Investigating advection control in competitive PDE systems and environmental transmission in Johne's Disease ODE models

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"It is better to conquer yourself than to win a thousand battles. Then the victory is yours. It cannot be taken from you, not by angels or by demons, heaven or hell."

- ***Gautama Buddha***

Abstract

We extend the work on optimal control of advective direction in a reaction-diffusion population model to a system representing two competing populations. We investigate the choice of movement direction to benefit a population. First, the advective direction in one of the populations in a competition model is the control. Next, we extend the work by taking the advective directions of both populations as controls. In both these cases the objective is to maximize a weighted combination of the two populations while minimizing the cost involved in the species movement. Mathematical analysis is completed to derive the optimality system and numerical results illustrating solutions of this system are presented.

Johne's disease is a bacterial infection caused by *Mycobacterium avium* subspecies *paratuberculosis* (MAP). It is a chronic, progressive, and infectious disease which has a long incubation period and probably not curable. The main problem with the disease is the reduction of milk production in infected dairy cows. We develop a deterministic model to describe the dynamics of the Johne's disease in a dairy farm. In this model we use a system of ordinary differential equations to describe the behavior of Johne's disease among dairy cows considering the progression of the disease and the age structure of the cows. We analyze the behavior of the Johne's disease by investigating the effects of the persistence of the bacteria in the environment. Stability and numerical results are computed.

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Chapter 1

Introduction

Mathematical modelling is a main branch in mathematics which interacts with different disciplines. It helps to capture and understand dynamics of the situations which would otherwise be impossible, difficult or could take a long time to observe in reality. In biology, mathematical modelling has been used widely from microscopic level experiments to macroscopic level experiments.

In this dissertation we consider mathematical modelling in two different areas: in population dynamics and in disease modeling. Also, we use optimal control theory, which is an important tool in mathematics for our studies.

1.1 Optimal control theory

Optimal control is an important tool in mathematics which can be used in many real world applications. It helps in making decisions by maximizing or minimizing a goal subject to dynamic equations influenced by control functions. Mainly when there is some risk or cost involved in using the control, the objective functional can be structured in a way to optimize our main goal while minimizing the risk or the cost in implementing that controls in the system.

The first step in an optimal control problem is to define the variables defining states of the system and the controls which can be used to change the states. Next, the

dynamics of these state variables and the control variables are represented in a system. The dynamics of this system can be defined using ordinary differential equations (ODEs), partial differential equations (PDEs), stochastic differential equations, discrete equations, integro-difference equations or a combination of above equations [26]. Further, we give the initial conditions and/or final conditions, boundary conditions according to the system. Next, using the state variables and the control variables we define the objective functional, which is usually given as an integral over the time. Here, we use different weight parameters to show the relative importance of the different goals versus the cost or the risk involved in implementing the controls on the system. Hence, the optimal control problem consists of state equations with the corresponding initial/final conditions, boundary conditions, a control set and the objective functional.

After the formulation of the optimal control problem, the next step is to solve it using the appropriate mathematical techniques. Here, in this introduction we briefly explain how to solve an optimal control problem with ODEs and PDEs. Detailed information for optimal control on ODEs can be found in [26, 38] and detailed information for optimal control on different PDE systems can be found in [31].

1.1.1 Optimal control for ODE's

After formulating the optimal control problem with state ODEs, controls and the objective functional, first step is to prove the existence of a solution for the ODE system with a given control. Next, we need to prove the existence of an optimal control using uniformly boundedness of finite states, controls and objective values. Then, we form the optimality system, characterize an optimal control and prove the uniqueness of the optimal control for a sufficiently small time.

About 1950 Pontryagin and his collaborators laid the foundation for optimal control theory for ordinary differential equations (ODEs). Pontryagin [38] introduced the adjoint variables to attach the differential equations to the objective functional.

Here, to solve the optimal control problem we attach the state ODEs to the objective functional using the adjoint variables [26, 38] by forming the Hamiltonian. By Pontryagin's [38] Maximum Principle, we maximize this Hamiltonian function pointwise with respect to the controls of the optimal controls. We differentiate the Hamiltonian with respect to the time and state variables and using these derivatives we explicitly define the adjoint ODEs and characterize the optimal control. This Adjoint system, the state ODEs and the optimal control characterization form the optimality system. Hence, finally we prove the uniqueness for the solution of the optimality system using some standard techniques, which gives uniqueness of the optimal control [17].

1.1.2 Optimal control for PDE's

To build some generalization of Pontryagin's Maximum Principle to partial differential equations (PDEs), but J. L. Lions [31] introduced the idea of optimal control of PDEs. After setting up a system of PDEs with one control or many controls in an appropriate weak solution space and an objective functional, proving existence of an optimal control is a main step.

Before proving the existence of an optimal control, we have to show that for a given control there exist a unique solution to the system of PDEs describing the dynamics of the system. Next, we show that there exists an optimal control maximizing or minimizing the objective functional and the corresponding optimal solutions for the state PDE system [30].

After proving existence of an optimal control, next we characterize this optimal control. In that regard we have to differentiate the objective functional with respect to each control [31]. Since the state variables depend on the control and are involved in the objective functional, we need to differentiate state variables with respect to control as well. These derivatives obtained in this process are called the sensitivity functions and they solve the linearized version of the state PDE system. The system

of adjoint PDEs is derived from the operators in the sensitivity equations. The non-homogeneous terms in the adjoint equations are found by differentiating the integrand of the objective functional with respect to the state variables. We use these weak solutions for the sensitivity and the adjoint PDEs to characterize an optimal control. The uniqueness of the optimal control is obtained by proving the uniqueness of solutions of the optimality system [17].

1.2 Population dynamics and optimal control in population models

The reaction of a population to resources is an important concern in ecology. Much work has been done to analyze the population dynamics under different resource conditions affecting the population [3, 6, 7, 8, 9, 10, 13, 16, 24]. Particularly, to capture, and explain more realistic dynamics of the populations as well as the persistence of species in the long run, reaction–diffusion PDEs with linear and nonlinear growth terms are used in many studies [3, 7, 8]. The effect of the resources on the population size is also investigated in [6, 13, 24] using reaction-diffusion models. For these reaction-diffusion models, the movement of a population from one place to another can occur in two different ways: random diffusion and directed movements. The reaction of the population to the surrounding environment and its dynamics has been studied with respect to the effects of diffusion coefficients and advection movements [3, 7, 9, 10, 16].

In the inhomogeneous environments, it is commonly believed that the population will move along direction of increase in the resources. With this assumption, work has been done to analyze the existence of steady state solutions by taking the advection term to be the gradient of the resource function [3, 9, 10].

With a different perspective, Phan et al. [16] investigated a general model of nonlinear advection reaction-diffusion equations, to find the optimal advection

coefficient using the optimal control theory. Precisely, they studied the following nonlinear parabolic equation:

$$\begin{aligned} u_t - \nabla \cdot [\mu \nabla u - u\vec{h}] &= u[m - f(x,t,u)], && Q_T \\ \mu \frac{\partial u}{\partial \nu} - u\vec{h} \cdot \nu &= 0, && \mathcal{S}_T \\ u(\cdot, 0) &= u_0 \geq 0, && \Omega \end{aligned}$$

where $Q_T = \Omega \times (0,T)$ for $T > 0$, $S_T = \partial\Omega \times (0,T)$ with Ω. The resource function is represented by $m(x,t)$, and $u(x,t)$ is the population density. Moreover, $\mu > 0$ is the diffusion rate, $\vec{h} : Q_T \to \mathbb{R}^n$ is the advection direction, and $f : Q_T \times \mathbb{R} \to \mathbb{R}$ is a non-negative function satisfying some natural smoothness and growth conditions. The advection direction $\vec{h}(x,t)$ is chosen to maximize the total population while minimizing the "cost" due to the risk of movement. Hence, by taking

$$U = \{\vec{h} \in L^2((0,T), L^2(\Omega)^n) : |h_k| \leq M, \quad \forall k = 1, 2, \cdots, n\}.$$

as the control set, they investigated the maximizer $\vec{h}^*$ of the problem

$$J(\vec{h}^*) = \max_U J(\vec{h}), \quad \text{where} \quad J(\vec{h}) = \int_{Q_T} [u(x,t) - B|\vec{h}(x,t)|^2] dxdt.$$

The numerical results in [16] indeed indicate that the optimal control $\vec{h}^*$ in some cases approximates the spatial gradient of the resource function $m(x,t)$.

Following the line of the research directions in [3, 9, 10, 13, 16], we move further on this work using optimal control techniques to investigate two competing species [25, 27, 28], and explore how the advection directions are chosen in such a way to maximize the total population while minimizing the associated control costs. As in [16], our cost represents the risk for the population when moving to a new location. Our goal is to use optimal control techniques to maximize the total population over the entire domain throughout the fixed period of time.

In Chapter 2 we consider two competing populations: one with both diffusion and advection movements and the other only with the diffusion movement. We investigate the choice of movement direction to benefit populations. Here, the advective directions of one population in a competition parabolic PDE system are the controls. The objective is to maximize a weighted combination of the two populations while minimizing the cost involved in the species movement. Then, we discuss the existence of state solution, and the existence of the optimal control. Further, we derive the optimality system, characterize the optimal control, and prove the uniqueness of this optimal control. Finally, we give some numerical results for the one dimensional case and conclude with some discussion and conclusions. Also, we will discuss further about the conclusions in [16] using some numerical simulations for a one population with advection.

Next, in Chapter 3, we consider two populations with both diffusion and advection movements. Here, we investigate given the chance for the directed movement, how would both populations choose to move in order to maximize their total abundance level. Again, the advective directions of the population in the competition system were taken as the controls. The objective is to maximize a weighted combination of the two populations while minimizing the cost involved in the movement. Hence, first we set up the advection reaction-diffusion equations to describe the dynamics of the two competitive species. Next, as similar to we did in Chapter 2 we discuss the existence of state solution, and the existence, uniqueness and the characterization of the optimal control. At the end again, we give some numerical results for the one dimensional case and concludes the results with some discussions, conclusions, and also some possible future research directions.

1.3 Mathematical modelling in epidemiology

Mathematical models has been developed as an important tool to understand the dynamics, incidence, prevalence of diseases as well as to understand the control of

diseases [12, 20, 23]. While in ecology, mathematical models are used to capture the dynamics of a certain species, in epidemiology, mathematical models can be used to capture the dynamics of a host population due to an infection. Models of infectious diseases can take direct and indirect paths of infection into account. Hence, in disease modeling, the host population is categorized according the stages of the disease and flow diagrams are used to illustrate the transition and the progress of a disease among different compartments in the host population. Usually the dynamics of each of these compartments are modeled by an ordinary differential equation.

The formulation of a compartment model depends on the nature and the characteristics of the disease [23]. When the time frame for an outbreak is short, the dynamics of the disease can be modeled without taking the population demography into account. But, on the other hand if the disease persists for a long time, population demography needs to be taken into account. For example, a simpler epidemic compartment model is the SIR model, with this name of the model is based on the stages of the disease (compartments) used in the model. Some other examples are SEIR, SI, SIRS models. Here, S, E, I, R stands for the susceptible, exposed, infected and recovered populations respectively. More detail on the terminologies, and different epidemic models can be found in [12, 20, 23].

One of the important terms in an epidemiological model is the transmission term. Here, to define the transmission we use the 'force of infection' which is defined as the per capita rate at which the susceptible individuals contract the infections [23]. The force of infection depends on the probability that a susceptible has contact with an infected and the probability that the contact will result in a successful transmission. There are two common ways to formulate the contact structure of the susceptible individuals with the infected individuals namely, density dependent transmission and frequency dependent transmission. The other important parameter in an epidemiological model is the recovery rate, which calculated as the inverse of the infectious period. This infectious period is usually calculated by the available epidemiological data.

1.4 Modeling disease transmission through the environment

When a pathogen cannot survive outside the host, the disease can be transmitted from one host to another only through direct contacts (for example measles) and these epidemiological models consider only the host–to–host transmission. But, there are many diseases which can be transmitted through both direct and indirect routes (for example influenza) due to bacteria and viruses which can survive in the environment for some time. As a result, in recent studies in disease modeling, researchers are analyzing the behavior of some diseases which can transmit through the environment.

There have been several studies modeling the effect of the environment on transmitting infectious diseases [1, 5, 29, 41]. In some studies the pathogen in the environment was treated as another compartment in the compartment models [5, 29, 41] and in others they were trying to model the disease dynamics using metapopulation models [1].

In [29], Li et al. developed a compartment model to analyze the dynamics of influenza. They treated the pathogen as one compartment and defined the transmission of the disease to susceptible individuals as a function of the intake of the pathogens from the environment. Further, using their mathematical model they studied the dependence of the frequency of the environment interaction of the susceptible individuals on the transmission of the disease. In [5] R. Breban studied SI models with the disease transmission through the environment. He treated the pathogens in the environment as a compartment and he defined the environment transmission rate as a function of pathogens in the environment. He studied the system of ODEs:

$$\begin{aligned}
\frac{dS}{dt} &= \pi - \mu S - \rho S f(V) \\
\frac{dI}{dt} &= \rho S f(V) - (\mu + \gamma) I \\
\frac{dV}{dt} &= \omega I - \eta V
\end{aligned}$$

where S is the number of susceptible individuals, I is the number of infected individuals, V is the number of pathogens in the environment, $1/\gamma$ is the recovery period, $1/\eta$ is the persistence time of the pathogen, ω is the shedding rate of the pathogens by the infected individuals, π is the susceptible inflow, μ is the natural death rate of the individuals and ρ is the contact rate with the environment. Breban analyzed the conditions needed to approximate the environment transmission as direct transmission.

Using a different approach to consider the pathogen in the environment, in [1] Ayescue et al. developed a mathematical model using metapopulation structure. They studied the dynamics of the *Escherichia coli* O157:H7 by considering water troughs, feed bunks, pen floors and the cattle hosts as the habitats for the pathogen. They described the dynamics of the bacteria in each location by using a system of ODEs.

1.5 Modeling the dynamics of the Johne's Disease

Johne's disease affects the small intestine of ruminants. It is a bacterial infection caused by *Mycobacterium avium* subspecies *paratuberculosis* (MAP). Johne's disease has a long incubation period and it is infectious, chronic, progressive and ultimately leads the animal to the death due to disease. Usually MAP spreads through oral transmission, MAP contaminated feces, water, soil, milk and colostrum and in utero

transmission was also reported [37, 40, 48]. Further, MAP can survive in the environment for a year or more [48, 50].

In recent years, much work has been done to understand and control the Johne's disease [33, 34, 35, 36, 37, 40]. Mathematical modeling has been used in many of these studies to capture the long term behavior of the disease. Some of these studies used discrete models and some of them have used continuous models with different biological hypotheses. Also, in some studies they have used deterministic models and sometimes stochastic models. In some of these studies they have considered the transmission of the disease through the environment [35, 36].

In Chapter 4 we construct a new deterministic model of Johnes disease to understand the mechanisms of disease transmission and progression, especially to include environment components. Here, we construct a compartment model by considering the bacteria in the environment as a compartment. We define a sigmoidal function to model the environmental bacterial transmission to the susceptible animals by considering the infectious dose of MAP and the probability of getting infected by the bacteria. To analyze the dynamics, first we considered a simplified model and later we analyzed our original model. We analyze the stability of the disease free equilibrium and discuss the basic reproduction number for both models. Further, we analyze how the testing and culling as well as cleaning the environment can help with controlling Johne's disease in a dairy farm.

1.6 Numerical methods

To solve the optimal control problems in Chapter 2 and Chapter 3 we have used the forward backward sweep method [21, 26]. To solve the ODE system in Chapter 4 we have used the ODE45 function in MATLAB and the first order Euler method.

Chapter 2

First Competition Model

2.1 Introduction

Population movement and its distribution in reaction to its surrounding environment are important concerns in ecology. Many efforts are devoted to analyze the population dynamics under different conditions affecting the population [3, 9, 6, 7, 8, 10, 13, 16, 24]. Particularly, to capture, and explain more realistic dynamics of the populations as well as the persistence of species in the long run, reaction–diffusion partial differential equations (PDEs) with linear and nonlinear growth terms have been used in many studies [3, 7, 8]. The effect of the resources on the population size was also investigated in [6, 13, 24] using such models. For these reaction-diffusion models, the spatial movement of a population can occur in two different ways: random diffusion and directed movements. The reaction of the population to the surrounding environments depends on the choice of diffusion and advection coefficients in the model [3, 7, 9, 10, 16].

In the inhomogeneous environments, it is commonly believed that the population will move along the direction of increasing resources. With that regard, Belgacem and Cosner [3] investigated an advection reaction-diffusion model in which the advection term is the gradient of the resource function. They investigated steady states solutions

of the reaction-diffusion equations with linear and nonlinear logistic growth terms:

$$u_t = \nabla \cdot [D\nabla u - \alpha u \nabla m(x)] + m(x)u, \quad \Omega \times (0, \infty)$$

and

$$u_t = \nabla \cdot [D\nabla u - \alpha u \nabla m(x)] + m(x)u - cu^2, \quad \Omega \times (0, \infty)$$

together with no flux boundary condition or Dirichlet boundary condition (lethal boundary). In these PDEs, the coefficient $m(x)$ for $x \in \Omega$ represents the intrinsic growth rate, and measures the availability of the resources. Moreover, $u(x, t)$ is the population density of the species at location $x \in \Omega$ and time $t \geq 0$. They studied the benefit to the population, [3] meaning the persistence of the population in the long run or the existence of a unique globally attracting positive steady state solution. An interesting observation in [3] was that the directed movement towards better resources could be beneficial or harmful to the population depending on the boundary conditions. In the case of no flux boundary conditions, the movement towards better resources could be beneficial to the population, while in the lethal boundary conditions, the movement towards better resources can be harmful if more favorable patches are closer to the boundary. In a further investigation [10], Cosner et al. studied the logistic reaction–diffusion models with the advection along the environmental gradient, with the no flux boundary conditions. Different from the results in [3], they found that even under the zero flux boundary conditions, the movement along the resource gradient may not always be beneficial to the population. Indeed, it turns out that the convexity of the domains plays a major role in this situation. If the domain is not convex, moving up along the resource gradient could be harmful to the population.

In a similar perspective as in [3], system of competing species have also been investigated. In particular, in [9], Lou et al. studied the persistence of two populations in a heterogeneous environment using Lotka–Volterra advection reaction–diffusion equations. They investigated two different movement strategies for the two completive

populations: one population only moves with random diffusion, while the other population is allowed to have both the random diffusion, and the directed movement along the resource gradient. A related work [10] shows that when the advection is strong, both competing populations can stably coexist.

In an alternate but related viewpoint, optimal control techniques were used in various work such as [13, 16, 24] to explore how different conditions such as limited resources, growth coefficient, advection movement, and harvesting can be optimized to be "beneficial" for populations. In particular, Ding et al. studied in [13] the effects of resource allocation on population size of the species. They investigated the problem in which the resource function $m(x)$ is the control, and the population abundance is a measurement of conservation effort. In other words, the work [13] explores the optimum resource allocation which maximizes the total population with the minimum cost for the resources.

In a similar framework as in [13], but different direction, Phan et al. [16] considered a general model of nonlinear advection reaction-diffusion equation. Precisely, they studied the following nonlinear parabolic equation:

$$\begin{cases} u_t - \nabla \cdot [\mu \nabla u - u\vec{h}] &= u[m - f(x,t,u)], & Q_T, \\ \mu \dfrac{\partial u}{\partial \nu} - u\vec{h} \cdot \nu &= 0, & \mathcal{S}_T, \\ u(\cdot, 0) &= u_0 \geq 0, & \Omega, \end{cases}$$

where $Q_T = \Omega \times (0,T)$ for $T > 0$, $S_T = \partial\Omega \times (0,T)$ with Ω, $m(x,t)$ and $u(x,t)$ as before. Moreover, $\mu > 0$ is the diffusion rate, $\vec{h} : Q_T \to \mathbb{R}^n$ is the advection direction, and $f : Q_T \times \mathbb{R} \to \mathbb{R}$ is a non-negative function satisfying some natural smoothness and growth conditions. In their study, the advection direction $\vec{h}$ is the control, and they seek for the optimal advection direction $\vec{h}(x,t)$ that maximizes the total population while minimizing the "cost" due to movement. In this work, the "cost" represents the risk of moving to a new location. The numerical results in [16] indeed indicate that the optimal control $\vec{h}^*$ in some cases approximates the spatial

gradient of the resource function $m(x,t)$. We will discuss this issue further in the numerical results of Chapter 2.

Following the line of the research directions in [3, 9, 10, 13, 16], we move further on this work using optimal control techniques to investigate two competing species[25, 27, 28], and explore how the advection directions are chosen to maximize the total population while minimizing the costs related to the risk of movement. Our goal is to analyze this playoff between benefits and cost using the optimal control techniques.

2.2 Problem formulation

Let Ω be an open bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, for $d \in \mathbb{N}$. For a given fixed time $0 < T < \infty$, we denote $Q = \Omega \times (0,T)$, and the lateral boundary of Q, $S = \partial\Omega \times (0,T)$. We denote $u(x,t), v(x,t)$ to be the population densities of two competing species in the spatial domain Ω with $x \in \Omega$ and $t \in [0,T)$.

Here, we consider the dynamics of the two populations using the following system of PDE's with the Robin boundary conditions:

$$\begin{aligned}
u_t - d_1\Delta u - \nabla\cdot(\vec{h}u) &= u[m - a_1u] - b_1uv \quad \text{in } Q \\
v_t - d_2\Delta v &= v[m - a_2v] - b_2uv \quad \text{in } Q \\
d_1\frac{\partial u}{\partial \eta} + u\vec{h}\cdot\eta &= 0 \quad \text{on } S \\
d_2\frac{\partial v}{\partial \eta} &= 0 \quad \text{on } S \\
u(x,0) &= u_0(x) \geq 0 \quad \text{for a.e. } x \in \Omega \\
v(x,0) &= v_0(x) \geq 0 \quad \text{for a.e. } x \in \Omega
\end{aligned} \tag{2.1}$$

The first population is given the opportunity to move (advection movement) in addition to random diffusion and the second population only has random diffusion. We consider Robin boundary condition for the first population and Neumann

boundary condition for the second population. Further, we assume the density dependent growth for both species. These two populations are competing with interaction coefficients $b_1(x,t)$ and $b_2(x,t)$.

We define the parameters:

d_1, d_2 – Constant positive diffusion coefficients

a_1, a_2 – Coefficients in the density dependent growth terms

m – Known resource function, $0 \leq m(x,t) \leq C_0$ on Q

b_1, b_2 – Positive interaction coefficients

$\vec{h}(x,t) = (h_1(x,t), h_2(x,t), ..., h_d(x,t))$ – Advective coefficient for the first species with $\vec{h} \in L^\infty(Q)^d$

η – Outward normal vector on $\partial\Omega$

u_0, v_0 – Initial population densities of the two populations with $u_0, v_0 \in H^1(\Omega)$

We use the following spaces:

$H^1(\Omega)$ – Sobolev space with $\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx\right)^{1/2}$

$H^1(\Omega))^*$ – Dual space of $H^1(\Omega)$

2.3 The optimal control problem formulation

We assume the first population can choose its advective direction $\vec{h}$ to maximize its abundance, but this advective movement may have a "cost" in terms of risk or energy. Viewing the advective coefficient as a control, the set of all controls is

$$U = \{\vec{h} \in (L^\infty(Q))^d : |h_i| \leq M \text{ for } i = 1, ..., d\}$$

for some known constant $M > 0$.

Then, for the known values of $A, B, C > 0$ we seek to find $\vec{h} \in U$ s.t.

$$J(\vec{h}^*) = \max_{\vec{h} \in U} J(\vec{h})$$

where, $J(\vec{h})$ is the objective functional given by,

$$J(\vec{h}) = \int_Q \left[Au + Bv - C|\vec{h}|^2 \right] \, dxdt \tag{2.2}$$

subject to the PDE system (2.1).

Our goal is to maximize a weighted sum of the two populations over the time while trying to minimize the total cost involved in this movement subject to the PDE system (2.1). In this case the cost is due to the "risk" in moving the population around by the advection as the movement to a new location may come with a risk.

2.4 Solving the optimal control problem

We first recall the definition of weak solutions for the state PDE system.

Definition (The weak solution for the state PDE system)

$u, v \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$ with $u_t, v_t \in L^2((0,T), (H^1(\Omega))^*)$ and $u(x,0) = u_0, v(x,0) = v_0$ is a weak solution for (2.1) if it satisfies,

$$\begin{aligned}
&\int_0^T < u_t, \phi_1 >_{((H^1)^*, H^1)} \, dt + \int_Q d_1 \nabla u \cdot \nabla \phi_1 \, dxdt + \int_Q u\vec{h} \cdot \nabla \phi_1 \, dxdt \\
&= \int_Q um\phi_1 \, dxdt - \int_Q a_1 u^2 \phi_1 \, dxdt - \int_Q b_1 uv\phi_1 \, dxdt \quad \text{in } Q, \text{ and} \\
&\int_0^T < v_t, \phi_2 >_{((H^1)^*, H^1)} \, dt + \int_Q d_2 \nabla v \cdot \nabla \phi_2 \, dxdt \\
&= \int_Q vm\phi_2 \, dxdt - \int_Q a_2 v^2 \phi_2 \, dxdt - \int_Q b_2 uv\phi_2 \, dxdt \quad \text{in } Q
\end{aligned} \tag{2.3}$$

for a.e. $x \in \Omega$ and for all $\phi_1\ ,\ \phi_2 \in L^2((0,T), H^1(\Omega))$.

We use $<,>_{((H^1)^*,H^1)}$ to denote the duality on Ω.
Note that we assume that $u, v \in L^\infty(Q)$, so that the terms with u^2, v^2 and uv make sense in $L^2(Q)$.

We give an explanation of this weak formulation.

Let $\phi_1\ ,\ \phi_2 \in L^2((0,T), H^1(\Omega))$ and we derive the weak formulation formally:

$$\begin{aligned}&\int_Q u_t\phi_1\ dxdt - \int_Q d_1\Delta u\phi_1\ dxdt - \int_Q \nabla\cdot(\vec{h}u)\phi_1\ dxdt\\ &= \int_Q um\phi_1\ dxdt - \int_Q a_1u^2\phi_1\ dxdt - \int_Q b_1uv\phi_1\ dxdt \qquad \text{in } Q\end{aligned}$$

and

$$\begin{aligned}&\int_Q v_t\phi_2\ dxdt - \int_Q d_2\Delta v\phi_2\ dxdt\\ &= \int_Q vm\phi_2\ dxdt - \int_Q a_2v^2\phi_2\ dxdt - \int_Q b_2uv\phi_2\ dxdt \qquad \text{in } Q.\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}&\int_0^T <u_t,\phi_1>_{((H^1)^*,H^1)}\ dt + \int_Q d_1\nabla u\cdot\nabla\phi_1\ dxdt - \int_S d_1\phi_1\frac{\partial u}{\partial\eta}\ dsdt\\ &+ \int_Q u\vec{h}\cdot\nabla\phi_1\ dxdt - \int_S u\phi_1\vec{h}\cdot\eta\ dsdt\\ &= \int_Q um\phi_1\ dxdt - \int_Q a_1u^2\phi_1\ dxdt - \int_Q b_1uv\phi_1\ dxdt \qquad \text{in } Q\end{aligned}$$

and

$$\begin{aligned}&\int_0^T <v_t,\phi_2>_{((H^1)^*,H^1)}\ dt + \int_Q d_2\nabla v\cdot\nabla\phi_2\ dxdt - \int_S d_2\phi_2\frac{\partial v}{\partial\eta}\ dsdt\\ &= \int_Q vm\phi_2\ dxdt - \int_Q a_2v^2\phi_2\ dxdt - \int_Q b_2uv\phi_2\ dxdt \qquad \text{in } Q.\end{aligned}$$

Then, by

$$d_1 \frac{\partial u}{\partial \eta} + u\vec{h} \cdot \eta \ = 0 \quad \text{and} \quad d_2 \frac{\partial v}{\partial \eta} \ = 0 \quad \text{on } S$$

we get the weak formulation (2.3).

Given $\vec{h} \in U$, we denote the corresponding states as $u(\vec{h}), v(\vec{h})$.

Lemma 2.0.1. *For any $0 \leq R < \infty$, $|\vec{h}| \in L^\infty(Q)^d$ and $u_0 \geq 0$ with $g(x,t) \geq 0$ on Q and $g \in L^2(Q)$, the weak solution $u \in L^2((0,T), H^1(\Omega))$ of*

$$\begin{aligned} u_t - d_1 \Delta u - \nabla \cdot (\vec{h} u) + Ru &= g \quad \text{in } Q \\ d_1 \frac{\partial u}{\partial \eta} + u\vec{h} \cdot \eta &= 0 \quad \text{on } S \\ u(x,0) &= u_0(x) \geq 0 \quad \text{for } x \in \Omega \end{aligned} \tag{2.4}$$

is non-negative on Q.

Proof. We use a similar proof as in Lemma 3.1 in [16].
Let $0 \leq s \leq T$ and $Q_s = \Omega \times (0,s)$. For $(x,t) \in Q_s$, let $w(x,t) = max\{-u(x,t), 0\}$.
Then, using w in the weak form of (2.4) we have

$$\int_\Omega u_t w \, dx + \int_\Omega d_1 \nabla u \cdot \nabla w \, dx + \int_\Omega u\vec{h} \cdot \nabla w \, dx + \int_\Omega Ruw \, dx \geq 0 \quad \text{for } 0 \leq t \leq s \, .$$

But, on the set $\{(x.t) \in Q_s : u(x,t) < 0\}$ we have

$$u = -w, u_t = -w_t, \nabla u = -\nabla w \, .$$

Hence, we get

$$\int_\Omega w_t w \, dx + \int_\Omega d_1 |\nabla w|^2 \, dx + \int_\Omega w\vec{h} \cdot \nabla w \, dx + \int_\Omega Rw^2 \, dx \leq 0 \quad \text{for } 0 \leq t \leq s \, .$$

Using

$$\int_\Omega w_t w dx = \int_\Omega \frac{1}{2}\frac{\partial(w^2)}{\partial t} dx \ ,$$

we have

$$\begin{aligned}
\int_\Omega \frac{1}{2}\frac{\partial(w^2)}{\partial t} dx + \int_\Omega d_1 |\nabla w|^2 \ dx &\leq -\int_\Omega w\vec{h}\cdot\nabla w \ dx - \int_\Omega Rw^2 \ dx \\
&\leq \int_\Omega w|\vec{h}|\cdot|\nabla w| \ dx + \int_\Omega Rw^2 \ dx \\
&\leq M\int_\Omega w|\nabla w| \ dx + R\int_\Omega w^2 \ dx \\
&\leq \frac{M}{4\epsilon}\int_\Omega w^2 dx + M\epsilon\int_\Omega |\nabla w|^2 \ dx + R\int_\Omega w^2 \ dx
\end{aligned}$$

using $w \geq 0$, $\vec{h} \in U$ and Cauchy's inequality.

This implies

$$\int_\Omega \frac{\partial(w^2)}{\partial t} dx + \int_\Omega 2d_1|\nabla w|^2 \ dx \leq 2R\int_\Omega w^2 \ dx + \frac{M}{2\epsilon}\int_\Omega w^2 dx + 2\epsilon M\int_\Omega |\nabla w|^2 \ dx \ .$$

Choosing

$$\epsilon \leq \frac{d_1}{M} \qquad \text{and} \qquad C = 2R + \frac{M}{2\epsilon} \ ,$$

we have

$$\int_\Omega \frac{\partial(w^2)}{\partial t} dx + \int_\Omega (2d_1 - 2\epsilon M)|\nabla w|^2 \ dx \leq C\int_\Omega w^2 \ dx$$

implies that

$$\frac{\partial}{\partial t}\int_\Omega w^2(x,t) dx \leq C\int_\Omega w^2(x,t) \ dx \quad \text{ for all } \ 0 \leq t \leq T.$$

Note that as $u(x,0) = u_0(x) \geq 0$ for $x \in \Omega$, $w(x,0) = 0$ for $x \in \Omega$ and hence, $\int_\Omega w^2(x,0) \ dx = 0$.

By Gronwall's inequality

$$\int_\Omega w^2(x,t)dx = 0 \quad \text{for all } \ 0 \le t \le T,$$

which implies $w(x,t) = 0$ for all $(x,t) \in Q_s$. Hence, we can see that $u \ge 0$ in Q.

□

Lemma 2.0.2. *Let $M > 0$ be defined in the definition of the control set, and $C_0 = \|m\|_{L^\infty(Q)}$. For a given control $(\vec{h_1}, \vec{h_2}) \in U$ let $u, v \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$ with $u_t, v_t \in L^2((0,T), (H^1(\Omega))^*)$ be a weak solution of (2.1) with $u \ge 0$, $v \ge 0$ a.e. on Q. Then, there exists constants K_1, K_2, K_3, K_4 depending only on M, C_0 and independent of $\vec{h}$ such that*

$$\begin{aligned} \|u(\vec{h})\|_{L^2((0,T),H^1(\Omega))} &\le K_1 \\ \|u_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\le K_2 \\ \|v(\vec{h})\|_{L^2((0,T),H^1(\Omega))} &\le K_3 \\ \|v_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\le K_4 \ . \end{aligned}$$

Proof. First, consider

$$\begin{aligned} u_t - d_1\Delta u - \nabla\cdot(\vec{h}u) &= u[m - a_1 u] - b_1 uv \quad \text{in } \ Q \\ d_1\frac{\partial u}{\partial \eta} + u\vec{h}\eta &= 0 \quad \text{on } \ S \\ u(x,0) &= u_0(x) \ge 0 \quad \text{on } \ \Omega. \end{aligned}$$

Then, for any time $0 \le s \le T$ taking $u \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$ as the test function in the weak formulation of the above PDE with $Q_s = \Omega \times (0,s)$ gives

$$\begin{aligned} &\int_0^s < u_t, u >_{((H^1)^*, H^1)} \ dt + \int_{Q_s} d_1 \nabla u \cdot \nabla u \ dxdt + \int_{Q_s} u\vec{h}\cdot\nabla u \ dxdt \\ &= \int_{Q_s} u^2 m \ dxdt - \int_{Q_s} a_1 u^3 \ dxdt - \int_{Q_s} b_1 u^2 v \ dxdt \quad \text{in } \ Q_s \ . \end{aligned}$$

The non-negativity of u, v implies

$$\int_0^s < u_t, u >_{((H^1)^*,H^1)} \; dt + \int_{Q_s} d_1 \nabla u \cdot \nabla u \; dxdt + \int_{Q_s} u\vec{h} \cdot \nabla u \; dxdt \\ \leq \int_{Q_s} u^2 m \; dxdt \qquad \text{in } \; Q_s.$$

Using

$$\begin{aligned} \int_0^s < u_t, u >_{((H^1)^*,H^1)} &= \int_0^s \int_\Omega u_t u dx dt \\ &= \int_0^s \int_\Omega \frac{1}{2} \frac{\partial u^2}{\partial t} dx dt \\ &= \frac{1}{2} \int_\Omega u^2(x,s) dx - \frac{1}{2} \int_\Omega u^2(x,0) dx \\ &= \frac{1}{2} \int_\Omega u^2(x,s) dx - \frac{1}{2} \int_\Omega u_0^2(x) dx \; , \end{aligned}$$

we begin to estimate:

$$\begin{aligned} &\frac{1}{2} \int_\Omega u^2(x,s) dx + \int_{Q_s} d_1 |\nabla u|^2 \; dxdt \\ &\leq \int_{Q_s} u^2 m \; dxdt - \int_{Q_s} u\vec{h} \cdot \nabla u \; dxdt + \frac{1}{2} \int_\Omega u_0(x)^2 dx \\ &\leq \int_{Q_s} |u^2 m| \; dxdt + \int_{Q_s} u|\vec{h}| \cdot |\nabla u| \; dxdt + \frac{1}{2} \int_\Omega u_0(x)^2 dx \\ &\leq C_0 \int_{Q_s} u^2 \; dxdt + M \int_{Q_s} u |\nabla u| \; dxdt + \frac{1}{2} \int_\Omega u_0(x)^2 dx \\ &\leq C_0 \int_{Q_s} u^2 \; dxdt + \frac{M}{4\epsilon} \int_{Q_s} u^2 dx dt + M\epsilon \int_{Q_s} |\nabla u|^2 \; dxdt + \frac{1}{2} \int_\Omega u_0(x)^2 dx \end{aligned}$$

using $\vec{h} \in U$, $m \in L^\infty(Q)$ and Cauchy's inequality.

This implies

$$\int_\Omega u^2(x,s)dx + \int_{Q_s} 2d_1|\nabla u|^2 \; dxdt$$
$$\leq 2C_0 \int_{Q_s} u^2 \; dxdt + \frac{M}{2\epsilon}\int_{Q_s} u^2 dxdt + 2\epsilon M \int_{Q_s} |\nabla u|^2 \; dxdt + \int_\Omega u_0(x)^2 dx \; .$$

Choosing $\epsilon \leq \frac{d_1}{M}$ and $C = 2C_0 + \frac{M}{2\epsilon}$,

$$\int_\Omega u^2(x,s)dx + \int_{Q_s} (2d_1 - 2M\epsilon)|\nabla u|^2 \; dxdt \leq C\int_{Q_s} u^2 \; dxdt + \int_\Omega u_0(x)^2 dx \quad (2.5)$$

implies that

$$\int_\Omega u^2(x,s)dx \leq C\int_{Q_s} u^2 \; dxdt + \int_\Omega u_0(x)^2 dx \; .$$

By taking,

$$Z_1(s) = \int_\Omega u^2(x,s) \; dx$$

we have

$$Z_1(s) \leq C\int_0^s Z_1(t) \; dt + A$$

where $A = \int_\Omega u_0(x)^2 dx = \|u_0\|^2_{L^2(\Omega)}$.

By Gronwall's Inequality, we have

$$Z_1(s) \leq A(1 + Cse^{Cs}) = \|u_0\|^2_{L^2(\Omega)}(1 + Cse^{Cs})$$

for almost every $0 \leq s \leq T$.

Hence, by taking maximum over the time we get

$$\max_{0\leq t\leq T} \int_\Omega u^2(x,t)dx \leq \|u_0\|^2_{L^2(\Omega)}(1 + CTe^{CT}) \; .$$

So, inequality (2.5) gives

$$\int_Q (2d_1 - 2M\epsilon)|\nabla u|^2 \, dxdt \leq C \int_Q u^2 \, dxdt + \int_\Omega u_0(x)^2 dx$$
$$\leq CT\|u_0\|^2_{L^2(\Omega)}(1 + CTe^{CT}) + \|u_0\|^2_{L^2(\Omega)}$$

which implies $\|\nabla u(x,t)\|^2_{L^2(Q)}$ is bounded.

Hence, we have

$$\|u(\vec{h})\|_{L^2((0,T),H^1(\Omega))} \quad \leq \quad K_1$$

where

$$(K_1)^2 = T\|u_0\|^2_{L^2(\Omega)}(1 + CTe^{CT}) + \frac{CT\|u_0\|^2_{L^2(\Omega)}(1 + CTe^{CT}) + \|u_0\|^2_{L^2(\Omega)}}{(2d_1 - 2M\epsilon)} \, .$$

Similarly, we also have

$$\|v(\vec{h})\|_{L^2((0,T),H^1(\Omega))} \quad \leq \quad K_3$$

where

$$(K_3)^2 = T\|v_0\|^2_{L^2(\Omega)}(1 + 2C_0Te^{2C_0T}) + \frac{2C_0T\|v_0\|^2_{L^2(\Omega)}(1 + 2C_0Te^{2C_0T}) + \|v_0\|^2_{L^2(\Omega)}}{2d_2} \, .$$

In the system

$$\begin{aligned} u_t &= d_1 \sum_{i=1}^n (u_{x_i})_{x_i} + \sum_{i=1}^n (h_i u)_{x_i} + um - a_1 u^2 - b_1 uv \qquad \text{in } Q \\ v_t &= d_2 \sum_{i=1}^n (v_{x_i})_{x_i} + vm - a_2 v^2 - b_2 uv \qquad \text{in } Q \, , \end{aligned}$$

note that all the terms in the right hand side of the u_t and v_t equations are either L^2 functions or the derivatives of an L^2 function. Using the $L^2((0,T),H^1(\Omega))$ bounds above, u_t and v_t are bounded in $L^2((0,T),H^1(\Omega)^*)$,

$$\begin{aligned} \|u_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_2 \\ \|v_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_4 \ . \end{aligned}$$

□

2.4.1 Existence and positivity of the state solution

In order to solve the optimal control problem (2.2), first we have to show that for a given control $\vec{h} \in U$ there exists a unique weak state solution $(u, v) = (u, v)(\vec{h})$ to (2.1), showing the dependence on $\vec{h}$. Further, we have to show this solution is non-negative and bounded above.

Theorem 2.1 (Existence and positivity of a weak solution for the state PDE system)**.** *Given $m \in L^\infty(Q)$, u_0, v_0 non-negative, $L^\infty(Q)$ bounded and in $H^1(\Omega)$. Then, for each $\vec{h} \in U$, there is a unique positive weak solution $(u, v) = (u(\vec{h}), v(\vec{h}))$ of (2.1).*

Proof. To show the existence of a weak solution to the state PDE system, we use an iterative scheme.

First to construct supersolutions of our system, consider

$$\begin{aligned} \overline{U}_t - d_1\Delta\overline{U} - \nabla\cdot(\vec{h}\overline{U}) &= m\overline{U} \quad \text{on } Q \\ d_1\frac{\partial\overline{U}}{\partial\eta} + \overline{U}\vec{h}\cdot\eta &= 0 \quad \text{in } \partial\Omega\times(0,T) \\ \overline{U}(x,0) &= u_0(x) \geq 0 \quad \text{for } x\in\Omega \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \overline{V}_t - d_2\Delta\overline{V} &= m\overline{V} \quad \text{on } Q \\ d_2\frac{\partial\overline{V}}{\partial\eta} &= 0 \quad \text{in } \partial\Omega\times(0,T) \\ \overline{V}(x,0) &= v_0(x) \geq 0 \quad \text{for } x\in\Omega \ . \end{aligned} \tag{2.7}$$

Then, by Theorem 3.1 in [16], problem (2.6) has a unique weak solution $\overline{U}(\vec{h})$ and a finite constant $C_1 > 0$ such that

$$0 \leq \overline{U}(\vec{h}) \leq C_1 \quad \text{for a.e. } (x,t) \in \Omega \times (0,T)$$

and problem (2.7) has a unique weak solution $\overline{V}(\vec{h})$ and a finite constant $C_2 > 0$ such that

$$0 \leq \overline{V}(\vec{h}) \leq C_2 \quad \text{for a.e. } (x,t) \in \Omega \times (0,T)$$

where, constants $C_1, C_2 > 0$ depending on M , $|Q|$, d_1 , d_2 , $\|m\|_{L^\infty}$ and the initial conditions.

Now, define $0 \leq R < \infty$ such that

$$R \geq \max\{-\inf_Q m + 2a_1 \sup_\Omega \overline{U} + b_1 \sup_\Omega \overline{V}, -\inf_Q m + 2a_2 \sup_\Omega \overline{V} + b_2 \sup_\Omega \overline{U}\} \ . \quad (2.8)$$

For $i = 2, 3, ...,$ consider the following iterative system of PDE's with $u_1 = \overline{U}$ and $v_1 = 0$ on Q

$$\begin{aligned}
L_1 u_i &= F(u_{i-1}, v_{i-1}) \quad \text{in } Q \\
L_2 v_i &= G(u_{i-1}, v_{i-1}) \quad \text{in } Q \\
d_1 \frac{\partial u_i}{\partial \eta} + u_i \vec{h} \cdot \eta &= 0 \quad \text{on } S \\
d_2 \frac{\partial v_i}{\partial \eta} &= 0 \quad \text{on } S \\
u_i(x,0) &= u_0(x) \geq 0 \quad \text{for } x \in \Omega \\
v_i(x,0) &= v_0(x) \geq 0 \quad \text{for } x \in \Omega \ .
\end{aligned} \quad (2.9)$$

The weak solution for (2.9) exist by the general results from [15], where,

$$
\begin{aligned}
L_1 u_i &= (u_i)_t - d_1 \Delta u_i - \nabla \cdot (\vec{h} u_i) + R u_i \\
L_2 v_i &= (v_i)_t - d_2 \Delta v_i + R v_i \\
&\text{and} \\
F(u_{i-1}, v_{i-1}) &= m u_{i-1} - a_1 (u_{i-1})^2 - b_1 u_{i-1} v_{i-1} + R u_{i-1} \\
G(u_{i-1}, v_{i-1}) &= m v_{i-1} - a_2 (v_{i-1})^2 - b_2 u_{i-1} v_{i-1} + R v_{i-1} \ .
\end{aligned}
$$

Choosing $u_1 = \overline{U}$ and $v_1 = 0$, then

$$
\begin{aligned}
L_1(u_1) &= m u_1 + R u_1 = m\overline{U} + R\overline{U} \\
L_2(v_1) &= m v_1 + R v_1 = 0 \ .
\end{aligned}
$$

Recall our super solution V satisfies $L_2(\overline{V}) = m\overline{V} + R\overline{V}$.

Claim 1

$0 \le u_i \le \overline{U}$ and $0 \le v_i \le \overline{V}$ for all $i = 1, 2, \ldots$.

Proof. We prove this by the mathematical induction.

Note that for $i = 1$, $0 \le u_1 \le \overline{U}$ and $0 \le v_1 \le \overline{V}$ since $u_1 = \overline{U}$ and $v_1 = 0$.

For $i = 2$,

$$
\begin{aligned}
L_1 u_2 &= F(u_1, v_1) \\
&= m u_1 - a_1 (u_1)^2 - b_1 u_1 v_1 + R u_1 \\
&= m\overline{U} - a_1 (\overline{U})^2 + R\overline{U} \qquad (\text{since } \ u_1 = \overline{U}, v_1 = 0) \\
&\ge \overline{U}(m - a_1 \overline{U} - \inf_Q m + 2a_1 \sup_\Omega \overline{U} + b_1 \sup_\Omega \overline{V}) \quad (\text{since } \ (2.8)) \\
&\ge a_1 \overline{U}^2 + b_1 \overline{U}\overline{V} \\
&\ge 0 \ ,
\end{aligned}
$$

with

$$d_1 \frac{\partial u_2}{\partial \eta} + u_2 \vec{h} \cdot \eta \ = 0 \qquad \text{on} \ \ S$$

and

$$u_2(x,0) \ = \ u_0(x) \geq 0 \qquad \text{in} \ \ \Omega \ .$$

Hence, by Lemma 2.0.1,

$$u_2 \geq 0 \ .$$

Further,

$$L_1(u_2) = m\overline{U} - a_1(\overline{U})^2 + R\overline{U}$$

implies

$$L_1(u_1 - u_2) \ = \ a_1(\overline{U})^2 \ \ \geq \ \ 0 \qquad \text{in} \ \ Q$$

with

$$d_1 \frac{\partial (u_1 - u_2)}{\partial \eta} + (u_1 - u_2) \vec{h} \cdot \eta \ = 0 \qquad \text{on} \ \ S$$

and

$$(u_1 - u_2)(x,0) \ = \ 0 \qquad \text{in} \ \ \Omega \ .$$

Hence, again by Lemma 2.0.1 gives $(u_1 - u_2) \geq 0$ and $0 \leq u_2 \leq U$.

Similarly, since

$$\begin{aligned} L_2 v_2 \ &= \ G(u_1, v_1) \\ &= \ m v_1 - a_2 (v_1)^2 - b_2 u_1 v_1 + R v_1 \\ &= \ 0 \qquad (\text{since} \ \ v_1 = 0), \end{aligned}$$

we have

$$L_2(v_2 - v_1) = 0 \qquad \text{on} \ \ Q$$

with

$$d_2 \frac{\partial(v_2 - v_1)}{\partial \eta} = 0 \qquad \text{in} \quad S$$

and

$$(v_2 - v_1)(x, 0) \;=\; v_0(x) \geq 0 \qquad \text{in} \quad \Omega \; .$$

Hence, again we have $(v_2 - v_1) \geq 0 \quad \text{and} \quad 0 \leq v_2$.

Also, we have

$$L_2(\overline{V} - v_2) = m\overline{V} + R\overline{V} \geq 0 \qquad \text{on} \quad Q$$

with

$$d_2 \frac{\partial(\overline{V} - v_2)}{\partial \eta} = 0 \qquad \text{in} \quad S$$

and

$$(\overline{V} - v_2)(x, 0) \;= 0 \qquad \text{in} \quad \Omega \; .$$

So, again by Lemma 2.0.1 we have, $\overline{V} \geq v_2$ and hence get $0 \leq v_2 \leq \overline{V}$.

Now, suppose $\quad 0 \leq u_j \leq \overline{U} \quad$ and $\quad 0 \leq v_j \leq \overline{V} \quad$ for $j = 1, 2, ..., i \quad$ to use induction.

Then,

$$\begin{aligned}
L_1 u_{i+1} &= F(u_i, v_i) \\
&= m u_i - a_1 (u_i)^2 - b_1 u_i v_i + R u_i \\
&= u_i (m - a_1 u_i - b_1 v_i + R) \\
&\geq u_i (m - a_1 u_i - b_1 v_i - \inf_Q m + 2a_1 \sup_\Omega \overline{U} + b_1 \sup_\Omega \overline{V}) \quad \text{(since (2.8))} \\
&\geq u_i [a_1 (2\overline{U} - u_i) + b_1 (\overline{V} - v_i)] \\
&\geq 0,
\end{aligned}$$

by the induction hypothesis.

Again with the boundary and initial conditions, we obtain $u_{i+1} \geq 0$.

Using

$$\begin{aligned} L_1(u_{i+1}) &= F(u_i, v_i) \\ &= mu_i - a_1(u_i)^2 - b_1 u_i v_i + Ru_i, \end{aligned}$$

we have

$$\begin{aligned} L_1(u_1 - u_{i+1}) &= m\overline{U} + R\overline{U} - mu_i + a_1(u_i)^2 + b_1 u_i v_i - Ru_i \\ &= m(\overline{U} - u_i) + R(\overline{U} - u_i) + a_1(u_i)^2 + b_1 u_i v_i \\ &\geq 0 \qquad \text{(by the induction hypothesis)} \end{aligned}$$

with

$$d_1 \frac{\partial(u_1 - u_{i+1})}{\partial \eta} + (u_1 - u_{i+1})\vec{h}\eta \ = 0 \qquad \text{on} \quad S$$

and

$$(u_1 - u_{i+1})(x, 0) \ = \ 0 \qquad \text{in} \quad \Omega \ .$$

Hence, again by Lemma 2.0.1 we have $u_{i+1} \leq u_1 = \overline{U}$.

