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CONSERVATION LAWS
IN
THREE DIMENSIONAL ELASTICITY

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Elizabeth Ann Robinson

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DEDICATION

This thesis is dedicated to my family
Mr. and Mrs. David W. Robinson, Janie, Fred, and Josh
and Mrs. Maxine Combs
without whom I could have never obtained this degree;

to my best friends
Valerie M. Beaman and Angela M. Scesa

WHATEVER;

and, to my Lord and Savior.

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ABSTRACT

In this thesis all of the conservation laws dependent only on ∇u for a system of equations are found. These laws are found from nontrivial solutions of the Euler-Lagrange equations $\mathcal{E}(u)=0$ of a linearized equation in three dimensional elasticity. The derivation of these equations from a physical situation is first shown. Next, the derivation of the laws linear and then quadratic in ∇u is developed followed by the proof that no other laws of higher order exist. The conservation laws found here will generalize previous results from a situation in classical linear elasticity. Lastly, characteristics of the system of equations and their relation to the conservation laws through the calculus of variations and Noether's Theorem are examined.

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I. Introduction

In this paper the conservation laws for a particular system of equations will be found. The system in question is a set of linear partial differential equations encountered in elasticity (cf. Simpson and Spector [12], [13]). Through the study of the calculus of variations and its primary theorem, Noether's Theorem, and underlying symmetries within the system of equations, conservation laws for the system can be derived.

Study along the same lines has been done by Olver [11]. He derived all conservation laws in the special case of classical linear elasticity in three dimensions (Gurtin [4]), this case being a special case of our results. The process and results in this thesis will parallel those of his.

In the chapter to follow, first we will derive the system of equations in question. Second, all conservation laws linear and quadratic in ∇u will be found under mild hypotheses. These include the Eshelby energy-momentum tensor (Eshelby [2], Knowles and Sternberg [5], Olver [9]). Third, a proof that no other conservation laws exist will follow. Lastly, the relationship of these laws to Noether's Theorem and to Olver's results [11] will be examined.

II. Results

We are looking for the conservation laws of the equations found by linearizing a boundary value problem in three-dimensional nonlinear elasticity. This is the problem in which a compressed cylinder loses its stability when the linearized equations have a nontrivial solution. Then we compute the conservation laws for these equations.

In its initial position, an elastic body occupies a region $\Omega \subset \mathbb{R}^3$ in three dimensions. At this initial position the body is unstressed and undeformed. When force is applied to the body, a point $x \in \Omega$ is moved. The movement of x is governed by a function ϕ called the deformation, $\phi: \Omega \rightarrow \mathbb{R}^3$, and x will move to $\phi(x) \in \mathbb{R}^3$. Here a requirement will be that $\phi \in C^2$ and $\det \nabla \phi > 0$ (the second requirement is necessary to insure ϕ is locally one-to-one and orientation preserving to prevent inversions and interpenetration of matter). $\nabla \phi$ is the deformation gradient which is a 3×3 matrix function of x whose components are

$$(\nabla \phi)_{ij} = \frac{\partial \phi_i}{\partial x_j} \quad \text{for } i, j = 1, 2, 3.$$

To measure the potential energy per unit volume after deformation ϕ , the stored energy function σ is used, and

$\sigma: M^3 \rightarrow \mathbb{R}$ where M^3 is the set of 3×3 matrices; $\sigma(\nabla\phi(x))$ measures this energy. In elasticity it is assumed that σ depends only on $\nabla\phi$ - the material is called hyperelastic, cf. Ciarlet [1]. The total potential or stored energy V is measured by $V = \int_{\Omega} \sigma(\nabla\phi(x)) dx$, where $dx = dx_1 dx_2 dx_3$ is the

volume element. The Principle of Minimum Energy (Ciarlet [1]) requires that V is minimized over all admissible ϕ described above.

The stress tensor T is defined by $T = [T_{ij}]$, where

$$T_{ij}(F) = \frac{\partial \sigma(F)}{\partial F_{ij}} \quad \text{for } F \in M^3 \quad i, j = 1, 2, 3; \quad (1)$$

here $T: M^3 \rightarrow M^3$. Under deformation ϕ , $T(\nabla\phi(x))$ will measure stress (force per unit area) in all directions in the body.

In a model problem (cf. Simpson and Spector [12] and [13]), an elastic cylinder occupying the space $\Omega = B \times (0, L)$, where $B = \{(x_1, x_2) : x_1^2 + x_2^2 < R^2\}$ is initially unstressed. In this cylinder, the bottom is on the frictionless $x_1 x_2$ -plane, the sides are stress free, and the top is compressed by a frictionless plane to a new position $x_3 = \lambda L$. At the initial position $x_3 = L$, λ is the compression ratio or load modulus

and $0 < \lambda < 1$. The equation of equilibrium is

$$\text{Div } T = 0 \quad (2)$$

in Ω with corresponding boundary conditions on $\partial\Omega$. These boundary conditions include $Tn = 0$ on the sides where n is the outward unit normal and $(Tn)_i = \sum_{j=1}^3 T_{ij} n_j$. On the top we

have that $(Tn)_1 = (Tn)_2 = 0$ and $\phi_3 = \lambda L$; and, on the bottom we have that $(Tn)_1 = (Tn)_2 = 0$ and $\phi_3 = 0$. Here the vector $\text{Div } T$ is defined by

$$(\text{Div } T)_i = \sum_{j=1}^3 \frac{\partial}{\partial x_j} T_{ij} (\nabla \phi(x)) \quad i=1,2,3. \quad (3)$$

We assume

$$\sigma(F) = \frac{1}{2} F \cdot F + \Psi(\det F) \quad \text{for } F \in M^3 \quad (4)$$

where Ψ is a convex function and $\Psi: (0, \infty) \rightarrow \mathbb{R}$ (cf. Simpson and Spector [12]). The function in (4) satisfies the condition of isotropy and Galilean-invariance

$$\sigma(QF) = \sigma(FQ) = \sigma(F) \quad \text{for all } F \in M^3,$$

where Q is a proper orthogonal matrix and $\det F > 0$ (cf. Ciarlet[1]). In (4) $A \cdot B$ is a scalar product in M^3 and

$$A \cdot B = \text{tr}(A^T B) = \sum_{i,j=1}^3 A_{ij} B_{ij}$$

where tr stands for trace, $A = [A_{ij}] \in M^3$, and $B = [B_{ij}] \in M^3$. The

stress tensor T can be computed from (4) via (1) and is

$$T(F) = F + \Psi'(\det F) \operatorname{cof} F \quad (5)$$

in which $\operatorname{cof} F = (F^{-T}) \det F$, the cofactor matrix ($F^{-T} = (F^{-1})^T$).

The above equations (2) have a solution of the form

$$\phi^0(x) = \begin{bmatrix} \mu x_1 \\ \mu x_2 \\ \lambda x_3 \end{bmatrix} \quad x = (x_1, x_2, x_3) \in \Omega, \mu > 0$$

with

(6)

$$\nabla \phi^0(x) = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \operatorname{diag}(\mu, \mu, \lambda).$$

Since $\nabla \phi^0$ is constant, ϕ^0 is called homogeneous. The side boundary conditions will give μ as a function of λ and the other conditions are satisfied trivially. Physically the cylinder is deformed by compression ϕ^0 to a new cylinder with height λL and radius μR .

As can be shown, the solution (6) becomes unstable when λ attains a critical value $\lambda_{cr} \in (0, 1)$ (cf. Simpson and Spector [12]). At this point, the cylinder buckles or barrels (flattens) to another deformed state. This is found by seeking nontrivial solutions of the linearized equations from (2). To find the linearized equations, replace ϕ in (2) with $\phi^0 + u$ for u a small perturbation function, where $u: \Omega \rightarrow \mathbb{R}^3$ and ∇u is also small (cf. Logan [6]). Next, we

expand T about $\nabla u=0$ in a Taylor series and keep those terms linear in ∇u . Using (5), T is replaced by $E(\nabla u)$, where $E(H)$, the elasticity tensor, is

$$E(H) = \left(\frac{\partial}{\partial F} T(F) \right) H \quad (7)$$

$$= H + \Psi'(\det F) (\det F) [(F^{-T} \cdot H) F^{-T} - F^{-T} H^T F^{-T}] + \Psi''(\det F) (\text{cof } F \cdot H) \text{cof } F;$$

$H \in M^3$ and $F = \text{diag}(\mu, \mu, \lambda)$ cf. Simpson and Spector [12]. Note

$E: M^3 \rightarrow M^3$ is linear and $E(H)_{ij} = \sum_{k,l=1}^3 \frac{\partial}{\partial F_{kl}} T_{ij} H_{kl}$ for $H \in M^3$. The

linearized equations are now

$$\text{Div } E(\nabla u) = 0 \text{ in } \Omega. \quad (8)$$

The boundary conditions will be linearized as well. Now, we must find nontrivial solutions u to (8). To do this first put

$$\mathcal{E}(u) = \text{Div } E(\nabla u), \quad (9)$$

(cf. equations (3), (7)), where $\mathcal{E}(u) = (\mathcal{E}(u)_1, \mathcal{E}(u)_2, \mathcal{E}(u)_3)$ and

$$\mathcal{E}(u)_i = \sum_{j=1}^3 \left[\frac{\partial^2 u_i}{\partial x_j \partial x_j} + k c_i c_j \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right] \text{ for } i=1,2,3. \quad (10)$$

Here $k = \Psi''(\mu^2 \lambda) \mu^2$, $c_1 = c_2 = \lambda$, $c_3 = \mu$. $\mathcal{E}(u)$ can thus be written as

$$\mathcal{E}(u) = \Delta u + k C \nabla (C \cdot \nabla u)$$

for $C = \text{diag}(\lambda, \lambda, \mu) = \frac{\text{cof } F}{\mu}$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$. Notice

that in the derivation of (10) -

$$\text{Div} [(F^{-T} \cdot \nabla u) F^{-T} - F^{-T} \nabla u^T F^{-T}] = 0 \quad (11)$$

i.e. $(F^{-T} \cdot \nabla u) F^{-T} - F^{-T} \nabla u^T F^{-T}$ is a null divergence (or trivial conservation law) as we define later. The proof of (11) goes as follows. We have

$$F^{-T} \cdot \nabla u = \text{tr} (F^{-1} \nabla u) .$$

Let $v = F^{-1} u$ and $I = 3 \times 3$ identity matrix. Then

$$F^{-T} \cdot \nabla u = \text{tr} \nabla v = \nabla v \cdot I$$

$$\Rightarrow (F^{-T} \cdot \nabla u) F^{-T} - F^{-T} \nabla u^T F^{-T} = F^{-T} [(\nabla v \cdot I) I - \nabla v^T]$$

$$\Rightarrow \text{Div} [(F^{-T} \cdot \nabla u) F^{-T} - F^{-T} \nabla u^T F^{-T}] = F^{-T} \text{Div} [(\nabla v \cdot I) I - \nabla v^T] . \quad (11')$$

Now,

$$\begin{aligned} \text{Div} [(\nabla v \cdot I) I - \nabla v^T]_i &= \sum_{j=1}^3 \frac{\partial}{\partial x_j} [(\nabla v \cdot I) \delta_j^i - (\nabla v^T)_{ij}] \\ &= \frac{\partial}{\partial x_i} (\nabla v \cdot I) - \sum_{j=1}^3 \frac{\partial}{\partial x_j} \left(\frac{\partial v_j}{\partial x_i} \right) , \end{aligned}$$

δ_j^i denotes the Kronecker delta. This becomes

$$\text{Div} [(\nabla v \cdot I) I - \nabla v^T] = \frac{\partial}{\partial x_i} \left(\sum_{j=1}^3 \frac{\partial v_j}{\partial x_j} \right) - \sum_{j=1}^3 \frac{\partial}{\partial x_j} \left(\frac{\partial v_j}{\partial x_i} \right)$$

since $\nabla v \cdot I = \text{tr} \nabla v = \nabla \cdot v = \sum_{j=1}^3 \frac{\partial v_j}{\partial x_j}$ implies

$$\text{Div}[(\nabla v \cdot I)I - \nabla v^T] = \sum_{j=1}^3 \frac{\partial^2 v_j}{\partial x_i \partial x_j} - \sum_{j=1}^3 \frac{\partial^2 v_j}{\partial x_j \partial x_i} = 0.$$

Substituting this into (11') yields (11). ■

We use the notation

$$u_j^i = \frac{\partial u_i}{\partial x_j}, \quad i, j = 1, 2, 3.$$

Also, ϵ_{ijk} denotes the Levi-Civita symbol. All functions that follow are assumed infinitely differentiable.

The equations $\mathcal{E}(u) = 0$ ((8), (9)) constitute the Euler-Lagrange equations for the quadratic functional

$$\mathcal{Q}(u) = \int_{\Omega} W(\nabla u(x)) dx \quad (12)$$

for

$$W(H) = \frac{1}{2} H \cdot H + \frac{1}{2} k [C \cdot H]^2, \quad H \in M^3. \quad (13)$$

We will take the general case of $C = \text{diag}(c_1, c_2, c_3)$, for $c_i \in \mathbb{R}$.

The objective is to find all the conservation laws for $\mathcal{E}(u) = 0$.

Definition: A triple $A = (A^1, A^2, A^3)$ is a conserved quantity for a differential equation $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)^T = 0$ if and only if

$Div A = \sum_{j=1}^3 \frac{\partial A^j}{\partial x_j} = 0$ is a consequence of $\mathcal{E}(u) = 0$.

Olver [9] proves that the statement in the definition is equivalent to the existence of some $\psi = (\psi_1, \psi_2, \psi_3)^T$ such that

$$Div A = \psi \cdot \mathcal{E}(u) = \sum_{i=1}^3 \psi_i \mathcal{E}(u)_i \quad (14)$$

identically; here, ψ is a function of $x, u, \nabla u$. We say a conservation law A is trivial (also called a null divergence) if and only if $Div A = 0$ holds identically irrespective of $\mathcal{E}(u) = 0$. For example, $A = (0, u_3^1, -u_2^1)$ has

$$Div A = \frac{\partial}{\partial x_2} u_3^1 - \frac{\partial}{\partial x_3} u_2^1 = \frac{\partial^2 u_1}{\partial x_2 \partial x_3} - \frac{\partial^2 u_1}{\partial x_3 \partial x_2} = 0 \text{ identically.}$$

These laws are

consequences of symmetry of second order partial derivatives, cf. Olver [9] for a characterization. We say a function $w(\nabla u)$ is a null Lagrangian if its Euler-Lagrange equation is identically zero. By Theorem 4.7, Olver [9], this is true if and only if $w = Div R$ for some $R = R(\nabla u)$.

