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Date 8/18/94

OVERDETERMINED BOUNDARY VALUE PROBLEMS FOR THE BIHARMONIC OPERATOR

A Thesis

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Scott E. Munn

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DEDICATION

This thesis is dedicated to my parents

Mrs. Diane Carlton

and

Mr. Harry Munn

who have given me invaluable support and educational opportunities.

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ABSTRACT

In the early seventies, Serrin studied an overdetermined boundary value problem and he determined that if a solution exists, then the domain must be a ball. Since then many authors have extended his results to other overdetermined boundary value problems. In this work we give a detailed development of the proof of the basic overdetermined problem first considered by Serrin as done by Weinberger using an elementary argument. We then consider five overdetermined boundary value problems for the equation $\Delta^2 u = f(r)$ in Ω. Two auxiliary conditions are given on interior surfaces and the extra condition is given on the boundary. We show that Ω must be a ball and, moreover, determine an integral representation of the solution for these problems.

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CHAPTER I

INTRODUCTION

In the study of boundary value problems in partial differential equations one of the major questions is whether the problem is well-posed. When encountering these types of problems there are some obvious questions. For example: Is there a solution and, if so, are there any restrictions in the problem?

In 1971, responding to a question posed by Fosdick, Serrin [5] showed that if the solution to the torsion problem has the property that the outward normal derivative is constant on the boundary, then the domain must be a ball. By using the Hopf maximum principle and by moving parallel planes to critical positions, he showed that the solution is radial about a point in the limiting plane and that consequently, the domain is a ball. Serrin then extended this result to more general elliptic problems. Because the normal derivative is also specified on the boundary, this type of problem is called an overdetermined boundary value problem.

Using a more elementary argument, Weinberger [6] obtained the same result for Serrin's basic problem. Weinberger also used the maximum principle together with Green's theorem to establish this result. His method, however, could not be extended as Serrin's method because it depended on the special form of the differential equation.

What about fourth order problems? Is there a similar result? Bennett [1] partly answered this question in 1986. He extended the problem studied by Serrin and Weinberger and also deduced that the domain must be a ball. His argument is very similar to that of Weinberger's, since he also defines a functional that obtains its maximum on the boundary. However, Bennett's work is more complicated since he must use fourth order Green identities.

Later, Payne and Schaefer[3] reformulated the Serrin/Weinberger problem in terms of an equivalent integral problem. By this new method, they also obtained the same result about the domain. With this approach, no use of the maximum principle was needed. They could then extend their argument from two to four to $2m^{th}$ order problems as well as problems for the Green's function for the Laplace and biharmonic operators.

Payne and Schaefer [4] also studied a different kind of fourth order overdetermined problem than was studied in the earlier problems mentioned here. Their equation with the special property of the inhomogenous term and the properties of the torsion function allowed them to reduce the problem to an equivalent second order problem not previously studied. After doing this, they were able to deduce that the domain was a ball.

There are many other papers concerning overdetermined boundary value problems that have appeared in the literature. We refer the reader to the references contained the papers already cited and to current periodicals that continue to report on research for such problems.

In Chapter II of this thesis, we give a detailed development of Weinberger's argument since both Bennett's and Payne and Schaefer's papers mentioned earlier depended on Weinberger's argument in some manner. We will break his argument down into three lemmas and then present his main result as a theorem.

In Chapter III, we will consider several fourth order problems for the biharmonic operator. The method used here differs from the methods mentioned earlier since we allow an inhomogeneous term of a more general type. Let Ω be a bounded domain in N-space with sufficiently smooth boundary and let $\Delta\Delta = \Delta^2$ denote the biharmonic operator. We consider the equation

$$\Delta^2 u = f(r) \quad \text{in } \Omega$$

along with three auxiliary conditions, two of which are given on inner surfaces and the third given on the boundary $\partial\Omega$. We first solve an interior problem and then extend this solution to the rest of the domain. Using the condition on $\partial\Omega$, we show that if there is indeed a solution, then the domain must be a ball.

CHAPTER II

WEINBERGER'S METHOD

In 1971 Weinberger proved that if Ω is a bounded domain in R^N with a smooth boundary and if the solution to the problem

$$\Delta u = -1 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{2.1}$$

has the property that its normal derivative is equal to a constant c on $\partial\Omega$, then Ω is a ball of radius $|Nc|$, where N denotes the dimension, and u is radially symmetric about the center. Weinberger's argument uses the Maximum Principle for second order elliptic partial differential equations and Rellich and Green's identities to establish certain properties that enable him to solve for all second derivatives of the solution of (2.1).

Bennett later extended Weinberger's argument to the fourth order problem:

$$\Delta^2 u = -1 \text{ in } \Omega, \quad u = 0 = \frac{\partial u}{\partial n}, \Delta u = c \text{ on } \partial\Omega, \tag{2.2}$$

and also determined that Ω must be a ball. Bennett's problem is the basic problem from which this thesis was developed. Since Bennett's result depends on Weinberger's method and Bennett's work is more involved because the equation is of fourth order, a complete development of the Weinberger argument is given in this chapter. For a complete development of Bennett's work, we refer the reader to Chapter II of Fouts' thesis[2].

We now consider the Weinberger argument, broken down into three lemmas and a theorem. A first result is

Lemma 1: If u depends only on the distance r from a fixed point and solves $\Delta u=-1$, then $\Delta\left(r\frac{\partial u}{\partial r}\right)=-2$.

Proof: By using the radial form of the Laplace operator in N dimensions, we have

$$
\begin{aligned}
\Delta\left(r\frac{\partial u}{\partial r}\right) &= \left(\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}\right)\left(r\frac{\partial u}{\partial r}\right) \\
&= \frac{\partial}{\partial r}\left(r\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r}\right)+\frac{N-1}{r}\left(r\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r}\right) \\
&= r\frac{\partial^3 u}{\partial r^3}+2\frac{\partial^2 u}{\partial r^2}+(N-1)\frac{\partial^2 u}{\partial r^2}+\left(\frac{N-1}{r}\right)\frac{\partial u}{\partial r} \\
&= r\frac{\partial^3 u}{\partial r^3}+N\frac{\partial^2 u}{\partial r^2}+\Delta u \\
&= 2\Delta u+r\frac{\partial^3 u}{\partial r^3}+(N-1)\frac{\partial^2 u}{\partial r^2}-\left(\frac{N-1}{r}\right)\frac{\partial u}{\partial r} \\
&= 2\Delta u+r\left\{\frac{\partial^3 u}{\partial r^3}+\left(\frac{N-1}{r}\right)\frac{\partial^2 u}{\partial r^2}-\left(\frac{N-1}{r^2}\right)\frac{\partial u}{\partial r}\right\} \\
&= 2\Delta u+r\frac{\partial}{\partial r}\{\Delta u\} \\
&= -2
\end{aligned}
$$

since $\Delta u = -1$.

In the proof of Lemma 2, we utilize the following Green identities:

$$\int_\Omega [U\Delta V+\nabla U\cdot\nabla V]dx = \int_{\partial\Omega} U\frac{\partial V}{\partial n}dS, \tag{2.3}$$

$$\int_\Omega [U\Delta V-V\Delta U]dx = \int_{\partial\Omega}\left[U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right]dS, \tag{2.4}$$

where dx denotes the element of volume in Ω and dS denotes the element of surface area.

Lemma 2: If u solves

$$\Delta u=-1 \text{ in } \Omega, \quad u=0, \frac{\partial u}{\partial n}=c \text{ on } \partial\Omega, \tag{2.5}$$

then $\int_\Omega u\,dx=\frac{Nc^2|\Omega|}{N+2}$, where $|\Omega|$ is the volume of Ω.

Proof: Using Lemma 1, we have

$$\int_\Omega \left[2u - r\frac{\partial u}{\partial r}\right]\mathrm{dx} = \int_\Omega \left[-u\Delta\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\Delta u\right]\mathrm{dx}$$
$$= \int_{\partial\Omega}\left[-u\frac{\partial}{\partial n}\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n}\right]\mathrm{dS}$$
$$= \int_{\partial\Omega} r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n}\mathrm{dS},$$

where we have used (2.4) together with the boundary conditions. Further, we can write

$$\int_{\partial\Omega} r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n}\mathrm{dS} = c\int_{\partial\Omega} r\left(\frac{\partial u}{\partial n}n_i\right)\cdot\frac{\partial r}{\partial x_i}\mathrm{dS}$$
$$= c\int_{\partial\Omega} r\frac{\partial u}{\partial n}\left(n_i\cdot\frac{\partial r}{\partial x_i}\right)\mathrm{dS}$$
$$= c^2\int_{\partial\Omega} r\frac{\partial r}{\partial n}\mathrm{dS}$$
$$= c^2\int_{\partial\Omega}\frac{\partial}{\partial n}\left(\frac{1}{2}r^2\right)\mathrm{dS}.$$

Now, by using (2.3) we have

$$c^2\int_{\partial\Omega}\frac{\partial}{\partial n}\left(\frac{1}{2}r^2\right)\mathrm{dS} = c^2\int_\Omega \Delta\left(\frac{1}{2}r^2\right)\mathrm{dx} = Nc^2|\Omega|,$$

where $|\Omega|$ denotes the volume of Ω. Thus

$$\int_\Omega\left[2u - r\frac{\partial u}{\partial r}\right]\mathrm{dx} = Nc^2|\Omega|. \tag{2.6}$$

We now consider

$$\int_\Omega r\frac{\partial u}{\partial r}\mathrm{dx} = \int_\Omega r(\nabla u\cdot\vec{r})\mathrm{dx} = \int_\Omega \nabla u\cdot\left(r\left(\frac{x_i}{r}\right)\right)\mathrm{dx}$$
$$= \int_\Omega \nabla u\cdot\frac{\partial}{\partial x_i}\left(\frac{1}{2}r^2\right)\mathrm{dx} = \int_\Omega \nabla u\cdot\nabla\left(\frac{1}{2}r^2\right)\mathrm{dx}.$$

By (2.3) and the boundary conditions on u, we have

$$\int_\Omega \nabla u\cdot\nabla\left(\frac{1}{2}r^2\right)\mathrm{dx} = -N\int_\Omega u\mathrm{dx}.$$

Hence,

$$\int_\Omega r\frac{\partial u}{\partial r}\mathrm{dx} = -N\int_\Omega u\mathrm{dx}. \tag{2.7}$$

The conclusion now follows when we substitute (2.7) into (2.6) and the proof is complete.

