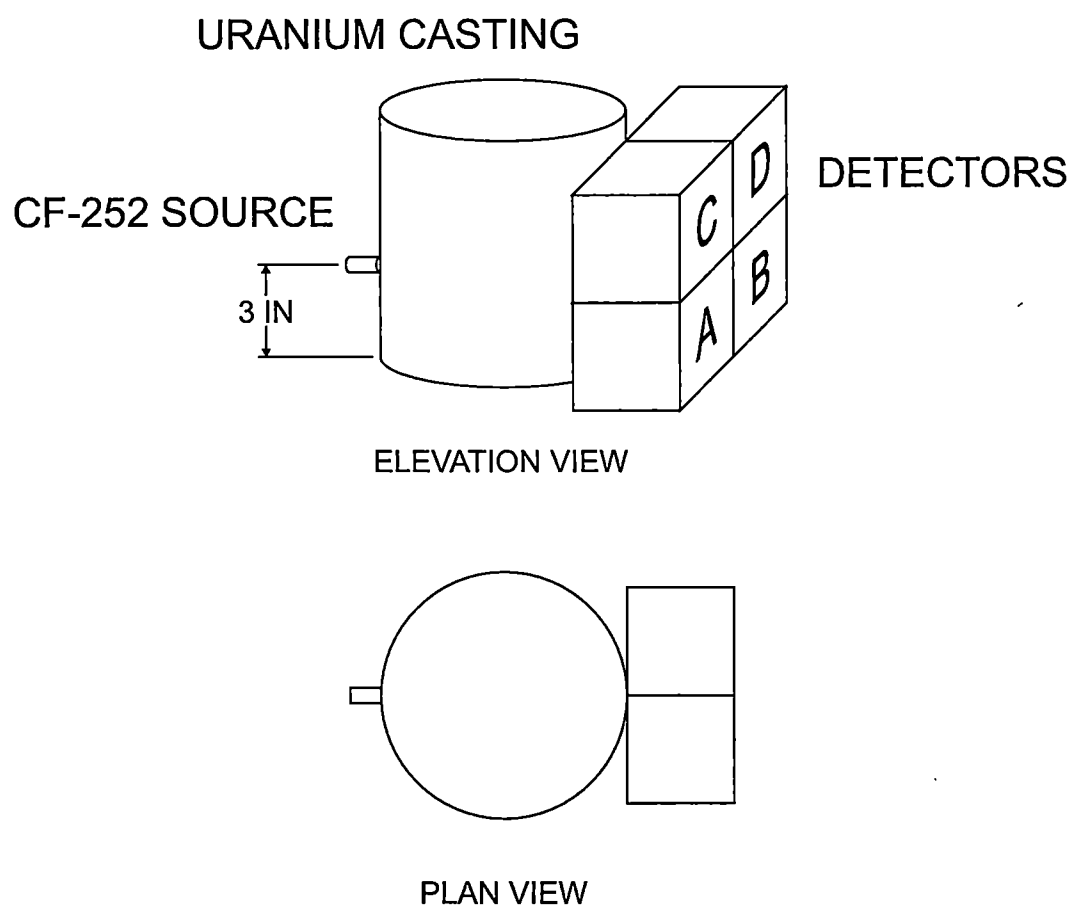


Figure 4.8 Elevation and plan views depicting the arrangement of the ^{252}Cf ion chamber, casting, and array of four plastic scintillators

meter above the concrete floor. The ^{252}Cf ion chamber was placed in contact with the can 76.2 mm above the table surface, which is approximately the vertical midplane of the casting. Prior to the measurement, the casting was shifted to the source side of the can by tilting the can until the casting was felt to slide to one side. This was done so that the location of the casting within the can during the measurement would be, more or less, known. The array of detectors rested on the table and were placed in contact with the can on the side opposite the ^{252}Cf ion chamber. Each detector was a Bicron type 420 (BC420) fast plastic scintillator with a sensitive volume of 69.85-mm $\times$ 69.85-mm $\times$ 101.6-mm-thick. The aluminum detector housings were 76.2-mm $\times$ 76.2-mm $\times$ 104.8-mm long, and each housing was surrounded by a 6.35-mm-thick shield fabricated from low-background lead. These shields do serve to reduce the background count rate, but more importantly they reduce the amount of "cross-talk" between detectors. This can occur due to gammas that either scatter from one detector to be counted in another or are produced by neutron capture in one detector but are counted in another. There is no practical way to completely prevent neutron cross-talk between detectors since they count fast neutrons. The lead shields probably mildly mitigate this effect via elastic and inelastic scattering, although inelastic scattering just serves to produce more gammas. However, neutron cross-talk is less significant than gamma cross-talk, in these detectors because they detect fast neutrons via scattering. Multiple counts due to a single neutron are rare because after the first scattering interaction the subsequent neutron energy is usually too low to produce any further counts.

Also note that the vertical midplane of the detector array was about 12 mm above the casting's vertical midplane, such that the detectors at the top of the array are slightly farther

away from the source and “see” a smaller fraction of the casting’s full height. This had a noticeable effect on the transmission of source particles to the detectors at the top of the array and also affected the number fission progeny counted by detectors at the top of the array. This difference between detectors at the top and bottom of the array will be evident in many of the following signatures.

The schematic shown in Fig. 4.9 depicts the layout of instrumentation employed in the measurement. Bias voltage was applied to the ^{252}Cf ion chamber by a bias voltage (BV) supply. The high gain fast amplifier designed and built at the Oak Ridge National Laboratory performs some pulse-shaping on the chamber output, amplifies the shaped output, and prevents significant contamination by electrical noise. Power necessary for the amplifier’s operation was provided by a stabilized +15 VDC supply (DC). The fast amplifier’s output was input to a constant fraction discriminator (CFD) whose output was in turn input to the first channel of the NWIS DCC.

Each scintillator was coupled to a twelve-stage photomultiplier tube (PMT). Bias voltage was supplied to each PMT by a custom tube base which in turn received power from a high voltage (HV) supply. The anode signal of each detector was directed into its own CFD, and each CFD’s output was in turn input to the remaining four channels of the NWIS DCC. Connections were made such that detectors A, B, C, and D mapped to DCC channels 2, 3, 4, and 5, respectively. The CFD’s were powered by an $\pm 6/12/24$ VDC NIM bin. Not shown in the schematic is a passive analog delay box that was used to increase the source channel’s transit time to the DCC. This was done to adjust for the longer transit time

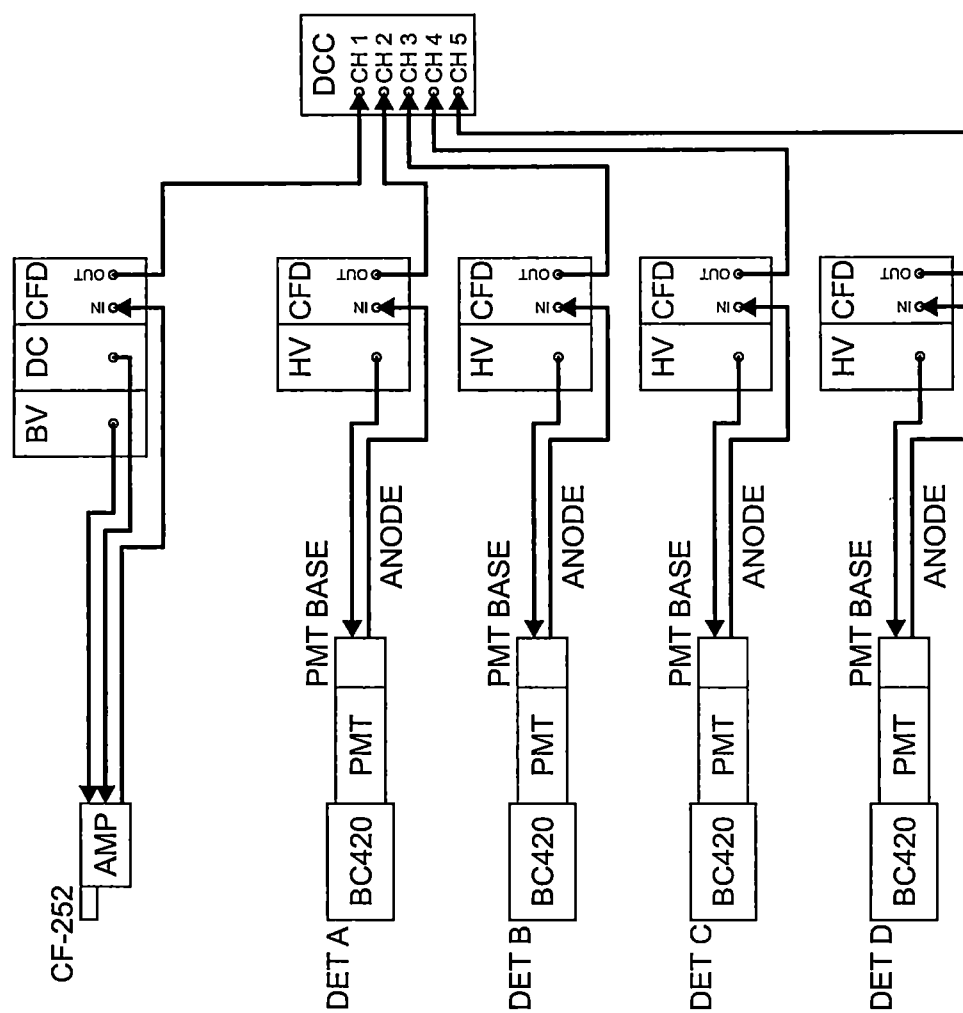


Figure 4.9 Schematic depicting layout of instrumentation employed in casting measurement

observed in the detector channels due to PMT electron multiplication and pulse generation, which takes approximately thirty nanoseconds.

Several weeks prior to the measurement, the CFD threshold for the ^{252}Cf source was set by pulse height analysis. The ion chamber is an extremely stable device, so it was unnecessary to repeat this setup immediately prior to the measurement. At the time of the measurement, the ion chamber count rate indicated that the source was approximately $0.76\ \mu\text{g}$ ^{252}Cf -mass, if it is assumed that the source CFD threshold was set such that nearly all fissions were counted and nearly all α -decays were discriminated.

A few days before the measurement, the detector bias voltages and CFD thresholds were coarsely set using a ^{137}Cs source. The bias voltages were set for each detector such that the Compton edge due to 662 keV gammas from ^{137}Cs corresponded to a pulse height of approximately 300 mV. The CFD thresholds were subsequently set at approximately 100 mV. This was done to ensure that the Compton edge due to 186 keV gammas emitted by ^{235}U during α -decay was below the threshold. Otherwise, the detectors would be overwhelmed by the deluge of 186 keV gammas emitted by the enriched uranium casting. Still, some elevated background due to pileup of 186 keV gammas was inevitable. In addition, the background was further elevated due to 1 MeV gammas emitted by ^{238}U during α -decay. At this time, each detector autocorrelation was measured with the ^{137}Cs source on the detector face to ensure that each detector and its associated PMT and base exhibited no temporal anomalies like ringing pulses or reflected pulses due to impedance mismatches. Just prior to the measurement, the voltage and threshold settings were fine-tuned by measuring each detector's neutron efficiency as a function of energy using ^{252}Cf neutrons.

By performing a time-of-flight measurement with the detectors separated from the source by one meter, the neutron counting efficiency as a function of energy may be determined as follows. Note that at one meter separation, gammas take approximately 3.3 nsec to travel from the source to the detector. The neutrons arrive at a distribution of times congruent to the ^{252}Cf fission neutron spectrum. By measuring the source-detector covariance for each detector, this distribution of arrival times folded with the detectors' intrinsic response, i.e., efficiency, is measured. Figure 4.10 shows one source-detector covariance derived from such a measurement. The narrow peak in the correlation is due to gamma counting events. The width of this peak indicates the timing resolution of the measurement. The broader peak appearing later in the correlation is due to neutron counting events. The gamma peak provides a known timing mark; deviations in the location of this peak from detector to detector are primarily due to differences in electronic transit time. Several days prior to the measurement, cable lengths were adjusted such that the gamma peak in each source-detector covariance measured in air appeared between 3 and 4 nsec. However, it's difficult to adjust the transit time more precisely than that because only two additional inches of cable can introduce a quarter nanosecond delay, and the optimal timing resolution at a sample rate of 1 GHz is at best one nanosecond.

Consider that each time "bin" under the neutron peak corresponds to the distribution of neutron arrival times folded with the detector's efficiency integrated from the beginning of the bin to its end. By converting this integrated distribution to reflect neutron kinetic energies, adjusting for the solid angle subtended by the detector, and dividing the resultant

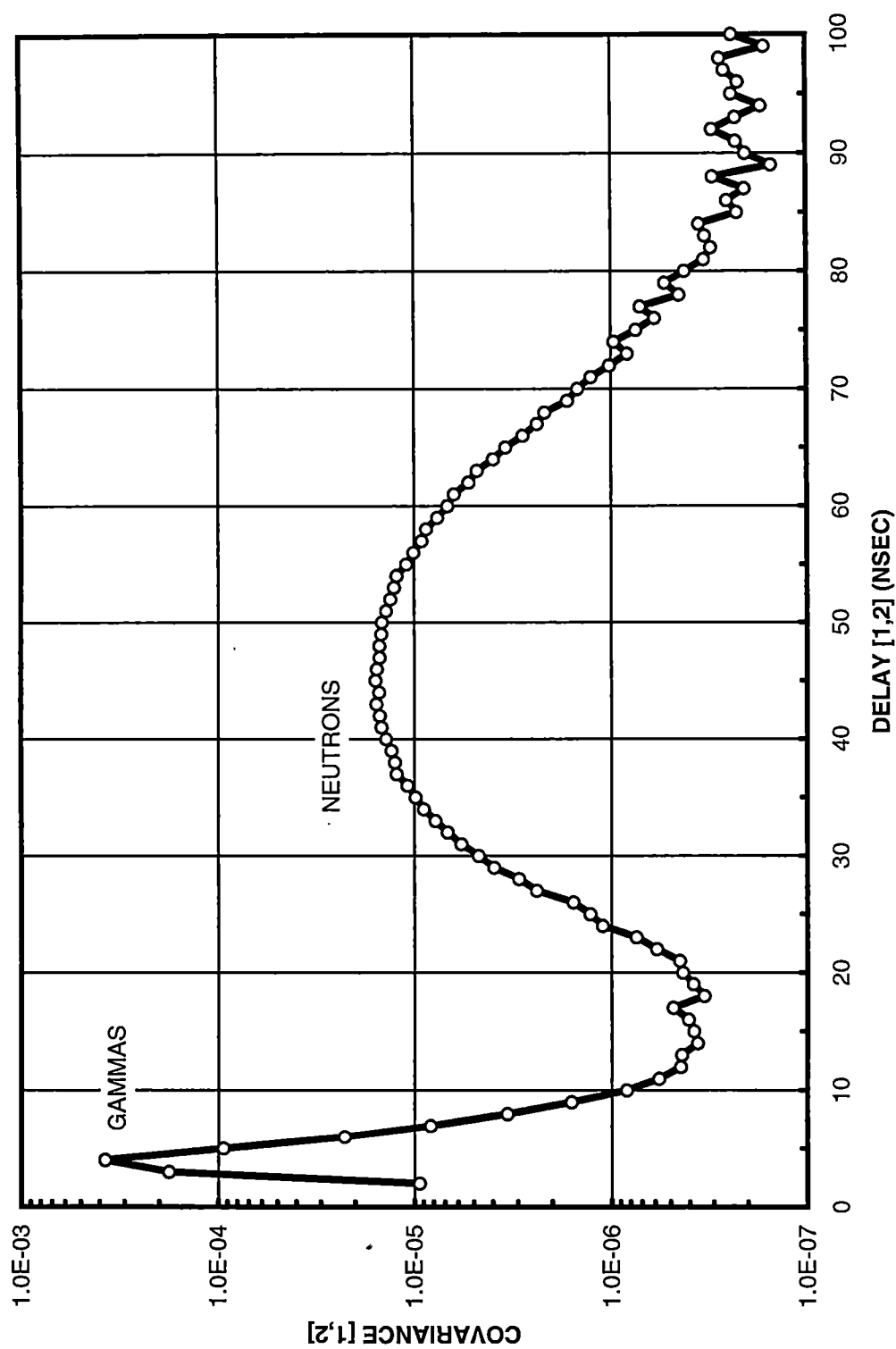


Figure 4.10 Source-detector covariance acquired during a time-of-flight measurement with the source and detector separated by one meter

distribution by the known ^{252}Cf fission spectrum, the detection efficiency in counts per incident neutron is determined as a function of incident neutron energy. This diagnostic function is one of the primary services provided by the NWIS analysis software, so it was applied iteratively until adjustments of detector voltages and thresholds yielded approximately the same efficiency and energy threshold for each detector. The peak neutron efficiency of each detector was observed to be approximately 45% at roughly 2.5 MeV. The neutron threshold, i.e., the energy at which the neutron efficiency was at half its maximum value, was observed to be at roughly 1.5 MeV for each detector.

The measurement of the casting itself was performed immediately after all voltage and threshold adjustments were completed. During the measurement of the casting, the ion chamber and detector signals were sampled by NWIS at a rate of 1 GHz, and all third-order HOS signatures – fifteen in all – were collected. The blocks size employed was 512 samples, and 5×10^9 blocks were collected. Consequently the following signatures were estimated from 2.56×10^{12} observations acquired at 1 nsec intervals. At maximum throughput, this measurement would require at least 512 nsec per block. Typical throughput with HOS estimation deactivated was observed to be 1.2 μsec per block, or about 42%. With HOS estimation activated, throughput was about 2.9 μsec per block – roughly 18%. The observed decrease in throughput was primarily due to the measurement of the source autocorrelation.

Figure 4.11 plots the measured source autocorrelation. All signatures are normalized to the measured source fission rate, hence the source autocorrelation is one at a delay of zero.

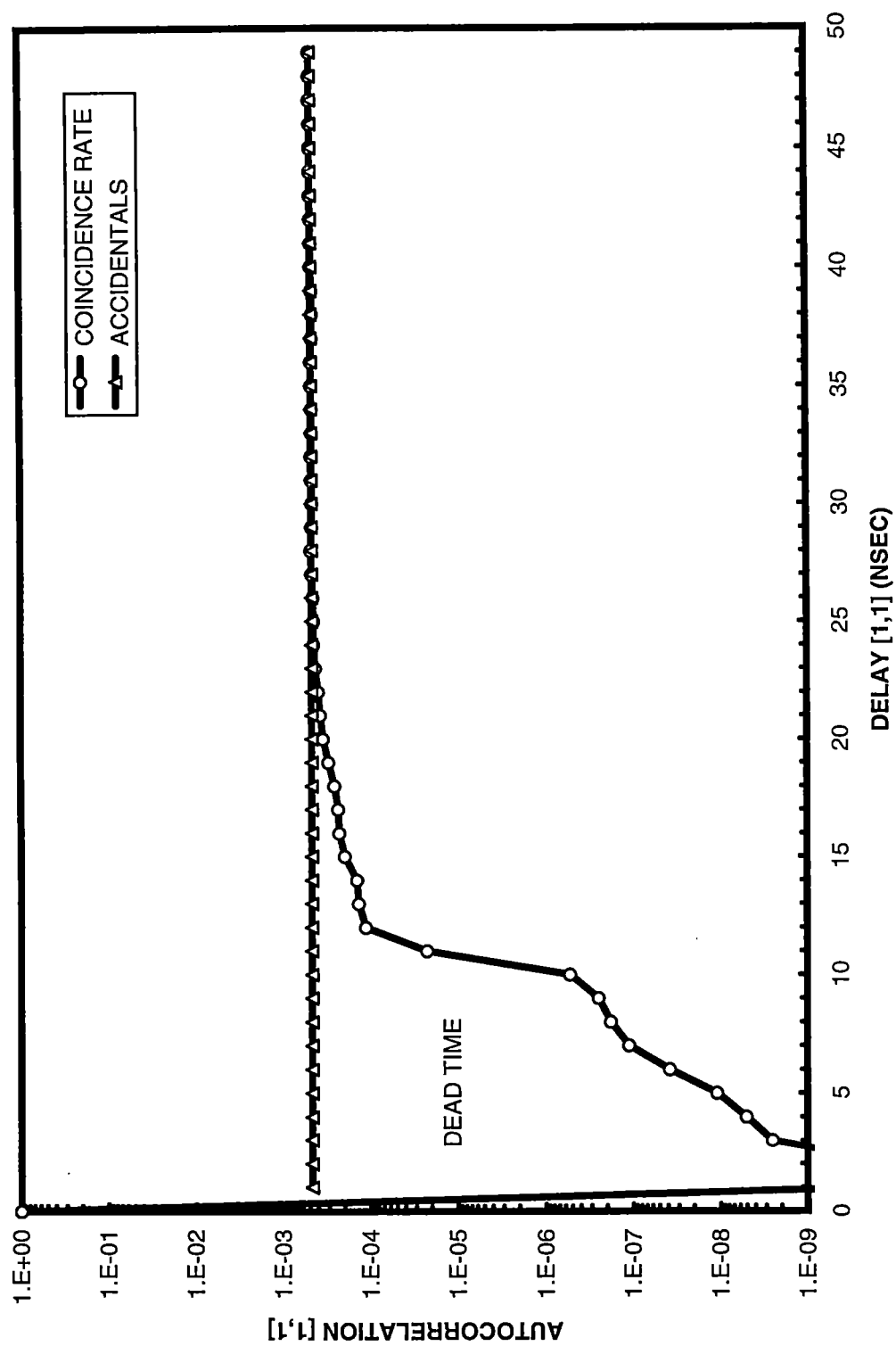


Figure 4.11 Measured source autocorrelation in coincident ion chamber counts per ion chamber count per nsec

The “well” immediately following zero delay is due to the dead time imposed by the nominal 20 nsec CFD pulse width set for the source channel. The distribution of observed dead times is due to variations in the CFD response time following each output pulse. At long delays the source autocorrelation is simply the accidental coincidence rate. Any temporal anomalies like ringing pulses, reflections due to impedance mismatches, or double-pulsing from the CFD would be evident as spikes in the source autocorrelation at long delays. No such anomalies appeared. Essentially the same information is revealed by the source autobicorrelation as Fig. 4.12 indicates. The “spike” at the origin is just the measured fission rate (normalized to itself). The “ridges” along the two axes and along the diagonal are due to accidental coincidence between two fissions. Note that the dead time distribution is again evident about the spike at the origin and along the accidental coincidence ridges. The “flat” areas on the rest of the plane reflect the rate of accidental bicoincidence between three fissions. Because both the source autocovariance and autobicovariance are zero everywhere except at the origin, the corresponding spectral signatures are completely featureless as Figs. 4.13 and 4.14 illustrate. Furthermore, even though the source autobispectrum is formally complex, its phase does not significantly deviate from zero, as Fig 4.15 illustrates.