For our purposes we will assume that $A = A(\nabla u)$, i.e. A is dependent only on ∇u . This in turn implies that $\psi = \psi(\nabla u)$ by Olver [10, p.122]. We note that Olver [11] examines the special case $\mu = \lambda = 1$, $C = I$. In his case, $k = \frac{\hat{\mu} + \hat{\lambda}}{\hat{\mu}}$ for his

notation where $\hat{\mu}, \hat{\lambda}$ are the Lamé moduli and it is assumed $k \neq 0, -1$. Then $W(\nabla u)$ reduces to $\frac{1}{2} \nabla u \cdot \nabla u + \frac{1}{2} k (I \cdot \nabla u)^2$ which

differs from $\frac{1}{4} (\nabla u + \nabla u^T) \cdot (\nabla u + \nabla u^T) + \frac{\hat{\lambda}}{2\hat{\mu}} (I \cdot \nabla u)^2$ in Olver's

notation by a null Lagrangian proportional to $\nabla u \cdot \nabla u^T - (I \cdot \nabla u)^2$

- note a misprint in equation (2.1) Olver [11] (multiply $\|\nabla u + \nabla u^T\|^2$ by $\frac{1}{4}$ and a factor of $\hat{\mu}$ has been divided out).

This is the situation of classical linear elasticity (cf. Gurtin [4], Muskhelishvili [7]) obtained by linearizing the equations about $\phi =$ identity map.

Some conservation laws can be derived from symmetries of (12). Suppose x, u are transformed to new variables $\tilde{x}, \tilde{u}$ by a one-parameter transformation group

$$\tilde{x} = \Xi(x, u, \varepsilon) \tag{15}$$

$$\tilde{u} = \Phi(x, u, \varepsilon)$$

for small ε with $\Xi = (\Xi_1, \Xi_2, \Xi_3)^T$ and $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T$, cf. Olver [9], Gelfand and Fomin [3]. We now examine when this represents a group of symmetries of (12). The resulting transformation of ∇u to $\nabla \tilde{u}$ is given by

$$(\nabla \tilde{u})_{ij} = \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \tilde{u}_j^i = \Theta_{ij}(x, u, \nabla u, \varepsilon)$$

where $\Theta_{ij} = \sum_{k=1}^3 (D_k \Phi_i) (J^{-1})_{kj}$. Here J is the 3×3 matrix with

components $J_{ij} = D_j \Xi_i$ and $D_j = \frac{\partial}{\partial x_j} + \sum_{m=1}^3 u_j^m \frac{\partial}{\partial u_m}$, for $i, j = 1, 2, 3$,

is the total derivative operator. Define

$\xi_i = \frac{\partial \Xi_i}{\partial \varepsilon} \big|_{\varepsilon=0}$ and $\phi_i = \frac{\partial \Phi_i}{\partial \varepsilon} \big|_{\varepsilon=0}$ for $i = 1, 2, 3$. $\mathcal{Q}(u)$ is said to be

invariant under the transformation group (15) if and only if $\mathcal{Q}(u)$ is unaffected by the previous change of variables $x, u, \nabla u$, i.e.,

$$\int_{\tilde{\Omega}} W(\tilde{\nabla} \tilde{u}(\tilde{x})) d\tilde{x} = \int_{\Omega} W(\nabla u(x)) dx;$$

$\tilde{\Omega}$ is the transform of Ω for ε small. This generates a conservation law according to (cf. Olver [9], Gelfand and Fomin [3], and Noether [8])

Noether's Theorem: Suppose the transformation group (15) is a one-parameter group of symmetries of $\mathcal{Q}(u)$, i.e. $\mathcal{Q}(u)$ is invariant under (15). Then there exists $A = (A^1, A^2, A^3)$ such that $\text{Div} A = 0$ is a conservation law for the Euler-Lagrange equations $\mathcal{E}(u)$; the components of A are given by

$$A^i = \sum_{j,k=1}^3 \xi_j u_j^k \frac{\partial W}{\partial u_i^k} - \xi_i W - \sum_{k=1}^3 \phi_k \frac{\partial W}{\partial u_i^k} \quad i=1, 2, 3 \quad (16)$$

and $\text{Div} A = \psi \cdot \mathcal{E}(u)$ from (14) where $\psi = (\psi_1, \psi_2, \psi_3)^T$,

$$\psi_k = \sum_{j=1}^3 \xi_j u_j^k - \phi_k \quad k=1,2,3. \quad (17)$$

Now W in (13) is independent of both x and u which implies that $\mathcal{Q}(u)$ is invariant under the translations $x \rightarrow x + \varepsilon a$ and $u \rightarrow u + \varepsilon b$ for arbitrary fixed $a, b \in \mathbb{R}^3$. From Noether's Theorem conservation laws for $\mathcal{E}(u) = 0$ can be derived via these symmetries. In what is to follow, all conservation laws will be calculated and their relation to symmetries will be examined.

The condition of ellipticity is sometimes placed upon the Euler-Lagrange equations $\mathcal{E}(u) = 0$. This condition requires the tensor E to be strongly elliptic according to the following definition and yields inequalities which must be satisfied by the coefficients.

Definition: The tensor E in (7) is strongly elliptic if and only if $(a \otimes b) \cdot E(a \otimes b) > 0$ for all $a, b \in \mathbb{R}^3$, $a, b \neq 0$ where $a \otimes b \in M^3$ is the matrix $\underline{ab}^T$ with components $(a \otimes b)_{ij} = a_i b_j$ (Ciarlet [1] and Simpson and Spector [12]).

Define

$$\alpha_i = 1 + k c_i^2,$$

$$\beta_{ij} = k c_i c_j, \quad i, j = 1, 2, 3.$$

Proposition: E is strongly elliptic if and only if $\alpha_i > 0, \quad i=1,2,3.$

Proof: To characterize the strong ellipticity of E , we compute $(a \otimes b) \cdot E(a \otimes b).$

Let $a, b \in \mathbb{R}^3.$ From (7), we have

$$\begin{aligned} (a \otimes b) \cdot E(a \otimes b) &= (a \otimes b) \{ (a \otimes b) + \Psi'(\det F) \det F [(F^{-T} \cdot (a \otimes b)) F^{-T} - F^{-T} (a \otimes b)^T F^{-T}] \\ &\quad + \Psi''(\det F) (\text{cof} F \cdot (a \otimes b)) \text{cof} F \} \end{aligned}$$

$$\begin{aligned} &= (a \otimes b) \cdot (a \otimes b) + \Psi'(\det F) \det F [(F^{-T} \cdot (a \otimes b)) \cdot (a \otimes b) \cdot F^{-T}] \\ &\quad - (a \otimes b) \cdot (F^{-T} (a \otimes b)^T F^{-T}) + \Psi''(\det F) (\text{cof} F \cdot (a \otimes b)) \cdot (a \otimes b) \cdot \text{cof} F. \end{aligned}$$

Now,

$$\begin{aligned} (a \otimes b) \cdot F^{-T} &= F^{-T} \cdot (a \otimes b) = \sum_{i,j=1}^3 (a \otimes b)_{ij} (F^{-T})_{ij} \\ &= (\mu \lambda)^{-1} \sum_{i,j=1}^3 a_i b_j \delta_j^i c_i = (\mu \lambda)^{-1} \sum_{i=1}^3 a_i b_i c_i \end{aligned}$$

and,

$$\begin{aligned} (a \otimes b) \cdot [F^{-T} (a \otimes b)^T F^{-T}] &= (\mu \lambda)^{-2} \sum_{i,j=1}^3 a_i b_j c_i a_j b_i c_j = (\mu \lambda)^{-2} \sum_{i=1}^3 a_i b_i c_i \sum_{j=1}^3 a_j b_j c_j \\ &= (\mu \lambda)^{-2} \left(\sum_{i=1}^3 a_i b_i c_i \right)^2. \end{aligned}$$

We substitute and obtain

$$\begin{aligned} & (\underline{a} \otimes \underline{b}) \cdot E(\underline{a} \otimes \underline{b}) \\ &= (\underline{a} \otimes \underline{b}) \cdot (\underline{a} \otimes \underline{b}) + \Psi'(\det F) \det F (\mu \lambda)^{-2} \left[\left(\sum_{i=1}^3 a_i b_i c_i \right)^2 - \left(\sum_{i=1}^3 a_i b_i c_i \right)^2 \right] \\ & \quad + \Psi''(\det F) (\operatorname{cof} F \cdot (\underline{a} \otimes \underline{b}))^2 \end{aligned}$$

$$= \sum_{i,j=1}^3 a_i^2 b_j^2 + \Psi''(\det F) ((\det F) F^{-T} \cdot (\underline{a} \otimes \underline{b}))^2.$$

$$\begin{aligned} \text{Thus, } (\underline{a} \otimes \underline{b}) \cdot E(\underline{a} \otimes \underline{b}) &= \|b\|^2 \sum_{i,j=1}^3 \delta_j^i a_i a_j + \Psi''(\det F) \mu^2 \left(\sum_{i=1}^3 b_i c_i a_i \right)^2 \\ &= \|b\|^2 \sum_{i,j=1}^3 \delta_j^i a_i a_j + k \sum_{i=1}^3 b_i c_i a_i \sum_{j=1}^3 b_j c_j a_j \\ &= \|b\|^2 \sum_{i,j=1}^3 \delta_j^i a_i a_j + k \sum_{i,j=1}^3 b_i b_j c_i c_j a_i a_j, \quad \|b\|^2 = b \cdot b. \end{aligned}$$

Let $M_{ij} = \delta_j^i \|b\|^2 + k b_i b_j c_i c_j$, so

$$(\underline{a} \otimes \underline{b}) \cdot E(\underline{a} \otimes \underline{b}) = \sum_{i,j=1}^3 M_{ij} a_i a_j.$$

Next we observe $(\underline{a} \otimes \underline{b}) \cdot E(\underline{a} \otimes \underline{b})$ is positive for all $\underline{a}, \underline{b} \neq 0$ if and only if $M = [M_{ij}]$ is positive definite for all $\underline{b} \neq 0$, i.e. all eigenvalues of M are positive. Now,

$$M = I \|b\|^2 + k (Cb \otimes Cb)$$

has eigenvalues

$$\|b\|^2, \quad \|b\|^2, \quad \|b\|^2 + k \|Cb\|^2$$

of which the first two eigenvalues are positive. Thus, E is strongly elliptic if

$$\|b\|^2 + k\|Cb\|^2 > 0 \text{ for all } b \in \mathbb{R}^3, b \neq 0,$$

i.e.,
$$\sum_{i=1}^3 b_i^2 + k \sum_{i=1}^3 (c_i b_i)^2 > 0.$$

We choose $b = (1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T$ to get

$$1 + kc_1^2 > 0, 1 + kc_2^2 > 0, 1 + kc_3^2 > 0$$

which implies
$$1 + kc_i^2 = \alpha_i > 0.$$

Conversely, these imply

$$\|b\|^2 + k\|Cb\|^2 \geq \min_i (1 + kc_i^2) \|b\|^2 > 0 \text{ for } b \neq 0.$$

■

Note that physically $c_1, c_2, c_3, k > 0$ ($k > 0$ since Ψ convex, $\Psi'' > 0$), so the strong ellipticity condition holds for this case. In our theorem, we will assume the weaker condition:

$$\alpha_i \neq 0, 1$$

$$\beta_{ij} \neq 0 \quad i, j = 1, 2, 3.$$

In the following theorem, we determine all conservation laws depending solely on ∇u for

$$\mathcal{E}(u) = \Delta u + kC\nabla(C \cdot \nabla u) = 0 \tag{18}$$

from (8), (9), and (10). Here, $C = \text{diag}(c_1, c_2, c_3)$, $k \in \mathbb{R}$.

Theorem: Suppose $c_i \in \mathbb{R}$, $\alpha_i \neq 0, 1$, $\beta_{ij} \neq 0$ for $i, j = 1, 2, 3$.

Suppose $\text{Div} A = 0$ is a conservation law for $\mathcal{E}(u) = 0$ in (18), $A = A(\nabla u)$ depending only on ∇u . Then A is a linear, constant coefficient combination of the following quantities:

a) **Nontrivial, linear**

$$S_j^i = u_j^i + \delta_j^i \sum_{m=1}^3 \beta_{im} u_m^m \quad (19)$$

b) **Trivial, linear**

$$B_j^{ik} = \sum_{l=1}^3 \epsilon_{ijl} u_l^k \quad (20)$$

c) **Nontrivial, quadratic**

$$P_j^i = \sum_{m=1}^3 u_i^m u_j^m + u_i^j \sum_{m=1}^3 \beta_{jm} u_m^m - \frac{1}{2} \delta_j^i \sum_{m,l=1}^3 [u_l^m u_l^m + \beta_{ml} u_m^m u_l^l] \quad (21)$$

$$Q_j^i = \sum_{k,l=1}^3 \left\{ \frac{\alpha_k \beta_{ik}}{\alpha_k - 1} u_k^k u_j^i + \frac{\beta_{ik}}{\alpha_k - 1} u_k^i u_j^k - \frac{\beta_{ij}}{\alpha_j - 1} u_k^i u_k^j - \frac{(\alpha_i \alpha_j - 1) \beta_{jk}}{\alpha_j - 1} u_k^k u_i^j \right.$$

$$\left. - \frac{\alpha_i \alpha_k - 1}{\alpha_k - 1} u_i^k u_j^k + \frac{1}{2} \delta_j^i \left[\frac{\alpha_i \alpha_k - 1}{\alpha_k - 1} u_i^k u_l^k + \gamma_{ikl} u_k^k u_l^l \right] \right\} \quad (22)$$

where

$$\gamma_{ikl} = \beta_{kl} \left(\frac{\alpha_i \alpha_k - 1}{\alpha_k - 1} + \frac{\alpha_i \alpha_l - 1}{\alpha_l - 1} - 1 \right)$$

d) **Trivial, quadratic**

$$\tilde{A}_j^i = \sum_{k,l,m,n=1}^3 \epsilon_{ikn} \epsilon_{jlm} u_l^k u_m^n \quad (23)$$

e) Constants.