In Lemma 3 we utilize the comma notation $\Psi,_i$ to denote partial differentiation with respect to x_i and the summation convention on repeated indices, i.e.,

$$u,_i u,_i = \sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^2.$$

Lemma 3: If u solves (2.5), then $|\nabla u|^2 + \frac{2}{N}u = const. \quad in\ \overline{\Omega}.$

Proof: By using the differential equation and the Schwarz inequality, we observe that

$$1 = (\Delta u)^2 \le N\sum_{i=1}^{N} u,_{ii}^2 \le N\sum_{i,j=1}^{N} u,_{ij}^2. \tag{2.8}$$

We now compute

$$\begin{aligned}\Delta\left[|\nabla u|^2 + \frac{2}{N}u\right] &= \Delta\left[u,_i u,_i + \frac{2}{N}u\right] = (u,_i u,_i),_{jj} + \frac{2}{N}u,_{jj}\\ &= (2u,_i u,_{ij}),_j - \frac{2}{N}\\ &= 2[u,_i u,_{ijj} + u,_{ij} u,_{ij}] - \frac{2}{N}\\ &= 2\left[u,_{ij} u,_{ij} - \frac{1}{N}\right].\end{aligned}$$

Thus, $\Delta\left[|\nabla u|^2 + \frac{2}{N}u\right] \ge 0$ by (2.8) so that $|\nabla u|^2 + \frac{2}{N}u$ is a subharmonic function.

Since $u = 0$ and $\frac{\partial u}{\partial n} = c$ on $\partial\Omega$, we know that

$$|\nabla u|^2 + \frac{2}{N}u = |\nabla u|^2 = \left(\frac{\partial u}{\partial n}\right)^2 = c^2 \text{ on } \partial\Omega.$$

We can conclude from the Strong Maximum Principle that either

(i) $|\nabla u|^2+\dfrac{2}{N}u<c^2$ in Ω or (ii) $|\nabla u|^2+\dfrac{2}{N}u\equiv c^2$ in Ω. We assume condition (i) holds. We integrate both sides and obtain

$$\int_\Omega |\nabla u|^2\,dx+\frac{2}{N}\int_\Omega u\,dx<\int_\Omega c^2 dx,$$

which by (2.3) is equivalent to

$$\int_\Omega u\,dx+\frac{2}{N}\int_\Omega u\,dx<\int_\Omega c^2 dx.$$

Thus

$$\frac{N+2}{N}\int_\Omega u\,dx<c^2|\Omega|$$

which contradicts Lemma 2. Hence we deduce that

$$|\nabla u|^2+\frac{2}{N}u\equiv c^2 \text{ in } \Omega.$$

We are now ready to prove Weinberger's main result which we formulate as a theorem.

Theorem 1: If u solves (2.5), then Ω is a ball of radius N|c| and $u=\dfrac{(Nc)^2-r^2}{2n}$.

Proof: From Lemma 3, we have

$$|\nabla u|^2+\frac{2}{N}u\equiv c^2 \text{ in } \Omega$$

so its Laplacian must vanish, i.e., we conclude

$$\Delta\left(|\nabla u|^2+\frac{2}{N}u\right)=2\sum_{i,j=1}^{N}u,_{ij}^2-\frac{2}{N}=0.$$

Thus,

$$\sum_{i,j=1}^{N}u,_{ij}^2=\frac{1}{N}. \tag{2.9}$$

Since

$$\frac{1}{N}\le\sum_{i=1}^{N}u,_{ii}^2\le\sum_{i,j=1}^{N}u,_{ij}^2,$$

from (2.8), along with (2.9) we have $u_{,ij} = 0$ for $i \neq j$. Hence, we have equality in (2.8). Now since $\Delta u = u_{,ii} = -1$ in Ω, and the fact that we have equality in (2.8) implies

$$u_{,ij} = \frac{-1}{N}\delta_{ij}$$

where δ_{ij} is the Kronecker delta symbol.

It follows that for N dimensions, we have

$$u = \frac{-x_i x_i}{2N} + \alpha_i x_i + \beta_i \beta_i$$

where α_i, β_i are arbitrary constants. By completing the square, we can write

$$u = \frac{-1}{2N}\left([x_i - N\alpha_i][x_i - N\alpha_i]\right) + \beta_i\beta_i + \frac{N}{2}\alpha_i\alpha_i$$

and letting $\frac{A}{2N} = \beta_i\beta_i + \frac{N}{2}\alpha_i\alpha_i$

$$u = \frac{-1}{2N}\left(r^2 - A\right) = \frac{1}{2N}\left(A - r^2\right)$$

where r denotes the distance from x to the center of Ω. Now, by applying the boundary conditions, we see that $r = \sqrt{A}$ since u=0 on the $\partial\Omega$. This implies that Ω is a ball of radius $\sqrt{A}$. Since $\partial u / \partial n = \partial u / \partial r$ on the surface of a ball, we obtain $\partial u / \partial n = -r / N$ and from $\partial u / \partial n = c$ on $\partial\Omega$ that $r^2 = (Nc)^2$. Hence,

$$u = \frac{1}{2N}\left(N^2c^2 - r^2\right).$$

Here we note that $\partial u / \partial n < 0$ on $\partial\Omega$.

As mentioned earlier, Bennett's result follows in a similar manner to Weinberger's, only it is more involved because of the fourth order differential equation in (2.2). We can outline his results as three lemmas and a theorem as in Weinberger's paper .

Lemma 4: If u depends only on the distance r from a fixed point and solves $\Delta^2 u = -1$, then $\Delta^2\left(r\frac{\partial u}{\partial r}\right) = -4$.

Lemma 5: If u solves (2.2), then $\int\limits_{\Omega} u\,\mathrm{dx} = \dfrac{-N}{N+4}c^2|\Omega|$, where $|\Omega|$ is the volume of Ω.

Lemma 6: If u solves (2.2) then

$$\Phi = \frac{N-4}{N+2}u + \frac{N-4}{2(N+2)}(\Delta u)^2 + \sum_{i,j=1}^{N} u,_{ij}^2 - u,_i(\Delta u),_i = \text{const.} \quad \text{in } \overline{\Omega}.$$

Theorem 2: Let Ω be a bounded domain in R^N where $\partial\Omega \in C^{4+\varepsilon}$ and $u \in C^4(\overline{\Omega})$. If u solves (2.2), then Ω is a ball of radius

$$r = \sqrt{[N(N+2)|c|]}$$

and

$$u = \frac{-1}{2N}\left\{\frac{1}{4}(N+2)(Nc)^2 + \frac{1}{2}Ncr^2 + \frac{1}{4(N+2)}r^4\right\}.$$

The problems we consider in Chapter III will require a different technique since the partial differential equation is more general and the auxiliary conditions are not all specified on $\partial\Omega$. Nevertheless, the examination of the Weinberger method and the Bennett problem is important to see how such overdetermined problems can be studied.

CHAPTER III

FIVE OVERDETERMINED PROBLEMS

3.0 MOTIVATION AND PRELIMINARIES

In this chapter we will consider five overdetermined problems where the three auxiliary conditions are given on three different surfaces. We find an integral representation of the solution of the partial differential equation and then, using the auxiliary conditions, we deduce that Ω is a ball. Moreover, we will have an integral representation of the solutions to our problems. These problems and this approach are motivated by an observation of Payne for a similar kind of problem.

L.E. Payne noticed that if u is a classical solution of

$$\begin{aligned} &\Delta^2 u = 1 \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial B_a, \\ &u = -c \text{ on } \partial B_b, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where c is a constant and the concentric balls B_a and B_b satisfy $B_a \subset B_b \subset \partial\Omega$, then Ω must be a ball. This follows since

$$u = \frac{(a^2 - r^2)}{4(N+2)}\Psi$$

solves the partial differential equation and the auxiliary conditions on ∂B_a and $\partial\Omega$, where Ψ is the torsion function for Ω, i.e., Ψ is the solution to the Dirichlet problem

$$\Delta\Psi = -1 \text{ in } \Omega, \;\; \Psi = 0 \text{ on } \partial\Omega.$$

This function u solves the partial differential equation since

$$\Delta u = \frac{r^2 - a^2}{4(N+2)} - \frac{N}{2(N+2)}\Psi + \frac{1}{2(N+2)}[2rr_{,i}\Psi_{,i}],$$

where we have used the comma notation and repeated index convention. Hence

$$\Delta u = \frac{r^2 - a^2}{4(N+2)} - \frac{N}{2(N+2)}\Psi - \frac{1}{N+2}x_i\Psi,_i$$

so that

$$\Delta^2 u = \frac{2N}{4(N+2)} + \frac{N}{2(N+2)} - \frac{1}{N+2}\left[2x_i,_j\,\Psi,_{ij}\right]$$
$$= \frac{N}{N+2} - \frac{1}{N+2}[2(-1)]$$
$$=1$$

since $x_i,_j = 0$ for $i \neq j$. The auxiliary conditions on $\partial \mathrm{B_a}$ and $\partial\Omega$ are clearly satisfied.