Then, by induction, we have $\quad 0 \leq u_i \leq \overline{U} \quad$ for all $\ i = 1, 2, \ldots \ .$

Similarly we can show that $\quad 0 \leq v_i \leq \overline{V} \quad$ for all $\ i = 1, 2, \ldots \ .$

□

Claim 2

The sequence $\{u_i\}$ is monotone decreasing and the sequence $\{v_i\}$ is monotone increasing :

$$\text{That is,} \ \ u_{i+1} \leq u_i \quad \text{and} \quad v_{i+1} \geq v_i \quad \text{for all } i = 1, 2, \ldots \ .$$

Proof. We prove this also by mathematical induction.

Using

$$L_1(u_1) = mu_1 + Ru_1$$

and

$$\begin{aligned} L_1(u_2) &= F(u_1, v_1) \\ &= mu_1 - a_1(u_1)^2 - b_1 u_1 v_1 + Ru_1 \\ &= mu_1 - a_1(u_1)^2 + Ru_1 \qquad (\text{Since} \quad v_1 = 0), \end{aligned}$$

we have

$$L_1(u_1 - u_2) \;=\; a_1(u_1)^2 \;\geq\; 0$$

with

$$d_1 \frac{\partial(u_1 - u_2)}{\partial \eta} + (u_1 - u_2)\vec{h}\eta \;= 0 \quad \text{on} \quad S$$

and

$$(u_1 - u_2)(x, 0) \;=\; 0 \qquad \text{in} \;\; \Omega \; .$$

This implies $(u_1 - u_2) \geq 0$ giving $u_1 \geq u_2$.

Similarly, we have

$$L_2(v_1) = mv_1 + Rv_1 = 0$$

and

$$\begin{aligned} L_2(v_2) &= G(u_1, v_1) \\ &= mv_1 - a_2(v_1)^2 - b_2 u_1 v_1 + Rv_1 \\ &= 0 \qquad (\text{Since} \quad v_1 := 0). \end{aligned}$$

So, we have

$$L_2(v_2 - v_1) = 0 \;\; \text{in} \;\; Q$$

with

$$d_2 \frac{\partial(v_2 - v_1)}{\partial \eta} = 0 \quad \text{on} \quad S$$

and

$$(v_2 - v_1)(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega \ .$$

Hence, we have $v_1 \leq v_2$.

Now, suppose that $u_j \leq u_{j-1}$ and $v_j \geq v_{j-1}$ for $j = 1, 2, ..., i$ for the induction hypothesis. Then, in Q

$$\begin{aligned} L_1(u_i) &= F(u_{i-1}, v_{i-1}) \\ L_1(u_{i+1}) &= F(u_i, v_i) \end{aligned}$$

implies

$$L_1(u_i - u_{i+1}) = F(u_{i-1}, v_{i-1}) - F(u_i, v_i) \ .$$

But, note that $u_j \leq u_{j-1}$, $v_j \geq v_{j-1}$ for $j = 1, 2, ..., i$ and we picked R such that $F(u, v)$ is increasing in u and decreasing in v.

Hence, we have

$$L_1(u_i - u_{i+1}) \geq 0 \quad \text{in } Q$$

with

$$d_1 \frac{\partial(u_i - u_{i+1})}{\partial \eta} + (u_i - u_{i+1})\vec{h} \cdot \eta = 0 \quad \text{on } S$$

and

$$(u_i - u_{i+1})(x, 0) = 0 \quad \text{in } \Omega,$$

which implies

$$(u_i - u_{i+1}) \geq 0 \quad \text{giving} \quad u_i \geq u_{i+1} \ .$$

Hence, by induction,

$$u_{i+1} \leq u_i \quad \text{for all } i = 1, 2, ... \ .$$

Using

$$L_2(v_i - v_{i+1}) \;=\; G(u_{i-1}, v_{i-1}) - G(u_i, v_i),$$

and $u_j \leq u_{j-1}$, $v_j \geq v_{j-1}$ for $j = 1, 2, ..., i$ and the choice of R such that $G(u, v)$ is decreasing in u and increasing in v, we have

$$L_2(v_i - v_{i+1}) \leq 0 \qquad \text{on} \;\; Q$$

with

$$d_2 \frac{\partial (v_i - v_{i+1})}{\partial \eta} \;= 0 \qquad \text{in} \;\; S$$

and

$$(v_i - v_{i+1})(x, 0) \;=\; 0 \qquad \text{in} \;\; \Omega \; .$$

Comparison results imply

$$(v_i - v_{i+1}) \leq 0 \quad \text{giving} \quad v_i \leq v_{i+1}.$$

Hence, by induction we have, $\quad v_{i+1} \geq v_i \quad$ for all $i = 1, 2, ...$.

□

Similarly as in Lemma 2.0.2, for some constants K_1, K_2, K_3, K_4 with $\|u_i\|_{L^\infty(Q)}$, $\|v_i\|_{L^\infty(Q)}$ bounds and $u_i \geq 0$, $v_i \geq 0$ a.e on Q, for all $i = 1, 2, ...$ we have:

$$\begin{aligned}
\|u_i(\vec{h})\|_{L^2((0,T),H^1(\Omega))} &\leq K_1 \\
\|(u_i)_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_2 \\
\|v_i(\vec{h})\|_{L^2((0,T),H^1(\Omega))} &\leq K_3 \\
\|(v_i)_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_4
\end{aligned}$$

So, by these *a priori* bounds for u_i, v_i, $(u_i)_t$, and $(v_i)_t$ and by the monotone property of $\{u_i\}$ and $\{v_i\}$ sequences, there exists $u, v \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$ with $u_t, v_t \in L^2((0,T), (H^1(\Omega))^*)$ with the weak convergences (on the whole sequence,

not just on a subsequences) below:

$$\begin{aligned} u_i &\rightharpoonup u \qquad \text{in} \qquad L^2((0,T),H^1(\Omega)) \\ (u_i)_t &\rightharpoonup u_t \qquad \text{in} \qquad L^2((0,T),(H^1(\Omega))^*) \\ v^i &\rightharpoonup v \qquad \text{in} \qquad L^2((0,T),H^1(\Omega)) \\ (v_i)_t &\rightharpoonup v_t \qquad \text{in} \qquad L^2((0,T),(H^1(\Omega))^*)\ . \end{aligned} \tag{2.10}$$

Then, by Simon's result [44] we have the strong convergence of the sequences $\{u_i\}$ and $\{v_i\}$ in $L^2(Q)$ as below:

$$\begin{aligned} u_i &\to u \qquad \text{in} \qquad L^2(Q) \\ v_i &\to v \qquad \text{in} \qquad L^2(Q)\ . \end{aligned} \tag{2.11}$$

Since there exists weak solutions to our iterative system (2.9), for the test functions $\phi_1\ ,\ \phi_2 \in L^2((0,T),H^1(\Omega))$, and $u_i, v_i \in L^2((0,T),H^1(\Omega)) \cap L^\infty(Q)$ with $(u_i)_t, (v_i)_t \in L^2((0,T),(H^1(\Omega))^*)$ and $u_i(x,0) = u_0, v_i(x,0) = v_0$ we have

$$\begin{aligned} &\int_0^T < (u_i)_t, \phi_1 >_{((H^1)^*,H^1)}\ dt + \int_Q d_1 \nabla u_i \cdot \nabla \phi_1\ dxdt + \int_Q u_i \vec{h} \cdot \nabla \phi_1\ dxdt \\ &+ \int_Q R u_i \phi_1\ dxdt \\ &= \int_Q u_{i-1} m \phi_1\ dxdt - \int_Q a_1 (u_{i-1})^2 \phi_1\ dxdt - \int_Q b_1 u_{i-1} v_{i-1} \phi_1\ dxdt \\ &+ \int_Q R u_{i-1} \phi_1 dxdt \qquad \text{in}\ \ Q \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\int_0^T < (v_i)_t, \phi_2 >_{((H^1)^*,H^1)}\ dt + \int_Q d_2 \nabla v_i \cdot \nabla \phi_2\ dxdt + \int_Q R v_i \phi_2 dxdt \\ &= \int_Q v_{i-1} m \phi_2\ dxdt - \int_Q a_2 (v_{i-1})^2 \phi_2\ dxdt - \int_Q b_2 u_{i-1} v_{i-1} \phi_2\ dxdt \\ &+ \int_Q R v_{i-1} \phi_2 dxdt \text{ in } Q\ . \end{aligned}$$

Since ϕ_1, ϕ_2 are in $L^2((0,T), H^1(\Omega))$, using the weak star convergence of $\{(u_i)_t\}$ and $\{(v_i)_t\}$ in $L^2((0,T), (H^1(\Omega))^*)$ we have

$$\lim_{i\to\infty} \int_0^T < (u_i)_t, \phi_1 >_{((H^1)^*, H^1)} \ dt = \int_0^T < u_t, \phi_1 >_{((H^1)^*, H^1)} \ dt$$

and

$$\lim_{i\to\infty} \int_0^T < (v_i)_t, \phi_2 >_{((H^1)^*, H^1)} \ dt = \int_0^T < v_t, \phi_2 >_{((H^1)^*, H^1)} \ dt \ .$$

Since d_1 and d_2 are constants and $\nabla u_i, \nabla \phi_1, \nabla v_i$ and $\nabla \phi_2$ are in $L^2(Q)$ by the weak convergence results of $\{u_i\}$ and $\{v_i\}$ in (2.10) we have

$$\lim_{i\to\infty} \int_Q d_1 \nabla u_i \cdot \nabla \phi_1 \ dxdt = \int_Q d_1 \nabla u \cdot \nabla \phi_1 \ dxdt$$

and

$$\lim_{i\to\infty} \int_Q d_2 \nabla v_i \cdot \nabla \phi_2 \ dxdt = \int_Q d_2 \nabla v \cdot \nabla \phi_2 \ dxdt \ .$$

Similarly as $\{u_i\}$ and $\{v_i\}$ are in $L^2(Q)$, using L^∞ bounds on $\vec{h}$, R and m and the weak convergence results of $\{u^i\}$, $\{v^i\}$ in (2.10) we have

$$\lim_{i\to\infty} \int_Q u_i \vec{h} \cdot \nabla \phi_1 \ dxdt = \int_Q u \vec{h} \cdot \nabla \phi_1 \ dxdt$$

$$\lim_{i\to\infty} \int_Q R u_i \phi_1 \ dxdt = \lim_{i\to\infty} \int_Q R u_{i-1} \phi_1 dxdt = \int_Q R u \phi_1 \ dxdt$$

$$\lim_{i\to\infty} \int_Q u_{i-1} m \phi_1 \ dxdt = \int_Q u m \phi_1 \ dxdt$$

$$\lim_{i\to\infty} \int_Q R v_i \phi_2 dxdt = \lim_{i\to\infty} \int_Q R v_{i-1} \phi_2 dxdt = \int_Q R v \phi_2 dxdt$$

$$\lim_{i\to\infty} \int_Q v_{i-1} m \phi_2 \ dxdt = \int_Q v m \phi_2 \ dxdt \ .$$

Note that

$$\begin{aligned}
&\lim_{i\to\infty}\int_Q a_1(u_{i-1})^2\phi_1 dxdt - \int_Q a_1u^2\phi_1\ dxdt\\
&= \lim_{i\to\infty}\int_Q a_1(u_{i-1}-u)(u_{i-1}+u)\phi_1\ dxdt\\
&\le a_1\|u_{i-1}+u\|_{L^\infty}\lim_{i\to\infty}\int_Q |u_{i-1}-u||\phi_1|\ dxdt\ .
\end{aligned}$$

Hence, by the Holder's inequality, strong convergences (2.11) and L^∞ bounds on u_i we have

$$\begin{aligned}
&\lim_{i\to\infty}\int_Q a_1(u_{i-1})^2\phi_1 dxdt - \int_Q a_1u^2\phi_1\ dxdt\\
&\le a_1\|u_{i-1}+u\|_{L^\infty}\lim_{i\to\infty}(\int_Q |u_{i-1}-u|^2\ dxdt)^{1/2}(\int_Q |\phi_1|^2\ dxdt)^{1/2}\\
&= 0
\end{aligned}$$

giving

$$\lim_{i\to\infty}\int_Q a_1(u_{i-1})^2\phi_1\ dxdt = \int_Q a_1u^2\phi_1\ dxdt\ .$$

Similarly by the Holder's inequality, strong convergences (2.11) and L^∞ bounds on v_i we have

$$\lim_{i\to\infty}\int_Q a_2(v_{i-1})^2\phi_2\ dxdt = \int_Q a_2v^2\phi_2\ dxdt\ .$$

Also, note that

$$\begin{aligned}
&\lim_{i\to\infty}\int_Q b_1u_{i-1}v_{i-1}\phi_1\ dxdt - \int_Q b_1uv\phi_1\ dxdt\\
&= \lim_{i\to\infty}\int_Q b_1[u_{i-1}(v_{i-1}-v)+v(u_{i-1}-u)]\phi_1\ dxdt\\
&\le b_1\|u_{i-1}\|_{L^\infty}\lim_{i\to\infty}\int_Q |v_{i-1}-v||\phi_1|\ dxdt + b_1\|v\|_{L^\infty}\lim_{i\to\infty}\int_Q |u_{i-1}-u||\phi_1|\ dxdt\ .
\end{aligned}$$

Hence, again by the Holder's inequality, strong convergences (2.11) and L^∞ bounds on u_i, v we have

$$\begin{aligned}
&\lim_{i\to\infty}\int_Q b_1 u_{i-1}v_{i-1}\phi_1 \; dxdt - \int_Q b_1 uv\phi_1 \; dxdt \\
&\leq b_1\|u_{i-1}\|_{L^\infty}\lim_{i\to\infty}(\int_Q |v_{i-1}-v|^2 \; dxdt)^{1/2}(\int_Q |\phi_1|^2 \; dxdt)^{1/2} \\
&\quad + b_1\|v\|_{L^\infty}\lim_{i\to\infty}(\int_Q |u_{i-1}-u|^2 \; dxdt)^{1/2}(\int_Q |\phi_1|^2 \; dxdt)^{1/2} \\
&= 0
\end{aligned}$$

giving

$$\lim_{i\to\infty}\int_Q b_1 u_{i-1}v_{i-1}\phi_1 \; dxdt = \int_Q b_1 uv\phi_1 \; dxdt$$

and similarly we get

$$\lim_{i\to\infty}\int_Q b_2 u_{i-1}v_{i-1}\phi_2 \; dxdt = \int_Q b_2 uv\phi_2 \; dxdt \; .$$

So, by taking as $i\to\infty$ in (2.12), we conclude that u, v are state solutions corresponding to control $\vec{h}$ given below:

$$\begin{aligned}
&\int_0^T < u_t, \phi_1 >_{((H^1)^*,H^1)} \; dt + \int_Q d_1\nabla u\cdot\nabla\phi_1 \; dxdt + \int_Q u\vec{h}\cdot\nabla\phi_1 \; dxdt \\
&= \int_Q um\phi_1 \; dxdt - \int_Q a_1u^2\phi_1 \; dxdt - \int_Q b_1uv\phi_1 \; dxdt \quad \text{in } \; Q
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T < v_t, \phi_2 >_{((H^1)^*,H^1)} \; dt + \int_Q d_2\nabla v\cdot\nabla\phi_2 \; dxdt \\
&= \int_Q vm\phi_2 \; dxdt - \int_Q a_2v^2\phi_2 \; dxdt - \int_Q b_2uv\phi_2 \; dxdt \quad \text{in } \; Q \; .
\end{aligned}$$

Next, to show that this weak solution (with L^∞ bounds) for (2.1) is unique suppose there are two solutions, (u,v) and $(\tilde{u},\tilde{v})$ for (2.1).

Now, with the test functions $(u-\tilde{u})$ and $(v-\tilde{v})$ in $L^2((0,T),H^1(\Omega))\cap L^\infty(Q)$ and $Q_s=\Omega\times(0,s)$ for any time $0\le s\le T$, the weak formulation of the solutions (u,v) and $(\tilde{u},\tilde{v})$ gives

$$\begin{aligned}
&\int_0^s <u_t,(u-\tilde{u})>_{((H^1)^*,H^1)}\,dt+\int_{Q_s} d_1\nabla u\cdot\nabla(u-\tilde{u})dxdt+\int_{Q_s} u\vec{h}\cdot\nabla(u-\tilde{u})dxdt\\
&=\int_{Q_s} um(u-\tilde{u})\,dxdt-\int_{Q_s} a_1u^2(u-\tilde{u})\,dxdt-\int_{Q_s} b_1uv(u-\tilde{u})\,dxdt \quad \text{in } Q_s
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&\int_0^s <v_t,(v-\tilde{v})>_{((H^1)^*,H^1)}\,dt+\int_{Q_s} d_2\nabla v\cdot\nabla(v-\tilde{v})\,dxdt\\
&=\int_{Q_s} vm(v-\tilde{v})\,dxdt-\int_{Q_s} a_2v^2(v-\tilde{v})\,dxdt-\int_{Q_s} b_2uv(v-\tilde{v})\,dxdt \quad \text{in } Q_s
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^s <\tilde{u}_t,(u-\tilde{u})>_{((H^1)^*,H^1)}\,dt+\int_{Q_s} d_1\nabla\tilde{u}\cdot\nabla(u-\tilde{u})dxdt+\int_{Q_s} \tilde{u}\vec{h}\cdot\nabla(u-\tilde{u})dxdt\\
&=\int_{Q_s} \tilde{u}m(u-\tilde{u})\,dxdt-\int_{Q_s} a_1\tilde{u}^2(u-\tilde{u})\,dxdt-\int_{Q_s} b_1\tilde{u}\tilde{v}(u-\tilde{u})\,dxdt \quad \text{in } Q_s
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
&\int_0^s <\tilde{v}_t,(v-\tilde{v})>_{((H^1)^*,H^1)}\,dt+\int_{Q_s} d_2\nabla\tilde{v}\cdot\nabla(v-\tilde{v})\,dxdt\\
&=\int_{Q_s} \tilde{v}m(v-\tilde{v})\,dxdt-\int_{Q_s} a_2\tilde{v}^2(v-\tilde{v})\,dxdt-\int_{Q_s} b_2\tilde{u}\tilde{v}(v-\tilde{v})\,dxdt \quad \text{in } Q_s
\end{aligned}$$

Then by taking the difference in the corresponding equations in (2.13) and (2.14),

$$\begin{aligned}
&\int_0^s <(u_t-\tilde{u}_t),(u-\tilde{u})>_{((H^1)^*,H^1)}\,dt+\int_{Q_s} d_1|\nabla(u-\tilde{u})|^2\,dxdt\\
&+\int_{Q_s}(u-\tilde{u})\vec{h}\cdot\nabla(u-\tilde{u})\,dxdt\\
&=\int_{Q_s}(u-\tilde{u})^2m\,dxdt-\int_{Q_s} a_1(u^2-\tilde{u}^2)(u-\tilde{u})\,dxdt-\int_{Q_s} b_1(uv-\tilde{u}\tilde{v})(u-\tilde{u})\,dxdt
\end{aligned}$$

and

$$\int_0^s < (v_t - \tilde{v}_t), (v - \tilde{v}) >_{((H^1)^*, H^1)} \ dt + \int_{Q_s} d_2 |\nabla(v - \tilde{v})|^2 \ dxdt$$
$$= \int_{Q_s} (v - \tilde{v})^2 m \ dxdt - \int_{Q_s} a_2(v^2 - \tilde{v}^2)(v - \tilde{v}) \ dxdt - \int_{Q_s} b_2(uv - \tilde{u}\tilde{v})(v - \tilde{v}) \ dxdt$$

in Q_s .

Rearranging yields

$$\int_0^s < (u_t - \tilde{u}_t), (u - \tilde{u}) >_{((H^1)^*, H^1)} \ dt + \int_{Q_s} d_1 |\nabla(u - \tilde{u})|^2 \ dxdt$$
$$+ \int_{Q_s} (u - \tilde{u}) \vec{h} \cdot \nabla(u - \tilde{u}) \ dxdt$$
$$= \int_{Q_s} (u - \tilde{u})^2 m \ dxdt - \int_{Q_s} a_1(u - \tilde{u})^2 (u + \tilde{u}) \ dxdt$$
$$- \int_{Q_s} b_1[(u - \tilde{u})v + (v - \tilde{v})\tilde{u}](u - \tilde{u}) \ dxdt$$

and

$$\int_0^s < (v_t - \tilde{v}_t), (v - \tilde{v}) >_{((H^1)^*, H^1)} \ dt + \int_{Q_s} d_2 |\nabla(v - \tilde{v})|^2 \ dxdt$$
$$= \int_{Q_s} (v - \tilde{v})^2 m \ dxdt - \int_{Q_s} a_2(v - \tilde{v})^2 (v + \tilde{v}) \ dxdt$$
$$- \int_{Q_s} b_2[(u - \tilde{u})v + (v - \tilde{v})\tilde{u}](v - \tilde{v}) \ dxdt \ .$$

Then, by the non-negativity of u, v, $\tilde{u}$ and $\tilde{v}$ we get

$$\begin{aligned}
&\int_0^s < (u_t - \tilde{u}_t), (u - \tilde{u}) >_{((H^1)^*, H^1)} \ dt + \int_{Q_s} d_1 |\nabla(u - \tilde{u})|^2 \ dxdt \\
&+ \int_{Q_s} (u - \tilde{u}) \vec{h} \cdot \nabla(u - \tilde{u}) \ dxdt \\
&\leq \int_{Q_s} (u - \tilde{u})^2 m \ dxdt - \int_{Q_s} b_1 [(u - \tilde{u})v + (v - \tilde{v})\tilde{u}](u - \tilde{u}) \ dxdt \qquad (2.15)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^s < (v_t - \tilde{v}_t), (v - \tilde{v}) >_{((H^1)^*, H^1)} \ dt + \int_{Q_s} d_2 |\nabla(v - \tilde{v})|^2 \ dxdt \\
&\leq \int_{Q_s} (v - \tilde{v})^2 m \ dxdt - \int_{Q_s} b_2 [(u - \tilde{u})v + (v - \tilde{v})\tilde{u}](v - \tilde{v}) \ dxdt \ .
\end{aligned}$$

But,

$$\begin{aligned}
\int_0^s < (u_t - \tilde{u}_t), (u - \tilde{u}) >_{((H^1)^*, H^1)} dt &= \int_0^s \int_\Omega (u_t - \tilde{u}_t)(u - \tilde{u}) dxdt \\
&= \int_0^s \int_\Omega \frac{1}{2} \frac{\partial (u - \tilde{u})^2}{\partial t} dxdt \\
&= \frac{1}{2} \int_\Omega (u - \tilde{u})^2 (x, s) dx - \frac{1}{2} \int_\Omega (u - \tilde{u})^2 (x, 0) dx \\
&= \frac{1}{2} \int_\Omega (u - \tilde{u})^2 (x, s) dx - \frac{1}{2} \int_\Omega (u_0 - \tilde{u}_0)(x)^2 dx \\
&= \frac{1}{2} \int_\Omega (u - \tilde{u})^2 (x, s) dx
\end{aligned}$$

and similarly,

$$\int_0^s < (v_t - \tilde{v}_t), (v - \tilde{v}) >_{((H^1)^*, H^1)} dt = \frac{1}{2} \int_\Omega (v - \tilde{v})^2 (x, s) dx \ .$$

We begin to estimate in (2.15):

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega}(u-\tilde{u})^2(x,s)dx+\int_{Q_s}d_1|\nabla(u-\tilde{u})|^2\ dxdt\\
&\leq\int_{Q_s}(u-\tilde{u})^2m\ dxdt-\int_{Q_s}(u-\tilde{u})\vec{h}\cdot\nabla(u-\tilde{u})\ dxdt\\
&\qquad-\int_{Q_s}b_1[(u-\tilde{u})v+(v-\tilde{v})\tilde{u}](u-\tilde{u})\ dxdt\\
&\leq\int_{Q_s}|(u-\tilde{u})^2m|\ dxdt+\int_{Q_s}|(u-\tilde{u})\vec{h}\cdot\nabla(u-\tilde{u})|\ dxdt+\int_{Q_s}b_1(u-\tilde{u})^2|v|\ dxdt\\
&\qquad+\int_{Q_s}b_1(v-\tilde{v})(u-\tilde{u})|\tilde{u}|\ dxdt\\
&\leq C_0\int_{Q_s}|(u-\tilde{u})^2|\ dxdt+M\int_{Q_s}|(u-\tilde{u})\cdot\nabla(u-\tilde{u})|\ dxdt+\int_{Q_s}b_1(u-\tilde{u})^2|v|\ dxdt\\
&\qquad+\int_{Q_s}b_1(v-\tilde{v})(u-\tilde{u})|\tilde{u}|\ dxdt\\
&\leq C_0\int_{Q_s}(u-\tilde{u})^2\ dxdt+\frac{M}{4\epsilon}\int_{Q_s}(u-\tilde{u})^2dxdt+\epsilon M\int_{Q_s}|\nabla(u-\tilde{u})|^2\ dxdt\\
&\qquad+b_1C_2\int_{Q_s}(u-\tilde{u})^2\ dxdt+\frac{1}{2}b_1C_1\int_{Q_s}(u-\tilde{u})^2\ dxdt+\frac{1}{2}b_1C_1\int_{Q_s}(v-\tilde{v})^2\ dxdt
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega}(v-\tilde{v})^2(x,s)dx+\int_{Q_s}d_2|\nabla(v-\tilde{v})|^2\ dxdt\\
&\leq\int_{Q_s}(v-\tilde{v})^2m\ dxdt-\int_{Q_s}b_2[(u-\tilde{u})v+(v-\tilde{v})\tilde{u}](v-\tilde{v})\ dxdt\\
&\leq\int_{Q_s}|(v-\tilde{v})^2m|\ dxdt+\int_{Q_s}b_2(v-\tilde{v})^2|\tilde{u}|\ dxdt+\int_{Q_s}b_2(v-\tilde{v})(u-\tilde{u})|v|\ dxdt\\
&\leq C_0\int_{Q_s}(v-\tilde{v})^2\ dxdt+b_2C_1\int_{Q_s}(v-\tilde{v})^2\ dxdt+\frac{1}{2}b_2C_2\int_{Q_s}(u-\tilde{u})^2\ dxdt\\
&+\frac{1}{2}b_2C_2\int_{Q_s}(v-\tilde{v})^2\ dxdt
\end{aligned}
$$

using $h\in U$, $m\in L^{\infty}Q$, $|\tilde{u}|\leq C_1$, $|v|\leq C_2$ and Cauchy's inequalities.

By adding these two inequalities we get

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (u-\tilde{u})^2(x,s)dx + \int_{Q_s} d_1|\nabla(u-\tilde{u})|^2\ dxdt + \frac{1}{2}\int_\Omega (v-\tilde{v})^2(x,s)dx \\
&+ \int_{Q_s} d_2|\nabla(v-\tilde{v})|^2\ dxdt \\
&\leq C_0\int_{Q_s}(u-\tilde{u})^2\ dxdt + \frac{M}{4\epsilon}\int_{Q_s}(u-\tilde{u})^2 dxdt + \epsilon M\int_{Q_s}|\nabla(u-\tilde{u})|^2\ dxdt \\
&\quad + b_1C_2\int_{Q_s}(u-\tilde{u})^2\ dxdt + \frac{1}{2}b_1C_1\int_{Q_s}(u-\tilde{u})^2\ dxdt + \frac{1}{2}b_1C_1\int_{Q_s}(v-\tilde{v})^2\ dxdt \\
&\quad + C_0\int_{Q_s}(v-\tilde{v})^2\ dxdt + \int_{Q_s} b_2C_1(v-\tilde{v})^2\ dxdt + \frac{1}{2}b_2C_2\int_{Q_s}(u-\tilde{u})^2\ dxdt \\
&\quad + \frac{1}{2}b_2C_2\int_{Q_s}(v-\tilde{v})^2\ dxdt
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_\Omega (u-\tilde{u})^2(x,s)dx + \int_\Omega (v-\tilde{v})^2(x,s)dx + \int_{Q_s}(2d_1-2\epsilon M)|\nabla(u-\tilde{u})|^2\ dxdt \\
&+ \int_{Q_s} 2d_2|\nabla(v-\tilde{v})|^2\ dxdt \\
&\leq (2C_0 + \frac{M}{2\epsilon} + 2b_1C_2 + b_1C_1 + b_2C_2)\int_{Q_s}(u-\tilde{u})^2 dxdt \\
&\qquad + (2C_0 + 2b_2C_1 + b_1C_1 + b_2C_2)\int_{Q_s}(v-\tilde{v})^2\ dxdt\ .
\end{aligned}
$$

Pick

$$
\epsilon \leq \frac{d_1}{M}\quad , \quad A = 2C_0 + \frac{M}{2\epsilon} + 2b_1C_1 + b_1C_1 + b_2C_2 \quad \text{and} \quad B = 2C_0 + 2b_2C_2 + b_1C_1 + b_2C_2.
$$

Then,

$$\int_\Omega (u-\tilde{u})^2(x,s) + (v-\tilde{v})^2(x,s)dx + \int_{Q_s} (2d_1 - 2\epsilon M)|\nabla(u-\tilde{u})|^2 \; dxdt$$
$$+ \int_{Q_s} 2d_2|\nabla(v-\tilde{v})|^2 \; dxdt$$
$$\leq A\int_{Q_s} (u-\tilde{u})^2 dxdt + B\int_{Q_s} (v-\tilde{v})^2 \; dxdt$$

and for positive constant $D \geq max\{A, B\}$

$$\int_\Omega [(u-\tilde{u})^2 + (v-\tilde{v})^2](x,s)dx \leq D\int_{Q_s} [(u-\tilde{u})^2 + (v-\tilde{v})^2] \; dxdt \; .$$

Hence, by taking,

$$Z_1(s) = \int_\Omega [(u-\tilde{u})^2 + (v-\tilde{v})^2](x,s)dx$$

we have

$$Z_1(s) \leq D\int_0^s Z_1(t) \; dt \; .$$

As, $D \geq 0$ is a constant , by Gronwall's Inequality we have for a.e. $0 \leq s \leq T$

$$Z_1(s) = 0 \quad \text{a.e..}$$

Hence, for a.e. $0 \leq s \leq T$ we have

$$\int_\Omega [(u-\tilde{u})^2 + (v-\tilde{v})^2](x,s)dx = 0$$

which implies

$$u = \tilde{u} \quad \text{and} \quad v = \tilde{v} \; ,$$

giving the uniqueness of our solution to (2.1).

□

2.4.2 Existence of an optimal control

In 2.1 we have showed that for a given control $\vec{h} \in U$ there exists a unique weak state solution $(u, v) = (u, v)(\vec{h})$, showing the dependence on $\vec{h}$. Next, we show that there exist an optimal control $\vec{h^*}$ and the corresponding state solutions $(u^*, v^*) = (u, v)(\vec{h^*})$ to the optimal control problem (2.2).

Theorem 2.2 (Existence of an optimal control $\vec{h^*}$)**.** *There exists an optimal control $\vec{h^*}$ maximizing the functional $J(\vec{h})$ over U.*

i.e.

$$J(\vec{h^*}) = \sup_{\vec{h} \in U} J(\vec{h})$$

Proof. Note that

$$J(\vec{h}) = \int_Q \left[Au + Bv - C|\vec{h}|^2 \right] \, dxdt$$

and as uniform L^∞ bounds on u, v and $\vec{h} \in (L^\infty)^d$, $\sup_{\vec{h} \in U} J(\vec{h})$ is finite.
Hence, we can choose a maximizing sequence, $\{\vec{h^n}\} \in U$ such that

$$\lim_{n \to \infty} J(\vec{h^n}) = \sup_{\vec{h} \in U} J(\vec{h})$$

and u^n, v^n the corresponding states.

By the *a priori* estimates in Lemma 2.0.2 there exists $u^*, v^* \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$, $u_t^*, v_t^* \in L^2((0,T), (H^1(\Omega))^*)$ and $\vec{h^*} \in U$ such that on a subsequence

$$\begin{aligned}
u^n &\rightharpoonup u^* && \text{in} && L^2((0,T), H^1(\Omega)) \\
u_t^n &\rightharpoonup u_t^* && \text{in} && L^2((0,T), (H^1(\Omega))^*) \\
v^n &\rightharpoonup v^* && \text{in} && L^2((0,T), H^1(\Omega)) \\
v_t^n &\rightharpoonup v_t^* && \text{in} && L^2((0,T), (H^1(\Omega))^*) \\
\vec{h^n} &\rightharpoonup \vec{h^*} && \text{in} && (L^2(Q))^d \, .
\end{aligned} \tag{2.16}$$

Then, again by Simon's result [44] we have the strong convergence of the sequences $\{u^n\}$ and $\{v^n\}$ in $L^2(\Omega \times [0,T])$ as below:

$$\begin{aligned} u^n &\to u^* \quad \text{in} \quad L^2(Q) \\ v^n &\to v^* \quad \text{in} \quad L^2(Q)\ . \end{aligned} \tag{2.17}$$

So, now we have to show that $\vec{h^*}$ is an optimal control and that $u^* = u(\vec{h^*})$ and $v^* = v(\vec{h^*})$.

For that consider the system of state PDEs for the subsequence $\{\vec{h^n}\}$ with the corresponding states $u(\vec{h^n}) = u^n$ and $v(\vec{h^n}) = v^n$ below:

$$\begin{aligned} &\int_0^T < u_t^n, \phi_1 >_{((H^1)^*, H^1)}\ dt + \int_Q d_1 \nabla u^n \cdot \nabla \phi_1\ dxdt + \int_Q u^n \vec{h^n} \cdot \nabla \phi_1\ dxdt \\ &= \int_Q u^n m \phi_1\ dxdt - \int_Q a_1 (u^n)^2 \phi_1\ dxdt - \int_Q b_1 u^n v^n \phi_1\ dxdt \quad \text{in } Q \\ &\int_0^T < v_t^n, \phi_2 >_{((H^1)^*, H^1)}\ dt + \int_Q d_2 \nabla v^n \cdot \nabla \phi_2\ dxdt \\ &= \int_Q v^n m \phi_2\ dxdt - \int_Q a_2 (v^n)^2 \phi_2\ dxdt - \int_Q b_2 u^n v^n \phi_2\ dxdt \quad \text{in } Q\ . \end{aligned} \tag{2.18}$$

Using the above weak convergences (2.16) and the L^∞ bounds on $\vec{h}$ and m we have

$$\begin{aligned} \lim_{n\to\infty} \int_0^T < u_t^n - u_t^*, \phi_1 >_{((H^1)^*, H^1)}\ dt &= \int_0^T < u_t^* - u_t^*, \phi_1 >_{((H^1)^*, H^1)}\ dt = 0 \\ \lim_{n\to\infty} \int_Q d_1 \nabla (u^n - u^*) \cdot \nabla \phi_1\ dxdt &= \int_Q d_1 \nabla (u^* - u^*) \cdot \nabla \phi_1\ dxdt = 0 \\ \lim_{n\to\infty} \int_Q (u^n - u^*) m \phi_1\ dxdt &= \int_Q (u^* - u^*) m \phi_1\ dxdt = 0 \\ \lim_{n\to\infty} \int_0^T < v_t^n - v_t^*, \phi_2 >_{((H^1)^*, H^1)}\ dt &= \int_0^T < v_t^* - v_t^*, \phi_2 >_{((H^1)^*, H^1)}\ dt = 0 \\ \lim_{n\to\infty} \int_Q d_2 \nabla (v^n - v^*) \cdot \nabla \phi_2\ dxdt &= \int_Q d_2 \nabla (v^* - v^*) \cdot \nabla \phi_2\ dxdt = 0 \end{aligned}$$

$$\lim_{n\to\infty}\int_Q (v^n - v^*)m\phi_2 \; dxdt = \int_Q (v^* - v^*)m\phi_2 \; dxdt = 0 \; .$$

Note that

$$\begin{aligned}
&\int_Q (u^n\vec{h^n} - u^*\vec{h^*})\cdot\nabla\phi_1 \; dxdt \\
&= \int_Q (u^n\vec{h^n} - u^*\vec{h^n} + u^*\vec{h^n} - u^*\vec{h^*})\cdot\nabla\phi_1 \; dxdt \qquad (2.19)\\
&= \int_Q (u^n - u^*)\vec{h^n}\cdot\nabla\phi_1 \; dxdt + \int_Q u^*(\vec{h^n} - \vec{h^*})\cdot\nabla\phi_1 \; dxdt
\end{aligned}$$

By the Cauchy's inequality, the strong convergence of $\{\vec{u^n}\}$, $\nabla\phi_1$ being in L^2 and the L^∞ bounds on $\{h^n\}$ the first term of (2.19) gives

$$\begin{aligned}
\lim_{n\to\infty}\int_Q (u^n - u^*)\vec{h^n}\cdot\nabla\phi_1 \; dxdt &\leq \lim_{n\to\infty}\int_Q |u^n - u^*||\vec{h^n}||\nabla\phi_1| \; dxdt \\
&\leq \lim_{n\to\infty} M(\int_Q (u^n - u^*)^2 dxdt)^{1/2}(\int_Q |\nabla\phi_1|^2 \; dxdt)^{1/2} \\
&= 0
\end{aligned}$$

and by the weak convergence of $\{\vec{h^n}\}$, $\nabla\phi_1$ being fixed in $L^2(Q)$ and $\{u^*\}$ being fixed with the L^∞ bounds, the second term in (2.19) gives

$$\lim_{n\to\infty}\int_Q u^*(\vec{h^n} - \vec{h^*})\cdot\nabla\phi_1 \; dxdt = 0 \; .$$

That is we have

$$\lim_{n\to\infty}\int_Q (u^n\vec{h^n} - u^*\vec{h^*})\cdot\nabla\phi_1 \; dxdt = 0 \; .$$

Since,

$$\begin{aligned}
&\lim_{n\to\infty}\int_Q a_1(u^n)^2\phi_1 \; dxdt - \int_Q a_1(u^*)^2\phi_1 \; dxdt \\
&= \lim_{n\to\infty}\int_Q a_1[(u^n)^2 - (u^*)^2]\phi_1 \; dxdt
\end{aligned}$$

by the Holder's inequality, L^∞ bounds on u and strong convergences (2.17) we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_Q a_1(u^n)^2\phi_1\ dxdt - \int_Q a_1(u^*)^2\phi_1\ dxdt \\
&\leq a_1\|u^n+u^*\|_{L^\infty}\lim_{n\to\infty}(\int_Q |u^n-u^*|^2\ dxdt)^{1/2}(\int_Q |\phi_1|^2\ dxdt)^{1/2} \\
&= 0
\end{aligned}$$

giving

$$\lim_{n\to\infty}\int_Q a_1(u^n)^2\phi_1\ dxdt = \int_Q a_1(u^*)^2\phi_1\ dxdt\ .$$

Similarly we have

$$\lim_{n\to\infty}\int_Q a_2(v^n)^2\phi_2\ dxdt = \int_Q a_2(v^*)^2\phi_2\ dxdt\ .$$

Again, since

$$\begin{aligned}
&\lim_{n\to\infty}\int_Q b_1u^nv^n\phi_1\ dxdt - \int_Q b_1u^*v^*\phi_1\ dxdt \\
&= \lim_{n\to\infty}\int_Q b_1v^n(u^n-u^*)\phi_1\ dxdt + \lim_{n\to\infty}\int_Q b_1u^*(v^n-v^*)\phi_1\ dxdt
\end{aligned}$$

by the Holder's inequality, L^∞ bounds on u, v and strong convergences (2.17) we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_Q b_1u^nv^n\phi_1\ dxdt - \int_Q b_1u^*v^*\phi_1\ dxdt \\
&\leq b_1\|v^n\|_{L^\infty}\lim_{n\to\infty}(\int_Q |u^n-u^*|^2dxdt)^{1/2}(\int_Q |\phi_1|^2\ dxdt)^{1/2} \\
&\quad + b_1\|u^*\|_{L^\infty}\lim_{n\to\infty}(\int_Q |v^n-v^*|^2dxdt)^{1/2}(\int_Q |\phi_1|^2\ dxdt)^{1/2} \\
&= 0
\end{aligned}$$

giving

$$\lim_{n\to\infty}\int_Q b_1u^nv^n\phi_1\ dxdt = \int_Q b_1u^*v^*\phi_1\ dxdt$$

and similarly

$$\lim_{n\to\infty}\int_Q b_2u^nv^n\phi_2\ dxdt=\int_Q b_2u^*v^*\phi_2\ dxdt\ .$$

Hence, as $n\to\infty$ in (2.18) we obtain

$$\begin{aligned}
&\int_0^T <u_t^*,\phi_1>_{((H^1)^*,H^1)}\ dt+\int_Q d_1\nabla u^*\cdot\nabla\phi_1\ dxdt+\int_Q u^*\vec{h^*}\cdot\nabla\phi_1\ dxdt\\
&=\int_Q u^*m\phi_1\ dxdt-\int_Q a_1(u^*)^2\phi_1\ dxdt-\int_Q b_1u^*v^*\phi_1\ dxdt\qquad \text{in } Q\\
&\int_0^T <v_t^*,\phi_2>_{((H^1)^*,H^1)}\ dt+\int_Q d_2\nabla v^*\cdot\nabla\phi_2\ dxdt\\
&=\int_Q v^*m\phi_2\ dxdt-\int_Q a_2(v^*)^2\phi_2\ dxdt-\int_Q b_2u^*v^*\phi_2\ dxdt\qquad \text{in } Q\ .
\end{aligned}\tag{2.20}$$

As u^*,v^* satisfies the system of state PDEs with the control $\vec{h^*}$ in (2.20), we can conclude that $(u^*,v^*)=(u,v)(\vec{h^*})$.

Again using the weak convergences (2.16) in the objective functional on the maximizing sequence $\{\vec{h^n}\}$ and the fact that

$$\int_Q(\vec{h^*})^2\ dxdt\le\liminf_{n\to\infty}\int_Q(\vec{h^n})^2\ dxdt$$

we get

$$\begin{aligned}
\sup_{\vec{h}\in U}J(\vec{h})=\lim_{n\to\infty}J(\vec{h^n})&=\lim_{n\to\infty}\int_Q\left[Au^n+Bv^n-C|\vec{h^n}|^2\right]\ dxdt\\
&\le\int_Q\left[Au^*+Bv^*-C|\vec{h^*}|^2\right]\ dxdt=J(\vec{h^*})\ .
\end{aligned}\tag{2.21}$$

Since we always have

$$\sup_{\vec{h}\in U}J(\vec{h})\ge J(\vec{h^*})\ ,$$

by (2.21) we conclude that $\vec{h^*}$ is an optimal control for the problem (2.2).

□

2.4.3 Derivation of the optimality system

The necessary conditions:

To characterize the optimal control we have to differentiate the $\vec{h} \to J(\vec{h})$ map with respect to the control $\vec{h}$. But, as $u = u(\vec{h})$ and $v = v(\vec{h})$, we also need to differentiate $\vec{h} \to (u, v)(\vec{h})$ map with respect to the control $\vec{h}$. These derivatives are called the sensitivity functions.

Lemma 2.2.1 (Finding the sensitivity PDE's)**.** *Let $\vec{h}$ be an optimal control for the problem (2.2) with the corresponding state solutions $u = u(\vec{h})$ and $v = v(\vec{h})$ and let $\vec{h}^\epsilon$ be another control with the corresponding state solutions $u^\epsilon = u(\vec{h} + \epsilon\vec{l})$ and $v^\epsilon = v(\vec{h} + \epsilon\vec{l})$ such that*

$$\vec{h}^\epsilon = \vec{h} + \epsilon\vec{l}$$

where $\vec{h} + \epsilon\vec{l} \in U$ for all sufficiently small $\epsilon > 0$ with $\vec{l} \in (L^\infty(\Omega))^d$.

Then, for any $\vec{h} \in U$ the mappings $\vec{h} \to u(\vec{h})$ and $\vec{h} \to v(\vec{h})$ are weakly differentiable in the directional derivative sense and there exists $\psi_1, \psi_2 \in L^2((0,T), H^1(\Omega))$ with $(\psi_1)_t, (\psi_2)_t \in L^2((0,T), (H^1(\Omega))^*)$ such that

$$\lim_{\epsilon \to 0^+} \frac{u(\vec{h} + \epsilon\vec{l}) - u(\vec{h})}{\epsilon} = \psi_1 \quad \text{in} \quad L^2(Q)$$

and

$$\lim_{\epsilon \to 0^+} \frac{v(\vec{h} + \epsilon\vec{l}) - v(\vec{h})}{\epsilon} = \psi_2 \quad \text{in} \quad L^2(Q)$$

Further, the sensitivity functions ψ_1 and ψ_2 satisfies

$$
\begin{aligned}
(\psi_1)_t - d_1\Delta\psi_1 - \nabla\cdot(\vec{h}\psi_1 + \vec{l}u) &= m\psi_1 - 2a_1u\psi_1 - b_1v\psi_1 - b_1u\psi_2 \qquad in\ \ Q \\
(\psi_2)_t - d_2\Delta\psi_2 &= m\psi_2 - 2a_2v\psi_2 - b_2v\psi_1 - b_2u\psi_2 \qquad in\ \ Q \\
\\
d_1\frac{\partial\psi_1}{\partial\eta} + \psi_1\vec{h}\cdot\eta &= -u\vec{l}\cdot\eta \qquad on\ \ S \\
d_2\frac{\partial\psi_2}{\partial\eta} &= 0 \qquad on\ \ S \\
\\
\psi_1(x,0) &= 0 \qquad for\ \ x\in\Omega \\
\psi_2(x,0) &= 0 \qquad for\ \ x\in\Omega\ .
\end{aligned}
\tag{2.22}
$$

Proof. Consider the state PDEs corresponding to the optimal control $\vec{h}$ and the PDEs corresponding to the control $\vec{h}^{\epsilon}$:

$$
\begin{aligned}
u_t - d_1\Delta u - \nabla\cdot(\vec{h}u) &= u[m - a_1u] - b_1uv \qquad \text{in}\ \ Q \\
v_t - d_2\Delta v &= v[m - a_2v] - b_2uv \qquad \text{in}\ \ Q \\
\\
d_1\frac{\partial u}{\partial\eta} + u\vec{h}\cdot\eta &= 0 \qquad \text{on}\ \ S \\
d_2\frac{\partial v}{\partial\eta} &= 0 \qquad \text{on}\ \ S \\
\\
u(x,0) &= u_0(x) \geq 0 \qquad \text{for}\ \ x\in\Omega \\
v(x,0) &= v_0(x) \geq 0 \qquad \text{for}\ \ x\in\Omega
\end{aligned}
$$

$$
\begin{aligned}
u^\epsilon_t - d_1\Delta u^\epsilon - \nabla\cdot(\vec{h}^\epsilon u^\epsilon) &= u^\epsilon[m - a_1 u^\epsilon] - b_1 u^\epsilon v^\epsilon \quad \text{in } Q \\
v^\epsilon_t - d_2\Delta v^\epsilon &= v^\epsilon[m - a_2 v^\epsilon] - b_2 u^\epsilon v^\epsilon \quad \text{in } Q
\end{aligned}
$$

$$
\begin{aligned}
d_1\frac{\partial u^\epsilon}{\partial \eta} + u^\epsilon \vec{h}^\epsilon\cdot\eta &= 0 \quad \text{on } S \\
d_2\frac{\partial v^\epsilon}{\partial \eta} &= 0 \quad \text{on } S
\end{aligned}
$$

$$
\begin{aligned}
u^\epsilon(x,0) &= u_0(x) \geq 0 \quad \text{for } x\in\Omega \\
v^\epsilon(x,0) &= v_0(x) \geq 0 \quad \text{for } x\in\Omega
\end{aligned}
$$

Then, forming the system solved by $\frac{(u^\epsilon-u)}{\epsilon}$ and $\frac{(v^\epsilon-v)}{\epsilon}$,

$$
\begin{aligned}
\frac{(u^\epsilon-u)_t}{\epsilon} - d_1\Delta\frac{(u^\epsilon-u)}{\epsilon} - \nabla\cdot\vec{h}\frac{(u^\epsilon-u)}{\epsilon} - \nabla\cdot\vec{l}u^\epsilon &= m\frac{(u^\epsilon-u)}{\epsilon} \\
-a_1\frac{(u^\epsilon-u)}{\epsilon}(u^\epsilon+u) - b_1 v\frac{(u^\epsilon-u)}{\epsilon} - b_1 u^\epsilon\frac{(v^\epsilon-v)}{\epsilon} &\quad \text{in } Q \\
\frac{(v^\epsilon-v)_t}{\epsilon} - d_2\Delta\frac{(v^\epsilon-v)}{\epsilon} &= m\frac{(v^\epsilon-v)}{\epsilon} \\
-a_2\frac{(v^\epsilon-v)}{\epsilon}(v^\epsilon+v) - b_2 v\frac{(u^\epsilon-u)}{\epsilon} - b_2 u^\epsilon\frac{(v^\epsilon-v)}{\epsilon} &\quad \text{in } Q \\
d_1\frac{\partial\frac{(u^\epsilon-u)}{\epsilon}}{\partial\eta} + \frac{(u^\epsilon-u)}{\epsilon}\vec{h}\cdot\eta + u^\epsilon\vec{l}\cdot\eta &= 0 \quad \text{on } S \qquad (2.23) \\
d_2\frac{\partial\frac{(v^\epsilon-v)}{\epsilon}}{\partial\eta} &= 0 \quad \text{on } S \\
\frac{(u^\epsilon-u)}{\epsilon}(x,0) &= 0 \quad \text{for } x\in\Omega \\
\frac{(v^\epsilon-v)}{\epsilon}(x,0) &= 0 \quad \text{for } x\in\Omega\ .
\end{aligned}
$$

By using similar calculations as in Lemma 2.0.2 for system (2.23) we can show that

$$\begin{aligned} \|\frac{u^\epsilon - u}{\epsilon}\|_{L^2((0,T),H^1(\Omega))} &\leq K_1' \\ \|\frac{(u^\epsilon - u)_t}{\epsilon}\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_2' \\ \|\frac{v^\epsilon - v}{\epsilon}\|_{L^2((0,T),H^1(\Omega))} &\leq K_3' \\ \|\frac{(v^\epsilon - v)_t}{\epsilon}\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_4' \end{aligned}$$

for some constants K_1', K_2', K_3' and K_4' .