Note that (19) and (21) can be written

$$S = [S_j^i] = \nabla u + k(C \cdot \nabla u) C$$

$$P = [P_j^i] = \nabla u^T \nabla u + k(C \cdot \nabla u) (C \cdot \nabla u)^T - \frac{1}{2} (\nabla u \cdot \nabla u + k(C \cdot \nabla u)^2) I$$

In (19), S_j^i stands for three conserved quantities S^1, S^2, S^3 with $S^i = (S_1^i, S_2^i, S_3^i)$, $i=1,2,3$; similarly for (21), (22), (23) and (20) yields nine conserved quantities B^{ik} , $i, k=1,2,3$.

Proof: As in Olver [11], let $A^{ij} = \frac{\partial A^i}{\partial u_k^j}$. Since we are assuming

$A = A(\nabla u)$, then

$$\text{Div} A = \sum_{i=1}^3 D_i A^i = \sum_{i,j,k=1}^3 \frac{\partial A^i}{\partial u_k^j} u_{ki}^j, \quad u_{ki}^j = \frac{\partial^2 u_j}{\partial x_k \partial x_i},$$

so,

$$\begin{aligned} \text{Div} A &= A^{11}_1 u_{11}^1 + A^{11}_2 u_{21}^1 + A^{11}_3 u_{31}^1 + A^{12}_1 u_{11}^2 + A^{12}_2 u_{21}^2 + A^{12}_3 u_{31}^2 + A^{13}_1 u_{11}^3 + A^{13}_2 u_{21}^3 + A^{13}_3 u_{31}^3 + A^{21}_1 u_{12}^1 \\ &+ A^{21}_2 u_{22}^1 + A^{21}_3 u_{32}^1 + A^{22}_1 u_{12}^2 + A^{22}_2 u_{22}^2 + A^{22}_3 u_{32}^2 + A^{23}_1 u_{12}^3 + A^{23}_2 u_{22}^3 + A^{23}_3 u_{32}^3 + A^{31}_1 u_{13}^1 + A^{31}_2 u_{23}^1 \\ &+ A^{31}_3 u_{33}^1 + A^{32}_1 u_{13}^2 + A^{32}_2 u_{23}^2 + A^{32}_3 u_{33}^2 + A^{33}_1 u_{13}^3 + A^{33}_2 u_{23}^3 + A^{33}_3 u_{33}^3. \end{aligned}$$

As noted earlier, A is a conserved quantity for $\mathcal{E} = \mathcal{E}(u) = 0$

if and only if there exists $\psi = (\psi_1(\nabla u), \psi_2(\nabla u), \psi_3(\nabla u))^T$ such that (cf. (10) and (14)).

$$\begin{aligned} \text{Div} A &= \psi \cdot \mathcal{E} = \psi_1 \mathcal{E}_1 + \psi_2 \mathcal{E}_2 + \psi_3 \mathcal{E}_3 \\ &= \psi_1 [u_{11}^1 + u_{22}^1 + u_{33}^1 + kC_1^2 u_{11}^1 + kC_1 C_2 u_{12}^2 + kC_1 C_3 u_{13}^3] \\ &\quad + \psi_2 [u_{11}^2 + u_{22}^2 + u_{33}^2 + kC_1 C_2 u_{21}^1 + kC_2^2 u_{22}^2 + kC_2 C_3 u_{23}^3] \\ &\quad + \psi_3 [u_{11}^3 + u_{22}^3 + u_{33}^3 + kC_3 C_1 u_{31}^1 + kC_3 C_2 u_{32}^2 + kC_3^2 u_{33}^3] . \end{aligned}$$

Now, equate coefficients of u_{jk}^i in $\text{Div} A = \psi \cdot \mathcal{E}$ and note

$u_{jk}^i = u_{kj}^i$. We obtain

$$u_{11}^1: A^{11} = \psi_1 + kC_1^2 \psi_1 = \psi_1 (1 + kC_1^2) ,$$

$$u_{22}^2: A^{22} = \psi_2 + kC_2^2 \psi_2 = \psi_2 (1 + kC_2^2) ,$$

$$u_{33}^3: A^{33} = \psi_3 + kC_3^2 \psi_3 = \psi_3 (1 + kC_3^2) .$$

These yield the equations

$$A^{i_i} = \psi_i (1 + kC_i^2) = \alpha_i \psi_i . \quad (24)$$

Similarly the coefficients of $u_{22}^1, u_{33}^1, u_{11}^2, u_{33}^2, u_{11}^3, u_{22}^3$ yield the equations

$$A^{j_i} = \psi_i \quad \text{for } i \neq j ; \quad (25)$$

the coefficients of

$$u_{21}^1 (= u_{12}^1) , u_{13}^1 (= u_{31}^1) , u_{21}^2 (= u_{12}^2) , u_{23}^2 (= u_{32}^2) , u_{31}^3 (= u_{13}^3) , u_{32}^3 (= u_{23}^3)$$

yield the equations

$$A^{ij}_j + A^{jj}_i = \psi_i k c_i c_j = \beta_{ij} \psi_i \quad \text{for } i \neq j; \quad (26)$$

and, the coefficients of $u_{23}^1 (=u_{32}^1)$, $u_{13}^2 (=u_{31}^2)$, $u_{12}^3 (=u_{21}^3)$ yield the equations

$$A^{ij}_k + A^{kj}_i = 0 \quad \text{for } i, j, k \text{ distinct.} \quad (27)$$

From (24)-(27), we get a system of 18 equations with 27 unknowns (A^{ij}_k).

$$A^{ii}_i = \alpha_i \psi_i, \quad (28a)$$

$$A^{jj}_j = \psi_j, \quad (28b)$$

$$A^{ij}_j + A^{jj}_i = \beta_{ij} \psi_i, \quad (28c)$$

$$A^{ij}_k + A^{kj}_i = 0, \quad (28d)$$

where $i, j, k \in \{1, 2, 3\}$ and i, j, k are unequal. From (28a)-(28d), we can derive all of the conservation laws (19)-(23) previously mentioned. Trivial conservation laws can be derived from $\psi_1 = \psi_2 = \psi_3 = 0$. We divide the proof into three parts: deriving the conserved quantities which are linear and then those which are quadratic in ∇u and showing all third order derivatives of A^i are identically zero completing the list of laws.

To find the conserved quantities which are linear in ∇u , we note that A^{ij}_k are constants. Since there are 18 equations with 27 unknowns, there are at least nine A^{ij}_k which are independent. We arbitrarily choose nine independent A^{ij}_k to be $A^{11}_2, A^{11}_3, A^{22}_3, A^{22}_1, A^{33}_1, A^{33}_2, A^{32}_1, A^{23}_1, A^{31}_2$ and show the other A^{ij}_k can be solved in terms of these and the ψ_i quantities. We have $A^i = \sum_{j,k=1}^3 A^{ij}_k u^j_k =$ linear combination of

$$\psi_1, \psi_2, \psi_3, A^{11}_2, A^{11}_3, A^{22}_3, A^{22}_1, A^{33}_1, A^{33}_2, A^{32}_1, A^{23}_1, A^{31}_2.$$

To find the coefficient of ψ_1 in this linear combination, let $\psi_1=1, \psi_2=\psi_3=0$, and the nine independent $A^{ij}_k=0$. Solving (28a)-(28d) yields the following

$$A^{11}_1=\alpha_1, A^{21}_2=1, A^{31}_3=1, A^{12}_2=\beta_{12}, A^{13}_3=\beta_{13}, \text{ and all other } A^{ij}_k=0.$$

Now,

$$A = (A^1, A^2, A^3) = (A^{11}_1 u^1_1 + A^{12}_2 u^2_2 + A^{13}_3 u^3_3, A^{21}_2 u^1_1, A^{31}_3 u^1_1)$$

$$A = (\alpha_1 u^1_1 + \beta_{12} u^2_2 + \beta_{13} u^3_3, u^1_2, u^1_3).$$

Let S^1 stand for this conserved quantity.

Similarly, to find the coefficient of ψ_2 , let $\psi_2=1$, $\psi_1=\psi_3=0$, and the nine independent $A^{ij}_k=0$. Solving (28a)-(28d) yields

$$A = (u_1^2, \beta_{21}u_1^1 + \alpha_2u_2^2 + \beta_{23}u_3^3, u_3^2).$$

Define this as S^2 .

To find the coefficient of ψ_3 , let $\psi_3=1$, $\psi_1=\psi_2=0$, and the nine independent $A^{ij}_k=0$, yielding

$$A = (u_1^3, u_2^3, \beta_{31}u_1^1 + \beta_{32}u_2^2 + \alpha_3u_3^3).$$

Define this as S^3 .

S^1, S^2, S^3 comprise the three nontrivial, linear conserved quantities (19). As can be seen using $\alpha_i=1+\beta_{ii}$

$$S_j^i = u_j^i + k \delta_j^i \sum_{m=1}^3 C_i C_m u_m^m = u_j^i + \delta_j^i \sum_{m=1}^3 \beta_{im} u_m^m.$$

Finding the coefficients of the nine independent A^{ij}_k terms will yield the nine trivial, linear conservation laws (20). We begin by finding the coefficient of A^{11}_2 . To do so, let $\psi_i=0$, $A^{11}_2=1$, and all other independent $A^{ij}_k=0$. Solving

(28a)-(28d) yields

$$A^{1_2} = -A^{2_1} = 1, \text{ all other } A^{ij} = 0.$$

Now,

$$A = (A^1, A^2, A^3) = (A^{1_2} u_2^1, A^{2_1} u_1^1, 0)$$

$$A = (u_2^1, -u_1^1, 0)$$

and we let B^{31} stand for this row vector.

Similarly, the coefficients of

$$A^{1_3}, A^{2_3}, A^{2_1}, A^{3_1}, A^{3_2}, A^{3_1}, A^{2_3}, A^{3_1}$$

yield

$$(u_3^1, 0, -u_1^1) = -B^{21}, (0, u_3^2, -u_2^2) = B^{12},$$

$$(-u_2^2, u_1^2, 0) = -B^{32}, (-u_3^3, 0, u_1^3) = B^{23},$$

$$(0, -u_3^3, u_2^3) = -B^{13}, (-u_3^2, 0, u_1^2) = B^{22},$$

$$(-u_2^3, u_1^3, 0) = -B^{33}, (0, -u_3^1, u_2^1) = -B^{11}.$$

As can be seen, each $B^{ik} = (B_1^{ik}, B_2^{ik}, B_3^{ik})$ can be derived from

the formula $B_j^{ik} = \sum_{l=1}^3 \epsilon_{ijl} u_l^k$. These nine B^{ik} constitute

trivial conservation laws (20). All conserved quantities linear in ∇u will thus be linear combinations of the S^i quantities and B^{ik} quantities (with constant coefficients).

To find the quadratic conservation laws, the same equations (28a)-(28d) will be used. We put

$$A^{ijl}_{km} = \frac{\partial^2 A^i}{\partial u_k^j \partial u_m^l}$$

which will be constants; note $A^{ijl}_{km} = A^{ijl}_{mk}$. There will be two classes of these second order derivatives A^{ijl}_{km} .

Class 1: i, j, k, l, m contain at most two different values,

$$\text{e.g. } A^{111}_{12}, A^{222}_{23}.$$

Class 2: i, j, k, l, m contain all three values, e.g.

$$A^{122}_{31}, A^{112}_{13}.$$

First, we examine those derivatives in Class 1. Differentiating equations (28a)-(28c) with respect to u_i^i yields the following ($i \neq j$):

$$A^{iii} = \alpha_i \frac{\partial \psi_i}{\partial u_i^i} \quad \text{from (28a),}$$

$$A^{jji} = \frac{\partial \psi_i}{\partial u_i^i} \quad \text{from (28b),}$$

$$A^{iji} + A^{jji} = \beta_{ij} \frac{\partial \psi_i}{\partial u_i^i} \quad \text{from (28c).}$$

Now, we eliminate $\frac{\partial \psi_i}{\partial u_i^i}$ from each of the above equations to

yield

$$A^{iii} = \alpha_i A^{jji} = \frac{\alpha_i}{\beta_{ij}} (A^{iji} + A^{jji}). \quad (29)$$

Differentiating equations (28a)-(28c) with respect to u_j^j yields the following:

$$\begin{aligned} A^{ijj} &= \alpha_i \frac{\partial \psi_i}{\partial u_j^j}, \\ A^{jij} &= \frac{\partial \psi_i}{\partial u_j^j}, \\ A^{ijj} + A^{jij} &= \beta_{ij} \frac{\partial \psi_i}{\partial u_j^j}. \end{aligned}$$

Eliminating $\frac{\partial \psi_i}{\partial u_j^j}$ from each of these yields

$$A^{ijj} = \alpha_i A^{jij} = \frac{\alpha_i}{\beta_{ij}} (A^{ijj} + A^{jij}). \quad (30)$$

Interchange indices i, j in (28a)-(28c) to arrive at the following ($i \neq j$):

$$A^{jj} = \alpha_j \psi_j, \quad (28a')$$

$$A^{ij} = \psi_j, \quad (28b')$$

$$A^{ji} + A^{ii} = \beta_{ij} \psi_j. \quad (28c')$$

Differentiating (28a')-(28c') with respect to u_j^i will give

$$\begin{aligned} A^{jji} &= \alpha_j \frac{\partial \psi_j}{\partial u_j^i}, \\ A^{iji} &= \frac{\partial \psi_j}{\partial u_j^i}, \\ A^{jii} + A^{iji} &= \beta_{ij} \frac{\partial \psi_j}{\partial u_j^i}. \end{aligned}$$

Eliminating $\frac{\partial \psi_j}{\partial u_j^i}$ from the previous equations gives

$$A^{jji} = \alpha_j A^{iji} = \frac{\alpha_j}{\beta_{ij}} (A^{jii} + A^{iji}). \quad (31)$$

Differentiating (28a')-(28c') again but with respect to u_i^j

and eliminating $\frac{\partial \psi_j}{\partial u_i^j}$ gives

$$A^{jji} = \alpha_j A^{iji} = \frac{\alpha_j}{\beta_{ij}} (A^{jii} + A^{iji}). \quad (32)$$