The remaining auxiliary condition allows one to determine Ψ on $\partial \mathrm{B_b}$, namely,

$$\Psi = \frac{4(N+2)}{b^2 - a^2}c \ \text{ on } \partial \mathrm{B_b}.$$

Thus we can determine the unique function Ψ from the Dirichlet problem

$$\Delta\Psi = -1 \ \text{ in } \mathrm{B_b}, \quad \Psi = \frac{4(N+2)}{b^2 - a^2}c \quad \text{on } \partial \mathrm{B_b}. \tag{3.2}$$

Since we are in a ball, we use the radial form of the Laplacian so that

$$r^{-(N-1)}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial\Psi}{\partial r}\right) = -1.$$

Multiplying by r^{N-1} and integrating with respect to r, we obtain,

$$r^{N-1}\frac{\partial\Psi}{\partial r} = \frac{-r^N}{N} + \mathrm{k}_1,$$

where k_1 is a constant of integration. Multiplying by $r^{-(N-1)}$ and integrating with respect to r, we obtain the solution of the differential equation in (3.2),

$$\Psi = \mathrm{k}_2 - \frac{r^2}{2N}$$

where k_2 is a constant of integration and we have chosen $\mathrm{k}_1 = 0$ to avoid a singularity at r = 0. Using the boundary condition in (3.2) we see that

$$\Psi = \frac{4(N+2)}{b^2 - a^2}c + \frac{b^2 - r^2}{2N}.$$

We now continue Ψ to the rest of Ω and since $\Psi = 0$ on $\partial\Omega$, we have an equation in r. When points are on the boundary $\partial\Omega$, r is a constant and it follows that Ω is a ball. In fact Ω is a ball of radius $r = \sqrt{b^2 + \dfrac{8N(N+2)c}{b^2 - a^2}}$. Thus, the solution of (3.1) is $u = \dfrac{a^2 - r^2}{4(N+2)}\left\{\dfrac{4(N+2)}{b^2 - a^2}\mathrm{c} + \dfrac{b^2 - r^2}{2N}\right\}$.

We extend this result to a more general class of problems by considering $\Delta^2 u = f(r)$, where, as in Payne's problem, the auxiliary data is given on three surfaces, but the data is allowed to be nonzero and not always of Dirichlet type. As a preliminary to this, however, we first seek a general radial solution of

$$\Delta^2 u = f(r). \tag{3.3}$$

As will become apparent later, we must consider the N=2 and N=4 cases separately.

Using the radial form of the Laplacian in N dimensions, namely,

$$\Delta u = r^{-(N-1)} \frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial u}{\partial r} \right),$$

we can write (3.3) as

$$r^{-(N-1)} \frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial \Delta u}{\partial r} \right) = f(r).$$

We multiply by r^{N-1} and integrate with respect to r to obtain

$$r^{N-1} \frac{\partial \Delta u}{\partial r} = \int_0^r p^{N-1} f(p) dp + \mathrm{k}_1$$

where k_1 is a constant of integration. Since

$$\frac{\partial \Delta u}{\partial r} = r^{-(N-1)} \int_0^r p^{N-1} f(p) dp + \mathrm{k}_1 r^{-(N-1)}$$

we choose $k_1 = 0$ to avoid a singularity at r = 0. Now integrating with respect to r, we have

$$\Delta u = \int_{t=0}^{t=r} t^{-(N-1)} \int_{p=0}^{p=t} p^{N-1} f(p) dp dt + \mathrm{k}_2$$

where k_2 is a constant of integration. Interchanging the order of integration, we have

$$\Delta u = \int_{p=0}^{p=r} p^{N-1} \int_{t=p}^{t=r} t^{-(N-1)} f(p)dtdp + \mathrm{k}_2$$

so that

$$\Delta u = \frac{1}{N-2}\int_0^r \left[p - \frac{p^{N-1}}{r^{N-2}}\right] f(p)dp + \mathrm{k}_2. \tag{3.4}$$

Using the radial form of the Laplacian, multiplication by r^{N-1}, and integration with respect to r, we obtain

$$r^{N-1}\frac{\partial u}{\partial r} = \frac{1}{N-2}\int_{t=0}^{t=r} t^{N-1}\int_{p=0}^{p=t}\left[p - \frac{p^{N-1}}{t^{N-2}}\right] f(p)dpdt + \mathrm{k}_2\frac{r^N}{N} + \mathrm{k}_3$$

where k_3 is a constant of integration. However, we assume $\mathrm{k}_3 = 0$ to avoid a singularity at r = 0. We again interchange the order of integration to achieve

$$r^{N-1}\frac{\partial u}{\partial r} = \frac{1}{N-2}\int_{p=0}^{p=r}\int_{t=p}^{t=r}\left[pt^{N-1} - p^{N-1}t\right] f(p)dtdp + \mathrm{k}_2\frac{r^N}{N}.$$

Now integrating with respect to t, we have

$$r^{N-1}\frac{\partial u}{\partial r} = \frac{1}{2N(N-2)}\int_0^r \left[2pr^N - Np^{N-1}r^2 + (N-2)p^{N+1}\right] f(p)dp + \mathrm{k}_2\frac{r^N}{N}.$$

Multiplication by $r^{-(N-1)}$ and integration with respect to r results in

$$u = \frac{1}{2N(N-2)}\int_{t=0}^{t=r} t^{-(N-1)}\int_{p=0}^{p=t}\left[2pt^N - Np^{N-1}t^2 + (N-2)p^{N+1}\right] f(p)dpdt + \mathrm{k}_2\frac{r^2}{2N} + \mathrm{k}_4$$

where k_4 is a constant of integration. Interchanging the order of integration, we have

$$u = \frac{1}{2N(N-2)}\int_{p=0}^{p=r}\int_{t=p}^{t=r}\left[2pt - N\frac{p^{N-1}}{t^{N-3}} + (N-2)\frac{p^{N+1}}{t^{N-1}}\right] f(p)dtdp + \mathrm{k}_2\frac{r^2}{2N} + \mathrm{k}_4. \tag{3.5}$$

Hence the classical solution of (3.3) is

$$u = \frac{1}{2N(N-2)}\int_0^r \left[pr^2 - \frac{p^{N+1}}{r^{N-2}} + \frac{N}{N-4}\left(\frac{p^{N-1}}{r^{N-4}} - p^3\right)\right] f(p)dp + \mathrm{k}_2\frac{r^2}{2N} + \mathrm{k}_4, \tag{3.6}$$

where the restriction $N \neq 2,4$ is evident.

We will now consider the N=2 case. We can write (3.3) as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Delta u}{\partial r}\right)=f(r).$$

We multiply by r and integrate with respect to r to obtain

$$r\frac{\partial \Delta u}{\partial r}=\int_0^r pf(p)dp+\mathrm{k}_1,$$

where k_1 is a constant of integration. Since

$$\frac{\partial \Delta u}{\partial r}=\frac{1}{r}\int_0^r pf(p)dp+\frac{\mathrm{k}_1}{r},$$

we choose $\mathrm{k}_1=0$ to avoid a singularity at r = 0. Now integrating with respect to r, we have

$$\Delta u=\int_{t=0}^{t=r}\frac{1}{t}\int_{p=0}^{p=t} pf(p)dpdt+\mathrm{k}_2,$$

where k_2 is a constant of integration. Interchanging the order of integration and integrating, we have

$$\Delta u=\int_0^r p\ln\left(\frac{r}{p}\right)f(p)dp+\mathrm{k}_2. \tag{3.7}$$

Using the radial form of the Laplacian, multiplication by r, and integration with respect to r, we obtain

$$r\frac{\partial u}{\partial r}=\int_{t=0}^{t=r}t\int_{p=0}^{p=t} p\ln\left(\frac{t}{p}\right)f(p)dpdt+\mathrm{k}_2\frac{r^2}{2}+\mathrm{k}_3$$

where k_3 is a constant of integration. However, we again assume $\mathrm{k}_3=0$ to avoid a singularity at r = 0. Again, we interchange the order of integration and integrate with respect to t so that

$$r\frac{\partial u}{\partial r}=\frac{1}{4}\int_0^r\left[2pr^2\ln\left(\frac{r}{p}\right)+p^3-pr^2\right]f(p)dp+\mathrm{k}_2\frac{r^2}{2}.$$

Multiplication by $1/r$ and integration with respect to r yields

$$u=\frac{1}{4}\int_{t=0}^{t=r}\frac{1}{t}\int_{p=0}^{p=t}\left[2pt^2\ln\left(\frac{t}{p}\right)+p^3-pt^2\right]f(p)dpdt+\mathrm{k}_2\frac{r^2}{4}+\mathrm{k}_4$$

where k_4 is a constant of integration. After interchanging the order of integration and simplifying, we see that

$$u=\frac{1}{4}\int_0^r\left[pr^2\ln\left(\frac{r}{p}\right)+p^3\ln\left(\frac{r}{p}\right)+p^3-pr^2\right]f(p)dp+\mathrm{k}_2\frac{r^2}{4}+\mathrm{k}_4 \qquad (3.8)$$

when N = 2.