Now note from Table 4.1 that the source count rate was more than 30 times greater than the average detector count rate. Consequently, the HOS accumulation procedures, as well as the standard second-order procedures, spent most of their time counting source fissions. Although the source autocorrelation measurement should probably be retained for its diagnostic value, the source autobicorrelation is a waste of valuable throughput; it

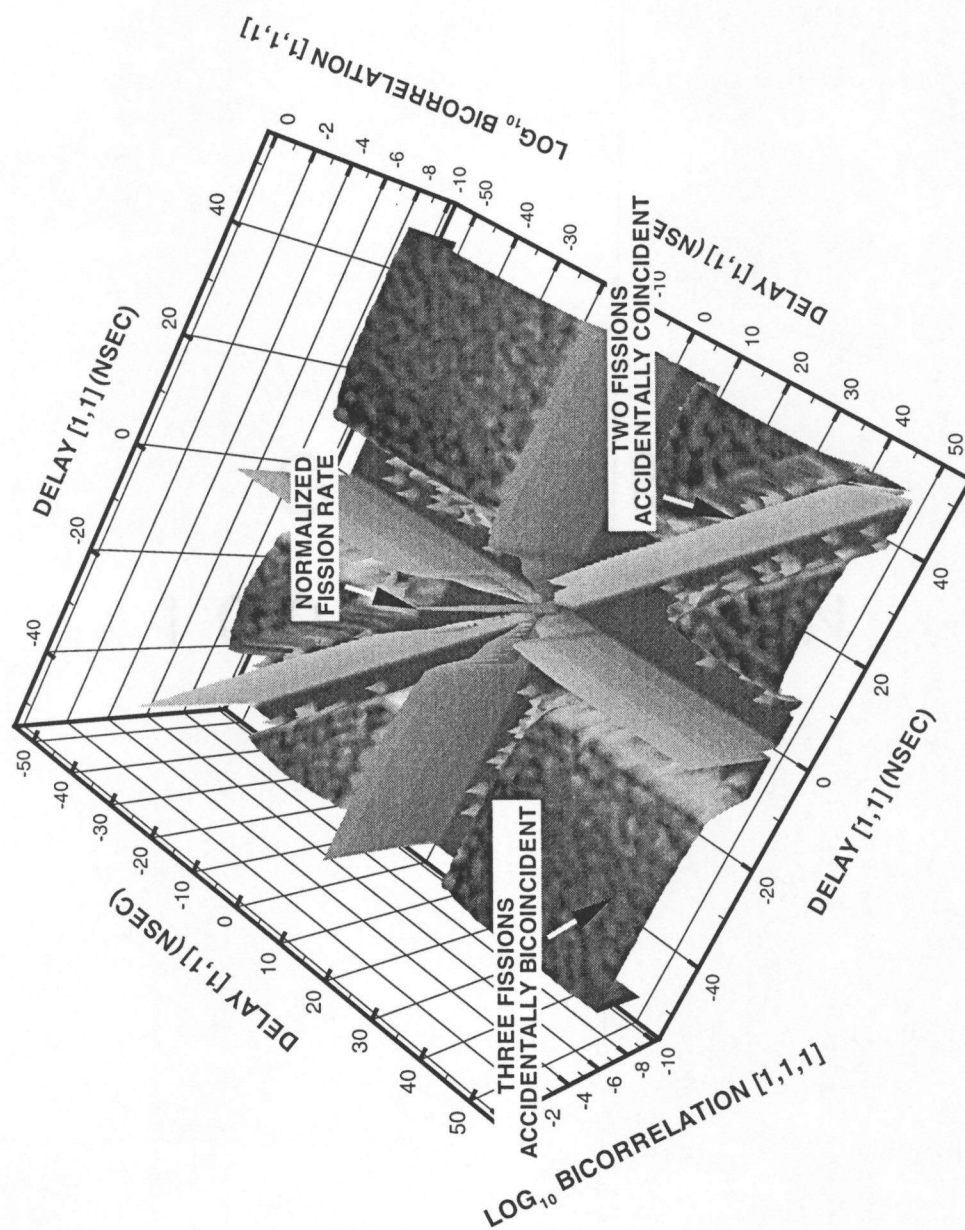


Figure 4.12 Measured source autocorrelation in bicoincident ion chamber counts per ion chamber count per nsec^2

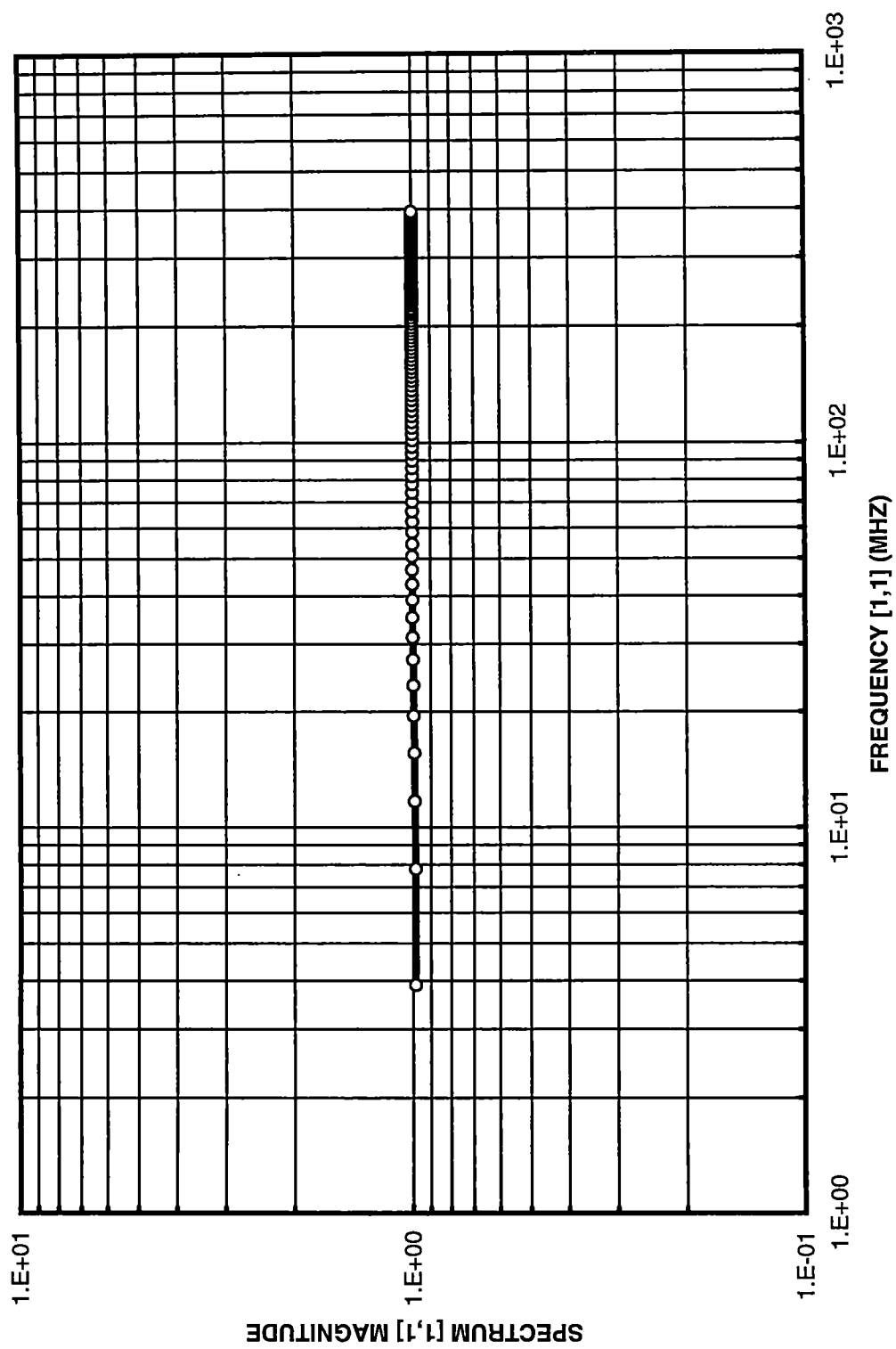


Figure 4.13 Measured source autospectrum in correlated ion chamber counts per ion chamber count

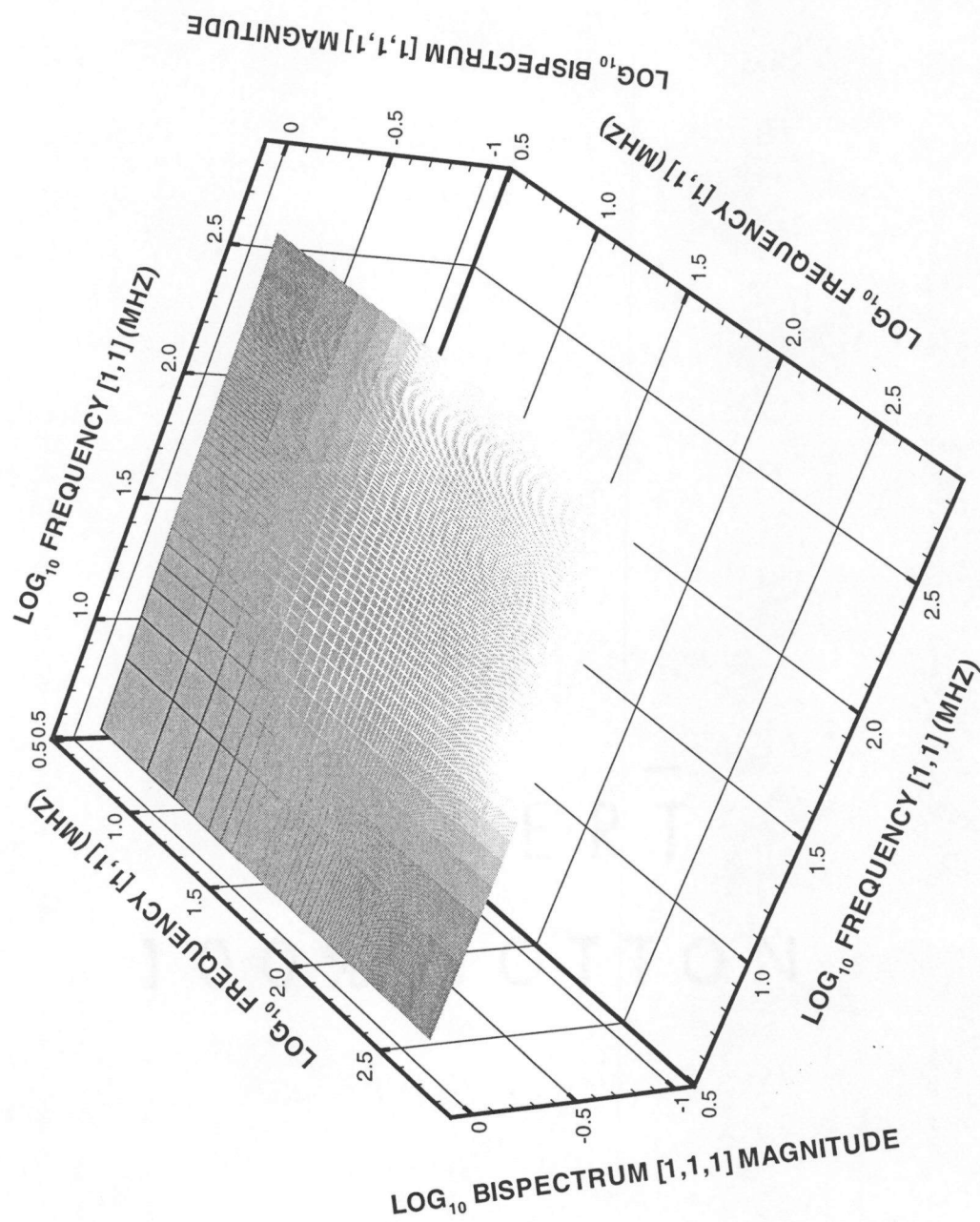


Figure 4.14 Measured source autobispectrum magnitude in bicorrelated ion chamber counts per ion chamber count

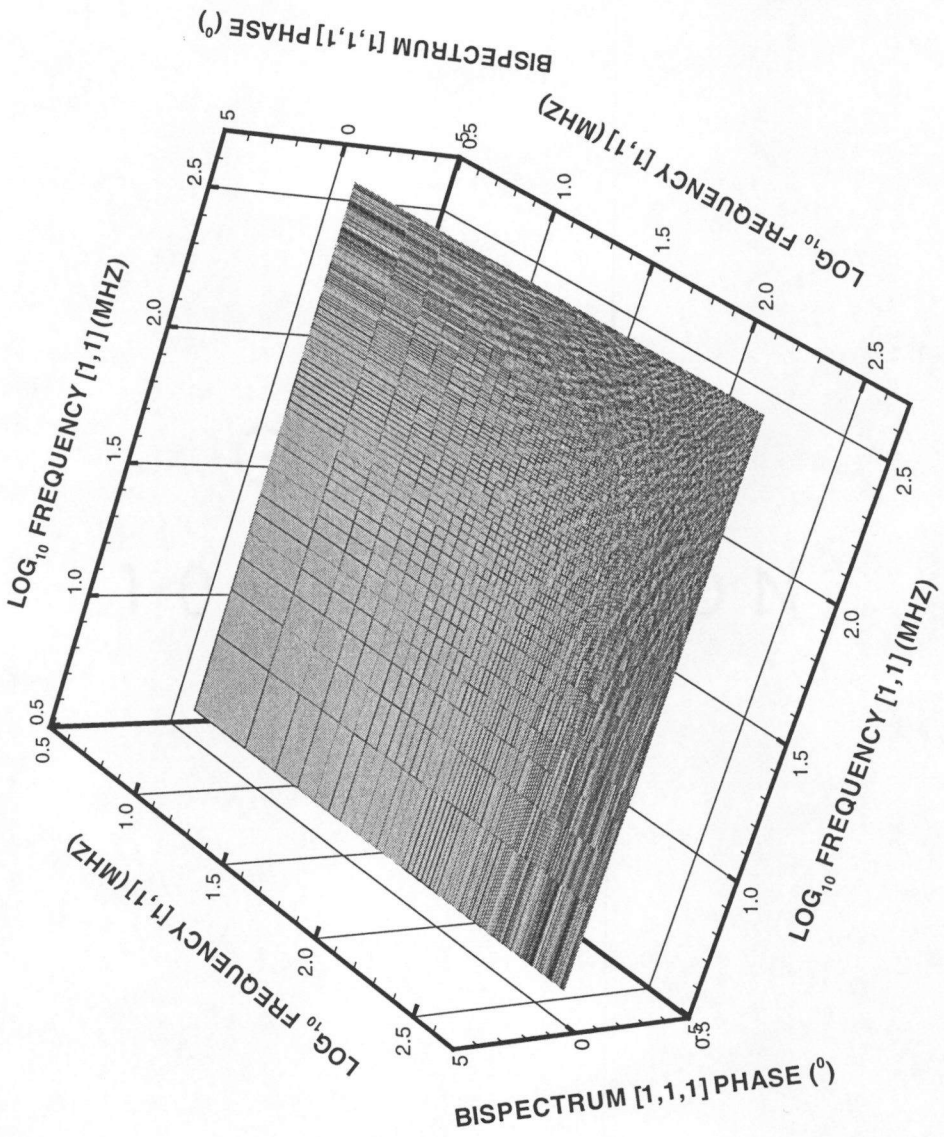


Figure 4.15 Measured source autobispectrum phase

Table 4.1 Count rates observed during measurement of uranium casting

	source	detectors
average counts per block	0.240	0.007

contains no more information than is revealed by the source autocorrelation. Elimination of the source autocorrelation would have dramatically increased throughput during this measurement.

The covariance between the source and detector 2 is plotted in Fig. 4.16, where several features of this signature are noted. But first observe that, relative to the time-of-flight measurement shown in Fig 4.10, the time-scale is significantly condensed. This is of course because the time-of-flight measurement was performed with the source and detector separated by one meter; the casting measurement was performed with the source and detector only 152 mm apart. Further observe that the casting strongly attenuates the transmission of gammas – they must pass through roughly 38 mm of uranium to reach the detectors. In contrast, fast neutrons pass through the casting with little interaction. At long delays, the distribution primarily consists of fission neutrons and gammas, though there is some contribution due to scattered neutrons and gammas and due to secondary gammas produced in capture and inelastic scattering events. However, the probability of these interactions is much lower than that of fission, so the distribution of scattered neutrons and gammas and secondary gammas is not visible in this signature.

Figure 4.17 depicts this signature compared to the source-detector covariance

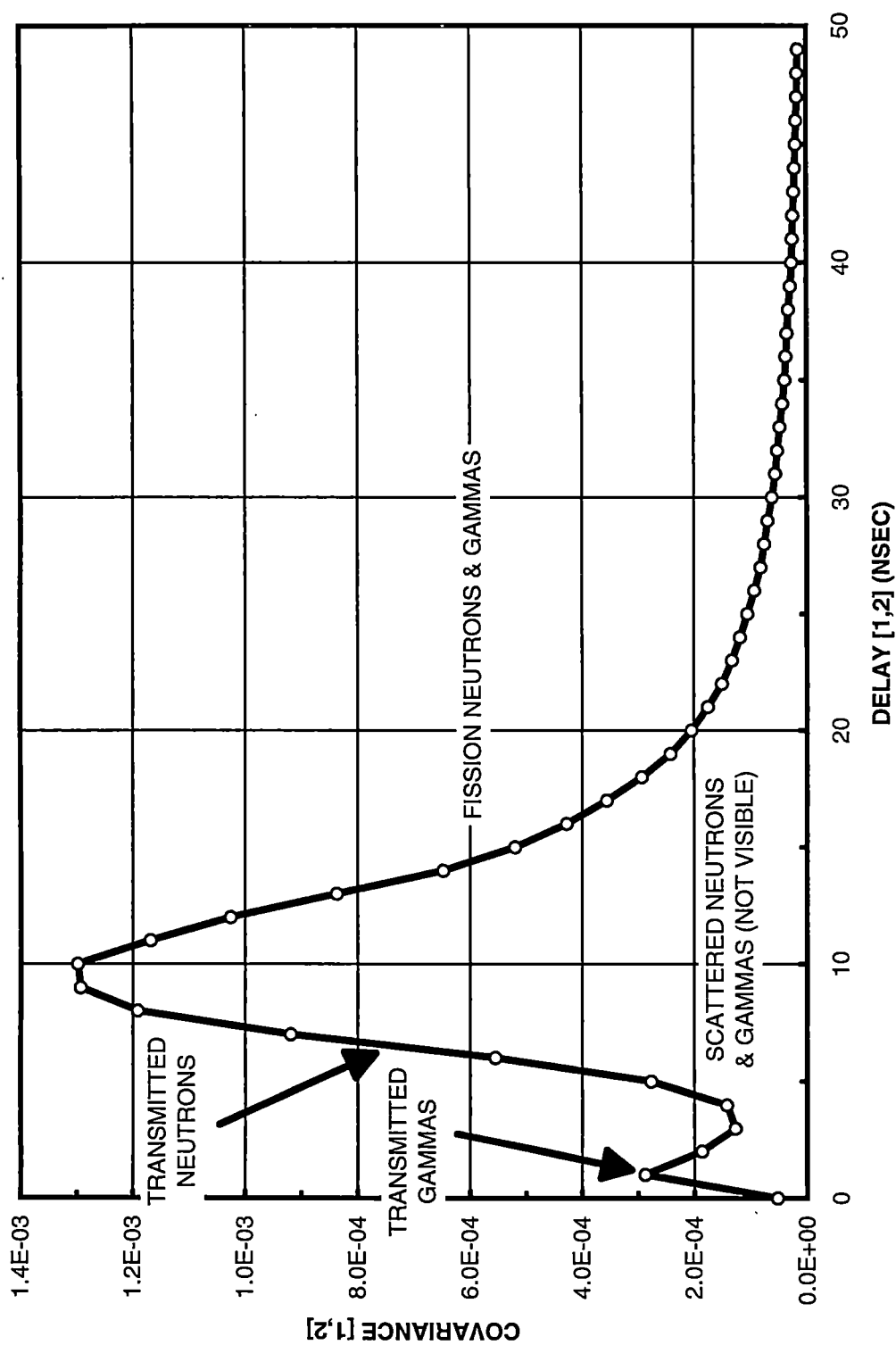


Figure 4.16 Measured source-detector covariance [1,2] in detector counts per ion chamber count per nsec (alternatively, counts/ ^{252}Cf -fission/nsec)

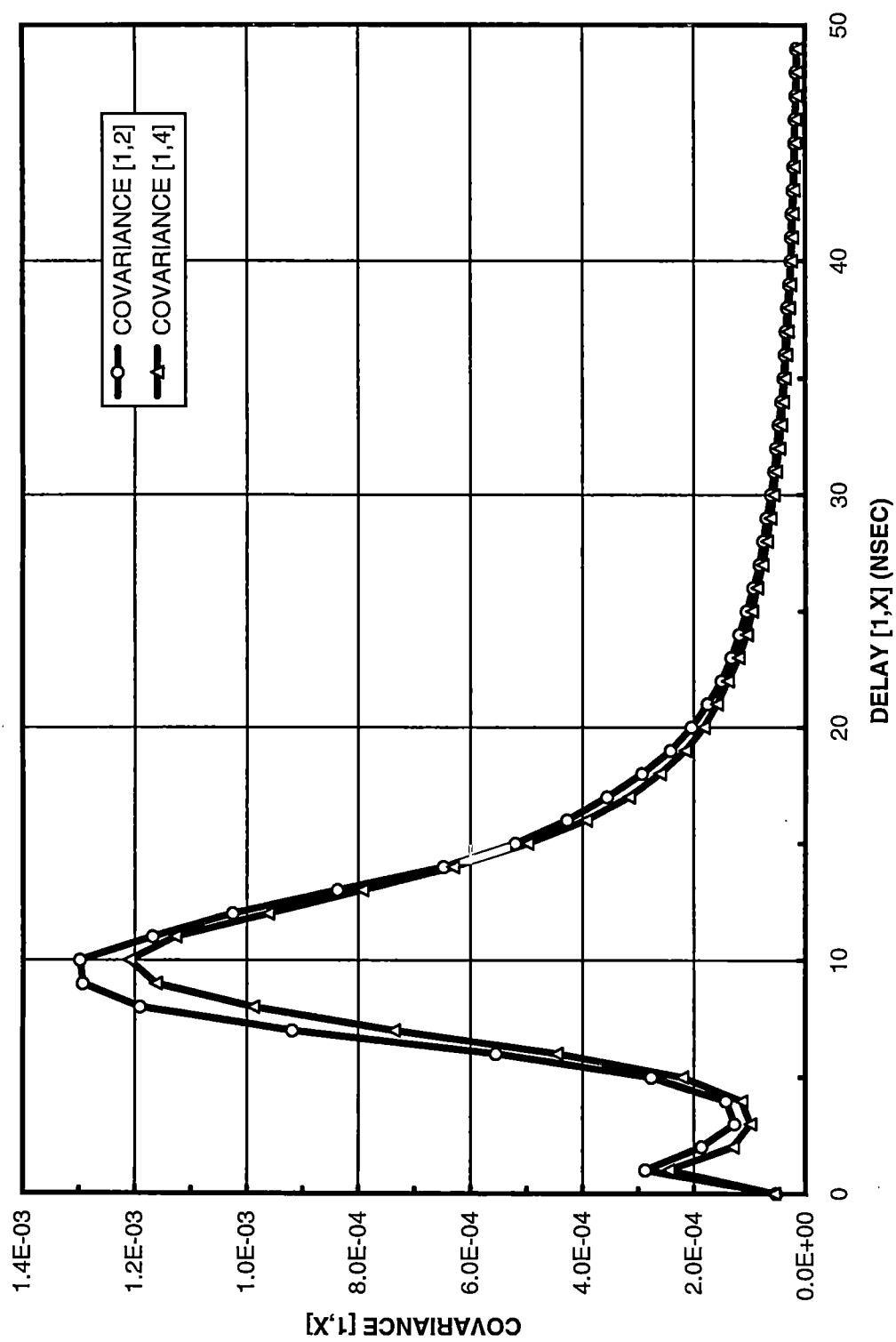


Figure 4.17 Comparison of source-detector covariances measured using detector 2 (at the top of the detector array) and detector 4 (at the bottom of the detector array)

acquired from detector 4, which was directly above detector 2. The measurement exhibited left-right symmetry such that covariances [1,2] and [1,3] were essentially identical, as were covariances [1,4] and [1,5]. However observe that the covariance between the source and a detector at the bottom of the array is noticeably different from that between the source and a detector at the top of the array. First, gamma transmission to the top detector is relatively lower. This is of course due to the slightly longer distance traveled to reach the top detector. This also has a noticeable effect on neutron transmission. Consequently it can be concluded that neutron scattering in the casting is not entirely negligible. However, observe that neutron and gamma production via fission results in similar distributions as “seen” by both the bottom and top detectors. Hence it may be concluded that the rate of induced fission was similar in both the bottom and top half of the casting.

