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I am submitting herewith a thesis written by Matthew P. Duran entitled "Modeling the ground state baryon octet using a generalization of the Lagrange triangle solution." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Masters of Science, with a major in Physics.

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Modeling the Ground State Baryon Octet Using a Generalization of the Lagrange Triangle Solution.

A Thesis

Presented for the

Masters of Science

Degree

The University of Tennessee, Knoxville

Matthew P. Duran

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DEDICATION

This thesis is dedicated to my family; particularly my parents, Tom and Patricia Duran, my elder brother Thomas for keeping me on track whenever I lost focus, and to my cousin Victor Fuentes whom I will always remember and miss.

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ABSTRACT

The baryon octet is modeled using relativistic and nonrelativistic Hamiltonians with interquark confining potentials. Using the works of Crater, a nonrelativistic three-body Hamiltonian is obtained where the corresponding eigenvalue equation will be shown to be separable and can be treated as three two-body like systems. Next, using relativistic constraint dynamics, our eigenvalue equation is treated relativistically following the works of Sazdjian and Yang. Using the free-mass-shell Hamiltonian constraints, Yang gives us an effective energy and reduced mass of relative motion for two particle systems which we expand to a three body case. With the help of work done by Sazdjian, we get a generalized 3-body eigenvalue equation in which the energy and reduced masses of effective particles of relative motion are expanded to obtain a form for the eigenvalue equation that can be solved analytically. Because we are dealing with only part of the entire baryon spectrum in which the particles are of the same spin and baryon number, attention will be focused upon confining forces and the development of potentials that are independent of spin and the relative momenta of the fermions.

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LIST OF SYMBOLS

c	speed of light
$\mathbf{R}$	position vector with respect to origin (MeV^{-1})
H	Hamiltonian (MeV)
T	Kinetic Energy (MeV)
V	Potential (MeV)
p_i	i-th particle momentum (MeV)
m_i	i-th particle mass (MeV)
$\mathbf{r}_i$	Position vector with respect to center of mass (MeV^{-1})
ϕ	Two-body interaction quasi potential (MeV)
M	Total mass of three body system (MeV)
$\mathbf{u}_i$	Relative coordinates (MeV^{-1})
Z	Separability condition (MeV^{-1})
μ_i	Reduced mass of Two body system (MeV)
P_i	Canonical Momentum (MeV)
κ	Gravitational constant
Λ	Power law constant (MeV)
ρ	power variable of potential model
Ψ	Wave function
E	Eigenenergy (MeV)
x	particle position (MeV^{-1})
α	unknown Λscaling parameter
β	unknown μ scaling parameter
n_r	number of nodes in wave function
l	Angular momentum quantum number
$\mathfrak{c}$	Expectation value
χ	Fitting Function (MeV^2)
p	proton mass (MeV)

n	neutron mass (MeV)
Σ^+	Sigma plus mass (MeV)
Σ^0	Sigma zero mass (MeV)
Σ^-	Sigma minus mass (MeV)
Ξ^0	Cascade zero mass (MeV)
Ξ^-	Cascade minus mass (MeV)
b_i	i-th experimental baryon mass (MeV)
$\mathbf{X}$	Step function (MeV)
u	Up quark mass (MeV)
d	Down quark mass (MeV)
s	Strange quark mass (MeV)
$\mathcal{H}_i$	i-th particle mass shell constraint (MeV)
λ_i	i-th Lagrange multiplier
$\mathcal{H}$	Interaction Hamiltonian (MeV)
τ	system proper time (sec)
w	total center of momentum energy (MeV)
$\boldsymbol{x}$	relative position coordinate in center of momentum system (MeV^{-1})
$x_\parallel$	mass point position parallel to total momentum (MeV^{-1})
$x_\perp$	mass point position perpendicular to total momentum (MeV^{-1})
$\eta^{\mu\nu}$	μ by ν basis for Minkowski space
$p_\perp$	Relative momentum conjugate to $x_\perp$ (MeV)
ε_i	energy of i'th particle in the c.m. system (MeV)
$\mathbf{p}_i$	i-th particle relative momentum (MeV)
b^2	on-shell value of relative momentum (MeV^2)
A	Time-like vector potential (MeV)
S	World scalar potential (MeV)
ε_w	energy of reduced particle of relative motion (MeV)
m_w	mass of reduced particle of relative motion (MeV)

f_i	Subfunction for total center of momentum energy
b_i	Subfunction for weighted relative momentum of i-th particle
F	Subfunction for total relativistic mass
M_i	Relativistic mass of i-th particle (MeV)
O_i	Subfunction for quasi potential of i'th particle
$\mathcal{M}_i$	Relativistic reduced mass (MeV)
β_j	Relativistic j-th baryon mass (MeV)
L	System Lagrangian (MeV)

LIST OF ABBREVIATIONS

C.M. center-of-mass

c.m. center-of-momentum

MeV Mega-Electron Volts

1 Introduction

1.1 Background

This research is in the area of theoretical physics and centers on relativistic, as well as nonrelativistic, classical and quantum mechanics. In particular, this research problem involves the study of the Lagrange and Hamiltonian formulation of the mechanics of three particles under direct mutual interactions.

The purpose of this paper is to investigate some of the basic building blocks of matter by modeling hadrons using the quark model for three-quark baryon systems, under direct mutual interactions. The question at issue is, how can baryons be modeled using a quantum mechanical generalization of the Lagrange triangle solution with a generalized power rule confining potential? Another way to ask this question is, how can a generalized power rule confining potential together with the generalized Lagrange point model give baryon mass values comparable to the experimental values given in Table 1 on page 2? Explicitly we desire results close to the listed observed values. Note in the quark content column the up quark is given the notation **u** and the down quark the notation **d**, while the strange quark is given the notation **s**; Y represents hypercharge.

The assumptions are that a generalization of the Lagrange equilateral triangle configuration exists for types of forces other than gravitational: for example, ones coming from harmonic and generalized power potentials. Evidence for Lagrange points are found from the Lagrange configuration in our solar system, which is composed of the equilateral configuration of the Sun, Jupiter, and the Trojan asteroids.

1.2 Scope of Present Work

We begin by first studying the baryon spectrum arising from a bound three-quark system in the framework of a nonrelativistic spin-independent phenomenological model. The confining potential of interest uses one of the general power laws such as the one used by Martin in his paper "A fit of upsilon and charmonium Spectra".[1]. By use of a generaliza-

tion of the Lagrange equilateral triangle solution as motivation to nongravitational forces [2], a confining potential is applied to obtain the corresponding baryon spectrum. What is found is that the nonrelativistic model suggests the need for a relativistic treatment. This will be done by use of previously published works of Sazdjian [3] and Yang [4].

1.3 Literature Review

A good motivational starting point for our model is provided by Crater [2]. Studying the dynamical interactions of three-bodies, Crater discovered that given special conditions the equations of motion satisfy what he calls a separability condition. With this condition, the corresponding Hamiltonian can be treated as three separate two-body like subsystems that have exact solutions, which can be generalized to other interactions involving world scalar and vector approximations for quark confinement other than gravitational forces. A power potential, $\Lambda(\Lambda r)^{\rho}$, can be applied to approximate the energies provided experimentally [5]. Extending into quantum mechanics, a nonrelativistic eigenvalue is derived with a separable Hamiltonian which can be solved numerically. However, a relativistic treatment is necessary. Considering only the spin one-half baryon octet described in [6], we attempt a spin-independent nonperturbative approximation. Therefore the Lambda particle is discarded from the approximation because its makeup is identical to the Sigma zero. The variance in mass stems from the differences in the individual spin values of their three constituent particles and is outside the scope of this thesis.

Crater and Yang [4] demonstrated how generalized mass shell constraints can provide a total system Hamiltonian that agrees with our previous conditions, provided the Hamiltonians of each body in the system satisfy the compatibility condition, $\{\mathcal{H}_1, \mathcal{H}_2\} \approx 0$. Crater and Alstine [7] showed how a relativistic description of two-body interactions can provide canonical momentums similar to those describing our classical Lagrangian system, independent of the particle spin. Todorov and Van Alstine [8] lead us to find the energies of the individual particles in the c.m. system and this is used for our relativistic corrections on the two-body kinematics, and dynamic constraint condition. By use of qua-

sipoteintials proposed by Yang [4], relativistic energies and masses of effective particles of relative motion were arrived at from constraint dynamics together with Wheeler-Feynman dynamics. Applied to our Hamiltonian, which satisfies the compatibility condition, the quasipotential may provide a simpler two-body Schrödinger-like equation that incorporates the relativistic corrections. It was Sazdjian's work on relativistic N-body mechanics [3] that gave us motivation to attempt a three-body adaptation of the 2-body constraint formulation. Coupled with Crater's separability condition and our previous constraints and potential models; it may be possible to derive a reasonable model that approximates the ground state baryon octet.

Table 1: Experimental baryon values.

Baryon	Quark Content	Rest mass MeV/c^2	Spin	Charge	Y
Proton	uud	938.3	1/2	1	1
Neutron	ddu	939.6	1/2	0	1
Sigma +	uus	1189.4	1/2	1	0
Sigma 0	uds	1192.5	1/2	0	0
Sigma -	dds	1197.3	1/2	-1	0
Cascade 0	uss	1315	1/2	0	-1
Cascade -	dss	1321	1/2	-1	-1
Lambda	uds	1115.6	1/2	0	0

2 Equations of motion for three interacting particles with arbitrary two-body scalar potentials

2.1 Motivation

It is known that one of the simplest solutions to the gravitational three-body problem is the Lagrange equilateral triangle. In the gravitational three-body problem there are special configurations that allow exact solutions. Using this as motivation, the goal of this paper will be to explore the possibility that such special configurations exist for the nongravitational strong force holding together three quarks inside baryons. The most general case for the Lagrange configuration is desired. Therefore a general triangle solution is considered.

2.2 General equations of motion

We wish to perform a symmetric reduction of the three-body problem. We will find that this can be accomplished by using the relative coordinates of the three particles. Let $\mathbf{R}$ be the position vector of the center of mass. The equations of motion will be written in the center-of-mass (C.M.) system $\dot{\mathbf{R}} = 0 = \mathbf{R} = \ddot{\mathbf{R}}$ and will be symmetrical in three variables. The following is a review of material in Ref.[2].

We begin with the Hamiltonian for a system of three mutually interacting point particles:

$$H = T + V = \sum_{i=1}^{3} \frac{\mathbf{p}_i^2}{2m_i} + \frac{1}{2} \sum_{i \neq j}^{3} \phi_{ij}(|\mathbf{r}_i - \mathbf{r}_j|). \tag{1}$$

The forces arise from action at a distance and are assumed to be velocity independent. The form of the potential in the Hamiltonian then leads to the equation of motion:

$$
\begin{aligned}
m_i\ddot{\mathbf{r}}_i &= -\boldsymbol{\nabla}_i\phi_{ij}(|\mathbf{r}_i-\mathbf{r}_j|) - \boldsymbol{\nabla}_i\phi_{ik}(|\mathbf{r}_i-\mathbf{r}_k|) \\
&= -\frac{\mathbf{r}_i-\mathbf{r}_j}{|\mathbf{r}_i-\mathbf{r}_j|}\phi_{ij}^{'}(|\mathbf{r}_i-\mathbf{r}_j|) - \frac{\mathbf{r}_i-\mathbf{r}_k}{|\mathbf{r}_i-\mathbf{r}_k|}\phi_{ik}^{'}(|\mathbf{r}_i-\mathbf{r}_k|); \quad \text{i,j,k cyclic,}
\end{aligned} \tag{2}
$$

where ϕ' is the derivative of ϕ with respect to its argument. The first integral of the equation of motion comes from summing these three equations:

$$
M\dot{\mathbf{R}} = \sum_{i=1}^{3} m_i\dot{\mathbf{r}}_i = const, \tag{3}
$$

where $M = m_1 + m_2 + m_3$. The constant can be set to zero with $\mathbf{R}$ representing the position of the center of mass with respect to some origin. This also can be set to zero, i.e. at the origin. Our first condition is imposed symmetrically by introducing the relative coordinates:

$$
\mathbf{u}_i = \mathbf{r}_j - \mathbf{r}_k, \quad i,j,k \text{ cyclic.} \tag{4}
$$

These variables can be related by a singular matrix:

$$
\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}. \tag{5}
$$

This stems from the fact that the variables $\mathbf{u}_i$ are not independent:

$$
\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = 0. \tag{6}
$$

Note here that Eq.(3) implies the $\mathbf{r}_i$ variables are not independent. Because the matrix in Eq.(5) is singular we cannot directly invert the equation to solve for $\mathbf{r}_i$ but we can use the identity

$$\mathbf{r}_i = \frac{m_j \mathbf{u}_k - m_k \mathbf{u}_j}{M} + \mathbf{R}. \tag{7}$$

To serve as the means to obtain the inverse, we use Eq.(7) and set $\mathbf{R}$ at the origin. The inverse of Eq.(5) is then:

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} = \frac{1}{M} \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix}. \tag{8}$$

Note that the matrix above has zero determinant. If we substitute Eq.(8) into Eq.(5) we obtain:

$$M \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} m_2 + m_3 & -m_1 & -m_1 \\ -m_1 & m_3 + m_1 & -m_2 \\ -m_3 & -m_3 & m_1 + m_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix}, \tag{9}$$

essentially a restatement of Eq. (6) .