For completeness, we now consider the N = 4 case of (3.3). Since all of the calculations are valid until the equation (3.5), we begin there. Substituting $N=4$ into (3.5) yields

$$u=\frac{1}{16}\int_{p=0}^{p=r}\int_{t=p}^{t=r}\left[2pt-4\frac{p^3}{t}+2\frac{p^5}{t^3}\right]f(p)dtdp+\mathrm{k}_2\frac{r^2}{8}+\mathrm{k}_4$$

so that

$$u=\frac{1}{16}\int_0^r\left[pr^2-\frac{p^5}{r^2}-4p^3\ln\left(\frac{r}{p}\right)\right]f(p)dp+\mathrm{k}_2\frac{r^2}{8}+\mathrm{k}_4. \qquad (3.9)$$

for N = 4.

3.1 Problem One

For our first problem, we consider

$$\Delta^2 u=f(r) \quad \text{in } \Omega, \qquad (3.10a)$$

$$u=\mathrm{c}_1 \quad \text{on } \partial\mathrm{B}_a, \qquad (3.10b)$$

$$u=\mathrm{c}_2 \quad \text{on } \partial\mathrm{B}_b, \qquad (3.10c)$$

$$u=\mathrm{c}_3 \quad \text{on } \partial\Omega, \qquad (3.10d)$$

where c_1, c_2, c_3 are constants and the concentric balls satisfy $\mathrm{B}_a\subset\mathrm{B}_b\subset\Omega$. Now using the solution of (3.10a) obtained earlier, i.e., (3.6), and the condition in (3.10b), we can find k_4 in terms of k_2, namely,

$$\mathrm{k}_4=\mathrm{c}_1-\frac{1}{2N(N-2)}\int_0^a\left[pa^2-\frac{p^{N+1}}{a^{N-2}}+\frac{N}{N-4}\left(\frac{p^{N-1}}{a^{N-4}}-p^3\right)\right]f(p)dp-\mathrm{k}_2\frac{a^2}{2N}.$$

Using k_4 and (3.10c) in (3.6) we find k_2,

$$\mathrm{k}_2=\frac{2N}{b^2-a^2}\left\{\mathrm{c}_2-\mathrm{c}_1+\frac{1}{2N(N-2)}\left[\int_0^a\left[pa^2-\frac{p^{N+1}}{a^{N-2}}+\frac{N}{N-4}\left(\frac{p^{N-1}}{a^{N-4}}-p^3\right)\right]f(p)dp\right.\right.$$

$$-\int_0^b\left[pb^2-\frac{p^{N+1}}{b^{N-2}}+\frac{N}{N-4}\left(\frac{p^{N-1}}{b^{N-4}}-p^3\right)\right]f(p)dp\Bigg]\Bigg\}.$$

Hence,

$$u=\frac{1}{2N(N-2)}\left\{\int_0^r\left[pr^2-\frac{p^{N+1}}{r^{N-4}}+\frac{N}{N-4}\left(\frac{p^{N-1}}{r^{N-4}}-p^3\right)\right]f(p)dp\right.$$
$$\left.-\int_0^a\left[pa^2-\frac{p^{N+1}}{a^{N-2}}+\frac{N}{N-4}\left(\frac{p^{N-1}}{a^{N-4}}-p^3\right)\right]f(p)dp\right\}+\frac{r^2-a^2}{b^2-a^2}\{c_2-c_1$$
$$+\frac{1}{2N(N-2)}\left\{\int_0^a\left[pa^2-\frac{p^{N+1}}{a^{N-2}}+\frac{N}{N-4}\left(\frac{p^{N-1}}{a^{N-4}}-p^3\right)\right]f(p)dp\right.$$
$$\left.-\int_0^b\left[pb^2-\frac{p^{N+1}}{b^{N-2}}+\frac{N}{N-4}\left(\frac{p^{N-1}}{b^{N-4}}-p^3\right)\right]f(p)dp\right\}\ \}+c_1.$$

This solution can be written in a simplified form, i.e.,

$$u=F(r)-F(a)+\frac{r^2-a^2}{b^2-a^2}\{c_2-c_1+F(a)-F(b)\}+c_1, \tag{3.11}$$

where

$$F(r)=\frac{1}{2N(N-2)}\int_0^r\left[pr^2-\frac{p^{N+1}}{r^{N-2}}+\frac{N}{N-4}\left(\frac{p^{N-1}}{r^{N-4}}-p^3\right)\right]f(p)dp. \tag{3.12}$$

Now by imposing (3.10d), we have an equation in r, namely,

$$F(r)-F(a)+\frac{r^2-a^2}{b^2-a^2}\{c_2-c_1+F(a)-F(b)\}+c_1-c_3=0. \tag{3.13}$$

Since we are under the assumption that (3.10) has a solution, we deduce that there is a root to equation (3.13) corresponding to points on $\partial\Omega$ and hence Ω is a ball. However, finding this root appears to be a fierce task since we are dealing with such complex functions.

Now, let us consider the N=2 case of (3.10). Using (3.8) and (3.10b), we find k_4 in terms of k_2:

$$k_4=c_1-\frac{1}{4}\int_0^a\left[pa^2\ln\frac{a}{p}+p^3\ln\frac{a}{p}-pa^2+p^3\right]f(p)dp-k_2\frac{a^2}{4}.$$

Using (3.10c) and our k_4 already determined, we find k_2 is given by

$$k_2 = \frac{4}{b^2 - a^2}\{c_2 - c_1 + \frac{1}{4}\left\{\int_0^a \left[pa^2 \ln\frac{a}{p} + p^3 \ln\frac{a}{p} - pa^2 + p^3\right] f(p)dp\right.$$

$$\left. - \int_0^b \left[pb^2 \ln\frac{b}{p} + p^3 \ln\frac{b}{p} - pb^2 + p^3\right] f(p)dp\right\} \}.$$

Hence, we can write the solution of (3.10a, b, c) when N = 2 as

$$u = \tilde{F}(r) - \tilde{F}(a) + \frac{r^2 - a^2}{b^2 - a^2}\left\{c_2 - c_1 + \tilde{F}(a) - \tilde{F}(b)\right\} + c_1, \qquad (3.14)$$

where

$$\tilde{F}(r) = \frac{1}{4}\int_0^r \left[pr^2 \ln\frac{r}{p} + p^3 \ln\frac{r}{p} - pr^2 + p^3\right] f(p)dp. \qquad (3.15)$$

We now continue u to the rest of Ω and impose (3.10d) to obtain an equation in r:

$$\tilde{F}(r) - \tilde{F}(a) + \frac{r^2 - a^2}{b^2 - a^2}\left\{c_2 - c_1 + \tilde{F}(a) - \tilde{F}(b)\right\} + c_1 - c_3 = 0. \qquad (3.16)$$

This implies that Ω is a ball or (3.10) has no solution.

Similarly for N=4, we find k_4 in terms of k_2 by (3.9) and (3.10b), i.e.,

$$k_4 = c_1 - \frac{1}{16}\int_0^a \left[pa^2 - \frac{p^5}{a^2} - 4p^3 \ln\frac{a}{p}\right] f(p)dp - k_2 \frac{a^2}{8}.$$

Using this k_4 and (3.10c), we find k_2 to be

$$k_2 = \frac{8}{b^2 - a^2}\left\{c_2 - c_1 + \frac{1}{16}\int_0^a \left[pa^2 - \frac{p^5}{a^2} - 4p^3 \ln\frac{a}{p}\right] f(p)dp\right.$$

$$\left. - \frac{1}{16}\int_0^b \left[pb^2 - \frac{p^5}{b^2} - 4p^3 \ln\frac{b}{p}\right] f(p)dp\right\}.$$

Thus the solution to (3.10a, b, c) is

$$u = \overline{F}(r) - \overline{F}(a) + \frac{r^2 - a^2}{b^2 - a^2}\left\{c_2 - c_1 + \overline{F}(a) - \overline{F}(b)\right\} + c_1 \qquad (3.17)$$

where

$$\overline{F}(r) = \frac{1}{16}\int_0^r \left[pr^2 - \frac{p^5}{r^2} - 4p^3 \ln\frac{r}{p}\right] f(p)dp. \qquad (3.18)$$

We now continue u to the rest of Ω and impose the extra condition (3.10d). Thus, we obtain an equation in r:

$$\overline{F}(r) - \overline{F}(a) + \frac{r^2 - a^2}{b^2 - a^2}\left\{c_2 - c_1 + \overline{F}(a) - \overline{F}(b)\right\} + c_1 - c_3 = 0. \qquad (3.19)$$

This implies that if (3.10) has a solution when N = 4, then Ω must be a ball.

For both equations (3.16) and (3.19) we again see that actually finding the roots to these equations appears quite foreboding, but implicitly, we know a root corresponding to points on $\partial\Omega$ must exist or there is no solution to our problem.