Hence, there exists $\psi_1(x,t)$ and $\psi_2(x,t)$ in $L^2((0,T),H^1(\Omega))$ such that as $\epsilon \to 0^+$

$$\begin{aligned} \frac{u(\vec{h}+\epsilon\vec{l}) - u(\vec{h})}{\epsilon} &\rightharpoonup \psi_1 \quad \text{in} \quad L^2((0,T),H^1(\Omega)) \\ \frac{v(\vec{h}+\epsilon\vec{l}) - v(\vec{h})}{\epsilon} &\rightharpoonup \psi_2 \quad \text{in} \quad L^2((0,T),H^1(\Omega)) \end{aligned}$$

and

$$\begin{aligned} \frac{(u(\vec{h}+\epsilon\vec{l}) - u(\vec{h}))_t}{\epsilon} &\rightharpoonup (\psi_1)_t \quad \text{in} \quad L^2((0,T),(H^1(\Omega))^*) \\ \frac{(v(\vec{h}+\epsilon\vec{l}) - v(\vec{h}))_t}{\epsilon} &\rightharpoonup (\psi_2)_t \quad \text{in} \quad L^2((0,T),(H^1(\Omega))^*) \ . \end{aligned}$$

Hence, again by the Simon's results [44], as $\epsilon \to 0^+$ we have the strong convergence of the different quotients $\{\frac{(u^\epsilon - u)}{\epsilon}\}$ and $\{\frac{(v^\epsilon - v)}{\epsilon}\}$ in $L^2(Q)$ given below:

$$\frac{u(\vec{h}+\epsilon\vec{l}) - u(\vec{h})}{\epsilon} \to \psi_1 \quad \text{in} \quad L^2(Q)$$

$$\frac{v(\vec{h}+\epsilon\vec{l}) - v(\vec{h})}{\epsilon} \to \psi_2 \quad \text{in} \quad L^2(Q) \ .$$

Then, using the weak and strong convergences above with techniques like in Theorem 2.2, as ϵ goes to zero in (2.23) we get the resulting system of PDEs (2.22) satisfied by ψ_1 and ψ_2.

□

Next, we give the weak solution for the sensitivity PDE's.

Definition (The weak solution for the sensitivity PDE's)

$\psi_1, \psi_2 \in L^2((0,T), H^1(\Omega))$ with $(\psi_1)_t, (\psi_2)_t \in L^2((0,T), (H^1(\Omega))^*)$ and $\psi_1(x,0) = 0 = \psi_2(x,0)$ is a weak solution for (2.22) if they satisfy,

$$\begin{aligned}
&\int_0^T < (\psi_1)_t, \phi_1 >_{((H^1)^*, H^1)} \ dt + \int_Q d_1 \nabla\psi_1 \cdot \nabla\phi_1 \ dxdt + \int_Q \psi_1 \vec{h} \cdot \nabla\phi_1 \ dxdt \\
&- \int_Q m\psi_1\phi_1 \ dxdt + \int_Q 2a_1 u\psi_1\phi_1 \ dxdt + \int_Q b_1 v\psi_1\phi_1 \ dxdt + \int_Q b_1 u\psi_2\phi_1 \ dxdt \\
&= - \int_Q u\vec{l} \cdot \nabla\phi_1 \ dxdt
\end{aligned}$$

and (2.24)

$$\begin{aligned}
&\int_0^T < (\psi_2)_t, \phi_2 >_{((H^1)^*, H^1)} \ dt + \int_Q d_2 \nabla\psi_2 \cdot \nabla\phi_2 \ dxdt \\
&- \int_Q m\psi_2\phi_2 \ dxdt + \int_Q 2a_2 v\psi_2\phi_2 \ dxdt + \int_Q b_2 v\psi_1\phi_2 \ dxdt + \int_Q b_2 u\psi_2\phi_2 \ dxdt \\
&= 0
\end{aligned}$$

for all ϕ_1 , $\phi_2 \in L^2((0,T), H^1(\Omega))$

We give an explanation of this weak formulation:

Let ϕ_1 , $\phi_2 \in L^2((0,T), H^1(\Omega))$.
Formally we illustrate the weak solution format:

$$\begin{aligned}
&\int_Q (\psi_1)_t\phi_1 \ dxdt - \int_Q d_1\Delta\psi_1\phi_1 \ dxdt - \int_Q \nabla\cdot(\vec{h}\psi_1)\phi_1 \ dxdt \\
&- \int_Q m\psi_1\phi_1 \ dxdt + \int_Q 2a_1 u\psi_1\phi_1 \ dxdt + \int_Q b_1 v\psi_1\phi_1 \ dxdt + \int_Q b_1 u\psi_2\phi_1 \ dxdt \\
&= \int_Q (\nabla\cdot\vec{l}u)\phi_1 \ dxdt \quad \text{in} \quad Q
\end{aligned}$$

and

$$\begin{aligned}
&\int_Q (\psi_2)_t \phi_2 \; dxdt - \int_Q d_2 \Delta \psi_2 \phi_2 \; dxdt \\
&- \int_Q m\psi_2\phi_2 \; dxdt + \int_Q 2a_2 v\psi_2\phi_2 \; dxdt + \int_Q b_2 v\psi_1\phi_2 \; dxdt + \int_Q b_2 u\psi_2\phi_2 \; dxdt \\
&= 0 \quad \text{in} \quad Q \; .
\end{aligned}$$

Using integration by parts on the d_1 and $\vec{h}$ terms we obtain

$$\begin{aligned}
&\int_0^T < (\psi_1)_t, \phi_1 >_{((H^1)^*, H^1)} \; dt + \int_Q d_1 \nabla\psi_1 \cdot \nabla\phi_1 \; dxdt - \int_S d_1\phi_1 \frac{\partial \psi_1}{\partial \eta} \; dsdt \\
&+ \int_Q \psi_1 \vec{h} \cdot \nabla\phi_1 \; dxdt - \int_S \psi_1\phi_1 \vec{h} \cdot \eta \; dsdt \\
&- \int_Q m\psi_1\phi_1 \; dxdt + \int_Q 2a_1 u\psi_1\phi_1 \; dxdt + \int_Q b_1 v\psi_1\phi_1 \; dxdt + \int_Q b_1 u\psi_2\phi_1 \; dxdt \\
&= - \int_Q u\vec{l} \cdot \nabla\phi_1 \; dxdt + \int_S u\phi_1 \vec{l} \cdot \eta \; dsdt \quad \text{in} \quad Q
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T < (\psi_2)_t, \phi_2 >_{((H^1)^*, H^1)} \; dt + \int_Q d_2 \nabla\psi_2 \cdot \nabla\phi_2 \; dxdt - \int_S d_2\phi_2 \frac{\partial \psi_2}{\partial \eta} \; dsdt \\
&- \int_Q m\psi_2\phi_2 \; dxdt + \int_Q 2a_2 v\psi_2\phi_2 \; dxdt + \int_Q b_2 v\psi_1\phi_2 \; dxdt + \int_Q b_2 u\psi_2\phi_2 \; dxdt \\
&= \; 0 \quad \text{in} \quad Q
\end{aligned}$$

Then, as

$$d_1 \frac{\partial \psi_1}{\partial \eta} + \psi_1 \vec{h} \cdot \eta \; + u\vec{l} \cdot \eta = 0 \quad \text{and} \quad d_2 \frac{\partial \psi_2}{\partial \eta} \; = 0 \quad \text{on} \quad S$$

we get the weak formulation (2.24).

Next, we formally illustrate how to derive the adjoint system:

$$\begin{aligned} -(\lambda_1)_t - d_1\Delta\lambda_1 + \vec{h}\cdot\nabla\lambda_1 - m\lambda_1 + 2a_1u\lambda_1 + b_1v\lambda_1 + b_2v\lambda_2 &= A \quad \text{in } Q \\ -(\lambda_2)_t - d_2\Delta\lambda_2 - m\lambda_2 + 2a_2v\lambda_2 + b_1u\lambda_1 + b_2u\lambda_2 &= B \quad \text{in } Q \end{aligned}$$

$$\begin{aligned} \frac{\partial\lambda_1}{\partial\eta} &= 0 \quad \text{on } S \quad (2.25) \\ \frac{\partial\lambda_2}{\partial\eta} &= 0 \quad \text{on } S \end{aligned}$$

$$\begin{aligned} \lambda_1(x,T) &= 0 \quad \text{for } x\in\Omega \\ \lambda_2(x,T) &= 0 \quad \text{for } x\in\Omega\ . \end{aligned}$$

Here, we want to find the adjoint equations such that

$$\begin{aligned} &\text{Sum of left hand sides of weak forms of sensitivity equations} \\ = \quad &\text{Sum of left hand sides of weak forms of the adjoint equations}\ . \end{aligned}$$

Now, by taking $\lambda_1, \lambda_2 \in L^2((0,T), H^1(\Omega))$ with $(\lambda_1)_t, (\lambda_2)_t \in L^2((0,T), (H^1(\Omega))^*)$ as the test functions, and using integration by parts on each left hand side term in (2.24):

$$\begin{aligned} \int_0^T < (\psi_1)_t, \lambda_1 >_{((H^1)^*, H^1)}\ dt &= \int_Q (\psi_1)_t\lambda_1\ dxdt \\ &= -\int_Q (\lambda_1)_t\psi_1\ dxdt + \int_{\Omega\times\{0\}} \lambda_1\psi_1\eta_t\ dxdt \\ &\quad + \int_{\Omega\times\{T\}} \lambda_1\psi_1\eta_t\ dxdt \\ &= -\int_Q (\lambda_1)_t\psi_1\ dxdt, \\ &\quad \text{since } \psi_1 = 0 \text{ at } t = 0 \text{ and } \lambda_1 = 0 \text{ at } t = T\ , \end{aligned}$$

$$\begin{aligned}\int_Q d_1\nabla\psi_1\cdot\nabla\lambda_1\ dxdt &= -\int_Q d_1\Delta\lambda_1\psi_1\ dxdt+\int_S d_1\psi_1\frac{\partial\lambda_1}{\partial\eta}\ dsdt\\ &= -\int_Q d_1\Delta\lambda_1\psi_1\ dxdt, \quad \text{since } \frac{\partial\lambda_1}{\partial\eta}=0 \text{ on } S\ ,\end{aligned}$$

$$\begin{aligned}\int_0^T <(\psi_2)_t,\lambda_2>_{((H^1)^*,H^1)}\ dt &= \int_Q(\psi_2)_t\lambda_2\ dxdt\\ &= -\int_Q(\lambda_2)_t\psi_2\ dxdt+\int_{\Omega\times\{0\}}\lambda_2\psi_2\eta_t\ dxdt\\ &\quad +\int_{\Omega\times\{T\}}\lambda_2\psi_2\eta_t\ dxdt\\ &= -\int_Q(\lambda_2)_t\psi_2\ dxdt,\\ &\qquad \text{since } \psi_2=0 \text{ at } t=0 \text{ and } \lambda_2=0 \text{ at } t=T\ ,\end{aligned}$$

and

$$\begin{aligned}\int_Q d_2\nabla\psi_2\cdot\nabla\lambda_2\ dxdt &= -\int_Q d_2\Delta\lambda_2\psi_2\ dxdt+\int_S d_2\psi_2\frac{\partial\lambda_2}{\partial\eta}\ dsdt\\ &= -\int_Q d_2\Delta\lambda_2\psi_2\ dxdt, \qquad \text{since } \frac{\partial\lambda_2}{\partial\eta}=0 \text{ on } S\ .\end{aligned}$$

So, collecting terms gives

$$\begin{aligned}&\int_0^T <(\psi_1)_t,\lambda_1>_{((H^1)^*,H^1)}\ dt+\int_Q d_1\nabla\psi_1\cdot\nabla\lambda_1\ dxdt+\int_Q\psi_1\vec{h}\cdot\nabla\lambda_1\ dxdt\\ &-\int_Q m\psi_1\lambda_1\ dxdt+\int_Q 2a_1u\psi_1\lambda_1\ dxdt+\int_Q b_1v\psi_1\lambda_1\ dxdt+\int_Q b_1u\psi_2\lambda_1\ dxdt\\ &=-\int_Q(\lambda_1)_t\psi_1\ dxdt-\int_Q d_1\Delta\lambda_1\psi_1\ dxdt+\int_Q\psi_1\vec{h}\cdot\nabla\lambda_1\ dxdt\\ &-\int_Q m\psi_1\lambda_1\ dxdt+\int_Q 2a_1u\psi_1\lambda_1\ dxdt+\int_Q b_1v\psi_1\lambda_1\ dxdt+\int_Q b_1u\psi_2\lambda_1\ dxdt\end{aligned}$$

and

$$\int_0^T < (\psi_2)_t, \lambda_2 >_{((H^1)^*, H^1)} \; dt + \int_Q d_2 \nabla \psi_2 \cdot \nabla \lambda_2 \; dxdt$$
$$- \int_Q m\psi_2\lambda_2 \; dxdt + \int_Q 2a_2 v\psi_2\lambda_2 \; dxdt + \int_Q b_2 v\psi_1\lambda_2 \; dxdt + \int_Q b_2 u\psi_2\lambda_2 \; dxdt$$
$$= - \int_Q (\lambda_2)_t\psi_2 \; dxdt - \int_Q d_2\Delta\lambda_2\psi_2 \; dxdt$$
$$- \int_Q m\psi_2\lambda_2 \; dxdt + \int_Q 2a_2 v\psi_2\lambda_2 \; dxdt + \int_Q b_2 v\psi_1\lambda_2 \; dxdt + \int_Q b_2 u\psi_2\lambda_2 \; dxdt \; .$$

Hence, by collecting terms with ψ_1 for the λ_1 equation and with ψ_2 for the λ_2 equation we get the left hand side of the adjoint equations to be

$$-(\lambda_1)_t - d_1\Delta\lambda_1 + \vec{h} \cdot \nabla\lambda_1 - m\lambda_1 + 2a_1 u\lambda_1 + b_1 v\lambda_1 + b_2 v\lambda_2$$

and

$$-(\lambda_2)_t - d_2\Delta\lambda_2 - m\lambda_2 + 2a_2 v\lambda_2 + b_1 u\lambda_1 + b_2 u\lambda_2 \; .$$

Note that the non-homogeneous terms on the RHS of the adjoint equations are obtained by differentiating the integrand of the objective functional with respect to the state variables u and v :

$$\begin{pmatrix} A \\ \\ B \end{pmatrix} = \begin{pmatrix} \frac{\partial(\text{Integrand of} J)}{\partial u} \\ \\ \frac{\partial(\text{Integrand of} J)}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial(Au+Bv-C|\vec{h}|^2)}{\partial u} \\ \\ \frac{\partial(Au+Bv-C|\vec{h}|^2)}{\partial v} \end{pmatrix} \; .$$

So, with appropriate boundary and final time conditions, we get the adjoint PDE system given in (2.25).

2.4.4 Characterizing the optimal control $\vec{h}^*$

Theorem 2.3 (Characterizing an optimal control $\vec{h}^*$)**.** *There exists adjoint functions $\lambda_1, \lambda_2 \in L^2((0,T), H^1(\Omega))$ with $(\lambda_1)_t, (\lambda_2)_t \in L^2((0,T), (H^1(\Omega))^*)$ which are weak solutions of (2.25).*
Let $\vec{h}^* \in U$ be an optimal control for the problem (2.2) with the corresponding state solutions $u^*, v^* \in L^2((0,T), H^1(\Omega))$. Then,

$$h_i^* = min\left(M, max\left(-\frac{u(\lambda_1)_{x_i}}{2C}, -M\right)\right)$$

for $i = 1, 2, ..., d$

Proof. Note that the adjoint system is linear and a weak solution exists by standard results.

For an optimal control $\vec{h}^* \in U$ we compute the directional derivative of the function $J(\vec{h^*})$ with respect to $\vec{h^*}$ in the direction $\vec{l}$ at u^* and v^* .

Then, as $J(\vec{h}^*)$ is the maximum value for the $J(\vec{h})$ we have

$$\begin{aligned}
0 &\geq \lim_{\epsilon \to 0^+} \frac{J(\vec{h}^* + \epsilon\vec{l}) - J(\vec{h}^*)}{\epsilon} \\
&= \lim_{\epsilon \to 0^+} \int_Q \frac{\left(Au^\epsilon + Bv^\epsilon - C|\vec{h^*} + \epsilon\vec{l}|^2\right) - \left(Au + Bv - C|\vec{h^*}|^2\right)}{\epsilon}\, dxdt \\
&= \int_Q (A\psi_1 + B\psi_2 - 2C\vec{h}^* \cdot \vec{l})\, dxdt \\
&= \int_Q A\psi_1\, dxdt + \int_Q B\psi_2\, dxdt - \int_Q 2C\vec{h}^* \cdot \vec{l}\, dxdt\ . \qquad (2.26)
\end{aligned}$$

Using the weak solution for the adjoint systems (2.25) to substitute for the first two terms in the right hand side of (2.26)

$$
\begin{aligned}
&\int_Q A\psi_1 \, dxdt + \int_Q B\psi_2 \, dxdt - \int_Q 2C\vec{h}^* \cdot \vec{l} \, dxdt \\
=& -\int_0^T <(\lambda_1)_t, \psi_1>_{((H^1)^*, H^1)} \, dt + \int_Q d_1 \nabla\lambda_1 \cdot \nabla\psi_1 \, dxdt + \int_Q \vec{h} \cdot \nabla\lambda_1\psi_1 \, dxdt \\
&- \int_Q m\lambda_1\psi_1 \, dxdt + \int_Q 2a_1 u\lambda_1\psi_1 \, dxdt + \int_Q b_1 v\lambda_1\psi_1 \, dxdt + \int_Q b_2 v\lambda_2\psi_1 \, dxdt \\
&- \int_0^T <(\lambda_2)_t, \psi_2>_{((H^1)^*, H^1)} \, dt + \int_Q d_2 \nabla\lambda_2 \cdot \nabla\psi_2 \, dxdt \\
&- \int_Q m\lambda_2\psi_2 \, dxdt + \int_Q 2a_2 v\lambda_2\psi_2 \, dxdt + \int_Q b_1 u\lambda_1\psi_2 \, dxdt + \int_Q b_2 u\lambda_2\psi_2 \, dxdt \\
&- \int_Q 2C\vec{h}^* \cdot \vec{l} \, dxdt \, .
\end{aligned}
$$

Using the weak solution for the sensitivity system (2.24)

$$
\begin{aligned}
&\int_Q A\psi_1 \, dxdt + \int_Q B\psi_2 \, dxdt - \int_Q 2C\vec{h}^* \cdot \vec{l} \, dxdt \\
=& \int_0^T <(\psi_1)_t, \lambda_1>_{((H^1)^*, H^1)} \, dt + \int_Q d_1 \nabla\psi_1 \cdot \nabla\lambda_1 \, dxdt + \int_Q \psi_1\vec{h} \cdot \nabla\lambda_1 \, dxdt \\
&- \int_Q m\psi_1\lambda_1 \, dxdt + \int_Q 2a_1 u\psi_1\lambda_1 \, dxdt + \int_Q b_1 v\psi_1\lambda_1 \, dxdt + \int_Q b_1 u\psi_2\lambda_1 \, dxdt \\
&+ \int_0^T <(\psi_2)_t, \lambda_2>_{((H^1)^*, H^1)} \, dt + \int_Q d_2 \nabla\psi_2 \cdot \nabla\lambda_2 \, dxdt \\
&- \int_Q m\psi_2\lambda_2 \, dxdt + \int_Q 2a_2 v\psi_2\lambda_2 \, dxdt + \int_Q b_2 v\psi_1\lambda_2 \, dxdt + \int_Q b_2 u\psi_2\lambda_2 \, dxdt \\
&- \int_Q 2C\vec{h}^* \cdot \vec{l} \, dxdt \, .
\end{aligned}
$$

Then, again by the right hand side of (2.24)

$$\begin{aligned}
&\int_Q A\psi_1 \, dxdt + \int_Q B\psi_2 \, dxdt - \int_Q 2C\vec{h}^* \cdot \vec{l} \, dxdt \\
&= - \int_Q u\vec{l} \cdot \nabla\lambda_1 \, dxdt - \int_Q 2C\vec{h}^* \cdot \vec{l} \, dxdt \\
&= - \int_Q \vec{l} \cdot (u\nabla\lambda_1 + 2C\vec{h}^*) \, dxdt \, .
\end{aligned}$$

Hence, for any $\vec{l}$ we have

$$0 \geq - \int_Q \vec{l} \cdot (u\nabla\lambda_1 + 2C\vec{h}^*) \, dxdt$$

which implies

$$0 \leq \int_Q \vec{l} \cdot (u\nabla\lambda_1 + 2C\vec{h}^*) \, dxdt \, . \tag{2.27}$$

Next to characterize this optimal control $\vec{h^*}$ consider (2.27) for different cases of h_i^* for $i = 1, 2, ..., d$.

Case I: On the set $\{(x,t) : \quad -M < h_i < M\}$, inequality (2.27) implies that l_i can have any sign. Hence, choosing $l_j = 0$ for $i \neq j$ we get

$$\begin{aligned}
u(\lambda_1)_{x_i} + 2Ch_i^* &= 0 \\
h_i^* &= -\frac{u(\lambda_1)_{x_i}}{2C} \, .
\end{aligned}$$

Case II: On the set $\{(x,t): \quad h_i = M\}$, similarly (2.27) implies that $l_i \leq 0$. Hence, choosing $l_j = 0$ for $i \neq j$ we get

$$\begin{aligned} u(\lambda_1)_{x_i} + 2Ch_i^* &\leq 0 \\ u(\lambda_1)_{x_i} + 2CM &\leq 0 \\ M &\leq -\frac{u(\lambda_1)_{x_i}}{2C} \\ h_i^* = M &= \min\left(-\frac{u(\lambda_1)_{x_i}}{2C}, M\right) \quad . \end{aligned}$$

Case III: On the set $\{(x,t): \quad h_i = -M\}$, again (2.27) implies that $l_i \geq 0$. Hence, choosing $l_j = 0$ for $i \neq j$ we get

$$\begin{aligned} u(\lambda_1)_{x_i} + 2Ch_i^* &\geq 0 \\ u(\lambda_1)_{x_i} + 2C(-M) &\geq 0 \\ -M &\geq -\frac{u(\lambda_1)_{x_i}}{2C} \\ h_i^* = -M &= \max\left(-\frac{u(\lambda_1)_{x_i}}{2C}, -M\right) \quad . \end{aligned}$$

Hence, we can characterize the optimal control $\vec{h^*}$ for the problem (2.2) as

$$h_i^* = \min\left(M, \max\left(-\frac{u(\lambda_1)_{x_i}}{2C}, -M\right)\right) \tag{2.28}$$

for $i = 1, 2, ..., d$.

□

2.4.5 Uniqueness of the optimal control

Theorem 2.4 (Uniqueness of the optimal control)**.** *For sufficiently small T and for cost coefficient C sufficiently large, the solution to the optimality system (2.1), (2.25) and (2.28) is unique.*

Proof. Suppose $(u,v),(\lambda_1,\lambda_2)$ and $(\bar{u},\bar{v}),(\bar{\lambda_1},\bar{\lambda_2})$ are two solutions to the state pde system (2.1) and (2.25). Hence the corresponding optimal control characterizations are

$$h_i^* = \min\left(M, \max\left(-\frac{u(\lambda_1)_{x_i}}{2C}, -M\right)\right) \tag{2.29}$$

for $i = 1, 2, ..., d$.

and

$$\bar{h_i^*} = \min\left(M, \max\left(-\frac{\bar{u}(\bar{\lambda_1})_{x_i}}{2C}, -M\right)\right) \tag{2.30}$$

for $i = 1, 2, ..., d$.

Consider the change of variables such that,

$$u = e^{at}p, \quad v = e^{at}q, \quad \lambda_1 = e^{-at}\xi_1, \quad \lambda_2 = e^{-at}\xi_2$$

$$\bar{u} = e^{at}\bar{p}, \quad \bar{v} = e^{at}\bar{q}, \quad \bar{\lambda_1} = e^{-at}\bar{\xi_1}, \quad \bar{\lambda_2} = e^{-at}\bar{\xi_2} \ ,$$

where $a \geq 0$ to be chosen and

$$p, q, \bar{p}, \bar{q}, \xi_1, \xi_2, \bar{\xi_1}, \bar{\xi_2} \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$$

with

$$p_t, q_t, \bar{p}_t, \bar{q}_t, (\xi_1)_t, (\xi_2)_t, (\bar{\xi_1})_t, (\bar{\xi_2})_t \in L^2((0,T), (H^1(\Omega))^*)$$

.

Then, substituting $u = e^{at}p, \quad v = e^{at}q$ in (2.1) and dividing by e^{at} we get,

$$\begin{aligned}
ap + p_t - d_1\Delta p - \nabla\cdot(\vec{h}p) &= p[m - a_1 p e^{at}] - b_1 pqe^{at} \quad \text{in } Q \\
aq + q_t - d_2\Delta q &= q[m - a_2 q e^{at}] - b_2 pqe^{at} \quad \text{in } Q \\
d_1\frac{\partial p}{\partial \eta} + p\vec{h}\cdot\eta &= 0 \quad \text{on } S \\
d_2\frac{\partial q}{\partial \eta} &= 0 \quad \text{on } S \\
p(x,0) &= u_0(x) \geq 0 \quad \text{for } x \in \Omega \\
q(x,0) &= v_0(x) \geq 0 \quad \text{for } x \in \Omega
\end{aligned} \tag{2.31}$$

By, substituting $\bar{u} = e^{at}\bar{p}, \quad \bar{v} = e^{at}\bar{q}$ in (2.1) and dividing by e^{at}, we get

$$\begin{aligned}
a\bar{p} + \bar{p}_t - d_1\Delta\bar{p} - \nabla\cdot(\vec{\bar{h}}\bar{p}) &= \bar{p}[m - a_1\bar{p}e^{at}] - b_1\bar{p}\bar{q}e^{at} \quad \text{in } Q \\
a\bar{q} + \bar{q}_t - d_2\Delta\bar{q} &= \bar{q}[m - a_2\bar{q}e^{at}] - b_2\bar{p}\bar{q}e^{at} \quad \text{in } Q \\
d_1\frac{\partial\bar{p}}{\partial\eta} + \bar{p}\vec{\bar{h}}\cdot\eta &= 0 \quad \text{on } S \\
d_2\frac{\partial\bar{q}}{\partial\eta} &= 0 \quad \text{on } S \\
\bar{p}(x,0) &= u_0(x) \geq 0 \quad \text{for } x \in \Omega \\
\bar{q}(x,0) &= v_0(x) \geq 0 \quad \text{for } x \in \Omega
\end{aligned} \tag{2.32}$$

Hence by subtracting the PDE system (2.32) from (2.31) we get

$$
\begin{aligned}
&a(p-\bar{p})+(p-\bar{p})_t-d_1\Delta(p-\bar{p})-\nabla\cdot(\vec{h}(p-\bar{p}))-\nabla\cdot((\vec{h}-\vec{\bar{h}})\bar{p})\\
&=(p-\bar{p})m-a_1(p-\bar{p})(p+\bar{p})e^{at}-b_1(p-\bar{p})\bar{q}e^{at}-b_1(q-\bar{q})pe^{at} \qquad \text{in } Q\\
&a(q-\bar{q})+(q-\bar{q})_t-d_2\Delta(q-\bar{q})\\
&=(q-\bar{q})m-a_2(q-\bar{q})(q+\bar{q})e^{at}-b_2(p-\bar{p})\bar{q}e^{at}-b_2(q-\bar{q})pe^{at} \qquad \text{in } Q
\end{aligned}
$$

$$
\begin{aligned}
d_1\frac{\partial(p-\bar{p})}{\partial\eta}+(p-\bar{p})\vec{h}\cdot\eta+\bar{p}(\vec{h}-\vec{\bar{h}})\cdot\eta &= 0 \qquad \text{on } S \qquad (2.33)\\
d_2\frac{\partial(q-\bar{q})}{\partial\eta} &= 0 \qquad \text{on } S
\end{aligned}
$$

$$
\begin{aligned}
(p-\bar{p})(x,0) &= 0 \qquad \text{for } x\in\Omega\\
(q-\bar{q})(x,0) &= 0 \qquad \text{for } x\in\Omega
\end{aligned}
$$

Substituting $\lambda_1=e^{-at}\xi_1,\quad \lambda_2=e^{-at}\xi_2$ in (2.25) and dividing by e^{-at} we get

$$
\begin{aligned}
&a\xi_1-(\xi_1)_t-d_1\Delta\xi_1+\vec{h}\cdot\nabla\xi_1-m\xi_1+2a_1p\xi_1e^{at}\\
&\qquad +b_1q\xi_1e^{at}+b_2q\xi_2e^{at} = Ae^{at} \qquad \text{in } Q\\
&a\xi_2-(\xi_2)_t-d_2\Delta\xi_2-m\xi_2+2a_2q\xi_2e^{at}\\
&\qquad +b_1p\xi_1e^{at}+b_2p\xi_2e^{at} = Be^{at} \qquad \text{in } Q
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\xi_1}{\partial\eta} &= 0 \qquad \text{on } S \qquad (2.34)\\
\frac{\partial\xi_2}{\partial\eta} &= 0 \qquad \text{on } S
\end{aligned}
$$

$$
\begin{aligned}
\xi_1(x,T) &= 0 \qquad \text{for } x\in\Omega\\
\xi_2(x,T) &= 0 \qquad \text{for } x\in\Omega\ .
\end{aligned}
$$

Similarly by substituting $\bar{\lambda_1} = e^{-at}\bar{\xi_1}, \quad \bar{\lambda_2} = e^{-at}\bar{\xi_2}$ in (2.25) and dividing by e^{-at} we get

$$\begin{aligned}
-(\bar{\xi_1})_t + a\bar{\xi_1} - d_1\Delta\bar{\xi_1} + \vec{\bar{h}}\cdot\nabla\bar{\xi_1} - m\bar{\xi_1} + 2a_1\bar{p}\bar{\xi_1}e^{at} & \\
+b_1\bar{q}\bar{\xi_1}e^{at} + b_2\bar{q}\bar{\xi_2}e^{at} &= Ae^{at} \quad \text{in } Q \\
-(\bar{\xi_2})_t + a\bar{\xi_2} - d_2\Delta\bar{\xi_2} - m\bar{\xi_2} + 2a_2\bar{q}\bar{\xi_2}e^{at} & \\
+b_1\bar{p}\bar{\xi_1}e^{at} + b_2\bar{p}\bar{\xi_2}e^{at} &= Be^{at} \quad \text{in } Q \\
\frac{\partial\bar{\xi_1}}{\partial\eta} &= 0 \quad \text{on } S \\
\frac{\partial\bar{\xi_2}}{\partial\eta} &= 0 \quad \text{on } S \\
\bar{\xi_1}(x,T) &= 0 \quad \text{for } x\in\Omega \\
\bar{\xi_2}(x,T) &= 0 \quad \text{for } x\in\Omega\ .
\end{aligned} \tag{2.35}$$

By subtracting (2.35) from (2.34) we get

$$\begin{aligned}
&a(\xi_1-\bar{\xi_1}) - (\xi_1-\bar{\xi_1})_t - d_1\Delta(\xi_1-\bar{\xi_1}) + \vec{h}\cdot\nabla(\xi_1-\bar{\xi_1}) + (\vec{h}-\vec{\bar{h}})\nabla\bar{\xi_1} \\
&= m(\xi_1-\bar{\xi_1}) - 2a_1(p-\bar{p})\bar{\xi_1}e^{at} - 2a_1(\xi_1-\bar{\xi_1})pe^{at} - b_1(q-\bar{q})\bar{\xi_1}e^{at} \\
&\quad - b_1(\xi_1-\bar{\xi_1})qe^{at} - b_2(q-\bar{q})\bar{\xi_2}e^{at} - b_2(\xi_2-\bar{\xi_2})qe^{at} \quad \text{in } Q \\
&a(\xi_2-\bar{\xi_2}) - (\xi_2-\bar{\xi_2})_t - d_2\Delta(\xi_2-\bar{\xi_2}) \\
&= m(\xi_2-\bar{\xi_2}) - 2a_2(q-\bar{q})\bar{\xi_2}e^{at} - 2a_2(\xi_2-\bar{\xi_2})qe^{at} - b_1(p-\bar{p})\bar{\xi_1}e^{at} \\
&\quad - b_1(\xi_1-\bar{\xi_1})pe^{at} - b_2(p-\bar{p})\bar{\xi_2}e^{at} - b_2(\xi_2-\bar{\xi_2})pe^{at} \quad \text{in } Q \\
&\frac{\partial(\xi_1-\bar{\xi_1})}{\partial\eta} = 0 \quad \text{on } S \\
&\frac{\partial(\xi_2-\bar{\xi_2})}{\partial\eta} = 0 \quad \text{on } S \\
&(\xi_1-\bar{\xi_1})(x,T) = 0 \quad \text{for } x\in\Omega \\
&(\xi_2-\bar{\xi_2})(x,T) = 0 \quad \text{for } x\in\Omega\ .
\end{aligned} \tag{2.36}$$

Now, considering the weak form of the first PDE in (2.33) we get

$$
\begin{aligned}
&a\int_Q (p-\bar{p})^2 dxdt + \int_0^T < (p-\bar{p})_t, (p-\bar{p}) >_{((H^1)^*,H^1)} dt + \int_Q d_1 |\nabla(p-\bar{p})|^2 \ dxdt \\
&+ \int_Q (p-\bar{p})\vec{h}\cdot\nabla(p-\bar{p}) \ dxdt + \int_Q \bar{p}(\vec{h}-\bar{\vec{h}})\cdot\nabla(p-\bar{p}) \ dxdt \\
&= \int_Q (p-\bar{p})^2 m dxdt - \int_Q a_1(p-\bar{p})^2(p+\bar{p})e^{at}dxdt - \int_Q b_1(p-\bar{p})^2\bar{q}e^{at}dxdt \\
&- \int_Q b_1(q-\bar{q})(p-\bar{p})pe^{at}dxdt \qquad \text{in } \ Q
\end{aligned}
\tag{2.37}
$$

Using,

$$
\int_0^T < (p_t-\bar{p}_t), (p-\bar{p}) >_{((H^1)^*,H^1)} dt = \frac{1}{2}\int_\Omega (p-\bar{p})^2(x,T)dx
$$

and the non negativity of $(p-\bar{p})^2$, (2.37) becomes

$$
\begin{aligned}
&a\int_Q (p-\bar{p})^2 dxdt + \frac{1}{2}\int_\Omega (p-\bar{p})^2(x,T)dx + \int_Q d_1 |\nabla(p-\bar{p})|^2 \ dxdt \\
&+ \int_Q (p-\bar{p})\vec{h}\cdot\nabla(p-\bar{p}) \ dxdt + \int_Q \bar{p}(\vec{h}-\bar{\vec{h}})\cdot\nabla(p-\bar{p}) \ dxdt \\
&\leq \int_Q (p-\bar{p})^2 m \ dxdt - \int_Q b_1(q-\bar{q})(p-\bar{p})pe^{at}dxdt \qquad \text{in } \ Q \ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&a\int_Q (p-\bar{p})^2 dxdt + \frac{1}{2}\int_\Omega (p-\bar{p})^2(x,T)dx + \int_Q d_1 |\nabla(p-\bar{p})|^2 \ dxdt \\
&\leq \int_Q (p-\bar{p})^2 m \ dxdt - \int_Q b_1(q-\bar{q})(p-\bar{p})pe^{at}dxdt \\
&- \int_Q (p-\bar{p})\vec{h}\cdot\nabla(p-\bar{p}) \ dxdt - \int_Q \bar{p}(\vec{h}-\bar{\vec{h}})\cdot\nabla(p-\bar{p}) \ dxdt \\
&\leq \int_Q (p-\bar{p})^2 |m| \ dxdt + \int_Q b_1 |(q-\bar{q})(p-\bar{p})p|e^{at}dxdt \\
&+ \int_Q |(p-\bar{p})\vec{h}\cdot\nabla(p-\bar{p})| \ dxdt + \int_Q |\bar{p}(\vec{h}-\bar{\vec{h}})\cdot\nabla(p-\bar{p})| \ dxdt \ ,
\end{aligned}
$$

using L^∞ bounds on $m, \vec{h}, p, \bar{p}$ and the Cauchy's inequalities,

$$
\begin{aligned}
&a\int_Q (p-\bar{p})^2 dxdt + \frac{1}{2}\int_\Omega (p-\bar{p})^2(x,T)dx + \int_Q d_1|\nabla(p-\bar{p})|^2\ dxdt \\
&\leq C_0\int_Q (p-\bar{p})^2\ dxdt + \frac{b_1\theta_1 e^{aT}}{2}\int_Q [(q-\bar{q})^2 + (p-\bar{p})^2]dxdt \\
&+ M\int_Q [\frac{1}{4\epsilon_1}(p-\bar{p})^2 + \epsilon_1|\nabla(p-\bar{p})|^2]\ dxdt + \theta_2\int_Q [\frac{1}{4\epsilon_2}(\vec{h}-\vec{\bar{h}})^2 + \epsilon_2|\nabla(p-\bar{p})|^2]\ dxdt
\end{aligned}
$$

we have

$$
\begin{aligned}
&a\int_Q (p-\bar{p})^2 dxdt + \frac{1}{2}\int_\Omega (p-\bar{p})^2(x,T)dx + (d_1 - M\epsilon_1 - \theta_2\epsilon_2)\int_Q |\nabla(p-\bar{p})|^2\ dxdt \\
&\leq (C_0 + \frac{M}{4\epsilon_1} + \frac{b_1\theta_1 e^{aT}}{2})\int_Q (p-\bar{p})^2\ dxdt + \frac{b_1\theta_1 e^{aT}}{2}\int_Q (q-\bar{q})^2 dxdt \\
&+ \frac{\theta_2}{4\epsilon_2}\int_Q (\vec{h}-\vec{\bar{h}})^2\ dxdt\ .
\end{aligned}
\tag{2.38}
$$

With similar calculations for the weak form of the second equation in (2.33) we get

$$
\begin{aligned}
&a\int_Q (q-\bar{q})^2 dxdt + \frac{1}{2}\int_\Omega (q-\bar{q})^2(x,T)dx + \int_Q d_2|\nabla(q-\bar{q})|^2\ dxdt \\
&\leq \int_Q (q-\bar{q})^2 m\ dxdt - \int_Q b_2(q-\bar{q})(p-\bar{p})\bar{q}e^{at}dxdt \qquad \text{in } Q \\
&\leq C_0\int_Q (q-\bar{q})^2\ dxdt + \frac{b_2\theta_4 e^{aT}}{2}\int_Q [(q-\bar{q})^2 + (p-\bar{p})^2]dxdt
\end{aligned}
$$

which implies,

$$
\begin{aligned}
&a\int_Q (q-\bar{q})^2 dxdt + \frac{1}{2}\int_\Omega (q-\bar{q})^2(x,T)dx + d_2\int_Q |\nabla(q-\bar{q})|^2\ dxdt \\
&\leq (C_0 + \frac{b_2\theta_4 e^{aT}}{2})\int_Q (q-\bar{q})^2\ dxdt + \frac{b_2\theta_4 e^{aT}}{2}\int_Q (p-\bar{p})^2 dxdt\ .
\end{aligned}
\tag{2.39}
$$

Now, similarly by considering the weak formulation of the first PDE in (2.36) we get

$$
\begin{aligned}
&a\int_Q(\xi_1-\bar{\xi_1})^2dxdt+\frac{1}{2}\int_\Omega(\xi_1-\bar{\xi_1})^2(x,0)dx+\int_Q d_1|\nabla(\xi_1-\bar{\xi_1})|^2\ dxdt\\
&+\int_Q(\xi_1-\bar{\xi_1})\vec{h}\cdot\nabla(\xi_1-\bar{\xi_1})\ dxdt+\int_Q(\xi_1-\bar{\xi_1})(\vec{h}-\bar{\vec{h}})\cdot\nabla\bar{\xi_1}\ dxdt\\
&\le\int_Q(\xi_1-\bar{\xi_1})^2m\ dxdt-\int_Q 2a_1(p-\bar{p})(\xi_1-\bar{\xi_1})\bar{\xi_1}e^{at}dxdt\\
&-\int_Q b_1(q-\bar{q})(\xi_1-\bar{\xi_1})\bar{\xi_1}e^{at}dxdt-\int_Q b_2(q-\bar{q})(\xi_1-\bar{\xi_1})\bar{\xi_2}e^{at}dxdt\\
&-\int_Q b_2(\xi_1-\bar{\xi_1})(\xi_2-\bar{\xi_2})qe^{at}dxdt\qquad\text{in }\ Q
\end{aligned}
$$

which implies

$$
\begin{aligned}
&a\int_Q(\xi_1-\bar{\xi_1})^2dxdt+\frac{1}{2}\int_\Omega(\xi_1-\bar{\xi_1})^2(x,0)dx+\int_Q d_1|\nabla(\xi_1-\bar{\xi_1})|^2\ dxdt\\
&\le\int_Q(\xi_1-\bar{\xi_1})^2|m|\ dxdt+\int_Q 2a_1|(p-\bar{p})(\xi_1-\bar{\xi_1})\bar{\xi_1}|e^{at}dxdt\\
&+\int_Q b_1|(q-\bar{q})(\xi_1-\bar{\xi_1})\bar{\xi_1}|e^{at}dxdt+\int_Q b_2|(q-\bar{q})(\xi_1-\bar{\xi_1})\bar{\xi_2}|e^{at}dxdt\\
&+\int_Q b_2|(\xi_1-\bar{\xi_1})(\xi_2-\bar{\xi_2})q|e^{at}dxdt\\
&+\int_Q|(\xi_1-\bar{\xi_1})\vec{h}\cdot\nabla(\xi_1-\bar{\xi_1})|\ dxdt+\int_Q|(\xi_1-\bar{\xi_1})(\vec{h}-\bar{\vec{h}})\cdot\nabla\bar{\xi_1}|\ dxdt\\
&\le C_0\int_Q(\xi_1-\bar{\xi_1})^2\ dxdt+\frac{2a_1\theta_6e^{aT}}{2}\int_Q[(p-\bar{p})^2+(\xi_1-\bar{\xi_1})^2]dxdt\\
&+\frac{b_1\theta_6e^{aT}}{2}\int_Q[(q-\bar{q})^2+(\xi_1-\bar{\xi_1})^2]dxdt+\frac{b_2\theta_8e^{aT}}{2}\int_Q[(q-\bar{q})^2+(\xi_1-\bar{\xi_1})^2]dxdt\\
&+\frac{b_2\theta_3e^{aT}}{2}\int_Q|(\xi_1-\bar{\xi_1})^2+(\xi_2-\bar{\xi_2})^2]dxdt+\frac{\theta_{10}}{2}\int_Q[(\xi_1-\bar{\xi_1})^2+(\vec{h}-\bar{\vec{h}})^2]\ dxdt\\
&+M\int_Q[\frac{1}{4\epsilon_3}(\xi_1-\bar{\xi_1})^2+\epsilon_3|\nabla(\xi_1-\bar{\xi_1})|^2]\ dxdt\ .
\end{aligned}
$$

Using standard regularity theory for the equations of $\xi_k,\bar{\xi_k}$ with $k=1,2$, and the boundedness of $p,\bar{p},q,\bar{q},\xi_1,\bar{\xi_1},\xi_2,\bar{\xi_2},\vec{h},m$, we can derive the L^∞-bounds on $|\nabla\xi_1|$ and $|\nabla\bar{\xi_1}|$, where the bounds only depends on M, the initial data, and the coefficients of

the system (2.1) (see for example, the proof [16, Lemma 5.2]). In above estimates note that we have constant $\theta_{10} > 0$, such that $|\nabla \bar{\xi_1}(x,t)| \leq \theta_{10}$ for all $(x,t) \in Q$.

Hence, now we have

$$\begin{aligned}
&a\int_Q (\xi_1 - \bar{\xi_1})^2 dxdt + \frac{1}{2}\int_\Omega (\xi_1 - \bar{\xi_1})^2(x,0)dx + (d_1 - M\epsilon_3)\int_Q |\nabla(\xi_1 - \bar{\xi_1})|^2 \ dxdt \\
&\leq (C_0 + a_1\theta_6 e^{aT} + \frac{b_1\theta_6 e^{aT}}{2} + \frac{b_2\theta_8 e^{aT}}{2} + \frac{b_2\theta_3 e^{aT}}{2} + \frac{\theta_{10}}{2} + \frac{M}{4\epsilon_3})\int_Q (\xi_1 - \bar{\xi_1})^2 \ dxdt \\
&+ a_1\theta_6 e^{aT}\int_Q (p - \bar{p})^2 dxdt + (\frac{b_1\theta_6 e^{aT}}{2} + \frac{b_2\theta_8 e^{aT}}{2})\int_Q (q - \bar{q})^2 dxdt \\
&+ \frac{b_2\theta_3 e^{aT}}{2}\int_Q (\xi_2 - \bar{\xi_2})^2 dxdt + \frac{\theta_{10}}{2}\int_Q (\vec{h} - \vec{\bar{h}})^2 \ dxdt \ .
\end{aligned} \tag{2.40}$$

Similarly considering the weak form of the second equation in (2.36) we get

$$\begin{aligned}
&a\int_Q (\xi_2 - \bar{\xi_2})^2 dxdt + \frac{1}{2}\int_\Omega (\xi_2 - \bar{\xi_2})^2(x,0)dx + \int_Q d_2|\nabla(\xi_2 - \bar{\xi_2})|^2 \ dxdt \\
&\leq \int_Q (\xi_2 - \bar{\xi_2})^2 m \ dxdt - \int_Q 2a_2(q - \bar{q})(\xi_2 - \bar{\xi_2})\bar{\xi_2} e^{at} dxdt \\
&- \int_Q b_1(p - \bar{p})(\xi_2 - \bar{\xi_2})\bar{\xi_1} e^{at} dxdt - \int_Q b_1 p(\xi_1 - \bar{\xi_1})(\xi_2 - \bar{\xi_2}) e^{at} dxdt \\
&- \int_Q b_2(p - \bar{p})(\xi_2 - \bar{\xi_2})\bar{\xi_2} e^{at} dxdt \qquad \text{in } \ Q \\
&\leq C_0\int_Q (\xi_2 - \bar{\xi_2})^2 \ dxdt + a_2\theta_8 e^{aT}\int_Q [(q - \bar{q})^2 + (\xi_2 - \bar{\xi_2})^2] dxdt \\
&+ \frac{b_1\theta_6 e^{aT}}{2}\int_Q [(p - \bar{p})^2 + (\xi_2 - \bar{\xi_2})^2] dxdt + \frac{b_1\theta_1 e^{aT}}{2}\int_Q [(\xi_1 - \bar{\xi_1})^2 + (\xi_2 - \bar{\xi_2})^2] dxdt \\
&+ \frac{b_2\theta_8 e^{aT}}{2}\int_Q [(p - \bar{p})^2 + (\xi_2 - \bar{\xi_2})^2] dxdt
\end{aligned}$$

which implies

$$
\begin{aligned}
& a\int_Q (\xi_2-\bar{\xi_2})^2 dxdt + \frac{1}{2}\int_\Omega (\xi_2-\bar{\xi_2})^2(x,0)dx + d_2\int_Q |\nabla(\xi_2-\bar{\xi_2})|^2\ dxdt \\
& \le (C_0 + a_2\theta_8 e^{aT} + \frac{b_1\theta_6 e^{aT}}{2} + \frac{b_1\theta_1 e^{aT}}{2} + \frac{b_2\theta_8 e^{aT}}{2})\int_Q (\xi_2-\bar{\xi_2})^2\ dxdt \\
& + a_2\theta_8 e^{aT}\int_Q (q-\bar{q})^2 dxdt + (\frac{b_1\theta_6 e^{aT}}{2} + \frac{b_2\theta_8 e^{aT}}{2})\int_Q (p-\bar{p})^2 dxdt \\
& + \frac{b_1\theta_1 e^{aT}}{2}\int_Q (\xi_1-\bar{\xi_1})^2 dxdt\ .
\end{aligned}
\tag{2.41}
$$

Now, by taking (2.38) + (2.39) + (2.40) + (2.41) we get

$$
\begin{aligned}
& a\int_Q [(p-\bar{p})^2 + (q-\bar{q})^2 + (\xi_1-\bar{\xi_1})^2 + (\xi_2-\bar{\xi_2})^2] dxdt + \\
& \frac{1}{2}\int_\Omega [(p-\bar{p})^2(x,T) + (q-\bar{q})^2(x,T) + (\xi_1-\bar{\xi_1})^2(x,0) + (\xi_2-\bar{\xi_2})^2(x,0)]dx \\
& + (d_1 - M\epsilon_1 - \theta_2\epsilon_2)\int_Q |\nabla(p-\bar{p})|^2\ dxdt + d_2\int_Q |\nabla(q-\bar{q})|^2\ dxdt \\
& + (d_1 - M\epsilon_3)\int_Q |\nabla(\xi_1-\bar{\xi_1})|^2\ dxdt + d_2\int_Q |\nabla(\xi_2-\bar{\xi_2})|^2\ dxdt \\
& \le (C_0 + \frac{M}{4\epsilon_1} + \frac{b_1\theta_1 e^{aT}}{2} + \frac{b_2\theta_4 e^{aT}}{2} + a_1\theta_6 e^{aT} + \frac{b_1\theta_6 e^{aT}}{2} \\
& \qquad + \frac{b_2\theta_8 e^{aT}}{2})\int_Q (p-\bar{p})^2\ dxdt \\
& + (C_0 + \frac{b_2\theta_4 e^{aT}}{2} + \frac{b_1\theta_1 e^{aT}}{2} + \frac{b_1\theta_6 e^{aT}}{2} + \frac{b_2\theta_8 e^{aT}}{2} + a_2\theta_8 e^{aT})\int_Q (q-\bar{q})^2\ dxdt \\
& + (C_0 + a_1\theta_6 e^{aT} + \frac{b_1\theta_6 e^{aT}}{2} + \frac{b_2\theta_8 e^{aT}}{2} + \frac{b_2\theta_3 e^{aT}}{2} + \frac{\theta_{10}}{2} \\
& \qquad + \frac{M}{4\epsilon_3} + \frac{b_1\theta_1 e^{aT}}{2})\int_Q (\xi_1-\bar{\xi_1})^2\ dxdt \\
& + (C_0 + a_2\theta_8 e^{aT} + \frac{b_1\theta_6 e^{aT}}{2} + \frac{b_1\theta_1 e^{aT}}{2} + \frac{b_2\theta_8 e^{aT}}{2} + \frac{b_2\theta_3 e^{aT}}{2})\int_Q (\xi_2-\bar{\xi_2})^2\ dxdt \\
& + (\frac{\theta_2}{4\epsilon_2} + \frac{\theta_{10}}{2})\int_Q (\vec{h}-\bar{\vec{h}})^2\ dxdt
\end{aligned}
\tag{2.42}
$$

Note that by (2.29) and (2.30) we have

$$\begin{aligned}(\vec{h}-\bar{\vec{h}})^2 &= (\frac{-u\nabla\lambda_1}{2C}+\frac{\bar{u}\nabla\bar{\lambda_1}}{2C})^2 = \frac{1}{4C^2}(p\nabla\xi_1-\bar{p}\nabla\bar{\xi_1})^2 \\ &= \frac{1}{4C^2}[(p-\bar{p})\nabla\xi_1-\bar{p}\nabla(\xi_1-\bar{\xi_1})]^2 \ .\end{aligned}$$