2.3 Separability condition

We may now rewrite our first equations of motion in terms of our new $\mathbf{u}$ variables by taking the second time derivative of Eq. (8) and substituting into Eq. (2). This leads to the new form

$$\ddot{\mathbf{u}}_i + \boldsymbol{\nabla}_i \frac{\phi_{jk}(u_i)}{\mu_i} = m_i \mathbf{Z}; \quad \textit{i,j,k} \text{ in cyclic order}, \tag{10}$$

where $\boldsymbol{\nabla}_i = \boldsymbol{\nabla}_{u_i}$. The three effective reduced masses are defined

$$\mu_i = \frac{m_j m_k}{M}; \text{(i,j,k cyclic)} \tag{11}$$

and

$$\mathbf{Z} = \frac{\boldsymbol{\nabla}_1 \phi_{23}(u_1)}{m_2 m_3} + \frac{\boldsymbol{\nabla}_2 \phi_{31}(u_2)}{m_3 m_1} + \frac{\boldsymbol{\nabla}_3 \phi_{12}(u_3)}{m_1 m_2}. \tag{12}$$

These new forms of equations of motion, proposed by Arenstorf in private communications with Crater[9], contain the center of mass restriction $\dot{\mathbf{R}} = 0$ and are symmetrical in three variables. Note also that they are independent of the nature of the potential. For orbital configurations in which $\mathbf{Z} = 0$, the equations of motion Eq.(10) separate into three uncoupled two-body-like equations.

The general equations of motion Eq.(10) also follow by applying Hamilton's equation to the Hamiltonian Ref.[2]:

$$\begin{aligned} H &= \frac{(\mathbf{P}_1 - \mathbf{P}_2)^2}{2m_3} + \frac{(\mathbf{P}_2 - \mathbf{P}_3)^2}{2m_1} + \frac{(\mathbf{P}_3 - \mathbf{P}_1)^2}{2m_2} \\ &\quad + \phi_{23}(\mathbf{u}_1) + \phi_{13}(\mathbf{u}_2) + \phi_{12}(\mathbf{u}_3). \end{aligned} \tag{13}$$

This Hamiltonian is an alternative expression of the Hamiltonian of Eq.(1). It is written in terms of the canonically conjugate variables $\mathbf{u}_i$ and $\mathbf{P}_i = m_j m_k \dot{\mathbf{u}}_i / M$, not to be confused with $\mathbf{p}_i$ used in Eq.(1). The potential $V(u_1, u_2, u_3)$ is the separable sum of our general potentials $\phi_{jk}(u_i)$. The second Hamiltonian Eq.(13) has the properties of being symmetrical in the relative variables $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$; the constraint $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = 0$ is a consequence of the equation of motion; it is in the center of mass frame. The derivation of Eq.(13) is given in Appendix A.

3 Gravitational interaction

3.1 Application

To demonstrate a realistic situation from known potentials let us apply this concept to the Lagrange equilateral triangle solution for gravitational interactions.

It is known that for gravitational forces the potential is of the form:

$$\phi_{ij}(u_k) = -\frac{\kappa m_i m_j}{|\mathbf{u}_k|},\ i,j,k \text{ cyclic.} \tag{14}$$

We apply our equation of motions in terms of variable $\mathbf{u}$ and obtain for the gravitational potential:

$$\ddot{\mathbf{u}}_i + \frac{M\kappa \mathbf{u}_i}{|\mathbf{u}_i|^3} = m_i \mathbf{Z},\ i = 1,2,3 \tag{15}$$

or

$$\mathbf{Z} = \kappa \left(\frac{\mathbf{u}_1}{|\mathbf{u}_1|^3} + \frac{\mathbf{u}_2}{|\mathbf{u}_2|^3} + \frac{\mathbf{u}_3}{|\mathbf{u}_3|^3} \right). \tag{16}$$

3.2 Separability satisfied

Let us assume the equilateral condition $|\mathbf{u}_1| = |\mathbf{u}_2| = |\mathbf{u}_3| \equiv |\mathbf{u}|$. Then Eq.(16) becomes

$$\mathbf{Z} = \frac{\kappa}{|\mathbf{u}|^3}(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) = 0, \tag{17}$$

and Eq.(15) separates into three two-body problems. The three sets of two-body equations indicate that the relative motion can be identical circles or ellipses. The motion will be parabolas and hyperbolas for unbounded motion. For our bound equilateral case the periodic motion will be described as a rotating equilateral triangle with either fixed sides or periodically varying sides. These are known as the Lagrange equilateral triangle solutions. The condition $\mathbf{Z} = 0$ is synonymous with the Lagrange solution for gravitational interactions of this specific three-body type system.

3.3 Hamiltonian

The triangle constraint $(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) = 0$ follows from the first of Hamilton's equations applied to Eq.(13). The separability of Eq.(15) is imposed by requiring the $\mathbf{Z} = \mathbf{0}$ condition. These same separable equations, Eq.(15) together with $\mathbf{Z} = \mathbf{0}$, can be obtained by applying Hamilton's equation to the following separable Hamiltonian in which the $\mathbf{u}_i$ are regarded as independent:

$$H = \frac{\mathbf{P}_1^2}{2\mu_1} + \frac{\mathbf{P}_2^2}{2\mu_2} + \frac{\mathbf{P}_3^2}{2\mu_3} - \frac{\kappa m_1 m_2}{|\mathbf{u}_3|} - \frac{\kappa m_2 m_3}{|\mathbf{u}_1|} - \frac{\kappa m_3 m_1}{|\mathbf{u}_2|} \tag{18}$$

and again $\mu_i = m_j m_k / M$ (i,j,k cyclic). Here the triangle constraint as well as the $\mathbf{Z} = \mathbf{0}$ condition is imposed on the solutions. These and the constraints of section 7.1 are the only conditions under which Eq.(18) yields the separable equations of motion, (see appendix A). Whereas the separability of the equations of motion Eq.(15) is imposed by requiring the $\mathbf{Z} = \mathbf{0}$ condition and combining this with the triangle constraint, the separability condition for the equations of motion that follow from Eq.(18) are automatic.

4 Harmonic and General Power Law Potentials

4.1 Confining Power Law Potentials

In this section the baryon spectrum of quark systems are studied in the framework of a nonrelativistic spin-independent phenomenological model. As a simple example we take the confining potential to be a power law potential, the harmonic oscillator potential being the most well known example. Our separable equations of motion allow us to obtain analytical solutions for the baryon spectrum. The interests of this section are the strong interaction of quarks forming bound states called hadrons and the possibilities of considering baryons as nonrelativistic bound states of quarks.

From the Virial theorem we have

$$\bar{T} = \frac{1}{2}\overline{r\frac{\partial V}{\partial r}}. \tag{19}$$

If the potential is a power law function of r, then

$$V = \Lambda(\Lambda r)^{\rho}, \rho > 0. \tag{20}$$

In natural units the constant Λ has units of mass or energy or inverse length so that V has the dimensions of energy. Now

$$\frac{\partial V}{\partial r}r = \rho V, \tag{21}$$

and we have the proportionality

$$\overline{T} = \frac{\rho+1}{2}\overline{V}. \tag{22}$$

Going back to our separable condition **Z**, our general potential has as its derivative with respect to its argument

$$\phi'_{ij}(\mathbf{u}_k) = \rho\Lambda^{\rho+1}\,|\mathbf{u}_k|^{\rho-1}\,. \tag{23}$$

Plugging into Eq.(12) gives

$$\mathbf{Z} = \frac{\mathbf{u}_1}{|\mathbf{u}_1|}\frac{\rho\Lambda^{\rho+1}\,|\mathbf{u}_1|^{\rho-1}}{m_2m_3} + \frac{\mathbf{u}_2}{|\mathbf{u}_2|}\frac{\rho\Lambda^{\rho+1}\,|\mathbf{u}_2|^{\rho-1}}{m_3m_1} + \frac{\mathbf{u}_3}{|\mathbf{u}_3|}\frac{\rho\Lambda^{\rho+1}\,|\mathbf{u}_3|^{\rho-1}}{m_1m_2} \tag{24}$$

or

$$\mathbf{Z} = \Lambda^{\rho+1}\rho[\frac{u_1^{\rho-2}}{m_2m_3}\mathbf{u}_1 + \frac{u_2^{\rho-2}}{m_3m_1}\mathbf{u}_2 + \frac{u_3^{\rho-2}}{m_1m_2}\mathbf{u}_3]. \tag{25}$$

If we set the ratio's of the three vector terms equal to each other, $u_1^{\rho-2}/m_2m_3 = u_2^{\rho-2}/m_3m_1 = u_3^{\rho-2}/m_1m_2$, then we obtain the separability condition $\mathbf{Z} = 0$. This ratio condition fixes the relative sizes of each triangle side, which is the same as defining the triangle[1]. The following separable Hamiltonian leads to three separable two-body equations Eq.(10) with $\mathbf{Z} = \mathbf{0}$:

$$H = \frac{\mathbf{P}_1^2}{2\mu_1} + \frac{\mathbf{P}_2^2}{2\mu_2} + \frac{\mathbf{P}_3^2}{2\mu_3} + \Lambda(\Lambda u_1)^\rho + \Lambda(\Lambda u_2)^\rho + \Lambda(\Lambda u_3)^\rho. \tag{26}$$

4.2 Harmonic oscillator potential

As a special case consider the harmonic oscillator potential

$$V = \frac{1}{2}kr^2 = \frac{1}{2}\Lambda(\Lambda r)^2. \tag{27}$$

The goal was the observation of the Lagrange-like point model with generalized confining potential. We generalize the confining potential to that of an unknown power ρ. For the harmonic oscillator case $\rho = 2$. It should also be pointed out that for the harmonic oscillator case, the separability condition follows from Eq.(25) only if all three masses are identical; i.e., $m_1 = m_2 = m_3$.

[1]Note, we will not be considering the stability of the triangular configuration.

In terms of the relative coordinates our potential is defined

$$\phi(\mathbf{r}_{ij}) = \frac{1}{2}k\left|\mathbf{u}_k\right|^2. \tag{28}$$

The derivative with respect to its argument is thus

$$\phi'_{ij}(\mathbf{u}_k) = k\left|\mathbf{u}_k\right|. \tag{29}$$

Here *i,j,k* are cyclic.