An example is now in order. We let f(r) = 1 in (3.10). We can now do the integrations and proceed as in the derivation of (3.6) to obtain a general regular form of the solution u of (3.10a) when f(r) = 1, i.e.,

$$u = \frac{r^4}{8N(N+2)} + k_2 r^2 + k_4.$$

Using (3.10b) we obtain

$$k_4 = c_1 - \frac{a^4}{8N(N+2)} - k_2 a^2.$$

Now using the auxiliary condition (3.10c), we find

$$k_2 = \frac{1}{b^2 - a^2}\left(c_2 - c_1 - \frac{b^4 - a^4}{8N(N+2)}\right).$$

Hence, the solution u of (3.10a, b, c) when f(r) = 1 is

$$u = \frac{r^4 - a^4}{8N(N+2)} + \frac{r^2 - a^2}{b^2 - a^2}\left\{c_2 - c_1 - \frac{b^4 - a^4}{8N(N+2)}\right\} + c_1. \tag{3.20}$$

We now continue u to the rest of Ω and by imposing (3.10d) our equation in r is

$$\frac{r^4 - a^4}{8N(N+2)} + \frac{r^2 - a^2}{b^2 - a^2}\left\{c_2 - c_1 - \frac{b^4 - a^4}{8N(N+2)}\right\} + c_1 - c_3 = 0. \tag{3.21}$$

This implies that Ω is a ball. In fact, by doing some tedious algebra, Ω is a ball of radius

$$r = \sqrt{\frac{1}{2}\left(-8N(N+2)\alpha + \sqrt{32N(N+2)\left[2N(N+2)\alpha^2 + \alpha a^2 + c_3 - c_1\right] - 4a^4}\right)}$$

where

$$\alpha = \frac{a^4 - b^4 + 8N(N+2)(c_2 - c_1)}{8N(N+2)(b^2 - a^2)}.$$

This illustrates that there are constraints in our problem. Obviously, c_1, c_2, and c_3 are not completely arbitrary because r must be positive and a real number. Thus, we caution the reader that there are some implicit restrictions on c_1, c_2, and c_3 which depend on what the function f(r) is in the differential equation. Clearly, two trivial requirements on the constants in this example are: $\alpha < 0$, i.e., $c_2 < c_1$, and also $c_3 > c_1$. These requirements are sufficient to guarantee that the radicand will remain positive. Hence, the basic constraints on the constants are $c_3 > c_1 > c_2$ in the special case f(r) = 1 of (3.10).

3.2 Problem 2

For our second problem, we change one of the interior boundary conditions, and, as will be shown, we arrive at a similar conclusion that Ω must be a ball. In particular we consider

$$\Delta^2 u = f(r) \quad \text{in } \Omega, \tag{3.22a}$$

$$\frac{\partial u}{\partial n} = c_1 \quad \text{on } \partial B_a, \tag{3.22b}$$

$$u = c_2 \quad \text{on } \partial B_b, \tag{3.22c}$$

$$u = c_3 \quad \text{on } \partial \Omega, \tag{3.22d}$$

where c_1, c_2, c_3 are constants and the same restrictions apply to the interior balls. Since we already know the classical solution (3.6) to (3.22a), we now calculate $\partial u/\partial n$ on ∂B_a. Since $\partial u/\partial n = \partial u/\partial r$ on ∂B_a, we have

$$\frac{\partial u}{\partial n} = \frac{1}{2N(N-2)} \int_0^r \left[2pr + (N-2)\frac{p^{N+1}}{r^{N-1}} - N\frac{p^{N-1}}{r^{N-3}} \right] f(p)dp + k_2 \frac{r}{N}.$$

From (3.22b) we find k_2, namely,

$$k_2 = \frac{N}{a}\left\{ c_1 - \frac{1}{2N(N-2)} \int_0^a \left[2pa + (N-2)\frac{p^{N+1}}{a^{N-1}} - N\frac{p^{N-1}}{a^{N-3}} \right] f(p)dp \right\}.$$

Now to find k_4 we use (3.6) and (3.22c) and obtain

$$k_4 = c_2 - k_2\frac{b^2}{2N} - \frac{1}{2N(N-2)} \int_0^b \left[pb^2 - \frac{p^{N+1}}{b^{N-2}} + \frac{N}{N-4}\left(\frac{p^{N-1}}{b^{N-4}} - p^3 \right) \right] f(p)dp,$$

where k_2 is as given above. Hence, using the simplifying function F in (3.12), we can write the function u which solves (3.22a, b, c), as

$$u = F(r) - F(b) + \frac{r^2 - b^2}{2a}\{c_1 - F'(a)\} + c_2. \tag{3.23}$$

We now continue u to the rest of Ω and impose the extra condition (3.22d). Thus, we obtain an equation in r, namely,

$$F(r) - F(b) + \frac{r^2 - b^2}{2a}\{c_1 - F'(a)\} + c_2 - c_3 = 0. \tag{3.24}$$

Again, under the assumption that (3.22) has a solution, there is a root of (3.24), which corresponds to points on ∂Ω. Hence, Ω is a ball. As before though, the task of finding this r, in the general case, appears formidable.

We now consider the N = 2 case of (3.22). We first compute ∂u/∂n on ∂B_a. Since ∂u/∂n = ∂u/∂r on ∂B_a, from (3.8) we have

$$\frac{\partial u}{\partial n} = \frac{1}{4}\int_0^r \left[2pr\ln\frac{r}{p} + \frac{p^3}{r} - pr\right] f(p)dp + k_2\frac{r}{2}.$$

Applying the condition (3.22b), k_2 is determined to be:

$$k_2 = \frac{2}{a}\left\{c_1 - \frac{1}{4}\int_0^a \left[2pa\ln\frac{a}{p} + \frac{p^3}{a} - pa\right] f(p)dp\right\}$$

and from (3.8) and (3.22c) we find

$$k_4 = c_2 - \frac{1}{4}\int_0^b \left[pb^2\ln\frac{b}{p} + p^3\ln\frac{b}{p} - pb^2 + p^3\right] f(p)dp - k_2\frac{b^2}{4}.$$

We can now write the solution u to (3.22a, b, c) for N = 2, using the simplifying function $\tilde{F}$ in (3.15), as

$$u = \tilde{F}(r) - \tilde{F}(b) + \frac{(r^2 - b^2)}{2a}\{c_1 - \tilde{F}'(a)\} + c_2. \tag{3.25}$$

We now continue u to the rest of Ω and impose the extra condition on ∂Ω. Thus, we obtain an equation in r,

$$\tilde{F}(r) - \tilde{F}(b) + \frac{r^2 - b^2}{2a}\{c_1 - \tilde{F}'(a)\} + c_2 - c_3 = 0. \tag{3.26}$$

Hence, there is a root of (3.26) corresponding to points on $\partial\Omega$ and we deduce that Ω is a disk.

For completeness we now consider the N = 4 case before we look at an example. Using (3.9) we compute $\partial u/\partial n$ on ∂B_a, i.e.,

$$\frac{\partial u}{\partial n} = \frac{1}{16}\int_0^r \left[2pr + 2\frac{p^5}{r^3} - 4\frac{p^3}{r}\right] f(p)dp + k_2\frac{r}{4}.$$

Now, by imposing (3.22b) we find

$$k_2 = \frac{4}{a}\left\{c_1 - \frac{1}{16}\int_0^a \left[2pa + \frac{2p^5}{a^3} - \frac{4p^3}{a}\right] f(p)dp\right\},$$

and from (3.22c) we obtain

$$k_4 = c_2 - \frac{1}{16}\int_0^b \left[pb^2 - \frac{p^5}{b^2} - 4p^3 \ln\frac{b}{p}\right] f(p)dp - k_2\frac{b^2}{8}.$$

We can now write the solution u to (3.22a, b, c) for N = 4, using the simplifying function $\overline{F}$ in (3.18) as

$$u = \overline{F}(r) - \overline{F}(b) + \frac{(r^2 - b^2)}{2a}\{c_1 - \overline{F}'(a)\} + c_2. \tag{3.27}$$

We now continue u to the rest of Ω and impose the extra condition (3.22d). Hence, we have an equation in r,

$$\overline{F}(r) - \overline{F}(b) + \frac{(r^2 - b^2)}{2a}\{c_1 - \overline{F}'(a)\} + c_2 - c_3 = 0. \tag{3.28}$$

Again, we have an equation in r after imposing the auxiliary conditions. Thus, there is a root of (3.28) corresponding to points on $\partial\Omega$. Hence, Ω is a ball.

We now consider a specific case of (3.22) when f(r) = 1. After doing the necessary integrations, we obtain in place of (3.6)

$$u = \frac{r^4}{8N(N+2)} + k_2 r^2 + k_4.$$

Now imposing (3.22b), we find

$$k_2 = \frac{1}{2a}\left\{c_1 - \frac{a^3}{2N(N+2)}\right\},$$

and from imposing (3.22c), we obtain

$$k_4 = c_2 - \frac{b^4}{8N(N+2)} - k_2 b^2.$$

Hence the solution u to (3.22a, b, c), when f(r) = 1, is

$$u = \frac{r^4 - b^4}{8N(N+2)} + \frac{r^2 - b^2}{2a}\left\{c_1 - \frac{a^3}{2N(N+2)}\right\} + c_2. \tag{3.29}$$

We now continue u to the rest of Ω and impose the extra condition (3.22d). Thus our equation in r is

$$\frac{r^4 - b^4}{8N(N+2)} + \frac{r^2 - b^2}{2a}\left\{c_1 - \frac{a^3}{2N(N+2)}\right\} + c_2 - c_3 = 0. \tag{3.30}$$

Now there is a root to (3.30) that corresponds to points on $\partial\Omega$. Hence Ω is a ball. In particular, Ω is a ball of radius

$$r = \sqrt{\frac{1}{2}\left(-8N(N+2)\beta + \sqrt{32N(N+2)\left[2N(N+2)\beta^2 + \beta b^2 + c_3 - c_2\right] + 4b^4}\right)}$$

where

$$\beta = \frac{2N(N+2)c_1 - a^3}{4N(N+2)a}.$$

As in the example of section (3.1) Problem One, we note that c_1, c_2, c_3 can not be completely arbitrary because the radicand must be a positive real number. Thus, we again have some implicit restrictions on c_1, c_2, c_3. In this particular problem, two trivial requirements on the constants, to ensure that r is a positive real number are: $\beta < 0$, i.e.,

$c_1 < \dfrac{a^3}{2N(N+2)}$, and also $c_3 > c_2 + \dfrac{a^3 - 2N(N+2)c_1}{4N(N+2)a} b^2$.