Now (29)-(32) yields

$$A^{iii} = \alpha_i A^{jji} = \frac{\alpha_i}{\beta_{ij}} (A^{iji} + A^{jji}), \quad (33a)$$

$$A^{iij} = \alpha_i A^{jjj} = \frac{\alpha_i}{\beta_{ij}} (A^{ijj} + A^{jjj}), \quad (33b)$$

$$A^{jjj} = \alpha_j A^{iji} = \frac{\alpha_j}{\beta_{ij}} (A^{jji} + A^{iji}), \quad (33c)$$

$$A^{jjj} = \alpha_j A^{iij} = \frac{\alpha_j}{\beta_{ij}} (A^{iij} + A^{iij}), \quad (33d)$$

with $i \neq j$. This is a system of eight equations in ten unknowns. Therefore, two A^{ijl} will be independent. We

choose these two to be A^{iji} and A^{jji} . Let

$A^{iji} = p_{ij}$, $A^{jji} = q_{ij}$, $i, j \in \{1, 2, 3\}$, $i \neq j$. Now, solve the system

(33a)-(33d) for all A^{ijl} (Class 1) in terms of p_{ij} , q_{ij} :

$$A^{iji} = p_{ij}, \quad (34-1)$$

$$A^{jji} = q_{ij}, \quad (34-2)$$

$$A^{iij} = \frac{q_{ij}}{\alpha_j}, \quad (34-3)$$

$$A^{ij}_{ii} = \frac{p_{ij}}{\alpha_j}, \quad (34-4)$$

$$A^{ij}_{ij} = \alpha_i q_{ij}, \quad (34-5)$$

$$A^{ij}_{jj} = \beta_{ij} q_{ij} - p_{ij}, \quad (34-6)$$

$$A^{ji}_{ii} = \frac{\beta_{ij} p_{ij} - q_{ij}}{\alpha_j}, \quad (34-7)$$

$$A^{ji}_{ji} = \frac{\beta_{ij} p_{ij} + (\alpha_i \alpha_j - 1) q_{ij}}{\alpha_j \beta_{ij}}, \quad (34-8)$$

$$A^{iii}_{ii} = \alpha_i \left(\frac{\beta_{ij} p_{ij} + (\alpha_i \alpha_j - 1) q_{ij}}{\alpha_j \beta_{ij}} \right), \quad (34-9)$$

$$A^{iii}_{jj} = \frac{((\beta_{ij})^2 - \alpha_i \alpha_j + 1) q_{ij} - \beta_{ij} p_{ij}}{\alpha_j \beta_{ij}}, \quad (34-10)$$

with $i \neq j$.

Now, let's examine Class 2 in which all three indices appear in A^{ijl}_{km} . In this class there are two subcases.

Subcase i: Two indices appear twice, e.g. A^{112}_{23} , A^{322}_{11} .

Subcase ii: One index appears three times, e.g.,

$$A^{112}_{13}, A^{212}_{23}.$$

The A^{ijl}_{km} in Subcase ii will later be shown to be zero. Those in Subcase i, though, will be needed to find the conservation laws quadratic in ∇u . All A^{ijl}_{km} in Subcase i are derived as follows. In the sequel i, j, k are distinct. First, in (28b) and (28c) replace i with k and j with i . Thus,

$$A^{jk}_{ij} = \psi_k, \quad (28b'')$$

$$A^{ki}_{ij} + A^{ji}_{ik} = \beta_{ik} \psi_k. \quad (28c'')$$

Differentiating both equations with respect to u_j^j and eliminating $\frac{\partial \psi_k}{\partial u_j^j}$ yields

$$A^{kij}_{ij} + A^{ijj}_{kj} = \beta_{ik} A^{jkj}_{jj}.$$

From (34-2), $A^{jkj}_{jj} = q_{kj}$. Substituting into this results

$$A^{kij}_{ij} + A^{ijj}_{kj} = \beta_{ik} q_{kj}. \quad (35)$$

Now, differentiating equation (28d) we have

$$A^{ijj}_{kj} + A^{kji}_{ij} = 0$$

$\Rightarrow$

$$A^{ijj}_{kj} = -A^{kji}_{ij}.$$

Also in equation (28d) switching i and j , we have

$$A^{j_i}_{k_i} + A^{k_i}_{j_i} = 0. \quad (28d')$$

Differentiating (28d') with respect to u_i^j gives

$$A^{j_{ij}}_{k_{ij}} + A^{k_{ij}}_{j_{ij}} = 0$$

and therefore,

$$A^{j_{ij}}_{k_{ij}} = -A^{k_{ij}}_{j_{ij}} = A^{i_{jj}}_{k_{jj}}.$$

Let the above be equal to t_k :

$$A^{j_{ij}}_{k_{ij}} = -A^{k_{ij}}_{j_{ij}} = A^{i_{jj}}_{k_{jj}} = t_k \quad (i, j, k \text{ distinct}). \quad (36)$$

Note t_k does not depend on i, j since $A^{k_{ij}}_{j_{ij}} = A^{k_{ji}}_{i_{ji}}$. Using

(28a) and (28c), we differentiate with respect to u_k^i and

eliminate $\frac{\partial \psi_i}{\partial u_k^i}$ to get

$$A^{i_{ji}}_{jk} + A^{j_{ji}}_{ik} = \frac{\beta_{ij}}{\alpha_i} A^{i_{ii}}_{ik}.$$

From (34-1), we know $A^{i_{ii}}_{ik} = p_{ki}$, so

$$A^{i_{ji}}_{jk} + A^{j_{ji}}_{ik} = \frac{\beta_{ij}}{\alpha_i} p_{ki}. \quad (37)$$

Thus, we have the following equations (from (35)-(37)):

$$A^{ij1}_{jk} + A^{jji}_{ik} = \frac{\beta_{ij}}{\alpha_i} p_{ki}, \quad (38-1)$$

$$A^{jij}_{ki} = -A^{kij}_{ji} = A^{iji}_{kj} = t_k, \quad (38-2)$$

$$A^{kij}_{ij} + A^{iji}_{kj} = \beta_{ik} q_{kj}, \quad (38-3)$$

i, j, k distinct.

We solve these equations. First

$A^{ijj}_{kj} = \frac{\beta_{ij}}{\alpha_i} p_{ki} - A^{jji}_{ik}$ from (38-1) and $A^{jji}_{ik} = t_k$ from (38-2) gives

$$A^{ijj}_{kj} = \frac{\beta_{ij}}{\alpha_i} p_{ki} - t_k. \quad (39)$$

Then, $A^{kij}_{ij} = \beta_{ik} q_{kj} - A^{iji}_{kj}$ from (38-3) and using (39) yields

$$A^{kij}_{ij} = \frac{\alpha_i \beta_{ik} q_{kj} - \beta_{ij} p_{ki}}{\alpha_i} + t_k. \quad (40)$$

To derive the rest of the A^{ij1}_{km} in this subcase, we begin with (28b) and replace j with i and i with k yielding

$$A^{ik}_{i} = \psi_k.$$

Differentiating with respect to u_j^j results in

$$A^{ikj}_{ij} = \frac{\partial \psi_k}{\partial u_j^j}.$$

From differentiation of (28b'') with respect to u_j^j , we know

$$A^{jkj}_{jj} = \frac{\partial \psi_k}{\partial u_j^j}. \quad \text{Combining these yields}$$

$$A^{ikj}_{ij} = A^{jkj}_{jj}.$$

$$\text{From (34-2), } A^{jkj}_{jj} = Q_{kj}$$

$$\Rightarrow A^{ikj}_{ij} = Q_{kj}. \quad (41)$$

Now, in equation (28d) replace i with k , j with i , k with j to get the following equation

$$A^{kji}_{ij} + A^{jii}_{ik} = 0.$$

Differentiating with respect to u_j^i gives

$$A^{kii}_{jj} + A^{jii}_{kj} = 0$$

$$\Rightarrow A^{kii}_{jj} = -A^{jii}_{kj}.$$

Differentiating equation (28b) with respect to u_k^i yields

$$A^{jii}_{kj} = \frac{\partial \psi_i}{\partial u_k^i}. \quad \text{Making this substitution we get}$$

$$A^{k_{ii}}_{jj} = -A^{j_{ii}}_{kj} = -\frac{\partial \psi_i}{\partial u_k^i}.$$

From differentiating equation (28a) with respect to u_k^i , we

know $\frac{\partial \psi_i}{\partial u_k^i} = \frac{1}{\alpha_i} A^{i_{ii}}_{ik}$. We make this substitution into the

previous equation to get

$$A^{k_{ii}}_{jj} = -A^{j_{ii}}_{kj} = -\frac{1}{\alpha_i} A^{i_{ii}}_{ik}.$$

From (34-2) $A^{i_{ii}}_{ik} = p_{ki}$, we get

$$A^{k_{ii}}_{jj} = -A^{j_{ii}}_{kj} = -\frac{p_{ki}}{\alpha_i}. \quad (42)$$

In (28d) interchange k, j to get

$$A^{i_k}_{j} + A^{j_k}_{i} = 0.$$

Differentiating with respect to u_j^i yields the following equation

$$A^{i_{ki}}_{jj} + A^{j_{ki}}_{ij} = 0$$

$\Rightarrow$

$$A^{i_{ki}}_{jj} = -A^{j_{ki}}_{ij}.$$

From (28b) differentiating with respect to u_i^k gives

$$A^{jki}_{ij} = \frac{\partial \psi_i}{\partial u_i^k}.$$

We substitute to get

$$A^{iki}_{jj} = -A^{jki}_{ij} = -\frac{\partial \psi_i}{\partial u_i^k}.$$

By differentiation of (28a) with respect to u_i^k , we know

$$\frac{\partial \psi_i}{\partial u_i^k} = \frac{1}{\alpha_i} A^{iik}_{ii}$$

⇒

$$A^{iki}_{jj} = -A^{jki}_{ij} = -\frac{1}{\alpha_i} A^{iik}_{ii}.$$

Utilizing the result in (34-2) gives

$$A^{iki}_{jj} = -A^{jki}_{ij} = -\frac{q_{ki}}{\alpha_i}. \quad (43)$$

Now, all possible A^{ijl}_{km} in this subcase have been found and solved in terms of p, q, t . They are (36), (39)-(43) below:

$$A^{jij}_{ki} = -A^{kij}_{ji} = A^{ijj}_{kj} = t_k, \quad (44-1)$$

$$A^{ijj}_{kj} = \frac{\beta_{ij} p_{ki}}{\alpha_i} - t_k, \quad (44-2)$$

$$A^{k_{ij}}_{ij} = \frac{\alpha_i \beta_{ik} q_{kj} - \beta_{ij} p_{ki}}{\alpha_i} + t_k, \quad (44-3)$$

$$A^{i_{kj}}_{ij} = q_{kj}, \quad (44-4)$$

$$A^{k_{ii}}_{jj} = -A^{j_{ii}}_{kj} = -\frac{p_{ki}}{\alpha_i}, \quad (44-5)$$

$$A^{i_{ki}}_{jj} = -A^{j_{ki}}_{ij} = -\frac{q_{ki}}{\alpha_i}, \quad (44-6)$$

i, j, k distinct.

From the ten equations (34-1)-(34-10) derived earlier and the eight equations (44-1)-(44-6) just derived, all the conservation laws quadratic in ∇u can be found (recall the derivatives in Case 2ii will be shown to be zero later). First, some identities must be established.

Lemma: $\frac{\beta_{ji}\beta_{ik}}{\beta_{jk}} = \alpha_i - 1, \quad (\beta_{ij})^2 = (\alpha_i - 1)(\alpha_j - 1), \quad i, j, k \in \{1, 2, 3\}$

Proof: From $\beta_{ji} = kc_j c_i$, permuting indices we get

$$\frac{\beta_{ji}\beta_{ik}}{\beta_{jk}} = \frac{(kc_j c_i)(kc_i c_k)}{(kc_j c_k)}$$

$$= kC_i^2 = \alpha_i - 1.$$

Also,

$$(\alpha_i - 1)(\alpha_j - 1) = (kC_i^2)(kC_j^2) = (kC_i C_j)^2 = (\beta_{ij})^2$$

■

Now, we establish a relationship among the q_{ij} quantities.

This relationship stems from consistency requirements which must be imposed upon equations (34-9) and (44-3). From (34-9)

$$A_{ii}^{iij} = \frac{\alpha_i(\beta_{ij}p_{ij} + (\alpha_i\alpha_j - 1)q_{ij})}{\alpha_j\beta_{ij}}, \quad i \neq j.$$

Look at the case $i=1$. Then

$$A_{11}^{111} = \frac{\alpha_1(\beta_{1j}p_{1j} + (\alpha_1\alpha_j - 1)q_{1j})}{\alpha_j\beta_{1j}}.$$

Now j is not specified, so $j=2$ or $j=3$ in this formula.