Assuming only the condition $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = 0$ and using

$$\mathbf{Z} = \frac{\mathbf{u}_1}{|\mathbf{u}_1|}\frac{\phi'_{23}(u_1)}{m_2 m_3} + \frac{\mathbf{u}_2}{|\mathbf{u}_2|}\frac{\phi'_{31}(u_2)}{m_3 m_1} + \frac{\mathbf{u}_3}{|\mathbf{u}_3|}\frac{\phi'_{12}(u_3)}{m_1 m_2}, \tag{30}$$

we have the separability condition

$$\mathbf{Z} = \frac{ku_1\mathbf{u}_1}{|\mathbf{u}_1|\,m_2 m_3} + \frac{ku_2\mathbf{u}_2}{|\mathbf{u}_2|\,m_3 m_1} + \frac{ku_3\mathbf{u}_3}{|\mathbf{u}_3|\,m_1 m_2}. \tag{31}$$

For equal masses

$$\mathbf{Z} = \frac{k}{m^2}(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3). \tag{32}$$

By the generalized triangular configuration $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$ the separability condition $\mathbf{Z} = \mathbf{0}$ is automatically satisfied.

Then the equations of motion are

$$\ddot{\mathbf{u}}_i + \mathbf{u}_i\frac{Mk}{m_j m_k} = 0;\ \text{where } i,j,k \text{ are cyclic.} \tag{33}$$

Treating the system as color independent, our equations of motion become

$$\mu\ddot{\mathbf{u}}_i = -k\mathbf{u}_i. \tag{34}$$

From the relation $\mathbf{F} = -\boldsymbol{\nabla}\phi$, we have $\mathbf{F} = -k\mathbf{u}$ such that $\phi(u) = ku^2/2$. The correspond-

ing Hamiltonian that leads to these equations of motion is

$$H = \frac{\mathbf{P}_1^2}{2\mu_1} + \frac{\mathbf{P}_2^2}{2\mu_2} + \frac{\mathbf{P}_3^2}{2\mu_3} + \frac{ku_1^2}{2} + \frac{ku_2^2}{2} + \frac{ku_3^2}{2}. \tag{35}$$

5 Quantum Mechanics for the three body problem

5.1 Separable Hamiltonian

We now wish to present the separable quantum equations resulting from the general power rule and determine from this the eigenvalues of the baryon spectrum. We use the generalized Lagrange point diagram, figure 1, page 16, for the basis of our nonrelativistic color and spin independent phenomenological model.

For the nonrelativistic potential the general power law is used:

$$\phi_{ij}(r_{ij}) = \Lambda(\Lambda r_{ij})^{\rho}. \tag{36}$$

Substitution into Eq.(10) gives the equation of motion

$$\ddot{\mathbf{u}}_i + \frac{\rho\Lambda^{\rho+1}\,|\mathbf{u}_k|^{\rho-1}}{\mu_i} = m_i\mathbf{Z}. \tag{37}$$

From this we rederive

$$\mathbf{Z} = \frac{\mathbf{u}_1}{|\mathbf{u}_1|}\frac{\rho\Lambda^{\rho+1}\,|\mathbf{u}_1|^{\rho-1}}{m_2m_3} + \frac{\mathbf{u}_2}{|\mathbf{u}_2|}\frac{\rho\Lambda^{\rho+1}\,|\mathbf{u}_2|^{\rho-1}}{m_3m_1} + \frac{\mathbf{u}_3}{|\mathbf{u}_3|}\frac{\rho\Lambda^{\rho+1}\,|\mathbf{u}_3|^{\rho-1}}{m_1m_2}, \tag{38}$$

which for certain ratios of the lengths of the sides has already been demonstrated to satisfy the separability condition $\mathbf{Z} = 0$.[2] Recall that the above equations of motion as well as the condition $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$ can be derived from the Hamiltonian

$$H = \frac{(\mathbf{P}_1 - \mathbf{P}_2)^2}{2m_3} + \frac{(\mathbf{P}_2 - \mathbf{P}_3)^2}{2m_1} + \frac{(\mathbf{P}_3 - \mathbf{P}_1)^2}{2m_2} + \Lambda(\Lambda u_1)^{\rho} + \Lambda(\Lambda u_2)^{\rho} + \Lambda(\Lambda u_3)^{\rho}. \tag{39}$$

Recall also that the separable equations of motion corresponding to $\mathbf{Z} = \mathbf{0}$ can be derived from the separable Hamiltonian

[2] See paragraph below Eq.(25).

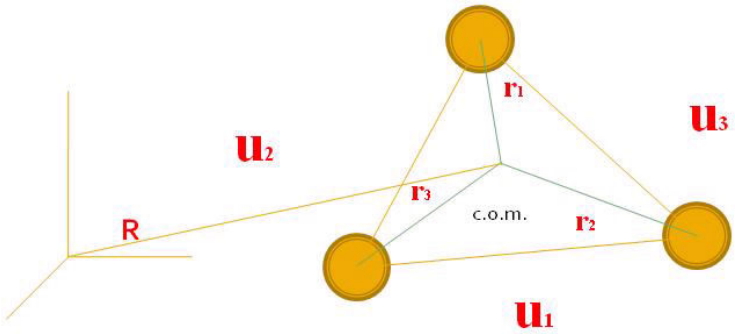

Figure 1: **Three body diagram.**

$$H = \frac{\mathbf{P}_1^2}{2\mu_1} + \frac{\mathbf{P}_2^2}{2\mu_2} + \frac{\mathbf{P}_3^2}{2\mu_3} + \Lambda(\Lambda u_1)^\rho + \Lambda(\Lambda u_2)^\rho + \Lambda(\Lambda u_3)^\rho. \tag{40}$$

This is just the sum of three independent two body like Hamiltonians $(\frac{\mathbf{P}_i^2}{2\mu_i} + \Lambda(\Lambda u_i)^\rho)$.

5.2 Deriving the binding energies

Attention will now go to deriving the binding energies beginning with the non-separable Schrödinger equation

$$E\Psi = (\frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{p}_3^2}{2m_3} + \Lambda(\Lambda u_1)^\rho + \Lambda(\Lambda u_2)^\rho + \Lambda(\Lambda u_3)^\rho)\Psi, \tag{41}$$

where

$$\mathbf{p}_i = -i\frac{\partial}{\partial \mathbf{r}_i}. \tag{42}$$

An equivalent non-separable Schrödinger equation is given by

$$\begin{aligned} E\Psi \quad = \quad & (\frac{(\mathbf{P}_1 - \mathbf{P}_2)^2}{2m_3} + \frac{(\mathbf{P}_2 - \mathbf{P}_3)^2}{2m_1} + \frac{(\mathbf{P}_3 - \mathbf{P}_1)^2}{2m_2} \\ & + \Lambda(\Lambda u_1)^\rho + \Lambda(\Lambda u_2)^\rho + \Lambda(\Lambda u_3)^\rho)\Psi, \end{aligned} \tag{43}$$

where

$$\mathbf{P}_i = -i\frac{\partial}{\partial \mathbf{u}_i}. \tag{44}$$

We wish to examine the separable Schrödinger equation corresponding to the separable classical Hamiltonian in Eq.(40)

$$E\Psi = (\frac{\mathbf{P}_1^2}{2\mu_1} + \frac{\mathbf{P}_2^2}{2\mu_2} + \frac{\mathbf{P}_3^2}{2\mu_3} + \Lambda(\Lambda u_1)^\rho + \Lambda(\Lambda u_2)^\rho + \Lambda(\Lambda u_3)^\rho)\Psi. \tag{45}$$

To do this we need to solve the separate Schrödinger equations[3]

$$(\frac{\mathbf{P}_i^2}{2\mu_i} + \Lambda(\Lambda u_i)^\rho)\Psi(\mathbf{u}_i) = E_i\Psi(\mathbf{u}_i) \tag{46}$$

We use dimensional arguments to obtain analytic eigenvalue solutions of the separate power law potentials resulting in considerable simplicity.

For the power law potential, we wish for the dimensions to match of both the kinetic and potential terms of the Schrödinger equation

$$(\frac{\mathbf{P}^2}{2\mu} + \Lambda(\Lambda u)^\rho)\Psi = E\Psi. \tag{47}$$

Thus with Λ having the dimension of energy , i.e., MeV's or mass or inverse length in natural units then Λr must be dimensionless. We let

$$x = r\Lambda^{-\alpha}\mu^{-\beta}. \tag{48}$$

In the equation above x is dimensionless so that when substituted into our Schrödinger eigenvalue equation it gives

$$E\Psi = [\frac{\mathbf{P}_x^2}{2}(\Lambda^{-2\alpha}\mu^{-2\beta-1}) + (\Lambda^{\rho+1+\rho\alpha}\mu^{\rho\beta})\,|\mathbf{x}|^\rho]\Psi. \tag{49}$$

Here,

$$\mathbf{P}_x = -i\frac{\partial}{\partial \mathbf{x}_i} \tag{50}$$

is a dimensionless operator.

To determine both α and β, we demand that the dimensions of each term in brackets be identical. This means that the powers of the first term are equal to the second. Therefore set the powers of Λ equal to each other as they are independent of the mass:

[3]Using natural units the measurements of time, energy, and space are related so that the constants $\hbar$ and c are unity.

$$\begin{aligned} -2\alpha &= \rho + 1 + \rho\alpha \\ -\alpha(2+\rho) &= \rho + 1 \end{aligned} \tag{51}$$

or

$$\alpha = -\frac{\rho+1}{\rho+2}. \tag{52}$$

Similarly, for the powers of the mass and μ,

$$\begin{aligned} -2\beta - 1 &= \rho\beta \\ -\beta(2+\rho) &= 1 \end{aligned} \tag{53}$$

or

$$\beta = -\frac{1}{2+\rho}. \tag{54}$$

Note that as expected from Eq.(48), $\alpha + \beta = -1$.

By substituting α and β into our energy equation Eq.(49) we obtain

$$E\Psi = \Lambda^{\frac{2\rho+2}{\rho+2}}/\mu^{\frac{\rho}{2+\rho}}(\frac{\mathbf{P}_x^2}{2} + \mathbf{x}^\rho)\Psi. \tag{55}$$

With $\Psi(\mathbf{x})$ the dimensionless eigenfunction (since $(\frac{\mathbf{P}_x^2}{2} + x^\rho)$ is dimensionless), we have

$$E(\Lambda, \mu, \rho) = \frac{\Lambda^{\frac{2\rho+2}{\rho+2}}}{\mu^{\frac{\rho}{2+\rho}}}\mathfrak{c}(\rho, n_r, l). \tag{56}$$

Here,

$$\mathfrak{c}(\rho, n_r, l) = \int d^3x \Psi^* (\frac{\mathbf{P}_x^2}{2} + |\mathbf{x}|^\rho)\Psi \tag{57}$$

is dimensionless and dependent only on the power ρ, the number of nodes n_r of the wave function and the angular momentum quantum number l. In this thesis we will work only on the ground states where $n_r = l = 0$. Since Λ is a free parameter, we absorb $\mathfrak{c}(\rho, 0, 0)$ into Λ. The resulting binding energy is

$$E(\Lambda, \mu, \rho) = \Lambda^{\frac{2\rho+2}{\rho+2}} / \mu^{\frac{\rho}{2+\rho}}. \tag{58}$$

The total ground state energy is then the rest mass energy of the three quarks plus the above binding energy. Since Λ has dimensions of mass, the above has the dimensions of mass or energy.

The total energy for the three independent two-body like baryon system is simply the rest mass energy of each quark plus the binding energies for the three separate two-body systems

$$\begin{aligned} w &= m_1 + m_2 + m_3 + \Lambda^{\frac{2\rho+2}{\rho+2}} (1/\mu_1^{\frac{\rho}{2+\rho}} + 1/\mu_2^{\frac{\rho}{2+\rho}} + 1/\mu_3^{\frac{\rho}{2+\rho}}), \\ \text{where } \mu_i &= m_j m_k / M. \end{aligned} \tag{59}$$

We now have a generalized model for the energy spectrum for a three quark baryon system. The question confronting us now is how to determine if the above nonrelativistic expression can reasonably fit the spin one-half baryon spectrum for appropriate choices of the three quark masses, the parameter Λ, and the power potential parameter ρ. To do this we use the method of least square fit.

