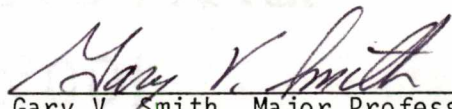


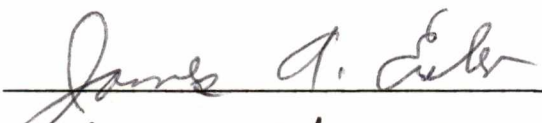
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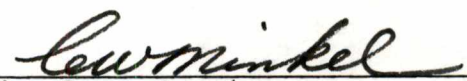
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DIGITAL CONTROL OF A NONLINEAR SYSTEM

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Jungkeun Rhee

August 1985

ACKNOWLEDGMENTS

The author would like to thank Dr. G. V. Smith for his efforts as advisor and major professor. Appreciation is also due to the members of the committee: Dr. J. A. Euler and Dr. J. W. Hodgson for their guidance and encouragement. Thanks are also given to graduate students for their friendship.

ABSTRACT

The use of digital computers in the real time control of dynamic systems has increased in recent years. With this increasing use of digital, as opposed to analog, controllers, there is a need for a better understanding of the interaction between a digital controller and a nonlinear plant. This thesis is directed toward performing a parametric study of the adjustable parameters in the digital equivalent of proportional, proportional plus integral, and proportional plus integral plus derivative controllers. The effects of these parameters are studied in regard to their influence on system stability, settling time, overshoot and steady state error. The nonlinear plant studied is a servomechanism with Coulomb friction.

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CHAPTER I

INTRODUCTION

In the control field, digital computers play two important roles. One important role is that digital computers are used extensively for simulation and computation of control systems dynamics. Since many physical systems are of a high order and contain nonlinear elements, digital computer simulations are often used to conduct the analysis and design of complex systems and to check the results obtained from closed-form solutions. Another important application of the digital computer in control systems is that it is often used as a controller or a processor in the feedback control of a plant.

In a feedback control system, control actions are taken to achieve and maintain the desired output and obtain better dynamic system performance. The basic control actions commonly used in industrial analog automatic controllers are proportional control (P-controller), integral control (I-controller), and proportional plus integral control (PI-controller), and proportional plus integral plus derivative control (PID-controller).

With the increasing use of digital controllers, as opposed to analog controllers, there has been a tendency to develop and market digital controllers which perform the digital equivalent of the analog operations of P-, PI-, and PID-controllers. This is likely due to the

fact that practicing control engineers can convert more easily to digital equipment which performs essentially the same tasks as the more familiar analog controllers. It is also likely that many, if not most, control engineers would not have the background necessary to develop and program unique algorithms on a digital controller. This is unfortunate since digital controllers offer the capability of using algorithms unique to the system being controlled and are not limited to the basic analog controller operations of proportional, integral, and derivative action.

The purpose of this thesis is to study the dynamic response of a nonlinear system using digital controllers. As the controlled process or the plant, a servomechanism which has a lumped mass, spring, and nonlinear (Coulomb) friction is used. A parametric study of the effects of the adjustable parameters in the digital equivalent of P-, PI-, and PID-controllers will be performed. The effects of these parameters on performance characteristics such as system stability, settling time, overshoot, and steady state error will be presented.

A. LITERATURE REVIEW

Over a period of three decades from about 1930 until 1960, control theory was developed primarily based on electronic feedback amplifier design modified for servomechanism problems. For these problems Eveleigh (1) and Shigley (2) have introduced the application of the analog computer for studying automatic control systems. The

use of Laplace transform representations of the system dynamics and the control specifications has been given by several authors including Dransfield (3), Kuo (4), Ogata (5), and Raven (6). The modeling of dynamic systems has been studied and presented by Palm (7) and Doebelin (8). The analysis of digital control systems for linear plants has been well studied and many authors have published books about the use of digital computers in the real time control of linear dynamic systems. Kuo (9) has applied the state diagram method for designing digital controllers while Phillips and Nagle (10) present the properties of the z-transform as well as the inverse of the z-transform which may be obtained by long division, by partial fraction expansion, or by evaluation of the inverse transform integral. Franklin and Powell (11) have studied approximation rules from the s-plane to the z-plane, which include the Forward rectangular rule, Backward rectangular rule, and Trapezoid rule which is also known as Tustin's rule. Jong (12) and Brogan (13) have introduced the application of the state-space method which is useful in modern control theory.

CHAPTER II

MATHEMATICAL ANALYSIS

A. CLASSICAL CONTROL THEORY

The closed-form solution for the output of a closed loop system with different digital controllers as a function of a reference and disturbance input of a system is developed in this chapter. After developing these solutions and transforming them to the z-plane, various methods for obtaining the inverse z-transform are discussed.