3.3 Problem Three

For our third problem, we introduce another type of the interior boundary condition and deduce that, again, Ω must be a ball. In particular, we now consider

$$\Delta^2 u = f(r) \quad \text{in } \Omega, \tag{3.31a}$$

$$u = c_1 \quad \text{on } \partial B_a, \tag{3.31b}$$

$$\Delta u = c_2 \quad \text{on } \partial B_b, \tag{3.31c}$$

$$u = c_3 \quad \text{on } \partial\Omega, \tag{3.31d}$$

where c_1, c_2, c_3 are constants and the same restrictions apply to the interior balls.

Since we already know the regular solution (3.6) to (3.31a), we impose (3.31b) to particularize k_4 in terms of k_2, i.e.,

$$k_4 = c_1 - k_2 \frac{a^2}{2N} - \frac{1}{2N(N-2)} \int_0^a \left[pa^2 - \frac{p^{N+1}}{a^{N-2}} + \frac{N}{N-4}\left(\frac{p^{N-1}}{a^{N-4}} - p^3 \right) \right] f(p)dp.$$

We now calculate Δu. We find that from (3.4),

$$\Delta u = \frac{1}{N-2} \int_0^r \left[p - \frac{p^{N-1}}{r^{N-2}} \right] f(p)dp + k_2 \text{ in } \Omega,$$

where k_2 is a constant of integration. By (3.31c), we obtain

$$k_2 = c_2 - \frac{1}{N-2} \int_0^b \left[p - \frac{p^{N-1}}{b^{N-2}} \right] f(p)dp.$$

Hence, the solution u of (3.31a, b, c) in the general dimensional case, using the simplifying function F in (3.12), can be written as

$$u = F(r) - F(a) + \frac{r^2 - a^2}{2N} \left\{ c_2 - \frac{1}{N-2} \int_0^b \left[p - \frac{p^{N-1}}{b^{N-2}} \right] f(p)dp \right\} + c_1. \tag{3.32}$$

We now continue u to the rest of Ω and impose the extra condition (3.31d). Thus, we obtain an equation in r, namely,

$$F(r) - F(a) + \frac{r^2 - a^2}{2N} \left\{ c_2 - \frac{1}{N-2} \int_0^b \left[p - \frac{p^{N-1}}{b^{N-2}} \right] f(p)dp \right\} + c_1 - c_3 = 0. \tag{3.33}$$

Since we are under the assumption that (3.31) has a solution, then (3.33) must have a root that corresponds to points on $\partial\Omega$. Hence, Ω is a ball. As in the previous two problems, actually finding this root appears to be very difficult.

Now let us consider the N = 2 case of (3.31). By (3.8) and by imposing (3.31b) we find k_4 in terms of k_2, i.e.,

$$k_4 = c_1 - k_2\frac{a^2}{4} - \frac{1}{4}\int_0^a \left[pa^2 \ln\frac{a}{p} + p^3 \ln\frac{a}{p} - pa^2 + p^3 \right] f(p)dp.$$

To use the condition (3.31c) we need to calculate Δu. From (3.7) when N = 2, the calculation yields

$$\Delta u = \int_0^r p \ln\frac{r}{p} f(p)dp + k_2 \text{ in } \Omega.$$

When the condition (3.31c) is imposed, the constant k_2 is determined to be

$$k_2 = c_2 - \int_0^b p \ln\frac{b}{p} f(p)dp.$$

Hence, we can write the solution u, of (3.31a, b, c) when N = 2, using the simplifying function $\tilde{F}$ in (3.15), as

$$u = \tilde{F}(r) - \tilde{F}(a) + \frac{r^2 - a^2}{4}\left\{ c_2 - \int_0^b p \ln\frac{b}{p} f(p)dp \right\} + c_1. \tag{3.34}$$

Continuing u to the rest of Ω and imposing the extra condition (3.31d), we obtain an equation in r, namely,

$$\tilde{F}(r) - \tilde{F}(a) + \frac{r^2 - a^2}{4}\left\{ c_2 - \int_0^b p \ln\frac{b}{p} f(p)dp \right\} + c_1 - c_3 = 0. \tag{3.35}$$

Again, under the assumption that (3.31) has a solution, (3.35) has a root that corresponds to points on $\partial\Omega$. Hence, Ω is a disk.

Now for the N = 4 case of (3.31). Using (3.9) and (3.31b) we find k_4 in terms of k_2, i.e.,

$$k_4 = c_1 - k_2\frac{a^2}{8} - \frac{1}{16}\int_0^a \left[pa^2 - \frac{p^5}{a^2} - 4p^3 \ln\frac{a}{p} \right] f(p)dp.$$

Substituting N = 4 into (3.4), we obtain

$$\Delta u = \frac{1}{2}\int_0^r \left[p - \frac{p^3}{r^2} \right] f(p)dp + \mathrm{k}_2 \text{ in } \Omega.$$

Imposing the condition (3.31c), we find

$$\mathrm{k}_2 = \mathrm{c}_2 - \frac{1}{2}\int_0^b \left[p - \frac{p^3}{b^2} \right] f(p)dp.$$

Thus, the solution u, of (3.31a, b, c) when N = 4, using the simplifying function $\overline{F}$ in (3.18) is

$$u = \overline{F}(r) - \overline{F}(a) + \frac{r^2 - a^2}{8}\left\{ \mathrm{c}_2 - \frac{1}{2}\int_0^b \left[p - \frac{p^3}{b^2} \right] f(p)dp \right\} + \mathrm{c}_1. \tag{3.36}$$

We, again, continue u to the rest of Ω and impose the extra condition (3.31d). Hence, we have an equation in r,

$$\overline{F}(r) - \overline{F}(a) + \frac{r^2 - a^2}{8}\left\{ \mathrm{c}_2 - \frac{1}{2}\int_0^b \left[p - \frac{p^3}{b^2} \right] f(p)dp \right\} + \mathrm{c}_1 - \mathrm{c}_3 = 0. \tag{3.37}$$

Using the same reasoning as in the general dimension case and the N = 2 case, we arrive at the conclusion that Ω must be a ball.

We now determine an explicit solution of (3.31) when f(r) = 1. The general regular solution of the differential equation is

$$u = \frac{r^4}{8N(N+2)} + \mathrm{k}_2 r^2 + \mathrm{k}_4.$$

Imposing (3.31b), we arrive at k_4 in terms of k_2, namely,

$$\mathrm{k}_4 = \mathrm{c}_1 - \mathrm{k}_2 a^2 - \frac{a^4}{8N(N+2)}.$$

Using the radial form of the Laplacian, we have

$$\Delta u = \frac{r^2}{2N} + 2N\mathrm{k}_2.$$

Hence,

$$\mathrm{k}_2 = \frac{1}{2N}\left(\mathrm{c}_2 - \frac{b^2}{2N} \right).$$

Thus, the solution of (3.31a, b, c) in the special case when f(r)=1 is

$$u = \frac{r^4 - a^4}{8N(N+2)} + \frac{r^2 - a^2}{2N}\left\{ \mathrm{c}_2 - \frac{b^2}{2N} \right\} + \mathrm{c}_1. \tag{3.38}$$

By continuing u to the rest of Ω and imposing the extra condition (3.31d) our equation in r is

$$\frac{r^4 - a^4}{8N(N+2)} + \frac{r^2 - a^2}{2N}\left\{c_2 - \frac{b^2}{2N}\right\} + c_1 - c_3 = 0. \tag{3.39}$$

Since there exists a root to (3.39) that corresponds to points on $\partial\Omega$, Ω is a ball. In particular, Ω is a ball of radius

$$r = \sqrt{\frac{1}{2}\left(-8N(N+2)\varphi + \sqrt{32N(N+2)\left[2N(N+2)\varphi^2 + \varphi a^2 + c_3 - c_1\right] + 4a^4}\right)},$$

where

$$\varphi = \frac{2Nc_2 - b^2}{4N^2}.$$

As in our previous examples, it is not surprising that c_1, c_2, c_3 can not be completely arbitrary since the radicand must be a positive real number. Consequently, there are some implicit constraints on c_1, c_2, c_3, e.g., $\varphi < 0$ which implies that $c_2 < \frac{b^2}{2N}$, and $c_3 > c_1 + \frac{2Na^2c_2 - a^2b^2}{4N^2}$ are basic conditions that are sufficient to guarantee the restriction on the radicand.