6 Least square

6.1 Method Outline

To attempt to fit the energies of our specific baryon octet we employ the method of least squares fit. The method of least squares uses the assumption that the best fit for a curve, with a given function, is a curve with minimal sum of deviations squared (*least square error*) from a set of given data points. Let the data points be $(x1, y1), (x2, y2), ..., (xn, yn)$, as a general example, where x is the independent variable and y the dependent variable. A fitting curve $f(x)$ has the deviation d from each point, $d1 = y1 - f(x1), d2 = y2 - f(x2), ..., dn = yn - f(xn)$. The parameter δ is the experimental error of each test run that will be ignored in our model since we are dealing only with an approximate equation. According to the method of least squares, the best fitting curve has the property that:

$$\chi^2 = d_1^2 + d_2^2 + ... + d_n^2 = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [\frac{f(x_i) - y_i}{\delta Experimental\ error}]^2 = \text{a minimum.} \tag{60}$$

For the baryon spectrum we use a χ^2 of the form[4]

$$\chi^2 = \sum_{i}^{n} [\frac{\text{Theoretical result} - \text{Experimental result}}{\text{Experimental result}}]_i^2 = \sum_{i}^{n} \left[\frac{f(x_i) - y_i}{y_i}\right]^2. \tag{61}$$

6.2 Fit to Baryon Spectrum

Let the theoretical results be our energy eigenvalues in Eq.(59) for seven baryons. We are not concerned with the Λ^0 particle in the baryon octet because its makeup is identical to that of the Σ^0. Its difference in mass stems from the spin of their constituent masses which is beyond the scope of this paper. The experimental results are the known baryon masses. This data is retrieved from a table of baryons taken from Serway chapter forty-seven [5]

[4] In ordinary least sqauare approximations the experimental error is accounted for. This ensures a high degree of accuracy. However, the model presented in this thesis is only a approximation and so we only desire to see if the results are reasonable and are not looking for an exact solution.

shown in Table 1.

The baryons of interest are the p, n, Σ^+, Σ^0, Σ^-, Ξ^0, and Ξ^-. The masses of these individual baryons we define as b_i. The individual quarks of interest in this specific model are the up, down, and strange quarks. The quark masses are labeled m_1, m_2, and m_3 for each of the seven baryon and represent the different combinations of up, down and strange quarks. The reduced masses are defined, $\mu_i = \frac{m_j m_k}{M}$. The proper expression for χ^2, our baryon spectrum model, is now, using Eq.(59),

$$\begin{aligned} & \chi^2(m_1, m_2, m_3, \lambda, \rho) \\ = & \sum_{i=1}^{7} [\frac{\Lambda^{\frac{2\rho+2}{\rho+2}}(1/\mu_1^{\frac{\rho}{2+\rho}} + 1/\mu_2^{\frac{\rho}{2+\rho}} + 1/\mu_3^{\frac{\rho}{2+\rho}}) + m_1 + m_2 + m_3. - b_i}{b_i}]^2. \end{aligned} \tag{62}$$

How do we find the values of our parameters m_1, m_2, m_3, Λ and ρ that give the smallest possible value of χ^2? By applying the gradient and creating a step function, we find the direction of greatest decrease in χ^2. The assumption is that χ^2 can be viewed as a hyperdimentional surface, hyper meaning more than three dimensions. $\nabla\chi^2$ will quickly find the local minimum. of χ^2. Thus our step function will be in the direction of the lowest point on the surface $\chi^2(m_1, m_2, m_3, \Lambda, \rho)$. See appendix B.

6.3 Grid Point Method

To avoid a local minimum, a grid of various parameter values along a reasonable span of $\chi^2(m_1, m_2, m_3, \Lambda, \rho)$ is taken in order to thoroughly cover the possible minimums. Various values of χ^2 are ranked so that we begin the gradient with the starting point of the ten best approximate values. It is after evaluating over a reasonable limit and narrowing our search that the top nine of many points were obtained. The results are summerized in table 2 on page 23.

Table 2: Nonrelativistic grid method results for baryon octet.

Rank	Λ	u	d	s	ρ	χ^2
1	128	77	74	355	1.25	.000207
2	128	77	77	355	1.25	.000223
3	115	77	74	370	1.75	.000293
4	128	74	74	355	1.25	.000297
5	128	74	77	355	1.25	.000298
6	1128	77	71	355	1.25	.000298
7	115	74	74	370	1.75	.0003
8	115	77	71	370	1.75	.000306
9	115	74	71	379	1.75	.000372

6.4 Stepping Method

Next we use a stepping function that will take us in the direction of greatest change of χ^2. This step function is defined in such a way that it will always take us in the direction of a local minimum of the function. We define this step function in vector notation

$$\mathbf{X}(u_0, d_0, s_0, \Lambda_0, \rho_0) = \mathbf{X}_0 + s\widehat{\mathbf{n}}. \tag{63}$$

The vector $\mathbf{X}_0$ is our starting position vector, determined by our grid method described above

$$\mathbf{X}_0 = u_0\widehat{\mathbf{i}}_1 + d_0\widehat{\mathbf{i}}_2 + s_0\widehat{\mathbf{i}}_3 + \Lambda_0\widehat{\mathbf{i}}_4 + \rho_0\widehat{\mathbf{i}}_5. \tag{64}$$

The second term is the gradient of χ^2 with respect to its arguments

$$\widehat{\mathbf{n}} = \frac{\partial_u\chi^2\widehat{\mathbf{i}}_1 + \partial_d\chi^2\widehat{\mathbf{i}}_2 + \partial_s\chi^2\widehat{\mathbf{i}}_3 + \partial_\Lambda\chi^2\widehat{\mathbf{i}}_4 + \partial_\rho\chi^2\widehat{\mathbf{i}}_5}{\sqrt{\partial_u\chi^{22} + \partial_d\chi^{22} + \partial_s\chi^{22} + \partial_\Lambda\chi^{22} + \partial_\rho\chi^{22}}}. \tag{65}$$

The scalar s is simply a constant which can be varied to increase or decrease the step size. In other words we can approach the local minimum faster with a larger s value[5]. The results of our step function is another position vector to be defined as our new starting position vector $\mathbf{X} = \mathbf{X}_{new}$. The process is repeated with $\mathbf{X}_0 = \mathbf{X}_{new}$. The parameters of both the position vector and the step function are the magnitudes of the resulting individual vector components of the previous step function $\mathbf{X}(u_1, d_1, s_1, \Lambda_1, \rho_1)$. The new position vector is denoted

$$\mathbf{X}'(u_1, d_1, s_1, \Lambda_1, \rho_1) = \mathbf{X}_{new} + s\widehat{\mathbf{n}}. \tag{66}$$

The point whose gradient gives the lowest value of $\chi^2(m_1, m_2, m_3, \Lambda, \rho)$ would be taken as the best approximation but it was found that the values were identical and independent

[5]It is a good point to note that if s is too large the step function will overstep the local min and it must be decreased till the step function begins to reach a constant value. When this happens we have reached the local min of χ.

of the points used. For each point χ^2 had the same results summarized in table 3, page 26. An averaged comparison of the results to known values are given in figure 2, page 27.

Though the model gives a good approximation of the baryon spectrum, the binding energies were found to be large compared to the rest masses which suggest the need for a relativistic treatment. With the up, down and strange masses being so low compared to the binding energies ranging from 450 MeV to 650 MeV, we must consider a small mass confined in a very small volume with a large potential. A small mass within a high binding energy implies that high speeds will occur within the system. In the following sections we will incorporate relativistic corrections. Starting with Einstein's mass shell constraint we construct an effective two-body Hamiltonian in the center of momentum frame (c.m.). We then give the conditions that must be satisfied for a separable three-body model to work. Reduced mass of relative motion and energy equations are obtained and then expanded into the generalized three body form to predict the energies of our seven chosen baryon systems.

Table 3: Nonrelativistic gradient method results.

χ^2	u	d	s	Λ	ρ
.000352	74	71	379	115	1.74

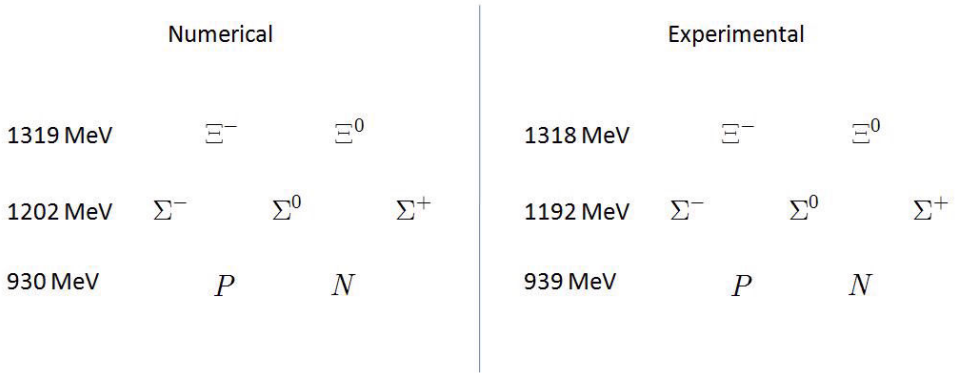

Figure 2: **Averaged nonrelativistic comparison spectrum.**

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7 Relativistic model of the ground state baryon octet

7.1 Generalized Mass Shell Constraints

Here we discuss how we extend the separable nonrelativistic three body model to the relativistic case. We use the methods of constraint dynamics. First we discuss the two-body case. Our discussion is a review of material presented in References [12],[4],and [10]. To see where the constraints for the relativistic case arise consider the Lagrange for a single free particle

$$L = -m\sqrt{-\dot{x}^2}. \tag{67}$$

Its momentum is written in terms of the partial of the Lagrangian with respect to the time derivative of the position

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{-\dot{x}^2}}, \tag{68}$$

so

$$p^2 + m^2 = 0. \tag{69}$$

However, the Legendre Hamiltonian $H_L = p \cdot \dot{x} - L = 0$. Thus since $H = H_L +$ constraints, the above functional constraint therefore serves as the Hamiltonian

$$\mathcal{H} = \lambda(p^2 + m^2) \tag{70}$$

for a single particle.

For two particles, using the free mass shell conditions $p_i^2 + m_i^2 \approx 0$ as a starting point, we define a two-body Hamiltonian

$$\mathcal{H} = \boldsymbol{\lambda}_1\mathcal{H}_1 + \boldsymbol{\lambda}_2\mathcal{H}_2, \tag{71}$$

where the coefficients $\boldsymbol{\lambda}_i$ are Lagrange multipliers and

$$\begin{aligned} \mathcal{H}_1 &= p_1^2 + m_1^2 + \Phi_1(x,P) \approx 0, \\ \mathcal{H}_2 &= p_2^2 + m_2^2 + \Phi_2(x,P) \approx 0. \end{aligned} \tag{72}$$

These are generalized mass shell constraints, where $P = p_1 + p_2$ is the total momentum with $w = \sqrt{-P^2}$ the total c.m. energy, and $\boldsymbol{x} = x_1 - x_2$ is the relative position coordinate.[6] The invariants $\Phi_i(x,P)$ are the c.m. energy-dependent potentials called quasipotentials and account for the deviations from the free mass shell conditions. These constraints must be compatible. This follows from the requirement that the time rate of change of the constraints be conserved in time.