The similarity between the distribution of fission progeny seen by both the bottom and top detectors is evident in the low-frequency magnitude of the corresponding source-detector spectra plotted in Fig 4.18. The visible difference is primarily due to the lower rate of neutron transmission observed by the top detector. In contrast, note that the lower rate of gamma transmission observed by the top detector is evident at high frequencies in the cross-spectrum magnitude. It might be concluded that the slightly longer transit time required for transmitted gammas to reach the top detector is also evident in the phase of the source-detector spectra plotted in Fig 4.19. However, such a conclusion should be viewed with skepticism – a small difference in the electronic transit time for channels 2 and 4, primarily due to differences in cable length, is the more probable cause of this effect.

Observe that a slight difference in electronic transit time is evident in between-

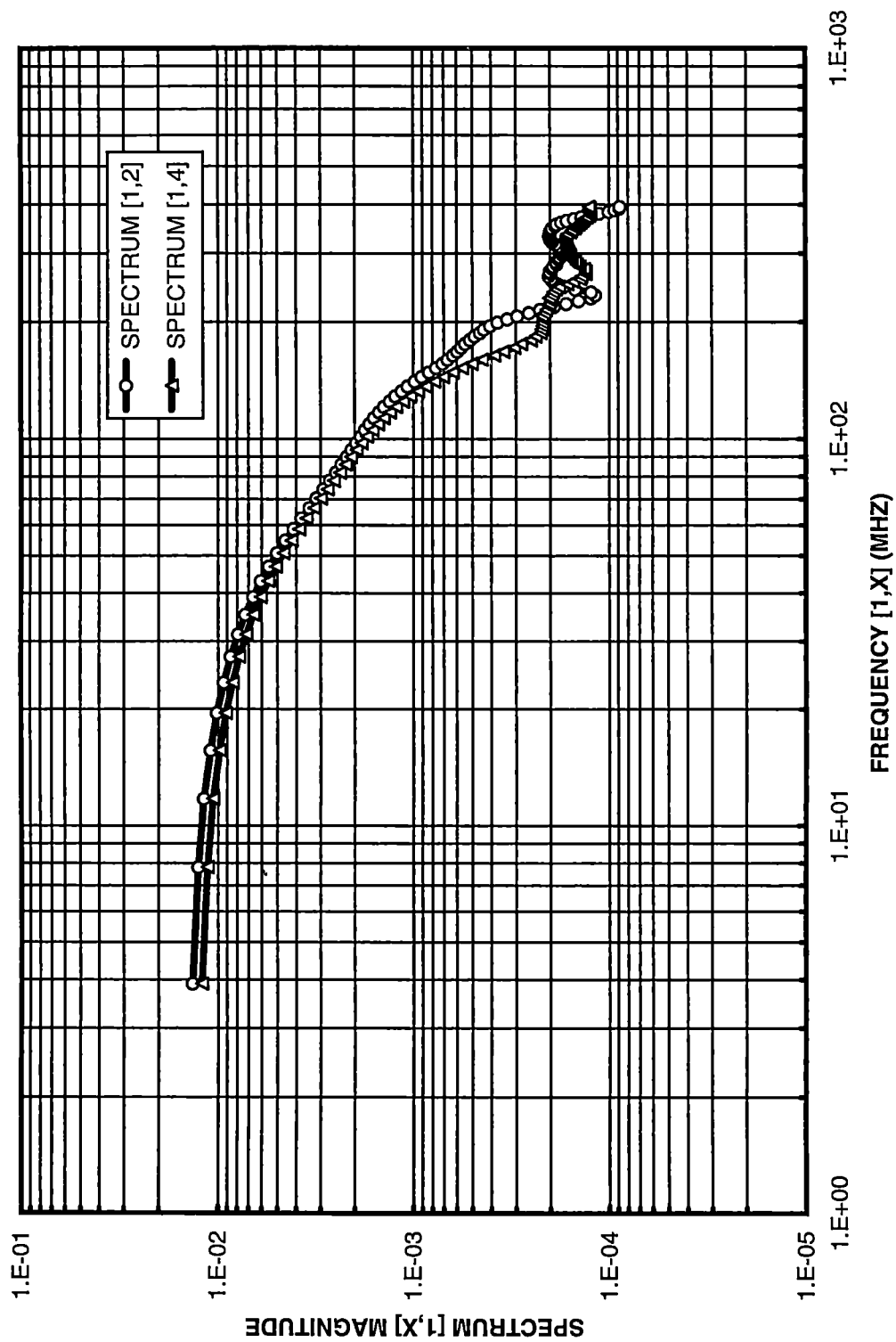


Figure 4.18 Comparison of source-detector spectra magnitudes (in counts/ ^{252}Cf -fission) measured using “bottom” detector 2 and “top” detector 4

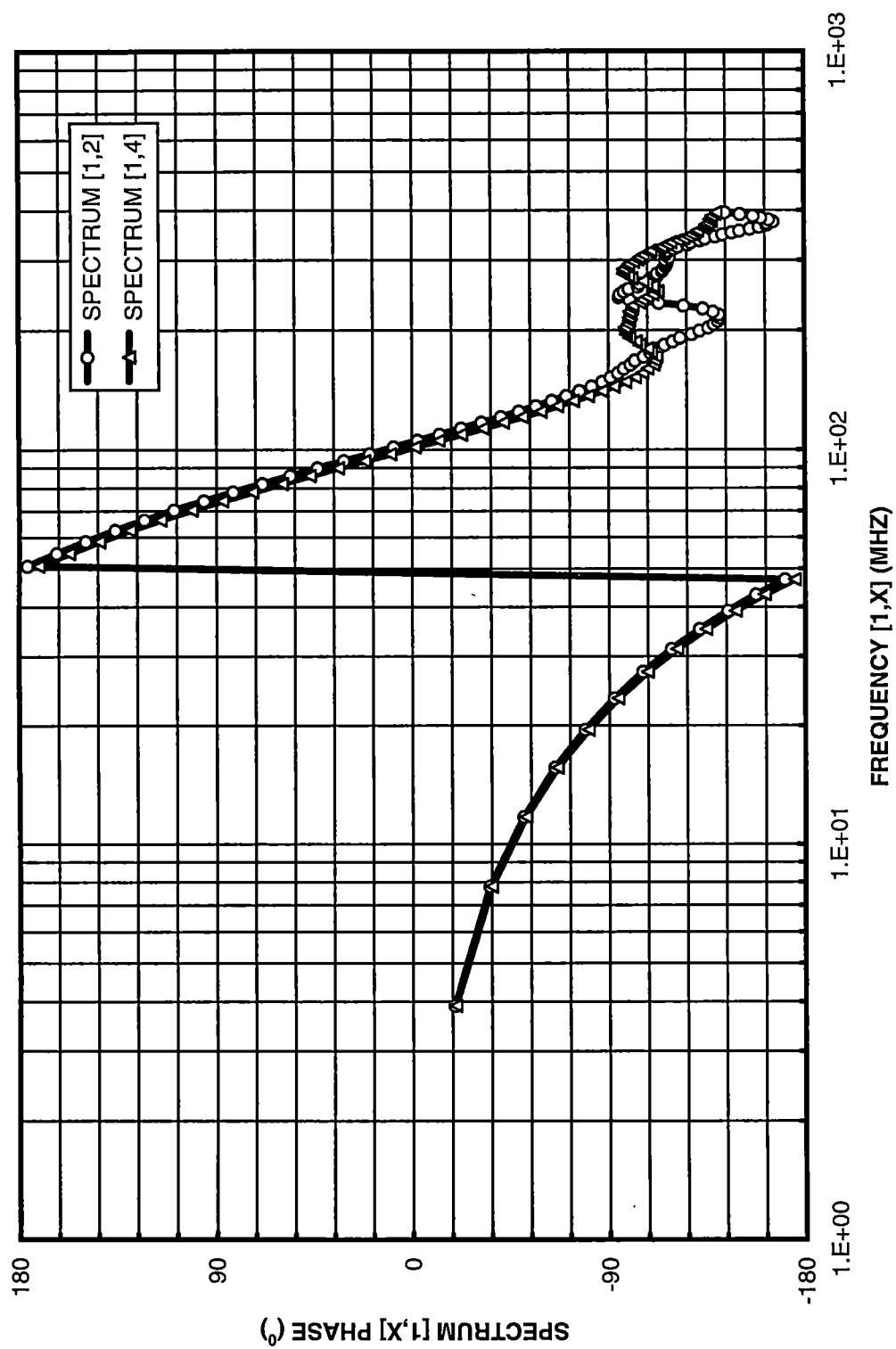


Figure 4.19 Comparison of source-detector spectra phases measured using bottom detector 2 and top detector 4

detectors covariance [2,3] plotted in Fig 4.20. These two detectors are left-right symmetric with respect to the source location, so the asymmetry evident in this signature is probably due to a minute difference in cable length. Observe that the signature exhibits two prominent features. The narrow distribution about zero delay is due to pairs of transmitted source gammas and fission gammas, which tend to arrive at these two detectors in rapid succession since they travel at the speed of light. The broader distribution about zero delay is primarily due to transmitted source neutrons and fission neutrons since these arrive at the detectors with a distribution of speeds. Furthermore, it is important to note that because a gamma and a neutron from the same source or induced fission event are correlated, the broader distribution also reflects the disparity between the speed of transmitted or fission gammas and neutrons. However, this signature provides no means to distinguish between contributions due to neutron-neutron pairs versus gamma-neutron pairs.

Observe that the relative positions of the source and detectors about the casting had a significant effect on the between-detectors covariance. This effect is clear in Fig. 4.21. The disparity between neutron and gamma transmission to the lower and upper detectors is evident in the difference between covariance [2,3], where both detectors are oriented on the lower half of the detector array, and covariance [4,5], where both detectors are oriented on the upper half of the array. Recall that because this signature arises from the detection of correlated pairs of events, it is more sensitive to the transmission and fission rates than is the source-detector signature. Note that the electronic transit time in channel 5 was roughly one-half nanosecond longer than that in channel 4, hence covariance [4,5] appears to be shifted to the right by this quantity. Further note that the covariance between detectors 3 and 4,

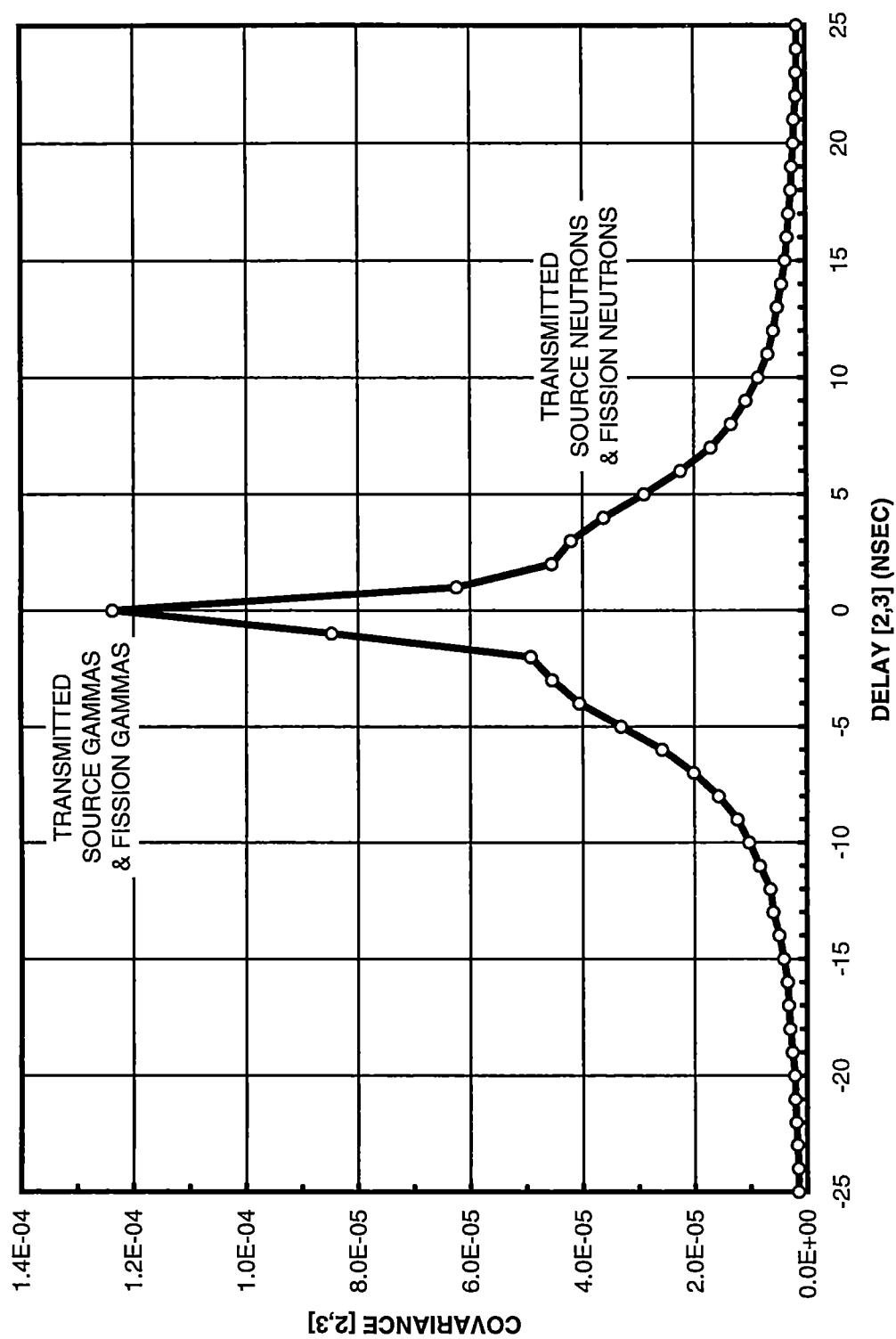


Figure 4.20 Measured between-detectors covariance [2,3] in correlated pairs/ ^{252}Cf -fission/nsec

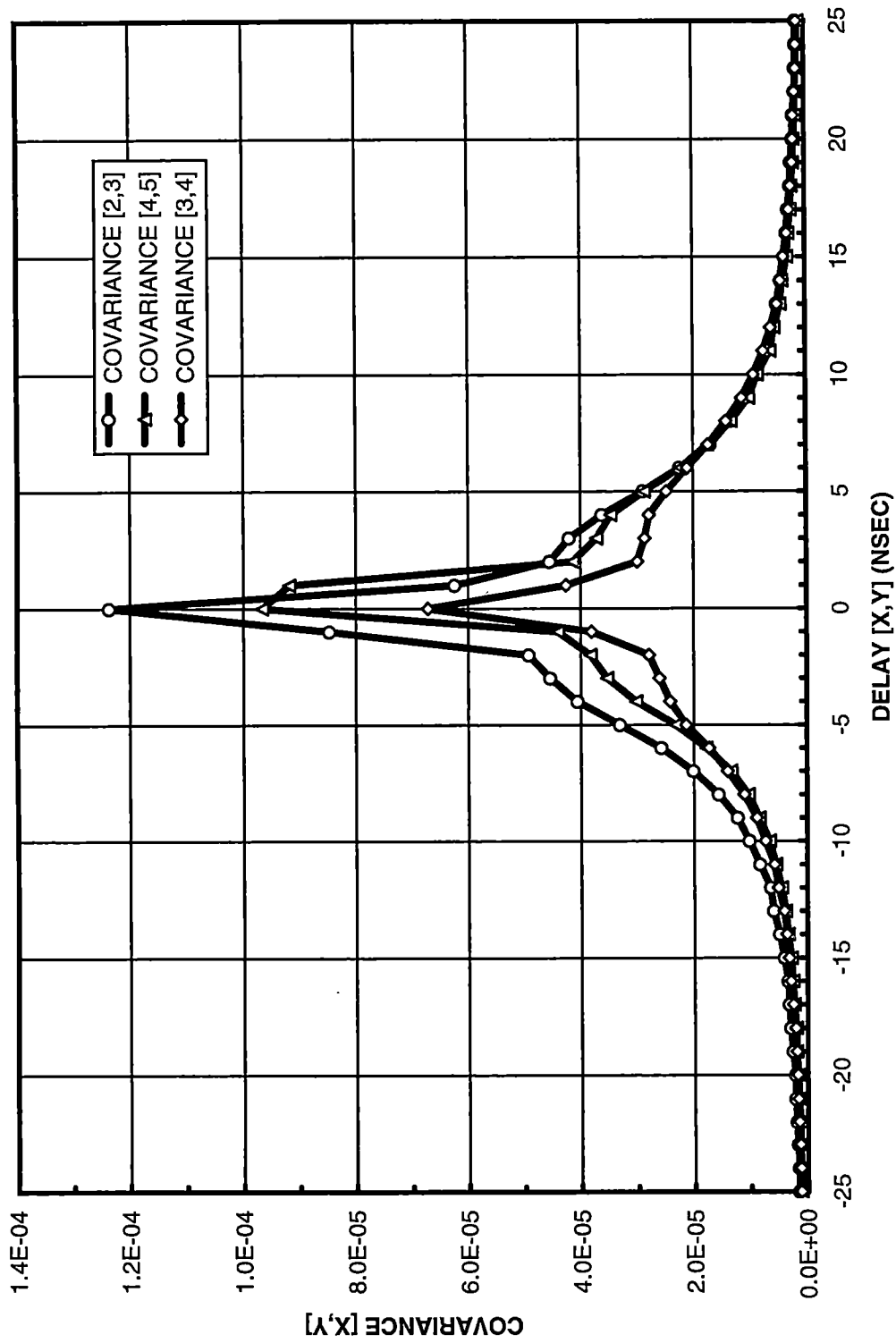


Figure 4.21 Comparison of between-detectors covariances measured using bottom detector pair [2,3], top detector pair [4,5], and detector pair [3,4] oriented on a diagonal of the detector array

which are located on a “diagonal” of the detector array, is much lower than that between detectors which are adjacent to one another either on the top or bottom half of the detector array. This tends to indicate that a significant amount of “cross-talk” between detectors occurred despite the lead shield that surrounded each detector. This conclusion may be drawn because detectors oriented on the diagonal have no adjacent faces, and there is a greater thickness of lead between them relative to adjacent detectors. In the future, a study of the magnitude of this effect versus lead shield thickness is advisable. Note that the integral differences between each of these covariance signatures is plain in the magnitude of the corresponding between-detectors spectra plotted in Fig. 4.22. Also observe that the relatively longer transit time in channel 5 is evident in the spectra phase plotted in Fig. 4.23.