3.4 Problem Four

We again consider an overdetermined problem where the three auxiliary conditions are prescribed on three different surfaces. For our fourth problem however, we assume the Laplacian and normal derivative of u are given as auxiliary conditions on the interior spheres and u is specified on $\partial\Omega$. Our new problem is formulated as

$$\Delta^2 u = f(r) \quad \text{in } \Omega, \tag{3.40a}$$

$$\Delta u = c_1 \quad \text{on } \partial B_a, \tag{3.40b}$$

$$\frac{\partial u}{\partial n} = c_2 \quad \text{on } \partial B_b, \tag{3.40c}$$

$$u = c_3 \quad \text{on } \partial\Omega, \tag{3.40d}$$

where c_1, c_2, c_3 are constants and B_a and B_b are concentric interior balls. By (3.4) and (3.40b) we determine that

$$k_2 = c_1 - \frac{1}{N-2}\int_0^a \left[p - \frac{p^{N-1}}{a^{N-2}} \right] f(p)dp.$$

Using (3.6), we calculate $\partial u / \partial n$ on ∂B_b. Since $\partial u / \partial n = \partial u / \partial r$ on ∂B_b, the calculation results in

$$\frac{\partial u}{\partial n} = \frac{1}{2N(N-2)}\int_0^r \left[2pr + (N-2)\frac{p^{N+1}}{r^{N-1}} - N\frac{p^{N-1}}{r^{N-3}} \right] f(p)dp + k_2 \frac{r}{N}.$$

The condition (3.40c) forces a constraint on c_1 and c_2, i.e.,

$$c_2 = \frac{1}{2N(N-2)}\int_0^b \left[2pb + (N-2)\frac{p^{N+1}}{b^{N-1}} - N\frac{p^{N-1}}{b^{N-3}} \right] f(p)dp + k_2 \frac{b}{N} \quad (3.41)$$

since k_2 was determined in terms of c_1. Hence, we can write the solution u of (3.40a, b, c) in the general dimension case, using the simplifying function F in (3.12), as

$$u = F(r) + \frac{r^2}{2N}\left\{ c_1 - \frac{1}{N-2}\int_0^a \left[p - \frac{p^{N-1}}{a^{N-2}} \right] f(p)dp \right\} + k_4, \quad (3.42)$$

where k_4 is still arbitrary. After continuing u to the rest of Ω and imposing the extra condition (3.40d), we obtain an equation in r, namely,

$$F(r) + \frac{r^2}{2N}\left\{ c_1 - \frac{1}{N-2}\int_0^a \left[p - \frac{p^{N-1}}{a^{N-2}} \right] f(p)dp \right\} + k_4 - c_3 = 0. \quad (3.43)$$

Under the assumption that (3.40) has a solution, we deduce that there is a root of (3.43) that corresponds to points on $\partial\Omega$. Hence, Ω is a ball. However, finding this root in terms of the arbitrary k_4 in the general case appears to be quite difficult.

Now let us consider the N = 2 case of (3.40). Using (3.7) and (3.40b) we find

$$k_2 = c_1 - \int_0^a p \ln \frac{a}{p} f(p)dp.$$

We calculate $\partial u / \partial n$ on ∂B_b using (3.8). Since $\partial u / \partial n = \partial u / \partial r$ on ∂B_b, we have

$$\frac{\partial u}{\partial n}=\frac{1}{4}\int_0^r\left[2pr\ln\frac{\mathrm{r}}{\mathrm{p}}+\frac{p^3}{r}-pr\right]f(p)dp+\mathrm{k}_2\frac{r}{2}.$$

Imposing (3.40c) yields a constraint on c_1 and c_2, i.e.,

$$\mathrm{c}_2=\frac{1}{4}\int_0^b\left[2pb\ln\frac{b}{p}+\frac{p^3}{b}-pb\right]f(p)dp+\mathrm{k}_2\frac{b}{2}.$$

Hence, the solution u of (3.40a, b, c) when N = 2 can be written as

$$u=\tilde{F}(r)+\frac{r^2}{4}\left\{\mathrm{c}_1-\int_0^a p\ln\frac{a}{p}f(p)dp\right\}+\mathrm{k}_4, \tag{3.44}$$

where k_4 is still an arbitrary constant and we have used (3.15). We continue u to the rest of Ω and impose the extra condition (3.40d). This yields an equation in r, namely,

$$\tilde{F}(r)+\frac{r^2}{2N}\left\{\mathrm{c}_1-\int_0^a p\ln\frac{a}{p}f(p)dp\right\}+\mathrm{k}_4-\mathrm{c}_3=0. \tag{3.45}$$

This equation implies that Ω is a disk, since (3.45) must have a root corresponding to points on $\partial\Omega$.

Before we investigate an example, let us consider the N = 4 case of (3.40) first. Again, using (3.4) and (3.40b) we find

$$\mathrm{k}_2=\mathrm{c}_1-\frac{1}{2}\int_0^a\left[p-\frac{p^3}{a^2}\right]f(p)dp.$$

Now we calculate $\partial u/\partial n$ on ∂B_b. Using (3.9) and since ∂B_b is a sphere, we find

$$\frac{\partial u}{\partial n}=\frac{1}{16}\int_0^r\left[2pr+2\frac{p^5}{r^3}-4\frac{p^3}{r}\right]f(p)dp+\mathrm{k}_2\frac{r}{4}.$$

Imposing (3.40c) yields a constraint on c_1 and c_2, namely,

$$\mathrm{c}_2=\frac{1}{16}\int_0^b\left[2pb+2\frac{p^5}{b^3}-4\frac{p^3}{b}\right]f(p)dp+\mathrm{k}_2\frac{b}{4}.$$

Hence, we can write the solution u of (3.40a, b, c) for the N = 4 case, using the simplifying function $\overline{F}$ in (3.18) as

$$u=\overline{F}(r)+\frac{r^2}{8}\left\{\mathrm{c}_1-\frac{1}{2}\int_0^a\left[p-\frac{p^3}{a^2}\right]f(p)dp\right\}+\mathrm{k}_4. \tag{3.46}$$

where k_4 is an arbitrary constant. After continuing u to the rest of Ω and using the condition (3.40d), we arrive at an equation in r, namely,

$$\overline{F}(r)+\frac{r^2}{8}\left\{c_1-\frac{1}{2}\int_0^a\left[p-\frac{p^3}{a^2}\right]f(p)dp\right\}+k_4-c_3=0. \tag{3.47}$$

This implies that Ω must be a ball since (3.47) must have a root that corresponds to points on $\partial\Omega$.

We now present an example letting f(r) = 1 in (3.40). The general classical solution of the differential equation is

$$u=\frac{r^4}{8N(N+2)}+k_2r^2+k_4. \tag{3.48}$$

Computing Δu, we have

$$\Delta u=\frac{r^2}{2N}+2Nk_2.$$

By (3.40b) we find

$$k_2=\frac{1}{2N}\left(c_1-\frac{a^2}{2N}\right).$$

Using (3.48), we compute $\partial u/\partial n$ on ∂B_b. Thus,

$$\frac{\partial u}{\partial n}=\frac{r^3}{2N(N+2)}+2k_2r,$$

Imposing (3.40c) yields a constraint on c_1 and c_2, i.e.,

$$c_2=\frac{b^3}{2N(N+2)}+\frac{b}{N}\left(c_1-\frac{a^2}{2N}\right).$$

Hence, we can write the solution u of (3.40a, b, c), when f(r) = 1, as

$$u=\frac{r^4}{8N(N+2)}+\frac{r^2}{2N}\left\{c_1-\frac{a^2}{2N}\right\}+k_4. \tag{3.49}$$

We now impose the extra condition (3.40d) and obtain an equation in r, i.e.,

$$\frac{r^4}{8N(N+2)}+\frac{r^2}{2N}\left\{c_1-\frac{a^2}{2N}\right\}+k_4-c_3=0. \tag{3.50}$$

This equation implies that Ω is a ball, because we know that (3.50) has a root that corresponds to points on $\partial\Omega$. In particular Ω is a ball of radius

$$r=\sqrt{\frac{1}{2}\left(-4(N+2)\gamma+\sqrt{16(N+2)^2\gamma^2-32N(N+2)(\mathrm{k}_4-\mathrm{c}_3)}\right)}$$

where

$$\gamma=\mathrm{c}_1-\frac{a^2}{2N}.$$

From this example we see that there is an explicit constraint on c_1 and c_2 along with some implicit constraints since r must be a positive real number. Two sufficient conditions on the constants here are: $c_1<\frac{a^2}{2N}$ and $c_3>k_4$, to ensure that the radicand does not become negative anywhere in Ω. We note that k_4 is still arbitrary, but it must still satisfy the condition with c_3.

3.5 Problem Five

For a final problem we interchange two conditions in (3.40). Here we consider

$$\Delta^2 u=f(r)\quad \text{in } \Omega, \tag{3.51a}$$

$$u=\mathrm{c}_1\quad \text{on } \partial\mathrm{B}_a, \tag{3.51b}$$

$$\frac{\partial u}{\partial n}=\mathrm{c}_2\quad \text{on } \partial\mathrm{B}_b, \tag{3.51c}$$

$$\Delta u=\mathrm{c}_3\quad \text{on } \partial\Omega, \tag{3.51d}$$

where c_1, c_2, c_3 are constants and B_a and B_b are again concentric interior balls.