The proper time derivatives are given by the Poisson bracket condition

$$\frac{df}{d\tau} = \{f, \mathcal{H}\}. \tag{73}$$

In the above equation, f is an arbitrary dynamical variable. The time rate of change of the constraints are

$$\begin{aligned} \frac{d\mathcal{H}_1}{d\tau} &= \{\mathcal{H}_1, \mathcal{H}\} \\ &= \{\mathcal{H}_1, \lambda_1\}\mathcal{H}_1 + \lambda_1\{\mathcal{H}_1, \mathcal{H}_1\} + \{\mathcal{H}_1, \lambda_2\}\mathcal{H}_2 + \lambda_2\{\mathcal{H}_1, \mathcal{H}_2\} \\ &\approx \lambda_2\{\mathcal{H}_1, \mathcal{H}_2\}, \end{aligned} \tag{74}$$

similarly,

$$\frac{d\mathcal{H}_2}{d\tau} \approx \lambda_1\{\mathcal{H}_2, \mathcal{H}_1\}. \tag{75}$$

[6] The symbol $\approx$ (called a weak equality) means that the constraint is to be imposed only after Poisson brackets have been worked out.

The constraints are thus constant in the system proper time if they satisfy the compatibility condition $\{\mathcal{H}_1, \mathcal{H}_2\} \approx 0$. In terms of the constraints in Eq.(72) we have

$$2p_1 \cdot \{p_1, \Phi_2\} + 2p_2 \cdot \{\Phi_1, p_2\} + \{\Phi_1, \Phi_2\} \approx 0. \tag{76}$$

Assume the quasipotentials to be of the form

$$\Phi_i = \Phi_i(\frac{x_\perp^2}{2}, \frac{x_\parallel^2}{2}, w). \tag{77}$$

The variables in the expression above have the definitions

$$\begin{aligned} x_\parallel^\mu &= -\frac{x \cdot P}{P^2} P^\mu \\ x_\perp^\mu &= x^\mu - x_\parallel^\mu = (\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu) x_\nu \\ P \cdot x_\perp &= 0. \end{aligned} \tag{78}$$

Substitution of the above relationships into Eq.(76) gives the compatibility condition

$$-4p_1 \cdot x_\perp \frac{\partial \Phi_2}{\partial x_\perp^2} - 4p_1 \cdot x_\parallel \frac{\partial \Phi_2}{\partial x_\parallel^2} - 4p_2 \cdot x_\perp \frac{\partial \Phi_1}{\partial x_\perp^2} - 4p_2 \cdot x_\parallel \frac{\partial \Phi_1}{\partial x_\parallel^2} + \{\Phi_1, \Phi_2\} \approx 0. \tag{79}$$

The simplest solution is

$$\Phi_1 = \Phi_2 = \Phi_{12}(\frac{x_\perp^2}{2}, w), \tag{80}$$

which is analogous to Newtons third law. The compatibility condition is now expressed in terms of the quasipotentials

$$\{\mathcal{H}_1, \mathcal{H}_2\} \approx -4P \cdot x_\perp \frac{\partial \Phi_{12}(\frac{x_\perp^2}{2}, w)}{\partial x_\perp^2} = 0. \tag{81}$$

In analogy to what is done in the nonrelativistic two-body problem, Todorov, Crater,

and Van Alstine developed a relativistic Hamiltonian approach based on the above constraints using a relative momentum conjugate to $x_\perp$[7]. It is given by

$$p_\perp^\mu = (\eta^{\mu\nu} + \hat{P}^\mu \hat{P}^\nu) p_\nu. \tag{82}$$

Here[7],

$$\begin{aligned} p &= \frac{(\varepsilon_2 p_1 - \varepsilon_1 p_2)}{w} \\ w &= \varepsilon_1 + \varepsilon_2 \\ \varepsilon_i &= \varepsilon_i(w). \end{aligned} \tag{83}$$

The variables ε_i are the energies of the individual particles in the c.m. system and since $\{x, P\} = 0$,

$$\eta_{\mu\nu} = \{x_\mu, p_\nu\}. \tag{84}$$

In terms of our relative momentum p we have the forms

$$\begin{aligned} p_1 &= \frac{\varepsilon_1}{w} P + p \\ p_2 &= \frac{\varepsilon_2}{w} P - p. \end{aligned} \tag{85}$$

The condition described in Eq.(80) serves as the relativistic counterpart of Newtons's third law and provides the kinematical constraint

$$\mathcal{H}_1 - \mathcal{H}_2 = 2P \cdot p + (\varepsilon_2 - \varepsilon_1) w + m_1^2 - m_2^2 \approx 0. \tag{86}$$

In the c.m. system

[7]Note this relative momentum is the relativistic counterpart of the nonrelativistic form $p = \frac{\mathrm{m_2 p_1 - m_1 p_2}}{\mathrm{m_1 + m_2}} = \frac{\mathrm{m_1 m_2}}{\mathrm{m_1 + m_2}}(\mathbf{v}_1 - \mathbf{v}_2) = \mu \mathbf{v}$

$$p^0 = (p_1^0\varepsilon_2 - \varepsilon_1 p_2^0)/w = (\varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_1)/w = 0, \tag{87}$$

and so

$$P \cdot p = wp^0 = 0. \tag{88}$$

On the constraint hypersurface

$$\varepsilon_1 - \varepsilon_2 = (m_1^2 - m_2^2)/w. \tag{89}$$

Alternatively we could define the constituent energies

$$\begin{aligned} \varepsilon_1 &= \frac{w^2 + m_1^2 - m_2^2}{2w} \\ \varepsilon_2 &= \frac{w^2 + m_2^2 - m_1^2}{2w}. \end{aligned} \tag{90}$$

Their relations with respect to the total center of momentum energy is thus

$$\begin{aligned} \varepsilon_1 - \varepsilon_2 &= (m_1^2 - m_2^2)/w \\ \varepsilon_1 + \varepsilon_2 &= w. \end{aligned} \tag{91}$$

Then,

$$\mathcal{H}_1 - \mathcal{H}_2 = 2P \cdot p \approx 0 \tag{92}$$

becomes a general constraint.

7.2 Hamiltonian

Going back to our constraints, Eq.(72) the Hamiltonian is Eq.(71) with the Lagrange multipliers λ_i determined so that with τ_i, the proper time of particle ι,

$$p_i^\mu = m_i \frac{dx_i^\mu}{d\tau_i} = m_i \frac{d\tau}{d\tau_i}\frac{dx_i^\mu}{d\tau} = \varepsilon_i \frac{dx_i^\mu}{d\tau}; i = 1, 2, \tag{93}$$

and the energy and mass are related by the ratio of the system proper time (= dt in c.m. frame) to the individual particle proper times

$$\varepsilon_i = m_i \frac{d\tau}{d\tau_i}. \tag{94}$$

Considering the simple case of $\Phi = 0$ to recover the free particle results

$$\varepsilon_i \frac{dx_i^\mu}{d\tau} = \varepsilon_i \{x_i^\mu, \mathcal{H}\} \approx \varepsilon_i 2\lambda_i p_i^\mu = p_i^\mu. \tag{95}$$

The Lagrange multipliers have the form Eq.(71)

$$\lambda_i = \frac{1}{2\varepsilon_i}, \tag{96}$$

so that

$$\mathcal{H} = \frac{p_1^2 + m_1^2 + \Phi_{12}}{2\varepsilon_1} + \frac{p_2^2 + m_2^2 + \Phi_{12}}{2\varepsilon_2} \approx 0. \tag{97}$$

Using Eq.(85) we obtain

$$\begin{aligned} \mathcal{H} &= \frac{P \cdot p}{w} + \frac{p^2 + \Phi_{12} + m_1^2 - \varepsilon_1^2}{2\varepsilon_1} + \frac{p^2 + \Phi_{12} + m_2^2 - \varepsilon_2^2}{2\varepsilon_2} \\ &\approx \frac{p^2 + \Phi_{12} + m_1^2 - \varepsilon_1^2}{2\varepsilon_1} + \frac{p^2 + \Phi_{12} + m_2^2 - \varepsilon_2^2}{2\varepsilon_2} \end{aligned} \tag{98}$$

where we have used the constraint (92). But using Eq.(90)

$$\varepsilon_1^2 - m_1^2 = \varepsilon_2^2 - m_2^2 = \frac{w^4 - 2(m_1^2 + m_2^2)w^2 + (m_1^2 - m_2^2)^2}{4w^2} \equiv b^2(w). \tag{99}$$

Thus our dynamic constraint condition $\mathcal{H} \approx 0$ becomes

$$\mathcal{H} = (p^2 + \Phi_{12} - b^2)\frac{w}{\varepsilon_1 \varepsilon_2}. \tag{100}$$

Note that for the free two-particle condition we have

$$p^2 \approx b^2. \tag{101}$$

Solving this algebraically leads to

$$w = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}. \tag{102}$$

In the c.m. system the constraint Eq.(92) implies $p = (0, \mathbf{p})$ so that the above equation expresses exact two-body relativistic kinematics, generalizing the total energy condition of

$$E = \sqrt{\mathbf{p}^2 + m^2}, \tag{103}$$

for a one body system.

7.3 Models for the Quasipotential Φ

In this section we use an analogy with the one-body system to obtain a model for Φ[4]. Introducing world scalar and vector interactions in the free mass shell constraint $\mathcal{H}_0 = p^2 + m^2 \approx 0$ for a single particle one finds

$$\mathcal{H} = \lambda[(p - A)^2 + (m + S)^2] = \lambda[p^2 + m^2 + \Phi] \approx 0. \tag{104}$$

Our one-body quasipotential can therefore be written as

$$\Phi = -2p \cdot A + A^2 + 2mS + S^2. \tag{105}$$

Using a time like vector model $A = (V, \mathbf{0})$ with $p = (E, \mathbf{p})$, we have by substitution the potential interaction

$$\Phi = 2EV - V^2 + 2mS + S^2, \tag{106}$$

and so

$$(E - V)^2 = \mathbf{p}^2 + (m + S)^2 \tag{107}$$

or

$$\begin{aligned} & \mathbf{p}^2 + 2EV - V^2 + 2mS + S^2 \\ = \quad & \mathbf{p}^2 + \Phi = E^2 - m^2. \end{aligned} \tag{108}$$

The next step is to derive the interaction Φ_{12} between two bodies analogous to the one body Φ. A two-body energy and mass expression must be obtained in order to proceed to the three body case. Recall that our nonrelativistic three body problem is formulated in terms of three two-body interactions. We wish to use constraint dynamics to arrive at a relativistic generalization of this nonrelativistic formulation of the three body problem.

In what has been called the classical counterpart of the two-body elastic sector of a quantum field theory taken from Wheeler-Feynman dynamics and in the form of the Fokker Tetrode action for world scalar and vector interactions, Yang [4] has developed the following two-body generalization in the c.m. frame of the one-body expression Eq.(107)

$$(\varepsilon_w - V)^2 = \mathbf{p}^2 + (m_w + S)^2, \tag{109}$$

where $(V, \mathbf{O})$ and S are world vector and scalar interactions. For time-like vector interactions (which are considered in this thesis) we have then

$$\begin{aligned} & \mathbf{p}^2 + 2\varepsilon_w V - V^2 + 2m_w S + S^2 \\ = \quad & \mathbf{p}^2 + \Phi_{12} = \varepsilon_w^2 - m_w^2 = b^2(w) \end{aligned} \tag{110}$$

with

$$\Phi_{12} = 2m_w S + S^2 + 2\varepsilon_w V - V^2. \tag{111}$$

The variables ε_w, m_w , introduced by Todorov and called the energy and mass of the effective particle of relative motion satisfy

$$\varepsilon_w^2 = b^2(w) + m_w^2 \tag{112}$$

and are given by

$$\begin{aligned} \varepsilon_w &= \frac{w^2 - m_1^2 - m_2^2}{2w} \\ m_w &= \frac{m_1 m_2}{w}. \end{aligned} \tag{113}$$

Reference [4] showed how their invariant forms could be derived by comparing constraint and Wheeler-Feynman dynamics. In the equation above $b^2(w)$ is the square of the relative momentum $\mathbf{p}$ of the two particles in the c.m. system in the absence of interactions.