So,

$$\frac{\alpha_1(\beta_{12}p_{12} + (\alpha_1\alpha_2 - 1)q_{12})}{\alpha_2\beta_{12}} = \frac{\alpha_1(\beta_{13}p_{13} + (\alpha_1\alpha_3 - 1)q_{13})}{\alpha_3\beta_{13}}$$

$$\Rightarrow \frac{p_{12}}{\alpha_2} + \frac{(\alpha_1\alpha_2 - 1)q_{12}}{\alpha_2\beta_{12}} = \frac{p_{13}}{\alpha_3} + \frac{(\alpha_1\alpha_3 - 1)q_{13}}{\alpha_3\beta_{13}}. \quad (45)$$

From (44-3)

$$A_{ij}^{kij} = \frac{\alpha_i\beta_{ik}q_{kj} - \beta_{ij}p_{ki}}{\alpha_i} + t_k, \quad i, j, k \text{ distinct.}$$

Examine the case $k=1$. Thus,

$$A_{ij}^{1ij} = \frac{\alpha_i \beta_{ii} q_{1j} - \beta_{ij} p_{1i}}{\alpha_i} + t_1.$$

Since $A_{ij}^{1ij} = A_{ji}^{1ji}$, interchanging i, j should result in equivalent equations. Putting $i=2, j=3$ and vice versa

$$\frac{\alpha_2 \beta_{21} q_{13} - \beta_{23} p_{12}}{\alpha_2} + t_1 = \frac{\alpha_3 \beta_{31} q_{12} - \beta_{32} p_{13}}{\alpha_3} + t_1$$

$$\Rightarrow \frac{\beta_{21}}{\beta_{23}} q_{13} - \frac{1}{\alpha_2} p_{12} = \frac{\beta_{31}}{\beta_{23}} q_{12} - \frac{1}{\alpha_3} p_{13}. \quad (46)$$

Add (45) and (46) and use the lemma to get

$$\left[\frac{\alpha_1 \alpha_2 - 1}{\alpha_2} - (\alpha_1 - 1) \right] \frac{q_{12}}{\beta_{12}} = \left[\frac{\alpha_1 \alpha_3 - 1}{\alpha_3} - (\alpha_1 - 1) \right] \frac{q_{13}}{\beta_{13}}$$

$$\Rightarrow \left(\frac{\alpha_2 - 1}{\beta_{12} \alpha_2} \right) q_{12} = \left(\frac{\alpha_3 - 1}{\beta_{13} \alpha_3} \right) q_{13}.$$

$$\text{Let } \phi_{ij} = \frac{\alpha_j - 1}{\beta_{ij} \alpha_j}. \quad (47)$$

We have just shown $\phi_{12} q_{12} = \phi_{13} q_{13}$, and we can show by analogous calculations that

$$\phi_{ij} q_{ij} = \phi_{ik} q_{ik} \quad (48)$$

holds for all i, j, k distinct. Thus, $\phi_{ij} q_{ij}$ is independent of j for fixed i , and we let

$$q_i = \phi_{ij} q_{ij} \quad i, j \in \{1, 2, 3\}, \quad i \neq j, \quad (49)$$

which yields

$$q_{ij} = \frac{q_i}{\phi_{ij}}.$$

We make this substitution into equations (34-1)-(34-10) and (44-1)-(44-6). Next we let

$$A_{ii}^{iij} = \alpha_i \left(\frac{\beta_{ij} p_{ij} + (\alpha_i \alpha_j - 1) q_{ij}}{\alpha_j \beta_{ij}} \right) = \alpha_i r_i, \quad i = 1, 2, 3 \quad (50)$$

which is valid since A_{ii}^{iij} depends only on i . This gives

$$r_i = \frac{\beta_{ij} p_{ij} + (\alpha_i \alpha_j - 1) q_{ij}}{\alpha_j \beta_{ij}} = \frac{\beta_{ij} p_{ij} + (\alpha_i \alpha_j - 1) \frac{q_i}{\phi_{ij}}}{\alpha_j \beta_{ij}}.$$

Solving for p_{ij} yields

$$p_{ij} = \alpha_j r_i - \frac{\alpha_j (\alpha_i \alpha_j - 1) q_i}{\alpha_j - 1}. \quad (51)$$

Make these substitutions as well. Thus, equations (34-1) - (34-10) and (44-1) - (44-6) can be written in terms of r_i , q_i , t_k :

$$A_{ij}^{jjj} = \alpha_j r_i - \frac{\alpha_j (\alpha_i \alpha_j - 1) q_i}{\alpha_j - 1}, \quad (52-1)$$

$$A^{jij} = \frac{\beta_{ij}\alpha_j q_i}{\alpha_j - 1}, \quad (52-2)$$

$$A^{iij} = \frac{\beta_{ij} q_i}{\alpha_j - 1}, \quad (52-3)$$

$$A^{iij} = r_i - \frac{(\alpha_i \alpha_j - 1)}{\alpha_j - 1} q_i, \quad (52-4)$$

$$A^{iij} = \frac{\alpha_i \alpha_j \beta_{ij} q_i}{\alpha_j - 1}, \quad (52-5)$$

$$A^{ijj} = \frac{\alpha_j ((\beta_{ij})^2 + \alpha_i \alpha_j - 1)}{\alpha_j - 1} q_i - \alpha_j r_i, \quad (52-6)$$

$$A^{jii} = \beta_{ij} r_i - \frac{\alpha_i \alpha_j \beta_{ij} q_i}{\alpha_j - 1}, \quad (52-7)$$

$$A^{jii} = r_i, \quad (52-8)$$

$$A^{iij} = \alpha_i r_i, \quad (52-9)$$

$$A^{iij} = \frac{(\beta_{ij})^2 q_i}{\alpha_j - 1} - r_i, \quad (52-10)$$

$$A^{jij} = -A^{kij} = A^{ijk} = t_k, \quad (52-11)$$

$$A^{ijk} = \beta_{ij} r_k - \frac{\beta_{ij} (\alpha_k \alpha_i - 1)}{\alpha_i - 1} q_k - t_k, \quad (52-12)$$

$$A^{k_{ij}}_{ij} = \beta_{ij} \left[\frac{\alpha_j (\alpha_k - 1)}{\alpha_j - 1} + \frac{\alpha_k \alpha_i - 1}{\alpha_i - 1} \right] q_k - \beta_{ij} r_k + t_k, \quad (52-13)$$

$$A^{i_{kj}}_{ij} = \frac{\alpha_j \beta_{kj} q_k}{\alpha_j - 1}, \quad (52-14)$$

$$A^{k_{ji}}_{jj} = -A^{j_{ii}}_{kj} = \frac{\alpha_k \alpha_i - 1}{\alpha_i - 1} q_k - r_k, \quad (52-15)$$

$$A^{i_{ki}}_{jj} = -A^{j_{ki}}_{ij} = -\frac{\beta_{ki} q_k}{\alpha_i - 1}, \quad (52-16)$$

with $i, j, k \in \{1, 2, 3\}$, i, j, k distinct. From these the nine conservation laws quadratic in ∇u can be found. To find the P^i quantities, we will substitute $r_i = 1$, $q_i = 0$, $t_k = 0$. To find the conserved quantity, P^1 , let $r_1 = 1$, $r_2 = r_3 = 0$, $q_i = 0$, $t_k = 0$. Solving equations (52-1) - (52-16) yields the following:

$$\begin{aligned} P^1 &= (P^1_1, P^1_2, P^1_3) \\ &= \left(\frac{1}{2} \alpha_1 u_1^1 u_1^1 - \frac{1}{2} u_2^1 u_2^1 - \frac{1}{2} u_3^1 u_3^1 - \frac{1}{2} \alpha_2 u_2^2 u_2^2 - \frac{1}{2} \alpha_3 u_3^3 u_3^3 + \frac{1}{2} u_1^2 u_1^2 + \frac{1}{2} u_1^3 u_1^3 - \beta_{23} u_2^2 u_3^3 \right. \\ &\quad \left. - \frac{1}{2} u_2^3 u_3^3 - \frac{1}{2} u_3^2 u_3^2, \alpha_2 u_1^2 u_2^2 + \beta_{12} u_1^1 u_1^2 + u_2^1 u_1^1 + \beta_{23} u_1^2 u_3^3 + u_1^3 u_2^3, \alpha_3 u_1^3 u_3^3 + \right. \\ &\quad \left. \beta_{13} u_1^1 u_1^3 + u_3^1 u_1^1 + \beta_{23} u_1^3 u_2^2 + u_1^2 u_3^2 \right). \end{aligned}$$

Similarly, we have (with $r_2 = 1$, $r_1 = r_3 = 0$, $q_i = 0$, $t_k = 0$)

$$\begin{aligned} P^2 &= (P^2_1, P^2_2, P^2_3) \\ &= \left(\alpha_1 u_2^1 u_1^1 + \beta_{21} u_2^2 u_2^1 + u_1^2 u_2^2 + \beta_{13} u_2^1 u_3^3 + u_2^3 u_1^3, \frac{1}{2} u_2^1 u_2^1 + \frac{1}{2} u_2^3 u_2^3 - \frac{1}{2} \alpha_1 u_1^1 u_1^1 \right. \\ &\quad \left. - \frac{1}{2} \alpha_3 u_3^3 u_3^3 + \frac{1}{2} \alpha_2 u_2^2 u_2^2 - \frac{1}{2} u_3^2 u_3^2 - \frac{1}{2} u_1^2 u_1^2 - \beta_{13} u_1^1 u_3^3 - \frac{1}{2} u_3^1 u_3^1 - \frac{1}{2} u_1^3 u_1^3, \alpha_3 u_2^3 u_3^3 \right. \\ &\quad \left. + \beta_{23} u_2^2 u_3^3 + u_3^2 u_2^2 + \beta_{31} u_2^3 u_1^1 + u_2^1 u_3^1 \right) \end{aligned}$$

and (with $r_3=1$, $r_1=r_2=0$, $q_i=0$, $t_k=0$)

$$P^3 = (P_1^3, P_2^3, P_3^3) \\ = (\alpha_1 u_3^1 u_1^1 + \beta_{31} u_3^3 u_3^1 + u_1^3 u_3^3 + \beta_{12} u_3^1 u_2^2 + u_3^2 u_1^2, \alpha_2 u_3^2 u_2^2 + \beta_{32} u_3^3 u_3^2 + u_2^3 u_3^3 \\ + \beta_{21} u_3^2 u_1^1 + u_3^1 u_2^1, \frac{1}{2} u_3^1 u_3^1 + \frac{1}{2} u_3^2 u_3^2 - \frac{1}{2} \alpha_1 u_1^1 u_1^1 - \frac{1}{2} \alpha_2 u_2^2 u_2^2 - \frac{1}{2} u_1^3 u_3^3 - \frac{1}{2} u_2^3 u_3^3 \\ + \frac{1}{2} \alpha_3 u_3^3 u_3^3 - \beta_{12} u_1^1 u_2^2 - \frac{1}{2} u_2^1 u_2^1 - \frac{1}{2} u_1^2 u_1^2).$$

As can be seen $P_j^i = \sum_{m=1}^3 u_i^m u_j^m + u_i^j \sum_{m=1}^3 \beta_{jm} u_m^m - \frac{1}{2} \delta_j^i \sum_{m,l=1}^3 [u_l^m u_l^m + \beta_{ml} u_m^m u_l^1]$

(21).

Likewise, the Q^i 's can be found. Letting $q_i=1$, $r_i=0$, $t_k=0$ will yield each Q^i conserved quantity.

To find Q^1 , let $q_1=1$ and all other variables equal zero.

The following conserved quantity results when equations (52-1)

-(52-16) are solved:

$$Q^1 = (Q_1^1, Q_2^1, Q_3^1) \\ = (\frac{\beta_{12}}{\alpha_2-1} u_1^2 u_2^1 + \frac{\beta_{13}}{\alpha_3-1} u_1^3 u_3^1 - \frac{1}{2} \frac{\alpha_1 \alpha_2 - 1}{\alpha_2 - 1} u_1^2 u_1^2 - \frac{1}{2} \frac{\alpha_1 \alpha_3 - 1}{\alpha_3 - 1} u_1^3 u_1^3 + \frac{\alpha_1 \alpha_2 \beta_{12}}{\alpha_2 - 1} u_1^1 u_2^2 + \frac{\alpha_1 \alpha_3 \beta_{13}}{\alpha_3 - 1} u_1^1 u_3^3 \\ + \frac{1}{2} \frac{\alpha_2 (\beta_{12}^2 + \alpha_1 \alpha_2 - 1)}{\alpha_2 - 1} u_2^2 u_2^2 + \frac{1}{2} \frac{\alpha_3 (\beta_{13}^2 + \alpha_1 \alpha_3 - 1)}{\alpha_3 - 1} u_3^3 u_3^3 + \frac{1}{2} \frac{\beta_{12}^2}{\alpha_2 - 1} u_2^1 u_2^1 + \frac{1}{2} \frac{\beta_{13}^2}{\alpha_3 - 1} u_3^1 u_3^1 \\ + \beta_{23} (\frac{\alpha_3 (\alpha_1 - 1)}{\alpha_3 - 1} + \frac{\alpha_1 \alpha_2 - 1}{\alpha_2 - 1}) u_2^2 u_3^3 + \frac{1}{2} \frac{\alpha_1 \alpha_2 - 1}{\alpha_2 - 1} u_3^2 u_3^2 + \frac{1}{2} \frac{\alpha_1 \alpha_3 - 1}{\alpha_3 - 1} u_2^3 u_2^3, - \frac{\alpha_2 (\alpha_1 \alpha_2 - 1)}{\alpha_2 - 1} u_1^2 u_2^2 \\ + \frac{\beta_{12} \alpha_2}{\alpha_2 - 1} u_2^1 u_2^2 - \frac{\alpha_1 \alpha_2 \beta_{12}}{\alpha_2 - 1} u_1^1 u_1^2 - \frac{\beta_{23} (\alpha_1 \alpha_2 - 1)}{\alpha_2 - 1} u_1^2 u_3^3 + \frac{\alpha_3 \beta_{13}}{\alpha_3 - 1} u_2^1 u_3^3 - \frac{\alpha_1 \alpha_3 - 1}{\alpha_3 - 1} u_1^3 u_2^3 \\ - \frac{\beta_{12}}{\alpha_2 - 1} u_3^1 u_3^2 + \frac{\beta_{13}}{\alpha_3 - 1} u_3^1 u_2^3, - \frac{\alpha_3 (\alpha_1 \alpha_3 - 1)}{\alpha_3 - 1} u_1^3 u_3^3 + \frac{\beta_{13} \alpha_3}{\alpha_3 - 1} u_3^1 u_3^3 - \frac{\alpha_1 \alpha_3 \beta_{13}}{\alpha_3 - 1} u_1^1 u_1^3 \\ - \frac{\beta_{23} (\alpha_1 \alpha_3 - 1)}{\alpha_3 - 1} u_1^3 u_2^2 + \frac{\alpha_2 \beta_{12}}{\alpha_2 - 1} u_3^1 u_2^2 - \frac{\alpha_1 \alpha_2 - 1}{\alpha_2 - 1} u_1^2 u_3^2 - \frac{\beta_{13}}{\alpha_3 - 1} u_2^1 u_2^3 + \frac{\beta_{12}}{\alpha_2 - 1} u_2^1 u_3^2).$$