Now observe in Fig 4.21 that the majority of correlated pairs occur with no more than 15 nsec delay between the two events that form the pair. Consequently, the majority of the correlated pairs that would have been observed in a detector autocorrelation are missed during the detector’s dead time of approximately 15 nsec, as Fig. 4.24 illustrates. The detector pulse width had been set to approximately 10 nsec; the distribution of observed dead times resulted from variations in the CFD response time following each output pulse. Do note, however, that some correlated pairs were observed at long delays. This is reflected by the disparity between the autocorrelation’s tail and the accidental coincidence rate. However, because the majority of correlated events are missed during the dead time, the detector autospectrum, plotted in Fig. 4.25, is essentially flat. Furthermore, due to low detection efficiency, its value is essentially the count rate normalized to the measured source fission rate. Consider that the solid angle between the source and each detector is a mere 2% of 4π ;

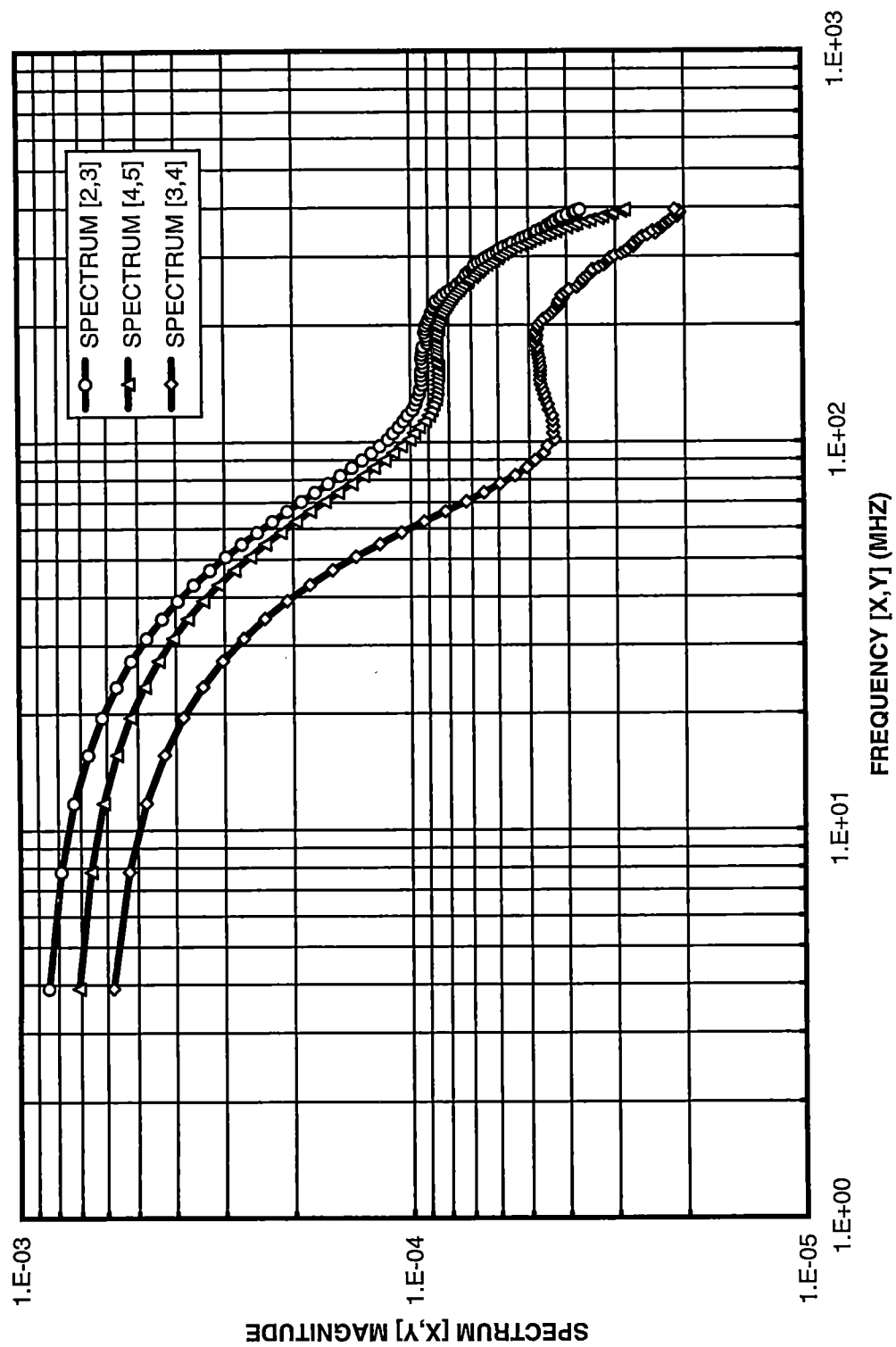


Figure 4.22 Comparison of between-detectors spectra magnitudes (in correlated pairs/ ^{252}Cf -fission) measured using bottom, top, and diagonal detectors pairs

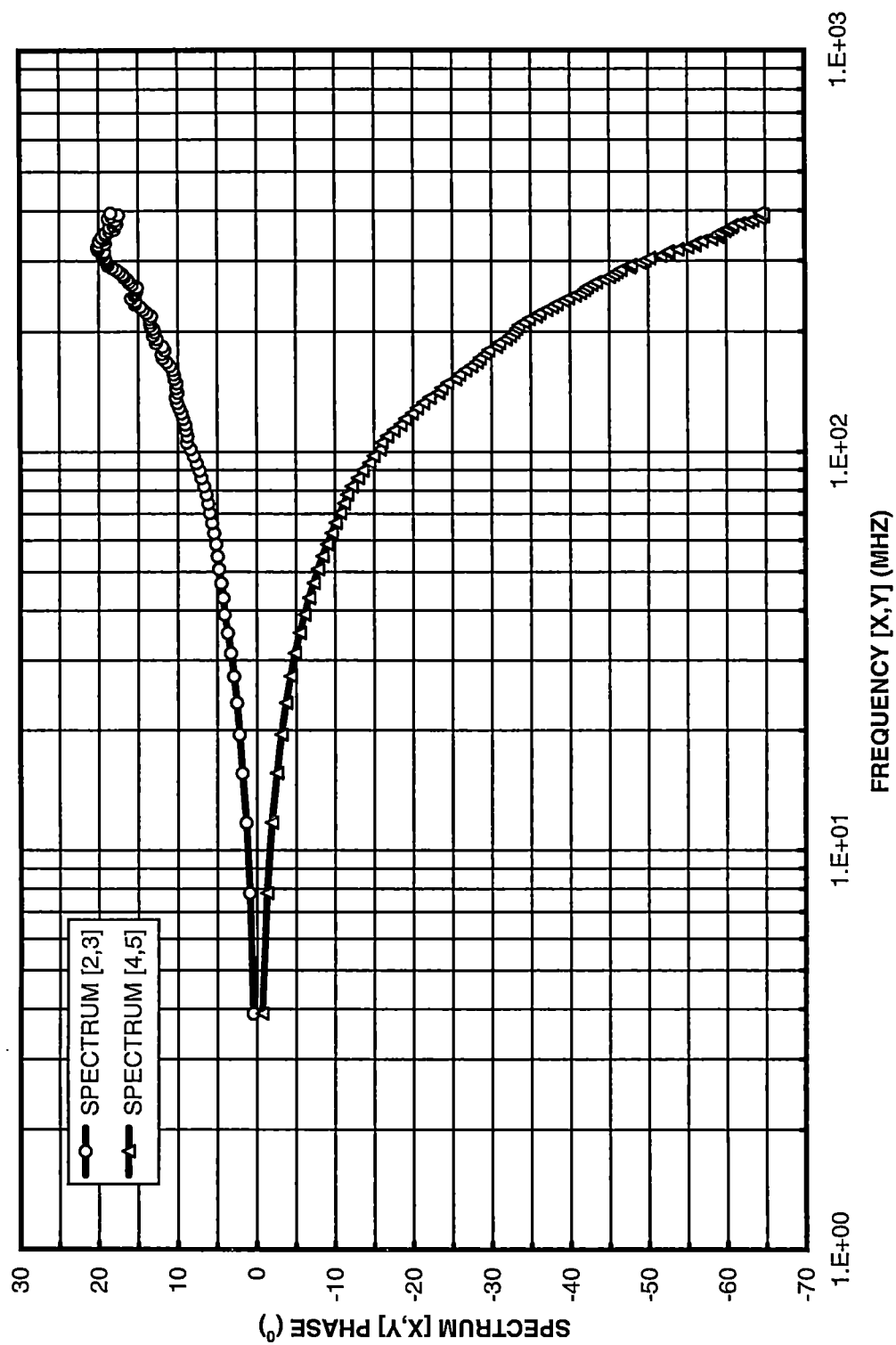


Figure 4.23 Comparison of between-detectors spectra phases measured using detector pairs [2,3] and [4,5]

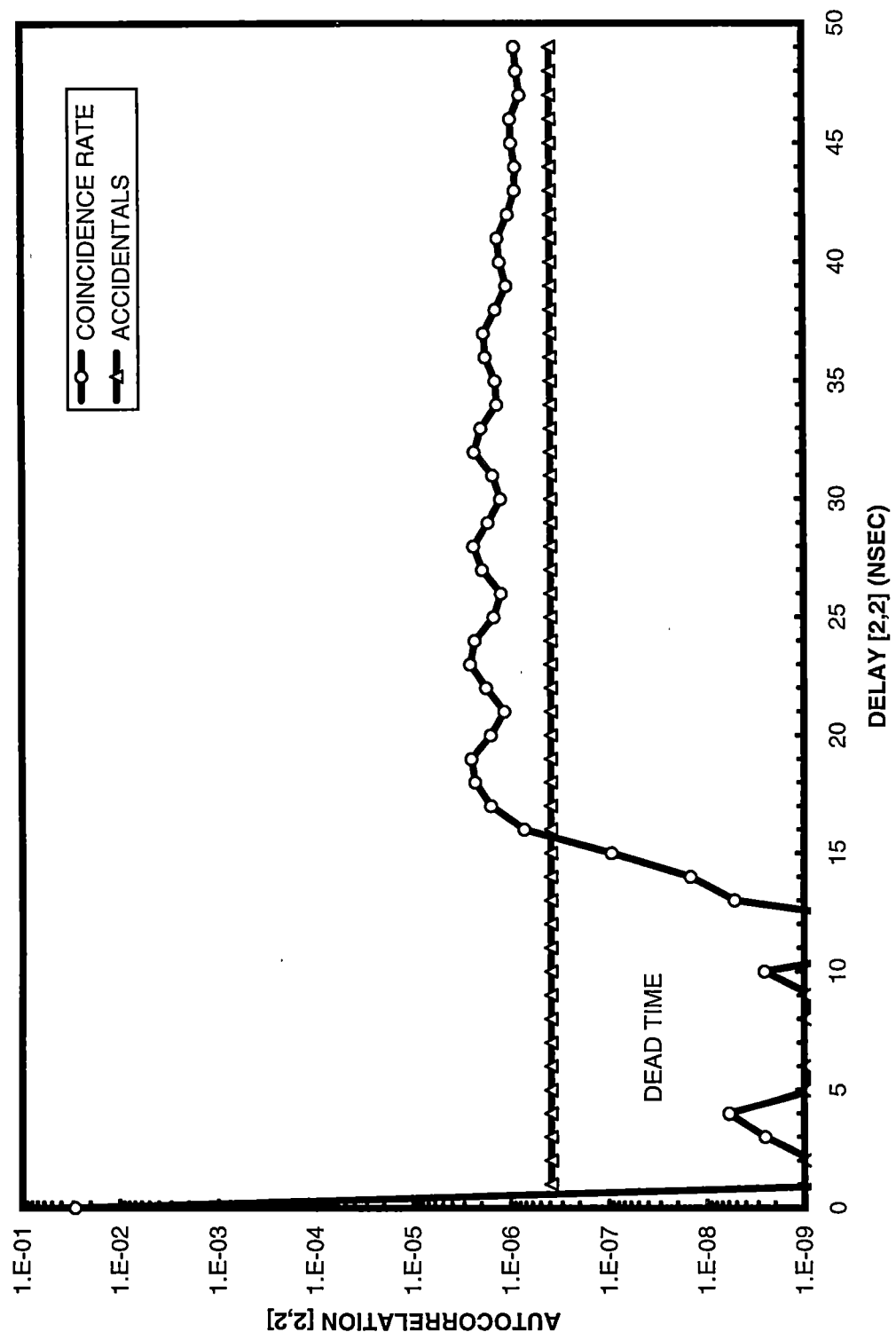


Figure 4.24 Measured detector autocorrelation [2,2] in coincident counts/ ^{252}Cf -fission/nsec

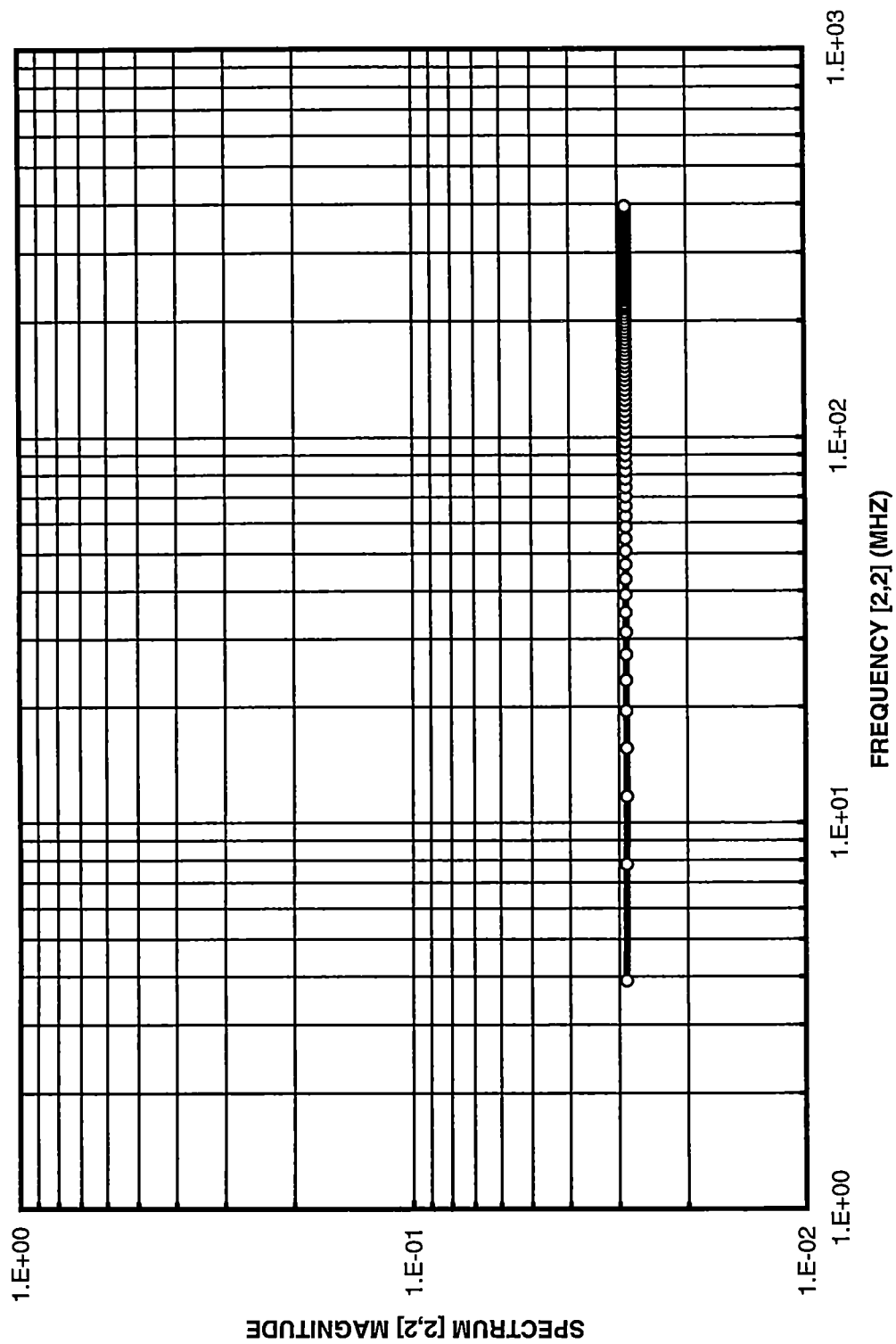


Figure 4.25 Measured detector autospectrum [2,2] in correlated counts/²⁵²Cf-fission

this coupled with each detector's intrinsic efficiency for counting neutrons ($\sim 50\%$) resulted in an overall neutron detection efficiency on the order of 1%. Recall that the count rate is proportional to detection efficiency, while correlated pairs are detected proportional to the efficiency squared. Consequently the probability of detecting a correlated pair is substantially lower than that of a single detection.

These facts are further illustrated by the detector autobicorrelation plotted in Fig. 4.26. The peak at the origin is the normalized count rate. Correlated pairs accidentally coincident with an independent count are distributed along the two axes and along the diagonal. The "noisy" region about the rest of the plane is primarily due to accidentally bicoincident triplets. At such long delays the probability of detecting a bicorrelated triplet is extremely low. Hence the detector autobicorrelation is as well merely another measure of detector count rate.

Consequently the detector autobispectrum is essentially flat as Fig. 4.27 illustrates. Furthermore, although the detector autobispectrum is formally complex, Fig. 4.28 shows that its phase does not deviate far from zero degrees. Therefore, because the detector autobicorrelation is no more informative than the detector autocorrelation, to enhance throughput it is probably advisable to eliminate the collection of detector autobicorrelation signatures in future measurements.

The bicovariance between the source and the two bottom detectors, 2 and 3, is plotted in Fig. 4.29. Note that, like the source-detector covariance, this signature exhibits a small peak corresponding to transmitted gammas. However, recognize that the peak in this bicovariance arises due to the transmission of pairs of gammas, as opposed to a single

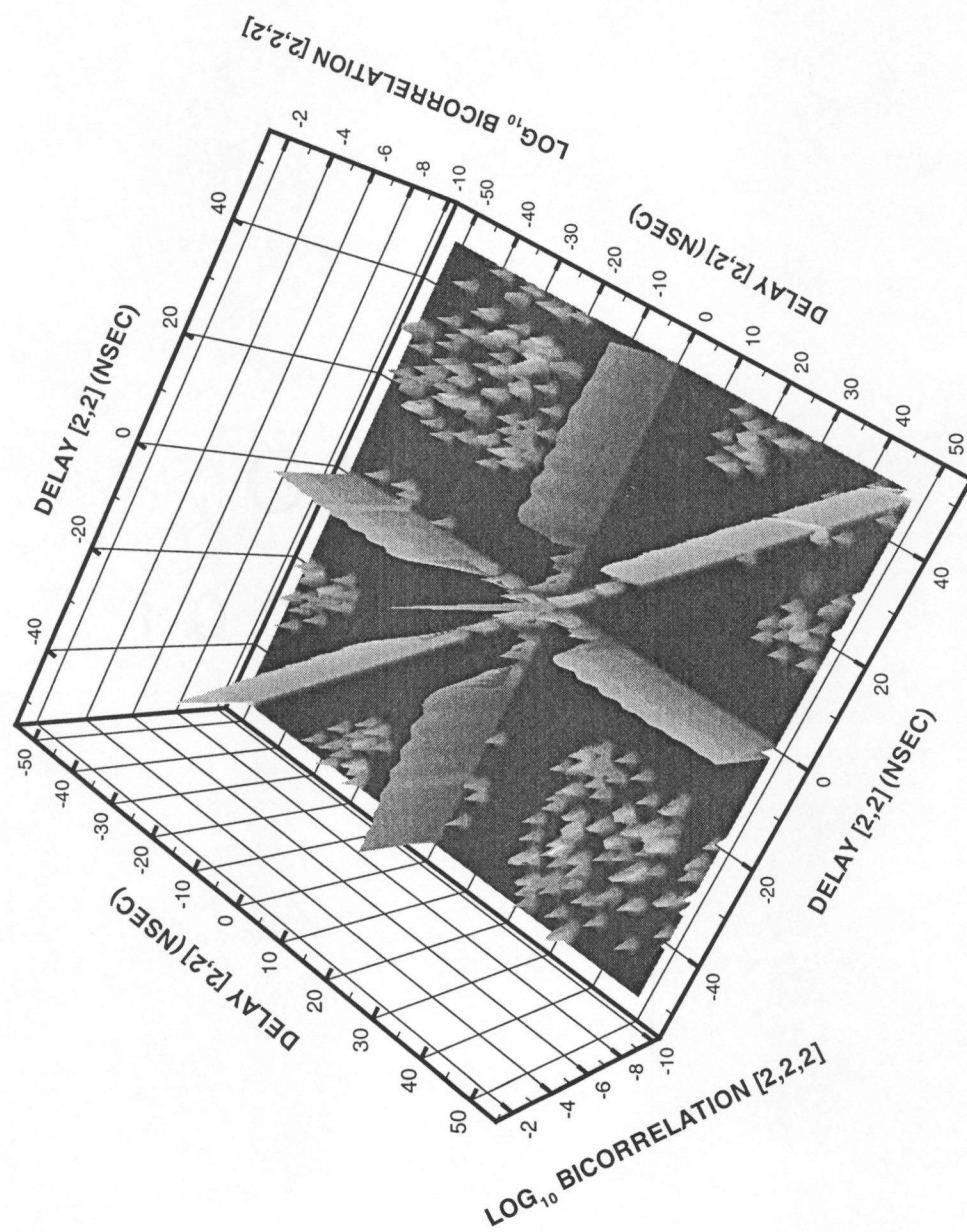


Figure 4.26 Measured detector autobicorrelation $[2,2,2]$ in bicoincident counts/ ^{252}Cf -fission/nsec²