7.4 Quantum Mechanical Relativistic Two-Body Problem

We first note that the quantum version of Hamiltonian Eq.(100) can be written in the form [8]

$$\mathcal{H}\Psi = (p^2 + \Phi_{12} - b^2)\Psi = 0. \tag{114}$$

Going back to Eq.(110) and substituting our potential model $V = S = \frac{\Lambda}{2}(\Lambda r)^\rho$**,** we have the quasipotential

$$\Phi_{12} = \frac{w^2 - (m_1 - m_2)^2}{2w}\Lambda(\Lambda r)^\rho = \frac{(\varepsilon_1^2 + \varepsilon_2^2) - (m_1 - m_2)^2}{2(\varepsilon_1 + \varepsilon_2)}\Lambda(\Lambda r)^\rho. \tag{115}$$

[8] In Dirac's theory of constraints quantization takes place by replacing constraints in the the classical dynamical variables with wave equations in the quantum dynamical variables ,i.e. $\mathcal{H}_i \approx 0$ replaced with $\mathcal{H}_i\Psi = 0$.

This is the relativistic version of Eq.(**??**). Note that with this choice of V, S the quasi-potential Φ_{12} has the same form as appears in the two-body nonrelativistic Schrödinger equation. Recall from the two-body nonrelativistic limit

$$\begin{aligned} w &= m_1 + m_2 + E, \\ E &< < m_1, m_2. \end{aligned} \tag{116}$$

In this limit we have a potential of the form

$$\Phi_{12} = \frac{2m_1 m_2}{m_1 + m_2}\Lambda(\Lambda r)^\rho = 2\mu\Lambda(\Lambda r)^\rho. \tag{117}$$

This limit tells us an appropriate designation is

$$\boldsymbol{\mu}_w = \frac{w^2 - (m_1 - m_2)^2}{4w}, \tag{118}$$

and so with

$$\Phi_{12} = 2\boldsymbol{\mu}_w\Lambda(\Lambda r)^\rho \tag{119}$$

our quantum two-body Schrödinger-like equation is

$$(p^2 + 2\boldsymbol{\mu}_w\Lambda(\Lambda r)^\rho)\Psi = b^2\Psi. \tag{120}$$

The equation is quite similar in form to the nonrelativistic two-body Schrödinger equation (47) used to describe the three body baryon system. In the following section we will begin with the general N body eigenvalue equation proposed by Sazdjian [3] and how our above eigenvalue equation can describe the three body problem as three two-body subsystems.

7.5 Adaption to the Three-Body problem

In order to motivate the Sazdjian relativistic N-body equation, we multiply Eq.(114) by $2\varepsilon_1\varepsilon_2/w$ to bring it to the form

$$[\frac{2(\frac{p^2+\Phi_{12}}{2\varepsilon_1})+2(\frac{p^2+\Phi_{12}}{2\varepsilon_2})}{\sum_{b=1}^{2}1/2\varepsilon_b}+m_1^2-\varepsilon_1^2+m_2^2-\varepsilon_2^2]\Psi(x_{12})=0. \tag{121}$$

This is the same form that Sazdjian has proposed for the general N body eigenvalue equation

$$\sum_{a=1}^{N}[N\frac{(p_{a\perp}^2+\Phi_a)/2\varepsilon_a}{\sum_{b=1}^{N}1/2\varepsilon_b}+m_a^2-\varepsilon_a^2]\Psi(x_{ab},...)=0. \tag{122}$$

This equation only applies to interacting particles in which each body cannot escape the system and is thus totally confined[3], which is the case we are considering here.

The eigenvalue equation for the 3 body problem can be written as

$$\begin{aligned}&[\varepsilon_1^2-m_1^2-3\frac{p_{1\perp}^2+\Phi_1}{\varepsilon_1\,(1/\varepsilon_1+1/\varepsilon_2+1/\varepsilon_3)}\\&+\varepsilon_2^2-m_2^2-3\frac{p_{2\perp}^2+\Phi_2}{\varepsilon_2\,(1/\varepsilon_1+1/\varepsilon_2+1/\varepsilon_3)}\\&+\varepsilon_3^2-m_3^2-3\frac{p_{3\perp}^2+\Phi_3}{\varepsilon_3\,(1/\varepsilon_1+1/\varepsilon_2+1/\varepsilon_3)}]\Psi(x_{12\perp},x_{23\perp},x_{31\perp}).\end{aligned} \tag{123}$$

The epsilons are found by Sazdjian [3] to be

$$\begin{aligned}
\varepsilon_1 &= \frac{w}{3}+\frac{1}{3}\frac{m_1-m_2}{1+(w-M)/(m_1m_2(1/m_1+1/m_2+1/m_3))} \qquad (124)\\
&+\frac{1}{3}\frac{m_1-m_3}{1+(w-M)/(m_1m_3(1/m_1+1/m_2+1/m_3))}\\
\varepsilon_2 &= \frac{w}{3}+\frac{1}{3}\frac{m_2-m_3}{1+(w-M)/(m_2m_3(1/m_1+1/m_2+1/m_3))} \qquad (125)\\
&+\frac{1}{3}\frac{m_2-m_1}{1+(w-M)/(m_2m_1(1/m_1+1/m_2+1/m_3))}\\
\varepsilon_3 &= \frac{w}{3}+\frac{1}{3}\frac{m_3-m_1}{1+(w-M)/(m_3m_1(1/m_1+1/m_2+1/m_3))} \qquad (126)\\
&+\frac{1}{3}\frac{m_3-m_2}{1+(w-M)/(m_3m_2(1/m_1+1/m_2+1/m_3))},
\end{aligned}$$

or

$$\begin{aligned}
\varepsilon_1 &= m_1+wf_1(E,m_1,m_2,m_3)\\
\varepsilon_2 &= m_2+Ef_2(E,m_1,m_2,m_3)\\
\varepsilon_3 &= m_3+Ef_3(E,m_1,m_2,m_3), \qquad (127)
\end{aligned}$$

where

$$\begin{aligned}
f_i &= \frac{3m_j^2m_k^2+3m_j^2m_km_i+3m_jm_k^2m_i+m_im_j^2E}{3(m_im_j+m_jm_k+m_km_i+m_kE)(m_im_j+m_jm_k+m_km_i+m_jE)}\\
&+\frac{2m_j^2m_kE+2m_k^2Em_j+m_k^2Em_i+m_kE^2m_j}{3(m_im_j+m_jm_k+m_km_i+m_kE)(m_im_j+m_jm_k+m_km_i+m_jE)}, \qquad (128)
\end{aligned}$$

with

$$\begin{aligned}
E &= w-M\\
M &= m_1+m_2+m_3, \qquad (129)
\end{aligned}$$

where w is the baryon energy and

$$\begin{aligned}
\Phi_1 &= \Phi_{12}(x_{12\perp}, \varepsilon_1, \varepsilon_2) + \Phi_{13}(x_{13\perp}, \varepsilon_2, \varepsilon_3) \\
\Phi_2 &= \Phi_{23}(x_{23\perp}, \varepsilon_2, \varepsilon_3) + \Phi_{21}(x_{21\perp}, \varepsilon_3, \varepsilon_1) \\
\Phi_3 &= \Phi_{31}(x_{31\perp}, \varepsilon_3, \varepsilon_1) + \Phi_{32}(x_{32\perp}, \varepsilon_1, \varepsilon_2).
\end{aligned} \tag{130}$$

For our model the variables $p_{i\perp}$,$x_{ij\perp}$ are relativistic analogues of the nonrelativistic momentum and position vectors and are essentially three dimensional since their components perpendicular to the total momentum are zero:

$$\begin{aligned}
p_{ij\perp} \cdot P &= 0 \\
x_{ij\perp} \cdot P &= 0.
\end{aligned} \tag{131}$$

For that to be true

$$\begin{aligned}
p_{i\perp} &= p_i + p_i \cdot \widehat{P}\widehat{P} = p_i - \varepsilon_i \widehat{P} \\
x_{ij\perp} &= x_{ij} + x_{ij} \cdot \widehat{P}\widehat{P} \\
\widehat{P} &= \frac{P}{\sqrt{-P^2}} \\
\text{and } P &= p_1 + p_2 + p_3.
\end{aligned} \tag{132}$$

A more familiar form of the eigenvalue equation similar to the nonrelativistic form Eq.(45) is desired.

Rewrite the bound state equation as

$$
\begin{aligned}
&[(\varepsilon_1^2 - m_1^2)(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3) - 3\frac{p_{1\perp}^2 + \Phi_1}{\varepsilon_1} \\
&+(\varepsilon_2^2 - m_2^2)(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3) - 3\frac{p_{2\perp}^2 + \Phi_2}{\varepsilon_2} \\
&+(\varepsilon_3^2 - m_3^2)(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3) - 3\frac{p_{3\perp}^2 + \Phi_3}{\varepsilon_3}]\Psi(x_{12\perp}, x_{23\perp}, x_{31\perp}) \\
= \;& 0. \qquad (133)
\end{aligned}
$$

Using our previous relations and substituting into our bound equation we are left with

$$
\begin{aligned}
&\{E[f_1 b_1 + f_2 b_2 + f_3 b_3] \\
&- \frac{p_{1\perp}^2 + \Phi_{12} + \Phi_{13}}{2\varepsilon_1} - \frac{p_{2\perp}^2 + \Phi_{23} + \Phi_{21}}{2\varepsilon_2} - \frac{p_{3\perp}^2 + \Phi_{31} + \Phi_{32}}{2\varepsilon_3} \\
&\}\Psi(x_{12\perp}, x_{23\perp}, x_{31\perp}) \\
= \;& 0, \qquad (134)
\end{aligned}
$$

where

$$
b_i = b_i(E, m_1, m_2, m_3) = \frac{\varepsilon_i + m_i}{6}(1/\varepsilon_1 + 1/\varepsilon_2 + 1/\varepsilon_3). \qquad (135)
$$

From our previous section the substitution $V = S = \frac{\Lambda}{2}(\Lambda r)^\rho$ gave us a correlation for the potential in terms of the two-body nonrelativistic limit[9]. In analogy to the two-body expression Eq.(115) for our three body case let

$$
\begin{aligned}
\Phi_{ij} &= 2\mu_{\varepsilon_i + \varepsilon_j}\Lambda(\Lambda r_{ij})^\rho \\
&= \frac{[\varepsilon_i + \varepsilon_j]^2 - (m_i - m_j)^2}{2[\varepsilon_i + \varepsilon_j]}\Lambda(\Lambda r_{ij})^\rho. \qquad (136)
\end{aligned}
$$

Thus the eigenenergy takes the form

[9] See Eq.(115)-Eq.(119) in the quantum mechanical relativistic two-body problem section.

$$
\begin{aligned}
&\frac{p_{1\perp}^2}{2\varepsilon_1}+\frac{p_{2\perp}^2}{2\varepsilon_2}+\frac{p_{3\perp}^2}{2\varepsilon_3}\\
&+\sum_{i,j=1}^{3}\frac{[\varepsilon_i+\varepsilon_j]^2-(m_i-m_j)^2}{2[\varepsilon_i+\varepsilon_j]\varepsilon_i}\Lambda(\Lambda r_{ij})^\rho\\
= \;& E[f_1b_1+f_2b_2+f_3b_3]\Psi(x_{12\perp},x_{23\perp},x_{31\perp}). \qquad (137)
\end{aligned}
$$

Simplifying further, let

$$
\begin{aligned}
M_i(E,m_1,m_2,m_3) &= \varepsilon_i F\\
O_{\,i}(E,m_1,m_2,m_3) &= \frac{[\varepsilon_j+\varepsilon_k]^2-(m_j-m_k)^2}{4\varepsilon_j\varepsilon_k F}, \qquad (138)
\end{aligned}
$$

where

$$
F(E,m_1,m_2,m_3)=f_1b_1+f_2b_2+f_3b_3. \qquad (139)
$$

Therefore ignoring spin and using time-like and scalar interactions similar to the two-body case we obtain our eigenvalue equation for the three body system

$$
\begin{aligned}
&[\frac{p_{1\perp}^2}{2M_1(E,m_1,m_2,m_3)}\\
&+\frac{p_{2\perp}^2}{2M_2(E,m_1,m_2,m_3)}\\
&+\frac{p_{3\perp}^2}{2M_3(E,m_1,m_2,m_3)}\\
&+O_1((E,m_1,m_2,m_3)\Lambda(\Lambda x_{12\perp})^\rho\\
&+O_2((E,m_1,m_2,m_3)\Lambda(\Lambda x_{23\perp})^\rho\\
&+O_3((E,m_1,m_2,m_3)\Lambda(\Lambda x_{31\perp})^\rho]\Psi(x_{12\perp},x_{23\perp},x_{31\perp})\\
= \;& E\Psi(x_{12\perp},x_{23\perp},x_{31\perp}). \qquad (140)
\end{aligned}
$$

What we are left with is a relativistic eigenvalue equation (140), which, unlike the nonrelativistic version, we need to solve iteratively since the eigenvalue appears in the potential and kinetic energy terms.