Hence,

$$\begin{aligned}&\int_Q(\vec{h}-\bar{\vec{h}})^2\ dxdt \\ &\leq \frac{1}{4C^2}\int_Q[(p-\bar{p})^2(\nabla\xi_1)^2+\bar{p}^2|\nabla(\xi_1-\bar{\xi_1})|^2+|\bar{p}\nabla\xi_1|(p-\bar{p})(\nabla(\xi_1-\bar{\xi_1}))]dxdt \\ &\leq \frac{1}{4C^2}\int_Q[\theta_9^2(p-\bar{p})^2+\theta_2^2|\nabla(\xi_1-\bar{\xi_1})|^2+\theta_2\theta_9(\frac{1}{4\epsilon_4}(p-\bar{p})^2+\epsilon_4[\nabla(\xi_1-\bar{\xi_1})]^2)]dxdt \\ &= (\frac{\theta_9^2}{4C^2}+\frac{\theta_2\theta_9}{16C^2\epsilon_4})\int_Q(p-\bar{p})^2dxdt+(\frac{\theta_2^2}{4C^2}+\frac{\theta_2\theta_9\epsilon_4}{4C^2})\int_Q|\nabla(\xi_1-\bar{\xi_1})|^2dxdt\end{aligned}$$

So that, (2.42) becomes

$$
\begin{aligned}
& a\int_Q [(p-\bar{p})^2+(q-\bar{q})^2+(\xi_1-\bar{\xi_1})^2+(\xi_2-\bar{\xi_2})^2]dxdt+ \\
& \frac{1}{2}\int_\Omega [(p-\bar{p})^2(x,T)+(q-\bar{q})^2(x,T)+(\xi_1-\bar{\xi_1})^2(x,0)+(\xi_2-\bar{\xi_2})^2(x,0)]dx \\
& +(d_1-M\epsilon_1-\theta_2\epsilon_2)\int_Q |\nabla(p-\bar{p})|^2\ dxdt+d_2\int_Q |\nabla(q-\bar{q})|^2\ dxdt \\
& +(d_1-M\epsilon_3-\frac{\theta_2^3}{16C^2\epsilon_2}-\frac{\theta_2^2\theta_9\epsilon_4}{16C^2\epsilon_2}-\frac{\theta_2^2\theta_{10}}{8C^2}-\frac{\theta_2\theta_9\theta_{10}\epsilon_4}{8C^2})\int_Q |\nabla(\xi_1-\bar{\xi_1})|^2\ dxdt \\
& +d_2\int_Q |\nabla(\xi_2-\bar{\xi_2})|^2\ dxdt \\
& \leq (C_0+\frac{M}{4\epsilon_1}+\frac{b_1\theta_4e^{aT}}{2}+\frac{b_2\theta_1e^{aT}}{2}+a_1\theta_6e^{aT}+\frac{b_1\theta_6e^{aT}}{2} \\
& \qquad +\frac{b_2\theta_8e^{aT}}{2}+\frac{\theta_2\theta_9^2}{16\epsilon_2C^2}+\frac{\theta_2^2\theta_9}{64C^2\epsilon_2\epsilon_4}+\frac{\theta_9^2\theta_{10}}{8C^2}+\frac{\theta_2\theta_9\theta_{10}}{32C^2\epsilon_4})\int_Q (p-\bar{p})^2\ dxdt \\
& +(C_0+\frac{b_2\theta_4e^{aT}}{2}+\frac{b_1\theta_1e^{aT}}{2}+\frac{b_1\theta_6e^{aT}}{2}+\frac{b_2\theta_8e^{aT}}{2}+a_2\theta_8e^{aT})\int_Q (q-\bar{q})^2\ dxdt \\
& +(C_0+a_1\theta_6e^{aT}+\frac{b_1\theta_6e^{aT}}{2}+\frac{b_2\theta_8e^{aT}}{2}+\frac{b_2\theta_3e^{aT}}{2}+\frac{\theta_{10}}{2} \\
& \qquad +\frac{M}{4\epsilon_3}+\frac{b_1\theta_1e^{aT}}{2})\int_Q (\xi_1-\bar{\xi_1})^2\ dxdt \\
& +(C_0+a_2\theta_8e^{aT}+\frac{b_1\theta_6e^{aT}}{2}+\frac{b_1\theta_1e^{aT}}{2}+\frac{b_2\theta_8e^{aT}}{2}+\frac{b_2\theta_3e^{aT}}{2})\int_Q (\xi_2-\bar{\xi_2})^2\ dxdt\ .
\end{aligned}
\tag{2.43}
$$

Pick ϵ_1 and ϵ_2 sufficiently small such that, $d_1-M\epsilon_1-\theta_2\epsilon_2>0$ and pick ϵ_3 and ϵ_4 sufficiently small and C sufficiently large such that, $d_1-M\epsilon_3-\frac{\theta_2^3}{16C^2\epsilon_2}-\frac{\theta_2^2\theta_9\epsilon_4}{16C^2\epsilon_2}-\frac{\theta_2^2\theta_{10}}{8C^2}-\frac{\theta_2\theta_9\theta_{10}\epsilon_4}{8C^2}>0$

Further, pick Γ such that

$$
\begin{aligned}
0\leq\Gamma\leq min[&(d_1-M\epsilon_1-\theta_2\epsilon_2), \\
&d_2, (d_1-M\epsilon_3-\frac{\theta_2^3}{16C^2\epsilon_2}-\frac{\theta_2^2\theta_9\epsilon_4}{16C^2\epsilon_2}-\frac{\theta_2^2\theta_{10}}{8C^2}-\frac{\theta_2\theta_9\theta_{10}\epsilon_4}{8C^2})]
\end{aligned}
$$

Also, by taking

$$\begin{aligned}
\rho_1 &= \frac{M}{4\epsilon_1} + \frac{\theta_2\theta_9^2}{16\epsilon_2 C^2} + \frac{\theta_2^2\theta_9}{64C^2\epsilon_2\epsilon_4} + \frac{\theta_9^2\theta_{10}}{8C^2} + \frac{\theta_2\theta_9\theta_{10}}{32C^2\epsilon_4} \\
\rho_2 &= \frac{b_1\theta_4}{2} + \frac{b_2\theta_1}{2} + a_1\theta_6 + \frac{b_1\theta_6}{2} + \frac{b_2\theta_8}{2} \\
\rho_3 &= \frac{b_2\theta_4}{2} + \frac{b_1\theta_1}{2} + \frac{b_1\theta_6}{2} + \frac{b_2\theta_8}{2} + a_2\theta_8 \\
\rho_4 &= \frac{\theta_{10}}{2} + \frac{M}{4\epsilon_3} \\
\rho_5 &= a_1\theta_6 + \frac{b_1\theta_6}{2} + \frac{b_2\theta_8}{2} + \frac{b_2\theta_3}{2} + \frac{b_1\theta_1}{2} \\
\rho_6 &= a_2\theta_8 + \frac{b_1\theta_6}{2} + \frac{b_1\theta_1}{2} + \frac{b_2\theta_8}{2} + \frac{b_2\theta_3}{2}
\end{aligned}$$

(2.43) becomes

$$\begin{aligned}
&(a - C_0 - \rho_1 - \rho_2 e^{aT}) \int_Q (p-\bar{p})^2 dxdt + (a - C_0 - \rho_3 e^{aT}) \int_Q (q-\bar{q})^2 dxdt \\
&+ (a - C_0 - \rho_4 - \rho_5 e^{aT}) \int_Q (\xi_1 - \bar{\xi_1})^2 dxdt + (a - C_0 - \rho_6 e^{aT}) \int_Q (\xi_2 - \bar{\xi_2})^2] dxdt + \\
&\frac{1}{2} \int_\Omega [(p-\bar{p})^2(x,T) + (q-\bar{q})^2(x,T) + (\xi_1 - \bar{\xi_1})^2(x,0) + (\xi_2 - \bar{\xi_2})^2(x,0)] dx \qquad (2.44) \\
&+ \Gamma \int_Q [|\nabla(p-\bar{p})|^2 + |\nabla(q-\bar{q})|^2 + |\nabla(\xi_1 - \bar{\xi_1})|^2 + |\nabla(\xi_2 - \bar{\xi_2})|^2] \; dxdt \\
&\leq 0 \; .
\end{aligned}$$

Now, fix a such that

$$a > max[(C_0 + \rho_1 + \rho_2) \; , \; (C_0 + \rho_3) \; , \; (C_0 + \rho_4 + \rho_5) \; , \; (C_0 + \rho_6)]$$

and by choosing T very small (2.44) implies that,

$$p = \bar{p} \; , \; q = \bar{q} \; , \; \xi_1 = \bar{\xi_1} \; , \; \xi_2 = \bar{\xi_2}$$

and hence we have

$$u = \bar{u} \; , \; v = \bar{v} \; , \; \lambda_1 = \bar{\lambda_1} \; , \; \lambda_2 = \bar{\lambda_2}$$

which implies the uniqueness of the optimum solution for a small time interval.

□

2.5 Numerical results

Numerical results for the 1–D problem

To study the numerical solutions first we consider the one dimensional problem corresponds to problem (2.2):

$$\begin{aligned}
u_t - d_1 u_{xx} - (hu)_x &= u[m - a_1 u] - b_1 uv \quad \text{on} \quad (0,L)\times(0,T) \\
v_t - d_2 v_{xx} &= v[m - a_2 v] - b_2 uv \quad \text{on} \quad (0,L)\times(0,T) \\
d_1 u_x \cdot \eta + uh \cdot \eta &= 0 \quad \text{for} \quad x = 0 \ \text{ or } \ L \quad \text{and} \quad t \in (0,T) \\
d_2 v_x \cdot \eta &= 0 \quad \text{for} \quad x = 0 \ \text{ or } \ L \quad \text{and} \quad t \in (0,T) \\
u(x,0) &= u_0(x) \geq 0 \quad \text{for} \quad x \in (0,L) \\
v(x,0) &= v_0(x) \geq 0 \quad \text{for} \quad x \in (0,L)
\end{aligned}$$

Hence from (2.28) we have

$$h^* = \min\left(M, \max\left(-\frac{u(\lambda_1)_x}{2C}, -M\right)\right) \ .$$

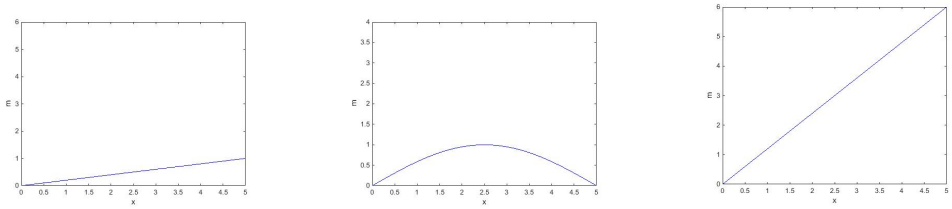
For our numerical simulations, we used parameter values

$$d_1 = d_2 = 0.2 \ , \ a_1 = a_2 = 1 \ , \ A = B = 1 \ , \ C = D = 0.5$$

and chose the optimal controls to satisfy

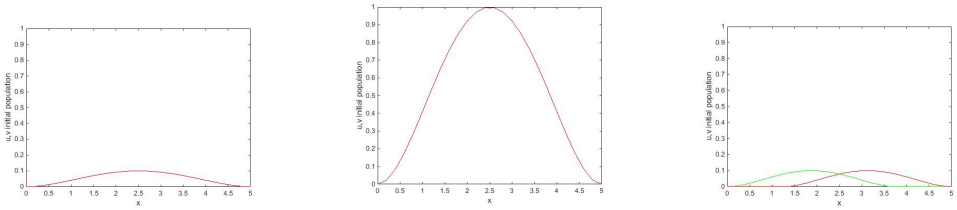
$$-4 \leq h_1 \leq 4$$

in a domain of spatial length of 5 units in 2 units of time. With our notation for the sign of advection terms, h_i being negative (positive) represents movement to right (left).



(a) $x/5$ **(b)** $sin(\pi x/5)$ **(c)** $6x/5$

Figure 2.1: Different resource functions $m(x)$



(a) **(b)** **(c)**

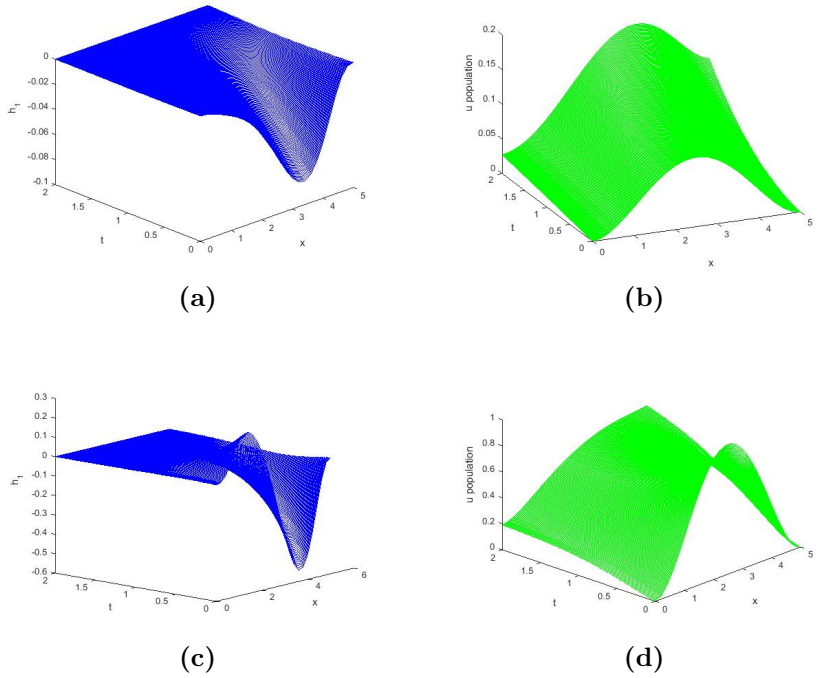
Figure 2.2: Different initial conditions: (2.2a) Smaller initial population at middle, (2.2b) Larger initial population at middle, (2.2c) Two smaller initial populations overlapping in the middle

We considered several cases with different resource functions, initial conditions and competition coefficients. For resource functions, we chose $m(x) = \frac{x}{5}$, $sin(\pi x/5)$ and $m = \frac{6x}{5}$ as shown in Figure 2.1. The initial conditions are shown in Figure 2.2. In our Figures 2.3 – 3.7, we represent u population in green, v population in red, optimal control h_1 in blue and the optimal control h_2 in magenta.

First, to study further the results in [16], we have considered one population with h_1 control under all three different resource functions in Figure 2.1 and the initial conditions in Figures 2.2a – 2.2b. All the figures in [16] have small initial conditions, which allowed the advection control terms to be dominated by the spatial gradient of the resource function. As we will see, diffusion and an initial condition with a lot of

variability may cause the populations to become more level and not to move always toward increasing resources.

The results we obtained for the u population only without competition are given in Figures 2.3 – 2.5.



(a) **(b)**

(c) **(d)**

Figure 2.3: One population only: Population dynamics and optimal control for u population with the resource function $m = x/5$ as in Figure 2.1a ; (2.3a) Optimal control h_1 with respect to the IC in Figure 2.2a , (2.3b) Population distribution of u with respect to the IC in Figure 2.2a , (2.3c) Optimal control h_1 with respect to the IC in Figure 2.2b , (2.3d) Population distribution of u with respect to the IC in Figure 2.2b

For the resource function given in Figure 2.1a and the initial conditions given in Figures 2.2a , 2.2b, we obtained the optimal control (blue) and the population distribution (green) in Figure 2.3. From Figure 2.3a, it can be seen that when the initial population level is smaller the population tends to move towards the higher resources with a small velocity. But, compared to Figure 2.3a, in Figure 2.3c when the initial population level is higher they tend to use the directed movement with larger magnitude than in Figure 2.3a to move away from the higher density areas to

lower density areas. Still the individuals moving to the boundary where the higher resources are located has a larger magnitude for the advection coefficient than for the individuals who are moving towards the boundary away from better resources.

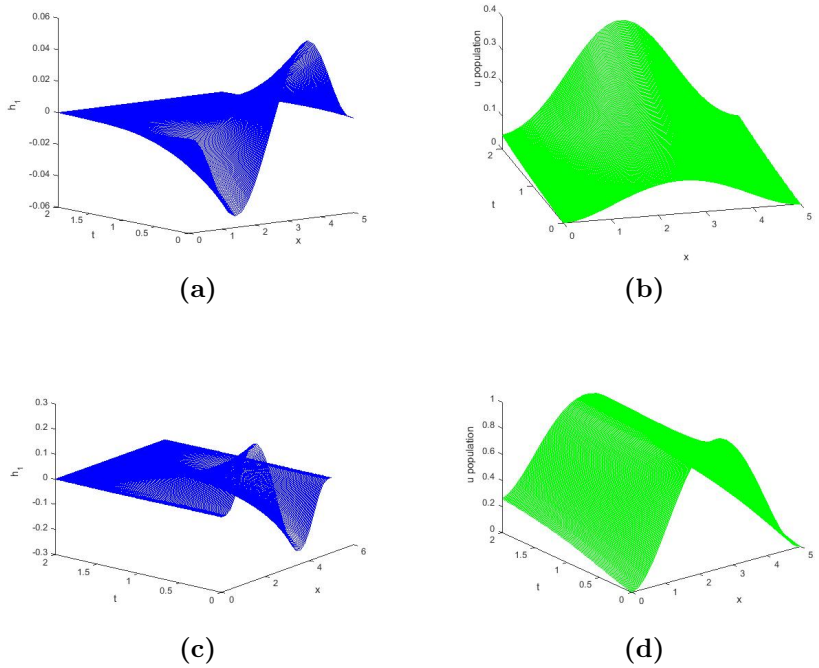
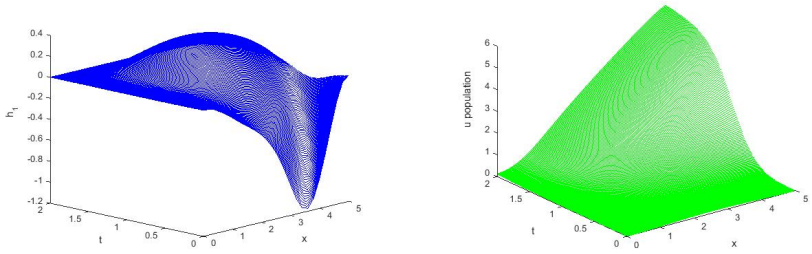


Figure 2.4: One population only: Population dynamics and optimal control for u population with the resource function $m = sin(\pi x/5)$ as in Figure 2.1b ; (2.4a) Optimal control h_1 with respect to the IC in Figure 2.2a , (2.4b) Population distribution of u with respect to the IC in Figure 2.2a , (2.4c) Optimal control h_1 with respect to the IC in Figure 2.2b , (2.4d) Population distribution of u with respect to the IC in Figure 2.2b

With the resource function (high resources in the middle) given in Figure 2.1b and the initial conditions given in Figures 2.2a , 2.2b, the optimal control and the population distribution are shown in Figure 2.4. When the resources are located at the center of the domain and the initial population is low, in Figure 2.4a, we see u tends to move towards the resources at the center from both sides with a smaller velocity. But, in contrast to Figure 2.4a, when the initial population is higher, Figure 2.4c shows that u tends to move towards the two boundaries even though the better resources are located at the center of the domain. Hence, it is clear that in spite

of being at the location of better resources, when the population level is higher, the directed movement moves towards low density areas away from the high density area at the center.



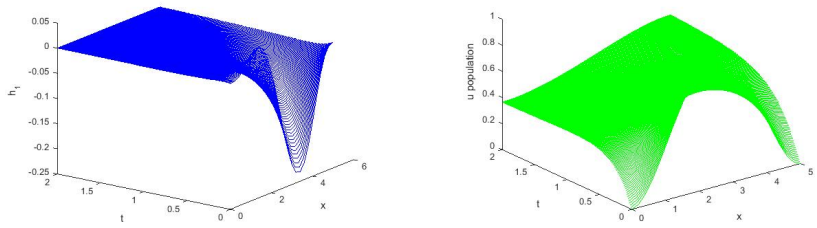
(a) **(b)**

Figure 2.5: One population only: Population dynamics and optimal control with a smaller IC as in Figure 2.2a and larger resources $m = 6x/5$ as in Figure 2.1c ; (2.5a) Optimal control h_1, (2.5b) Population distribution of u

We ran the simulations to see how a smaller population may act in a higher resource environment. In Figure 2.5 we have the optimal control h_1 and the population distribution of u when the resource function is higher as in Figure 2.1c with the smaller initial condition in Figure 2.2a. In Figure 2.5 simulations with higher resources, the population grows faster in a shorter time. Hence, compared to Figure 2.3a, in Figure 2.5a, when the population level is smaller at the beginning, they tend to move towards the resources, and once the population reaches a higher level they move towards low density areas in spite of the location of the better resources.

Next to see the effect of diffusion coefficient on the directed movement we have considered the higher initial condition as in Figure 2.2b with the resource functions as in Figure 2.1a and Figure 2.1b. Here we have taken $d = 2$, which is a very large diffusion coefficient. The results are given in Figures 2.6 and 2.7.

In Figure 2.6a, compared to Figure 2.3c we can see that when the diffusion coefficient is higher, the population tend to move towards better resources. Due to the higher diffusion coefficient, the initial population can level out faster and evenly distribute over the space. Hence, the directed movement will be dominated by the



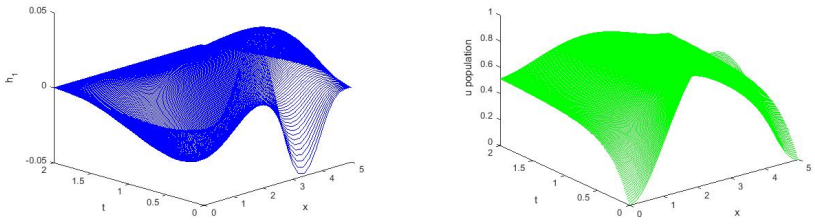
(a) **(b)**

Figure 2.6: One population only with: Population dynamics and optimal control for u population with $d = 2$ and the resource function $m = x/5$ as in Figure 2.1a ; (2.6a) Optimal control h_1 with respect to the IC in Figure 2.2b , (2.6b) Population distribution of u with respect to the IC in Figure 2.2b

resource level. Similarly, from Figure 2.7a, compared to the Figure 2.4c we can see that at the beginning when the initial population level is higher, the individuals start moving to level out and then start moving towards better resources at the center. Hence, from the results in Figures 2.6 – 2.7 we can see that the population size is important in their movement decisions. When the population has low variation in space they tend to move towards better resources.

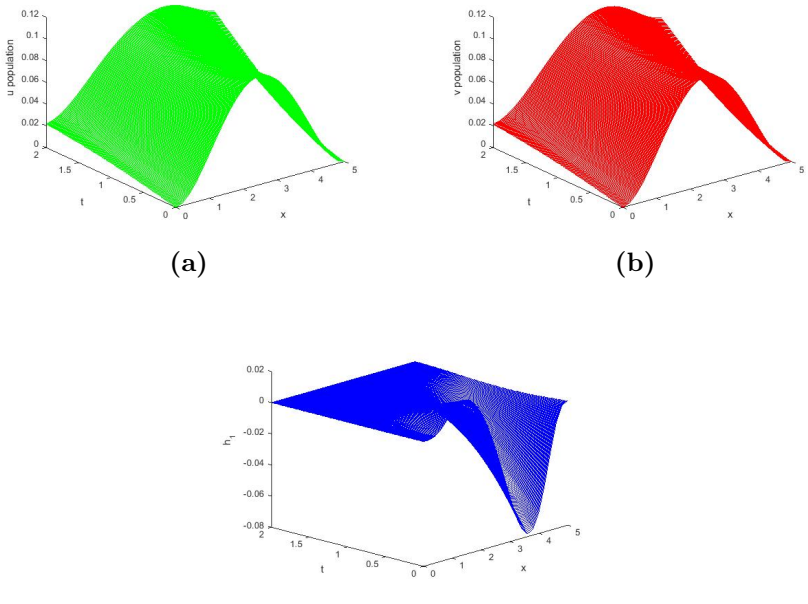
Thus, from the numerical simulations in Figure 2.3 – 2.7, we can see that given the opportunity for directed movement, a population may not always move towards the resources but their movement depends on the initial population also. Note that this feature was not illustrated in the numerical results of [16].

Next, to see the effect when one population is given the opportunity for the directed movement under the competition, first we have considered the resource function as in Figure 2.1a with the initial condition as in Figure 2.2a. The results are given under the Figure 2.8. Compared to the Figure 2.3a, we can see that under the competition when given the opportunity to move, more u individuals tend to move towards the better resources and some u individuals tend to move towards left, away from resources to avoid the higher density of competing population v. Hence, we can see that the directed movement choice also depends on the competition.



(a) **(b)**

Figure 2.7: One population only with: Population dynamics and optimal control for u population with $d = 2$ and the resource function $m = sin(\pi x/5)$ as in Figure 2.1b ; (2.7a) Optimal control h_1 with respect to the IC in Figure 2.2b , (2.7b) Population distribution of u with respect to the IC in Figure 2.2b



(a) **(b)**

(c)

Figure 2.8: Dynamics of two populations and optimal control for u population with the resource function $m = x/5$ as in Figure 2.1a and the initial condition as in Figure 2.2a ; (2.8a) Population distribution of u, (2.8b) Population distribution of v, (2.8c) Optimal control h_1

2.6 Conclusions

Optimal control results and the needed PDE estimates for controlling the advection directions in one parabolic PDE in a system of parabolic PDEs for two competing populations was completed to obtain existence, uniqueness and characterization of the optimal control. Thus characterization shows the explicit dependence on the coefficients, A and B in the objective functional.

With numerical simulations for one population only, we were able to show the population does not always choose the advection direction to move toward increasing resources. When the initial condition has sufficiently high variation, the movement may be chosen to move to level the population, sometimes instead of moving toward increasing resources. This numerical work indicates some strong relationships between the optimal advective directions and the initial conditions and diffusion coefficients. In the systems case, the level of the competition coefficients can also influence the choice of movement direction.

There are various generalizations in the PDEs that could be considered. An important feature of allowing both populations to choose advection movements will be treated in the next chapter.

Chapter 3

Second competition model

3.1 Introduction

In many biological scenarios, one would expect that both populations would have direct movements. Improving the first competition model from Chapter 2, we now discuss the dynamics of two populations, both having diffusive and advective movements.

3.2 Problem formulation

Let Ω be an open bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, for $d \in \mathbb{N}$. For a given fixed time $0 < T < \infty$, we denote $Q = \Omega \times (0, T)$, and the lateral boundary of Q, $S = \partial\Omega \times (0, T)$. We denote $u(x, t), v(x, t)$ to be the population densities of two competing species in the spatial domain Ω with $x \in \Omega$ and $t \in [0, T)$.

Here, we consider the dynamics of the two populations using the following system of PDE's with the Robin boundary conditions:

$$
\begin{aligned}
u_t - d_1\Delta u - \nabla\cdot(\vec{h_1}u) &= u[m - a_1u] - b_1uv \quad \text{in } Q \\
v_t - d_2\Delta v - \nabla\cdot(\vec{h_2}v) &= v[m - a_2v] - b_2uv \quad \text{in } Q \\
d_1\frac{\partial u}{\partial \eta} + u\vec{h_1}\cdot\eta &= 0 \quad \text{on } S \\
d_2\frac{\partial v}{\partial \eta} + v\vec{h_2}\cdot\eta &= 0 \quad \text{on } S \\
u(x,0) &= u_0(x) \geq 0 \quad \text{for a.e. } x \in \Omega \\
v(x,0) &= v_0(x) \geq 0 \quad \text{for a.e. } x \in \Omega
\end{aligned}
\tag{3.1}
$$

Now, in contrast to in Chapter 2 PDE model 2.1, both populations are given the opportunity to move (advection movement) in addition to random diffusion and we consider Robin boundary condition for both populations. Further, we assume the density dependent growth for both species. These two populations are competing with interaction coefficients $b_1(x,t)$ and $b_2(x,t)$.

We define the parameters:

d_1, d_2 – Constant positive diffusion coefficients

a_1, a_2 – Coefficients in the density dependent growth terms

m – Known resource function, $0 \leq m(x,t) \leq C_0$ on Q

b_1, b_2 – Positive interaction coefficients

$\vec{h_1}(x,t) = ((h_1)_1(x,t), (h_1)_2(x,t), ..., (h_1)_d(x,t))$ – Advective direction for the first species with $\vec{h} \in L^\infty(Q)^d$

$\vec{h_2}(x,t) = ((h_2)_1(x,t), (h_2)_2(x,t), ..., (h_2)_d(x,t))$ – Advective direction for the second species with $\vec{h} \in L^\infty(Q)^d$

η – Outward normal vector on $\partial\Omega$

u_0, v_0 – Initial population densities of the two populations with $u_0, v_0 \in H^1(\Omega)$

Similar to as in Chapter 2, to find the solutions to the problem we use the spaces

$H^1(\Omega)$ – Sobolev space with $\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx\right)^{1/2}$ and

$H^1(\Omega))^*$ – Dual space of $H^1(\Omega)$

3.3 The optimal control problem formulation

We assume both populations can choose their advective directions $\vec{h_1}$ and $\vec{h_2}$ to maximize their abundance, but this advective movements may have a "cost" in terms of risk or energy.

Viewing the advective coefficients as controls, the set of all controls is

$$U = \{(\vec{h_1}, \vec{h_2}) \in ((L^\infty(Q))^d, (L^\infty(Q))^d) : |(h_1)_i| \leq M_1, |(h_2)_i| \leq M_2 \text{ for } i = 1, ..., d\}$$

for some known constants $M_1, M_2 > 0$.

Then, for given non-negative constants A, B, C, D, we seek to find $(\vec{h_1}, \vec{h_2}) \in U$ such that

$$J((\vec{h_1}, \vec{h_2})^*) = \max_{(\vec{h_1}, \vec{h_2}) \in U} J(\vec{h_1}, \vec{h_2})$$

where, $J(\vec{h_1}, \vec{h_2})$ is the objective functional given by,

$$J(\vec{h_1}, \vec{h_2}) = \int_Q \left[Au + Bv - C|\vec{h_1}|^2 - D|\vec{h_2}|^2\right] \, dxdt \tag{3.2}$$

subject to the PDE system (3.1).

Our goal is to maximize a weighted sum of the two populations over the time while trying to minimize the total cost involved in this movement subject to the PDE system (3.1). In this case the cost is due to the "risk" in moving the two populations around by the advection as the movement to a new location may come with a risk.

3.4 Solving the optimal control problem

We first recall the definition of weak solutions for the state PDE system.

Definition (The weak solution for the state PDE system)

$u, v \in L^2((0,T), H^1(\Omega)) \bigcap L^\infty(Q)$ with $u_t, v_t \in L^2((0,T), (H^1(\Omega))^*)$ and $u(x,0) = u_0, v(x,0) = v_0$ is a weak solution for (3.1) if it satisfies,

$$\begin{aligned}
&\int_0^T < u_t, \phi_1 >_{((H^1)^*, H^1)} \ dt + \int_Q d_1 \nabla u \cdot \nabla \phi_1 \ dxdt + \int_Q u\vec{h_1} \cdot \nabla \phi_1 \ dxdt \\
&= \int_Q um\phi_1 \ dxdt - \int_Q a_1 u^2 \phi_1 \ dxdt - \int_Q b_1 uv\phi_1 \ dxdt \quad \text{in } Q \\
&\text{and} \qquad (3.3) \\
&\int_0^T < v_t, \phi_2 >_{((H^1)^*, H^1)} \ dt + \int_Q d_2 \nabla v \cdot \nabla \phi_2 \ dxdt + \int_Q u\vec{h_2} \cdot \nabla \phi_2 \ dxdt \\
&= \int_Q vm\phi_2 \ dxdt - \int_Q a_2 v^2 \phi_2 \ dxdt - \int_Q b_2 uv\phi_2 \ dxdt \quad \text{in } Q
\end{aligned}$$

for a.e. $x \in \Omega$ and for all ϕ_1 , $\phi_2 \in L^2((0,T), H^1(\Omega))$.

As we did in Chapter 2, we can give an explanation to this weak formulation.

Lemma 3.0.1. *Let $M_1, M_2 > 0$ be defined in the definition of the control set, and $C_0 = \|m\|_{L^\infty(Q)}$. For a given control $(\vec{h_1}, \vec{h_2}) \in U$ let $u, v \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$ with $u_t, v_t \in L^2((0,T), (H^1(\Omega))^*)$ be a weak solution of (3.1) with $u \geq 0$, $v \geq 0$ a.e. on Q. Then, there exists constants K_1, K_2, K_3, K_4 depending only on M_1, M_2, C_0 and independent of $\vec{h}$ such that*

$$\begin{aligned}
\|u(\vec{h})\|_{L^2((0,T),H^1(\Omega))} &\leq K_1 \\
\|u_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_2 \\
\|v(\vec{h})\|_{L^2((0,T),H^1(\Omega))} &\leq K_3 \\
\|v_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_4 \ .
\end{aligned}$$

Proof. As in Lemma 2.0.2, taking u , v as test functions and considering the weak formulation of (3.1) with the standard techniques and Gronwall's inequality we can get the above *a prior* estimates.

□

3.4.1 Existence and positivity of the state solution

In order to solve the optimal control problem (3.2), first we have to show that for a given control $(\vec{h_1}, \vec{h_2}) \in U$ there exists a unique weak state solution $(u, v) = (u, v)(\vec{h_1}, \vec{h_2})$ to (3.1), showing the dependence on $\vec{h_1}, \vec{h_2}$. Further, we have to show this solution is non-negative and bounded above.

Theorem 3.1 (Existence and positivity of a weak solution for the state PDE system)**.** *Given $m \in L^\infty(Q)$, u_0, v_0 non-negative, $L^\infty(Q)$ bounded and in $H^1(\Omega)$. Then, for each $(\vec{h_1}, \vec{h_2}) \in U$, there is a unique positive weak solution $(u, v) = (u(\vec{h_1}, \vec{h_2}), v(\vec{h_1}, \vec{h_2}))$ of (3.1).*

Proof. As in the proof of Theorem 2.1, using an iterative system of PDEs, we can show the existence and positivity of a weak solution to the state PDE system (3.1). First to construct supersolutions of our system, consider

$$\begin{aligned} \overline{U}_t - d_1\Delta\overline{U} - \nabla\cdot(\vec{h_1}\overline{U}) &= m\overline{U} \quad \text{on } Q \\ d_1\frac{\partial\overline{U}}{\partial\eta} + \overline{U}\vec{h_1}\cdot\eta &= 0 \quad \text{in } \partial\Omega\times(0,T) \\ \overline{U}(x,0) &= u_0(x) \geq 0 \quad \text{for } x\in\Omega \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \overline{V}_t - d_2\Delta\overline{V} - \nabla\cdot(\vec{h_2}\overline{V}) &= m\overline{V} \quad \text{on } Q \\ d_2\frac{\partial\overline{V}}{\partial\eta} + \overline{V}\vec{h_2}\cdot\eta &= 0 \quad \text{in } \partial\Omega\times(0,T) \\ \overline{V}(x,0) &= v_0(x) \geq 0 \quad \text{for } x\in\Omega\ . \end{aligned} \tag{3.5}$$

Then, by Theorem 3.1 in [16], problem (3.4) has a unique weak solution $\overline{U}(\vec{h})$ and a finite constant $C_1 > 0$ such that

$$0 \leq \overline{U}(\vec{h}) \leq C_1 \qquad \text{for a.e. } (x,t) \in \Omega \times (0,T)$$

and problem (3.5) has a unique weak solution $\overline{V}(\vec{h})$ and a finite constant $C_2 > 0$ such that

$$0 \leq \overline{V}(\vec{h}) \leq C_2 \qquad \text{for a.e. } (x,t) \in \Omega \times (0,T)$$

where, constants $C_1, C_2 > 0$ depending on M_1 , M_2 , $|Q|$, d_1 , d_2 , $\|m\|_{L^\infty}$ and the initial conditions.

Now, define $0 \leq R < \infty$ such that

$$R \geq \max\{-\inf_Q m + 2a_1 \sup_\Omega \overline{U} + b_1 \sup_\Omega \overline{V}, -\inf_Q m + 2a_2 \sup_\Omega \overline{V} + b_2 \sup_\Omega \overline{U}\} \ . \quad (3.6)$$

For $i = 2, 3, ...,$ consider the following iterative system of PDE's with $u_1 = \overline{U}$ and $v_1 = 0$ on Q.

$$\begin{aligned}
L_1 u_i &= F(u_{i-1}, v_{i-1}) \qquad \text{in } Q \\
L_2 v_i &= G(u_{i-1}, v_{i-1}) \qquad \text{in } Q \\
d_1 \frac{\partial u_i}{\partial \eta} + u_i \vec{h_1} \cdot \eta &= 0 \qquad \text{on } S \\
d_2 \frac{\partial v_i}{\partial \eta} + v_i \vec{h_2} \cdot \eta &= 0 \qquad \text{on } S \\
u_i(x,0) &= u_0(x) \geq 0 \qquad \text{for } x \in \Omega \\
v_i(x,0) &= v_0(x) \geq 0 \qquad \text{for } x \in \Omega
\end{aligned} \quad (3.7)$$

The weak solution for (3.7) exist by the general results from [15]. where,

$$\begin{aligned} L_1 u_i &= (u_i)_t - d_1 \Delta u_i - \nabla \cdot (\vec{h_1} u_i) + R u_i \\ L_2 v_i &= (v_i)_t - d_2 \Delta v_i - \nabla \cdot (\vec{h_2} v_i) + R v_i \end{aligned}$$

and

$$\begin{aligned} F(u_{i-1}, v_{i-1}) &= m u_{i-1} - a_1 (u_{i-1})^2 - b_1 u_{i-1} v_{i-1} + R u_{i-1} \\ G(u_{i-1}, v_{i-1}) &= m v_{i-1} - a_2 (v_{i-1})^2 - b_2 u_{i-1} v_{i-1} + R v_{i-1} \ . \end{aligned}$$

Choosing $u_1 = \overline{U}$ and $v_1 = 0$, then

$$\begin{aligned} L_1(u_1) &= m u_1 + R u_1 = m\overline{U} + R\overline{U} \\ L_2(v_1) &= m v_1 + R v_1 = 0 \ . \end{aligned}$$

Recall our super solution V satisfies

$$L_2(\overline{V}) = m\overline{V} + R\overline{V} \ .$$

Claim 1

$0 \leq u_i \leq \overline{U}$ and $0 \leq v_i \leq \overline{V}$ for all $i = 1, 2, ...$.

Proof. We can prove this using mathematical induction and Lemma 2.0.1.

□

Claim 2

The sequence $\{u_i\}$ is monotone decreasing and the sequence $\{v_i\}$ is monotone increasing :

$$u_{i+1} \leq u_i \quad \text{and} \quad v_{i+1} \geq v_i \quad \text{for all} \quad i = 1, 2, ... \ .$$

Proof. Again, we can prove this using mathematical induction, Lemma 2.0.1 and the choice of R such that $F(u,v)$ is increasing in u and decreasing in v and $G(u,v)$ is decreasing in u and increasing in v for $(u,v) \in [0, \|U_1\|_{L^\infty(Q)}] \times [0, \|V_1\|_{L^\infty(Q)}]$ as in (3.6).

□

Now, similarly as in Lemma 2.0.2, for some constants K_1, K_2, K_3, K_4 with $\|u_i\|_{L^\infty(Q)}$, $\|v_i\|_{L^\infty(Q)}$ bounds and $u_i \geq 0$, $v_i \geq 0$ a.e on Q we have:

$$\begin{aligned} \|u_i(\vec{h})\|_{L^2((0,T),H^1(\Omega))} &\leq K_1 \\ \|(u_i)_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_2 \\ \|v_i(\vec{h})\|_{L^2((0,T),H^1(\Omega))} &\leq K_3 \\ \|(v_i)_t(\vec{h})\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_4 \end{aligned}$$

for all $i = 1, 2, ...$

So, by these *a priori* bounds for u_i, v_i, $(u_i)_t$, and $(v_i)_t$ and by the monotone property of $\{u_i\}$ and $\{v_i\}$ sequences, there exists $u, v \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$ with $u_t, v_t \in L^2((0,T), (H^1(\Omega))^*)$ with the weak convergences (on the whole sequence, not just on a subsequences) below:

$$\begin{aligned} u_i &\rightharpoonup u && \text{in} && L^2((0,T), H^1(\Omega)) \\ (u_i)_t &\rightharpoonup u_t && \text{in} && L^2((0,T), (H^1(\Omega))^*) \\ v^i &\rightharpoonup v && \text{in} && L^2((0,T), H^1(\Omega)) \\ (v_i)_t &\rightharpoonup v_t && \text{in} && L^2((0,T), (H^1(\Omega))^*) \ . \end{aligned} \tag{3.8}$$

Then, by Simon's result [44] we have the strong convergence of the sequences $\{u_i\}$ and $\{v_i\}$ in $L^2(Q)$ as below:

$$\begin{aligned} u_i &\to u \quad \text{in} \quad L^2(Q) \\ v_i &\to v \quad \text{in} \quad L^2(Q)\ . \end{aligned} \tag{3.9}$$

Since there exists weak solutions to our iterative system (3.7), for the test functions $\phi_1\ ,\ \phi_2 \in L^2((0,T),H^1(\Omega))$, and $u_i, v_i \in L^2((0,T),H^1(\Omega)) \cap L^\infty(Q)$ with $(u_i)_t, (v_i)_t \in L^2((0,T),(H^1(\Omega))^*)$ and $u_i(x,0) = u_0, v_i(x,0) = v_0$ we have

$$\begin{aligned} &\int_0^T < (u_i)_t, \phi_1 >_{((H^1)^*,H^1)}\ dt + \int_Q d_1 \nabla u_i \cdot \nabla \phi_1\ dxdt + \int_Q u_i \vec{h_1} \cdot \nabla \phi_1\ dxdt \\ &+ \int_Q R u_i \phi_1\ dxdt \\ &= \int_Q u_{i-1} m \phi_1\ dxdt - \int_Q a_1 (u_{i-1})^2 \phi_1\ dxdt - \int_Q b_1 u_{i-1} v_{i-1} \phi_1\ dxdt \\ &+ \int_Q R u_{i-1} \phi_1 dxdt \quad \text{in} \quad Q \\ &\text{and} \\ &\int_0^T < (v_i)_t, \phi_2 >_{((H^1)^*,H^1)}\ dt + \int_Q d_2 \nabla v_i \cdot \nabla \phi_2\ dxdt + \int_Q v_i \vec{h_2} \cdot \nabla \phi_2\ dxdt \\ &+ \int_Q R v_i \phi_2 dxdt \\ &= \int_Q v_{i-1} m \phi_2\ dxdt - \int_Q a_2 (v_{i-1})^2 \phi_2\ dxdt - \int_Q b_2 u_{i-1} v_{i-1} \phi_2\ dxdt \\ &+ \int_Q R v_{i-1} \phi_2 dxdt \quad \text{in} \quad Q\ . \end{aligned} \tag{3.10}$$

Since ϕ_1, ϕ_2 are in $L^2((0,T),H^1(\Omega))$, using the weak star convergence of $\{(u_i)_t\}$ and $\{(v_i)_t\}$ in $L^2((0,T),(H^1(\Omega))^*)$ we have

$$\lim_{i\to\infty} \int_0^T < (u_i)_t, \phi_1 >_{((H^1)^*,H^1)}\ dt = \int_0^T < u_t, \phi_1 >_{((H^1)^*,H^1)}\ dt$$

and

$$\lim_{i\to\infty}\int_0^T < (v_i)_t, \phi_2 >_{((H^1)^*,H^1)}\ dt = \int_0^T < v_t, \phi_2 >_{((H^1)^*,H^1)}\ dt\ .$$

Since d_1 and d_2 are constants and $\nabla u_i, \nabla\phi_1, \nabla v_i$ and $\nabla\phi_2$ are in $L^2(Q)$ by the weak convergence results of $\{u_i\}$ and $\{v_i\}$ in (3.8) we have

$$\lim_{i\to\infty}\int_Q d_1\nabla u_i\cdot\nabla\phi_1\ dxdt = \int_Q d_1\nabla u\cdot\nabla\phi_1\ dxdt$$

and

$$\lim_{i\to\infty}\int_Q d_2\nabla v_i\cdot\nabla\phi_2\ dxdt = \int_Q d_2\nabla v\cdot\nabla\phi_2\ dxdt\ .$$

Similarly as $\{u_i\}$ and $\{v_i\}$ are in $L^2(Q)$, using L^∞ bounds on $\vec{h}$, R and m and the weak convergence results of $\{u^i\}$, $\{v^i\}$ in (3.8) we have

$$\lim_{i\to\infty}\int_Q u_i\vec{h_1}\cdot\nabla\phi_1\ dxdt = \int_Q u\vec{h_1}\cdot\nabla\phi_1\ dxdt$$

$$\lim_{i\to\infty}\int_Q Ru_i\phi_1\ dxdt = \lim_{i\to\infty}\int_Q Ru_{i-1}\phi_1 dxdt = \int_Q Ru\phi_1\ dxdt$$

$$\lim_{i\to\infty}\int_Q u_{i-1}m\phi_1\ dxdt = \int_Q um\phi_1\ dxdt$$

$$\lim_{i\to\infty}\int_Q v_i\vec{h_2}\cdot\nabla\phi_2\ dxdt = \int_Q v\vec{h_2}\cdot\nabla\phi_2\ dxdt$$

$$\lim_{i\to\infty}\int_Q Rv_i\phi_2 dxdt = \lim_{i\to\infty}\int_Q Rv_{i-1}\phi_2 dxdt = \int_Q Rv\phi_2 dxdt$$

$$\lim_{i\to\infty}\int_Q v_{i-1}m\phi_2\ dxdt = \int_Q vm\phi_2\ dxdt\ .$$

Note that

$$\begin{aligned}
&\lim_{i\to\infty}\int_Q a_1(u_{i-1})^2\phi_1 dxdt - \int_Q a_1u^2\phi_1\ dxdt\\
&= \lim_{i\to\infty}\int_Q a_1(u_{i-1}-u)(u_{i-1}+u)\phi_1\ dxdt\\
&\le a_1\|u_{i-1}+u\|_{L^\infty}\lim_{i\to\infty}\int_Q |u_{i-1}-u||\phi_1|\ dxdt\ .
\end{aligned}$$

Hence, by the Holder's inequality, strong convergences (3.9) and L^∞ bounds on u_i we have

$$\begin{aligned}
&\lim_{i\to\infty}\int_Q a_1(u_{i-1})^2\phi_1 dxdt - \int_Q a_1u^2\phi_1\ dxdt\\
&\le a_1\|u_{i-1}+u\|_{L^\infty}\lim_{i\to\infty}(\int_Q |u_{i-1}-u|^2\ dxdt)^{1/2}(\int_Q |\phi_1|^2\ dxdt)^{1/2}\\
&= 0
\end{aligned}$$

giving

$$\lim_{i\to\infty}\int_Q a_1(u_{i-1})^2\phi_1\ dxdt = \int_Q a_1u^2\phi_1\ dxdt\ .$$

Similarly by the Holder's inequality, strong convergences (3.9) and L^∞ bounds on v_i we have

$$\lim_{i\to\infty}\int_Q a_2(v_{i-1})^2\phi_2\ dxdt = \int_Q a_2v^2\phi_2\ dxdt\ .$$

Also, note that

$$\begin{aligned}
&\lim_{i\to\infty}\int_Q b_1u_{i-1}v_{i-1}\phi_1\ dxdt - \int_Q b_1uv\phi_1\ dxdt\\
&= \lim_{i\to\infty}\int_Q b_1[u_{i-1}(v_{i-1}-v)+v(u_{i-1}-u)]\phi_1\ dxdt\\
&\le b_1\|u_{i-1}\|_{L^\infty}\lim_{i\to\infty}\int_Q |v_{i-1}-v||\phi_1|\ dxdt + b_1\|v\|_{L^\infty}\lim_{i\to\infty}\int_Q |u_{i-1}-u||\phi_1|\ dxdt
\end{aligned}$$

Hence, again by the Holder's inequality, strong convergences (3.9) and L^∞ bounds on u_i, v we have

$$\begin{aligned}
&\lim_{i\to\infty}\int_Q b_1 u_{i-1}v_{i-1}\phi_1 \; dxdt - \int_Q b_1 uv\phi_1 \; dxdt \\
&\leq b_1\|u_{i-1}\|_{L^\infty}\lim_{i\to\infty}(\int_Q |v_{i-1}-v|^2 \; dxdt)^{1/2}(\int_Q |\phi_1|^2 \; dxdt)^{1/2} \\
&\quad + b_1\|v\|_{L^\infty}\lim_{i\to\infty}(\int_Q |u_{i-1}-u|^2 \; dxdt)^{1/2}(\int_Q |\phi_1|^2 \; dxdt)^{1/2} \\
&= 0
\end{aligned}$$

giving

$$\lim_{i\to\infty}\int_Q b_1 u_{i-1}v_{i-1}\phi_1 \; dxdt = \int_Q b_1 uv\phi_1 \; dxdt$$

and similarly we get

$$\lim_{i\to\infty}\int_Q b_2 u_{i-1}v_{i-1}\phi_2 \; dxdt = \int_Q b_2 uv\phi_2 \; dxdt \; .$$

So, by taking as $i\to\infty$ in (3.10), we conclude that u, v are state solutions corresponding to control $\vec{h}$ given below:

$$\begin{aligned}
&\int_0^T < u_t, \phi_1 >_{((H^1)^*,H^1)} \; dt + \int_Q d_1\nabla u\cdot\nabla\phi_1 \; dxdt + \int_Q u\vec{h_1}\cdot\nabla\phi_1 \; dxdt \\
&= \int_Q um\phi_1 \; dxdt - \int_Q a_1u^2\phi_1 \; dxdt - \int_Q b_1uv\phi_1 \; dxdt \quad \text{in } \; Q
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T < v_t, \phi_2 >_{((H^1)^*,H^1)} \; dt + \int_Q d_2\nabla v\cdot\nabla\phi_2 \; dxdt + \int_Q v\vec{h_2}\cdot\nabla\phi_2 \; dxdt \\
&= \int_Q vm\phi_2 \; dxdt - \int_Q a_2v^2\phi_2 \; dxdt - \int_Q b_2uv\phi_2 \; dxdt \quad \text{in } \; Q \; .
\end{aligned}$$

Next, to show that this weak solution (with L^∞ bounds) for (3.1) is unique suppose there are two solutions, (u,v) and $(\tilde{u},\tilde{v})$ for (3.1).