From (3.6) and (3.51b) we find k_4 in terms of k_2, namely,

$$\mathrm{k}_4=\mathrm{c}_1-\frac{1}{2N(N-2)}\int_0^a\left[pa^2-\frac{p^{N+1}}{r^{N-2}}+\frac{N}{N-4}\left(\frac{p^{N-1}}{a^{N-4}}-p^3\right)\right]f(p)dp-\mathrm{k}_2\frac{a^2}{2N}.$$

Using (3.6) we now calculate $\partial u/\partial n$ on ∂B_b. Since $\partial u/\partial n=\partial u/\partial r$ on ∂B_b we have

$$\frac{\partial u}{\partial n}=\frac{1}{2N(N-2)}\int_0^r\left[2pr+(N-2)\frac{p^{N+1}}{r^{N-1}}-N\frac{p^{N-1}}{r^{N-3}}\right]f(p)dp+\mathrm{k}_2\frac{r}{N},$$

and imposing (3.51c), we obtain

$$k_2 = \frac{N}{b}\left\{c_2 - \frac{1}{2N(N-2)}\int_0^b \left[2pb + (N-2)\frac{p^{N+1}}{b^{N-1}} - N\frac{p^{N-1}}{b^{N-3}}\right] f(p)dp\right\}.$$

Hence, we can write the solution u of (3.51a, b, c), using the simplifying function F in (3.12) as

$$u = F(r) - F(a) + \frac{r^2 - a^2}{2b}\{c_2 - F'(b)\} + c_1. \qquad (3.52)$$

After continuing u to the rest of Ω, we impose the condition (3.51d). First though, from (3.4), we have

$$\Delta u = \frac{1}{N-2}\int_0^r \left[p - \frac{p^{N-1}}{r^{N-2}}\right] f(p)dp + \frac{N}{b}\{c_2 - F'(b)\}$$

Now imposing (3.51d), we obtain the following equation in r

$$\frac{1}{N-2}\int_0^r \left[p - \frac{p^{N-1}}{r^{N-2}}\right] f(p)dp + \frac{N}{b}\{c_2 - F'(b)\} - c_3 = 0, \qquad (3.53)$$

when the points are on the $\partial\Omega$. Equation (3.53) must have a root that corresponds to points on $\partial\Omega$. Hence, we deduce that Ω is a ball and the solution is given by (3.52), since we assumed (3.51) is solvable.

We now consider the N = 2 case of (3.51). Using (3.8) and (3.51b) we find k_4 in terms of k_2 to be

$$k_4 = c_1 - k_2\frac{a^2}{4} - \frac{1}{4}\int_0^a \left[pa^2\ln\frac{a}{p} + p^3\ln\frac{a}{p} + p^3 - pa^2\right] f(p)dp.$$

We must compute $\partial u / \partial n$ on ∂B_b. From (3.8), we have

$$\frac{\partial u}{\partial n} = \frac{1}{4}\int_0^r \left[2pr\ln\frac{r}{p} + \frac{p^3}{r} - pr\right] f(p)dp + k_2\frac{r}{2}.$$

Imposing (3.51c), we find

$$k_2 = \frac{2}{b}\left\{c_2 - \frac{1}{4}\int_0^b \left[2pb\ln\frac{b}{p} + \frac{p^3}{b} - pb\right] f(p)dp\right\}.$$

Hence, we can write the solution u, of (3.51a, b, c) for the N = 2 case, using the simplifying function $\tilde{F}$ in (3.15) as

$$u = \tilde{F}(r) - \tilde{F}(a) + \frac{r^2 - a^2}{2b}\left\{c_2 - \tilde{F}'(b)\right\} + c_1. \tag{3.54}$$

After continuing u to the rest of Ω, we calculate

$$\Delta u = \int_0^r p \ln\frac{r}{p} f(p)dp + k_2.$$

When the extra condition (3.51d) is imposed the result is an equation in r, namely,

$$\int_0^r p \ln\frac{r}{p} f(p)dp + \frac{2}{b}\left\{c_2 - \tilde{F}'(b)\right\} - c_3 = 0. \tag{3.55}$$

We now deduce that Ω is a disk, since (3.55) has a root that corresponds to points on $\partial\Omega$.

For completeness, we now consider the N = 4 case of (3.51) before we give an example. Using (3.9) and (3.51b) we find k_4, i.e.,

$$k_4 = c_1 - k_2\frac{a^2}{8} - \frac{1}{16}\int_0^a\left[pa^2 - \frac{p^5}{a^2} - 4p^3\ln\frac{a}{p}\right]f(p)dp.$$

Computing $\partial u / \partial n$ on ∂B_b , we have

$$\frac{\partial u}{\partial n} = \frac{1}{16}\int_0^r\left[2pr + 2\frac{p^5}{r^3} - 4\frac{p^3}{r}\right]f(p)dp + k_2\frac{r}{4},$$

and imposing (3.51c), we find

$$k_2 = \frac{4}{b}\left\{c_2 - \frac{1}{16}\int_0^b\left[2pb + 2\frac{p^5}{b^3} - 4\frac{p^3}{b}\right]f(p)dp\right\}.$$

Hence, we can write the solution u, of (3.51a, b, c) when N =4, using the simplifying function $\overline{F}$ in (3.18) as

$$u = \overline{F}(r) - \overline{F}(a) + \frac{r^2 - a^2}{2b}\left\{c_2 - \overline{F}'(b)\right\} + c_1. \tag{3.56}$$

After continuing u to the rest of Ω, we calculate

$$\Delta u = \frac{1}{2}\int_0^r\left[p - \frac{p^3}{r^2}\right]f(p)dp + k_2.$$

From the condition (3.51d), we obtain an equation in r, namely,

$$\frac{1}{2}\int_0^r\left[p - \frac{p^3}{r^2}\right]f(p)dp + \frac{4}{b}\left\{c_2 - \overline{F}'(b)\right\} - c_3 = 0. \tag{3.57}$$

From (3.57), we deduce that Ω must be a ball since (3.57) has a root that corresponds to points on $\partial\Omega$ because we assumed that problem (3.51) had a solution.

Now let us consider a particular case of (3.51). Letting f(r) = 1 in (3.51), the general regular solution is

$$u = \frac{r^4}{8N(N+2)} + k_2 r^2 + k_4.$$

Now using (3.51b) we find k_4 in terms of k_2,

$$k_4 = c_1 - k_2 a^2 - \frac{a^4}{8N(N+2)}.$$

We must compute $\partial u / \partial n$ on ∂B_b:

$$\frac{\partial u}{\partial n} = \frac{r^3}{2N(N+2)} + 2k_2 r.$$

From (3.51c), we find

$$k_2 = \frac{1}{2b}\left\{c_2 - \frac{b^3}{2N(N+2)}\right\}.$$

Thus the solution u of (3.51a, b, c) , in the special case when f(r) = 1, is

$$u = \frac{r^4 - a^4}{8N(N+2)} + \frac{r^2 - a^2}{2b}\left\{c_2 - \frac{b^3}{2N(N+2)}\right\} + c_1. \tag{3.58}$$

After continuing u to the rest of Ω, we calculate

$$\Delta u = \frac{r^2}{2N} + \frac{N}{b}\left(c_2 - \frac{b^3}{2N(N+2)}\right).$$

Now imposing the extra condition (3.51d), we have for points on $\partial\Omega$ that

$$\frac{r^2}{2N} + \frac{N}{b}\left[c_2 - \frac{b^3}{2N(N+2)}\right]. \tag{3.59}$$

Since we are under the assumption that (3.51) has a solution, then there exists a root of (3.59) that corresponds to points on $\partial\Omega$. In fact, Ω is a ball of radius

$$r = \sqrt{2Nc_3 - \frac{2N^2}{b}\left\{c_2 - \frac{b^3}{2N(N+2)}\right\}}.$$

Here we notice that there is no implicit constraint on c_1. However, two trivial constraints on c_2 and c_3, which are sufficient to guarantee that the radicand will remain positive are: $c_3 > 0$ and $c_2 < \frac{b^3}{2N(N+2)}$.

3.6 Concluding Remarks

In this thesis, we have considered a number of fourth order overdetermined problems in which one concludes that if a solution exists, then the domain Ω must be a ball. Here we remark on the similarity of the first three problems. In those problems we specified u on two surfaces and used a different type of auxiliary condition on the third surface. We first obtained the general solution to the differential equation and then used the data on the two inner surfaces. After continuing this solution to the rest of Ω and imposing the extra condition, we arrived at an equation in r. Our assumption that a solution exists implied that there was a root corresponding to points on the boundary and hence, Ω is a ball. In our example, we noticed that there are some implicit constraints on the constants. These constraints will depend on what the function f(r) is chosen to be. Our example was important because this was not readily apparent before we solved a particular case.

Problem Four was different than the previous three in that we had three different auxiliary conditions specified on the three surfaces. Another difference was that we found an explicit constraint between c_1 and c_2. Also, the solution was not unique, due to an arbitrary constant . However, our approach was the same, as was the conclusion about the geometry of Ω.

Problem Five was similar to Problem Four in that the auxiliary conditions were the same type only we required them on different surfaces. This problem was also similar to the first three problems in that the solution is unique, if indeed a solution exists.

An important note is that these results hold only if the inhomogeneous term in (3.3) is a function of the distance from a fixed point. A function of any other argument(s) would need further study. Thus, we conclude that if the problem is of the form (3.10), (3.18), (3.24), (3.29), or (3.36) and a solution exists, then the domain must be a ball.

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