In analogy to the steps that led in the nonrelativistic case from the three body Eq.(43) to the cluster of nonrelativistic two-body equations (**??**), we now solve for the individual clusters of two-body equations

$$[\frac{P_{k\perp}^2}{2\mathcal{M}_k(E,m_1,m_2,m_3)} + O_k(E,m_1,m_2,m_3)\Lambda(\Lambda x_{ij\perp})^\rho]\Psi(x_{ij\perp}) = E_k\Psi(x_{ij\perp}), \qquad (141)$$

where in analogy to Eq.(11) and Eq.(40)

$$\begin{aligned} & \mathcal{M}_i(E,m_1,m_2,m_3) \qquad (142) \\ = \quad & \frac{M_j(E,m_1,m_2,m_3)M_k(E,m_1,m_2,m_3)}{M_1(E,m_1,m_2,m_3)+M_2(E,m_1,m_2,m_3)+M_3(E,m_1,m_2,m_3)}, \end{aligned}$$

and the energy $E = E_1 + E_2 + E_3$.

Stepping back and observing our eigenvalue value equation, it is of the form

$$[\frac{\mathbf{p}^2}{2\mathcal{M}} + O\Lambda(\Lambda r)^\rho]\Psi(\mathbf{r}) = E\Psi(\mathbf{r}). \qquad (143)$$

This bears a striking resemblance to Eq.(47), which is nothing less than the nonrelativistic limit of Eq.(143). For our relativistic case, we absorb the dimensionless variable O into Λ since Λ is a variable parameter. That is we define $\overline{\Lambda}^{\rho+1} = O\Lambda^{\rho+1}$ or $\overline{\Lambda} = O^{1/(1+\rho)}\Lambda$ so that

$$\overline{\Lambda}^{(2+2\rho)/(2+\rho)} = O^{2/(2+2\rho)}\Lambda^{(2+2\rho)/(2+\rho)}. \qquad (144)$$

By using dimensional arguments similar to those used in the nonrelativistic case that led to Eq.(58), the energy eigenvalue for our two-body Hamiltonian equation is now

$$E_i = O_i^{2/(2+\rho)} \frac{\Lambda^{(2+2\rho)/(2+\rho)}}{\mathcal{M}_i^{\rho/(\rho+2)}}. \tag{145}$$

The parameters $\Lambda, m_1, m_2, m_3, \rho$ shall be determined from the minimization of the $n-1$ iteration with respect to their initial values. This is done by applying two methods, a random point and single variable step method. Our fitting function is the sum of seven squared terms, each composed of the difference between our derived theoretical function w and the rest energies ε of seven baryons from the $+(1/2)$ baryon octet, so that

$$\chi^2 = \sum_{j=1}^{7} \left(\frac{w_{nj} - \beta_j}{\beta_j} \right)^2. \tag{146}$$

The theoretical function w_n is

$$\begin{aligned} w_n &= m_1 + m_2 + m_3 + E \\ E &= E_1 + E_2 + E_3 = \Lambda^{(2+2\rho)/(2+\rho)} \sum_{i=1}^{3} \frac{O_i^{2/(2+\rho)}(E, m_1, m_2, m_3)}{\mathcal{M}_i(E, m_1, m_2, m_3)^{\rho/\rho+2}}. \end{aligned} \tag{147}$$

The β_j's are the masses of the $P, N, \Sigma's$, and $\Xi's$. The rest mass energies m_1, m_2, m_3 are the corresponding quark combinations of quarks up, down, and strange.

8 Numerical Fit

8.1 Random point minimization

For this method we begin with a reasonable range of five parameters similar to the grid point method. As the random method suggests, random points in a set range of parameters are chosen, and then the binding energies are used to compute a value of χ^2. This process is repeated and each value of χ^2 is ranked and the top ten points chosen to proceed with our final method, the single variable step method.

What sets this apart from the previous methods is that the points randomly chosen can lead us out of any local minimum we may be in and pursue the global minimum we desire. Our random number generator is defined RD, where $0 \leq RD \leq 1$. The reason we use this method in the relativistic case is because we are zoning in on the global minimum. We no longer need a wide grid as we should be within the vicinity of our desired point of minimization.

Given a set of ranges for our variables, we define the values of each according to the following. For the following our variables (u, d, s, Λ, p) are denoted $(x_1, x_2, x_3, x_4, x_5)$. Let the range of variable x_i be defined from $x_{i\,\min}$ to $x_{i\,\max}$. Our variable is defined

$$x_i = x_{i\,\min} + RD \cdot (x_{i\,\max} - x_{i\,\min}); \quad \text{i = 1,2,3,4,5.} \tag{148}$$

These variables are then used to compute the nonrelativistic energies

$$\begin{aligned} E_i^{(0)}(x_1, x_2, x_3, x_4, x_5) &= \sum_{j=1}^{3} \frac{x_4^{(2x_5+2)/(x_5+2)}}{\mu_j^{x_5/(2+x_5)}}; \quad \text{i = 1,...,7,} \\ \text{where } \mu_j &= \frac{x_j x_k}{x_1 + x_2 + x_3}. \end{aligned}$$

Using this to express our relativistic energies

$$E_i^{(n)}(E_i, x_1, x_2, x_3, x_4, x_5) = O_i^{2/(2+x_5)} \frac{x_4^{(2x_5+2)/(x_5+2)}}{\mathcal{M}_i^{x_5/(x_5+2)}}; \quad \text{i = 1,...,7, n = 1,2,3...,} \quad (149)$$

which are then averaged

$$E_i = \frac{E_i^{(n)} + E_i^{(0)}}{2},$$

and the averaging continues until it converges using the previous energy value instead of the nonrelativistic energy value

$$E_i = \frac{E_i^{(n)} + E_i^{(n-1)}}{2}.$$

This process continues generating various values of χ^2 which are ranked and then used as starting points for the single variable step method.

8.2 Single Variable Step Method

Starting with our initial parameter values obtained above, $(mo_1, mo_2, mo_3, \lambda o, \rho o)$, we attempt to minimize χ^2 further by cycling through each parameter x_i, $i = 1, 2, 3, 4, 5$, and stepping in whichever direction, $x_{io} \pm \Delta x_i$, that takes us towards a local min. For instance take our first parameter m_1. To minimize, simply vary the first parameter, m_1, by , $m_1 + \Delta m_1$. We determine if this "step" decreases the function by evaluating $\chi^2(m_1 + \Delta m_1, m_2, m_3, \lambda, \rho)$. If it does not, we return to our original value $m_1$and try decreasing by $m_1 - \Delta m_1$ and evaluate the function with this value to determine if the function minimizes. We continue in whichever minimizing direction $m_1 \pm \Delta m_1$ until we come toward a local minimum. Once the parameter step method reaches the local min. we continue to repeat the process for the next parameter all they way through to the final parameter ρ. Our final results are shown and ranked in table 5, page 47. For a quick reference the random point generator results are given in Table 4, page 46.

Table 4: Random point generator results.

Rank	χ^2	u	d	s	Λ	ρ
1	.00511	179	166	422	112	4.41
2	.00513	82.9	109	406	127	4.73
3	.00526	124	150	413	120	3.96
4	.00528	152	148	418	117	3.56
5	.00556	224	226	454	89	1.97
6	.02	175	176	356	115	4.29
7	0.1	236	136	583	75.9	4.69
8	.169	272	134	291	93.9	4.51
9	.274	153	240	167	107	2.77
10	.3	52.5	404	635	67.7	4.41

Table 5: Relativistic single variable step results.

Rank	χ^2	u	d	s	Λ	ρ
1	.00492	130.32	139.8	413	120	10.39
2	.00495	165	168	422	112	8.51
3	.00505	146.04	151.35	413.3	117.31	6.57
4	.00505	147.48	152.48	413.82	117	6.59
5	.00506	221.76	223.74	449.46	89	2.29
6	.00526	90.67	108.99	410.94	127	6.44
7	.0085	255.53	202.48	458	91.7	7929.38
8	.01	314.45	272.99	474.26	43.94	1.51
9	.0411	285.87	132.4	408.14	105.79	4.54
10	.0849	170.69	254.76	335.12	125.46	2.77

8.3 Simplex Method

The Simplex, or downhill simplex method, was designed by Nelder and Mead [13]. It is the fastest way to get results when faced with a Hyperdimentional function. This is because the method does not require derivatives, only function evaluations. The Simplex method takes advantage of the geometry of a simplex. Let N denote the number of dimensions the simplex encloses. The simplex is a geometrical figure that consist of N dimensions and has N+1 vertices. a two dimensional simplex is a triangle, a three dimensional simplex is a tetrahedron.

To start the method we require N+1 points to define our simplex. Each point in the simplex is defined by our five parameters with a computed value of χ^2. For the baryon fit N = 5 and the number of vertices is then 5+1=6. The algorithm is designed to make its own way to a minimum. Here the initial points are denoted $\mathbf{P}_{0i}$ and the step in the direction of the minimum is denoted $\mathbf{P}_i$

$$\mathbf{P}_i = P_{0i} + \lambda_i \mathbf{e}_i. \tag{150}$$

Above equation describes the step as the initial point $\mathbf{P}_0$ plus a unit vector $\mathbf{e}_i$ times a scalare λ_i which determines the direction and step size. Much similar to the step in the gradient method. The steps continue towards a minimum, reversing directions when the function χ^2 increases. These steps are called reflections and are disengaged to conserve the volume of the simplex. If a maximum is contained within the volume the simplex contracts in all directions and centers around the local minimum. Each time this occurs the points converge to a minimum point as the simples decreases in volume. In cases where non-convergence occurs it may be useful to change the directions of the unit vectors, rotate them, and restart the procedure. The results are outlined in Table 6, page 49. Note that they are comparable but better than the variable step method results.

Table 6: Simplex Results.

χ^2	u	d	s	Λ	ρ
.00499	174	180	429	108	3.91

9 Conclusion

For the basis of our conclusions; the best approximation from the Variable step method is used, see Table 7 page 52. The accepted experimental values used throughout the thesis has been that of the baryon values taken from the hyperphysics website. Figure 3, page 51, allows for a comparison of the experimental and theoretical averaged values of the ground state baryon octet [10].

The percent errors range from between zero and two point eight. The higher of the errors coming from the Σ baryons. This is due to the small contributions from the relativistic center of energy and reduced mass terms. The difference between the up/down masses and the strange mass is significant enough to cause the theoretical model to fall short of the experimental values**.** The power ρ ranges from 2 to 10.4 with the latter giving slightly better results. This would suggest the need for a better approximation that accounts for differences in power in the near and far field ranges. No one has ever seen an isolated quark. The binding force appears to increase rapidly when one of the three baryons are pulled away from the system. The behavior corresponds to other potential approximations [1] and the long ranged nonperturbative quantum chromo dynamics potentials proposed for the strong force.