Now, with the test functions $(u-\tilde{u})$ and $(v-\tilde{v})$ in $L^2((0,T),H^1(\Omega))\cap L^\infty(Q)$ and $Q_s=\Omega\times(0,s)$ for any time $0\le s\le T$, by considering the weak formulation of the solutions (u,v) and $(\tilde{u},\tilde{v})$ and then by taking the difference in the corresponding equations in Q_s we have

$$\begin{aligned}
&\int_0^s <(u_t-\tilde{u}_t),(u-\tilde{u})>_{((H^1)^*,H^1)}\ dt+\int_{Q_s} d_1|\nabla(u-\tilde{u})|^2\ dxdt\\
&+\int_{Q_s}(u-\tilde{u})\vec{h_1}\cdot\nabla(u-\tilde{u})\ dxdt\\
&=\int_{Q_s}(u-\tilde{u})^2 m\ dxdt-\int_{Q_s}a_1(u^2-\tilde{u}^2)(u-\tilde{u})\ dxdt-\int_{Q_s}b_1(uv-\tilde{u}\tilde{v})(u-\tilde{u})\ dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^s <(v_t-\tilde{v}_t),(v-\tilde{v})>_{((H^1)^*,H^1)}\ dt+\int_{Q_s} d_2|\nabla(v-\tilde{v})|^2\ dxdt\\
&+\int_{Q_s}(v-\tilde{v})\vec{h_2}\cdot\nabla(v-\tilde{v})\ dxdt\\
&=\int_{Q_s}(v-\tilde{v})^2 m\ dxdt-\int_{Q_s}a_2(v^2-\tilde{v}^2)(v-\tilde{v})\ dxdt-\int_{Q_s}b_2(uv-\tilde{u}\tilde{v})(v-\tilde{v})\ dxdt
\end{aligned}$$

Rearranging yields and then, by the non-negativity of u, v, $\tilde{u}$ and $\tilde{v}$ we get

$$\begin{aligned}
&\int_0^s <(u_t-\tilde{u}_t),(u-\tilde{u})>_{((H^1)^*,H^1)}\ dt+\int_{Q_s} d_1|\nabla(u-\tilde{u})|^2\ dxdt\\
&+\int_{Q_s}(u-\tilde{u})\vec{h_1}\cdot\nabla(u-\tilde{u})\ dxdt\\
&\le\int_{Q_s}(u-\tilde{u})^2 m\ dxdt-\int_{Q_s}b_1[(u-\tilde{u})v+(v-\tilde{v})\tilde{u}](u-\tilde{u})\ dxdt
\end{aligned}$$

and (3.11)

$$\begin{aligned}
&\int_0^s <(v_t-\tilde{v}_t),(v-\tilde{v})>_{((H^1)^*,H^1)}\ dt+\int_{Q_s} d_2|\nabla(v-\tilde{v})|^2\ dxdt\\
&+\int_{Q_s}(v-\tilde{v})\vec{h_2}\cdot\nabla(v-\tilde{v})\ dxdt\\
&\le\int_{Q_s}(v-\tilde{v})^2 m\ dxdt-\int_{Q_s}b_2[(u-\tilde{u})v+(v-\tilde{v})\tilde{u}](v-\tilde{v})\ dxdt\ .
\end{aligned}$$

Since,

$$\int_0^s < (u_t - \tilde{u}_t), (u - \tilde{u}) >_{((H^1)^*, H^1)} \, dt = \frac{1}{2} \int_\Omega (u - \tilde{u})^2(x, s) dx,$$

$$\int_0^s < (v_t - \tilde{v}_t), (v - \tilde{v}) >_{((H^1)^*, H^1)} \, dt = \frac{1}{2} \int_\Omega (v - \tilde{v})^2(x, s) dx$$

and using $h \in U$, $m \in L^\infty Q$, $|\tilde{u}| \leq C_1$, $|v| \leq C_2$ and Cauchy's inequalities in (3.11)we get:

$$\begin{aligned}
&\frac{1}{2} \int_\Omega (u - \tilde{u})^2(x, s) dx + \int_{Q_s} d_1 |\nabla(u - \tilde{u})|^2 \, dxdt \\
&\leq C_0 \int_{Q_s} (u - \tilde{u})^2 \, dxdt + \frac{M}{4\epsilon_1} \int_{Q_s} (u - \tilde{u})^2 dxdt + \epsilon_1 M \int_{Q_s} |\nabla(u - \tilde{u})|^2 \, dxdt \\
&\quad + b_1 C_2 \int_{Q_s} (u - \tilde{u})^2 \, dxdt + \frac{1}{2} b_1 C_1 \int_{Q_s} (u - \tilde{u})^2 \, dxdt + \frac{1}{2} b_1 C_1 \int_{Q_s} (v - \tilde{v})^2 \, dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2} \int_\Omega (v - \tilde{v})^2(x, s) dx + \int_{Q_s} d_2 |\nabla(v - \tilde{v})|^2 \, dxdt \\
&\leq C_0 \int_{Q_s} (v - \tilde{v})^2 \, dxdt + \frac{M}{4\epsilon_2} \int_{Q_s} (v - \tilde{v})^2 dxdt + \epsilon_2 M \int_{Q_s} |\nabla(v - \tilde{v})|^2 \, dxdt \\
&\quad + b_2 C_1 \int_{Q_s} (v - \tilde{v})^2 \, dxdt + \frac{1}{2} b_2 C_2 \int_{Q_s} (u - \tilde{u})^2 \, dxdt + \frac{1}{2} b_2 C_2 \int_{Q_s} (v - \tilde{v})^2 \, dxdt
\end{aligned}$$

Hence, by adding these two inequalities and picking

$$\epsilon_1 \leq \frac{d_1}{M} \quad , \quad \epsilon_2 \leq \frac{d_2}{M} \quad , \quad A = A(\epsilon, C_0, C_1, C_2, b_1, b_2) \text{ and } B = B(C_0, C_1, C_2, b_1, b_2)$$

we get,

$$\int_\Omega (u-\tilde{u})^2(x,s) + (v-\tilde{v})^2(x,s)dx + \int_{Q_s} (2d_1 - 2\epsilon_1 M)|\nabla(u-\tilde{u})|^2 \; dxdt$$
$$+ \int_{Q_s} (2d_2 - 2\epsilon_2 M)|\nabla(v-\tilde{v})|^2 \; dxdt$$
$$\leq A\int_{Q_s} (u-\tilde{u})^2 dxdt + B\int_{Q_s} (v-\tilde{v})^2 \; dxdt$$

and for positive constant $D \geq max\{A, B\}$

$$\int_\Omega [(u-\tilde{u})^2 + (v-\tilde{v})^2](x,s)dx \leq D\int_{Q_s} [(u-\tilde{u})^2 + (v-\tilde{v})^2] \; dxdt \; .$$

Hence, by taking,

$$Z_1(s) = \int_\Omega [(u-\tilde{u})^2 + (v-\tilde{v})^2](x,s)dx$$

we have

$$Z_1(s) \leq D\int_0^s Z_1(t) \; dt \; .$$

As, $D \geq 0$ is a constant , by Gronwall's Inequality we have for a.e. $0 \leq s \leq T$

$$Z_1(s) = 0 \quad \text{a.e..}$$

Hence, for a.e. $0 \leq s \leq T$ we have

$$\int_\Omega [(u-\tilde{u})^2 + (v-\tilde{v})^2](x,s)dx = 0$$

which implies

$$u = \tilde{u} \quad \text{and} \quad v = \tilde{v} \; ,$$

giving the uniqueness of our solution to (3.1). □

3.4.2 Existence of an optimal control

From Theorem (3.1) we can see that for a given control $(\vec{h_1}, \vec{h_2}) \in U$ there exists a unique weak state solution $(u, v) = (u, v)(\vec{h_1}, \vec{h_2})$, showing the dependence on $(\vec{h_1}, \vec{h_2})$. Next, we show that there exist an optimal control $(\vec{h_1}, \vec{h_2})^*$ and the corresponding state solutions $(u^*, v^*) = (u, v)((\vec{h_1}, \vec{h_2})^*)$ to the optimal control problem (3.2).

Theorem 3.2 (Existence of an optimal control $(\vec{h_1}, \vec{h_2})^*$)**.** *There exists an optimal control $(\vec{h_1}, \vec{h_2})^*$ maximizing the functional $J(\vec{h_1}, \vec{h_2})$ over U.*
i.e.

$$J(\vec{h_1}, \vec{h_2})^* = \sup_{(\vec{h_1}, \vec{h_2}) \in U} J(\vec{h_1}, \vec{h_2}) \ .$$

Proof. Similar to the proof of Theorem 2.2 we can prove the Theorem 3.2 as given below:

Note that

$$J(\vec{h_1}, \vec{h_2}) = \int_Q \left[Au + Bv - C|\vec{h_1}|^2 - D|\vec{h_2}|^2 \right] \, dxdt$$

and as uniform L^∞ bounds on u, v and $\vec{h_1}, \vec{h_2} \in (L^\infty)^d$, $\sup_{\vec{h} \in U} J(\vec{h})$ is finite.
Hence, we can choose a maximizing sequence, $\{(\vec{h_1}, \vec{h_2})^n\} \in U$ such that

$$\lim_{n \to \infty} J(\vec{h_1}, \vec{h_2})^n = \sup_{(\vec{h_1}, \vec{h_2}) \in U} J(\vec{h_1}, \vec{h_2})$$

and u^n, v^n the corresponding states.

By the *a priori* estimates in Lemma 3.0.1 there exists $u^*, v^* \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$, $u_t^*, v_t^* \in L^2((0,T), (H^1(\Omega))^*)$ and $(\vec{h_1}, \vec{h_2}) \in U$ such that on a subsequence

$$
\begin{aligned}
u^n &\rightharpoonup u^* \qquad \text{in} \qquad L^2((0,T),H^1(\Omega)) \\
u_t^n &\rightharpoonup u_t^* \qquad \text{in} \qquad L^2((0,T),(H^1(\Omega))^*) \\
v^n &\rightharpoonup v^* \qquad \text{in} \qquad L^2((0,T),H^1(\Omega)) \\
v_t^n &\rightharpoonup v_t^* \qquad \text{in} \qquad L^2((0,T),(H^1(\Omega))^*) \\
\vec{h_1^n} &\rightharpoonup \vec{h^*} \qquad \text{in} \qquad (L^2(Q))^d \\
\vec{h_2^n} &\rightharpoonup \vec{h^*} \qquad \text{in} \qquad (L^2(Q))^d \ .
\end{aligned}
\tag{3.12}
$$

Then, by Simon's result [44] we have the strong convergence of the sequences $\{u^n\}$ and $\{v^n\}$ in $L^2(\Omega \times [0,T])$ as below:

$$
\begin{aligned}
u^n &\to u^* \qquad \text{in} \qquad L^2(Q) \\
v^n &\to v^* \qquad \text{in} \qquad L^2(Q).
\end{aligned}
\tag{3.13}
$$

So, now we have to show that $(\vec{h_1},\vec{h_2})^*$ is an optimal control and that $u^* = u((\vec{h_1},\vec{h_2})^*)$ and $v^* = v((\vec{h_1},\vec{h_2})^*)$.

For that consider the system of state PDEs for the subsequence $\{(\vec{h_1},\vec{h_2})^n\}$ with the corresponding states $u((\vec{h_1},\vec{h_2})^n) = u^n$ and $v((\vec{h_1},\vec{h_2})^n) = v^n$ below:

$$
\begin{aligned}
&\int_0^T < u_t^n, \phi_1 >_{((H^1)^*,H^1)} \ dt + \int_Q d_1 \nabla u^n \cdot \nabla \phi_1 \ dxdt + \int_Q u^n \vec{h_1^n} \cdot \nabla \phi_1 \ dxdt \\
&= \int_Q u^n m \phi_1 \ dxdt - \int_Q a_1 (u^n)^2 \phi_1 \ dxdt - \int_Q b_1 u^n v^n \phi_1 \ dxdt \qquad \text{in} \ \ Q \\
&\int_0^T < v_t^n, \phi_2 >_{((H^1)^*,H^1)} \ dt + \int_Q d_2 \nabla v^n \cdot \nabla \phi_2 \ dxdt + \int_Q v^n \vec{h_2^n} \cdot \nabla \phi_2 \ dxdt \\
&= \int_Q v^n m \phi_2 \ dxdt - \int_Q a_2 (v^n)^2 \phi_2 \ dxdt - \int_Q b_2 u^n v^n \phi_2 \ dxdt \qquad \text{in} \ \ Q \ .
\end{aligned}
\tag{3.14}
$$

Using the above weak convergences (3.12) and the L^∞ bounds on $\vec{h_1}, \vec{h_2}$ and m we have

$$\lim_{n\to\infty}\int_0^T < u_t^n - u_t^*, \phi_1 >_{((H^1)^*,H^1)}\ dt = \int_0^T < u_t^* - u_t^*, \phi_1 >_{((H^1)^*,H^1)}\ dt = 0$$

$$\lim_{n\to\infty}\int_Q d_1\nabla(u^n - u^*)\cdot\nabla\phi_1\ dxdt = \int_Q d_1\nabla(u^* - u^*)\cdot\nabla\phi_1\ dxdt = 0$$

$$\lim_{n\to\infty}\int_Q (u^n - u^*)m\phi_1\ dxdt = \int_Q (u^* - u^*)m\phi_1\ dxdt = 0$$

$$\lim_{n\to\infty}\int_0^T < v_t^n - v_t^*, \phi_2 >_{((H^1)^*,H^1)}\ dt = \int_0^T < v_t^* - v_t^*, \phi_2 >_{((H^1)^*,H^1)}\ dt = 0$$

$$\lim_{n\to\infty}\int_Q d_2\nabla(v^n - v^*)\cdot\nabla\phi_2\ dxdt = \int_Q d_2\nabla(v^* - v^*)\cdot\nabla\phi_2\ dxdt = 0$$

$$\lim_{n\to\infty}\int_Q (v^n - v^*)m\phi_2\ dxdt = \int_Q (v^* - v^*)m\phi_2\ dxdt = 0\ .$$

Note that

$$\begin{aligned}
&\int_Q (u^n\vec{h_1^n} - u^*\vec{h_1^*})\cdot\nabla\phi_1\ dxdt \\
&= \int_Q (u^n\vec{h_1^n} - u^*\vec{h_1^n} + u^*\vec{h_1^n} - u^*\vec{h_1^*})\cdot\nabla\phi_1\ dxdt \qquad (3.15)\\
&= \int_Q (u^n - u^*)\vec{h_1^n}\cdot\nabla\phi_1\ dxdt + \int_Q u^*(\vec{h_1^n} - \vec{h_1^*})\cdot\nabla\phi_1\ dxdt.
\end{aligned}$$

By the Cauchy's inequality, the strong convergence of $\{\vec{u^n}\}$, $\nabla\phi_1$ being in L^2 and the L^∞ bounds on $\{h_1^n\}$ the first term of (3.15) gives

$$\begin{aligned}
\lim_{n\to\infty}\int_Q (u^n - u^*)\vec{h_1^n}\cdot\nabla\phi_1\ dxdt &\leq \lim_{n\to\infty}\int_Q |u^n - u^*||\vec{h_1^n}||\nabla\phi_1|\ dxdt \\
&\leq \lim_{n\to\infty} M(\int_Q (u^n - u^*)^2 dxdt)^{1/2}(\int_Q |\nabla\phi_1|^2\ dxdt)^{1/2} \\
&= 0
\end{aligned}$$

and by the weak convergence of $\{\vec{h_1^n}\}$, $\nabla\phi_1$ being fixed in $L^2(Q)$ and $\{u^*\}$ being fixed with the L^∞ bounds, the second term in (3.15) gives

$$\lim_{n\to\infty}\int_Q u^*(\vec{h_1^n}-\vec{h_1^*})\cdot\nabla\phi_1\ dxdt=0\ .$$

That is we have

$$\lim_{n\to\infty}\int_Q (u^n\vec{h_1^n}-u^*\vec{h_1^*})\cdot\nabla\phi_1\ dxdt=0$$

and similarly we have

$$\lim_{n\to\infty}\int_Q (v^n\vec{h_2^n}-v^*\vec{h_2^*})\cdot\nabla\phi_2\ dxdt=0\ .$$

Since,

$$\begin{aligned}&\lim_{n\to\infty}\int_Q a_1(u^n)^2\phi_1\ dxdt-\int_Q a_1(u^*)^2\phi_1\ dxdt\\&=\lim_{n\to\infty}\int_Q a_1[(u^n)^2-(u^*)^2]\phi_1\ dxdt\end{aligned}$$

by the Holder's inequality, L^∞ bounds on u and strong convergences (3.13) we have

$$\begin{aligned}&\lim_{n\to\infty}\int_Q a_1(u^n)^2\phi_1\ dxdt-\int_Q a_1(u^*)^2\phi_1\ dxdt\\&\le a_1\|u^n+u^*\|_{L^\infty}\lim_{n\to\infty}(\int_Q|u^n-u^*|^2\ dxdt)^{1/2}(\int_Q|\phi_1|^2\ dxdt)^{1/2}\\&=0\end{aligned}$$

giving

$$\lim_{n\to\infty}\int_Q a_1(u^n)^2\phi_1\ dxdt=\int_Q a_1(u^*)^2\phi_1\ dxdt\ .$$

Similarly we have

$$\lim_{n\to\infty}\int_Q a_2(v^n)^2\phi_2\ dxdt=\int_Q a_2(v^*)^2\phi_2\ dxdt\ .$$

Again, since

$$\lim_{n\to\infty}\int_Q b_1u^nv^n\phi_1\ dxdt-\int_Q b_1u^*v^*\phi_1\ dxdt$$
$$=\lim_{n\to\infty}\int_Q b_1v^n(u^n-u^*)\phi_1\ dxdt+\lim_{n\to\infty}\int_Q b_1u^*(v^n-v^*)\phi_1\ dxdt$$

by the Holder's inequality, L^∞ bounds on u, v and strong convergences (3.13) we have

$$\lim_{n\to\infty}\int_Q b_1u^nv^n\phi_1\ dxdt-\int_Q b_1u^*v^*\phi_1\ dxdt$$
$$\le b_1\|v^n\|_{L^\infty}\lim_{n\to\infty}(\int_Q|u^n-u^*|^2dxdt)^{1/2}(\int_Q|\phi_1|^2\ dxdt)^{1/2}$$
$$+b_1\|u^*\|_{L^\infty}\lim_{n\to\infty}(\int_Q|v^n-v^*|^2dxdt)^{1/2}(\int_Q|\phi_1|^2\ dxdt)^{1/2}$$
$$=0$$

giving

$$\lim_{n\to\infty}\int_Q b_1u^nv^n\phi_1\ dxdt=\int_Q b_1u^*v^*\phi_1\ dxdt$$

and similarly

$$\lim_{n\to\infty}\int_Q b_2u^nv^n\phi_2\ dxdt=\int_Q b_2u^*v^*\phi_2\ dxdt\ .$$

Hence, as $n\to\infty$ in (3.14) we obtain

$$\begin{aligned}
&\int_0^T<u_t^*,\phi_1>_{((H^1)^*,H^1)}\ dt+\int_Q d_1\nabla u^*\cdot\nabla\phi_1\ dxdt+\int_Q u^*\vec{h_1^*}\cdot\nabla\phi_1\ dxdt\\
&=\int_Q u^*m\phi_1\ dxdt-\int_Q a_1(u^*)^2\phi_1\ dxdt-\int_Q b_1u^*v^*\phi_1\ dxdt\qquad\text{in }\ Q\\
&\int_0^T<v_t^*,\phi_2>_{((H^1)^*,H^1)}\ dt+\int_Q d_2\nabla v^*\cdot\nabla\phi_2\ dxdt+\int_Q v^*\vec{h_2^*}\cdot\nabla\phi_2\ dxdt\\
&=\int_Q v^*m\phi_2\ dxdt-\int_Q a_2(v^*)^2\phi_2\ dxdt-\int_Q b_2u^*v^*\phi_2\ dxdt\qquad\text{in }\ Q\ .
\end{aligned}\tag{3.16}$$

and as u^*, v^* satisfies the system of state PDEs with the control $(\vec{h_1}, \vec{h_2})^*$ in (3.16), we can conclude that $(u^*, v^*) = (u, v)((\vec{h_1}, \vec{h_2})^*)$.

Again using the weak convergences (3.12) in the objective functional on the maximizing sequence $\{(\vec{h_1}, \vec{h_2})^n\}$ and the fact that

$$\int_Q (\vec{h_1^*})^2 \, dxdt \leq \liminf_{n\to\infty} \int_Q (\vec{h_1^n})^2 \, dxdt$$

and

$$\int_Q (\vec{h_2^*})^2 \, dxdt \leq \liminf_{n\to\infty} \int_Q (\vec{h_2^n})^2 \, dxdt$$

we get

$$\begin{aligned} \sup_{(\vec{h_1},\vec{h_2})\in U} J(\vec{h_1}, \vec{h_2}) &= \lim_{n\to\infty} J((\vec{h_1}, \vec{h_2})^n) \\ &= \lim_{n\to\infty} \int_Q \left[Au^n + Bv^n - C|\vec{h_1^n}|^2 - D|\vec{h_2^n}|^2 \right] \, dxdt \\ &\leq \int_Q \left[Au^* + Bv^* - C|\vec{h_1^*}|^2 - D|\vec{h_1^*}|^2 \right] \, dxdt \\ &= J((\vec{h_1}, \vec{h_2})^*) \ . \end{aligned} \tag{3.17}$$

Since we always have

$$\sup_{(\vec{h_1},\vec{h_2})\in U} J(\vec{h_1}, \vec{h_2}) \geq J((\vec{h_1}, \vec{h_2})^*) \ ,$$

by (3.17) we conclude that $(\vec{h_1}, \vec{h_2})^*$ is an optimal control for the problem (3.2).

□

3.4.3 Derivation of the optimality system

The necessary conditions:

To characterize the optimal control we have to differentiate the $(\vec{h_1}, \vec{h_2}) \to J(\vec{h_1}, \vec{h_2})$ map with respect to the controls $\vec{h_1}$ and $\vec{h_2}$. But, as $u = u(\vec{h_1}, \vec{h_2})$ and $v = v(\vec{h_1}, \vec{h_2})$, we also need to differentiate $(\vec{h_1}, \vec{h_2}) \to (u, v)(\vec{h_1}, \vec{h_2})$ map with respect to the controls $\vec{h_1}$ and $\vec{h_2}$. These derivatives are called the sensitivity functions.

Lemma 3.2.1 (Finding the sensitivity PDE's)**.** *Let* $(\vec{h_1}, \vec{h_2})$ *be an optimal control for the problem (3.2) with the corresponding state solutions* $u = u(\vec{h_1}, \vec{h_2})$ *and* $v = v(\vec{h_1}, \vec{h_2})$ *and let* $(\vec{h_1}, \vec{h_2})^\epsilon$ *be another control with the corresponding state solutions* $u^\epsilon = u(\vec{h_1} + \epsilon\vec{l_1}, \vec{h_2} + \epsilon\vec{l_2})$ *and* $v^\epsilon = v(\vec{h_1} + \epsilon\vec{l_1}, \vec{h_2} + \epsilon\vec{l_2})$ *such that*

$$(\vec{h_1}, \vec{h_2})^\epsilon = (\vec{h_1} + \epsilon\vec{l_1}, \vec{h_2} + \epsilon\vec{l_2})$$

where $(\vec{h_1}, \vec{h_2}) + \epsilon(\vec{l_1}, \vec{l_2}) \in U$ *for all sufficiently small* $\epsilon > 0$ *with* $(\vec{l_1}, \vec{l_2}) \in ((L^\infty(\Omega))^d, (L^\infty(\Omega))^d)$.

Then, for any $(\vec{h_1}, \vec{h_2}) \in U$ *the mappings* $(\vec{h_1}, \vec{h_2}) \to u(\vec{h_1}, \vec{h_2})$ *and* $(\vec{h_1}, \vec{h_2}) \to v(\vec{h_1}, \vec{h_2})$ *are weakly differentiable in the directional derivative sense and there exists* $\psi_1, \psi_2 \in L^2((0,T), H^1(\Omega))$ *with* $(\psi_1)_t, (\psi_2)_t \in L^2((0,T), (H^1(\Omega))^*)$ *such that*

$$\lim_{\epsilon \to 0^+} \frac{u(\vec{h_1} + \epsilon\vec{l_1}, \vec{h_2} + \epsilon\vec{l_2}) - u(\vec{h_1}, \vec{h_2})}{\epsilon} = \psi_1 \quad \text{in} \quad L^2(Q)$$

and

$$\lim_{\epsilon \to 0^+} \frac{v(\vec{h_1} + \epsilon\vec{l_1}, \vec{h_2} + \epsilon\vec{l_2}) - v(\vec{h_1}, \vec{h_2})}{\epsilon} = \psi_2 \quad \text{in} \quad L^2(Q)\ .$$

Further, the sensitivity functions ψ_1 *and* ψ_2 *satisfy*

$$\begin{aligned}
(\psi_1)_t - d_1\Delta\psi_1 - \nabla\cdot(\vec{h_1}\psi_1 + \vec{l_1}u) &= m\psi_1 - 2a_1u\psi_1 - b_1v\psi_1 - b_1u\psi_2 \quad \text{in } Q \\
(\psi_2)_t - d_2\Delta\psi_2 - \nabla\cdot(\vec{h_2}\psi_2 + \vec{l_2}v) &= m\psi_2 - 2a_2v\psi_2 - b_2v\psi_1 - b_2u\psi_2 \quad \text{in } Q
\end{aligned}$$

$$\begin{aligned}
d_1\frac{\partial\psi_1}{\partial\eta} + \psi_1\vec{h_1}\cdot\eta &= -u\vec{l_1}\cdot\eta \quad \text{on } S \\
d_2\frac{\partial\psi_2}{\partial\eta} + \psi_2\vec{h_2}\cdot\eta &= -v\vec{l_2}\cdot\eta \quad \text{on } S
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
\psi_1(x,0) &= 0 \quad \text{for } x \in \Omega \\
\psi_2(x,0) &= 0 \quad \text{for } x \in \Omega\ .
\end{aligned}$$

Proof. Similar to the proof of Lemma 2.2.1, by considering the state PDEs corresponding to the optimal control $(\vec{h_1}, \vec{h_2})$, the control $(\vec{h_1}, \vec{h_2})^\epsilon$ and then forming the system solves by $\frac{(u^\epsilon - u)}{\epsilon}$, $\frac{(v^\epsilon - v)}{\epsilon}$, we get:

$$\begin{aligned}
&\frac{(u^\epsilon - u)_t}{\epsilon} - d_1\Delta\frac{(u^\epsilon - u)}{\epsilon} - \nabla\cdot\vec{h_1}\frac{(u^\epsilon - u)}{\epsilon} - \nabla\cdot\vec{l_1}u^\epsilon \\
&\quad = m\frac{(u^\epsilon - u)}{\epsilon} - a_1\frac{(u^\epsilon - u)}{\epsilon}(u^\epsilon + u) - b_1 v\frac{(u^\epsilon - u)}{\epsilon} - b_1 u^\epsilon\frac{(v^\epsilon - v)}{\epsilon} \qquad \text{in } Q \\
&\frac{(v^\epsilon - v)_t}{\epsilon} - d_2\Delta\frac{(v^\epsilon - v)}{\epsilon} - \nabla\cdot\vec{h_2}\frac{(v^\epsilon - v)}{\epsilon} - \nabla\cdot\vec{l_2}v^\epsilon \\
&\quad = m\frac{(v^\epsilon - v)}{\epsilon} - a_2\frac{(v^\epsilon - v)}{\epsilon}(v^\epsilon + v) - b_2 v\frac{(u^\epsilon - u)}{\epsilon} - b_2 u^\epsilon\frac{(v^\epsilon - v)}{\epsilon} \qquad \text{in } Q \\
&d_1\frac{\partial\frac{(u^\epsilon - u)}{\epsilon}}{\partial\eta} + \frac{(u^\epsilon - u)}{\epsilon}\vec{h_1}\cdot\eta + u^\epsilon\vec{l_1}\cdot\eta = 0 \qquad \text{on } S \\
&d_2\frac{\partial\frac{(v^\epsilon - v)}{\epsilon}}{\partial\eta} + \frac{(u^\epsilon - u)}{\epsilon}\vec{h_2}\cdot\eta + u^\epsilon\vec{l_2}\cdot\eta = 0 \qquad \text{on } S \\
&\frac{(u^\epsilon - u)}{\epsilon}(x,0) = 0 \qquad \text{for } x\in\Omega \\
&\frac{(v^\epsilon - v)}{\epsilon}(x,0) = 0 \qquad \text{for } x\in\Omega\ .
\end{aligned} \tag{3.19}$$

For (3.19), using Cauchy's inequality and the Gronwall's inequality with similar techniques used for the proof of Lemma 2.0.2 we can show that

$$\begin{aligned}
\|\frac{u^* - u^\epsilon}{\epsilon}\|_{L^2((0,T),H^1(\Omega))} &\leq K_1'' \\
\|\frac{(u^* - u^\epsilon)_t}{\epsilon}\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_2'' \\
\|\frac{v^* - v^\epsilon}{\epsilon}\|_{L^2((0,T),H^1(\Omega))} &\leq K_3'' \\
\|\frac{(v^* - v^\epsilon)_t}{\epsilon}\|_{L^2((0,T),(H^1(\Omega))^*)} &\leq K_4''
\end{aligned}$$

for some constants K_1'', K_2'', K_3'' and K_4'' .

Hence, there exists $\psi_1(x,t)$ and $\psi_2(x,t)$ in $L^2((0,T),H^1(\Omega))$ such that as $\epsilon \to 0^+$

$$\frac{u(\vec{h_1}+\epsilon\vec{l_1},\vec{h_2}+\epsilon\vec{l_2})-u(\vec{h_1},\vec{h_2})}{\epsilon} \rightharpoonup \psi_1 \quad \text{in} \quad L^2((0,T),H^1(\Omega))$$
$$\frac{v(\vec{h_1}+\epsilon\vec{l_1},\vec{h_2}+\epsilon\vec{l_2})-v(\vec{h_1},\vec{h_2})}{\epsilon} \rightharpoonup \psi_2 \quad \text{in} \quad L^2((0,T),H^1(\Omega))$$

and

$$\frac{((\vec{h_1}+\epsilon\vec{l_1},\vec{h_2}+\epsilon\vec{l_2})-u(\vec{h_1},\vec{h_2}))_t}{\epsilon} \rightharpoonup (\psi_1)_t \quad \text{in} \quad L^2((0,T),(H^1(\Omega))^*)$$
$$\frac{((\vec{h_1}+\epsilon\vec{l_1},\vec{h_2}+\epsilon\vec{l_2})-v(\vec{h_1},\vec{h_2}))_t}{\epsilon} \rightharpoonup (\psi_2)_t \quad \text{in} \quad L^2((0,T),(H^1(\Omega))^*)\ .$$

Hence, again by the Simon's results [44], as $\epsilon \to 0^+$ we have the strong convergence of the different quotients $\{\frac{(u^\epsilon-u)}{\epsilon}\}$ and $\{\frac{(v^\epsilon-v)}{\epsilon}\}$ in $L^2(Q)$ given below:

$$\frac{u(\vec{h_1}+\epsilon\vec{l_1},\vec{h_2}+\epsilon\vec{l_2})-u(\vec{h_1},\vec{h_2})}{\epsilon} \to \psi_1 \quad \text{in} \quad L^2(Q)$$

$$\frac{v(\vec{h_1}+\epsilon\vec{l_1},\vec{h_2}+\epsilon\vec{l_2})-u(\vec{h_1},\vec{h_2})}{\epsilon} \to \psi_2 \quad \text{in} \quad L^2(Q)\ .$$

Then, using the above convergences with techniques as in the proof of Theorem 2.2, as ϵ goes to zero in (3.19) we get the resulting PDEs (3.18) satisfied by ψ_1 and ψ_2.

□

Definition (The weak solution for the sensitivity PDE')

$\psi_1, \psi_2 \in L^2((0,T),H^1(\Omega))$ with $(\psi_1)_t, (\psi_2)_t \in L^2((0,T),(H^1(\Omega))^*)$ and $\psi_1(x,0) = 0 = \psi_2(x,0)$ is a weak solution for (3.18) if it satisfies,

$$\int_0^T < (\psi_1)_t, \phi_1 >_{((H^1)^*, H^1)} \; dt + \int_Q d_1 \nabla\psi_1 \cdot \nabla\phi_1 \; dxdt + \int_Q \psi_1 \vec{h_1} \cdot \nabla\phi_1 \; dxdt$$
$$- \int_Q m\psi_1\phi_1 \; dxdt + \int_Q 2a_1 u\psi_1\phi_1 \; dxdt + \int_Q b_1 v\psi_1\phi_1 \; dxdt + \int_Q b_1 u\psi_2\phi_1 \; dxdt$$
$$= - \int_Q u\vec{l_1} \cdot \nabla\phi_1 \; dxdt$$

and (3.20)

$$\int_0^T < (\psi_2)_t, \phi_2 >_{((H^1)^*, H^1)} \; dt + \int_Q d_2 \nabla\psi_2 \cdot \nabla\phi_2 \; dxdt + \int_Q \psi_2 \vec{h_2} \cdot \nabla\phi_2 \; dxdt$$
$$- \int_Q m\psi_2\phi_2 \; dxdt + \int_Q 2a_2 v\psi_2\phi_2 \; dxdt + \int_Q b_2 v\psi_1\phi_2 \; dxdt + \int_Q b_2 u\psi_2\phi_2 \; dxdt$$
$$= - \int_Q v\vec{l_2} \cdot \nabla\phi_2 \; dxdt$$

for all ϕ_1 , $\phi_2 \in L^2((0,T), H^1(\Omega))$.

Here again we can give an explanation of this weak formulation as we did for the weak formulation of (2.24).

Next, we want to find the adjoint equations such that

Sum of left hand sides of weak forms of sensitivity equations

$=$ Sum of left hand sides of weak forms of the adjoint equations

and we can derive this adjoint system given below in the same way as we did in (2.25). The non-homogeneous terms on the RHS of the adjoint equations are obtained by differentiating the integrand of the objective functional with respect to the state variables u and v.

The adjoint system:

$$\begin{aligned}
-(\lambda_1)_t - d_1\Delta\lambda_1 + \vec{h_1}\cdot\nabla\lambda_1 - m\lambda_1 + 2a_1u\lambda_1 + b_1v\lambda_1 + b_2v\lambda_2 &= A \quad \text{in } Q \\
-(\lambda_2)_t - d_2\Delta\lambda_2 + \vec{h_2}\cdot\nabla\lambda_2 - m\lambda_2 + 2a_2v\lambda_2 + b_1u\lambda_1 + b_2u\lambda_2 &= B \quad \text{in } Q \\
\frac{\partial\lambda_1}{\partial\eta} &= 0 \quad \text{on } S \\
\frac{\partial\lambda_2}{\partial\eta} &= 0 \quad \text{on } S \\
\lambda_1(x,T) &= 0 \quad \text{for } x\in\Omega \\
\lambda_2(x,T) &= 0 \quad \text{for } x\in\Omega\ .
\end{aligned} \tag{3.21}$$

3.4.4 Characterizing the optimal control $\vec{h}^*$

Theorem 3.3 (Characterizing an optimal control $\vec{h}^*$)**.** *There exists adjoint functions $\lambda_1, \lambda_2 \in L^2((0,T), H^1(\Omega))$ with $(\lambda_1)_t, (\lambda_2)_t \in L^2((0,T), (H^1(\Omega))^*)$ which are weak solutions of (3.21).*
Let $(\vec{h_1}, \vec{h_2})^* \in U$ be an optimal control for the problem (3.2) with the corresponding state solutions $u^*, v^* \in L^2((0,T), H^1(\Omega))$. Then,

$$(h_1)_i^* = min\left(M_1, max\left(-\frac{u(\lambda_1)_{x_i}}{2C}, -M_1\right)\right)$$

$$(h_2)_i^* = min\left(M_2, max\left(-\frac{v(\lambda_2)_{x_i}}{2D}, -M_2\right)\right)$$

for $i = 1, 2, ..., d$

Proof. Note that the adjoint system is linear and a weak solution exists by standard results.

For an optimal control $(\vec{h_1}, \vec{h_2})^* \in U$ we compute the directional derivative of the function $J((\vec{h_1}, \vec{h_2})^*)$ with respect to $(\vec{h_1}, \vec{h_2})^*$ in the directions $(\vec{l_1}, \vec{l_2})$ at u^* and v^* .

Then, as $J(\vec{h_1}, \vec{h_2})^*$ is the maximum value for the $J(\vec{h_1}, \vec{h_2})$ we have

$$
\begin{aligned}
0 \geq \lim_{\epsilon \to 0^+} & \frac{J((\vec{h_1}, \vec{h_2})^* + \epsilon(\vec{l_1}, \vec{l_2})) - J((\vec{h_1}, \vec{h_2})^*)}{\epsilon} \\
&= \lim_{\epsilon \to 0^+} \int_{Q_T} \frac{\begin{array}{r}\left(Au^\epsilon + Bv^\epsilon - C|\vec{h_1}^* + \epsilon\vec{l_1}|^2 - D|\vec{h_2}^* + \epsilon\vec{l_2}|^2\right) \\ - \left(Au + Bv - C|\vec{h_1}^*|^2 - D|\vec{h_2}^*|^2\right)\end{array}}{\epsilon} \, dxdt \\
&= \int_Q (A\psi_1 + B\psi_2 - 2C\vec{h_1}^* \cdot \vec{l_1} - 2D\vec{h_2}^* \cdot \vec{l_2}) \, dxdt \\
&= \int_Q A\psi_1 \, dxdt + \int_Q B\psi_2 \, dxdt - \int_Q 2C\vec{h_1}^* \cdot \vec{l_1} \, dxdt - \int_Q 2D\vec{h_2}^* \cdot \vec{l_2} \, dxdt \qquad (3.22)
\end{aligned}
$$

Using the adjoint systems (3.21) to substitute for the first two terms in the right hand side of (3.22)

$$
\begin{aligned}
&\int_Q A\psi_1 \, dxdt + \int_Q B\psi_2 \, dxdt - \int_Q 2C\vec{h_1}^* \cdot \vec{l_1} \, dxdt - \int_Q 2D\vec{h_2}^* \cdot \vec{l_2} \, dxdt \\
&= -\int_0^T < (\lambda_1)_t, \psi_1 >_{((H^1)^*, H^1)} \, dt + \int_Q d_1 \nabla\lambda_1 \cdot \nabla\psi_1 \, dxdt + \int_Q \vec{h_1} \cdot \nabla\lambda_1\psi_1 \, dxdt \\
&\quad - \int_Q m\lambda_1\psi_1 \, dxdt + \int_Q 2a_1 u\lambda_1\psi_1 \, dxdt + \int_Q b_1 v\lambda_1\psi_1 \, dxdt + \int_Q b_2 v\lambda_2\psi_1 \, dxdt \\
&\quad - \int_0^T < (\lambda_2)_t, \psi_2 >_{((H^1)^*, H^1)} \, dt + \int_Q d_2 \nabla\lambda_2 \cdot \nabla\psi_2 \, dxdt + \int_Q \vec{h_2} \cdot \nabla\lambda_2\psi_2 \, dxdt \\
&\quad - \int_Q m\lambda_2\psi_2 \, dxdt + \int_Q 2a_2 v\lambda_2\psi_2 \, dxdt + \int_Q b_1 u\lambda_1\psi_2 \, dxdt + \int_Q b_2 u\lambda_2\psi_2 \, dxdt \\
&\quad - \int_Q 2C\vec{h_1}^* \cdot \vec{l_1} \, dxdt - \int_Q 2D\vec{h_2}^* \cdot \vec{l_2} \, dxdt
\end{aligned}
$$

Using the sensitivity system (3.20)

$$
\begin{aligned}
&\int_Q A\psi_1 \; dxdt + \int_Q B\psi_2 \; dxdt - \int_Q 2C\vec{h_1}^* \cdot \vec{l_1} \; dxdt - \int_Q 2D\vec{h_2}^* \cdot \vec{l_2} \; dxdt \\
&= \int_0^T < (\psi_1)_t, \lambda_1 >_{((H^1)^*, H^1)} \; dt + \int_Q d_1 \nabla\psi_1 \cdot \nabla\lambda_1 \; dxdt + \int_Q \psi_1 \vec{h_1} \cdot \nabla\lambda_1 \; dxdt \\
&\quad - \int_Q m\psi_1\lambda_1 \; dxdt + \int_Q 2a_1 u\psi_1\lambda_1 \; dxdt + \int_Q b_1 v\psi_1\lambda_1 \; dxdt + \int_Q b_1 u\psi_2\lambda_1 \; dxdt \\
&\quad + \int_0^T < (\psi_2)_t, \lambda_2 >_{((H^1)^*, H^1)} \; dt + \int_Q d_2 \nabla\psi_2 \cdot \nabla\lambda_2 \; dxdt + \int_Q \psi_2 \vec{h_2} \cdot \nabla\lambda_2 \; dxdt \\
&\quad - \int_Q m\psi_2\lambda_2 \; dxdt + \int_Q 2a_2 v\psi_2\lambda_2 \; dxdt + \int_Q b_2 v\psi_1\lambda_2 \; dxdt + \int_Q b_2 u\psi_2\lambda_2 \; dxdt \\
&\quad - \int_Q 2C\vec{h_1}^* \cdot \vec{l_1} \; dxdt - \int_Q 2D\vec{h_2}^* \cdot \vec{l_2} \; dxdt \\
&= - \int_Q u\vec{l_1} \cdot \nabla\lambda_1 \; dxdt - \int_Q v\vec{l_2} \cdot \nabla\lambda_2 \; dxdt - \int_Q 2C\vec{h_1}^* \cdot \vec{l_1} \; dxdt - \int_Q 2D\vec{h_2}^* \cdot \vec{l_2} \; dxdt \\
&= \int_Q -\vec{l_1} \cdot (u\nabla\lambda_1 + 2C\vec{h_1}^*) - \vec{l_2} \cdot (v\nabla\lambda_2 + 2D\vec{h_2}^*) \; dxdt \\
&= - \int_Q (\vec{l_1}, \vec{l_2}) \cdot (u\nabla\lambda_1 + 2C\vec{h_1}^*, v\nabla\lambda_2 + 2D\vec{h_2}^*) \; dxdt
\end{aligned}
$$

Hence, for any $(\vec{l_1}, \vec{l_2})$ we have

$$
0 \geq - \int_Q (\vec{l_1}, \vec{l_2}) \cdot (u\nabla\lambda_1 + 2C\vec{h_1}^*, v\nabla\lambda_2 + 2D\vec{h_2}^*) \; dxdt
$$

which implies

$$
0 \leq \int_Q (\vec{l_1}, \vec{l_2}) \cdot (u\nabla\lambda_1 + 2C\vec{h_1}^*, v\nabla\lambda_2 + 2D\vec{h_2}^*) \; dxdt \tag{3.23}
$$

Next to characterize this optimal control $(\vec{h_1}, \vec{h_2})^*$ consider (3.23) for different cases of $(h_1)_i^*, (h_2)_i^*$ for $i = 1, 2, ..., d$.

Case I: On the set $\{(x,t): \quad -M_1 < (h_1)_i < M_1\}$, inequality (3.23) implies $(l_1)_i$ can have any sign and choosing $(l_1)_j = 0; i \neq j$ and $\vec{l_2} = 0$ we get

$$\begin{aligned} u(\lambda_1)_{x_i} + 2C(h_1)_i^* &= 0 \\ (h_1)_i^* &= -\frac{u(\lambda_1)_{x_i}}{2C} \ . \end{aligned}$$

Case II: On the set $\{(x,t): \quad (h_1)_i = M_1\}$, similarly (3.23) implies $(l_1)_i \leq 0$ and choosing $(l_1)_j = 0; i \neq j$ and $\vec{l_2} = 0$ we get

$$\begin{aligned} u(\lambda_1)_{x_i} + 2C(h_1)_i^* &\leq 0 \\ u(\lambda_1)_{x_i} + 2CM_1 &\leq 0 \\ M_1 &\leq -\frac{u(\lambda_1)_{x_i}}{2C} \\ (h_1)_i^* &= \min\left(-\frac{u(\lambda_1)_{x_i}}{2C}, M_1\right) \ . \end{aligned}$$

Case III: On the set $\{(x,t): \quad (h_1)_i = -M_1\}$, again inequality (3.23) implies $(l_1)_i \geq 0$ and choosing $(l_1)_j = 0; i \neq j$ and $\vec{l_2} = 0$ we have

$$\begin{aligned} u(\lambda_1)_{x_i} + 2C(h_1)_i^* &\geq 0 \\ u(\lambda_1)_{x_i} + 2C(-M_1) &\geq 0 \\ M_1 &\leq \frac{u(\lambda_1)_{x_i}}{2C} \\ (h_1)_i^* &= \max\left(-\frac{u(\lambda_1)_{x_i}}{2C}, -M_1\right) \ . \end{aligned}$$

Case IV: On the set $\{(x,t): \quad -M_2 < (h_2)_i < M_2\}$, inequality (3.23) implies $(l_2)_i$ can have any sign and choosing $(l_2)_j = 0; i \neq j$ and $\vec{l_1} = 0$ implies

$$\begin{aligned} v(\lambda_2)_{x_i} + 2D(h_2)_i^* &= 0 \\ (h_2)_i^* &= -\frac{v(\lambda_2)_{x_i}}{2D} \ . \end{aligned}$$

Case V: On the set $\{(x,t): \quad (h_2)_i = M_2\}$, (3.23) implies $(l_2)_i \leq 0$ and choosing $(l_2)_j = 0; i \neq j$ and $\vec{l_1} = 0$ we get

$$\begin{aligned} v(\lambda_2)_{x_i} + 2D(h_2)_i^* &\leq 0 \\ v(\lambda_2)_{x_i} + 2DM_2 &\leq 0 \\ M_2 &\leq -\frac{v(\lambda_2)_{x_i}}{2D} \\ (h_2)_i^* &= \min\left(-\frac{v(\lambda_2)_{x_i}}{2D}, M_2\right) . \end{aligned}$$

Case VI: On the set $\{(x,t): \quad (h_1)_i = -M_1\}$, similarly inequality (3.23) implies $(l_2)_i \geq 0$ and choosing $(l_2)_j = 0; i \neq j$ and $\vec{l_1} = 0$ we have

$$\begin{aligned} v(\lambda_1)_{x_i} + 2D(h_2)_i^* &\geq 0 \\ v(\lambda_1)_{x_i} + 2D(-M_2) &\geq 0 \\ M_2 &\leq \frac{v(\lambda_2)_{x_i}}{2D} \\ (h_2)_i^* &= \max\left(-\frac{v(\lambda_2)_{x_i}}{2D}, -M_2\right) . \end{aligned}$$

Hence, we can characterize the optimal control $(\vec{h_1}, \vec{h_2})^*$ for the problem (3.2) as

$$\begin{aligned} (h_1)_i^* &= \min\left(M_1, \max\left(-\frac{u(\lambda_1)_{x_i}}{2C}, -M_1\right)\right) \\ (h_2)_i^* &= \min\left(M_2, \max\left(-\frac{v(\lambda_2)_{x_i}}{2D}, -M_2\right)\right) \end{aligned} \tag{3.24}$$

for $i = 1, 2, ..., d$.

□

3.4.5 Uniqueness of the optimal control

Theorem 3.4 (Uniqueness of the optimal control)**.** *For sufficiently small T, and for cost coefficients C and D sufficiently large, the solution to the optimality system (3.1), (3.21) and (3.24) is unique.*

Proof. Similar to the proof of Theorem 2.4, we can prove Theorem 3.4 as given below:

Suppose $(u, v), (\lambda_1, \lambda_2)$ and $(\bar{u}, \bar{v}), (\bar{\lambda_1}, \bar{\lambda_2})$ are two solutions to the state pde system (3.1) and (3.21). Hence the corresponding optimal control characterizations are

$$\begin{aligned}(h_1)_i^* &= \min\left(M_1, \max\left(-\frac{u(\lambda_1)_{x_i}}{2C}, -M_1\right)\right)\\ (h_2)_i^* &= \min\left(M_2, \max\left(-\frac{v(\lambda_2)_{x_i}}{2D}, -M_2\right)\right)\end{aligned} \tag{3.25}$$

for $i = 1, 2, ..., d$.

and

$$\begin{aligned}(\bar{h_1})_i^* &= \min\left(M_1, \max\left(-\frac{\bar{u}(\bar{\lambda_1})_{x_i}}{2C}, -M_1\right)\right)\\ (\bar{h_2})_i^* &= \min\left(M_2, \max\left(-\frac{\bar{v}(\bar{\lambda_2})_{x_i}}{2D}, -M_2\right)\right)\end{aligned} \tag{3.26}$$

for $i = 1, 2, ..., d$.

Consider the change of variables such that,

$$u = e^{at}p, \quad v = e^{at}q, \quad \lambda_1 = e^{-at}\xi_1, \quad \lambda_2 = e^{-at}\xi_2$$

$$\bar{u} = e^{at}\bar{p}, \quad \bar{v} = e^{at}\bar{q}, \quad \bar{\lambda_1} = e^{-at}\bar{\xi_1}, \quad \bar{\lambda_2} = e^{-at}\bar{\xi_2} \ ,$$

where $a \geq 0$ to be chosen and

$$p, q, \bar{p}, \bar{q}, \xi_1, \xi_2, \bar{\xi_1}, \bar{\xi_2} \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$$

with

$$p_t, q_t, \bar{p}_t, \bar{q}_t, (\xi_1)_t, (\xi_2)_t, (\bar{\xi_1})_t, (\bar{\xi_2})_t \in L^2((0,T),(H^1(\Omega))^*)$$

.