Similarly with $q_2=1$ and all other variables equal zero, we get

$$\begin{aligned}
Q^2 &= (Q_1^2, Q_2^2, Q_3^2) \\
&= \left(-\frac{\alpha_1(\alpha_2\alpha_1-1)}{\alpha_1-1} u_2^1 u_1^1 + \frac{\beta_{21}\alpha_1}{\alpha_1-1} u_1^2 u_1^1 - \frac{\alpha_2\alpha_1\beta_{21}}{\alpha_1-1} u_2^2 u_2^1 - \frac{\beta_{13}(\alpha_2\alpha_1-1)}{\alpha_1-1} u_2^1 u_3^3 + \frac{\alpha_3\beta_{23}}{\alpha_3-1} u_1^2 u_3^3 \right. \\
&\quad - \frac{\alpha_2\alpha_3-1}{\alpha_3-1} u_2^3 u_1^3 - \frac{\beta_{21}}{\alpha_1-1} u_3^2 u_3^1 + \frac{\beta_{23}}{\alpha_3-1} u_3^2 u_1^3, \quad \frac{\beta_{21}}{\alpha_1-1} u_2^1 u_1^2 + \frac{\beta_{23}}{\alpha_3-1} u_2^3 u_3^2 - \frac{1}{2} \frac{\alpha_2\alpha_1-1}{\alpha_1-1} u_2^1 u_2^1 \\
&\quad - \frac{1}{2} \frac{\alpha_2\alpha_3-1}{\alpha_3-1} u_2^3 u_2^3 + \frac{\alpha_2\alpha_1\beta_{21}}{\alpha_1-1} u_2^2 u_1^1 + \frac{\alpha_2\alpha_3\beta_{23}}{\alpha_3-1} u_2^2 u_3^3 + \frac{1}{2} \frac{\alpha_1(\beta_{21}^2 + \alpha_1\alpha_2-1)}{\alpha_1-1} u_1^1 u_1^1 \\
&\quad + \frac{1}{2} \frac{\alpha_3(\beta_{23}^2 + \alpha_2\alpha_3-1)}{\alpha_3-1} u_3^3 u_3^3 + \frac{1}{2} \frac{\beta_{21}^2}{\alpha_1-1} u_1^2 u_1^2 + \frac{1}{2} \frac{\beta_{23}^2}{\alpha_3-1} u_3^2 u_3^2 + \beta_{13} \left[\frac{\alpha_3(\alpha_2-1)}{\alpha_3-1} + \frac{\alpha_2\alpha_1-1}{\alpha_1-1} \right] u_1^1 u_3^3 \\
&\quad \frac{1}{2} \frac{\alpha_2\alpha_1-1}{\alpha_1-1} u_3^1 u_3^1 + \frac{1}{2} \frac{\alpha_2\alpha_3-1}{\alpha_3-1} u_1^3 u_1^3, \quad -\frac{\alpha_3(\alpha_2\alpha_3-1)}{\alpha_3-1} u_2^3 u_3^3 + \frac{\beta_{23}\alpha_3}{\alpha_3-1} u_3^2 u_3^3 - \frac{\alpha_2\alpha_3\beta_{23}}{\alpha_3-1} u_2^2 u_3^3 \\
&\quad \left. - \frac{\beta_{13}(\alpha_2\alpha_3-1)}{\alpha_3-1} u_2^3 u_1^1 + \frac{\alpha_1\beta_{21}}{\alpha_1-1} u_3^2 u_1^1 - \frac{\alpha_2\alpha_1-1}{\alpha_1-1} u_2^1 u_3^1 - \frac{\beta_{23}}{\alpha_3-1} u_1^2 u_1^3 + \frac{\beta_{21}}{\alpha_1-1} u_1^2 u_3^1 \right)
\end{aligned}$$

and with $q_3=1$ instead

$$\begin{aligned}
Q^3 &= (Q_1^3, Q_2^3, Q_3^3) \\
&= \left(-\frac{\alpha_1(\alpha_3\alpha_1-1)}{\alpha_1-1} u_3^1 u_1^1 + \frac{\beta_{31}\alpha_1}{\alpha_1-1} u_1^3 u_1^1 - \frac{\alpha_3\alpha_1\beta_{31}}{\alpha_1-1} u_3^3 u_3^1 - \frac{\beta_{12}(\alpha_3\alpha_1-1)}{\alpha_1-1} u_3^1 u_2^2 + \frac{\alpha_2\beta_{32}}{\alpha_2-1} u_1^3 u_2^2 \right. \\
&\quad - \frac{\alpha_3\alpha_2-1}{\alpha_2-1} u_3^2 u_1^2 - \frac{\beta_{31}}{\alpha_1-1} u_2^3 u_2^1 + \frac{\beta_{32}}{\alpha_2-1} u_2^3 u_1^2, \quad -\frac{\alpha_2(\alpha_3\alpha_2-1)}{\alpha_2-1} u_3^2 u_2^2 + \frac{\beta_{32}\alpha_2}{\alpha_2-1} u_2^3 u_2^2 - \frac{\alpha_2\alpha_3\beta_{32}}{\alpha_2-1} u_3^3 u_3^2 \\
&\quad - \frac{\beta_{12}(\alpha_3\alpha_2-1)}{\alpha_2-1} u_3^2 u_1^1 + \frac{\alpha_1\beta_{31}}{\alpha_1-1} u_2^3 u_1^1 - \frac{\alpha_3\alpha_1-1}{\alpha_1-1} u_3^1 u_2^1 - \frac{\beta_{32}}{\alpha_2-1} u_1^3 u_1^2 + \frac{\beta_{31}}{\alpha_1-1} u_1^3 u_2^1, \quad \frac{\beta_{31}}{\alpha_1-1} u_3^1 u_1^3 \\
&\quad + \frac{\beta_{32}}{\alpha_2-1} u_2^3 u_2^3 - \frac{1}{2} \frac{\alpha_3\alpha_1-1}{\alpha_1-1} u_3^1 u_3^1 - \frac{1}{2} \frac{\alpha_3\alpha_2-1}{\alpha_2-1} u_3^2 u_3^2 + \frac{\alpha_3\alpha_1\beta_{31}}{\alpha_1-1} u_3^3 u_1^1 + \frac{\alpha_3\alpha_2\beta_{32}}{\alpha_2-1} u_3^3 u_2^2 \\
&\quad + \frac{1}{2} \frac{\alpha_1(\beta_{31}^2 + \alpha_3\alpha_3-1)}{\alpha_1-1} u_1^1 u_1^1 + \frac{1}{2} \frac{\alpha_2(\beta_{32}^2 + \alpha_3\alpha_3-1)}{\alpha_2-1} u_2^2 u_2^2 + \frac{1}{2} \frac{\beta_{31}^2}{\alpha_1-1} u_1^3 u_1^3 + \frac{1}{2} \frac{\beta_{32}^2}{\alpha_2-1} u_2^3 u_2^3 \\
&\quad \left. + \beta_{12} \left[\frac{\alpha_2(\alpha_3-1)}{\alpha_2-1} + \frac{\alpha_3\alpha_1-1}{\alpha_1-1} \right] u_1^1 u_2^2 + \frac{1}{2} \frac{\alpha_3\alpha_1-1}{\alpha_1-1} u_2^1 u_2^1 + \frac{1}{2} \frac{\alpha_3\alpha_2-1}{\alpha_2-1} u_1^2 u_1^2 \right).
\end{aligned}$$

As can be seen Q^i can be found from

$$\begin{aligned}
Q_j^i &= \sum_{k,l=1}^3 \left(\frac{\alpha_k\beta_{ik}}{\alpha_k-1} u_k^k u_j^i + \frac{\beta_{ik}}{\alpha_k-1} u_k^i u_j^k - \frac{\beta_{ij}}{\alpha_j-1} u_k^i u_k^j - \frac{(\alpha_i\alpha_j-1)\beta_{jk}}{\alpha_j-1} u_k^k u_i^j - \frac{\alpha_i\alpha_k-1}{\alpha_k-1} u_i^k u_j^k \right. \\
&\quad \left. + \frac{1}{2} \delta_j^i \left[\frac{\alpha_i\alpha_k-1}{\alpha_k-1} u_l^k u_l^k + \gamma_{ikl} u_k^k u_l^1 \right] \right)
\end{aligned}$$

where $\gamma_{ikl} = \beta_{kl} \left(\frac{\alpha_i\alpha_k-1}{\alpha_k-1} + \frac{\alpha_i\alpha_l-1}{\alpha_l-1} - 1 \right)$ (22) (cf. Lemma).

The only laws left to derive in this case are the trivial quadratic ones. These can be found by letting

$t_k=2, r_i=0, q_i=0$. These yield

$$\tilde{A}^1 = (\tilde{A}_1^1, \tilde{A}_2^1, \tilde{A}_3^1) = 2 (-u_3^2 u_2^3 + u_2^2 u_3^3, u_1^3 u_3^2 - u_1^2 u_3^3, u_1^2 u_2^3 - u_1^3 u_2^2),$$

$$\tilde{A}^2 = (\tilde{A}_1^2, \tilde{A}_2^2, \tilde{A}_3^2) = 2 (u_2^3 u_3^1 - u_2^1 u_3^3, -u_3^1 u_1^3 + u_1^1 u_3^3, u_2^1 u_1^3 - u_2^3 u_1^1),$$

$$\tilde{A}^3 = (\tilde{A}_1^3, \tilde{A}_2^3, \tilde{A}_3^3) = 2 (u_3^2 u_2^1 - u_3^1 u_2^2, u_3^1 u_1^2 - u_3^2 u_1^1, -u_2^1 u_1^2 + u_1^1 u_2^2).$$

Thus, $\tilde{A}^i = \sum_{k,l,m,n=1}^3 \epsilon_{ikn} \epsilon_{jlm} u_l^k u_m^n$ (23). These nine laws are all

the possible conservation laws quadratic in ∇u (21) - (23).

At this point we must show that in Class 2, subcase ii:

$A^{i_{j1}}_{km} = 0$ where one index appears three times and the others once each. In what is to follow i, j, k are distinct.

Differentiating (28a) and (28c) with respect to u_k^j and eliminating $\frac{\partial \psi_i}{\partial u_k^j}$ gives

$$A^{i_{ij}}_{ik} = \frac{\alpha_i}{\beta_{ik}} (A^{i_{kj}}_{kk} + A^{k_{kj}}_{ik}). \quad (53)$$

Now, differentiating (28b) and (28d) with respect to u_i^k and u_k^k respectively gives the following:

$$A^{k_{jk}}_{ki} = \frac{\partial \psi_j}{\partial u_i^k} = A^{i_{jk}}_{ii},$$

$$A^{i_{jk}}_{kk} + A^{k_{jk}}_{ik} = 0$$

$$\Rightarrow A^{ijk}_{kk} = -A^{kjk}_{ik}.$$

We make these two substitutions into (53) to get

$$A^{ijj}_{ik} = \frac{\alpha_i}{\beta_{ik}} (-A^{kjk}_{ik} + A^{ijj}_{ii}). \quad (54)$$

Differentiating (28a) and (28b) with respect to u_i^j and

eliminating $\frac{\partial \psi_k}{\partial u_i^j}$ yields

$$A^{kij}_{ki} = \alpha_k A^{ikj}_{ii}.$$

We make this substitution into (54):

$$A^{ijj}_{ik} = \frac{\alpha_i}{\beta_{ik}} (-\alpha_k A^{ikj}_{ii} + A^{ijj}_{ii})$$

$$\Rightarrow A^{ijj}_{ik} = \frac{\alpha_i (1 - \alpha_k)}{\beta_{ik}} A^{ijj}_{ii}. \quad (55)$$

Now, we will establish a second equation using (28a), (28b) and differentiating with respect to u_k^j ,

$$A^{ijj}_{ik} = \alpha_i A^{kij}_{kk}. \quad (56)$$

Differentiation of (28b) and (28c) with respect to u_k^i gives

$$A^{jki}_{kk} + A^{kji}_{jk} = \beta_{jk} \frac{\partial \psi_j}{\partial u_k^i}$$

and

$$A^{kij}_{kk} = \frac{\partial \psi_j}{\partial u_k^i}$$

$\Rightarrow$

$$A^{kij}_{kk} = \frac{1}{\beta_{jk}} (A^{jki}_{kk} + A^{kji}_{jk}) .$$

We make this substitution into (56) to get

$$A^{iij}_{ik} = \frac{\alpha_i}{\beta_{jk}} (A^{jki}_{kk} + A^{kji}_{jk}) . \quad (57)$$

We differentiate (28d) with respect to u_k^k and (28a)

and (28b) with respect to u_j^k to get

$$A^{kik}_{jk} + A^{jik}_{kk} = 0$$

$\Rightarrow$

$$-A^{kik}_{jk} = A^{jik}_{kk} ,$$

and

$$A^{kik}_{kj} = \frac{1}{\alpha_i} A^{iik}_{ij} .$$

We make these substitutions into (57) to get

$$A^{iij}_{ik} = -\frac{\alpha_i}{\beta_{jk}} A^{kik}_{jk} + \frac{1}{\beta_{jk}} A^{iik}_{ij} . \quad (58)$$

We use (28a) and (28b) and differentiate with respect to u_j^i

and eliminate $\frac{\partial \psi_k}{\partial u_j^i}$ to get

$$A^{k_{kj}} = \alpha_k A^{j_{kj}}.$$

Substitute this result into (58) to get

$$A^{i_{ij}} = -\frac{\alpha_i \alpha_k}{\beta_{jk}} A^{j_{kj}} + \frac{1}{\beta_{jk}} A^{i_{ik}}. \quad (59)$$

We use (28a) and (28b) again but differentiate with respect to

u_j^k and eliminate $\frac{\partial \psi_i}{\partial u_j^k}$,

$$A^{j_{ik}} = \frac{1}{\alpha_i} A^{i_{ik}}.$$

We make this substitution into (59):

$$A^{i_{ij}} = -\frac{\alpha_k}{\beta_{jk}} A^{i_{ik}} + \frac{1}{\beta_{jk}} A^{i_{ik}}$$

$$\Rightarrow A^{i_{ij}} = \frac{1 - \alpha_k}{\beta_{jk}} A^{i_{ik}}. \quad (60)$$

In (55) $A^{i_{jk}} = \frac{\beta_{ik}}{\alpha_i (1 - \alpha_k)} A^{i_{ij}}$. $A^{i_{ij}}$ is symmetric in j, k which

means that interchanging j, k will yield an equivalent result

$$\frac{\beta_{ij}}{\alpha_i (1 - \alpha_j)} A^{i_{ik}} = \frac{\beta_{ik}}{\alpha_i (1 - \alpha_k)} A^{i_{ij}}$$

$$\Rightarrow A^{iik} = \frac{(1-\alpha_j)\beta_{ik}}{(1-\alpha_k)\beta_{ij}} A^{iij}.$$

We substitute this result into (60) to get

$$A^{iij} = \frac{(1-\alpha_j)\beta_{ik}}{\beta_{ij}\beta_{jk}} A^{iik}.$$

We use the lemma to get

$$A^{iik} = \frac{1-\alpha_j}{\alpha_j-1} A^{iij}$$

$$\Rightarrow 2A^{iij} = 0$$

$$\Rightarrow A^{iij} = 0.$$

This and calculations above also show

$$A^{ijk} = A^{kji} = A^{kjk} = A^{kik} = 0.$$

There are still two in this subcase left to show zero: A^{kii}

and A^{iij} . We differentiate (28b) and (28c) with respect to

u_j^i

$$\Rightarrow A^{iij} + A^{kii} = \beta_{ik} A^{ikj}.$$

From previous results $A^{kii} = 0$, so

$$A^{iij} + A^{kii} = 0.$$

We use (28d) and differentiate with respect to u_i^i to get

$$A^{k_{ii}}_{ji} = -A^{j_{ii}}_{ki}.$$

These two equations give

$$A^{i_{ii}}_{kj} - A^{j_{ii}}_{ik} = 0. \quad (61)$$

Now, differentiate (28a) and (28c) with respect to u_k^i

$$\Rightarrow A^{j_{ii}}_{ik} + A^{i_{ii}}_{jk} = \frac{\beta_{ij}}{\alpha_j} A^{j_{ji}}_{jk}.$$

Again from above we have $A^{j_{ji}}_{jk} = 0$, so

$$A^{j_{ii}}_{ik} + A^{i_{ii}}_{jk} = 0. \quad (62)$$

Add (61) and (62) to get $A^{i_{ii}}_{jk} = 0$. Then (61) yields $A^{j_{ii}}_{ki} = 0$.