The deviations in this approximation reflect the fact that it is a very simplified model, ignoring a lot of quantum numbers and inter-particle dynamics. It should also be noted that the nonrelativistic model better approximates the spectrum than the relativistic one. This would suggest that the relativistic dynamics of three body interactions is still not fully understood. For a more general baryon spectrum modeling one must deal with several more isomultiplets. This will result in a more thorough model with dependence of various quantum numbers. The results of this paper are good enough to encourage further investigation into a more detailed model in the future.

[10] Recall that there are only seven baryons in the figure because the Λ baryon has the exact makeup as the Σ^0 which is due to spin, a variable that is ignored in the approximation, and is thus excuded.

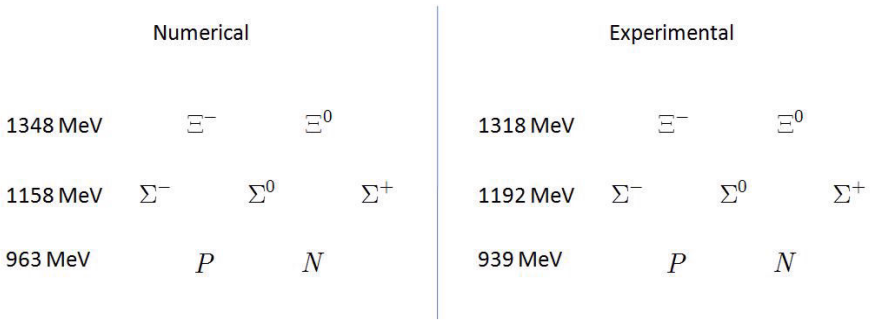

Figure 3: **Averaged final results comparison of ground state baryon octet.**

Table 7: Comparison of simplex, step and experimental baryon masses.

Baryon	Simplex	Step	Experimental
P	953.34	960.96	938.3
N	956.92	965.24	939.6
Σ^+	1151.98	1154.47	1189.4
Σ^0	1155.53	1158.66	1192.5
Σ^-	1159.07	1162.85	1197.3
Ξ^0	1351.16	1346.7	1315
Ξ^-	1354.68	1350.78	1321

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Appendices

Appendix A: Symmetric center of momentum Hamiltonian derivation

The Lagrangian corresponding to our first Hamiltonian Eq.(1) is,

$$L = \frac{1}{2}\sum_{i=1}^{3} m_i \dot{\mathbf{r}}_i^2 - \phi(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_2 - \mathbf{r}_3, \mathbf{r}_3 - \mathbf{r}_1). \tag{151}$$

Using Eq. (7) yields,

$$L = T - \phi = L(\dot{\mathbf{R}}, \dot{\mathbf{u}}_1, \dot{\mathbf{u}}_2, \dot{\mathbf{u}}_3, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3). \tag{152}$$

Without imposing the constraints $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = 0$ and $\dot{\mathbf{R}} = 0 = \mathbf{R}$ the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2 (m_1 + m_2)}{M^2}\dot{\mathbf{u}}_3^2 + \frac{1}{2}\frac{m_2 m_3 (m_2 + m_3)}{M^2}\dot{\mathbf{u}}_1^2 \\ &+ \frac{1}{2}\frac{m_3 m_1 (m_3 + m_1)}{M^2}\dot{\mathbf{u}}_2^2 \\ &- \frac{m_1 m_2 m_3}{M^2}(\dot{\mathbf{u}}_1 \cdot \dot{\mathbf{u}}_2 + \dot{\mathbf{u}}_2 \cdot \dot{\mathbf{u}}_3 + \dot{\mathbf{u}}_3 \cdot \dot{\mathbf{u}}_1). \end{aligned} \tag{153}$$

Assume now $\dot{\mathbf{R}} = 0$, but the constraints $\sum_{i=1}^{3} \dot{\mathbf{u}}_i = 0 = \sum_{i=1}^{3} \mathbf{u}_i$ are not imposed. We can now define the canonical momentum conjugate to $\mathbf{u}_i$ as

$$\mathbf{P}_i = \boldsymbol{\nabla}_{u_2} L = \frac{m_j m_k \dot{\mathbf{u}}_i}{M} - \frac{m_1 m_2 m_3}{M^2}(\dot{\mathbf{u}}_1 + \dot{\mathbf{u}}_2 + \dot{\mathbf{u}}_3),\ i,j,k \text{ cyclic.} \tag{154}$$

If the second constraints are imposed then $\mathbf{P}_i = m_j m_k \dot{\mathbf{u}}_i / M$. Note that for this case, these are not independent canonical momenta.

Express the Legendre Hamiltonian as

$$H = \sum_{i=1}^{3} \mathbf{P}_i \cdot \dot{\mathbf{u}}_i - L. \tag{155}$$

The Lagrangian with use of Eq(152) gives the total Hamiltonian,

$$H = T + \phi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \tag{156}$$

as before, T is given by Eq(153). On the other hand, from Eq(154) follows

$$\begin{bmatrix} \mathbf{P}_1 - \mathbf{P}_2 \\ \mathbf{P}_2 - \mathbf{P}_3 \\ \mathbf{P}_3 - \mathbf{P}_1 \end{bmatrix} = \frac{1}{M} \begin{bmatrix} m_2 m_3 & -m_3 m_1 & 0 \\ 0 & m_1 m_3 & -m_2 m_1 \\ -m_3 m_2 & 0 & m_1 m_2 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_1 \\ \dot{\mathbf{u}}_2 \\ \dot{\mathbf{u}}_3 \end{bmatrix}. \tag{157}$$

From these vectors we obtain our second Hamiltonian:

$$H = \frac{(\mathbf{P}_1 - \mathbf{P}_2)^2}{2m_3} + \frac{(\mathbf{P}_2 - \mathbf{P}_3)^2}{2m_1} + \frac{(\mathbf{P}_3 - \mathbf{P}_1)^2}{2m_2} + \phi(u_1, u_2, u_3). \tag{158}$$

This is the Hamiltonian in the c.m. frame because we have made the assumption $\dot{\mathbf{R}} = 0$.

Appendix B: Non relativistic flow charts

Below is a flow chart outlining the numerical methods used to approximate χ^2 nonrelativistically.

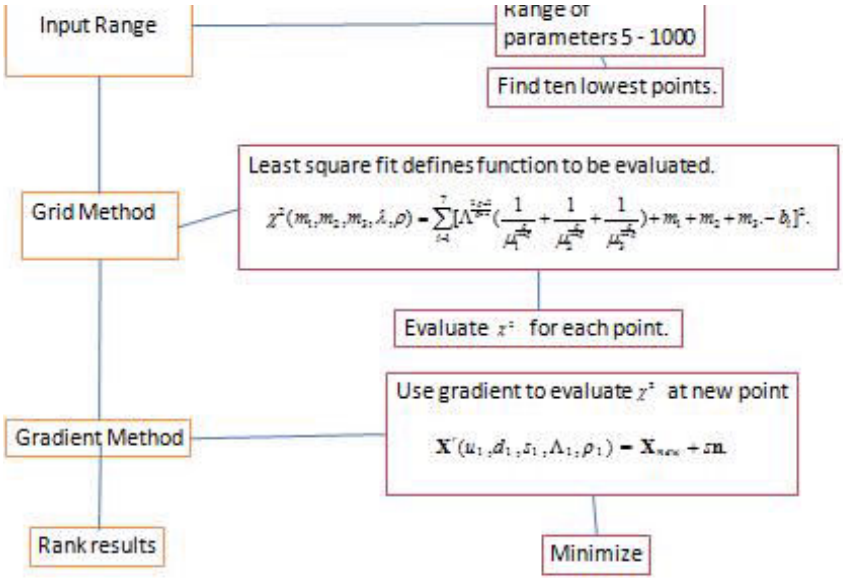

Figure 4: **Non-relativistic program outline.**

Appendix C: Flow chart for random method

Below is a flow chart outlining the random method fit for χ^2.

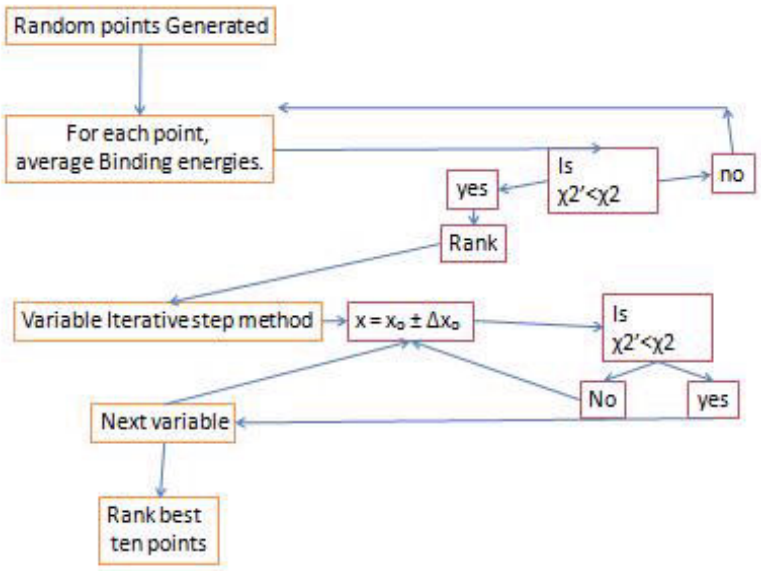

Figure 5: **Random method outline.**

Appendix D: Variable Step Method Flow Chart

Below is a flow chart outlining the variable step method fit for χ^2.

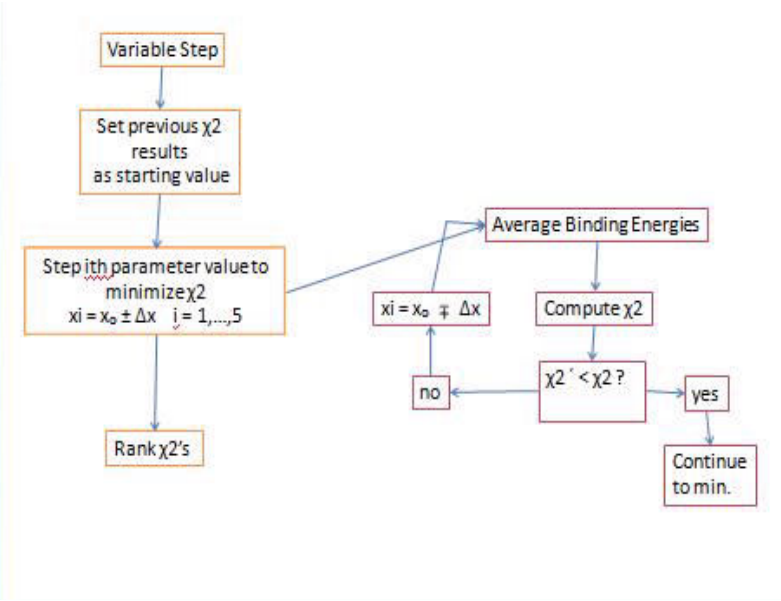

Figure 6: **Variable step method outline.**

Vita

Matthew Patrick Duran was born in Kansas City, Missouri on July 11, 1982. He received his undergraduate degree attaining a double major in Physics and Mathematics with minor in Philosophy at Rockhurst University, Kansas City, Missouri in May 2004. From Spring 2003 to Spring 2004 he worked as a Lab Technician for Quintiles' physical stability team, Missouri. In June 2004 he joined the University of Tennessee Space Institute as Graduate Research Assistant to pursue a Doctor of Philosophy degree in Theoretical Physics.

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