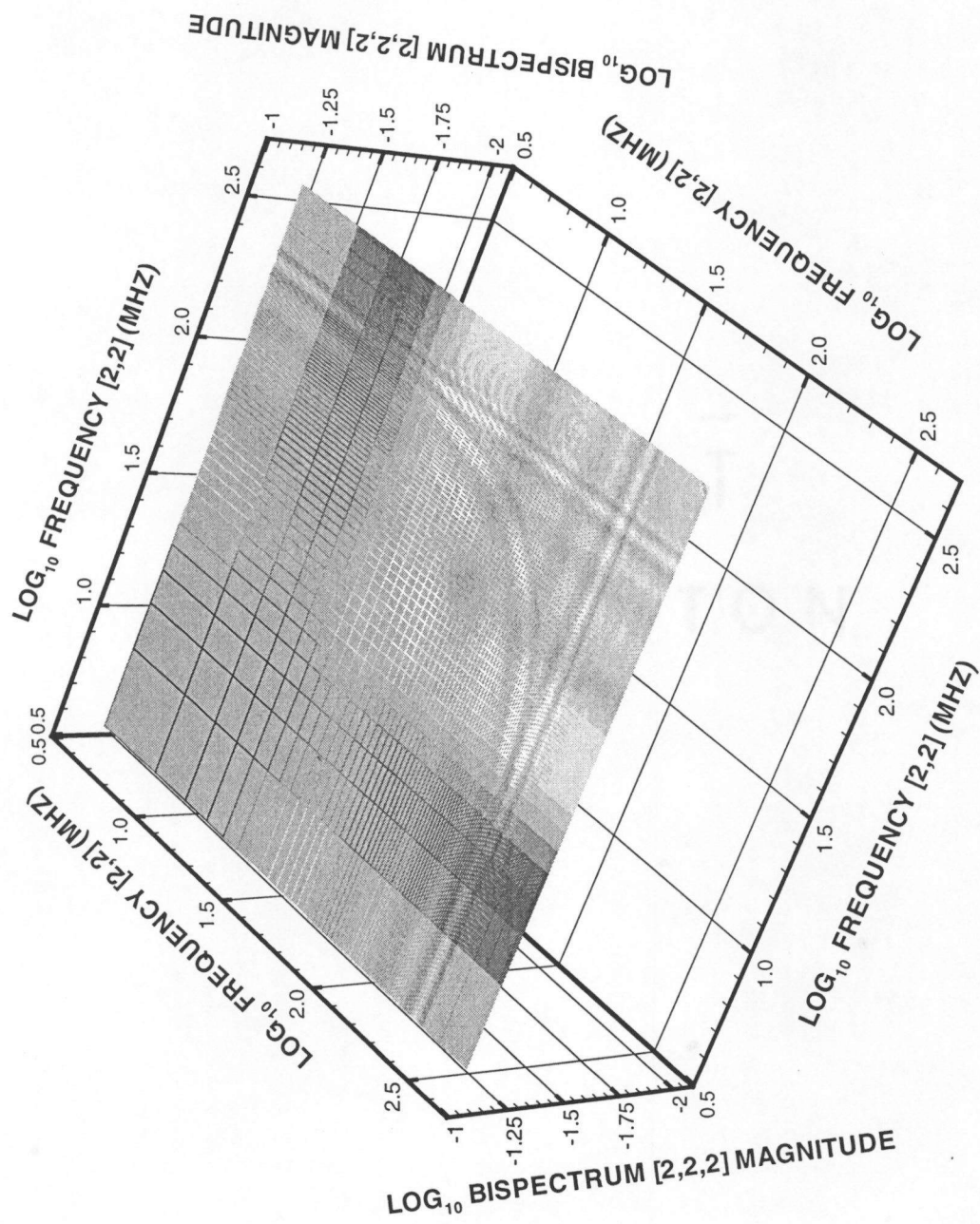


Figure 4.27 Measured detector autobispectrum [2,2,2] magnitude in biccrrrelated counts/ ^{252}Cf -fission

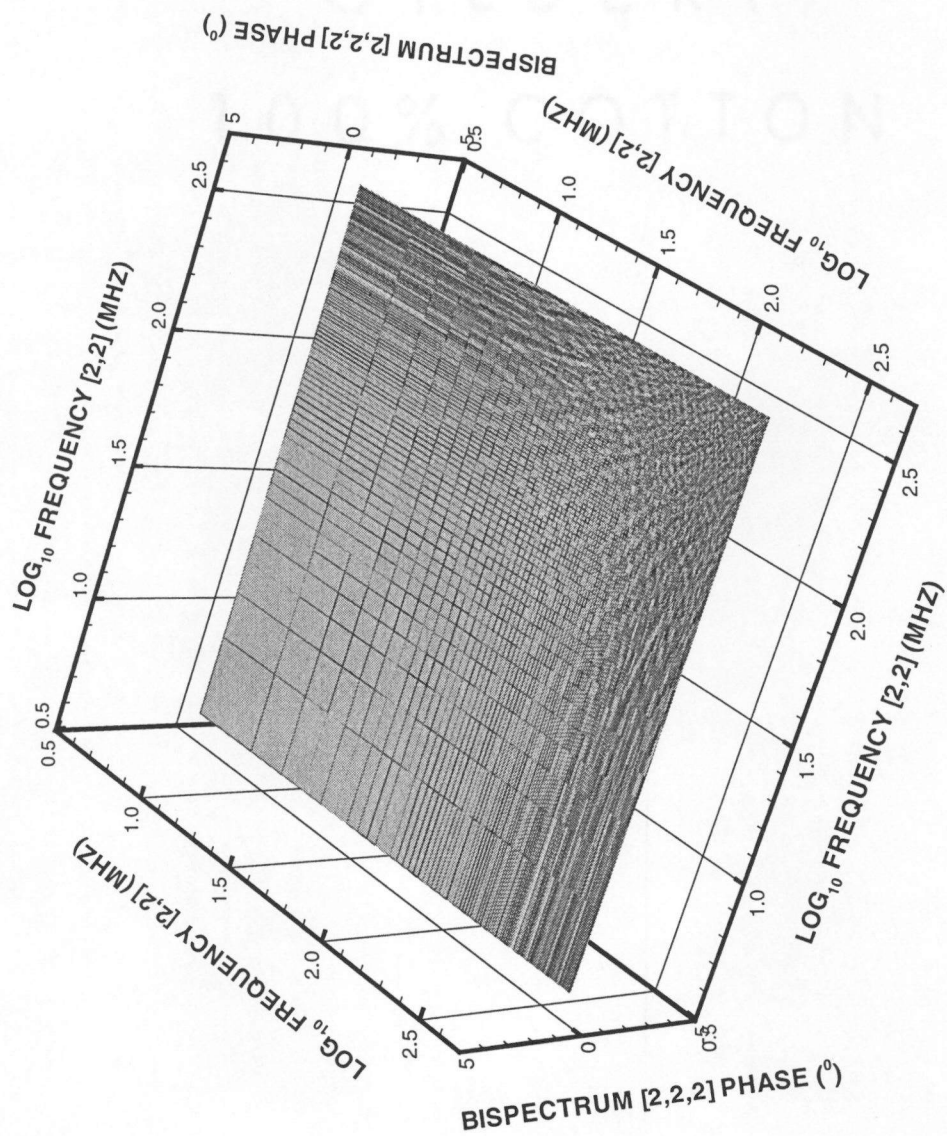


Figure 4.28 Measured detector autobispectrum [2,2,2] phase

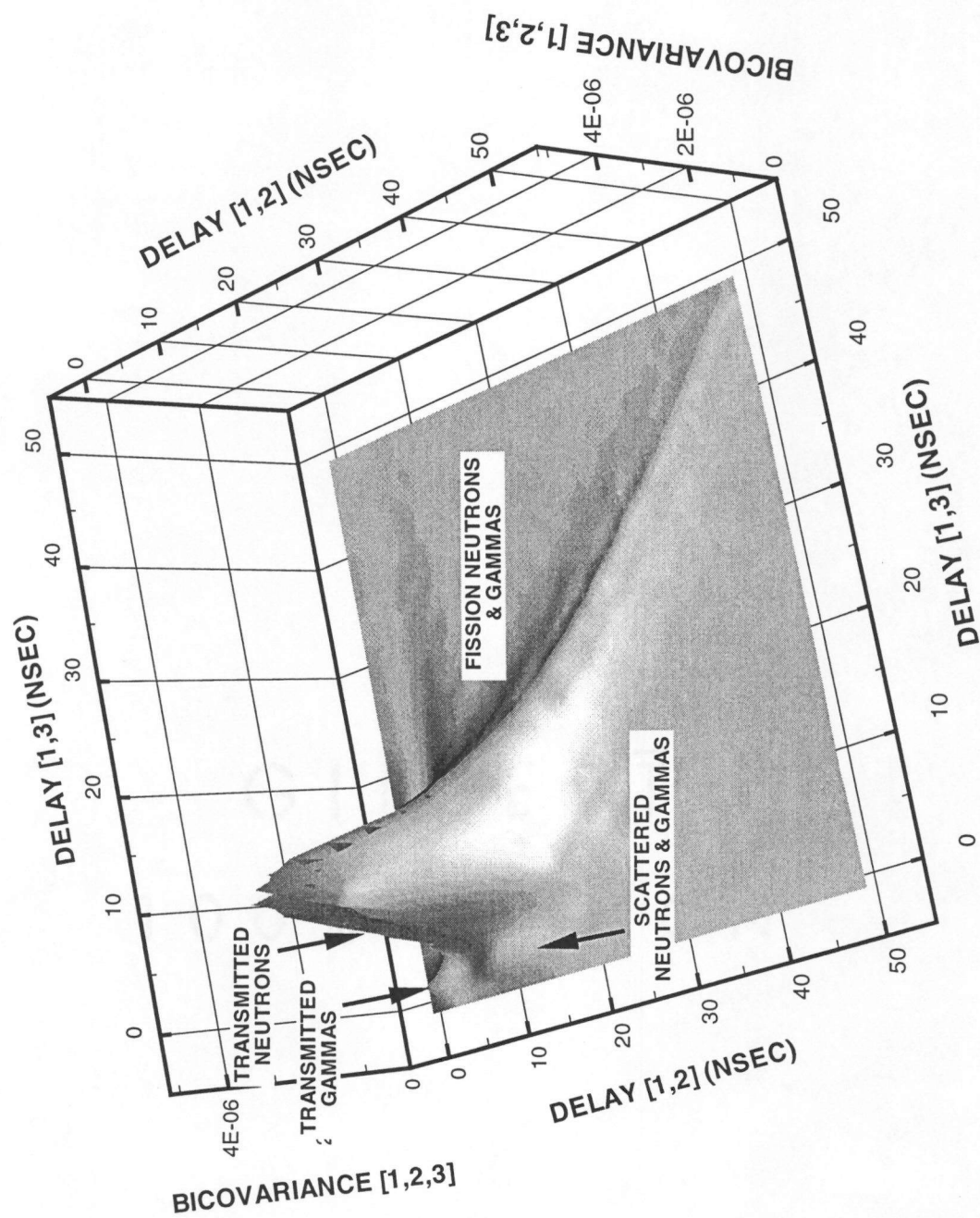


Figure 4.29 Measured source-detector bicovariance [1,2,3] in correlated pairs/ ^{252}Cf -fission/nsec²

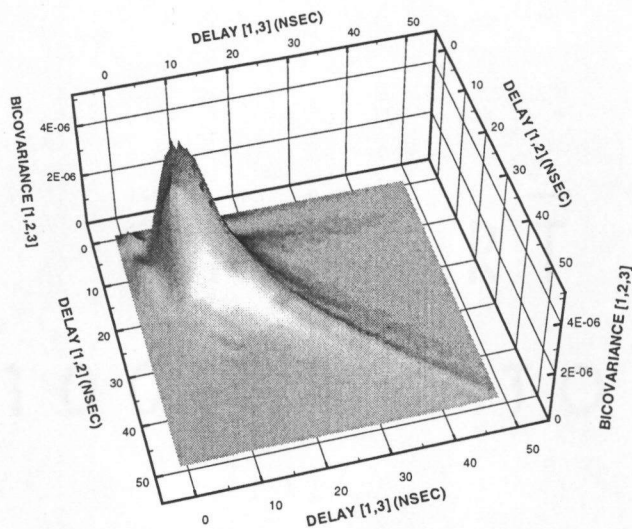
gamma, since correlated gammas must be counted in both detectors to contribute to this bicovariance. For the same reason, the rising portion of the larger peak corresponds to the transmission of pairs of neutrons. The distribution of source neutron kinetic energies is reflected in the width of the rising portion of this peak. Further note that the ridge along the diagonal is due to the detection of pairs of gammas from induced fission; the broader underlying distribution is primarily due to pairs of neutrons from induced fission, although in part it is also due to gamma-neutron pairs from induced fission. Underlying this broad distribution are contributions due to correlated pairs of particles resulting from the less probable reactions including elastic and inelastic scattering and neutron capture. However, note that a small "bump" distributed along the direction of each axis is evident to either side of the broad distribution. The alignment of this distribution along each axis indicates that one particle of a pair was detected much earlier than the second particle in the pair. Consequently it may be concluded that this distribution is due to gamma-neutron pairs and same-particle pairs composed of one transmitted and another interacted or secondary particle. Hence, it is probably primarily due to gamma-gamma, neutron-neutron, and gamma-neutron pairs where one particle has scattered and gamma-neutron pairs arising from inelastic scatter and capture. Future measurements of other materials where these interactions are more probable will be needed to better characterize these features of the source-detector bicovariance, but it is already clear that some information is revealed by this signature that is hidden in the second-order signatures.

Further note that the greater distance required to travel between the source and a detector at the top of the array, which results in some disparity between the transmission rates

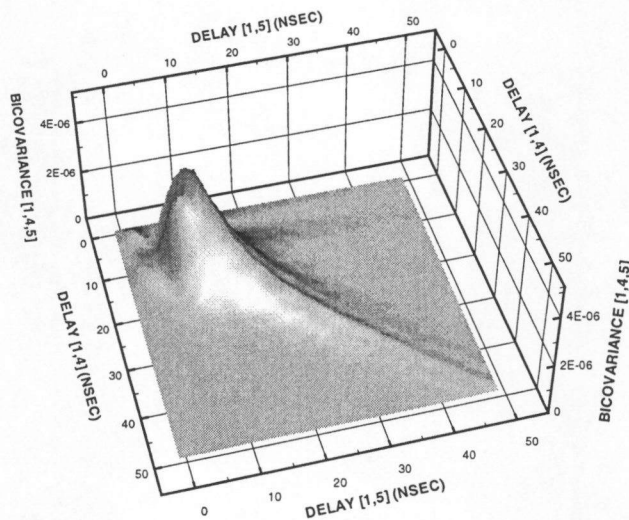
to the upper and lower halves of the detector array, is evident in bicovariance between the source and the two top detectors, 4 and 5. This signature is plotted in Fig. 4.30. Relative to the bicovariance between the source and the two bottom detectors, this source-detector bicovariance has lower amplitude everywhere, even on the long tail, which also indicates that fewer pairs resulting from fission are counted by the detectors at the top of the array. Recall that, unlike the source-detector covariance, which arises from the detection of a single source-correlated particle, the source-detector bicovariance requires the detection of a pair of particles bicorrelated with the source. Hence, like the between-detector's covariance, it is more sensitive to changes in transmission and fission than is the source-detector covariance. The source-detector bicovariance also responds to cross-talk between detectors with sensitivity similar to the between-detectors covariance. This is also evident in Fig. 4.30 where the source-detector covariance involving detectors 3 and 4 is plotted. Because these two detectors are oriented on a diagonal of the detector array, the scatter of particles back and forth between them is less likely than it is for a pair of adjacent detectors.

The integral differences between these three source-detector bicovariances is evident in the magnitude of the corresponding source-detector bispectra, which are shown “edge-on” in Fig. 4.31 so that the difference in magnitude is clear. The following figure, 4.32, shows more of the shape of bispectrum [1,2,3]; the other source-detector bispectra are similar in shape. At low frequencies, the source-detector bispectrum magnitude reflects the distribution of source-bicorrelated pairs detected at widely separated times. Hence, this portion of the signature primarily reflects the distribution neutron-neutron pairs and gamma-neutron pairs. At higher frequencies, a rough “plateau” is evident. This portion of the bispectrum

(a)



(b)



(c)

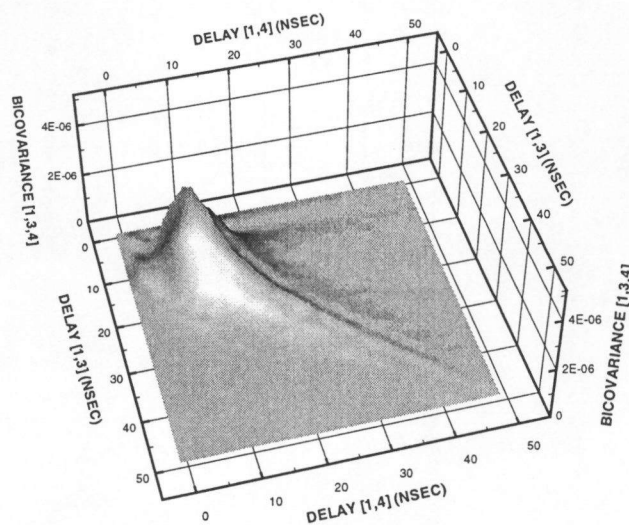


Figure 4.30 Comparison of source-detector bicovariances measured using (a) top, (b) bottom, and (c) diagonal detector pairs

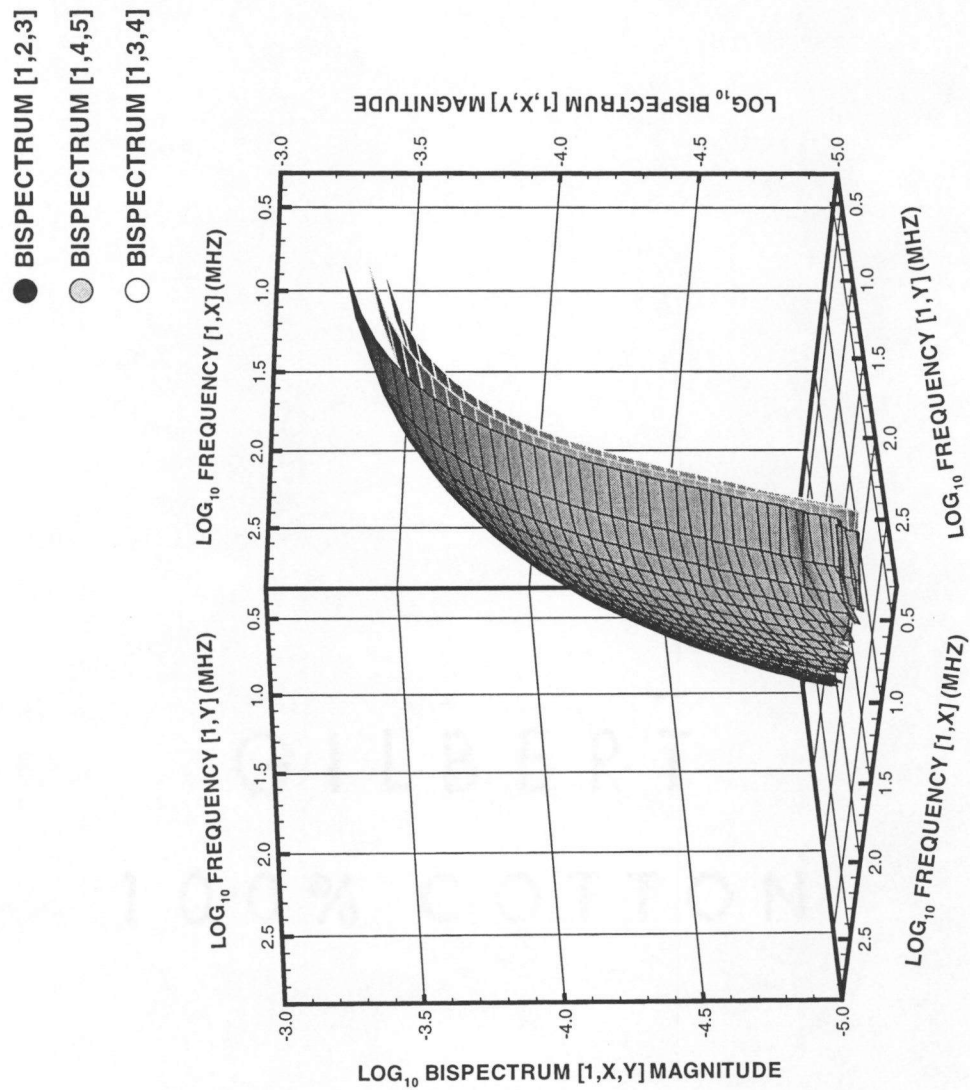


Figure 4.31 Comparison of source-detector bispectra magnitudes (in correlated pairs/ ^{252}Cf -fission) measured using bottom, top, and diagonal detector pairs

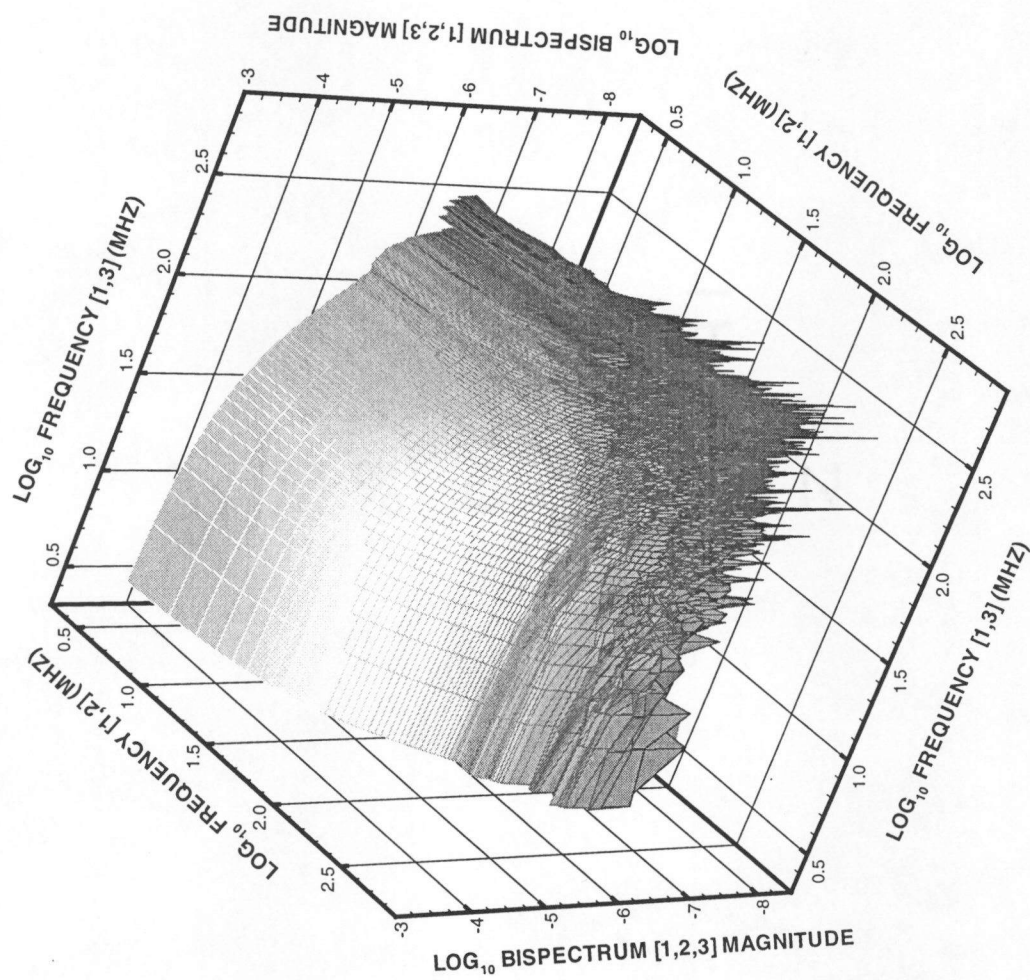


Figure 4.32 Measured source-detector bispectrum $[1,2,3]$ magnitude

magnitude reflects the distribution of pairs detected in rapid succession; these are primarily correlated pairs of gammas. The bispectrum phase, plotted in Fig. 4.33, slopes downward from zero at the origin to pass through several revolutions. Unfortunately, the large number of 360° revolutions makes it virtually impossible to observe the additional negative phase shift introduced into bispectrum [1,4,5] due to the additional half nanosecond delay in channel 5.