Among the techniques available for analyzing sampled-data systems, classical control theory and modern control theory are most widely used. Generally speaking, classical control theory is coupled with electronic technology suitable for implementing the required dynamic compensators to give a set of approaches to solve control problems. The transform method is usually employed to accomplish this. The concept of modern control theory, which is also referred to as the state-space method, will be discussed in the next section.

The z-transform is an important concept in a discrete-time system. In contrast to the Laplace transform which can be used to represent or solve a set of differential equations in a continuous-time system, the z-transform is more appropriate for use in a discrete-time system which occurs with the addition of an analog/digital sampler.

In order to put control theory into practice, the process of modeling must be built between the real system and mathematical theory. Consider a servomechanism which has a plant modeled as a lumped mass, spring, and nonlinear (Coulomb) friction, shown schematically in Figure II-1. The block diagram of the entire sampled data system is presented in Figure II-2 in terms of the Laplace transform. The variables are defined as:

1. R = reference input;
2. $E = R - Y$ = system error;
3. $E^* = R^* - Y^*$ = sampled or discrete error;
4. D = digital controller;
5. M^* = discrete output signal from D ;
6. X = control input signal to the plant;
7. V = disturbance input;
8. Y = controlled output signal;
9. z.o.h. = zero-order hold device.

In this thesis the assumption will be made that all the discrete signals arrive with the same fixed period T , called the sampling period while $*$ indicates the sampled signals or the discrete signals represented by a train of impulses. Since the controlled process (i.e., the plant) contains analog elements, the output signal of the controller should be converted to a continuous signal. For this reason, a zero-order hold device is used in order to hold the signal until the next impulse signal arrives from the sampler. The transfer function of the zero-order hold (9) is given by

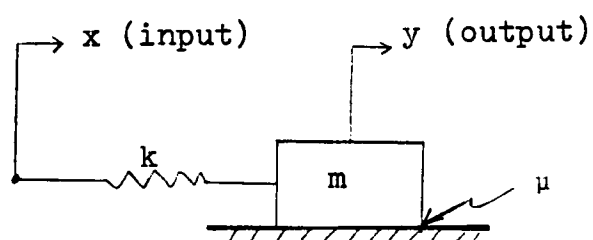


Figure II-1. Schematic diagram of a mass-spring system with Coulomb friction.

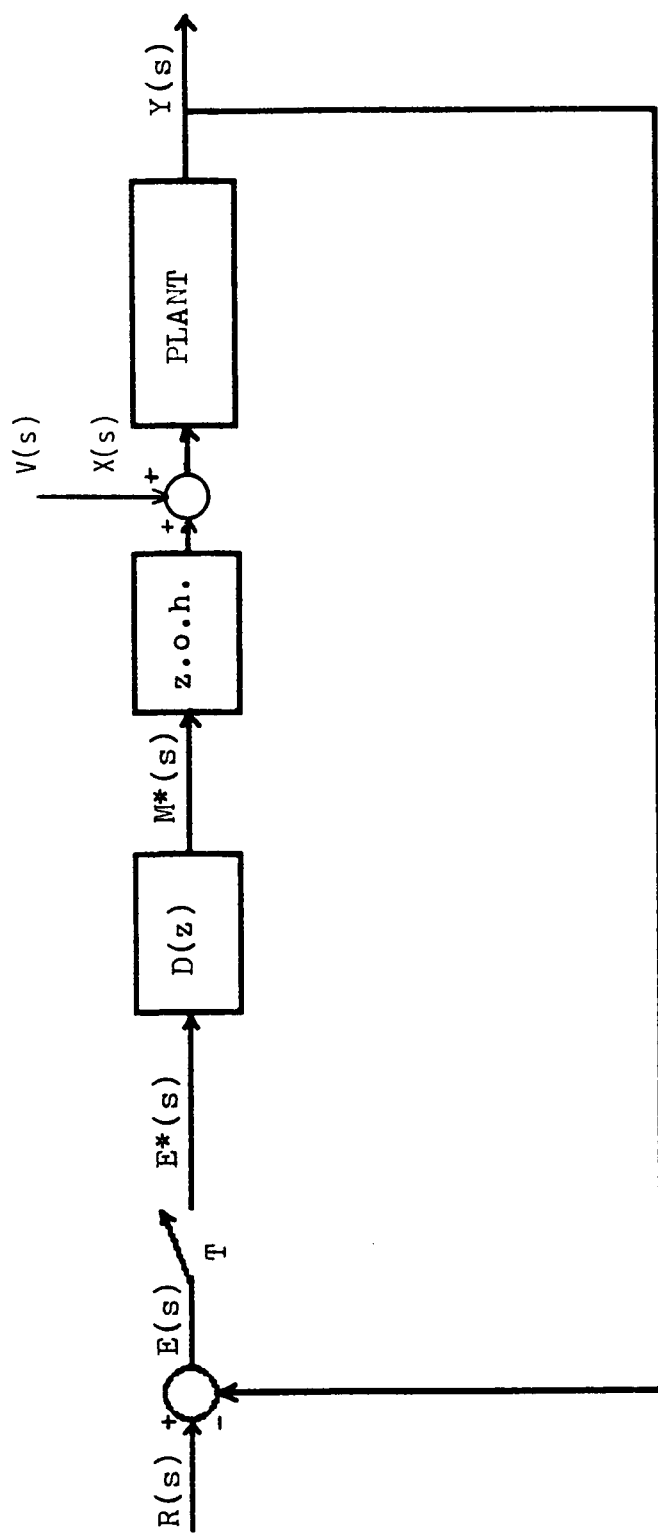


Figure II-2. Block diagram of digital control system.