This establishes that all $A^{i_{jl}}_{km}$ in this subcase are equal to zero.

To prove there are no other conservation laws for this system, we show that all third-order derivatives $A^{i_{jln}}_{kmp}$ are

zero, where $A^{i_{jln}}_{kmp} = \frac{\partial A^{i_{jl}}_{km}}{\partial u_p^n}$. We adopt the symbol $A^{i_{jln}}_{kmp} = (i|jln)_{kmp}$,

cf. Olver [11].

Case I: The first case is when at most two index values appear in i, j, k, l, m, n, p . If we have such a symbol, it can be written as a linear combination of symbols with all three indices appearing.

Here is an example:

$$\begin{aligned} \left(1 \left| \begin{smallmatrix} 1 & 2 & n \\ 2 & 1 & p \end{smallmatrix} \right. \right) &= \frac{\partial}{\partial u_p^n} A^{\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}} \\ &= \frac{\partial}{\partial u_p^n} \left(\frac{\beta_{12} q_1}{\alpha_2 - 1} \right) && \text{from (52-3)} \\ &= \frac{\beta_{12}}{\alpha_2 - 1} \frac{\partial}{\partial u_p^n} (q_1). \end{aligned}$$

Using (52-14) we get

$$\begin{aligned} \left(1 \left| \begin{smallmatrix} 1 & 2 & n \\ 2 & 1 & p \end{smallmatrix} \right. \right) &= \frac{\beta_{12}}{\alpha_2 - 1} \frac{\partial}{\partial u_p^n} \left(\frac{\alpha_2 - 1}{\alpha_2 \beta_{12}} A^{\begin{smallmatrix} 3 & 1 \\ 2 & 2 \end{smallmatrix}} \right) \\ &= \frac{1}{\alpha_2} \left(3 \left| \begin{smallmatrix} 1 & 2 & n \\ 3 & 2 & p \end{smallmatrix} \right. \right). \end{aligned}$$

Likewise, if r_k appears we use (52-14) and (52-15). Thus, all derivatives in this case can be written as linear combinations of derivatives in which all three indices appear. It suffices to show all of the latter are zero which is the next case.

Case II: This is the case where all three indices appear in the symbol. The term inversion will mean that column entries are interchanged, i.e., the column $\begin{pmatrix} j \\ k \end{pmatrix}$ would become $\begin{pmatrix} k \\ j \end{pmatrix}$.

Here a column lies to the right of the bar in the symbol.

The "3-1-1 rule" will be used extensively throughout this proof. Suppose that in i, j, k, l, m, m, n, p among i and two columns, one index appears three times and the other two once each, for example in $i, \binom{j}{k}, \binom{l}{m}$. Then we utilize Case 2ii previously examined which gives $A^{ijl}_{km} = 0$. This implies $\left(i \middle| \binom{j}{k} \binom{l}{m} n\right)$

must also be zero since $\left(i \middle| \binom{j}{k} \binom{l}{m} n\right) = \frac{\partial}{\partial u_p^n} A^{ijl}_{km}$.

A direct result of this rule is that if any index appears five or more times then that symbol is zero.

Case IIa: In this case we will consider those symbols in which one index value occurs exactly four times in i, j, k, l, m, n, p . These symbols are :

$$\left(i \middle| \binom{i}{i} \binom{i}{k} j\right), \left(i \middle| \binom{i}{i} \binom{i}{j} k\right), \left(i \middle| \binom{i}{j} \binom{i}{j} k\right), \left(j \middle| \binom{i}{i} \binom{i}{i} k\right), \left(j \middle| \binom{i}{i} \binom{i}{j} k\right), \left(k \middle| \binom{i}{i} \binom{i}{i} j\right), \left(k \middle| \binom{i}{i} \binom{i}{j} j\right),$$

and also those symbols found through inversions of columns.

Here and throughout the rest of the proof, i, j, k are distinct. First, we have $\left(i \middle| \binom{i}{i} \binom{i}{j} k\right) = \left(i \middle| \binom{i}{j} \binom{i}{j} k\right) = \left(j \middle| \binom{i}{i} \binom{i}{i} k\right) = \left(k \middle| \binom{i}{i} \binom{i}{j} j\right) = 0$ due

to the 3-1-1 rule in II above. To prove the rest are zero, we use equations (28a) - (28d). Now,

$$\left(i \middle| \binom{i}{i} \binom{i}{k} j\right) = \frac{\partial}{\partial u_j^j} A^{iik}_{jk}.$$

Differentiating (28a) and (28b) with respect to u_k^i gives

$$A^{i i i}_{i k} = \alpha_i \frac{\partial \psi_i}{\partial u_k^i},$$

$$A^{j j i}_{j k} = \frac{\partial \psi_i}{\partial u_k^i}$$

$$\Rightarrow A^{i i i}_{i k} = \alpha_i A^{j j i}_{j k}.$$

Then,

$$\left(i \left| \begin{smallmatrix} i & i & j \\ i & k & j \end{smallmatrix} \right. \right) = \alpha_i \frac{\partial}{\partial u_j^j} A^{j j i}_{j k}$$

$$\Rightarrow \left(i \left| \begin{smallmatrix} i & i & j \\ i & k & j \end{smallmatrix} \right. \right) = \alpha_i \left(j \left| \begin{smallmatrix} i & i & j \\ j & k & j \end{smallmatrix} \right. \right).$$

We use the 3-1-1 rule to get

$$\left(j \left| \begin{smallmatrix} i & i & j \\ j & k & j \end{smallmatrix} \right. \right) = 0$$

$$\Rightarrow \left(i \left| \begin{smallmatrix} i & i & j \\ i & k & j \end{smallmatrix} \right. \right) = 0. \quad (63)$$

This also holds if columns are inverted.

Next,

$$\left(j \left| \begin{smallmatrix} i & i & j \\ i & i & k \end{smallmatrix} \right. \right) = \frac{\partial}{\partial u_k^j} A^{j i i}_{i i}.$$

We use (28c) and differentiate with respect to u_i^i to get

$$A^{jii} = -A^{iji} + \beta_{ij} \frac{\partial \psi_j}{\partial u_i^i}.$$

Then,

$$\begin{aligned} (j|i \ i \ i \ j) &= -\frac{\partial}{\partial u_k^j} A^{iji} + \beta_{ij} \frac{\partial}{\partial u_k^j} \left(\frac{\partial \psi_j}{\partial u_i^i} \right) \\ &= -(i|j \ i \ i \ k) + \beta_{ij} \frac{\partial^2 \psi_j}{\partial u_k^j \partial u_i^i}. \end{aligned}$$

We differentiate (28a) with respect to u_i^i to get

$$\frac{\partial \psi_j}{\partial u_i^i} = \frac{1}{\alpha_j} A^{jji};$$

so,

$$\begin{aligned} (j|i \ i \ i \ j) &= -(i|j \ i \ i \ k) + \frac{\beta_{ij}}{\alpha_j} \frac{\partial}{\partial u_k^j} A^{jji} \\ &= -(i|j \ i \ i \ k) + \frac{\beta_{ij}}{\alpha_j} (j|j \ i \ i \ k) \end{aligned}$$

$\Rightarrow$

$$(j|i \ i \ i \ j) = 0$$

by the 3-1-1 rule and (63). This also hold with inversions.

Lastly,

$$(k|i \ i \ i \ j) = \frac{\partial}{\partial u_j^j} A^{kii}.$$

By (28b) and (28c) differentiating with respect to u_i^i yields

$$A^{k_{ii}}_{ii} = -A^{i_{ii}}_{ki} + \beta_{ik} \frac{\partial \psi_k}{\partial u_i^i},$$

$$A^{i_{ki}}_{ii} = \frac{\partial \psi_k}{\partial u_i^i}$$

$$\Rightarrow A^{k_{ii}}_{ii} = -A^{i_{ii}}_{ki} + \beta_{ik} A^{i_{ki}}_{ii}.$$

We make this substitution into the above to get

$$\begin{aligned} \left(k \left| \begin{smallmatrix} i & i & j \\ i & i & j \end{smallmatrix} \right.\right) &= -\frac{\partial}{\partial u_j^j} A^{i_{ii}}_{ki} + \beta_{ik} \frac{\partial}{\partial u_j^j} A^{i_{ki}}_{ii} \\ &= -\left(i \left| \begin{smallmatrix} i & i & j \\ k & i & j \end{smallmatrix} \right.\right) + \beta_{ik} \left(i \left| \begin{smallmatrix} k & i & j \\ i & i & j \end{smallmatrix} \right.\right) \\ &= 0 \end{aligned}$$

by (63). This proves all symbols in Case IIa are zero.

Case IIb: In this case we consider those symbols in which two indices appear exactly three times and the third index once.

These symbols are

$$\left(i \left| \begin{smallmatrix} i & i & j \\ j & j & k \end{smallmatrix} \right.\right), \left(i \left| \begin{smallmatrix} i & i & j \\ j & k & j \end{smallmatrix} \right.\right), \left(i \left| \begin{smallmatrix} i & j & j \\ i & j & k \end{smallmatrix} \right.\right), \left(k \left| \begin{smallmatrix} i & i & i \\ j & j & j \end{smallmatrix} \right.\right), \left(k \left| \begin{smallmatrix} i & i & j \\ i & j & j \end{smallmatrix} \right.\right),$$

and their inversions. Using the 3-1-1 rule in the above group,

$$\left(i \left| \begin{smallmatrix} i & i & j \\ j & j & k \end{smallmatrix} \right.\right) = \left(i \left| \begin{smallmatrix} i & j & j \\ i & j & k \end{smallmatrix} \right.\right) = \left(k \left| \begin{smallmatrix} i & i & j \\ i & j & j \end{smallmatrix} \right.\right) = 0.$$

Now for the rest. We start with

$$\left(i \left| \begin{smallmatrix} i & i & j \\ j & j & k \end{smallmatrix} \right.\right) = \frac{\partial}{\partial u_k^j} A^{i i i}_{j j j}. \quad (64)$$

By (52-10)

$$A^{i i i}_{j j j} = \frac{\beta_{ij}^2}{\alpha_j - 1} q_i - r_i. \quad (65)$$

From (52-2) and (52-1)

$$q_i = \frac{\alpha_j - 1}{\beta_{ij} \alpha_j} A^{j j j}_{i i i},$$

$$r_i = \frac{1}{\alpha_j} A^{j j j}_{i i i} + \frac{\alpha_i \alpha_j - 1}{\alpha_j - 1} q_i.$$

We substitute these into (65) to get

$$A^{i i i}_{j j j} = \frac{\beta_{ij}^2 - \alpha_i \alpha_j + 1}{\alpha_j \beta_{ij}} A^{j j j}_{i i i} - \frac{1}{\alpha_j} A^{j j j}_{i i i}.$$

So, by (64) and then using the 3-1-1 rule

$$\left(i \left| \begin{smallmatrix} i & i & j \\ j & j & k \end{smallmatrix} \right.\right) = \frac{\beta_{ij}^2 - \alpha_i \alpha_j + 1}{\alpha_j \beta_{ij}} \left(j \left| \begin{smallmatrix} i & j & j \\ j & j & k \end{smallmatrix} \right.\right) - \frac{1}{\alpha_j} \left(j \left| \begin{smallmatrix} j & j & j \\ i & j & k \end{smallmatrix} \right.\right) = 0. \quad (66)$$

Inversions of this symbol are also zero by a similar argument.

Last in this case is

$$\left(k \left| \begin{smallmatrix} i & i & i \\ j & j & j \end{smallmatrix} \right.\right) = \frac{\partial}{\partial u_j^i} A^{k i i}_{j j j}.$$

From (28d) we get

$$A^{k_{ii}}_{jj} = -A^{j_{ii}}_{kj}.$$

So, by substitution and (66) we get

$$\left(k \middle| \begin{smallmatrix} i & i & i \\ j & j & j \end{smallmatrix} \right) = -\frac{\partial}{\partial u_j^i} A^{j_{ii}}_{kj} = -\left(j \middle| \begin{smallmatrix} i & i & i \\ k & j & j \end{smallmatrix} \right) = 0.$$

The same holds for inversions. Thus, all symbols in Case IIb have been shown to be zero.