Then, by substituting $u = e^{at}p, \quad v = e^{at}q$ and $\bar{u} = e^{at}\bar{p}, \quad \bar{v} = e^{at}\bar{q}$ in (3.1) and then subtracting the corresponding two systems of equations we get,

$$\begin{aligned}
&a(p-\bar{p}) + (p-\bar{p})_t - d_1\Delta(p-\bar{p}) - \nabla\cdot(\vec{h_1}(p-\bar{p})) - \nabla\cdot((\vec{h_1}-\vec{\bar{h}_1})\bar{p}) \\
&= (p-\bar{p})m - a_1(p-\bar{p})(p+\bar{p})e^{at} - b_1(p-\bar{p})\bar{q}e^{at} - b_1(q-\bar{q})pe^{at} \qquad \text{in } Q \\
&a(q-\bar{q}) + (q-\bar{q})_t - d_2\Delta(q-\bar{q}) - \nabla\cdot(\vec{h_2}(q-\bar{q})) - \nabla\cdot((\vec{h_q}-\vec{\bar{h}_2})\bar{q}) \\
&= (q-\bar{q})m - a_2(q-\bar{q})(q+\bar{q})e^{at} - b_2(p-\bar{p})\bar{q}e^{at} - b_2(q-\bar{q})pe^{at} \qquad \text{in } Q
\end{aligned}$$

$$\begin{aligned}
d_1\frac{\partial(p-\bar{p})}{\partial\eta} + (p-\bar{p})\vec{h_1}\cdot\eta + \bar{p}(\vec{h_1}-\vec{\bar{h}_1})\cdot\eta &= 0 \qquad \text{on } S \\
d_2\frac{\partial(q-\bar{q})}{\partial\eta} + (q-\bar{q})\vec{h_2}\cdot\eta + \bar{q}(\vec{h_2}-\vec{\bar{h}_2})\cdot\eta &= 0 \qquad \text{on } S
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
(p-\bar{p})(x,0) &= 0 \qquad \text{for } x\in\Omega \\
(q-\bar{q})(x,0) &= 0 \qquad \text{for } x\in\Omega
\end{aligned}$$

Then, by substituting $\lambda_1 = e^{-at}\xi_1$, $\lambda_2 = e^{-at}\xi_2$ and $\bar{\lambda_1} = e^{-at}\bar{\xi_1}$, $\bar{\lambda_2} = e^{-at}\bar{\xi_2}$ in (3.21) and subtracting the corresponding systems of equations we get,

$$\begin{aligned}
&a(\xi_1 - \bar{\xi_1}) - (\xi_1 - \bar{\xi_1})_t - d_1\Delta(\xi_1 - \bar{\xi_1}) + \vec{h}_1 \cdot \nabla(\xi_1 - \bar{\xi_1}) + (\vec{h}_1 - \vec{\bar{h}}_1)\nabla\bar{\xi_1} \\
&= \; m(\xi_1 - \bar{\xi_1}) - 2a_1(p - \bar{p})\bar{\xi_1}e^{at} + 2a_1(\xi_1 - \bar{\xi_1})pe^{at} - b_1(q - \bar{q})\bar{\xi_1}e^{at} \\
&\quad - b_1(\xi_1 - \bar{\xi_1})qe^{at} - b_2(q - \bar{q})\bar{\xi_2}e^{at} - b_2(\xi_2 - \bar{\xi_2})qe^{at} \qquad \text{in } \; Q, \\
&a(\xi_2 - \bar{\xi_2}) - (\xi_2 - \bar{\xi_2})_t - d_2\Delta(\xi_2 - \bar{\xi_2}) + \vec{h}_2 \cdot \nabla(\xi_2 - \bar{\xi_2}) + (\vec{h}_2 - \vec{\bar{h}}_2)\nabla\bar{\xi_2} \\
&= \; m(\xi_2 - \bar{\xi_2}) - 2a_2(q - \bar{q})\bar{\xi_2}e^{at} - 2a_2(\xi_2 - \bar{\xi_2})qe^{at} - b_1(p - \bar{p})\bar{\xi_1}e^{at} \\
&\quad - b_1(\xi_1 - \bar{\xi_1})pe^{at} - b_2(p - \bar{p})\bar{\xi_2}e^{at} - b_2(\xi_2 - \bar{\xi_2})pe^{at} \qquad \text{in } \; Q,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
\frac{\partial(\xi_1 - \bar{\xi_1})}{\partial\eta} &= 0 \qquad \text{on } \; S \\
\frac{\partial(\xi_2 - \bar{\xi_2})}{\partial\eta} &= 0 \qquad \text{on } \; S
\end{aligned}$$

$$\begin{aligned}
(\xi_1 - \bar{\xi_1})(x, T) &= 0 \qquad \text{for } \; x \in \Omega \\
(\xi_2 - \bar{\xi_2})(x, T) &= 0 \qquad \text{for } \; x \in \Omega \; .
\end{aligned}$$

Note that by (3.25) and (3.26) we have

$$\begin{aligned}
(\vec{h_1} - \vec{\bar{h}_1})^2 &= (\frac{-u\nabla\lambda_1}{2C} + \frac{\bar{u}\nabla\bar{\lambda_1}}{2C})^2 \\
&= \frac{1}{4C^2}(p\nabla\xi_1 - \bar{p}\nabla\bar{\xi_1})^2 \\
&= \frac{1}{4C^2}[(p - \bar{p})\nabla\xi_1 - \bar{p}\nabla(\xi_1 - \bar{\xi_1})]^2
\end{aligned} \tag{3.29}$$

and

$$(\vec{h_2} - \vec{\bar{h}_2})^2 = \frac{1}{4D^2}[(q - \bar{q})\nabla\xi_2 - \bar{q}\nabla(\xi_2 - \bar{\xi_2})]^2 \; .$$

By using standard regularity theory for the equations of $\xi_k, \bar{\xi_k}$ with $k = 1, 2$, and the boundedness of $p, \bar{p}, q, \; \bar{q}, \xi_1, \bar{\xi_1}, \; \xi_2, \bar{\xi_2}, \vec{h}_1, \vec{h}_2, m$, we can derive the L^∞-bounds on

$|\nabla\xi_k|$ and $|\nabla\bar{\xi_k}|$, where the bounds only depends on M_1, M_2, the initial data, and the coefficients of the system (3.1) [16, Lemma 5.2].

Then, by the standard estimation techniques, L^∞ bounds on $p, \bar{p}, q, \bar{q}, \xi_1, \bar{\xi_1}, \xi_2, \bar{\xi_2}, \vec{h_1}, \vec{h_2}, m$, L^∞ bounds on $|\nabla\xi_1(x,t)|$ and $|\nabla\bar{\xi_1}(x,t)|$ for all $(x,t) \in Q$, Cauchy's inequality and Holder's inequality on the first and second PDEs in (3.27), (3.28) and (3.29) we get

$$
\begin{aligned}
&(a - C_0 - \rho_1 - \rho_2 e^{aT}) \int_Q (p-\bar{p})^2 dxdt \\
&+ (a - C_0 - \rho_3 - \rho_4 e^{aT}) \int_Q (q-\bar{q})^2 dxdt \\
&+ (a - C_0 - \rho_5 - \rho_6 e^{aT}) \int_Q (\xi_1 - \bar{\xi_1})^2 dxdt \\
&+ (a - C_0 - \rho_7 - \rho_8 e^{aT}) \int_Q (\xi_2 - \bar{\xi_2})^2 dxdt \\
&+ \frac{1}{2} \int_\Omega [(p-\bar{p})^2(x,T) + (q-\bar{q})^2(x,T) + (\xi_1 - \bar{\xi_1})^2(x,0) + (\xi_2 - \bar{\xi_2})^2(x,0)] dx \\
&+ \Gamma \int_Q [|\nabla(p-\bar{p})|^2 + |\nabla(q-\bar{q})|^2 + |\nabla(\xi_1 - \bar{\xi_1})|^2 + |\nabla(\xi_2 - \bar{\xi_2})|^2] \, dxdt \\
&\leq 0 \ .
\end{aligned}
\tag{3.30}
$$

where, $\Gamma, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7$ and ρ_8 depending on the L^∞ bounds of $p, \bar{p}, q, \bar{q}, \xi_1, \bar{\xi_1}, \xi_2, \bar{\xi_2}, \vec{h_1}, \vec{h_2}, m, |\nabla\xi_1(x,t)|$, $|\nabla\bar{\xi_1}(x,t)|$ and sufficiently large cost coefficients C and D values .

Now, fixing a such that

$$
a > max[(C_0 + \rho_1 + \rho_2) \ , \ (C_0 + \rho_3 + \rho_4) \ , \ (C_0 + \rho_5 + \rho_6) \ , \ (C_0 + \rho_7 + \rho_8)]
$$

and by choosing T very small (3.30) implies that,

$$
p = \bar{p} \ , \ q = \bar{q} \ , \ \xi_1 = \bar{\xi_1} \ , \ \xi_2 = \bar{\xi_2}
$$

and hence we have

$$u = \bar{u} \ , \ v = \bar{v} \ , \ \lambda_1 = \bar{\lambda_1} \ , \ \lambda_2 = \bar{\lambda_2}$$

which implies the uniqueness of the optimum solution for a small time interval.

□

3.5 Numerical results

Numerical results for the 1–D problem

To study the numerical solutions first we have considered the one dimensional problem corresponds to problem (3.2):

$$\begin{aligned}
u_t - d_1 u_{xx} - (h_1 u)_x &= u[m - a_1 u] - b_1 uv \quad \text{on} \quad (0, L) \times (0, T) \\
v_t - d_2 v_{xx} - (h_2 v)_x &= v[m - a_2 v] - b_2 uv \quad \text{on} \quad (0, L) \times (0, T) \\
d_1 u_x \cdot \nu + u h_1 \cdot \nu &= 0 \quad \text{for} \quad x = 0 \ \text{ or } \ L \quad \text{and} \quad t \in (0, T) \\
d_2 v_x \cdot \nu + u h_1 \cdot \nu &= 0 \quad \text{for} \quad x = 0 \ \text{ or } \ L \quad \text{and} \quad t \in (0, T) \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{for} \quad x \in (0, L) \\
v(x, 0) &= v_0(x) \geq 0 \quad \text{for} \quad x \in (0, L)
\end{aligned} \tag{3.31}$$

Hence from (3.24) we have

$$\begin{aligned}
h_1^* = &\quad \min\left(M_1, \max\left(-\frac{u(\lambda_1)_x}{2C}, -M_1\right)\right) \\
h_2^* = &\quad \min\left(M_2, \max\left(-\frac{v(\lambda_2)_x}{2D}, -M_2\right)\right) \ .
\end{aligned}$$

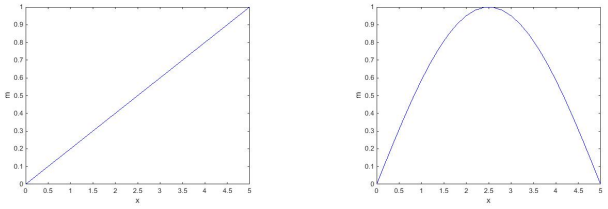
For our numerical simulations, we used parameter values

$$d_1 = d_2 = 0.2 \ , \ a_1 = a_2 = 1 \ , \ A = B = 1 \ , \ C = D = 0.5$$

and chose the optimal controls to satisfy

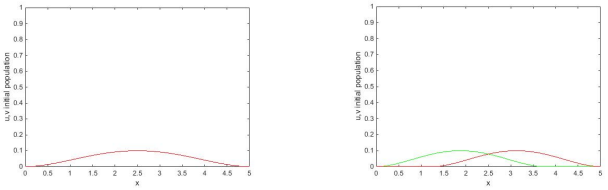
$$-4 \leq h_1 \leq 4 \text{ and } -4 \leq h_2 \leq 4$$

in a domain of spatial length of 5 units in 2 units of time. With our notation for the sign of advection terms, h_i being negative (positive) represents movement to right (left).



(a) $x/5$ **(b)** $sin(\pi x/5)$

Figure 3.1: Different resource functions $m(x)$



(a) **(b)**

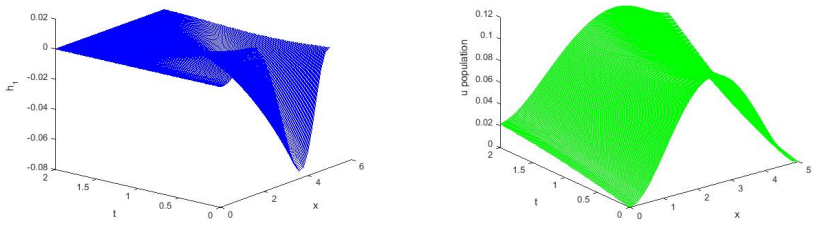
Figure 3.2: Different initial conditions: (3.2a) Smaller initial population at middle, (3.2b) Two smaller initial populations overlapping in the middle

Similar to what we did in Chapter 2, we considered several cases with different resource functions, initial conditions and competition coefficients. For resource functions, we chose $m(x) = \frac{x}{5}$, $sin(\pi x/5)$ and $m = \frac{6x}{5}$ as shown in Figure 3.1. The initial conditions are shown in Figure 3.2. In our Figures 3.3 – 3.7, we represent u population in green, v population in red, optimal control h_1 in blue and the optimal control h_2 in magenta.

From the numerical results in Figure 2.3 – 2.5 in Chapter 2, we can see that when the population level is higher at some location, their directed movement may also be

affected by diffusion. For this reason to show the competition effect separate from the diffusion effect, in Figures 3.3 – 3.7 we have used smaller initial conditions as in Figure 3.2a , 3.2b in our numerical simulations for the competition system (3.31).

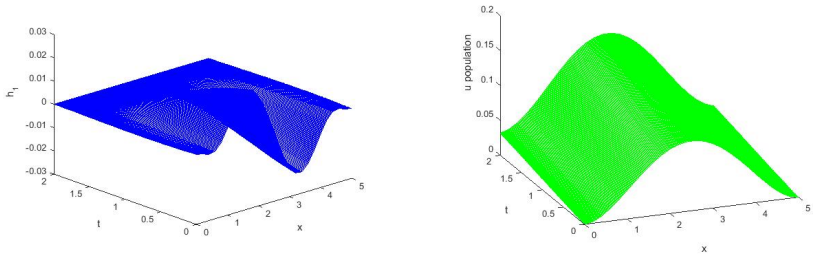
Using the initial conditions given in Figure 3.2a, with the resource functions given in Figure 3.1a, 3.1b, the numerical simulation results are in Figures 3.3 – 3.4. Here, in both figures we have taken $b_1 = b_2 = 4$ and the both populations were taken to be with the same initial condition.



(a) **(b)**

Figure 3.3: Two populations with competition: Population dynamics and optimal control with a smaller IC as in Figure 3.2a and the resource function $m = x/5$ as given in Figure 3.1a ; (3.3a) Optimal control h_1, (3.3b) Population distribution of u

Figure 3.3 gives the optimal control h_1 and the population distribution of u when the resources are at one boundary as in Figure 3.1a and the initial populations levels are smaller and located at the center of the spatial domain as in Figure 3.2a. As we have taken the same parameter values and the initial conditions for the two populations both u and v populations react in the same way and optimal control h_2 is as same as the optimal control h_1 in Figure 3.3a and the population distribution of v is as same as the population distribution of u in Figure 3.3b. In Figure 3.3a compared to the Figure 2.3a, we can see that when there is a competition in spite of the smaller population abundance level both populations tend to move away from each competing population's high density areas. Yet the individuals closer to the higher resources move a little faster toward resources than the individuals moving toward the other direction.



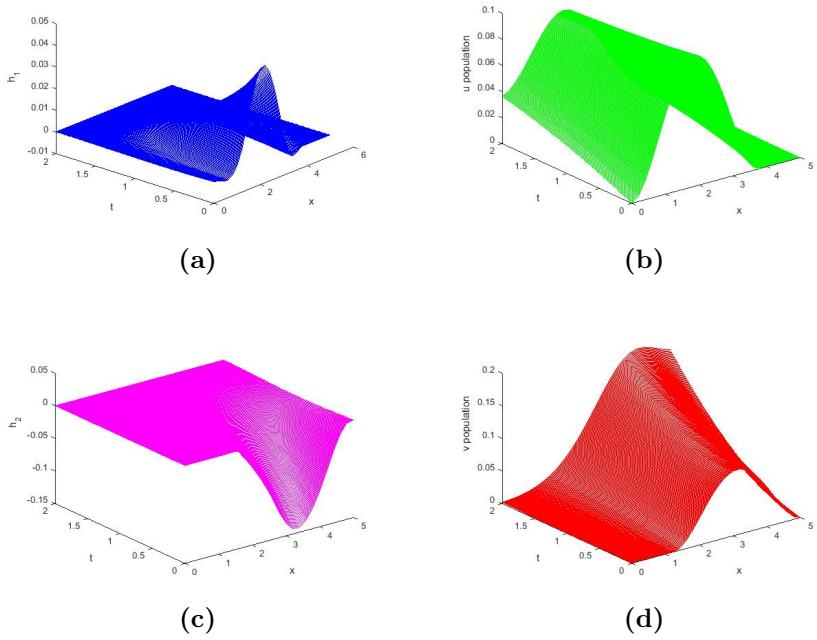
(a) **(b)**

Figure 3.4: Two populations with competition: Population dynamics and optimal control with a smaller IC as in Figure 3.2a and the resource function $m = sin(\pi x/5)$ as given in Figure 3.1b ; (3.4a) Optimal control h_1, (3.4b) Population distribution of u

In Figure 3.4 we give the optimal control h_1 and the population distribution of u over Q when the resources are located at the center of the domain as in Figure 3.1b and the initial populations levels are smaller and located at the center of the spatial domain as in Figure 3.2a. Again, we need to show only one population. The population moves in an opposite way in Figure 3.4a as compared with Figure 2.4a. In spite of the smaller population level, both populations tend to move away from each other but not towards the resources.

Further, to see the dynamics of two competing populations, we have run the numerical simulations with the initial condition in Figure 3.2b with the resource functions in Figure 3.1a, 3.1b. In Figure 3.5 we have the two optimal controls and the population distributions of the two populations over Q when the resources are higher at one boundary as in Figure 3.1a and the initial populations levels are smaller and are overlapping only at the center of the spatial domain as in Figure 3.2b. From the results we can see the two populations (green, red) tend to move towards the two opposite boundaries to move away from each other.

In Figure 3.6 we give the two optimal controls and the population distributions of the two populations over Q when the resources are higher at the center as in Figure 3.1b and the initial populations levels are smaller and are overlapping only at the



(a) **(b)**

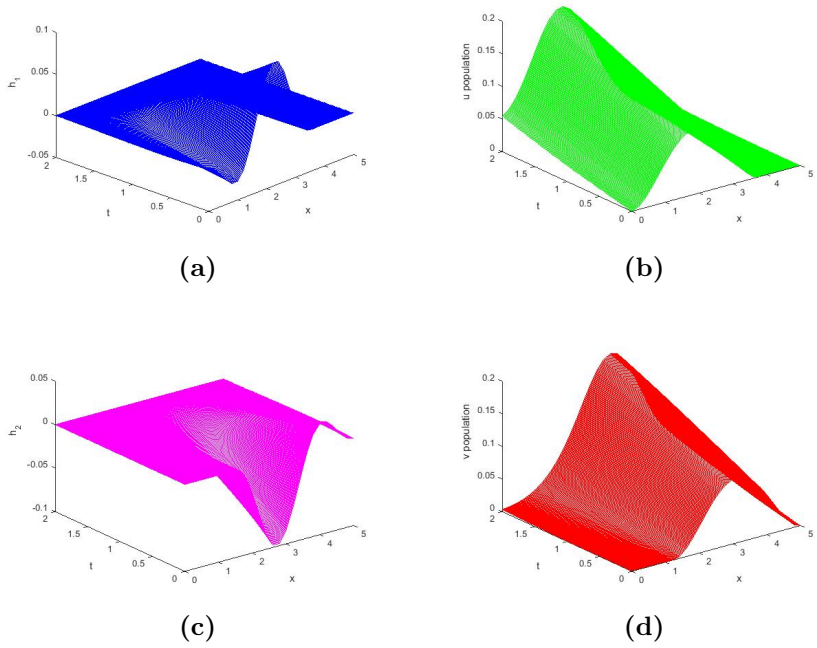
(c) **(d)**

Figure 3.5: Two populations with competition: Population dynamics and optimal control with a smaller IC as in Figure 3.2b and the resource function $m = x/5$ as given in Figure 3.1a ; (3.5a) Optimal control h_1, (3.5b) Population distribution of u, (3.5c) Optimal control h_2, (3.5d) Population distribution of v

center of the spatial domain as in Figure 3.2b. Again, similar to Figure 3.5, we can see the two populations tend to move towards the two opposite boundaries to move away from each other and not towards the better resources.

Hence, from the numerical results in Figures 3.3 – 3.6 we can see that when there is a high competition ($b_1 = b_2 = 4$), in spite of the smaller population abundance level and the favorable resource location, they tend to move away from each other. But, this reaction could depend on the competition coefficients b_1 and b_2 and we need further analysis to see any relationship.

To see how each population will react when they have different competition coefficients, in Figure 3.7, we have taken the competition coefficients to be $b_1 = 4$ and $b_2 = 0.5$ and considered the population dynamics and the optimal controls with a smaller initial condition in Figure 3.2a and resources in Figure 3.1b. Note that here



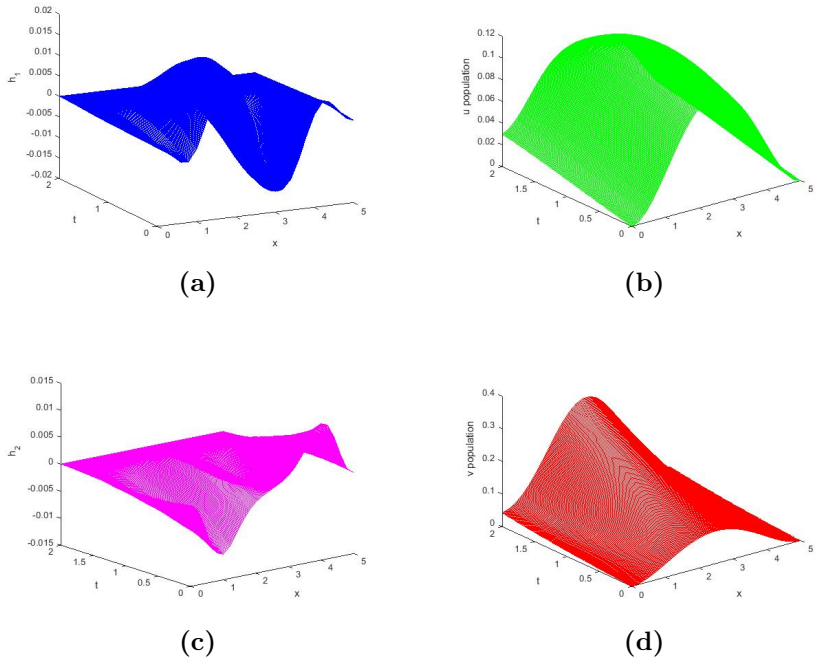
(a) **(b)**

(c) **(d)**

Figure 3.6: Two populations with competition: Population dynamics and optimal control with a smaller IC as in Figure 3.2b and the resource function $m = sin(\pi x/5)$ as given in Figure 3.1b ; (3.6a) Optimal control h_1, (3.6b) Population distribution of u, (3.6c) Optimal control h_2, (3.6d) Population distribution of v

again we take initial populations to be the same for both u and v. Compared to the results in Figure 3.4, in Figure 3.7 we see that when the competition coefficients are different two populations move in different ways. From Figure 3.7a we can see that u population (green) tend to move towards the two boundaries to move away from v population (red). But, in contrast to Figure 3.7a, in Figure 3.7c we can see that v population tends to move towards the center, with better resources.

Hence, from the numerical results in Figure 3.7 we can see that the directed movements of the competing populations can also depend on the competition rates.



(a) **(b)**

(c) **(d)**

Figure 3.7: Two populations with different competition rates: Population dynamics and optimal control with a smaller IC as in Figure 3.2a and the resource function $m = sin(\pi x/5)$ as given in Figure 3.1b and $b_1 = 4$, $b_2 = 0.5$; (3.7a) Optimal control h_1, (3.7b) Population distribution of u, (3.7c) Optimal control h_2, (3.7d) Population distribution of v

3.6 Conclusions

Optimal control analysis for controlling the advection directions in a parabolic PDE systems for competing populations was completed to obtain existence, uniqueness and characterization of the optimal control. The characterization involving the adjoint functions shows the explicit dependence on the coefficients, A, B, C and D, in the objective functional.

With numerical simulations, we were able to show the competing population does not always choose the advection direction to move toward increasing resources. When the initial condition has a sufficiently high population with some variation, the movement may be chosen to move to level each population, sometimes instead of moving toward increasing resources and avoiding competing population. With the

use of almost flat initial conditions, the level of the competition coefficients can also influence the choice of movement direction.

This numerical work indicates some interesting relationships between the optimal advective directions and the initial conditions, diffusion effects, competition rates, and resources; further work to study these relationship is warranted. The features of these PDEs and the corresponding numerical simulations need to be generalized for further investigation. Having variable diffusion, growth and competition rates will also have an impact on the optimal controls. The parameters A, B in the objective functional could be functions to force different choices on the populations. It would also be interesting to do some simulations in two space dimensions. Population interactions besides competition may also be considered. In our future work, we expect to continue this investigation with more generality.

Chapter 4

Johne's Disease problem

4.1 Introduction

Johne's disease is a bacterial infection caused by *Mycobacterium avium* subspecies *paratuberculosis* (MAP). It is a chronic, progressive, and infectious disease which has a long incubation period. It affects the small intestine of ruminants and ultimately leads to the death of the animal. Usually MAP spreads through fecal-oral transmission, MAP contaminated feces, water, soil, milk and colostrum [37, 40, 48]. In utero transmission was also reported [48].

MAP is a bacterium with a very slow growth rate. MAP is resistent to the heat and cold and hence, it can survive outside in the environment for a year or longer [48, 50]. But, in nature, MAP needs a host cell to grow and it extracts the iron from the host cells. Hence, MAP cannot replicate outside a host in the environment. [48]

Johne's disease is important to study as its economic impact on the dairy industry and its possible connection to the human disease, Crohn's disease. MAP had been estimated to be present in about 68% of US dairy operations [47]. The main problem of Johne's disease to the dairy industry is the reduction in milk production and the premature culling of the infected animals. This causes about $200 million to $250 million annual loss in the productivity of the dairy industry [47]. Although, Johne's

disease is curable by antibiotics, the treatment is not economically feasible to dairy farmers. Further most importantly there is a controversy that the MAP could be be a cause for the Crohn's disease [42, 45, 46].

An important way of controlling Johne's disease is through proper management of animals in the farm [48]. The use of antibiotics has not been successful in controlling the disease [48]. There are two main ways to diagnosis Johne's disease. One is by testing for the bacteria (MAP) and the other is by testing for the antibodies. In the first method of testing, the fecal oral samples collected from the animals or from the surrounding environment is tested, and due to the slow growth of the MAP these tests could take about two months for the results. In the second method of testing, either blood or milk is tested for the antibodies and the results can be collected much faster than in the first case. ELISA (enzyme-linked immunosorbent assays) is an antibody test which is widely used for Johne's disease [48]. But, recently Eda and his collaborates have developed a test called EVELISA (ELISA using ethanol extract of MAP) with better sensitivity [37, 43] than the standard ELISA test. Culling the animals with positive test results has been studied as a control of the Johne's disease in dairy farms. In [32], it is shown that with proper farm management, the culling the high shedding animals can be used to control the disease. Further, it showed that for better results the testing has to be done more frequently than once a year to identify the high shedders.

As the Johne's disease has a long incubation period, mathematical models can do a great deal of work to understand the dynamics of the disease and to predict the effect of different control strategies before implementing and collecting the data in the field. In that regard, much work has been done on modeling the dynamics of the Johne's disease. In a review paper Marce et al. [33] described eight existing epidemiological models describing the within-herd MAP transmission in dairy farms. Out of those models, four models were discrete-time stochastic models, one was a discrete-time deterministic model and three were continuous-time deterministic models. Some of these models considered the horizontal transmission. But only one, Humpry et

al.,[22] considered the ingestion of feces through the environment in their discrete-time stochastic model investigating the dynamics of the Johne's disease in a suckler-beef herd.

In [35], Marce et al. developed a stochastic model including both the vertical transmission and the horizontal transmission through the ingestion of colostrum, milk or feces in the environment, and they also considered the possibility of calf–to–calf disease transmission. Here, they studied the dynamics of MAP when no control measures are implemented. For their model they divided the calves in to six categories as susceptible, resistant, transiently infectious, latently infected, subclinically infected and clinically infected. Further, they considered several contaminated environments for each cow category, the environment of the whole farm and the outdoor environment of calves when they graze.

Then, in [36] Marce et al. studied the effect of the contact structure of animals in a continually infected dairy farm on the transmission of MAP using the stochastic model developed in [35].

In [46], Massaro et al. developed a discrete mathematical model to describe the dynamics of the Johne's disease in a dairy farm. Here they have considered the contact structure of the animals in the farm and categorize the animal compartments according to the stage of the disease as well as the age. In their work, they studied the cost effectiveness of using the EVELISA testing.

Recently, Robins et al. [40] developed an agent based model to study the dynamics of the Johne's disease in an dairy farm. Here, they implicitly incorporated the the effect of the bacteria in the environment. Further, they have studied the effect of different disease transmission routes on the prevalence of the disease in the system.

4.2 Problem formulation

4.2.1 Formulation of the main model

In our study we developed a continuous deterministic model to describe the dynamics of the Johne's disease in a dairy farm. Here, we analyze the dynamics of the Johne's disease prevalence by taking the environmental persistence of the bacteria into account.In this model we use a system of ordinary differential equations to describe the behavior of Johne's disease among dairy cows considering the progression of the disease and the age structure of the cows. Here we categorize the cows into three age groups as 0 - 2 months as calves, 2 - 24 months as heifers and above 24 months as adults, similar to our previous papers [37, 40]. Then, we further divide these age categories according to the stage of the disease as explained later.

Here, only the female cows are taken into account and male newborn calves would be removed from the dairy farm. In common dairy farm management, the cows are grouped in different locations according to the age and the milk production. Shortly after the birth the calves are moved into individual hutches and the heifers and adults are kept at separate locations. Usually the adult cows can be in four different locations (housing) as pasture, pregnancy pen, lactation barn and maternity barn. But, in our model for simplicity we consider one location for all the adult cows.

We assume that the calves born to susceptible and exposed adults to be not infected with MAP at the birth. But, soon after the birth, some calves born to susceptible or exposed adults can be infected with the MAP in the adult's environment (direct contact or from MAP in environment). Also, we assume that some calves born to infected adults can be infected in-utero (vertical transmission) and some will born without the infection. Further, the susceptible calves can be infected by the MAP contaminated colostrum and milk. Hence, we subdivided the calves into two categories as susceptible and exposed. We did not consider the calves

progressing stages, as the calves are kept in individual hutches and MAP shed into the environment does not contribute to the MAP transmission.

As we assume the heifers can only progress in disease to the level of low shedding, the heifers are sub-divided into three categories as susceptible, exposed and low shedding. Since, there are low shedding animals among heifers, the heifer group housing is considered to have bacteria in that environment (called environment 1 in the model). As a result susceptible heifers not only get infected by direct interaction with low shedding heifers but also by the bacteria in their housing environment.

In the adult category, infected adults can further progress in disease to the level of high shedding. Thus, the adults were subdivided into four categories as susceptible, exposed, low shedding and high shedding. Also, due to the low shedding and high shedding adults, their barns were considered to be contaminated with the bacteria (called environment 2 in the model). Hence, the susceptible adults can be infected due to the direct interaction with low shedding and high shedding cows and also due to the interaction with the environment. In a dairy system, testing and culling can be used to control the Johne's disease [32]. In common management the cows are tested for the Johne's disease once or twice a year and if the cows are tested positive as a high shedder they are likely removed from the system.

With all the above assumptions and the two environments, we use a system of 11 ordinary differential equations to describe the dynamics of the Johne's disease in a dairy farm. In our system we do not consider a category for resistant animals. We assume that the units of time are per day.

Here, we assumed the same birth rates for the susceptible and exposed adults and a different birth rate for the infected adults. Also, we assume two different vertical transmission rates for low shedding and high shedding animals. Further, different death rates are used for each category and we assume that the culled animals are replaced by susceptible heifers. Also, we use different infection rates for the calves due to the colostrum and milk from low shedders and high shedders. Further, we use different contact rates for the susceptible heifers with the low shedding heifers and for

the susceptible adults with the low shedding adults and high shedding adults. In the two environments we assumed the same death rate for the bacteria in the environment 1 and environment 2 and three different values for the rates at which the bacteria is added to the corresponding environment from low shedding heifers, low shedding adults and high shedding adults. Further, we assumed two different values for the probability of getting infected from the bacteria in the environment for the calves and for the heifers and adults.

In this study our main goal is to analyse the role of the environments in disease transmission and to analyze the effect of cleaning the environment in controlling the disease. Due to the infrequent testing and low diagnostic test sensitivity, later we wish to add these features as a control into our ODE system.

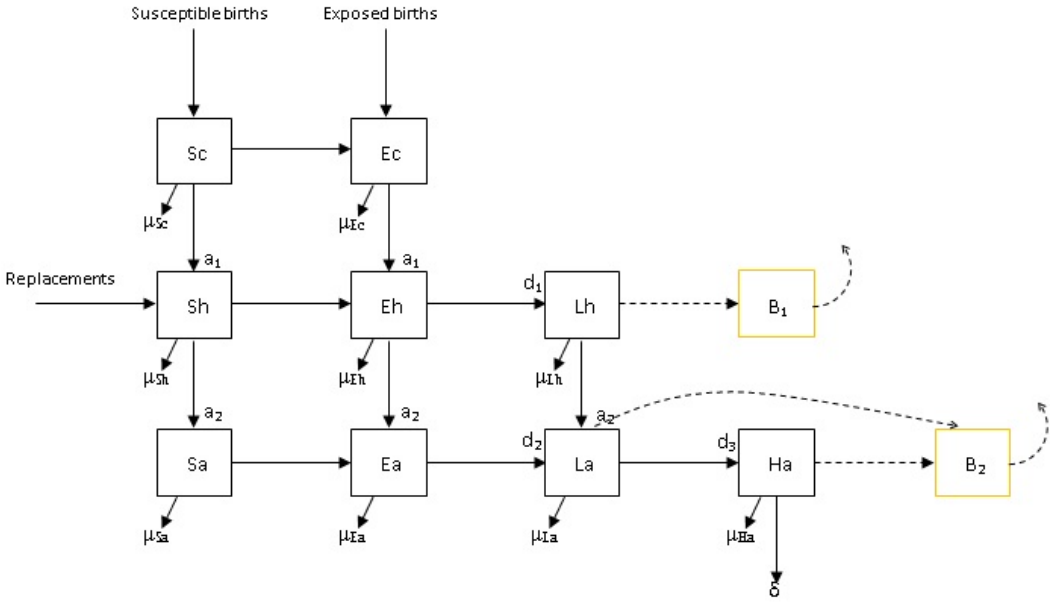


Figure 4.1: Flow diagram of the transition of Johnes disease

The flow diagram of our model of the Johne's disease is given in Figure 4.1. In Figure 4.1, the horizontal arrows represent the progression in disease stages and the vertical arrows represent the age transmission. The horizontal dashed arrows represent the sources that the amount of bacteria in the environment 1 and

environment 2 has added to the environment 1 and environment 2. Further, the curved dashed arrows indicate that the bacteria in the environment 1 and 2 interact with the susceptible animals.Our compartments and parameters are listed below:

S_c = Number of susceptible calves

S_h = Number of susceptible heifers

S_a = Number of susceptible adults

E_c = Number of exposed calves

E_h = Number of exposed heifers

E_a = Number of exposed adults

L_h = Number of low shedding heifers

L_a = Number of low shedding adults

H_a = Number of high shedding adults

B_1 = Amount of bacteria (MAP) in the environment 1 (Scaled in 10^8)

B_2 = Amount of bacteria (MAP) in the environment 2 (Scaled in 10^8)

b = Birth rate of calves from susceptible and exposed adults

b_{L_a} = Birth rate of calves from low-shedding adults

b_{H_a} = Birth rate of calves from high-shedding adults

μ_{S_c} = Death rate of susceptible calves

μ_{S_h} = Death rate of susceptible heifers

μ_{S_a} = Death rate of susceptible adults

μ_{E_c} = Death rate of exposed calves

μ_{E_h} = Death rate of exposed heifers

μ_{E_a} = Death rate of exposed adults

μ_{L_h} = Death rate of low shedding heifers

μ_{L_a} = Death rate of low shedding adults

μ_{H_a} = Death rate of high shedding adults

μ_{B_1} = Decay rate of bacteria (MAP) in the heifers environment

μ_{B_2} = Decay rate of bacteria (MAP) in the adults environment

δ = Culling rate of high-shedding adults

ν_L = Probability of getting infected through vertical transmission from low-shedding adults

ν_H = Probability of getting infected through vertical transmission from high-shedding adults

a_1 = Transfer rate from calves to heifers due to age progression

a_2 = Transfer rate from heifers to adults due to age progression

d_1 = Transfer rate from exposed heifers to low shedding heifers

d_2 = Transfer rate from exposed adults to low shedding adults

d_3 = Transfer rate from low shedding adults to high shedding adults

β_1 = Transmission rate for susceptible calves due to the colostrum and milk from low shedding adults

β_2 = Transmission rate for susceptible calves due to the colostrum and milk from high shedding adults

γ_1 = Transmission rate for susceptible heifers due to direct contact with low-shedding heifers

γ_2 = Transmission rate for susceptible adults due to direct contact with low-shedding adults

γ_3 = Transmission rate for susceptible adults due to direct contact with high-shedding adults

λ_1 = Rate at which the bacteria is added to the heifers environment from the low-shedding heifers

λ_2 = Rate at which the bacteria is added to the adults environment from the low-shedding adults

λ_3 = Rate at which the bacteria is added to the adults environment from the high-shedding adults

The bacteria MAP enters the environment mainly from the feces of the shedding cows. The low shedding cows shed less MAP (1-50 MAP/gram of feces)[39] compared

to the MAP shed by high shedding cows (100-10,000,000 MAP/gram of feces)[49]. In average a cow produces about 29.5 kg of feces per day [18].

To define the transmission of the Johne's disease due to the bacteria in the environment we need a function with a threshold level before its transition to maximum. Thus, to find our infection terms with B_1 and B_2 we use the function given below with a threshold effect

$$f(B) = \frac{1 - e^{-\frac{B}{K_1}}}{1 + K_2 e^{-\frac{B}{K_1}}} \ . \tag{4.1}$$

In (4.1), B is the total amount of bacteria taken by an animal and K_1 and K_2 are two constants representing the effect of the infectious nature of bacteria. More precisely, K_1 reflects the heterogeneity of infectiousness in the bacteria population and K_2 reflects the infecting ability of the bacteria.

To transmit MAP, there is an infection dose at 10,000 MAP [19]. So, ingesting a lower level of MAP may not result in transmission. Hence, first, we figure out K_1, and K_2 values to capture this infectiousness of the MAP.

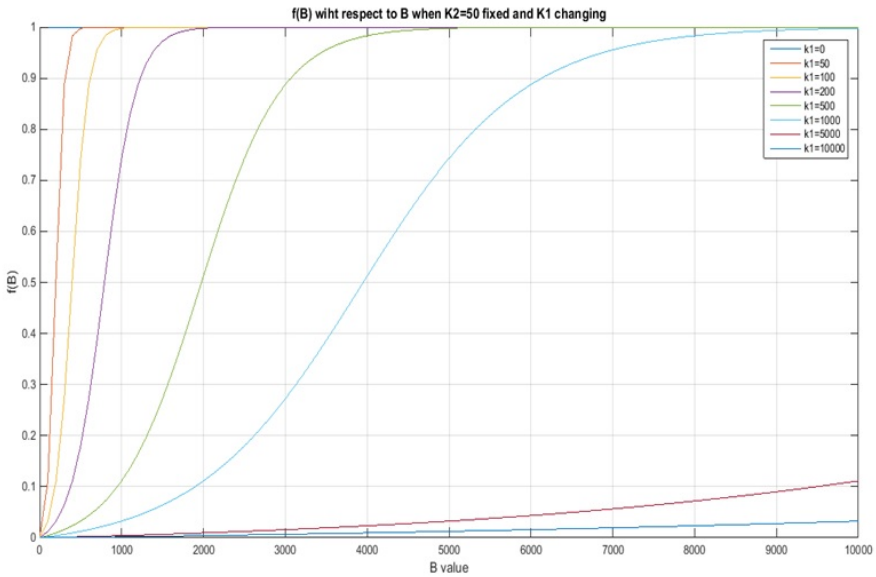


Figure 4.2: $F(B)$ for different K_1 values with a fixed K_2 value ($K_2 = 50$)

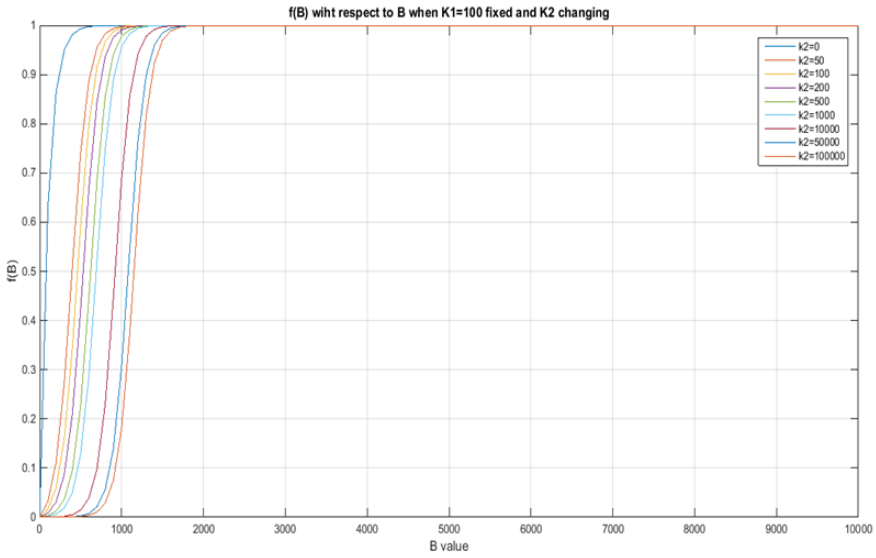


Figure 4.3: $F(B)$ for different K_2 values with a fixed K_1 value ($K_1 = 100$)

Figure 4.2 shows how the $f(B)$ function changes with respect to B and K_1 with a fixed K_2 value. From the Figure 4.2 we can see that K_1 changes the slope, shape and the threshold value of the function.

Figure 4.3 shows how the $f(B)$ function changes with respect to B and K_2 with a fixed K_1 value. We can see that K_2 changes the threshold value of the function and does not affect the slope of the graph.

Hence, for our problem, we have picked $K_1 = 1000$ and $K_2 = 100$ such that $f(10000) = 1$ and $f(5000) \simeq 0.5$, since ingesting 10000 MAP bacteria is very likely to result in an infection [19]. The graph of the $F(B)$ function with $K_1 = 1000$ and $K_2 = 100$ is given in Figure 4.4.

In our ODE model, we take B_1 and B_2 to represent the total amount of MAP in the heifers environment and the adults environment. Now, that B represents the amount of bacteria taken by an animal, we use a scaling factor C to reflect that the animal consumes only a proportion of bacteria in the environment such that $B = B_i \times C$.

To calculate this proportion of bacteria taken by an animal (C), here we assumes that the bacteria is evenly distributed in the feces and the surrounding environment.

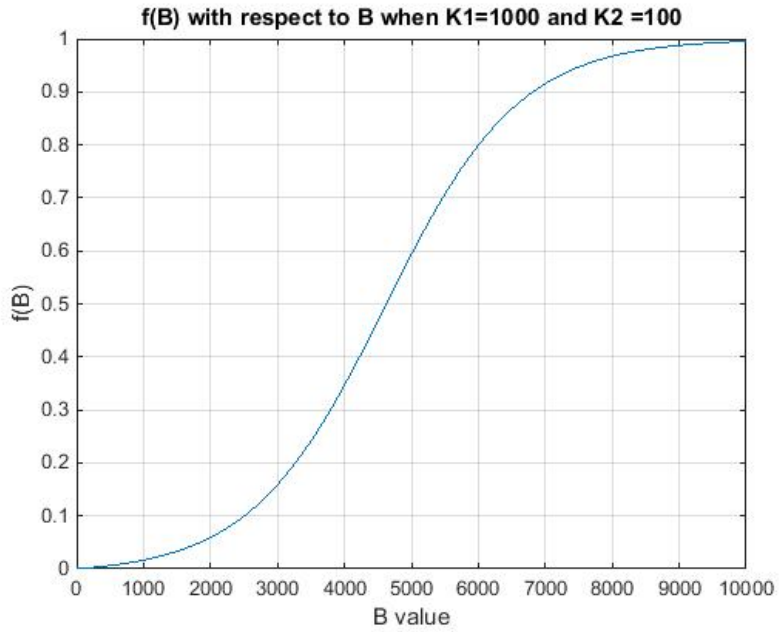


Figure 4.4: $F(B)$ with $K_1 = 1000$ and $K_2 = 100$

Since feces are the main route for MAP to enter and survive in the surrounding environment, in our calculations, the proportion of bacteria taken by and animal per day is approximated by the proportion of the feces taken by an animal per day.

We approximate the amount of new feces enter the environment each day by assuming 800 heifers and 1000 adults in the farm. We calculate the total amount of new feces in the environment 1 and environment 2 per day as below. Here we take in average a cow produces about 29.5 kg of feces per day [18]

$$\begin{aligned} \text{Total amount of new feces in the environment 1 per day} &= 800 \times 29.5 \times 10^6 mg \\ &= 2.36 \times 10^{10} mg \\ \text{Total amount of new feces in the environment 2 per day} &= 1000 \times 29.5 \times 10^6 mg \\ &= 2.95 \times 10^{10} mg \end{aligned}$$

We assume that a cow will take about 1 mg of feces per day. Hence the proportion of feces taken by an animal per day (C_F) for two environments can be calculated as

given below. Here, we approximate the proportion of the feces taken by an animal per day, by the proportion of the new feces taken by an animal per day (C_F).

For environment 1:

$$C_{1F} = \frac{1}{2.36 \times 10^{10}} = 4.23 \times 10^{-11}$$

For environment 2:

$$C_{2F} = \frac{1}{2.95 \times 10^{10}} = 3.39 \times 10^{-11}$$

Hence, for simplicity, we average C_{1F} and C_{2F} and use $C_F = 3.8 \times 10^{-11}$ for both environments.

In our numerical simulations we scale B_1 and B_2 in ten millions (10^8). Hence, to adjust for the B_1, B_2 units in 10^8, the C_F above gives $C = 0.0038$ in our $F(B_i)$ function for the model.

Next to represent that the calves are more vulnerable to disease than the heifers and adults we use some parameters p, r_1, r_2 to represent the probability of getting infected by the MAP in the environment for calves heifers and adults respectively, with $p > r_1 = r_2$

p = Probability of new born susceptible calves getting infected by the MAP in the adult environment

r_1 = Probability of susceptible heifers getting infected by the MAP in the heifers environment

r_2 = Probability of susceptible adults getting infected by the MAP in the adult environment .

We describe the dynamics of the system with above assumptions using the following system of ODEs.

$$
\begin{aligned}
\frac{dS_c}{dt} &= b(S_a+E_a)(1-p\frac{1-e^{-\frac{B_2C}{K_1}}}{1+K_2e^{-\frac{B_2C}{K_1}}}) + b_{L_a}(1-\nu_L)L_a + b_{H_a}(1-\nu_H)H_a \\
&\quad - a_1S_c - \beta_1S_cL_a - \beta_2S_cH_a - \mu_{S_c}S_c \\
\frac{dE_c}{dt} &= b(S_a+E_a)p\frac{1-e^{-\frac{B_2C}{K_1}}}{1+K_2e^{-\frac{B_2C}{K_1}}} + b_{L_a}\nu_LL_a + b_{H_a}\nu_HH_a + \beta_1S_cL_a + \beta_2S_cH_a \\
&\quad - a_1E_c - \mu_{E_c}E_c \\
\frac{dS_h}{dt} &= a_1S_c + \delta H_a - \gamma_1S_hL_h - r_1S_h\frac{1-e^{-\frac{B_1C}{K_1}}}{1+K_2e^{-\frac{B_1C}{K_1}}} - a_2S_h - \mu_{S_h}S_h \\
\frac{dE_h}{dt} &= a_1E_c + \gamma_1S_hL_h + r_1S_h\frac{1-e^{-\frac{B_1C}{K_1}}}{1+K_2e^{-\frac{B_1C}{K_1}}} - a_2E_h - d_1E_h - \mu_{E_h}E_h \\
\frac{dL_h}{dt} &= d_1E_h - a_2L_h - \mu_{L_h}L_h \\
\frac{dS_a}{dt} &= a_2S_h - \gamma_2S_aL_a - \gamma_3S_aH_a - r_2S_a\frac{1-e^{-\frac{B_2C}{K_1}}}{1+K_2e^{-\frac{B_2C}{K_1}}} - \mu_{S_a}S_a \\
\frac{dE_a}{dt} &= a_2E_h + \gamma_2S_aL_a + \gamma_3S_aH_a + r_2S_a\frac{1-e^{-\frac{B_2C}{K_1}}}{1+K_2e^{-\frac{B_2C}{K_1}}} - d_2E_a - \mu_{E_a}E_a \\
\frac{dL_a}{dt} &= a_2L_h + d_2E_a - d_3L_a - \mu_{L_a}L_a \\
\frac{dH_a}{dt} &= d_3L_a - \mu_{H_a}H_a - \delta H_a \\
\frac{dB_1}{dt} &= \lambda_1L_h - \mu_{B_1}B_1 \\
\frac{dB_2}{dt} &= \lambda_2L_a + \lambda_3H_a - \mu_{B_2}B_2 \ .
\end{aligned}
$$

We explain the terms in the model below.

$b(S_a + E_a)$ – Number of calves born to susceptible and exposed adults

$b_{L_a}(1-\nu_L)L_a$ – Number of susceptible calves born to low shedding adults

$\nu_L L_a b_{L_a}$ – Number of exposed calves born to low shedding adults (vertical transmission)

$b_{H_a}(1-\nu_H)H_a$ – Number of susceptible calves born to high shedding adults

$\nu_H H_a b_{H_a}$ – Number of exposed calves born to high shedding adults (vertical transmission)

$\beta_1 S_c L_a$ – Number of susceptible calves become exposed calves due to the colostrum and milk from low shedding adults

$\beta_2 S_c H_a$ – Number of susceptible calves become exposed calves due to the colostrum and milk from high shedding adults

$\gamma_1 S_h L_h$ – Number of susceptible heifers become exposed heifers due to direct interactions with low shedding heifers

$\gamma_2 S_a L_a$ – Number of susceptible adults become exposed adults due to direct interactions with low shedding adults

$\gamma_3 S_a H_a$ – Number of susceptible adults become exposed adults due to direct interactions with high shedding adults

$p\dfrac{1-e^{-\frac{B_2 C}{K_1}}}{1+K_2 e^{-\frac{B_2 C}{K_1}}} b(S_a + E_a)$ – From the calves born to susceptible and exposed adults the number of calves become exposed due to the bacteria in the adults environment.