Now recall that the between-detectors covariance provided no means to distinguish between neutron-neutron pairs and gamma-neutron pairs arising from induced fission. Further recall that these tend to contribute to the between-detector's covariance at long delays because they arrive at the detectors dispersed in time due to the distribution of fission neutron kinetic energies and the disparity between the average fission neutron speed and the speed of light. Now examine between-detectors bicovariance [2,3,4] plotted in Fig 4.34. Like the between-detectors covariance, the between-detectors bicovariance exhibits a narrow distribution about the origin. Since this distribution must arise from particles that are detected in rapid succession, it can be concluded that it arises primarily due to triplets of transmitted source gammas and induced fission gammas. The broader underlying distribution arises from triplets of particles that are more widely dispersed in arrival time, hence it is probably primarily due to transmitted source neutrons and induced fission neutrons because these are born with a distribution of speeds. Some portion of this wide distribution is also due to gamma-neutron-neutron triplets since these should also be detected at widely dispersed times. However, observe that a narrow ridge along each axis is evident in this signature. This ridge indicates that a pair of correlated particles was detected in rapid

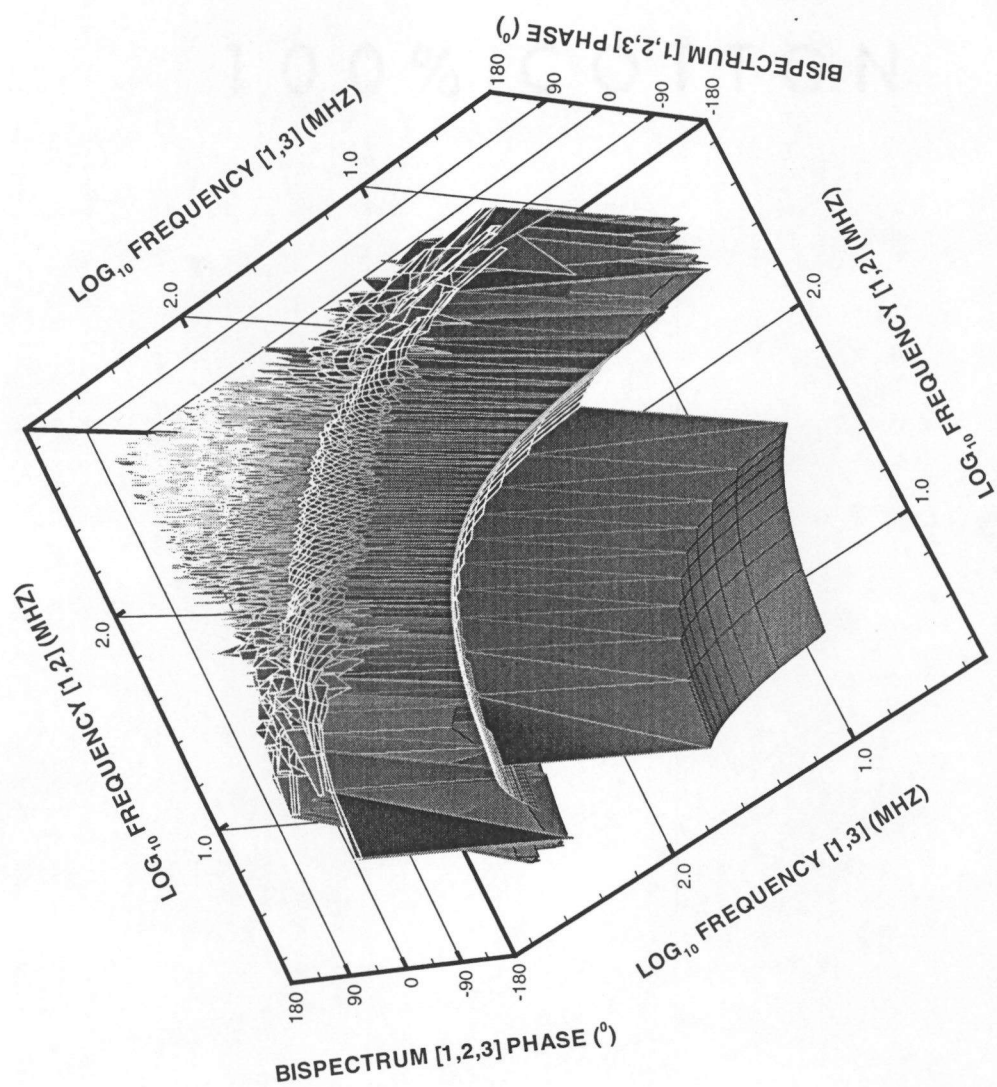


Figure 4.33 Measured source-detector bispectrum [1,2,3] phase

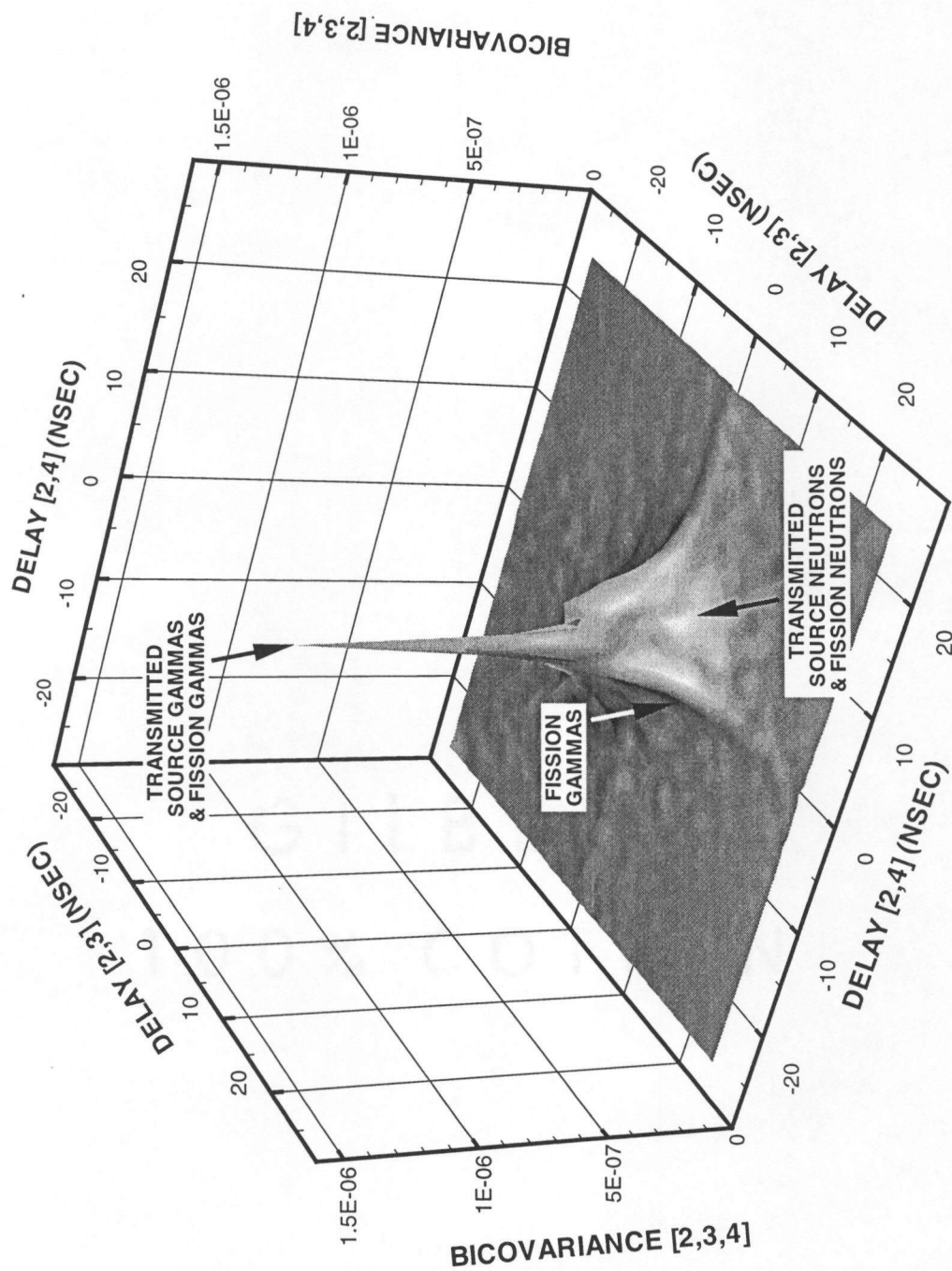


Figure 4.34 Measured between-detectors bicovariance [2,3,4] in bicornelated triplets/ ^{252}Cf -fission/nsec²

succession relative to the third particle in the triplet that was detected sometime later. Consequently it may be deduced that these ridges arise from gamma-gamma pairs resulting from induced fission; the third particle in such triplets is probably usually a neutron since these travel more slowly than gammas. Some of these triplets will involve a third gamma arising from an inelastic scatter or neutron capture subsequent to the initial fission. Again, the scope of this experiment is not sufficient to demonstrate these deductions in any quantitative way, but it is clear that the between-detectors bicovariance reveals information that is hidden in the between-detectors covariance.

Next observe in Fig. 4.35 that the additional half nanosecond delay present in channel 5 is evident in bicovariance [3,4,5]. Further note that because this signature involves two detectors at the top of the array, whereas bicovariance [2,3,4] involves two detectors at the bottom of the array, the amplitude of bicovariance [3,4,5] is relatively lower due to the fewer transmitted particles and fission progeny counted by the upper half of the detector array. However, because each between-detector bicovariance involves detectors oriented on the diagonals of the array, no additional observations regarding the rate of cross-talk between detectors can be made from these signatures. Further note that because each between-detectors bicovariance involves at least one bottom detector, the corresponding between-detectors bispectra, relative to the source-detector bispectra, are actually less sensitive to the fewer transmitted and fission particles counted by detectors at the top of the array. This is illustrated by Fig. 4.36, where the magnitude of bispectrum [2,3,4] is compared to that of bispectrum [3,4,5]. The magnitude of the second bispectrum, which involves only one bottom detector, is only slightly lower than the magnitude of the first, which involves two

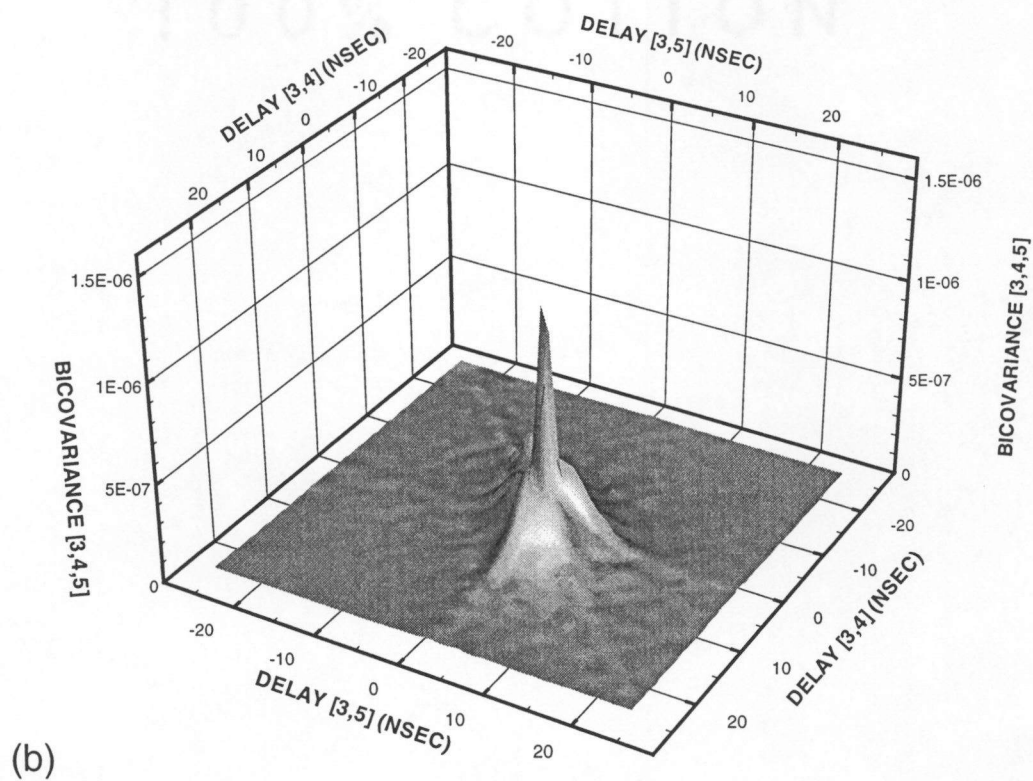
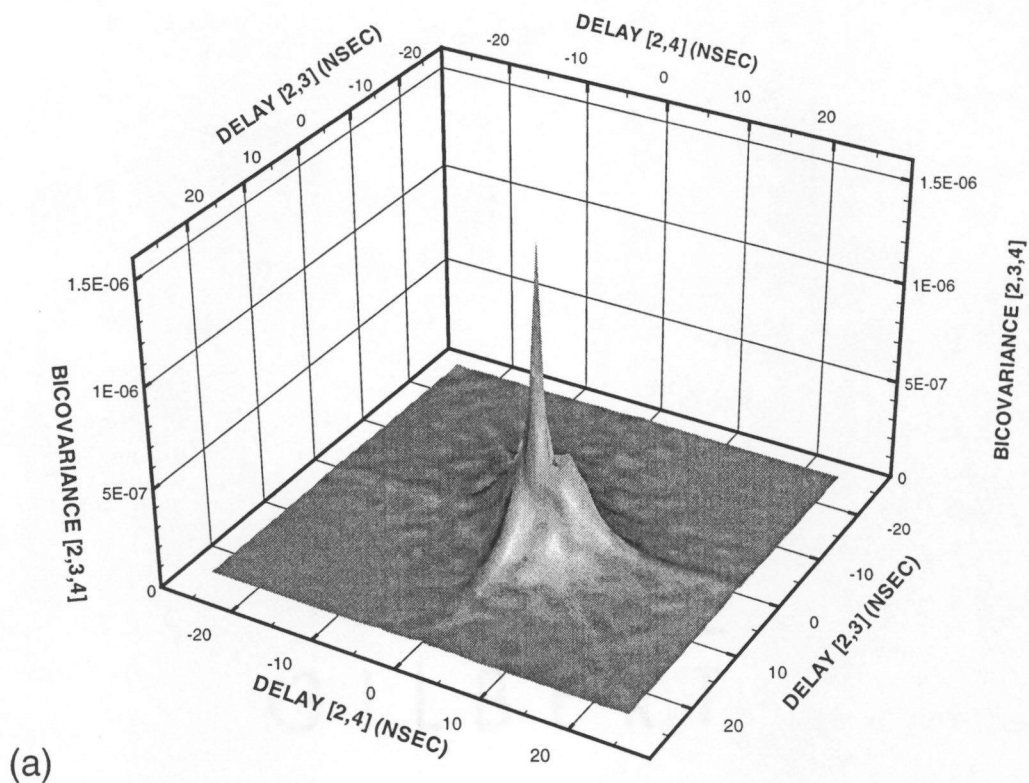


Figure 4.35 Comparison of between-detectors bicovariances measured using detector triplets (a) [2,3,4] and (b) [3,4,5]

● BISPECTRUM [2,3,4]

○ BISPECTRUM [3,4,5]

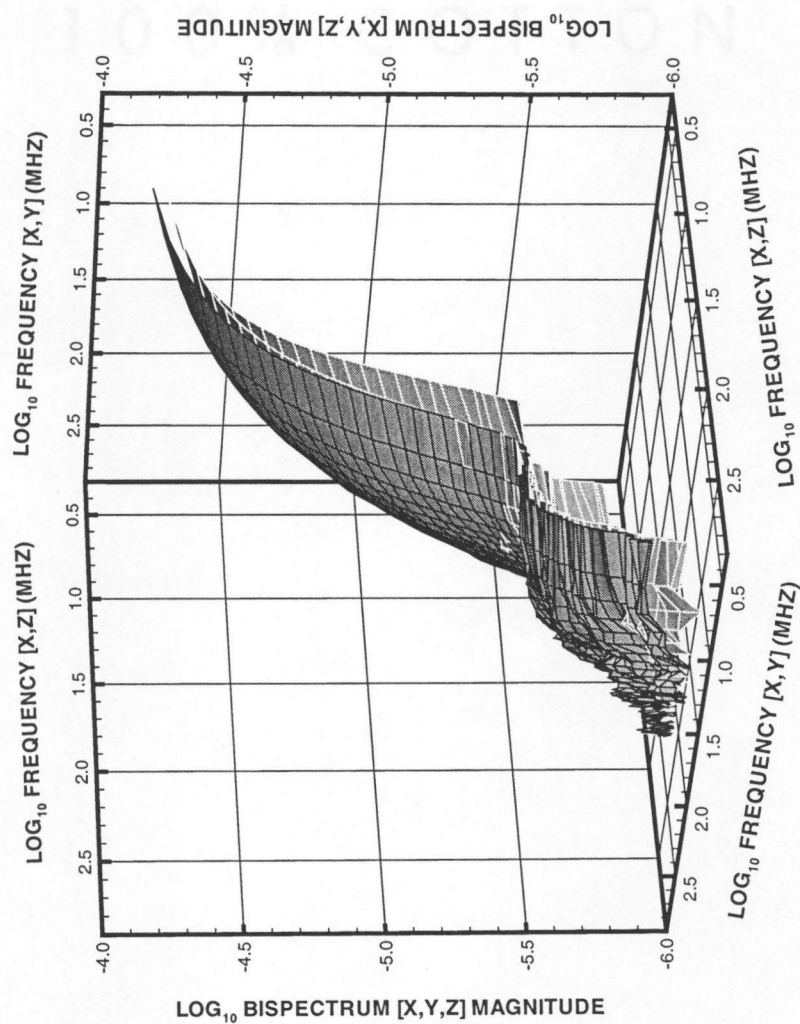


Figure 4.36 Comparison of between-detectors bispectra magnitudes (in bicornelated triplets/ ^{252}Cf -fission) measured using detector triplets [2,3,4] and [3,4,5]

bottom detectors. The shape of the between-detectors bispectra is illustrated by Fig. 4.37, where the magnitude of bispectrum [2,3,4] is plotted. Again, the low-frequency behavior of the magnitude reflects the distribution of particles that are detected widely dispersed in time, so it arises primarily due to triplets of neutrons and gamma-neutron-neutron triplets. The magnitude at high frequencies is more indicative of particles that are detected in rapid succession, so it reflects the distribution of triplets of gammas and gamma-gamma-neutron triplets. Finally, recall that the between-detectors bispectrum is formally complex, although the phase shift tends to be fairly small as was observed for the detector autobispectrum. Hence the phase of bispectrum [2,3,4], plotted in Fig. 4.38, does not pass through even one full revolution. However, observe that the additional half-nanosecond delay present in channel 5 is evident in the phase of bispectrum [3,4,5] – it becomes negative along frequency [3,5] as noted in Fig. 4.38.