Case IIc: The only case left to consider is that in which one index appears three times and the others twice each. These symbols are as follows:

$$\left(j \middle| \begin{smallmatrix} i & i & k \\ i & j & k \end{smallmatrix} \right), \left(j \middle| \begin{smallmatrix} i & i & j \\ i & k & k \end{smallmatrix} \right), \left(j \middle| \begin{smallmatrix} i & i & i \\ j & k & k \end{smallmatrix} \right), \left(i \middle| \begin{smallmatrix} i & i & j \\ k & k & j \end{smallmatrix} \right), \left(i \middle| \begin{smallmatrix} i & i & j \\ j & k & k \end{smallmatrix} \right), \left(i \middle| \begin{smallmatrix} i & j & j \\ i & k & k \end{smallmatrix} \right), \left(i \middle| \begin{smallmatrix} i & j & k \\ i & j & k \end{smallmatrix} \right),$$

and their inversions. By the 3-1-1 rule we get that the following and their inversions are zero:

$$\left(j \middle| \begin{smallmatrix} i & i & j \\ i & k & k \end{smallmatrix} \right) = \left(i \middle| \begin{smallmatrix} i & i & j \\ j & k & k \end{smallmatrix} \right) = \left(i \middle| \begin{smallmatrix} i & j & j \\ i & k & k \end{smallmatrix} \right) = 0.$$

Considering the rest, we start with

$$\left(j \middle| \begin{smallmatrix} i & i & k \\ i & j & k \end{smallmatrix} \right) = \frac{\partial}{\partial u_j^i} A^{j_{ik}}_{ik}. \quad (67)$$

From (28b) and (28c), we get

$$A^{j_{ik}}_{ik} = -A^{k_{ki}}_{ji} + \beta_{kj} \frac{\partial \psi_j}{\partial u_i^i},$$

$$A^{kji} = \frac{\partial \psi_j}{\partial u_i^i}.$$

So (67) yields

$$\begin{aligned} (j | \begin{smallmatrix} i & i & k \\ i & j & k \end{smallmatrix}) &= -\frac{\partial}{\partial u_j^i} A^{kji} + \beta_{kj} \frac{\partial}{\partial u_j^i} A^{kji} \\ &= -\left(k | \begin{smallmatrix} k & i & i \\ j & i & j \end{smallmatrix}\right) + \beta_{kj} \left(k | \begin{smallmatrix} j & i & i \\ k & i & j \end{smallmatrix}\right). \end{aligned}$$

Using the 3-1-1 rule we get

$$(j | \begin{smallmatrix} i & i & k \\ i & j & k \end{smallmatrix}) = 0. \quad (68)$$

The same holds for inversions. Next we examine

$$(i | \begin{smallmatrix} i & j & k \\ i & j & k \end{smallmatrix}) = \frac{\partial}{\partial u_k^k} A^{iij}.$$

By (28a) and (28b)

$$A^{iij} = \alpha_i A^{jij}.$$

We make this substitution to get

$$\begin{aligned} (i | \begin{smallmatrix} i & j & k \\ i & j & k \end{smallmatrix}) &= \alpha_i \frac{\partial}{\partial u_k^k} A^{jij} \\ &= \alpha_i (j | \begin{smallmatrix} i & j & k \\ j & j & k \end{smallmatrix}) \\ &= 0 \end{aligned}$$

since j appears four times (Case IIa). Now,

$$(i | \begin{smallmatrix} i & i & j \\ k & k & j \end{smallmatrix}) = \frac{\partial}{\partial u_j^j} A^{\begin{smallmatrix} i & i & i \\ k & k & k \end{smallmatrix}}.$$

By (28b) and (28c)

$$A^{\begin{smallmatrix} i & i & i \\ k & k & k \end{smallmatrix}} = -A^{\begin{smallmatrix} k & i & i \\ i & k & \end{smallmatrix}} + \beta_{ik} A^{\begin{smallmatrix} j & k & i \\ j & k & \end{smallmatrix}},$$

so

$$\begin{aligned} (i | \begin{smallmatrix} i & i & j \\ k & k & j \end{smallmatrix}) &= -\frac{\partial}{\partial u_j^j} A^{\begin{smallmatrix} k & i & i \\ i & k & \end{smallmatrix}} + \beta_{ik} \frac{\partial}{\partial u_j^j} A^{\begin{smallmatrix} j & k & i \\ j & k & \end{smallmatrix}} \\ &= -\left(k | \begin{smallmatrix} i & i & j \\ i & k & j \end{smallmatrix}\right) + \beta_{ik} \left(j | \begin{smallmatrix} k & i & j \\ j & k & j \end{smallmatrix}\right) \end{aligned}$$

→

$$(i | \begin{smallmatrix} i & i & j \\ k & k & j \end{smallmatrix}) = 0 \quad (69)$$

by (68) and the 3-1-1 rule. This argument allows inversions in the second and third columns but not in the first. This symbol must be examined separately:

$$(i | \begin{smallmatrix} k & i & j \\ i & k & j \end{smallmatrix}) = \frac{\partial}{\partial u_j^j} A^{\begin{smallmatrix} i & k & i \\ i & k & \end{smallmatrix}}.$$

From (28a) and (28b) we get

$$A^{\begin{smallmatrix} i & k & i \\ i & k & \end{smallmatrix}} = \frac{1}{\alpha_k} A^{\begin{smallmatrix} k & k & i \\ k & k & \end{smallmatrix}};$$

therefore,

$$\begin{aligned}
(i|_{i k j}^{k i j}) &= \frac{1}{\alpha_k} \frac{\partial}{\partial u_j^j} A^{k k i}_{k k} \\
&= \frac{1}{\alpha_k} (k|_{k k j}^{k i j}) \\
&= 0
\end{aligned}$$

since k occurs four times (Case IIa). The only symbol remaining is $(j|_{j k k}^{i i i})$ and its inversions $(j|_{j k i}^{i i k})$ (with inversions of the first two columns), and $(j|_{k k i}^{i i j})$.

We have

$$(j|_{j k i}^{i i k}) = \frac{\partial}{\partial u_j^i} A^{j i k}_{k i}.$$

From (28d)

$$A^{j k i}_{i k} = -A^{i k i}_{j k}$$

$$\begin{aligned}
\Rightarrow (j|_{j k i}^{i i k}) &= -\frac{\partial}{\partial u_j^i} A^{i k i}_{j k} \\
&= -(i|_{j k j}^{k i i}).
\end{aligned}$$

Using the 3-1-1 rule we get

$$(j|_{j k i}^{i i k}) = 0. \quad (70)$$

The first two columns may be inverted. Now,

$$(j|_{k k i}^{i i j}) = \frac{\partial}{\partial u_i^j} A^{j i i}_{k k}.$$

From (28d) we get

$$A^{jii}_{kk} = -A^{kii}_{jk}.$$

Making this substitution yields

$$\begin{aligned} (j|\frac{i}{k}\frac{i}{k}\frac{j}{i}) &= -\frac{\partial}{\partial u_i^j} A^{kii}_{jk} \\ &= -(k|\frac{i}{j}\frac{i}{k}\frac{j}{i}); \end{aligned}$$

and by (70)

$$(j|\frac{i}{k}\frac{i}{k}\frac{j}{i}) = 0. \quad (71)$$

Lastly,

$$(j|\frac{i}{j}\frac{i}{k}\frac{i}{k}) = \frac{\partial}{\partial u_k^i} A^{jii}_{jk},$$

and (28b) and (28c) yield

$$A^{jii}_{jk} = \frac{1}{\beta_{ij}} (A^{ijj}_{jk} + A^{jji}_{ik}).$$

Substituting, we get

$$\begin{aligned} (j|\frac{i}{j}\frac{i}{k}\frac{i}{k}) &= \frac{1}{\beta_{ij}} \frac{\partial}{\partial u_k^i} A^{ijj}_{jk} + \frac{1}{\beta_{ij}} \frac{\partial}{\partial u_k^i} A^{jji}_{ik} \\ &= \frac{1}{\beta_{ij}} (i|\frac{j}{j}\frac{i}{k}\frac{i}{k}) + \frac{1}{\beta_{ij}} (j|\frac{j}{i}\frac{i}{k}\frac{i}{k}). \end{aligned}$$

Using (69) and (71),

$$(j|\frac{i}{j}\frac{i}{k}\frac{i}{k}) = 0.$$

Thus, all third order and higher derivatives are zero. Consequently, there are no other conservation laws.

■

The conserved quantities $S^i, P^i, B^{ik}, \tilde{A}^i$ and their relation to Noether's Theorem will now be examined. First, $\mathcal{L}(u)$ in (12) is invariant under the group of translations in u since $W(\nabla u) = \frac{1}{2} \sum_{i,j=1}^3 [u_j^i u_j^i + \beta_{ij} u_i^i u_j^j]$ (13) is independent of

u . This translation is defined as $u \rightarrow u + \epsilon \underline{b}$ for arbitrary $\underline{b} \in \mathbb{R}^3$. From this and (15), $\Xi(x, u, \epsilon) = x$, $\Phi(x, u, \epsilon) = u + \epsilon \underline{b}$, which in turn implies $\xi_i = 0$, $\phi_k = b_k$. From Noether's Theorem there exists a conserved quantity A with $\text{Div } A = 0$, $A^i = - \sum_{k=1}^3 b_k \frac{\partial W}{\partial u_i^k}$ (16). Choose

$$\underline{b} = (-1, 0, 0)^T, (0, -1, 0)^T, (0, 0, -1)^T$$

giving conserved quantities

$$A^1, A^2, A^3, \quad A^i = (A_1^i, A_2^i, A_3^i), \quad A_j^i = \frac{\partial W}{\partial u_j^i}, \quad \text{for } i, j = 1, 2, 3.$$

These are exactly $S_j^i = \frac{\partial W}{\partial u_j^i} = u_j^i + \delta_j^i \sum_{m=1}^3 \beta_{im} u_m^m$ in (19). Then

$\text{Div} S = 0$ coincides with the Euler-Lagrange equation $\mathcal{E}(u) = 0$ (cf. (10) and Olver [9, pp.281-3]). Note $S(\nabla u)$ differs from $E(\nabla u)$ (7) by a constant multiple of the null divergence $(F^{-T} \cdot \nabla u) F^{-T} - F^{-T} \nabla u^T F^{-T}$, cf. (11) ($\text{Div} S = \mathcal{E}(u) = \text{Div} E(\nabla u)$).

Second, $\mathcal{Q}(u)$ is invariant under translations in x since $\mathcal{Q}(u)$ is independent of x . This translation is given by $x \rightarrow x + \varepsilon \underline{a}$ for any $\underline{a} \in \mathbb{R}^3$. This gives $\Xi(x, u, \varepsilon) = x + \varepsilon \underline{a}$, $\Phi(x, u, \varepsilon) = u$ (15), which implies $\xi_i = a_i$, $\phi_k = 0$. Noether's Theorem will now give

$$A_i = \sum_{j=1}^3 a_j \left[\sum_{k=1}^3 u_j^k \frac{\partial W}{\partial u_i^k} - \delta_j^i W \right].$$

Choose $\underline{a} = (1, 0, 0)^T$, $(0, 1, 0)^T$, $(0, 0, 1)^T$ giving conserved quantities

$$P^1, P^2, P^3, \quad P^i = (P_1^i, P_2^i, P_3^i)^T, \quad P_j^i = \sum_{k=1}^3 [u_i^k u_j^k + \beta_{jk} u_k^k u_i^j] - \delta_j^i W$$

for $i, j = 1, 2, 3$. These form the Eshelby energy-momentum tensor

$$\text{with } \text{Div} P^i = \sum_{j=1}^3 \frac{\partial P_j^i}{\partial x_j} = 0 \quad i = 1, 2, 3, \quad \text{cf. Olver [9, pp.281-3],}$$

Eshelby [2], Knowles and Sternberg [5].

Lastly, we examine the trivial conservation laws B^{ik} , $\tilde{A}^i$ which are null divergences. Proposition 4.24 in Olver [9] states that A is a null divergence if and only if

$$A_i = \sum_{j=1}^3 D_j V_{ij} \quad \text{for some } V(x, u, \nabla u), \quad V_{ij} = -V_{ji}, \quad \text{for } i, j = 1, 2, 3.$$

Example: $B^{11} = (0, u_3^1, -u_2^1)$ satisfies this when $V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & u_1 \\ 0 & -u_1 & 0 \end{bmatrix}$ and

$\tilde{A}^1$ has $V = 2 \begin{bmatrix} 0 & -u^3 u_3^2 & u^3 u_2^2 \\ u^3 u_3^2 & 0 & -u^3 u_1^2 \\ -u^3 u_2^2 & u^3 u_1^2 & 0 \end{bmatrix}$. Notice that

$$\text{Div} \tilde{A}^1 = 2 \left[\frac{\partial}{\partial x_1} \frac{\partial(u_2, u_3)}{\partial(x_2, x_3)} + \frac{\partial}{\partial x_2} \frac{\partial(u_2, u_3)}{\partial(x_3, x_1)} + \frac{\partial}{\partial x_3} \frac{\partial(u_2, u_3)}{\partial(x_1, x_2)} \right] = 0 \quad \text{is a Jacobi-}$$

type identity where $\frac{\partial(u_2, u_3)}{\partial(x_i, x_j)} = \det \begin{bmatrix} u_i^2 & u_i^3 \\ u_j^2 & u_j^3 \end{bmatrix}$.

A final note: letting $c_1 = c_2 = c_3 = 1$ ($C = I$) reduces $\mathcal{E}(u)$ to a constant multiple of the Lamé-Navier equations in Olver[11, (2.2)]. S_j^i, P_j^i reduce to those in Olver [11, (3.6), (3.4)] except for a factor of $\hat{\mu}$. However, Q in (22) becomes $\hat{\mu}^{-2} [Q_j^i - (3\hat{\mu} + \hat{\lambda}) P_j^i]$ in Olver's notation. Thus, the conserved quantities (19)-(23) include those of Olver as a special case (cf. Proposition 3.1 in Olver [11]).

III. Conclusion

As has been shown, using the results of the calculus of variations the conservation laws of a system of equations can be found. Here all such laws depending only on ∇u have been derived for a system in which the equations governing the compression of a cylinder have been linearized.

The results found here generalize those in Olver [11]. The parallel argument yields conserved quantities equivalent to those found in Olver [11]. These are the tensor S^i , the Eshelby tensor P^i , the trivial or null divergences $\tilde{A}^i$ and B^{ik} , and the tensor Q^i .

These, as has been shown, are the only such laws dependent only on ∇u corresponding to a functional quadratic in ∇u arising from the equations of elasticity.

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VITA

Elizabeth Ann Robinson was born in Hazard, Kentucky on August 20, 1970. After eight years, she moved with her family to Middlesboro, Kentucky. She attended school in the Middlesboro Independent School System and graduated from Middlesboro High School in May, 1988. In September, 1988, she began studies at Transylvania University in Lexington, Kentucky. In May, 1992, she received a Bachelor of Arts degree in Mathematics and a Bachelor of Arts degree in Secondary Education. In August, 1992, she began studies at The University of Tennessee, Knoxville and received a Master of Science degree in Mathematics in August, 1994.