$r_1 S_h \dfrac{1-e^{-\frac{B_1 C}{K_1}}}{1+K_2 e^{-\frac{B_1 C}{K_1}}}$ – Number of susceptible heifers become exposed heifers due to the bacteria in the environment 1

$r_2 S_a \frac{1 - e^{-\frac{B_2 C}{K_1}}}{1 + K_2 e^{-\frac{B_2 C}{K_1}}}$ – Number of susceptible adults become exposed adults due to the bacteria in the environment 2

$a_1 S_c$ – Number of susceptible calves progress to susceptible heifers due to age progression

$a_1 E_c$ – Number of exposed calves progress to exposed heifers due to age progression

$a_2 S_h$ – Number of susceptible heifers progress to susceptible adults due to age progression

$a_2 E_h$ – Number of exposed heifers progress to exposed adults due to age progression

$a_2 L_h$ – Number of low shedding heifers progress to low shedding adults due to age progression

$d_1 E_h$ – Number of exposed heifers become low shedding heifers due to disease progression

$d_2 E_a$ – Number of exposed adults become low shedding adults due to disease progression

$d_3 L_a$ – Number of low shedding adults become high shedding adults due to disease progression

$\mu_{S_c} S_c$ – Number of susceptible calves remove from the class due to natural death

$\mu_{E_c} E_c$ – Number of exposed calves remove from the class due to natural death

$\mu_{S_h} S_h$ – Number of susceptible heifers remove from the class due to natural death

$\mu_{E_h} E_h$ – Number of exposed heifers remove from the class due to natural death

$\mu_{L_h} L_h$ – Number of low shedding heifers remove from the class due to natural death

$\mu_{S_a} S_a$ – Number of susceptible adults remove from the class due to natural death

$\mu_{E_a} E_a$ – Number of exposed adults remove from the class due to natural death

$\mu_{L_a} L_a$ – Number of low shedding adults remove from the class due to natural death

$\mu_{H_a} H_a$ – Number of high shedding adults remove from the class due to natural death

δH_a – Culled number of high shedding adults

$\lambda_1 L_h$ – Amount of MAP added to the environment 1 from the low shedding heifers

$\lambda_2 L_a$ – Amount of MAP added to the environment 2 from the low shedding adults

$\lambda_3 H_a$ – Amount of MAP added to the environment 2 from the high shedding adults

$\mu_{B_1} B_1$ – Amount of MAP removed from the environment 1 due to natural decay of the bacteria

$\mu_{B_2} B_2$ – Amount of MAP removed from the environment 2 due to natural decay of the bacteria .

4.2.2 Formulation of the simplified model

Before analyzing the above model in section 4.2, first we consider a simplified model with one environment compartment for stability analysis. In this simplified version

we divide the cows into only two categories according to the age; we take 1–2 months old animals as calves and the animals above 2 months as adults. Further, we combine both low and high shedders as one shedders compartment and consider only the adult environment for the disease transmission through the environment and the environment gets the bacteria (MAP) from the feces of the shedding cows.

Similarly as in the complete model, the effect of the MAP in the environment is described by

$$f(B_e) = \frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}}$$

with the same interpretation and values for the constants K_1, K_2 and C.

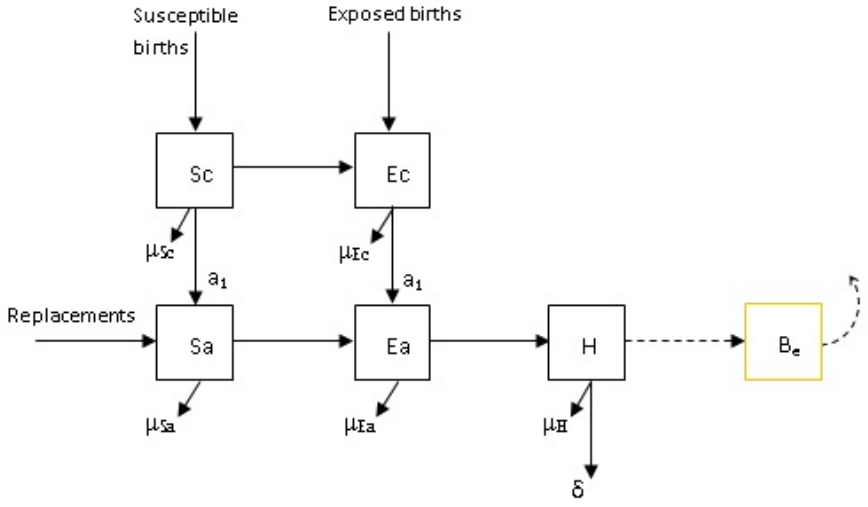


Figure 4.5: Flow diagram of the transition of Johne's disease in the simplified case

The flow diagram of the Johne's disease in the simplified model is displayed in Figure 4.5. Similarly as in Figure 4.1, in Figure 4.5, the horizontal arrows represent the progression in disease stages and the vertical arrows represent the age transmission. The horizontal dashed arrows represent the sources that the amount of bacteria has added to the environment. Further, the curved dashed arrows indicate that the bacteria in the environment interact with the susceptible adults.

We list the compartments and parameters in our simplified model:

S_c = Number of susceptible calves

S_a = Number of susceptible adults

E_c = Number of exposed calves

E_a = Number of exposed adults

H = Number of shedding adults

B_e = Amount of bacteria (MAP) in the environment

b = Birth rate of calves from susceptible and exposed adults

b_H = Birth rate of calves from shedding adults

μ_{S_c} = Death rate of susceptible calves

μ_{S_a} = Death rate of susceptible adults

μ_{E_c} = Death rate of exposed calves

μ_{E_a} = Death rate of exposed adults

μ_H = Death rate of shedding adults

μ_{B_e} = Death rate of the bacteria in the environment

a_1 = Transfer rate from calves to adults due to age progression

d = Transfer rate from exposed to shedding

ν_H = Probability of getting infected through vertical transmission from shedding adults

β = Contact rate of H with Sc (For the colostrum and milk)

γ = Contact rate of susceptible adults with shedding adults

p = Probability of new born susceptible calves getting infected by the MAP in the adult environment

r = Probability of susceptible adults getting infected by the MAP in the environment

λ = Rate at which the bacteria is added to the environment from the shedding adults

δ = Culling rate of shedding cows

$K_1 = 1000 -$ reflects the heterogeneity of infectiousness in the bacteria population

$K_2 = 100 -$ reflects the infecting ability of the bacteria (representing threshold)

$C = 0.0038 -$ a scaling factor to reflect that the animal picks only a proportion of feces in the environment

We describe the dynamics of the simplified model using the following system of ODEs.

$$\begin{aligned}
\frac{dS_c}{dt} &= b(S_a + E_a)(1 - p\frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}}) + b_H(1 - \nu_H)H - a_1 S_c - \mu_{S_c} S_c - \beta S_c H \\
\frac{dE_c}{dt} &= b(S_a + E_a)p\frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}} + b_H \nu_H H + \beta S_c H - a_1 E_c - \mu_{E_c} E_c \\
\frac{dS_a}{dt} &= a_1 S_c + \delta H - \gamma S_a H - r S_a \frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}} - \mu_{S_a} S_a \\
\frac{dE_a}{dt} &= a_1 E_c + \gamma S_a H + r S_a \frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}} - \mu_{E_a} E_a - d E_a \\
\frac{dH}{dt} &= d E_a - \mu_H H - \delta H \\
\frac{dB_e}{dt} &= \lambda H - \mu_{B_e} B_e
\end{aligned}$$

Again we explain the terms in the model.

$b(S_a + E_a)$ – Total number of calves born to susceptible and exposed adults

$b_H(1 - \nu_H)H$ – Total number of susceptible calves born to shedding adults

$\nu_H b_H H$ – Number of exposed calves born to shedding adults (vertical transmission)

$\beta S_c H$ – Number of susceptible calves become exposed calves due to the colostrum and milk from shedding adults

$\gamma S_a H$ – Number of susceptible adults become exposed adults due to direct interactions with shedding adults

$r S_a \frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}}$ – Number of susceptible adults become exposed adults due to the bacteria in the adults environment

$p \frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}} b(S_a + E_a)$ – From the calves born to susceptible and exposed adults the number of calves become exposed due to the bacteria in the adults environment

$a_1 S_c$ – Number of susceptible calves progress to susceptible adults due to age progression

$a_1 E_c$ – Number of exposed calves progress to exposed adults due to age progression

$d E_a$ – Number of exposed adults become shedding adults due to disease progression

$\mu_{S_c} S_c$ – Number of susceptible calves remove from the class due to natural death

$\mu_{E_c} E_c$ – Number of exposed calves remove from the class due to natural death

$\mu_{S_a} S_a$ – Number of susceptible adults remove from the class due to natural death

$\mu_{E_a} E_a$ – Number of exposed adults remove from the class due to natural death

$\mu_H H$ – Number of shedding adults remove from the class due to natural death

δH – Culled number of shedding adults

λH – Amount of MAP added to the environment from the shedding adults

$\mu_{B_e} B_e$ – Amount of MAP removed from the environment due to natural decay of the bacteria .

4.2.3 Stability of the disease free equilibrium of the simplified dynamic system

To use the next generation matrix method [2, 4, 11, 12, 14], we use the ODEs for the infected classes and the noninfected classes separately. The equations for infected classes are:

$$\begin{aligned}
\frac{dE_c}{dt} &= b(S_a + E_a)p\frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}} + b_H \nu_H H + \beta S_c H - a_1 E_c - \mu_{E_c} E_c \\
\frac{dE_a}{dt} &= a_1 E_c + \gamma S_a H + r S_a \frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}} - \mu_{E_a} E_a - d E_a \\
\frac{dH}{dt} &= d E_a - \mu_H H - \delta H \\
\frac{dB_e}{dt} &= \lambda H - \mu_{B_e} B_e
\end{aligned}$$

and the equations for uninfected classes are:

$$\frac{dS_c}{dt} = b(S_a + E_a)(1 - p\frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}}) + b_H(1 - \nu_H)H - a_1 S_c - \mu_{S_c} S_c - \beta S_c H \tag{4.2}$$

$$\frac{dS_a}{dt} = a_1 S_c + \delta H - \gamma S_a H - r S_a \frac{1 - e^{-\frac{B_e C}{K_1}}}{1 + K_2 e^{-\frac{B_e C}{K_1}}} - \mu_{S_a} S_a \ . \tag{4.3}$$

Disease free equilibrium (DFE):

Disease free equilibrium is the equilibrium point where the disease does not exist in the system. Hence, at the disease free equilibrium we have $E_c = E_a = H = B_e = 0$. By setting

$$\frac{dE_c}{dt} = \frac{dE_a}{dt} = \frac{dH}{dt} = \frac{dB}{dt} = \frac{dS_c}{dt} = \frac{dS_a}{dt} = 0$$

we find the conditions for DFE to exist.

Hence, equation (4.2) implies $bS_a = (a_1 + \mu_{S_c})Sc$ and we obtain

$$\frac{S_a}{S_c} = \frac{a_1 + \mu_{S_c}}{b}.$$

Similarly, equation (4.3) implies $a_1(S_c) = \mu_{S_a} S_a$ and

$$\frac{S_a}{S_c} = \frac{a_1}{\mu_{S_a}}.$$

Thus, for a DFE to exist, we assume

$$\frac{a_1 + \mu_{S_c}}{b} = \frac{a_1}{\mu_{S_a}}\ , \tag{4.4}$$

which means the rate of calves leaving the calves compartment per birth rate to be equal to the rate of adults entering the adults environment per death rate of adults.

Further, by (4.4), we have

$$a = \frac{\mu_{S_c}\mu_{S_a}}{(b - \mu_{S_a})}\ .$$

Hence, further we assume $b > \mu_{S_a}$.

Thus, we have our DFE,

$$X_0 = (0,\ 0,\ 0,\ 0,\ S_c^*,\ S_a^*) = (0,\ 0,\ 0,\ 0,\ \frac{(b - \mu_{S_a})}{\mu_{S_c}} S_a^*,\ S_a^*)\ .$$

We analyze the stability of the DFE using the next generation matrix method [4, 11, 12, 14] as explained below.

Consider $x_i' = f_i(x)$ where $i = 1, \ldots 6$ and $x = (E_c, E_a, H, B_e, S_c, S_a)$ with
$f_i(x) = \mathbb{F}_i(x) - \mathbb{V}_i(x) = \mathbb{F}_i(x) - (\mathbb{V}_i^-(x) - \mathbb{V}_i^+(x))$ with

$\mathbb{F}_i(x)$ = Rate of appearance of new infections in compartment i
$\mathbb{V}_i^+(x)$ = Rate of transfer of individuals into compartment i by all other means
$\mathbb{V}_i^-(x)$ = Rate of transfer of individuals out of compartment i by all other means.

Hence, for the above system of equations, we have

$$\mathbb{F} = \begin{bmatrix} b(S_a + E_a)p\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}} + b_H\nu_H H + \beta S_c H \\ \gamma S_a H + rS_a\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathbb{V} = \begin{bmatrix} a_1E_c + \mu_{E_c}E_c \\ \mu_{E_a}E_a + dE_a - a_1E_c \\ \mu_H H + \delta H - dE_a \\ \mu_{B_e}B_e - \lambda H \\ a_1S_c + \mu_{S_c}S_c + \beta S_c H - b(S_a + E_a)(1 - p\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}}) - b_H(1-\nu_H)H \\ \gamma S_a H + rS_a\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}} + \mu_{S_a}S_a - a_1S_c - \delta H \end{bmatrix}.$$

We denote

$$F \;=\; [\frac{\partial \mathbb{F}_i}{\partial x_j}(X_0)] \quad \text{and} \quad V \;=\; [\frac{\partial \mathbb{V}_i}{\partial x_j}(X_0)]$$

where i, x_j are for the infected classes. Hence we have

$$F(X) \;=\; \begin{bmatrix} 0 & bp\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}} & b_H\nu_H+\beta S_c & b(S_a+E_a)p\ \frac{\partial}{\partial B_e}\left(\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}}\right) \\ 0 & 0 & \gamma S_a & rS_a\ \frac{\partial}{\partial B_e}\left(\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}}\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and evaluating $F(X)$ at X_0 gives

$$F(X_0) \;=\; \begin{bmatrix} 0 & 0 & b_H\nu_H+\beta S_c^* & \frac{bS_apC}{K_1(1+K_2)} \\ 0 & 0 & \gamma S_a^* & \frac{rS_aC}{K_1(1+K_2)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

and

$$V(X) \;=\; \begin{bmatrix} a_1+\mu_{E_c} & 0 & 0 & 0 \\ -a_1 & d+\mu_{E_a} & 0 & 0 \\ 0 & -d & \mu_H+\delta & 0 \\ 0 & 0 & -\lambda & \mu_{B_e} \end{bmatrix} \;=\; V(X_0)\ .$$

Next, taking the inverse of V we get

$$V^{-1}(X_0) = \begin{bmatrix} \frac{1}{(a_1+\mu_{E_c})} & 0 & 0 & 0 \\ \frac{a_1}{(a_1+\mu_{E_c})(d+\mu_{E_a})} & \frac{1}{(d+\mu_{E_a})} & 0 & 0 \\ \frac{a_1 d}{(a_1+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)} & \frac{d}{(d+\mu_{E_a})(\mu_H+\delta)} & \frac{1}{(\mu_H+\delta)} & 0 \\ \frac{a_1 d\lambda}{(a_1+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)\mu_{B_e}} & \frac{d\lambda}{(d+\mu_{E_a})(\mu_H+\delta)\mu_{B_e}} & \frac{\lambda}{(\mu_H+\delta)\mu_{B_e}} & \frac{1}{\mu_{B_e}} \end{bmatrix}.$$

Hence, we get the next generation matrix

$$(FV^{-1})(X_0) = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.5}$$

where,

$$A = \frac{a_1 d(b_H\nu_H + \beta S_c^*)}{(a_1+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)} + \frac{a_1 dbp\lambda S_a^* C}{K_1(1+K_2)(a_1+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)\mu_{B_e}}$$

$$B = \frac{d(b_H\nu_H + \beta S_c^*)}{(d+\mu_{E_a})(\mu_H+\delta)} + \frac{dbp\lambda S_a^* C}{K_1(1+K_2)(d+\mu_{E_a})(\mu_H+\delta)\mu_{B_e}}$$

$$C = \frac{(b_H\nu_H + \beta S_c^*)}{(\mu_H+\delta)} + \frac{bp\lambda S_a^* C}{K_1(1+K_2)(\mu_H+\delta)\mu_{B_e}}$$

$$D = \frac{bpS_a^* C}{\mu_{B_e} K_1(1+K_2)}$$

$$E = \frac{a_1 d\gamma S_a^*}{(a_1+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)} + \frac{a_1 dr\lambda S_a^* C}{K_1(1+K_2)(a_1+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)\mu_{B_e}}$$

$$F = \frac{d\gamma S_a^*}{(d+\mu_{E_a})(\mu_H+\delta)} + \frac{dr\lambda S_a^* C}{K_1(1+K_2)(d+\mu_{E_a})(\mu_H+\delta)\mu_{B_e}}$$

$$G = \frac{\gamma S_a^*}{(\mu_H+\delta)} + \frac{r\lambda S_a^* C}{K_1(1+K_2)(\mu_H+\delta)\mu_{B_e}}$$

$$H = \frac{rS_a^* C}{\mu_{B_e} K_1(1+K_2)} .$$

Next, we calculate the eigenvalues of $(FV^{-1})(X_0)$. From

$$\det \begin{vmatrix} A-\lambda & B & C & D \\ E & F-\lambda & G & H \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

we obtain

$$\lambda^2[(A-\lambda)(F-\lambda) - BE] = 0$$

and

$$\lambda^2[\lambda^2 - (A+F)\lambda + (AF-BE)] = 0$$

with the eigenvalues

$$\lambda_1 = 0,\ \lambda_2 = 0,\quad \lambda_3\ , \lambda_4 = \frac{(A+F) \pm \sqrt{(A+F)^2 - 4(AF-BE)}}{2} .$$

But, from the terms in the matrix (4.5), we have $AF - BE = 0$ and

$$\begin{aligned} A+F &= \frac{a_1 d(b_H \nu_H + \beta S_c^*)}{(a_1+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)} + \frac{a_1 dbp\lambda S_a^* C}{K_1(1+K_2)(a_1+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)\mu_{B_e}} \\ &\quad + \frac{d\gamma S_a^*}{(d+\mu_{E_a})(\mu_H+\delta)} + \frac{dr\lambda S_a^* C}{K_1(1+K_2)(d+\mu_{E_a})(\mu_H+\delta)\mu_{B_e}} \\ &> 0 \end{aligned}$$

Hence, the eigenvalues of $(FV^{-1})(X_0)$ are:

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

$$\lambda_4 = \frac{a_1 d(b_H \nu_H + \beta S_c^*)}{(a_1 + \mu_{E_c})(d + \mu_{E_a})(\mu_H + \delta)} + \frac{a_1 dbp\lambda S_a^* C}{K_1(1+K_2)(a_1 + \mu_{E_c})(d + \mu_{E_a})(\mu_H + \delta)\mu_{B_e}}$$
$$+ \frac{d\gamma S_a^*}{(d + \mu_{E_a})(\mu_H + \delta)} + \frac{dr\lambda S_a^* C}{K_1(1+K_2)(d + \mu_{E_a})(\mu_H + \delta)\mu_{B_e}} .$$

Thus, the spectral radius of FV^{-1} gives the basic reproduction number

$$\mathbb{R}_0 = \frac{a_1 d(b_H \nu_H + \beta S_c^*)}{(a_1 + \mu_{E_c})(d + \mu_{E_a})(\mu_H + \delta)} + \frac{a_1 dbp\lambda S_a^* C}{K_1(1+K_2)(a_1 + \mu_{E_c})(d + \mu_{E_a})(\mu_H + \delta)\mu_{B_e}}$$
$$+ \frac{d\gamma S_a^*}{(d + \mu_{E_a})(\mu_H + \delta)} + \frac{dr\lambda S_a^* C}{K_1(1+K_2)(d + \mu_{E_a})(\mu_H + \delta)\mu_{B_e}} \tag{4.6}$$

where S_c^* , S_a^* refers to the disease free equilibrium.

Hence, by using the results from [14], in (4.8) if $\mathbb{R}_0 < 1$, then the DFE is locally stable and if $\mathbb{R}_0 > 1$, then the DFE is unstable.

Recall that the basic reproduction number $\mathbb{R}_0$ is the average number of new cases of an infection caused by one typical infected individual during its entire infectious period in a completely susceptible population. Here in (4.8), it shows that the $\mathbb{R}_0$ of the simplified model is the expected number of secondary infections in E_c compartment produced by individuals initially in E_c compartment plus the expected number of secondary infections in E_a compartment produced by individuals initially in E_a compartment. Further, the terms in $\mathbb{R}_0$ expression shows all possible transmission routes as shown below.

$$\frac{ad(b_H\nu_H + \beta S_c^*)}{(a+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)} \quad - \quad \text{Calf infection from the vertical transmission and drinking colostrum and milk}$$

$$\frac{adbp\lambda S_a^* C}{K_1(1+K_2)(a+\mu_{E_c})(d+\mu_{E_a})(\mu_H+\delta)\mu_B} \quad - \quad \text{Calf infection from the bacteria in the adults environment}$$

$$\frac{d\gamma S_a^*}{(d+\mu_{E_a})(\mu_H+\delta)} \quad - \quad \text{Adults infection from the direct interactions with the shedding adults}$$

$$\frac{dr\lambda S_a^* C}{K_1(1+K_2)(d+\mu_{E_a})(\mu_H+\delta)\mu_B} \quad - \quad \text{Adults infection from the bacteria in the adults environment .}$$

To include a continuous bacterial removal term we can modify the simplified ODE system.

$$\begin{aligned}
\frac{dS_c}{dt} &= b(S_a+E_a)(1-p\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}}) + b_H(1-\nu_H)H - a_1S_c - \mu_{S_c}S_c - \beta S_cH \\
\frac{dE_c}{dt} &= b(S_a+E_a)p\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}} + b_H\nu_HH + \beta S_cH - a_1E_c - \mu_{E_c}E_c \\
\frac{dS_a}{dt} &= a_1S_c + \delta H - \gamma S_aH - rS_a\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}} - \mu_{S_a}S_a \\
\frac{dE_a}{dt} &= a_1E_c + \gamma S_aH + rS_a\frac{1-e^{-\frac{B_eC}{K_1}}}{1+K_2e^{-\frac{B_eC}{K_1}}} - \mu_{E_a}E_a - dE_a \\
\frac{dH}{dt} &= dE_a - \mu_HH - \delta H \\
\frac{dB_e}{dt} &= \lambda H - \mu_{B_e}B_e - RB_e
\end{aligned} \tag{4.7}$$

where R is the environmental bacteria removal rate and RB_e is the amount of bacteria removed per day from the environment.

Then, the reproduction number $\mathbb{R}_0$ corresponding to system (4.7) is as given below.

$$\begin{aligned}\mathbb{R}_0 = & \frac{a_1 d(b_H \nu_H + \beta S_c^*)}{(a_1 + \mu_{E_c})(d + \mu_{E_a})(\mu_H + \delta)} \\ & + \frac{a_1 dbp\lambda S_a^* C}{K_1(1+K_2)(a_1 + \mu_{E_c})(d + \mu_{E_a})(\mu_H + \delta)(\mu_{B_e} + R)} \\ & + \frac{d\gamma S_a^*}{(d + \mu_{E_a})(\mu_H + \delta)} + \frac{dr\lambda S_a^* C}{K_1(1+K_2)(d + \mu_{E_a})(\mu_H + \delta)(\mu_{B_e} + R)}\end{aligned} \tag{4.8}$$

where S_c^* , S_a^* refers to the disease free equilibrium.

Note that, in the terms of $\mathbb{R}_0$, the culling rate of the high shedders and the bacteria removal term appear in the denominator. Hence, we can assume that by increasing the culling rate of the high shedders and cleaning the bacteria from the environment can reduce the $\mathbb{R}_0$ value.

4.2.4 Stability of the disease free equilibrium of the main model

Similarly as in subsection 4.2.3, we use the next generation matrix method [2, 4, 11, 12, 14] to analyze the stability of the disease free equilibrium for the bigger model.

The equations for infected classes are:

$$\begin{aligned}\frac{dE_c}{dt} &= b(S_a + E_a)p\frac{1 - e^{-\frac{B_2 C}{K_1}}}{1 + K_2 e^{-\frac{B_2 C}{K_1}}} + b_{L_a}\nu_L L_a + b_{H_a}\nu_H H_a + \beta_1 S_c L_a + \beta_2 S_c H_a \\ &\quad - a_1 E_c - \mu_{E_c} E_c \\ \frac{dE_h}{dt} &= a_1 E_c + \gamma_1 S_h L_h + r_1 S_h \frac{1 - e^{-\frac{B_1 C}{K_1}}}{1 + K_2 e^{-\frac{B_1 C}{K_1}}} - a_2 E_h - d_1 E_h - \mu_{E_h} E_h\end{aligned}$$

$$
\begin{aligned}
\frac{dL_h}{dt} &= d_1E_h - a_2L_h - \mu_{L_h}L_h \\
\frac{dE_a}{dt} &= a_2E_h + \gamma_2S_aL_a + \gamma_3S_aH_a + r_2S_a\frac{1-e^{-\frac{B_2C}{K_1}}}{1+K_2e^{-\frac{B_2C}{K_1}}} - d_2E_a - \mu_{E_a}E_a \\
\frac{dL_a}{dt} &= a_2L_h + d_2E_a - d_3L_a - \mu_{L_a}L_a \\
\frac{dH_a}{dt} &= d_3L_a - \mu_{H_a}H_a - \delta H_a \\
\frac{dB_1}{dt} &= \lambda_1L_h - \mu_{B_1}B_1 \\
\frac{dB_2}{dt} &= \lambda_2L_a + \lambda_3H_a - \mu_{B_2}B_2 \ .
\end{aligned}
$$

The equations for uninfected classes are:

$$
\begin{aligned}
\frac{dS_c}{dt} &= b(S_a+E_a)(1-p\frac{1-e^{-\frac{B_2C}{K_1}}}{1+K_2e^{-\frac{B_2C}{K_1}}}) + b_{L_a}(1-\nu_L)L_a + b_{H_a}(1-\nu_H)H_a \\
&\quad - a_1S_c - \beta_1S_cL_a - \beta_2S_cH_a - \mu_{S_c}S_c \qquad (4.9) \\
\frac{dS_h}{dt} &= a_1S_c + \delta H_a - \gamma_1S_hL_h - r_1S_h\frac{1-e^{-\frac{B_1C}{K_1}}}{1+K_2e^{-\frac{B_1C}{K_1}}} - a_2S_h - \mu_{S_h}S_h \qquad (4.10) \\
\frac{dS_a}{dt} &= a_2S_h - \gamma_2S_aL_a - \gamma_3S_aH_a - r_2S_a\frac{1-e^{-\frac{B_2C}{K_1}}}{1+K_2e^{-\frac{B_2C}{K_1}}} - \mu_{S_a}S_a \ . \qquad (4.11)
\end{aligned}
$$

Disease free equilibrium:

Let X_0 = Disease Free Equilibrium (DFE). At the DFE we have

$$E_c = E_h = L_h = E_a = L_a = H_a = B_1 = B_2 = 0$$

and

$$\frac{dE_c}{dt} = \frac{dE_h}{dt} = \frac{dL_h}{dt} = \frac{dE_a}{dt} = \frac{dL_a}{dt} = \frac{dH_a}{dt} = \frac{dB_1}{dt} = \frac{dB_2}{dt} = \frac{dS_c}{dt} = \frac{dS_H}{dt} = \frac{dS_a}{dt} = 0 \ .$$

From (4.9) we have

$$b(S_a) \;=\; (a_1+\mu_{S_c})S_c \;\text{ giving }\; \frac{S_a}{S_c} \;=\; \frac{a_1+\mu_{S_c}}{b}\;.$$

From (4.10) we have

$$a_1(S_c) \;=\; (a_2+\mu_{S_h})S_h \;\text{ giving }\; \frac{S_c}{S_h} \;=\; \frac{a_2+\mu_{S_h}}{a_1}\;.$$

From (4.11) we have

$$a_2(S_h) \;=\; \mu_{S_a}S_a \;\text{ giving }\; \frac{S_h}{S_a} \;=\; \frac{\mu_{S_a}}{a_2}\;.$$

Hence, we need

$$(\frac{a_1+\mu_{S_c}}{b})(\frac{a_2+\mu_{S_h}}{a_1})(\frac{\mu_{S_a}}{a_2}) \;=\; 1\;,$$

which implies

$$(a_1+\mu_{S_c})(a_2+\mu_{S_h})(\mu_{S_a}) \;=\; ba_1a_2\;.$$

Hence, we have

$$\begin{aligned} X_0 \;&= (0,\;0,\;0,\;0,\;0,\;0,\;0,\;0,\;S_c^*,\;S_h^*,\;S_a^*) \\ &= (0,\;0,\;0,\;0,\;0,\;0,\;0,\;0,\;\frac{b}{a_1+\mu_{S_c}}S_a^*,\;\frac{\mu_{S_a}}{a_2}S_a^*,\;S_a^*)\;. \end{aligned}$$

Analyzing the stability of the DFE using the next generation matrix:

Our system can be written as:

$$x' = f_i(x) \;=\; \mathbb{F}_i(x)-\mathbb{V}_i(x) \;=\; \mathbb{F}_i(x)-(\mathbb{V}_i^-(x)-\mathbb{V}_i^+(x))$$

with

$\mathbb{F}_i(x)$ = Rate of appearance of new infections in compartment i.

$\mathbb{V}_i^+(x)$ = Rate of transfer of individuals into compartment i by all other means.

$\mathbb{V}_i^-(x)$ = Rate of transfer of individuals out of compartment i by all other means.

Then,

$$F \;=\; [\frac{\partial \mathbb{F}_i}{\partial x_j}(X_0)] \text{ and } V \;=\; [\frac{\partial \mathbb{V}_i}{\partial x_j}(X_0)] \;;$$

where we only use i, x_j for the infected classes.

Hence, for the above system of equations,

$$\mathbb{F} \;=\; \begin{bmatrix} b(S_a + E_a)pf(B_2) + b_{L_a}\nu_L L_a + b_{H_a}\nu_H H_a + \beta_1 S_c L_a + \beta_2 S_c H_a \\ \gamma_1 S_h L_h + r_1 S_h f(B_1) \\ 0 \\ \gamma_2 S_a L_a + \gamma_3 S_a H_a + r_2 S_a f(B_2) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.12}$$

and

$$\mathbb{V} = \begin{bmatrix} a_1E_c + \mu_{E_c}E_c \\ a_2E_h + d_1E_h + \mu_{E_h}E_h - a_1E_c \\ a_2L_h + \mu_{L_h}L_h - d_1E_h \\ d_2E_a + \mu_{E_a}E_a - a_2E_h \\ d_3L_a + \mu_{L_a}L_a - a_2L_h - d_2E_a \\ \mu_{H_a}H_a + \delta H_a - d_3L_a \\ \mu_{B_1}B_1 - \lambda_1 L_h \\ \mu_{B_2}B_2 - \lambda_2 L_a - \lambda_3 H_a \\ a_1S_c + \beta_1 S_c L_a + \beta_2 S_c H_a + \mu_{S_c}S_c \\ -b(S_a + E_a)(1 - pf(B_2)) - b_{L_a}(1 - \nu_L)L_a - b_{H_a}(1 - \nu_H)H_a \\ \gamma_1 S_h L_h + r_1 S_h f(B_1) + a_2 S_h + \mu_{S_h}S_h - a_1 S_c - \delta H_a \\ \gamma_2 S_a L_a + \gamma_3 S_a H_a + r_2 S_a f(B_2) + \mu_{S_a}S_a - a_2 S_h \end{bmatrix} \tag{4.13}$$

where,

$$f(B_1) = \frac{1 - e^{-\frac{B_1 C}{K_1}}}{1 + K_2 e^{-\frac{B_1 C}{K_1}}}$$

and

$$f(B_2) = \frac{1 - e^{-\frac{B_2 C}{K_1}}}{1 + K_2 e^{-\frac{B_2 C}{K_1}}} .$$

$$F(X) =$$

$$\begin{bmatrix}
0 & 0 & 0 & bpf(B_2) & b_{L_a}\nu_L + \beta_1 S_c & b_{H_a}\nu_H + \beta_2 S_c & 0 & b(S_a + E_a)p\ \frac{\partial(f(B_2))}{\partial B_2} \\
0 & 0 & \gamma_1 S_h & 0 & 0 & 0 & r_1 S_h\ \frac{\partial(f(B_1))}{\partial B_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_2 S_a & \gamma_3 S_a & 0 & r_2 S_a\ \frac{\partial(f(B_2))}{\partial B_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Note that

$$\frac{\partial\left(f(B_1)\right)}{\partial B_1} = \frac{(1+K_2)\ e^{-\frac{B_1 C}{K_1}}}{K_1\ \left(1+K_2 e^{-\frac{B_1 C}{K_1}}\right)^2}\quad,\quad \frac{\partial\left(f(B_2)\right)}{\partial B_2} = \frac{(1+K_2)\ e^{-\frac{B_2 C}{K_1}}}{K_1\ \left(1+K_2 e^{-\frac{B_2 C}{K_1}}\right)^2}$$

and at X_0

$$f(B_2) = 0\quad,\quad \frac{\partial\left(f(B_1)\right)}{\partial B_1} \quad = \quad \frac{C}{K_1\ (1+K_2)} \quad = \quad \frac{\partial\left(f(B_2)\right)}{\partial B_2}$$

and we have

$$F(X_0) = \begin{bmatrix} 0 & 0 & 0 & 0 & b_{L_a}\nu_L + \beta_1 S_c^* & b_{H_a}\nu_H + \beta_2 S_c^* & 0 & \frac{bS_a^* pC}{K_1(1+K_2)} \\ 0 & 0 & \gamma_1 S_h^* & 0 & 0 & 0 & \frac{r_1 S_h^* C}{K_1(1+K_2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2 S_a^* & \gamma_3 S_a^* & 0 & \frac{r_2 S_a^* C}{K_1(1+K_2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We denote by

$$F(X_0) = \begin{bmatrix} 0 & 0 & 0 & 0 & P & Q & 0 & R \\ 0 & 0 & S & 0 & 0 & 0 & T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & U & V & 0 & W \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where,

$$P = b_{L_a}\nu_L + \beta_1 S_c^* \qquad T = \frac{r_1 S_h^* C}{K_1(1+K_2)}$$
$$Q = b_{H_a}\nu_H + \beta_2 S_c^* \qquad U = \gamma_2 S_a^*$$
$$R = \frac{bS_a^* pC}{K_1(1+K_2)} \qquad V = \gamma_3 S_a^*$$
$$S = \gamma_1 S_h^* \qquad W = \frac{r_2 S_a^* C}{K_1(1+K_2)}$$

and

$V(X_0) =$

$$\begin{bmatrix} a_1+\mu_{E_c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1 & a_2+d_1+\mu_{E_h} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -d_1 & a_2+\mu_{L_h} & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & d_2+\mu_{E_a} & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & -d_2 & d_3+\mu_{L_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_3 & \mu_{H_a}+\delta & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 & \mu_{B_1} & 0 \\ 0 & 0 & 0 & 0 & -\lambda_2 & -\lambda_3 & 0 & \mu_{B_2} \end{bmatrix}$$

Hence, we get

$V^{-1}(X_0) =$

$$\begin{bmatrix} \frac{1}{(a_1+\mu_{E_c})} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A & \frac{1}{(a_2+d_1+\mu_{E_h})} & 0 & 0 & 0 & 0 & 0 & 0 \\ B & C & \frac{1}{(a_2+\mu_{L_h})} & 0 & 0 & 0 & 0 & 0 \\ D & E & 0 & \frac{1}{(d_2+\mu_{E_a})} & 0 & 0 & 0 & 0 \\ F & G & H & I & \frac{1}{(d_3+\mu_{L_a})} & 0 & 0 & 0 \\ J & K & L & M & N & \frac{1}{(\mu_{H_a}+\delta)} & 0 & 0 \\ P & Q & R & 0 & 0 & 0 & \frac{1}{(\mu_{B_1})} & 0 \\ S & T & U & V & W & X & 0 & \frac{1}{(\mu_{B_2})} \end{bmatrix} \quad (4.14)$$

where,

$$\begin{aligned} A &= \frac{a_1}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})} \\ B &= \frac{a_1 d_1}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})} \end{aligned}$$

$$
\begin{aligned}
C &= \frac{d_1}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})} \\
D &= \frac{a_1a_2}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(d_2+\mu_{E_a})} \\
E &= \frac{a_2}{(a_2+d_1+\mu_{E_h})(d_2+\mu_{E_a})} \\
F &= \frac{a_1a_2\left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})} \\
G &= \frac{a_2\left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})} \\
H &= \frac{a_2}{(a_2+\mu_{L_h})(d_3+\mu_{L_a})} \\
I &= \frac{d_2}{(d_2+\mu_{E_a})(d_3+\mu_{L_a})} \\
J &= \frac{a_1a_2d_3\left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\
K &= \frac{a_2d_3\left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\
L &= \frac{a_2d_3}{(a_2+\mu_{L_h})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\
M &= \frac{d_2d_3}{(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\
N &= \frac{d_3}{(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\
P &= \frac{a_1d_1\lambda_1}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})\mu_{B_1}} \\
Q &= \frac{d_1\lambda_1}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})\mu_{B_1}} \\
R &= \frac{\lambda_1}{(a_2+\mu_{L_h})\mu_{B_1}} \\
S &= \frac{a_1a_2\left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)\left(\lambda_2(\mu_{H_a}+\delta)+d_3\lambda_3\right)}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}} \\
T &= \frac{a_2\left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)\left(\lambda_2(\mu_{H_a}+\delta)+d_3\lambda_3\right)}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}} \\
U &= \frac{a_2(\lambda_2(\mu_{H_a}+\delta)+d_3\lambda_3)}{(a_2+\mu_{L_h})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}} \\
V &= \frac{d_2(\lambda_2(\mu_{H_a}+\delta)+d_3\lambda_3)}{(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}} \\
W &= \frac{(\lambda_2(\mu_{H_a}+\delta)+d_3\lambda_3)}{(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}}
\end{aligned}
$$

$$X = \frac{\lambda_3}{(\mu_{H_a}+\delta)\mu_{B_2}}.$$

Hence, we obtain the next generation matrix

$$(FV^{-1})(X_0) = \begin{bmatrix} A & B & C & D & E & F & 0 & G \\ H & I & J & 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L & M & N & P & Q & R & 0 & S \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.15}$$

where,

$$\begin{aligned} A =& \frac{a_1 a_2 \left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)\left(b_{L_a}\nu_L+\beta_1 S_c^*\right)}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})} \\ &+ \frac{a_1 a_2 d_3 \left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)\left(b_{H_a}\nu_H+\beta_2 S_c^*\right)}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\ &+ \frac{a_1 a_2 \left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)\left(\lambda_2(\mu_{H_a}+\delta)+d_3\lambda_3\right)bS_a^* pC}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}K_1(1+K_2)} \end{aligned}$$

$$\begin{aligned} B =& \frac{a_2 \left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)\left(b_{L_a}\nu_L+\beta_1 S_c^*\right)}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})} \\ &+ \frac{a_2 d_3 \left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)\left(b_{H_a}\nu_H+\beta_2 S_c^*\right)}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\ &+ \frac{a_2 \left(d_1(d_2+\mu_{E_a})+d_2(a_2+\mu_{L_h})\right)\left(\lambda_2(\mu_{H_a}+\delta)+d_3\lambda_3\right)bS_a^* pC}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}K_1(1+K_2)} \end{aligned}$$

$$C = \frac{a_2(b_{L_a}\nu_L + \beta_1 S_c^*)}{(a_2 + \mu_{L_h})(d_3 + \mu_{L_a})} + \frac{a_2 d_3(b_{H_a}\nu_H + \beta_2 S_c^*)}{(a_2 + \mu_{L_h})(d_3 + \mu_{L_a})(\mu_{H_a} + \delta)} + \frac{a_2(\lambda_2(\mu_{H_a} + \delta) + d_3\lambda_3)bS_a^* pC}{(a_2 + \mu_{L_h})(d_3 + \mu_{L_a})(\mu_{H_a} + \delta)\mu_{B_2}K_1(1 + K_2)}$$

$$D = \frac{d_2(b_{L_a}\nu_L + \beta_1 S_c^*)}{(d_2 + \mu_{E_a})(d_3 + \mu_{L_a})} + \frac{d_2 d_3(b_{H_a}\nu_H + \beta_2 S_c^*)}{(d_2 + \mu_{E_a})(d_3 + \mu_{L_a})(\mu_{H_a} + \delta)} + \frac{d_2(\lambda_2(\mu_{H_a} + \delta) + d_3\lambda_3)bS_a^* pC}{(d_2 + \mu_{E_a})(d_3 + \mu_{L_a})(\mu_{H_a} + \delta)\mu_{B_2}K_1(1 + K_2)}$$

$$E = \frac{(b_{L_a}\nu_L + \beta_1 S_c^*)}{(d_3 + \mu_{L_a})} + \frac{d_3(b_{H_a}\nu_H + \beta_2 S_c^*)}{(d_3 + \mu_{L_a})(\mu_{H_a} + \delta)} + \frac{(\lambda_2(\mu_{H_a} + \delta) + d_3\lambda_3)bS_a^* pC}{(d_3 + \mu_{L_a})(\mu_{H_a} + \delta)\mu_{B_2}K_1(1 + K_2)}$$

$$F = \frac{(b_{H_a}\nu_H + \beta_2 S_c^*)}{(\mu_{H_a} + \delta)} + \frac{\lambda_3 bS_a^* pC}{(\mu_{H_a} + \delta)\mu_{B_2}K_1(1 + K_2)}$$

$$G = \frac{bS_a^* pC}{(\mu_{B_2})K_1(1 + K_2)}$$

$$H = \frac{a_1 d_1 \gamma_1 S_h^*}{(a_1 + \mu_{E_c})(a_2 + d_1 + \mu_{E_h})(a_2 + \mu_{L_h})} + \frac{a_1 d_1 \lambda_1 r_1 S_h C}{(a_1 + \mu_{E_c})(a_2 + d_1 + \mu_{E_h})(a_2 + \mu_{L_h})\mu_{B_1}K_1(1 + K_2)}$$

$$I = \frac{d_1 \gamma_1 S_h^*}{(a_2 + d_1 + \mu_{E_h})(a_2 + \mu_{L_h})} + \frac{d_1 \lambda_1 r_1 S_h^* C}{(a_2 + d_1 + \mu_{E_h})(a_2 + \mu_{L_h})\mu_{B_1}K_1(1 + K_2)}$$

$$J = \frac{\gamma_1 S_h^*}{(a_2 + \mu_{L_h})} + \frac{\lambda_1 r_1 S_h^* C}{(a_2 + \mu_{L_h})\mu_{B_1}K_1(1 + K_2)}$$

$$K = \frac{r_1 S_h^* C}{\mu_{B_1}K_1(1 + K_2)}$$

$$
\begin{aligned}
L =& \frac{a_1 a_2 \gamma_2 S_a^* \left(d_1(d_2+\mu_{E_a}) + d_2(a_2+\mu_{L_h})\right)}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})} \\
&+ \frac{a_1 a_2 d_3 \gamma_3 S_a^* \left(d_1(d_2+\mu_{E_a}) + d_2(a_2+\mu_{L_h})\right)}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\
&+ \frac{a_1 a_2 r_2 S_a^* C \left(d_1(d_2+\mu_{E_a}) + d_2(a_2+\mu_{L_h})\right)\left(\lambda_2(\mu_{H_a}+\delta) + d_3\lambda_3\right)}{(a_1+\mu_{E_c})(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}K_1(1+K_2)}
\end{aligned}
$$

$$
\begin{aligned}
M =& \frac{a_2 \gamma_2 S_a^* \left(d_1(d_2+\mu_{E_a}) + d_2(a_2+\mu_{L_h})\right)}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})} \\
&+ \frac{a_2 d_3 \gamma_3 S_a^* \left(d_1(d_2+\mu_{E_a}) + d_2(a_2+\mu_{L_h})\right)}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\
&+ \frac{a_2 r_2 S_a^* C \left(d_1(d_2+\mu_{E_a}) + d_2(a_2+\mu_{L_h})\right)\left(\lambda_2(\mu_{H_a}+\delta) + d_3\lambda_3\right)}{(a_2+d_1+\mu_{E_h})(a_2+\mu_{L_h})(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}K_1(1+K_2)}
\end{aligned}
$$

$$
\begin{aligned}
N =& \frac{a_2 \gamma_2 S_a^*}{(a_2+\mu_{L_h})(d_3+\mu_{L_a})} + \frac{a_2 d_3 \gamma_3 S_a^*}{(a_2+\mu_{L_h})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\
&+ \frac{a_2 r_2 S_a^* C(\lambda_2(\mu_{H_a}+\delta) + d_3\lambda_3)}{(a_2+\mu_{L_h})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}K_1(1+K_2)}
\end{aligned}
$$

$$
\begin{aligned}
P =& \frac{d_2 \gamma_2 S_a^*}{(d_2+\mu_{E_a})(d_3+\mu_{L_a})} + \frac{d_2 d_3 \gamma_3 S_a^*}{(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} \\
&+ \frac{d_2 r_2 S_a^* C(\lambda_2(\mu_{H_a}+\delta) + d_3\lambda_3)}{(d_2+\mu_{E_a})(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}K_1(1+K_2)}
\end{aligned}
$$

$$
Q = \frac{\gamma_2 S_a^*}{(d_3+\mu_{L_a})} + \frac{d_3 \gamma_3 S_a^*}{(d_3+\mu_{L_a})(\mu_{H_a}+\delta)} + \frac{r_2 S_a^* C(\lambda_2(\mu_{H_a}+\delta) + d_3\lambda_3)}{(d_3+\mu_{L_a})(\mu_{H_a}+\delta)\mu_{B_2}K_1(1+K_2)}
$$

$$
R = \frac{\gamma_3 S_a^*}{(\mu_{H_a}+\delta)} + \frac{\lambda_3 r_2 S_a^* C}{(\mu_{H_a}+\delta)\mu_{B_2}K_1(1+K_2)}
$$

$$
S = \frac{r_2 S_a^* C}{(\mu_{B_2})K_1(1+K_2)}
$$

The next generation matrix (4.15) has 3 nonzero eigenvalues and it is too complicated to calculate the dominant eigenvalue, which would give the reproduction number ($\mathbb{R}_0$) explicitly. Hence, later we calculate $\mathbb{R}_0$ using specific values for the variables and the parameters.

4.3 Numerical results

For the numerical simulations we have used the following initial values and parameter values given below.

As seen in Tables 4.1 and 4.2, we used the same initial values for the total number of calves, heifers, adults and same initial prevalence for the infected animals as in [40]. The birth rates, death rates, vertical transmission rates, disease progression rates were calculated using the parameter values in [40]. The contact rates β_1 and β_2 were calculated using the parameter values in [37, 40]. Parameters related to the environmental transmission were decided on the knowledge of farm practice as described in section 4.2.

Table 4.1: Initial number of animals in each compartment

	Susceptible	Exposed	Low–shedding	High–shedding
Calves	130	70	0	0
Heifers	520	248	32	0
Adults	650	250	80	20

Table 4.2: Initial prevalence of the disease in the system

	Susceptible	Exposed	Low–shedding	High–shedding
Calves	65%	35%	0%	0%
Heifers	65%	31%	4%	0%
Adults	65%	25%	8%	2%

Amount of initial MAP in the two environments

$B_1 = 0.2 \times 10^8$

$B_2 = 590 \times 10^8$

The parameter values we have used in the numerical simulations are given below:

Birth rates

$b = b_{La} = b_{Ha} = 0.00127$

Death rates

$\mu_{S_c} = \mu_{E_c} = 0.00028$

$\mu_{S_h} = \mu_{E_h} = \mu_{L_h} = 0.000063$

$\mu_{S_a} = \mu_{E_a} = \mu_{L_a} = \mu_{H_a} = 0.0012$

$\mu_{B_1} = \mu_{B_1} = 0.0027$

Vertical transmission rates

$\nu_L = 0 \qquad \nu_H = 0.22$

Age transfer rates

$a_1 = 0.0168 \qquad a_2 = 0.00151$

Disease progression rates

$d_1 = 0.0014 \qquad d_2 = 0.0014 \qquad d_3 = 0.00078$

Contact rates

$\beta_1 = 0.0014 \qquad \beta_2 = 0.001066$

$\gamma_1 = 0.0000024 \qquad \gamma_2 = 0.0000012 \qquad \gamma_3 = 0.0000018$

$p = 0.3 \quad r_1 = 0.06 \quad r_2 = 0.06$

$\lambda_1 = 0.007 \quad \lambda_2 = 0.007 \quad \lambda_3 = 29.5$

First to see the dynamics of the disease with no testing or culling we have taken $\delta = 0$ in the numerical simulations given in Figures 4.6 – 4.8 and Table 4.3.

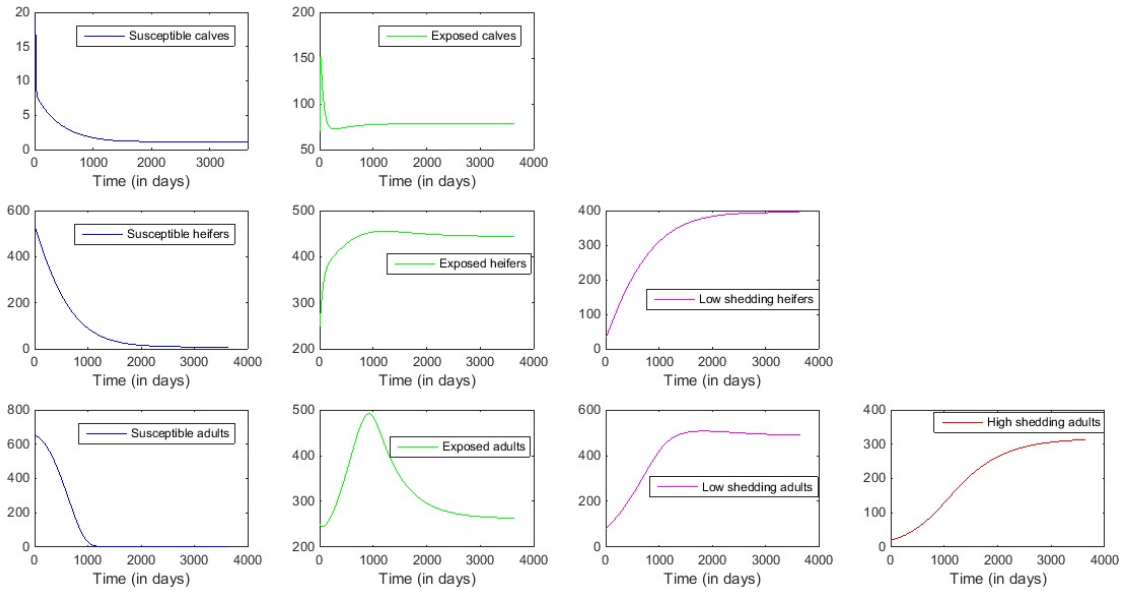


Figure 4.6: Dynamics of the animals in each compartment with no testing or culling

The dynamics of the total animals in each class with no testing or culling is given in the Figure 4.6. From the numerical results, we can see that the population approaches an equilibrium. With no testing or culling, in 1000 days, the number of susceptible cows approach zero and in 2000 days, susceptible calves and heifers also are at very low level. At the end of 10 years there will be only one susceptible calf and 8 susceptible heifers in the system. On the other hand, diseased animals increase over time and at the end of 10 years there will be 395 low shedding heifers, 489 low shedding adults and 313 high shedding adults.