The integral of each signature (except the source and detector autocorrelation and autobicorrelation) is listed in Table 4.2 cross-referenced to the orientation of the detectors involved in each signature. Alternatively, these may be viewed as the low-frequency asymptote of each corresponding spectral signature. Where multiple signatures were available, the numbers in the table denote their average integral. For example, the integral of the source-detector covariance for detectors on the bottom of the array is the average of covariance [1,2] and [1,3]. Since the between-detectors bicovariance signatures involve three detectors, the orientation given in the table indicates the orientation of the majority of the detectors involved in the signature. For example, the between-detector bicovariance for detectors oriented on the left of the array is bicovariance [2,4,5], since both detectors 2 and

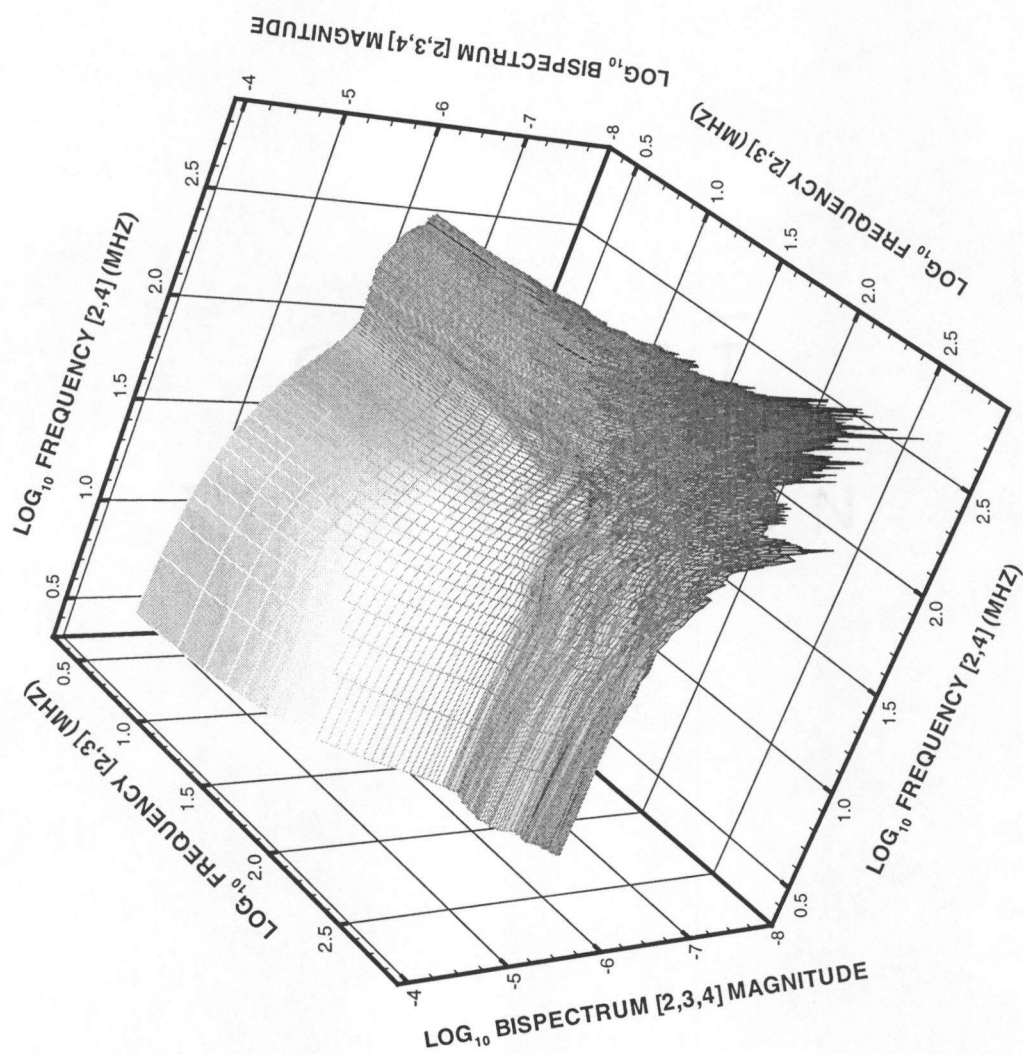
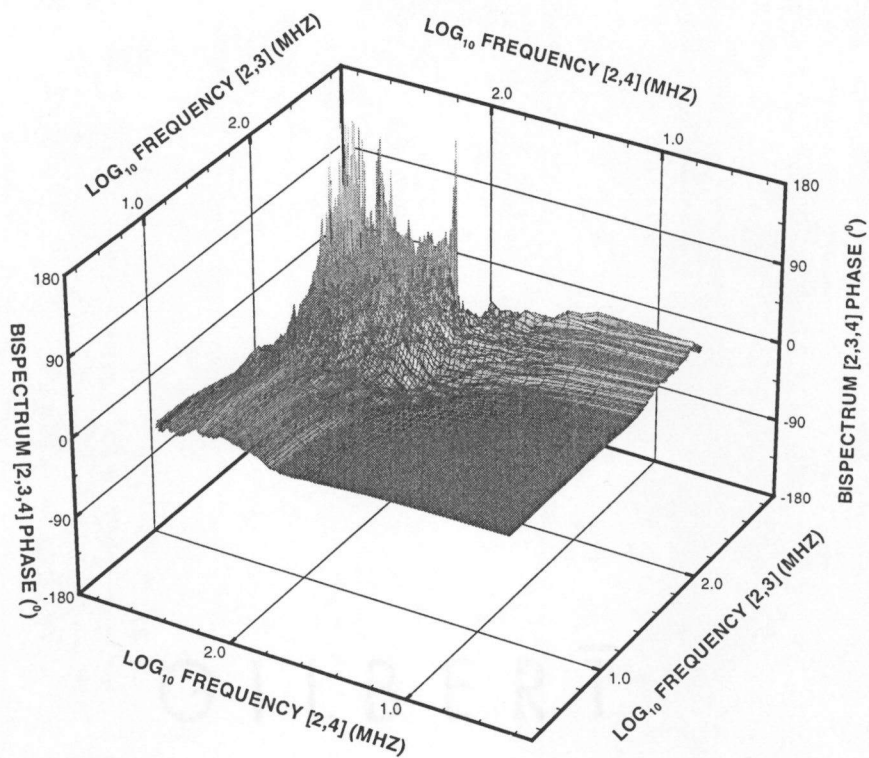
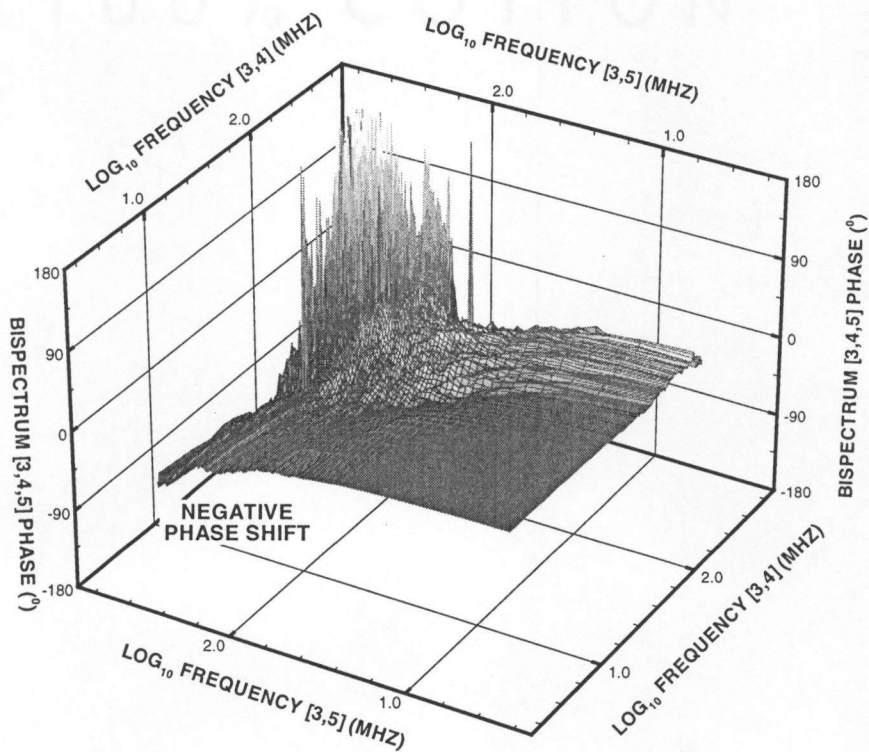


Figure 4.37 Measured between-detectors bispectrum [2,3,4] magnitude



(a)



(b)

Figure 4.38 Comparison of between-detectors bispectra phases measured using detector triplets (a) [2,3,4] and (b) [3,4,5]

Table 4.2 Statistics measured from the uranium casting

	orientation						
	bottom	top	left	right	diagonal	all	all except diagonal
detector count rate (counts/ ²⁵² Cf-fission)	2.9×10 ⁻²	2.6×10 ⁻²	2.7×10 ⁻²	2.7×10 ⁻²	–	2.7×10 ⁻²	–
source-detector covariance (counts/ ²⁵² Cf-fission)	1.4×10 ⁻²	1.2×10 ⁻²	1.3×10 ⁻²	1.3×10 ⁻²	–	1.3×10 ⁻²	–
between-detectors covariance (pairs/ ²⁵² Cf-fission)	9.1×10 ⁻⁴	7.6×10 ⁻⁴	8.5×10 ⁻⁴	8.2×10 ⁻⁴	6.2×10 ⁻⁴	7.6×10 ⁻⁴	8.3×10 ⁻⁴
source-detector bico variance (pairs/ ²⁵² Cf-fission)	7.1×10 ⁻⁴	5.8×10 ⁻⁴	6.5×10 ⁻⁴	6.3×10 ⁻⁴	5.1×10 ⁻⁴	6.0×10 ⁻⁴	6.4×10 ⁻⁴
between-detectors bico variance (triplets/ ²⁵² Cf-fission)	8.1×10 ⁻⁵	7.4×10 ⁻⁵	7.8×10 ⁻⁵	7.7×10 ⁻⁵	–	7.8×10 ⁻⁵	–
fraction of counts correlated with source	0.49	0.48	0.49	0.48	–	0.48	–
fraction of pairs bico related with source	0.78	0.76	0.76	0.76	0.82	0.79	0.77
bispectral ratio (independent pairs per correlated pair)	0.27	0.27	0.27	0.27	0.34	0.29	0.27

4 are on the left side of the array.

First note that each signature more or less exhibits right-left symmetry indicating that the casting was essentially centered horizontally in the can. Next observe that each “top” signature is significantly lower than its “bottom” counterpart. This is due to the disparity between the top and bottom detectors’ views of the source and casting; fewer transmitted particles and fission progeny were counted by detectors at the top of the array. Also observe that signatures involving a pair of detectors oriented on the diagonal of the array are significantly lower than those involving adjacent detectors. This is an indication of the rate of cross-talk between detectors, though it is not a quantitative measure of this effect. Finally observe that the difference between signatures involving “top” detectors and those involving “bottom” detectors follows a pattern that conforms to earlier observations regarding the sensitivity of these signatures. That is, the change in the source-detector covariance is essentially the same as the change in count rate. Similarly, the change in source-detector bicovariance is the same as the change in between-detector bicovariance. Furthermore, the change in the two signatures involving a pair of detectors is larger than the change in those signatures that involve only a single detector. The change in between-detector bicovariance does not follow this pattern of increasing sensitivity because each between-detector bicovariance involves at least one detector at the top of the array.

Now observe that the fraction of counts correlated with the source is only about one-half for each detector. Recall that this fraction is estimated by the ratio $\sum_k C_{1x}(k) / \bar{x}$. The remainder of the counts are most likely due to background, which was elevated in the presence of the casting due to 1 MeV gammas emitted during ^{238}U α -decay and due to pileup

of 186 keV gammas emitted during ^{235}U α -decay. More noteworthy is the fact that the fraction of pairs attributable to the source is only about three-quarters for each pairing of adjacent detectors. Recall that this fraction is estimated by $\sum_{k_1} \sum_{k_2} C_{lxy}(k_1, k_2) / \sum_k C_{xy}(k)$. To what source is the remaining quarter of pairs related?

There are a number of possibilities. First recognize that a ^{252}Cf source that has aged sufficiently will emit a delayed neutron for every hundred or so prompt neutrons. Delayed neutrons conform to a softer spectrum, so they constitute a source that is effectively worth a few percent of the ^{252}Cf prompt neutron source. Furthermore, for this fast system, delayed neutrons and gammas have no “memory” of their origin – they will appear to be uncorrelated with prompt neutrons born of the same fission event. So particles born from fission chains initiated by delayed neutrons will appear uncorrelated with particles born from fission chains initiated by prompt source neutrons. In addition, some neutron background was observed, and though the background rate was extremely low compared to the count rate measured in the presence of the source and casting, the spectrum of these neutrons should be softer than that of the ^{252}Cf prompt neutrons. Consequently background may account for one or two percent of the pairs that appear to be uncorrelated with the source. However, the most likely source is in fact no source at all. It is most likely that the ^{252}Cf ion chamber efficiency had degraded such that only 85% to 90% of fission fragments were counted. Those fissions that were not counted can still produce correlated pairs, but to the source-detector signatures these pairs will appear to be uncorrelated with the source because no initial source fission was observed. Consequently both the number of counts correlated with the source and the number of pairs bicorrelated with the source is possibly quite higher than indicated in the

table. This could be confirmed by measuring the fraction of pairs bicornelated with the source in the absence of the casting or any other neutron sources. The measured fraction would actually indicate the fraction of ^{252}Cf fission fragments counted in the ion chamber. Further note that this conclusion is supported by the substantially higher fraction of source-bicornelated pairs observed by detectors oriented on the diagonals of the array. These are subjected to a substantially lower rate of cross-talk. It is possible that a significant amount of the observed cross-talk is due to 1 MeV gammas from ^{238}U that scatter between detectors. In any circumstance, more measurements are required to substantiate, or refute, these hypotheses.

At first the relatively higher magnitude of the bispectral ratio observed for detector pairs on a diagonal in the array appears to contradict the previous observations, since a larger fraction of pairs detected by non-adjacent detectors are source-bicornelated relative to the fraction of pairs detected in adjacent detectors. However, this impression is illusory. Note that the total number of source-bicornelated pairs, measured by $S_{lxy}(0, 0)$, is substantially higher in adjacent detectors. The total number of source-correlated counts is similar in all detectors. Consequently, the bispectral ratio measured by adjacent detectors is lower than that measured by a pair of detectors on the diagonal. The fact that the same bispectral ratio is measured by any pair of adjacent detectors, regardless of their orientation, is remarkable. This implies that the bispectral ratio is less sensitive to small changes in the arrangement of source and detectors than are the other signatures.

Of course many of the preceding observations are qualitative in nature. This measurement is not intended to fully demonstrate the sensitivity of higher order signatures

as were the calculations presented in the previous chapter. However, the measurement does go a long way toward demonstrating the practicality and utility of measuring higher order signatures. Furthermore, this measurement, like any good experiment, asks at least as many questions as it answers. Hopefully this will serve to encourage further experimentation with these techniques.

5 Summary

The point models applied to predict the behavior of the first- through fourth-order signatures are admittedly limited. However, they do illustrate the processes contributing to each of these signatures and the relative importance of these processes. Furthermore, despite the simplicity of the point model, a number of general conclusions that transcend the point model's limitations may be drawn. First, it can be concluded that in general the covariance signatures obey the bounding relationships

$$\int_0^{\infty} d\tau C_{wx}(\tau) \leq \bar{x} , \quad (5.1)$$

$$\int_0^{\infty} d\tau_1 \int_0^{\infty} d\tau_2 C_{wxy}(\tau_1, \tau_2) \leq \int_{-\infty}^{\infty} d\tau C_{xy}(\tau) , \quad (5.2)$$

and

$$\int_0^{\infty} d\tau'_1 \int_0^{\infty} d\tau'_2 \int_0^{\infty} d\tau'_3 C_{wxyz}(\tau'_1, \tau'_2, \tau'_3) \leq \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 C_{xyz}(\tau_1, \tau_2) , \quad (5.3)$$

where w denotes the introduced observable source and x , y , and z denote detectors. The conditions required for the equality to hold are the same in each case. First, each source fission event must be counted. If only a fraction of source fission events are counted, then detector counting events that are in fact correlated with a source fission will not appear to be so to the source-detector signatures C_{wx} , C_{wxy} , and C_{wxyz} . Next, no sources besides the observable source may interact with the fissile system, because the count rate $\bar{x}$, between-

detectors covariance C_{xy} , and between-detectors bicovariance C_{xyz} will respond to these sources while the source-detector signatures will not. Among these are any external sources as well as sources that the system itself may contain, such as spontaneously fissioning isotopes and single-neutron sources, e.g., a combination of isotopes that generates neutrons due to (α, n) reactions. Furthermore, the inherent source that arises due to fission induced by delayed neutrons must be negligible compared to the rate of induced fission by prompt neutrons. This inherent delayed neutron source is most significant in systems closely approaching delayed critical.

In the event that all source fissions are counted, but the system is also driven by unobservable external or internal sources or a substantial inherent source, the preceding integrals respectively yield that portion of counts, correlated doublets, and bicorrelated triplets that are chain-related to the introduced source. Consequently,

$$\int_0^{\infty} d\tau C_{wx}(\tau) = \overline{x|w}, \quad (5.4)$$

i.e., the source-detector covariance reflects the distribution of source-correlated counts following source fission. Similarly, the source-detector bicovariance reflects the distribution of source-bicorrelated doublets following source fission,

$$\int_0^{\infty} d\tau_1 C_{wxy}(\tau_1, \tau_1 + \tau) = \int_0^{\infty} d\tau_2 C_{wxy}(\tau_2 - \tau, \tau_2) = C_{xy|w}(\tau), \quad (5.5)$$

and the source-detector tricovariance reflects the distribution of source-tricorrelated triplets following source-fission,

$$\begin{aligned}
\int_0^{\infty} d\tau'_1 C_{wxyz}(\tau'_1, \tau'_1 + \tau_1, \tau'_1 + \tau_2) &= \\
\int_0^{\infty} d\tau'_2 C_{wxyz}(\tau'_2 - \tau_1, \tau'_2, \tau'_2 + \tau_2 - \tau_1) &= \\
\int_0^{\infty} d\tau'_3 C_{wxyz}(\tau'_3 - \tau_2, \tau'_3 - \tau_2 + \tau_1, \tau'_3) &= C_{xyz|w}(\tau_1, \tau_2) .
\end{aligned} \tag{5.6}$$

Furthermore, following (5.1), (5.2), and (5.3), the zero-frequency asymptotes of each corresponding spectral signature will consequently obey similar bounding relationships:

$$S_{wx}(0) \leq \bar{x} , \tag{5.7}$$

$$S_{wxy}(0, 0) \leq S_{xy}(0) , \tag{5.8}$$

and

$$S_{wxyz}(0, 0, 0) \leq S_{xyz}(0, 0) . \tag{5.9}$$

Finally, it can be concluded that the source-detector spectral signatures can be used to determine that portion of the between-detectors spectral signatures that arise due to events chain-related to the introduced source:

$$S_{wx}(0) = \overline{x|w} , \tag{5.10}$$

$$S_{wxy}(-s, s) = S_{xy|w}(s) , \tag{5.11}$$

and

$$S_{wxyz}(-s_1 - s_2, s_1, s_2) = S_{xyz|w}(s_1, s_2) . \tag{5.12}$$

These follow directly from (5.4), (5.5), and (5.6).

In weakly multiplicative systems, the majority of singlets (counts), correlated

doublets, and bicorrelated triplets will arise from branching fission chains that originate at the source. That is, most correlated doublets will occur due to source emissions of two or more neutrons, and most bicorrelated triplets will occur due to source emissions of three or more neutrons. However, in more reactive systems, fission-chain multiplication is the primary process producing singlets, doublets, and triplets. In other words, in highly reactive subcritical systems, the injection of only a single source neutron will produce chain-related neutrons in abundance due to branching induced fission-chains. Consequently, the number of singlets, the number of correlated doublets per singlet, and the number of bicorrelated triplets per doublet all tend to be equally sensitive to reactivity. Hence the total number of correlated doublets tends to be more sensitive to reactivity than the number of singlets, and the total number of bicorrelated triplets tends to be more sensitive to reactivity than the number of doublets. An immediate result of this trend is the following: signatures that count bicorrelated triplets are more sensitive to changes in reactivity than those that count correlated doublets, which are in turn more sensitive to changes in reactivity than signatures that count only singlets. Therefore, the second-order source-detector signature tends to have the same sensitivity as the detector count rate, the third-order source-detector signature tends to have the same sensitivity as the second-order between-detectors signature, and the fourth-order source-detector signature tends to have the same sensitivity as the third-order between-detectors signature. Furthermore, because these signatures respectively count singlets, correlated doublets, and bicorrelated triplets, each successive pair of signatures tends to exhibit higher sensitivity to reactivity than the preceding pair. This progression in sensitivity should be observed in yet higher order signatures. Finally, the source-detector signatures add

to their sensitivity the ability to distinguish between events chain-related to the introduced source and those chain-related to other sources.

This feature of the source-detector signatures can be applied to construct a bispectral ratio

$$\Psi_{x \cdot y | w} = \frac{\frac{1}{\bar{w}} \int d\tau C_{wx}(\tau) \cdot \frac{1}{\bar{w}} \int d\tau' C_{wy}(\tau')}{\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 C_{wxy}(\tau_1, \tau_2)} \quad (5.13)$$

that is a measure of the number of independent pairs resulting from source-correlated counts relative to the number of source-bicorrelated doublets. Since the number of doublets grows more rapidly than the number of singlets as a fissile system approaches critical, the bispectral ratio tends to zero as a system approaches prompt critical, and it tends to approach this lower asymptote linearly. The bispectral ratio is insensitive to changes in detection efficiency and does not depend upon the fraction of source fissions that are counted. Furthermore, because each quantity in the bispectral ratio is correlated with the introduced source, the bispectral ratio does not reflect the strength of the inherent source or of any other unobservable sources. Consequently, it may prove to have significant value in interpreting subcritical measurements where the inherent source is substantial and difficult to estimate. For example, the bispectral ratio may prove particularly useful for measuring spent nuclear fuel or for measuring systems containing ^{240}Pu .

Similar reasoning can be applied to construct two trispectral ratios,

$$\Psi_{x \cdot y \cdot z | w} = \frac{\frac{1}{\bar{w}} \int d\tau C_{wx}(\tau) \cdot \frac{1}{\bar{w}} \int d\tau' C_{wy}(\tau') \cdot \frac{1}{\bar{w}} \int d\tau'' C_{wz}(\tau'')}{\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 \int d\tau_3 C_{wxyz}(\tau_1, \tau_2, \tau_3)}, \quad (5.14)$$

and

$$\Psi_{xy \cdot z | w} = \frac{\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 C_{wxy}(\tau_1, \tau_2) \cdot \frac{1}{\bar{w}} \int d\tau C_{wz}(\tau)}{\frac{1}{\bar{w}} \int d\tau_1 \int d\tau_2 \int d\tau_3 C_{wxyz}(\tau_1, \tau_2, \tau_3)} . \quad (5.15)$$

These respectively represent the ratio of independent triplets to bicorrelated triplets and the ratio of correlated doublets to bicorrelated triplets, where each count is chain-related to some initial source event. They both also tend to zero as a fissile system approaches prompt critical, but the first trispectral ratio approaches this lower asymptote at a faster rate since the number of bicorrelated triplets grows much more rapidly than the number of singlets as a system approaches critical. Furthermore, as is true of the bispectral ratio, neither of the trispectral ratios depends upon detection efficiency or the fraction of source fissions counted. Neither do the trispectral ratios respond to the inherent delayed neutron source or any other unobservable sources. Consequently, these signatures may also prove useful in interpreting subcritical measurements.

The generality of these conclusions is supported by the results of Monte Carlo calculations performed on a set of uranium-metal spheres of varying enrichment, which, simple as they are, still do not behave as point reactors. These results showed that the zero-frequency asymptote of the source-detector spectrum had the same sensitivity to reactivity as the detector count rate. The zero-frequency asymptote of the source-detector bispectrum had higher sensitivity to reactivity, and this higher sensitivity was shared by the zero-frequency asymptote of the between-detectors spectrum. Finally, the highest sensitivity to reactivity was observed in the zero-frequency asymptotes of the source-detector trispectrum and between-detectors bispectrum. Furthermore, the calculational results showed that the first trispectral ratio had higher sensitivity to reactivity than did the bispectral ratio and

second trispectral ratio. The latter two ratios appeared to have roughly the same sensitivity to reactivity.