The dynamics of the total animals in each disease class with no culling is given in Figure 4.7. It is clear that over time the total susceptible animals are very low and the total exposed, low shedding and high shedding animals increase in the system. At the end of 10 years there will be only 9 susceptible animals left in the system. There will be about 785 exposed animals, 884 low shedding animals and 313 animals. Hence, only 0.45 % of the total animals will be uninfected with the Johne's disease in the system at the end of 10 years.

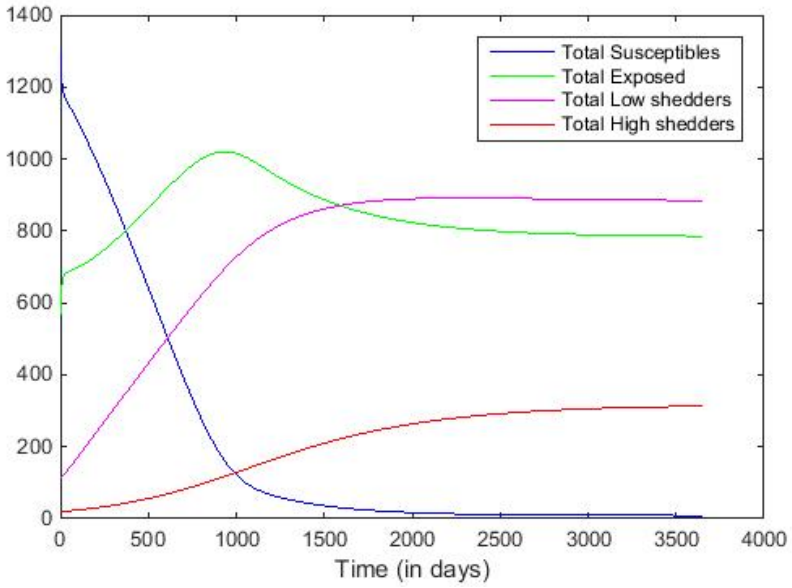


Figure 4.7: Dynamics of the total animals in each disease class with no testing or culling

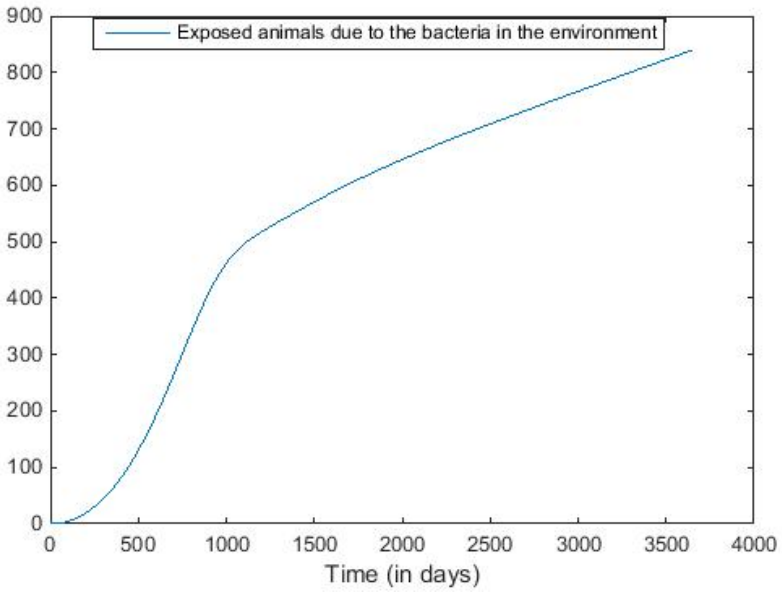


Figure 4.8: Number of exposed cows from the bacteria in the environment when $p = 0.3, r_1 = 0.06, r_2 = 0.06$ with no testing or culling

The number of exposed cows from the bacteria in the environment when $p = 0.3, r_1 = 0.06, r_2 = 0.06$ without testing or culling is given in the Figure 4.8. It can be seen that at the end of 10 years, about 840 animals get infected from the bacteria in the environment. Hence, the bacteria surviving in the environment is also an important route in Johne's disease transmission.

Table 4.3: Prevalence of the disease at the end of 10 years without testing or culling

	Susceptible	Exposed	Low–shedding	High–shedding
Calves	1%	99%	0%	0%
Heifers	1%	52%	47%	0%
Adults	0%	25%	46%	29%

With no testing or culling, the prevalence of the disease at the end of 10 years is given in Table 4.3. Compared to Table 4.2, it is clear that over 10 years the disease prevalence reaches a very higher value. Starting with 35% of exposed calves, at the end of 10 years, 99% of the calves will be exposed. Starting with 31% exposed heifers, at the end of 10 years, there will be 52% exposed heifers. The exposed adults prevalence at the end of 10 years will be as same as the initial prevalence of the exposed adults. The low shedding heifers goes from 4% to 47% at the end of 10 years. Starting from 8% low shedding adults, they become 46% of the total adults. Starting from 2%, there will be 29% of high shedding adults among total adults, after 10 years.

Next to see the dynamics of the disease in the system with testing and culling, we first have considered the continuous daily culling with $\delta = 0.01$ corresponding to continuous testing. The numerical results are given in Figures 4.9 – 4.11. Later we improve this by considering more realistic discrete testing and culling and the numerical results are given in Figures 4.12– 4.14 and the Table 4.4.

The dynamics of the total animals in each class with continuous culling $\delta = 0.01$ is given in the Figures 4.9. Over time, the population approaches an equilibrium. Compared to Figure 4.6, here we can see that it takes more time for the susceptible

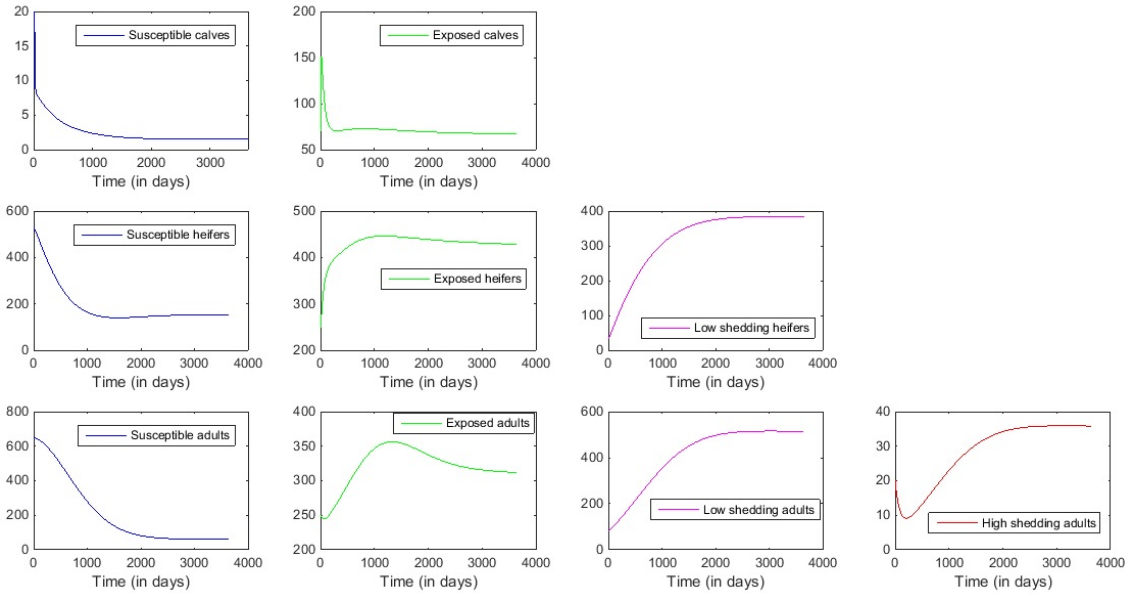


Figure 4.9: Dynamics of the animals in each compartment with continuous culling

adults to reach a low level, yet they do not reach zero and there will be 60 susceptible adults in the system. Due to the continuous culling, at the end of 10 years there will be only 36 high shedders and the culled high shedders are replaced by the susceptible heifers. Hence with continuous culling there will be 154 susceptible heifers at the the end of 10 years. There will be more exposed adults in this case than with the no culling situation. But, the susceptible calves, exposed calves, exposed heifers and low shedding heifers shows similar dynamics in both cases. But, it takes a longer time for the low shedding adults to reach its equilibrium value than in the no culling case.

The dynamics of the total animals in each disease class with continuous culling is given in Figure 4.10. Even with continuous testing and culling, the total susceptible animals still decrease over the time and diseased animals increase. But, at the end of 10 years still there will be about 215 susceptible animals in the system. Compared to the no culling case, this is good improvement.

The number of exposed cows from the bacteria in the environment when $p = 0.3, r_1 = 0.06, r_2 = 0.06$ is given in the Figure 4.11. This graph shows that over 10 years about 469 animals get infected through the bacteria in the environment.

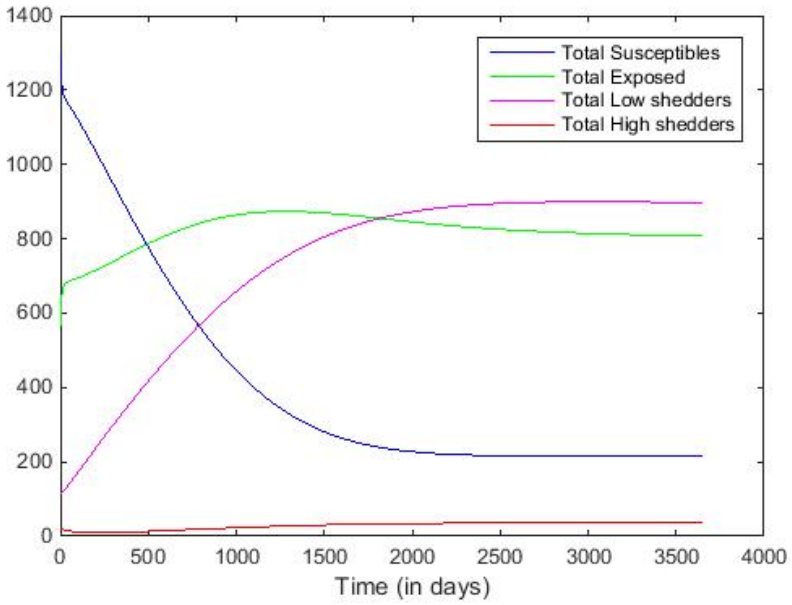


Figure 4.10: Dynamics of the total animals in each disease class with continuous culling

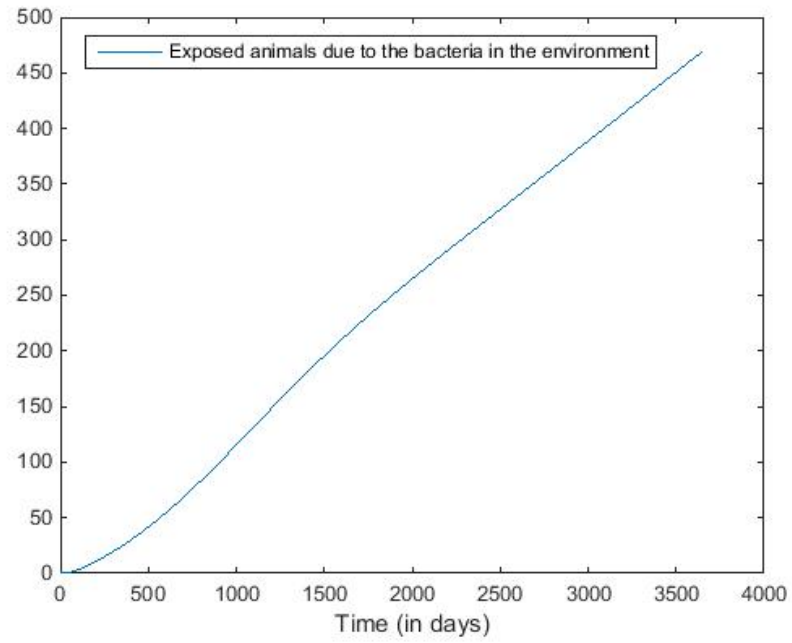


Figure 4.11: Number of exposed cows from the bacteria in the environment when $p = 0.3, r_1 = 0.06, r_2 = 0.06$ with continuous culling

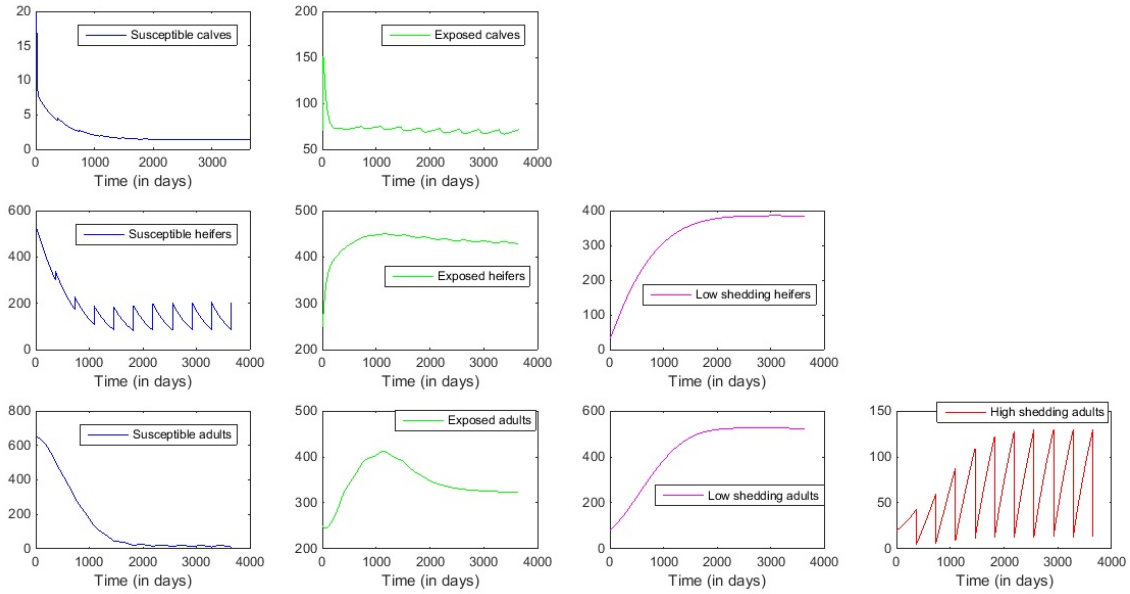


Figure 4.12: Dynamics of the animals in each compartment with discrete culling

Next, to simulate more realistic dynamics in a dairy farm we run numerical simulations with discrete culling. Here, we assume the testing occurs once a year and depending on the sensitivity of the test, some proportion of animals with positive test results are removed from the system. From [43], the sensitivity of the EVELISA test is 97.1%. Hence, for an approximation we remove 90% of the high shedders at the end of each year and add that number of animals into the susceptible heifers, and then the ODE simulation is restarted. The simulation results are given in Figures 4.12–4.14.

The dynamics of the animals in each compartment with discrete culling is given in the Figure 4.12. With more realistic discrete culling, we can see that over the time the population approaches an equilibrium. Compared to Figure 4.6, with discrete culling, it takes more time for the susceptible adults to approach a low number, and the susceptible adults do not approach zero and they will approach about 11, approximately in 2000 days. But, with no culling, susceptible adults become zero in about 1000 days. With discrete annual culling, but just before testing and culling at the end of 10th year, there will be about 129 high shedders. This is a better case

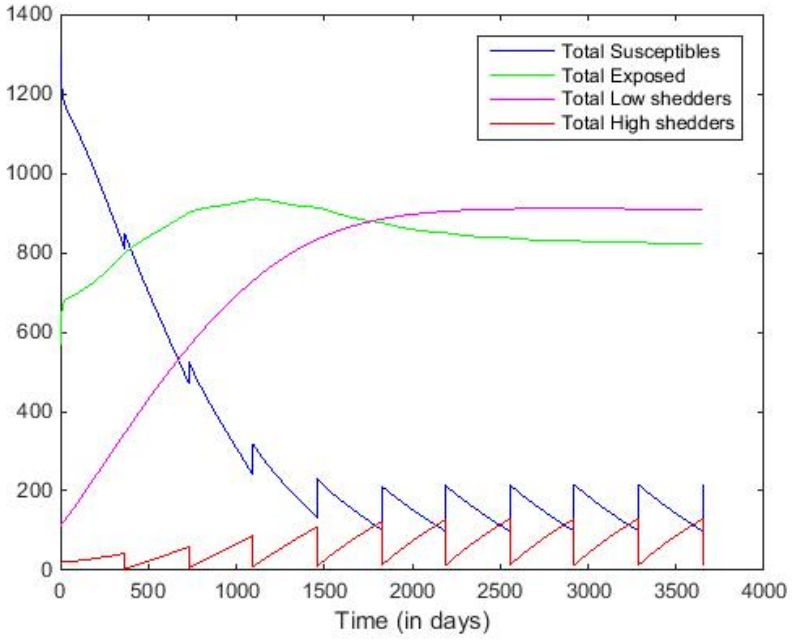


Figure 4.13: Dynamics of the total animals in each disease class with discrete culling

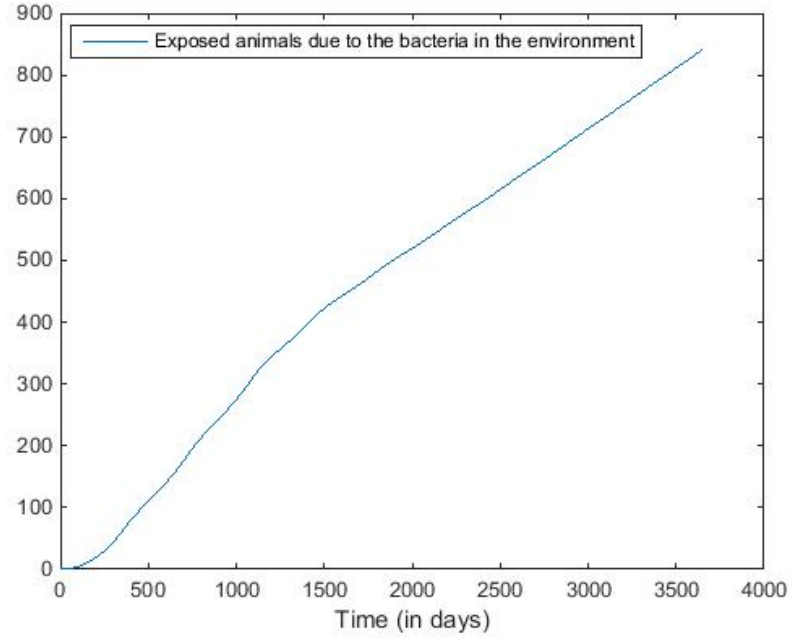


Figure 4.14: Number of exposed cows from the bacteria in the environment with discrete culling when $p = 0.3, r_1 = 0.06, r_2 = 0.06$

compared to having 313 high shedders at the end of 10 years in the no culling case. The removed high shedders are replaced by the susceptible heifers, and just before this new replacements at the end of the 10th year, there will be about 87 susceptible heifers in the system. Compared to having only 8 susceptible heifers in the system with no culling, discrete culling gives good improvement. Also, with the discrete culling, the equilibrium level of the exposed adults over the time is higher than the equilibrium level of the exposed adults in the no culling case. But, similar to the case with continuous culling, the susceptible calves, exposed calves, exposed heifers and low shedding heifers shows similar dynamics as in no culling case. But, it takes a longer time for the low shedding adults to reach its equilibrium value than in the no culling case. Further, here we can clearly see that the decrease in high shedders and the corresponding increase in susceptible heifers.

The dynamics of the total animals in each disease class with discrete culling is given in the Figure 4.13. Over the time even with annual testing and culling, we can see that the total susceptible animals decrease and the infected animals increase. At the end of 10 years, just before testing and culling and the new replacements, there will be about 99 susceptible animals left in the system. There will be about 823 exposed animals, 907 low shedding animals and 129 high shedding animals. Hence, with the annual testing and culling, about 5.05 % of the total animals will be uninfected with the Johne's disease in the system at the end of 10 years. This is about 11% increase in the susceptible animals from no culling case.

Table 4.4: Prevalence of the disease at the end of 10 years with annual culling

	Susceptible	Exposed	Low–shedding	High–shedding
Calves	1%	99%	0%	0%
Heifers	20%	42%	38%	0%
Adults	1%	37%	60%	2%

With the annual testing and culling, the number of exposed cows from the bacteria in the environment when $p = 0.3, r_1 = 0.06, r_2 = 0.06$ is given in the Figure 4.14.

From this figure we can see that even with annual testing and culling, still about 840 animals get infected from the bacteria in the environment.

With discrete culling, the prevalence of the disease at the end of 10 years is given in Table 4.4. Compared to Table 4.2, from Table 4.4, it is clear that over 10 years even with testing and culling, the disease prevalence reaches a higher value. Starting with 35% exposed calves, at the end of 10 years, 99% of the calves will be exposed. Starting with 31% exposed heifers, at the end of 10 years there will be 42% exposed heifers. Starting with 25% exposed adults, at the end of 10 years there will be about 37% of exposed adults. The low shedding heifers go from 4% to 38% in 10 years. Starting from 8% low shedding adults, they become 60% among total adults. The high shedding adults stay 2% of the total adults. But, compared to the results in Table 4.3 for the no culling case, discrete culling decreases the prevalence of the exposed heifers, low shedding heifers and high shedding adults. Further the discrete testing and culling increases the uninfected heifers and adults in the system compared to the no culling case.

Table 4.5: Comparison of the number of animals in each compartment at the end of 10 years without culling and with annual culling

Compartment	Without culling	With annual testing & culling
S_c	1	1
E_c	78	71
S_h	8	204
E_h	444	429
L_h	395	384
S_a	0	11
E_a	263	323
L_a	489	523
H_a	313	13

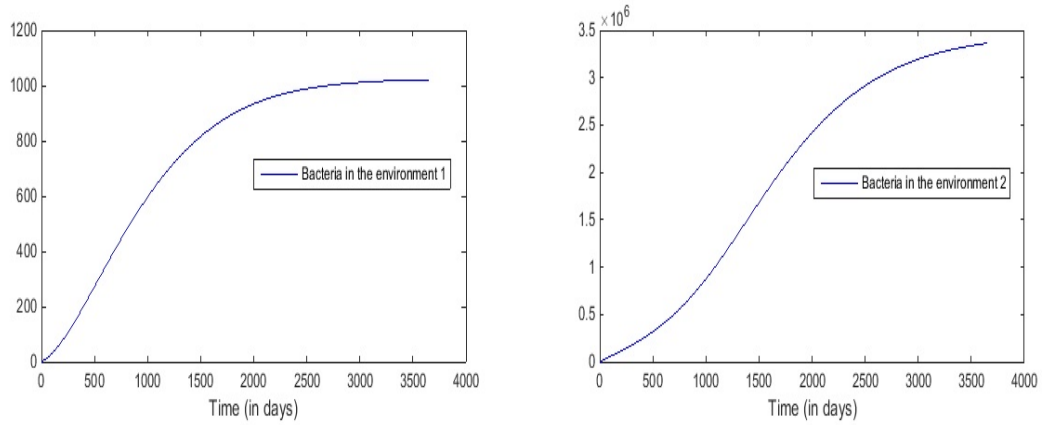


Figure 4.15: Dynamics of the bacteria in the two environments with no culling

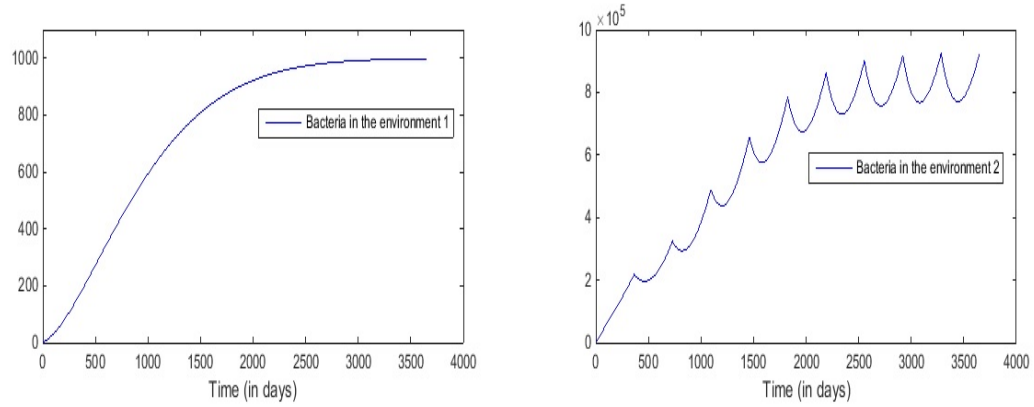


Figure 4.16: Dynamics of the bacteria in the two environments with discrete culling

The effect of culling is given in the Table 4.5. From Table 4.5 we can see that testing and culling can increase the susceptible animals in each susceptible animal compartment and reduce the infected animals in the all diseased compartments except exposed adults and low shedding adults. At the end of 10 years after the annual testing and culling, there will be 216 susceptible animals, 823 exposed animals, 907 low shedding animals and 13 high shedding animals. Hence, about 11.03% of the total animals will be uninfected at the end of the 10 years. Hence, it is clear that with annual testing and culling, one can reduce the prevalence of the diseased animals in the system and increase the uninfected animals in the system by about 25%.

From Figures 4.8, 4.11 and 4.14, it is clear that over the time of 10 years many animals can be infected due to the bacteria in the surrounding environment. Hence, more studies should be conducted to understand the disease transmission through the bacteria in the environment and to estimate better values for the parameters related to the environmental transmission of MAP.

The dynamics of the bacteria in the two environments without and with culling is give in Figures 4.15 and 4.16 respectively.

From Figure 4.15, we can see that over the 10 years, with no testing or culling MAP in the heifers environment reaches an equilibrium of 1×10^{11}. Further, the MAP in the adults environment reaches about 3.4×10^{14} at the end of 10 years. From Figure 4.16, we can see that with annual testing and culling of the high shedders, MAP in the heifers environment reaches an equilibrium about 1×10^{11}. Further, the MAP in the adults environment reaches about 9.2×10^{13} at the end of 10 years. Hence, compared to the Figure 4.15, from Figure 4.16 we see that the annual removal of the 90% of high shedders reduces the total MAP in the adults environment.

Hence, using the numerical simulation results given in Figures 4.6 – 4.16 and Tables 4.3 – 4.5 we can see that annual testing and culling can reduce the prevalence of the Johne's disease in the system.

Next, using the parameter values in the model, we calculate $\mathbb{R}_0$ corresponding to the next generation matrix (4.15) using $S_a^* = 650$, our initial value for S_a. With no culling ($\delta = 0$),

$$\mathbb{R}_0 = 21.5081$$

and with the continuous culling ($\delta = 0.01$),

$$\mathbb{R}_0 = 14.2318.$$

Hence, we can see that using testing and culling we can reduce the disease progression in the system. Also, it is clear that even with the removal of the high shedders, the disease can persist in the system.

4.4 Conclusions

A deterministic model for the Johne's disease was developed considering possible disease transmission through the environment. First, a more realistic model was developed and later it was simplified to study the stability of the disease free equilibrium in the system. Compared with previous work [5, 33], this work has a novel way of representing the transmission from the environment by using a more realistic approximation of bacteria in feces in the farm pasture and barn. This work models testing and culling using pulse actions which corresponds to actual farm practices.

We explicitly found the basic reproduction number ($\mathbb{R}_0$) for the simplified model and it clearly shows all possible routes of infection of the Johne's disease. From, the expression for $\mathbb{R}_0$, it is clear that cleaning the environment can reduce the basic reproduction number and hence the prevalence of the disease in the system. Also, $\mathbb{R}_0$ for the more realistic model was calculated using the parameter values and it showed that the disease free equilibrium is unstable, even with a high level of removal of high shedding animals from the system.

Numerical simulations were run for the more realistic model. Some of the parameters were calculated from the available data and literature and some were determined through knowledge of farm practices. From the numerical simulation results we can conclude that the bacteria in the environment is an important route for the disease transmission. We have analyzed and recorded the effect of discrete (annual) culling on the prevalence of the disease.

Further modeling and simulation work should be done to see the effect of more realistic cleaning of the environment on the prevalence of the Johne's disease in a farm. Also, we plan to investigate the effect of testing and culling two or three times a year on reducing the prevalence of the Johne's disease in the system.

Bibliography

[1] P. Ayscue, C. Lanzas, R. Ivanek, and Y. T. Gröhn. Modeling on-farm *Escherichia coli* o157:h7 population dynamics. *Foodborne Pathogens and Disease*, 6(4):461–470, 2009. doi: 10.1089/fpd.2008.0235. 8, 9

[2] M. Bani-Yaghoub, R. Gautam, Z. Shuai, P. V. D. Driessche, and R. Ivanek. Reproduction numbers for infections with freeliving pathogens growing in the environment. *Journal of Biological Dynamics*, 6(2):923–940, 2012. doi: 10.1080/17513758.2012.693206. 143, 151

[3] F. Belgacem and C. Cosner. The effects of dispersal along environmental gradients on the dynamics of populations in heterogeneous environment. *Canadian Applied Mathematics Quarterly*, 3(4):379–397, 1995. 4, 5, 11, 12, 14

[4] F. Brauer, P. V. D. Driessche, and J. (Eds.) Wu. *Mathematical Epidemiology*. Springer Berlin Heidelberg, 2008. 143, 145, 151

[5] R. Breban. Role of environmental persistence in pathogen transmission: a mathematical modeling approach. *Journal of Mathematical Biology*, 66:535–546, 2013. doi: 10.1007/s00285-012-0520-2. 8, 176

[6] R. S. Cantrell and C. Cosner. The effects of spatial heterogeneity in population dynamics. *Journal of Mathematical Biology*, 29(1991):315–338, 1990. 4, 11

[7] R. S. Cantrell and C. Cosner. Diffusive logistic equations with indefinite weights: population models in disrupted environments ii. *Society for Industrial and Applied Mathematics*, 22(4):1043–1064, 1991. 4, 11

[8] R. S. Cantrell and C. Cosner. *Spatial Ecology via Reaction-Diffusion Equations*. Wiley series in mathematical and computational biology, 2003. 4, 11

[9] R. S. Cantrell, C. Cosner, and Y. Lou. Advection–mediated coexistence of cometing species. *Proceedings of the Royal Society of Edinburgh*, 137 A:497–518, 2007. 4, 5, 11, 12, 14

[10] C. Cosner and Y. Lou. Does movement towards better environments always benifit a population? *Journel of Mathematical analysis and Applications*, 277 (2003):489–503, 2001. doi: 10.1016/S0022-247X(02)00575-9. 4, 5, 11, 12, 13, 14

[11] O. Diekmann, J. A. P. Heesterbeek, and Roberts M. G. The construction of next-generation matrices for compartmental epidemic models. *Journal of the Royal Society Interface*, 7(47):873–885, 2010. doi: 10.1098/rsif.2009.0386. 143, 145, 151

[12] O. Diekmann, H. Heesterbeek, and T. Britton. *Mathematical Tools for Understanding Infectious Disease Dynamics: (Princeton Series in Theoretical and Computational Biology)*. Princeton University Press, 2013. 7, 143, 145, 151

[13] W. Ding, H. Finotti, S. Lenhart, Y. Lou, and Q. Ye. Optimal control of growth coefficient on a steady-state population model. *Nonlinear Analysis: Real World Applications*, 11(2010):688–704, 2009. doi: 10.1016/j.nonrwa.209.01.015. 4, 5, 11, 13, 14

[14] P. V. D. Driessche and J. Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1–2):29–48, 2002. doi: 10.1016/S0025-5564(02)00108-6. 143, 145, 149, 151

[15] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, 1998. 25, 87

[16] H. Finotti, S. Lenhart, and T. V. Phan. Optimal control of advective direction in reaction-diffusion population models. *Evolution equations and control theory*, 1(1):81–107, 2012. doi: 10.3934/eect.2012.1.81. 4, 5, 6, 11, 13, 14, 18, 25, 68, 74, 78, 86, 114

[17] K. R. Fister. *Applications of optimal control*. Ph.D thesis, University of Tennessee, Knoxville, 1996. 3, 4

[18] Alice Ferguson Foundation. Cow in and out game. http://fergusonfoundation.org/lessons/cow_in_out/cowmoreinfo.shtml, 2012. 131, 133

[19] Gerlach. Paratuberculosis: the pathogen and routes of infection. paratuberkulose: Ereeger und ubertragungswege. *Dtsch Tierarztl Wochenschr*, 109(12):504–506, 2002. 131, 132

[20] J. Giesecke. *Modern Infectious Disease Epidemiology*. Taylor Francis Ltd, 2002. 7

[21] W. Hackbush. A numerical method for solving parabolic equations with opposite orientations. *Computing*, 20:229–240, 1978. 10, 192

[22] R. W. Humphry, A. W. Stott, C. Adams, and G. J. Gunn. A model of the relationship between the epidemiology of johne's disease and the environment in suckler-beef herds. *The Veterinary Journal*, 172:432–445, 2006. doi: 10.1016/j.tvjl.2005.07.017. 125

[23] M. J. Keeling and P. Rohani. *Modeling Infectious Diseases in Humans and Animals*. Princeton University Press, 2007. 7

[24] Jr. M. R. Kelly. *Spatial Dynamic Models for Fishery Management and Waterborne Disease Control*. Ph.D thesis, University of Tennessee, Knoxville, 2014. 4, 11, 13

[25] S. Lenhart and V. Protopopescu. Optimal control for parabolic systems with competitive interactions. *Mathematical Methods in the Applied Sciences*, 17(7): 509–524, 1994. doi: 10.1002/mma.1670170703. 5, 14

[26] S. Lenhart and T. J. Workman. *Optimal control applied to biological models*. Chapman and Hall/CRC Mathematical and Computational Biology Series, 2007. 2, 3, 10, 192

[27] S. Lenhart, V. Protopopescu, and A. A. Szpiro. Optimal control for competing coalitions. *Nonlinear Analysis-theory Methods & Applications*, 28(8):1411–1428, 1997. doi: 10.1016/0362-546X(95)00227-M. 5, 14

[28] S. Lenhart, M. Liang, and V. Protopopescu. Optimal control of boundary habitat hostility for interacting species. *Mathematical Methods in the Applied Sciences*, 22(13):1061–1077, 1999. doi: 10.1002/(SICI)1099-1476(19990910)22:13⟨1061::AID-MMA70⟩3.0.CO;2-I. 5, 14

[29] S. Li, J. N. S. Eisenberg, I. H. Spicknall, and J. S. Koopman. Dynamics and control of infections transmitted from person to person through the environment. *American Journal of Epidemiology*, 170(2):257–265, 2009. doi: 10.1093/aje/kwp116. 8

[30] X. Li and J. Yong. *Optimal Control Theory for Infinite Dimensional Systems*. Birkhäuser, Boston, 1995. 3

[31] J. L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations*. Grundlehren der mathematischen Wissenschaften, Band 170, Springer-Verlag, New York-Berlin, 1971. 2, 3

[32] Z. Lu, R. M. Mitchell, R. L. Smith, J. S. Van Kessel, P. P. Chapagain, Y. H. Schukken, and Y. T. Grohn. The importance of culling in Johne's disease control. *Journal of Theoretical Biology*, 254(1):135–146, 2008. doi: 10.1016/j.jtbi.2008.05.008. 124, 127

[33] C. Marcé, P. Ezanno, M. F. Weber, H. Seegers, D. U. Pfeiffer, and C. Fourichon. Invited review: Modeling within-herd transmission of *Mycobacterium avium* subspecies *paratuberculosis* in dairy cattle: A review. *Journal of Dairy Science*, 93(10):4455–4470, 2010. doi: 10.3168/jds.2010-3139. 10, 124, 176

[34] C. Marcé, R. Guatteo1, N. Bareille, and C. Fourichon. Dairy calf housing systems across europe and risk for calf infectious diseases. *Animal*, 4(9):1588–1596, 2010. doi: 10.1017/S1751731110000650. 10

[35] C. Marcé, P. Ezanno, H. Seegers, D. U. Pfeiffer, and C. Fourichon. Predicting fadeout versus persistence of paratuberculosis in a dairy cattle herd for management and control purposes: a modelling study. *Veterinary Research*, 42(1):36, 2011. doi: 10.1186/1297-9716-42-36. 10, 125

[36] C. Marcé, P. Ezanno, H. Seegers, D. U. Pfeiffer, and C. Fourichon. Within-herd contact structure and transmission of *Mycobacterium avium* subspecies *paratuberculosis* in a persistently infected dairy cattle herd. *Preventive Veterinary Medicine*, 100(2):116–125, 2011. doi: 10.1016/j.prevetmed.2011.02.004. 10, 125

[37] T. Massaro, S. Lenhart, M. Spence, C. Drakes, G. Yang, F. Agusto, R. Johnson, B. Whitlock, A. Wadhwa, and S. Eda. Modeling for cost analysis of Johne'S disease control based on evelisa testing. *Journal of Biological Systems*, 21(4):1340010, 2013. doi: 10.1142/S021833901340010X. 10, 123, 124, 126, 163

[38] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelize, and E. F. Mishchenoko. *The Mathematical Theory of Optimal Processes*. Wiley, New York, 1962. 2, 3

[39] E.A. Raizman, J. Fetrow, S.J. Wells, S.M. Godden, M.J. Oakes, and G. Vazquez. The association between *Mycobacterium avium* subsp. *paratuberculosis* fecal shedding or clinical Johne's disease and lactation performance on two minnesota,

usa dairy farms. *Preventive veterinary medicine*, 78:179–195, 2007. doi: 10.1016/j.prevetmed.2006.10.006. 130

[40] J. Robins, S. Bogen, A. Francis, A. Westhoek, A. Kanarek, S. Lenhart, and S. Eda. Agent-based model for Johne's disease dynamics in a dairy herd. *Veterinary Research*, 46:68, 2015. doi: 10.1186/s13567-015-0195-y. 10, 123, 125, 126, 163

[41] P. Rohani, R. Breban, D. E. Stallknecht, and Drake J. M. Environmental transmission of low pathogenicity avian influenza viruses and its implications for pathogen invasion. *PNAS*, 106(25):10365–10369, 2009. doi: 10.1073/pnas.0809026106. 8

[42] A. M. Scanu, T. J. Bull, S. Cannas, J. D. Sanderson, L. A. Sechi, G. Dettori, S. Zanetti, and J. H. Taylor. *Mycobacterium avium* Subspecies *paratuberculosis* Infection in Cases of Irritable Bowel Syndrome and Comparison with Crohns Disease and Johne's Disease: Common Neural and Immune Pathogenicities. *JOURNAL OF CLINICAL MICROBIOLOGY*, 45(12):3883–3890, 2007. doi: 10.1128/JCM.01371-07. 124

[43] M. C. Scott, J. P. Bannantine, Y. Kaneko, A. J. Branscum, R. H. Whitlock, Y. Mori, C. A. Speer, and S Eda. Absorbed EVELISA: A Diagnostic Test with Improved Specificity for Johne's Disease in Cattle. *Foodborne Pathogens and Disease*, 7:1291–1296, 2010. doi: 10.1089=fpd.2010.0541. 124, 170

[44] J. Simon. Compact sets in the $l^p(0,t;b)$. *Annali di Matematica Pure ed Applicata*, 146(4):65–96, 1987. 33, 44, 51, 89, 97, 104

[45] S. Singh and K. Gopinath. *Mycobacterium avium* subspecies *Paratuberculosis* and Crohn's Regional Ileitis: How Strong is Association? *Journal of Laboratory Physicians*, 3(2):69–74, 2011. doi: 10.4103/0974-2727.86836. 124

[46] J. H. Taylor. Review *Mycobacterium avium* subspecies *paratuberculosis*, Crohn's disease and the Doomsday Scenario. *Gut Pathogens*, 1, 2009. doi: 10.1186/1757-4749-1-15. 124, 125

[47] National Johne's Education Initiative, National Institute for Animal Agriculture. JOHNE'S INFORMATION CENTRAL. http://www.johnesdisease.org/index.html. 123

[48] The University of Wisconsin – Madison School of Veterinary Medicine. Johne's Information Center. http://www.johnes.org/general/faqs.html, 2001. 10, 123, 124

[49] R. H. Whitlock, R. W. Sweeney, T. L. Fyock, and J. Smith. Map supershedders: another factor in the control of johnes disease. page 164. In E. J. B. Manning and S. S. Nielsen (ed.), Proceedings of the 8th International Colloquium on Paratuberculosis. International Association for Paratuberculosis, Madison, WI., 2005. 131

[50] R. J. Whittington, I. B. Marsh, and L. A. Reddacliff. Survival of *Mycobacterium avium* subsp. *paratuberculosis* in dam water and sediment. *Journal of Laboratory Physicians*, 71(9):5304–5308, 2005. doi: 10.1128/AEM.71.9.53045308.2005. 10, 123

Appendix

Appendix A

PDE approximation scheme

A.1 The optimal control of the advection direction in a competition model - numerical analysis

Here, we present some of the numerical techniques we used to analyze the optimal control problem 2.2 and 3.2.

The one - dimensional problem:

One system for the simulation is:

$$\begin{aligned} u_t - d_1 u_{xx} - (hu)_x &= u[m - a_1 u] - b_1 uv \text{ in } \quad Q \\ v_t - d_2 v_{xx} &= v[m - a_2 v] - b_2 uv \text{ in } \quad Q \end{aligned}$$

$$d_1 u_x \cdot \eta + uh \cdot \eta = 0 \text{ on } \quad (0,t) \cup (L,t) \text{ for all } \ t \in (0,T) \tag{A.1}$$

$$d_2 v_x \cdot \eta = 0 \text{ on } \quad (0,t) \cup (L,t) \text{ for all } \ t \in (0,T) \tag{A.2}$$

$$\begin{aligned} u(x,0) &= u_0(x) \geq 0 \quad \text{for a.e. } \ x \in [0,L] \\ v(x,0) &= v_0(x) \geq 0 \quad \text{for a.e. } \ x \in [0,L] \end{aligned}$$

That is here,

$$Q = (0, L) \times (0, T)$$

and

$$S = (0, t) \cup (L, t) \text{ for all } t \in (0, T)$$

Numerical method:

The numerical simulations were done in MATLAB and using a finite difference scheme. To approximate the first order derivatives, forward differences or the backward differences were used and for the second order derivatives, the central difference method was used.

Forward difference:

$$u_t(i, j) = \frac{u(i, j+1) - u(i, j)}{\Delta t}$$

$$v_t(i, j) = \frac{v(i, j+1) - v(i, j)}{\Delta t}$$

Central difference:

$$\begin{aligned} u_{xx}(i, j) =& \frac{u_x(i + \frac{1}{2}, j) - u_x(i - \frac{1}{2}, j)}{\Delta x} \\ =& \frac{\frac{u(i+1,j)-u(i,j)}{\Delta x} - \frac{u(i,j)-u(i-1,j)}{\Delta x}}{\Delta x} \\ =& \frac{u(i+1, j) - 2u(i, j) + u(i-1, j)}{(\Delta x)^2} . \end{aligned}$$

Similarly, we have

$$v_{xx}(i, j) = \frac{v(i+1, j) - 2v(i, j) + v(i-1, j)}{(\Delta x)^2} .$$

For the advection terms we have used the up wind method:

If $h(i,j) > 0$

$$(hu)_x(i,j) = \frac{(h_1(i+1,j)u(i+1,j) - h_1(i,j)u(i,j))\Delta t}{\Delta x} \ ,$$

if $h(i,j) > 0$

$$(hu)_x(i,j) = \frac{(h_1(i,j)u(i,j) - h_1(i-1,j)u(i-1,j))\Delta t}{\Delta x} \ .$$

Hence, the above state equations can be approximated as:

If $h(i,j) > 0$

$$\begin{aligned}
\frac{u(i,j+1) - u(i,j)}{\Delta t} &= d_1 \frac{(u(i+1,j) - 2u(i,j) + u(i-1,j))}{(\Delta x)^2} \\
&\quad + \frac{(h(i+1,j)u(i+1,j) - h(i,j)u(i,j))}{\Delta x} \\
&\quad + u(i,j)[m(i,j) - a_1 u(i,j)] - b_1 u(i,j)v(i,j) \\
\frac{v(i,j+1) - v(i,j)}{\Delta t} &= d_2 \frac{v(i+1,j) - 2v(i,j) + u(i-1,j)}{(\Delta x)^2} \\
&\quad + v(i,j)[m(i,j) - a_2 v(i,j)] - b_2 u(i,j)v(i,j) \ ,
\end{aligned}$$

if $h(i,j) < 0$

$$\begin{aligned}
\frac{u(i,j+1) - u(i,j)}{\Delta t} &= d_1 \frac{(u(i+1,j) - 2u(i,j) + u(i-1,j))}{(\Delta x)^2} \\
&\quad + \frac{(h(i,j)u(i,j) - h(i-1,j)u(i-1,j))}{\Delta x} \\
&\quad + u(i,j)[m(i,j) - a_1 u(i,j)] - b_1 u(i,j)v(i,j) \\
\frac{v(i,j+1) - v(i,j)}{\Delta t} &= d_2 \frac{v(i+1,j) - 2v(i,j) + u(i-1,j)}{(\Delta x)^2} \\
&\quad + v(i,j)[m(i,j) - a_2 v(i,j)] - b_2 u(i,j)v(i,j) \ .
\end{aligned}$$

After doing some algebra we have the numerical system:

If $h(i,j) > 0$

$$\begin{aligned} u(i,j+1) &= u(i,j) + \frac{d_1(u(i+1,j) - 2u(i,j) + u(i-1,j))\Delta t}{(\Delta x)^2} \\ &+ \frac{(h(i+1,j)u(i+1,j) - h(i,j)u(i,j))\Delta t}{\Delta x} \\ &+ u(i,j)(m(i,j) - a_1 u(i,j))\Delta t - b_1 u(i,j)v(i,j)\Delta t \end{aligned}$$

$$\begin{aligned} v(i,j+1) &= v(i,j) + \frac{d_2(v(i+1,j) - 2v(i,j) + v(i-1,j))\Delta t}{(\Delta x)^2} \\ &+ v(i,j)(m(i,j) - a_2 v(i,j))\Delta t - b_2 u(i,j)v(i,j)\Delta t \ , \end{aligned}$$

if $h(i,j) < 0$

$$\begin{aligned} u(i,j+1) &= u(i,j) + \frac{d_1(u(i+1,j) - 2u(i,j) + u(i-1,j))\Delta t}{(\Delta x)^2} \\ &+ \frac{(h(i,j)u(i,j) - h(i-1,j)u(i-1,j))\Delta t}{\Delta x} \\ &+ u(i,j)(m(i,j) - a_1 u(i,j))\Delta t - b_1 u(i,j)v(i,j)\Delta t \end{aligned}$$

$$\begin{aligned} v(i,j+1) &= v(i,j) + \frac{d_2(v(i+1,j) - 2v(i,j) + v(i-1,j))\Delta t}{(\Delta x)^2} \\ &+ v(i,j)(m(i,j) - a_2 v(i,j))\Delta t - b_2 u(i,j)v(i,j)\Delta t \ . \end{aligned}$$

Approximating boundary conditions

At the boundary points (0,t), the outward normal is $\eta = -1$ and at the boundary points (L,t), $\eta = +1$

Hence, we get the boundary conditions as:

$$\begin{aligned} -d_1 u_x \cdot -uh\cdot &= 0 \quad at(0,t) \\ -d_2 v_x\cdot &= 0 \quad at(0,t) \end{aligned}$$

and

$$\begin{aligned} d_1 u_x \cdot +uh\cdot &= 0 \quad at(L,t) \\ d_2 v_x\cdot &= 0 \quad at(L,t)\ . \end{aligned}$$

Then, using the forward difference at the points (0,t) we can write:

$$-d_1 \frac{u(2,j)-u(1,j)}{\Delta x} - h(1,j)u(i,j) = 0$$

and

$$-d_2 \frac{v(2,j)-v(1,j)}{\Delta x} = 0$$

(Note that in MATLAB notation the initial nodes will not be defined with zero. Hence, in the above formula u(1,j) represents all the (0,t) points.)

After doing some algebra we can write these as:

$$u(1,j) = \frac{d_1 u(2,j)}{d_1 - h(1,j)\Delta x}$$

and

$$v(1,j) = v(2,j)\ .$$

Similarly using the backward difference at the points (L,t) we can write:

$$d_1 \frac{u(N,j)-u(N-1,j)}{\Delta x} + h(N,j)u(N,j) = 0$$

and

$$d_2 \frac{v(N,j)-v(N-1,j)}{\Delta x} = 0\ .$$

(Note that by assuming there are N nodes representing the differential system in MATLAB, in the above formula u(N,j) represents all the (L,t) points.)

Again after doing some algebra, we can write these as:

$$u(N,j) = \frac{d_1 u(N-1,j)}{d_1 + h(N,j)\Delta x}$$

and

$$v(N,j) = v(N-1,j) \ .$$

To solve the optimal control problems in (2.45) and (3.31) we have used the forward backward sweep method [21, 26].

Vita

Kokum Reha De Silva was born in Matara, Sri Lanka on 24th of Noember, 1983. She is the eldest and the only daughter of Quton De Silva and Daya Ranjanie. She started her studies at St. Joseph's Girls' School in Gampola and achieving the grade 5 scholarship, she started her studies at Mahamaya Girls' College in Kandy. In 2003 she entered the Faculty of Science, University of Peradeniya where, in July, 2007, she received her Bachelor of Science, Special Degree in Mathematics with first class honours. In July 2007, Kokum joined the Department of Mathematics, Faculty of Science, University of Peradeniya as a temporary lecturer and then she was hired as a probationary lecturer in 2009. In August 2010 she was granted study leave to pursue her higher studies and she started her graduate studies at the University of Tennessee, Knoxville, USA. There she was supported by a graduate teaching assistantship from the Department of Mathematics throughout her graduate studies. In December 2013, she graduated with a concurrent Masters Degree and in December 2015, she graduated with her Ph.D in Mathematics.

Kokum Rekha De Silva will continue her work in Mathematical Biology and teaching Mathematics at the Department of Mathematics, University of Peradeniya, Sri Lanka.