The high-order statistical (HOS) estimation procedures developed to calculate these signatures from Monte Carlo simulation of detector signals are generally applicable and highly efficient. However, a set of HOS estimation procedures optimized for binary detector signals were developed and implemented in a high speed data acquisition system designed specifically for source-driven measurements. These alternative estimation procedures are more efficient than conventional procedures when applied to binary detector signals provided that the sampling rate is sufficiently high that blocks of samples are sparse in counts. Signatures acquired from a uranium-metal casting driven by a ^{252}Cf ion chamber and observed by an array of four plastic scintillators showed a number of features characterizing both the neutron and gamma response of the casting. In particular, the source-detector bicovariance and between-detectors bicovariance revealed information that was hidden in the lower order source-detector covariance and between-detectors covariance. Furthermore, this measurement demonstrated the signatures' sensitivity to neutron and gamma transmission and induced fission. This sensitivity was observed to follow a trend analogous to that predicted by the previous theoretical developments and calculational results. Finally, the bispectral ratio measured from each pair of adjacent detectors yielded the same value independent of the pair's orientation. This suggests that the bispectral ratio may be less sensitive to variations in source and detector positions than the other signatures, which are very sensitive to such changes. For this reason, the bispectral ratio may prove more robust in field applications where precise source and detector positioning is difficult. However, it

is an integral signature, whereas the others depict the distribution of correlated events. Consequently, the bispectral ratio does not reveal as much information as can be observed in the other signatures.

However, the measurement of a single casting does not provide enough information to draw quantitative conclusions regarding these signatures' sensitivity to reactivity under the less than ideal conditions frequently encountered in measurements. Nor do the calculations performed on a set of five uranium spheres even approach a complete characterization of these signatures. The five highly multiplicative uranium spheres barely covered a single decade's variation in reactivity, and the calculations employed high-efficiency detectors and an optimally located source. Computational studies on less reactive systems should be performed to characterize the signatures' behavior in weakly multiplicative systems, and the effects of varying source and detector locations need to be characterized. Furthermore, the best evaluation of the practicality and utility of these signatures will necessarily come from further experiment. Measurements performed on a set of uranium castings of varying enrichment would be a good start – these would more adequately demonstrate the signatures sensitivity to reactivity. There are numerous other subcritical systems, including those composed of fissile oxides and liquids, that have not even been addressed. Nor have reflected or moderated systems been explored. In point of fact, the present work really only introduces the application of high order statistical signatures in nuclear measurements. Only long experience with these methods will allow them to be applied with the sophistication currently employed in the analysis of second-order signatures. But at least now the tools are available to explore these methods.

6 Conclusions and Recommendations

The higher order signatures described in this research measure the time- and frequency-distributions of fission-chain-related doublets and triplets of detection events. The distribution (and hence the total number) of doublets tends to be more sensitive to reactivity than the distribution of singlets. Progressively, the distribution of triplets tends to be more sensitive to reactivity than that of doublets. This trend is expected to continue, i.e., the distribution of $(n+1)$ -tuplets is expected to be more sensitive to reactivity than the distribution of n -tuplets. Consequently, successively higher order signatures possess progressively higher sensitivity to reactivity and therefore are more sensitive to changes in composition, fissile mass, and configuration.

The signatures that correlate with the source introduced to drive the subcritical fissile system are insensitive to any sources that are internal or inherent to the system. Furthermore, they can be applied in tandem with signatures that do not correlate with the introduced source to distinguish between the distribution of singlets, doublets, and triplets that are chain-related to the introduced source and the distribution of those that are chain-related to internal or inherent sources. This capability combined with the enhanced sensitivity of the higher order signatures suggest that they will be of significant utility in a number of applications.

The enhanced sensitivity of these new signatures may be useful in systems identification applications since they should be able to distinguish between very similar, but still different, fissile assemblies. For example, these techniques may be used to enhance radiation signature fingerprints acquired for weapons identification purposes to aid in nuclear

disarmament and safeguards efforts. Furthermore, their ability to effectively separate the contributions of induced versus spontaneous fission to such radiation signatures implies that they may be extremely useful for identifying plutonium weapons components and that they may improve the ability to identify uranium components in the presence of plutonium components. This same ability to distinguish between the introduced and inherent source suggests that these techniques may be applied to the nondestructive analysis of spent nuclear fuels without precise knowledge of the fuels' content of radioactive fission products, (α ,n) sources, and spontaneously fissioning isotopes. These techniques may also prove useful for nondestructive analysis of non-fissile assemblies containing radioactive sources. In point of fact, these are only a few potential applications for higher order signatures acquired from radiation measurements; the future will undoubtedly reveal others. In general, these techniques expand present capabilities in the analysis of subcritical measurements.

List of References

List of References

- [Bel] G. I. Bell, "On the Stochastic Theory of Neutron Transport," *Nucl. Sci. Eng.* **21**, 390, 1965.
- [Bend1] J. S. Bendat and A. G. Piersol, "Spectral Analysis of Non-Linear Systems Involving Square-Law Operations," *J. Sound Vib.* **81**(2), 199, 1982.
- [Bend2] J. S. Bendat and A. G. Piersol, *Random Data: Analysis and Measurement Procedures*, 2nd ed., 252-302, John Wiley and Sons, 1986.
- [Benn] E. F. Bennett, "The Rice Formulation of Pile Noise," *Nucl. Sci. Eng.* **8**, 53, 1960.
- [Brie] J. F. Briesmeister, Ed., "MCNP-4A: A Monte Carlo N-Particle Transport Code," **LA-12625-M**, Los Alamos National Laboratory, 1993.
- [Bril1] D. R. Brillinger, "The Identification of Polynomial Systems by Means of Higher Order Spectra," *J. Sound Vib.* **12**(3), 301, 1970.
- [Bril2] D. R. Brillinger and M. Rosenblatt, "Computation and Interpretation of k-Th Order Spectra," in *Spectral Analysis of Time Series*, B. Harris, Ed., 189, John Wiley and Sons, 1967.
- [Bru] G. S. Brunson, "Measuring the Prompt Period of a Reactor," *Nucleonics* **15**(11), 132, 1957.
- [Bud] C. Budtz-Jorgensen and H. H. Knitter, "Simultaneous investigation of Fission Fragments and Neutrons in ²⁵²Cf (SF)," *Nucl. Phys.* **A490**, 27, 1976.
- [Coh1] C. E. Cohn, "A Simplified Theory of Pile Noise," *Nucl. Sci. Eng.* **7**, 472, 1960.
- [Coh2] C. E. Cohn, "Determination of Reactor Kinetics Parameters by Pile Noise Analysis," *Nucl. Sci. Eng.* **5**, 331, 1959.
- [Coo] G. R. Cooper and C. D. McGillem, *Probabilistic Methods of Signal and System Analysis*, 2nd ed., Harcourt-Brace-Jovanovich, 1986.
- [Cor] N. Corngold, "Theoretical Interpretation of Pulsed Neutron Phenomena," in *Pulsed Neutron Research Vol. I*, 199, Intl. Atomic Energy Agency, May 1965.
- [Cou] E. D. Courant and P. R. Wallace, "Fluctuations in the Number of Neutrons in a Pile," *Phys. Rev.* **72**, 1038, 1947.

- [Cov] R. R. Coveyou, V. R. Cain, and K. J. Yost, "Adjoint and Importance in Monte Carlo Application," *Nucl. Sci. Eng.* **27**, 219, 1967.
- [Dav] W. G. Davey and W. C. Redman, *Techniques in Fast Reactor Critical Experiments*, ANS-USAEC Monograph, Gordon and Breach, 1970.
- [deH1] F. deHoffmann, "Intensity Fluctuations of a Neutron Chain Reactor," **MDCC-382, LADC-256**, 1946.
- [deH2] F. deHoffmann, "Statistical Aspects of Pile Theory," Chap. 9 in *The Science and Engineering of Nuclear Power* Vol. II, C. Goodman, Ed., Addison-Wesley, 1949.
- [Dif] F. C. Difilippo, "Stochastic Processes in a Subcritical Nuclear Reactor in the Presence of a Fission Source," *Nucl. Sci. Eng.* **90**, 13, 1985.
- [Div] B. C. Diven, et al., "Multiplicities of Fission Neutrons", *Phys. Rev.* **101**, 1012, 1956.
- [Ede] M. Edelmann, "New Rossi- α -Measurement Methods," **INR-4/68-15**, Karlsruhe Nuclear Research Center, January 1968.
- [Ell] D. F. Elliot and K. R. Rao, *Fast Transforms: Algorithms, Analyses, Applications*, Academic Press, 1982.
- [Ens] N. Ensslin, et al., "Expected Precision for Neutron Multiplicity Assay Using Higher Order Moments," *Proc. Inst. Nucl. Materials Mgmt.* **26**, 1997.
- [Fey] R. Feynman, F. deHoffmann, and R. Serber, "Dispersion of the Neutron Emission in ^{235}U Fission", *J. Nucl. Energy* **3**, 64, 1956.
- [Fic1] E. P. Ficaro, *KENO-NR: A Monte Carlo Code Simulating the ^{252}Cf -Source-Driven Noise Analysis Experimental Method for Determining Subcriticality*, Ph.D. Dissertation, Univ. of Michigan, 1991.
- [Fic2] E. P. Ficaro and D. K. Wehe, "Monte Carlo Simulations of the ^{252}Cf -Source-Driven Noise Analysis Measurements for Determining Subcriticality," in *Proc. of the Intl. Topical Mtg. on Advances in Mathematics, Computations, and Reactor Physics Vol. 1*, p. 5.22.1, American Nucl. Soc., 1991.
- [Fri] O. R. Frisch and D. J. Littler, "Pile Modulation and Statistical Fluctuation in Piles," *Phil. Mag.* **45**, 126, 1954.

- [Goe] G. Goertzel and M. H. Kalos, "Monte Carlo Methods in Transport Problems," in *Progress in Nuclear Energy II, Series 1, Physics and Mathematics*, D. J. Hughes, J. E. Sanders, and J. Horowitz, Eds., 315, Pergamon Press, 1958.
- [Hag] W. Hage and D. M. Cifarelli, "Correlation Analysis with Neutron Count Distributions in Randomly or Signal Triggered Time Intervals for Assay of Special Fissile Materials," *Nucl. Sci. Eng.* **89**, 75, 1985.
- [Han] G. E. Hansen, H. H. Helmick, and J. D. Orndoff, "Neutron Counting Statistics in Basic Fast Critical Assemblies," Proc. Joint Japan-United States Seminar on Reactor Noise Analysis, 1968.
- [Has] K. Hasselmann, W. Munk, and G. MacDonald, "Bispectra of Ocean Waves," in *Proc. of the Symposium on Time Series Analysis*, M. Rosenblatt, Ed., 125, John Wiley and Sons, 1962.
- [Hen] A. F. Henry, *Nuclear Reactor Analysis*, Massachusetts Inst. Technology, 1975.
- [Hin] M. J. Hinich and C. S. Clay, "The Application of the Discrete Fourier Transform in the Estimation of Power Spectra, Coherence, and Bispectra of Geophysical Data," *Reviews of Geophysics* **6**(3), 347, August 1968.
- [Kal] Malvin H. Kalos and Paula A. Whitlock, *Monte Carlo Methods*, John Wiley and Sons, 1986.
- [Kay] Steven M. Kay, *Modern Spectral Estimation: Theory and Application*, Prentice-Hall, 1988.
- [Kee] G. R. Keepin, *Physics of Nuclear Kinetics*, Addison-Wesley, 1965.
- [Kno] G. F. Knoll, *Radiation Detection and Measurement*, 2nd ed., John Wiley and Sons, 1989.
- [Lii1] K. S. Lii, M. Rosenblatt, and C. Van Atta, "Bispectral Measurements in Turbulence," *J. Fluid Mech.* **77** part I, 45, 1976.
- [Lii2] K. S. Lii and M. Rosenblatt, "Deconvolution and Estimation of Transfer Function Phase and Coefficients for Nongaussian Linear Processes," *Annals of Statistics* **10**(4), 1195, 1982.
- [Loh] A. W. Lohmann and B. Wirnitzer, "Triple Correlations," *Proc. IEEE* **72**(7), 889, July 1984.

- [Man] W. Manhart, "Evaluation of Cf-252 Fission Neutron Spectrum Between 0 MeV and 20 MeV," **IAEA-TECDOC-410**, Intl. Atomic Energy Agency, 1987.
- [Mar] S. Lawrence Marple, Jr., *Digital Signal Analysis with Applications*, Prentice-Hall, 1987.
- [Mats] Toshifumi Matsuoka and T. J. Ulrych, "Phase Estimation Using the Bispectrum," *Proc. IEEE* **72**(10), 1403, October 1984.
- [MATL] MATLAB HOSA Toolbox, <http://www.mathworks.com/products/hosa/>, maintained by The MathWorks, Inc. (info@mathworks.com), 1998.
- [Men] Jerry M. Mendel, "Tutorial on Higher-Order Statistics (Spectra) in Signal Processing and System Theory: Theoretical Results and Some Applications," *Proc. IEEE* **79**(3), 278, March 1991.
- [Mih1] J. T. Mihalczo, "Use of ^{252}Cf as a Randomly Pulsed Neutron Source for Prompt Neutron Decay Measurements," **Y-DR-41**, UCC-ND Oak Ridge Y-12 Plant, 1970.
- [Mih2] J. T. Mihalczo and V. K. Paré, "Theory of Correlation Measurements in Time and Frequency Domains with ^{252}Cf ," **ORNL/TM-4732**, Oak Ridge National Laboratory, 1974.
- [Mih3] J. T. Mihalczo, V. K. Paré, G. L. Ragan, M. V. Mathis, and G. C. Tillet, "Determination of Reactivity from Power Spectral Density Measurements with Californium-252," *Nucl. Sci. Eng.* **66**, 29, 1978.
- [Mih4] J. T. Mihalczo and W. T. King, "Evaluation of an Alternate Three-Detector Spectral Density Measurement for Determination of Subcriticality," *American Nucl. Soc. Trans.* **43**, 702, 1982.
- [Mih5] J. T. Mihalczo, R. C. Kryter, W. T. King, and E. D. Blakeman, "Cf-Source-Driven Neutron-Noise Measurements of Subcriticality for a 4.95 wt% ^{235}U -Enriched Uranyl Fluoride Solution Cylinder," *Ann. Nuclear Energy* **13**(7), 351, 1986.
- [Mih6] J. T. Mihalczo and T. E. Valentine, "Calculational Verification and Process Control Applications Utilizing the High Sensitivity of Noise Measurement Parameters to Fissile System Configuration," *Nucl. Sci. Eng.* **121**, 286, 1995.

- [Mih7] J. T. Mihalczo, "Prompt Neutron Decay for Very Loosely Coupled Delayed Critical 38.1-cm-diam Uranium (93.2) Metal Cylinders," *American Nucl. Soc. Trans.* **75**, 350, 1996.
- [Mih8] J. T. Mihalczo, T. E. Valentine, and J. K. Mattingly, "NWIS Methodology," **Y/LB-15,953**, Oak Ridge Y-12 Plant, October 1997.
- [Mih9] J. T. Mihalczo, personal communication.
- [Moo] M. N. Moore, "The Determination of Reactor Transfer Functions from Measurements at Steady Operation," *Nucl. Sci. Eng.* **3**, 387, 1958.
- [Muñ1] J. L. Muñoz-Cobo, R. B. Perez, and G. Verdú, "Stochastic Neutron Transport Theory: Neutron Counting Statistics in Nuclear Assemblies," *Nucl. Sci. Eng.* **95**, 83, 1987.
- [Muñ2] J. L. Muñoz-Cobo and G. Verdú, "Neutron Stochastic Transport Theory with Delayed Neutrons," *Ann. Nucl. Energy* **14**(7), 327, 1987.
- [Net] J. Neter, W. Wasserman, and M. H. Kutner, *Applied Linear Statistical Models*, 3rd ed., 6, Richard D. Irwin, Inc., 1990.
- [Nik] C. L. Nikias and M. R. Raghuveer, "Bispectrum Estimation: A Digital Signal Processing Framework," *Proc. IEEE* **75**(7), 869, 1987.
- [Nom] Tsutomu Nomura, Seiichiro Gotoh, Kozo Yamaki, "Reactivity Measurements by Neutron Noise Analysis Using Two-Detector Correlation Method and Supercritical Reactor Noise Analysis," Nippon Atomic Industry Group Co., Ltd.
- [Noo] J. Noonan, V. Premus, and J. Izra, "AR Model Selection Based on Bispectral Cross Correlation," *IEEE Trans. Signal Processing* **39**(6), 1440, June 1991.
- [Opp] A. V. Oppenheim and R. W. Schaffer, *Discrete-Time Signal Processing*, Prentice-Hall, 1989.
- [Orn] J. D. Orndoff, "Prompt Neutron Periods of Metal Critical Assemblies," *Nucl. Sci. Eng.* **2**, 450, 1957.
- [Pál] L. Pál, *Il Nuovo Cimento Suppl.* **VII**, 25, 1958.
- [Pap] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, 2nd ed., McGraw-Hill, 1984.

- [Pax] H. C. Paxton and N. L. Pruvost, "Critical Dimensions of Systems Containing ^{235}U , ^{239}Pu , and ^{233}U ," **LA-10860-MS**, Los Alamos National Laboratory, 1987.
- [Per] R. B. Perez, personal communication.
- [Pet] L. M. Petrie, personal communication.
- [Piñ] J. Piñeyro and K. Behringer, "Bispectrum Estimation Using Overlapped Segments," in *Noise and Nonlinear Phenomena in Nuclear Systems*, J. L. Muñoz-Cobo and F. C. Difilippo, Eds., NATO ASI Series B: Physics Vol. 192, 275, Plenum Press, 1988.
- [Pre] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Programming*, 2nd ed., 496-608, Cambridge University Press, 1992.
- [Rag] M. R. Raghuveer and C. L. Nikias, "Bispectrum Estimation: A Parametric Approach," *IEEE Trans. ASSP* **ASSP-33**(4), 1213, October 1985.
- [Rei] D. Reilly, N. Ensslin, and H. Smith, Jr., Eds., *Passive Nondestructive Assay of Nuclear Materials*, **NUREG/CR-5550**, **LA-UR-90-732**, 352-353, Los Alamos National Laboratory, March 1991.
- [Rob] A. Robba, E. Dowdy, and H. Atwater, "Neutron Multiplication Measurements Using Moments of the Neutron Counting Distribution," *Nucl. Instr. and Meth.* **215**, 473, 1983.
- [Ros1] M. Rosenblatt and J. W. Van Ness, "Estimation of the Bispectrum," *Ann. Math Stat.* **36**, 1120, 1965.
- [Ros2] M. Rosenblatt, "Linear Processes and Bispectra," *J. Appl. Prob.* **17**, 265-270, 1980.
- [Sei] W. Seifritz, D. Stegemann, and W. Văth, "Two-Detector Crosscorrelation Experiments in the Fast-Thermal Argonaut Reactor STARK," *Intl. Symposium on Neutron Noise, Wave and Pulse Propagation*, February 14-16, 1966.
- [Spe] R. R. Spencer, R. Gwin, and R. Ingle, "A Measurement of the Average Number of Prompt Neutrons for Spontaneous Fission of Californium-252," *Nucl. Sci. Eng.* **80**, 1982.
- [Thi] J. A. Thie, *Reactor Noise*, Rowman and Littlefield, 1963.

- [Uhr] R. Uhrig, *Random Noise Techniques in Nuclear Reactor Systems*, Ronald Press, 1970.
- [Upa1] B. R. Upadhyaya, M. Kitamura, and T. W. Kerlin, "Multivariate Signal Analysis Algorithms for Process Monitoring and Parameter Estimation in Nuclear Reactors," *Ann. Nucl. Energy* **7**, 1, 1980.
- [Upa2] B. R. Upadhyaya, O. Glöckler, and J. Ekland, "Multivariate Statistical Signal Processing Techniques for Fault Detection and Diagnostics," *ISA Trans.* **29**(4), 79, 1990.
- [Val1] T. E. Valentine, *MCNP-DSP: A Neutron and Gamma Ray Monte Carlo Calculation of Source-Driven Noise-Measured Parameters*, Ph.D. Dissertation, Univ. of Tennessee, 1995.
- [Val2] T. E. Valentine and J. T. Mihalczo, "MCNP-DSP: A Neutron and Gamma Ray Monte Carlo Calculation of Source Driven Noise-Measured Parameters," *Ann. Nucl. Energy* **23**, 1271, 1996.
- [Val3] T. E. Valentine and J. T. Mihalczo, "Validation of the Monte Carlo Code MCNP-DSP," *Ann. Nucl. Energy* **24**(2), 79, 1997.
- [Val4] T. E. Valentine, personal communication.
- [Wag] C. Wagemans, *The Nuclear Fission Process*, 329-346, CRC Press, 1991.
- [Wya] M. S. Wyatt, *Characterization of an Enriched Uranyl Fluoride Deposit at the Former K-25 Enrichment Plant Using Active Neutron and Gamma Interrogation Techniques*, M.S. Thesis, Univ. of Tennessee, 1997.
- [Zuc] M. S. Zucker and N. E. Holden, "Energy Dependence of the Neutron Multiplicity P_v in Fast-Neutron-Induced Fission for $^{235,238}\text{U}$ and ^{239}Pu ," *American Nucl. Soc. Trans.* **52**, 630, 1986.

Vita

John Kelly Mattingly was born the first son of Barbara Ann McClimans Mattingly and John Alfred Mattingly in Louisville, Kentucky on June 15, 1970. In August 1988, he enrolled in the nuclear engineering undergraduate program at the University of Tennessee, Knoxville, and in December 1992, he graduated Summa Cum Laude with a Bachelor of Science in nuclear engineering. That semester, he was awarded the title "top graduating senior" by the dean of the college of engineering. In May 1995, John received a Master of Science from the same university, and began his Ph.D. research presented in the preceding. This is just the history of his life; its meaning is embodied in his love for his wife, Laura.

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