$$G_{ho}(s) = \frac{1 - e^{-Ts}}{s} .$$

Block diagram algebra can be applied to find the relation between the output Y and inputs R and V .

From the free body diagram of the plant (Figure II-1), application of Newton's law of motion yields the differential equation of motion as

$$\Sigma F = m \ddot{y} = -k(y - x) - \mu m g \operatorname{sgn}(\dot{y}) \quad (\text{II-1})$$

or

$$m \ddot{y} + k y = k x - \mu m g \operatorname{sgn}(\dot{y}) \quad (\text{II-2})$$

where m is the mass, k is the spring stiffness, μ is the coefficient of sliding friction, x is an input to the spring, and y is the output displacement of the mass. The function $\operatorname{sgn}(\dot{y})$ has the value ± 1 , depending upon the sign of the velocity. Dividing Eq. II-2 through by m and substituting $\omega_n^2 = k/m$ yields

$$\ddot{y} + \omega_n^2 y = \omega_n^2 x - \mu g \operatorname{sgn}(\dot{y}) \quad (\text{II-3})$$

where the parameter ω_n is defined as the natural frequency of the mass. Taking the Laplace transform of each term yields

$$\mathcal{L}(\ddot{y}) = s^2 Y(s) - s y(0) - \dot{y}(0)$$

$$\mathcal{L}(\omega_n^2 y) = \omega_n^2 Y(s)$$

$$\mathcal{L}(\omega_n^2 x) = \omega_n^2 X(s)$$

$$\mathcal{L}(\mu g \operatorname{sgn}(\dot{y})) = \frac{\mu g \operatorname{sgn}(\dot{y})}{s}.$$

Notice that the Laplace of $\operatorname{sgn}(\dot{y})$ given above can be interpreted as $\operatorname{sgn}(\dot{y})$ is a constant in the regions where $\dot{y}$ is positive and negative. The Laplace transform of Eq. II-3 can then be written as

$$s^2 Y(s) - s y(0) - \dot{y}(0) + \omega_n^2 Y(s) = \omega_n^2 X(s) - \frac{\mu g \operatorname{sgn}(\dot{y})}{s}.$$

Solving for $Y(s)$ gives

$$(s^2 + \omega_n^2) Y(s) = \omega_n^2 X(s) + s y(0) + \dot{y}(0) - \frac{\mu g \operatorname{sgn}(\dot{y})}{s} \quad (\text{II-4})$$

or

$$Y(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} X(s) + \frac{A s + B}{s^2 + \omega_n^2} - \frac{\mu g \operatorname{sgn}(\dot{y})}{s(s^2 + \omega_n^2)} \quad (\text{II-5})$$

where $A = y(0)$ and $B = \dot{y}(0)$. By defining

$$G_1(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$G_2(s) = \frac{n}{s^2 + \omega_n^2}$$

$$G_3(s) = - \frac{\mu g \operatorname{sgn}(\dot{y})}{s(s^2 + \omega_n^2)}$$

and

$$G_{ho}(s) = \frac{1 - e^{-Ts}}{s} ,$$

the block diagram in Figure II-2 indicates that

$$Y(s) = G_1(s) X(s) + G_2(s) + G_3(s)$$

$$X(s) = G_{ho}(s) M^*(s) + V(s)$$

$$M^*(s) = D(z) E^*(s)$$

and

$$E^*(s) = R^*(s) - Y^*(s) .$$

Combining these equations yields

$$\begin{aligned} Y(s) &= G_1(s) [G_{ho}(s)M^*(s) + V(s)] + G_2(s) + G_3(s) \\ &= G_{ho}(s)G_1(s)M^*(s) + G_1(s)V(s) + G_2(s) + G_3(s) \\ &= G_{ho}(s)G_1(s)D(z)E^*(s) + G_1(s)V(s) + G_2(s) + G_3(s) \\ &= G_{ho}(s)G_1(s)D(z) [R^*(s) - Y^*(s)] + G_1(s)V(s) \\ &\quad + G_2(s) + G_3(s) . \end{aligned}$$

In terms of the sampled signal form, it becomes

$$Y^*(s) = G_{ho}G_1^*(s)D(z) [R^*(s) - Y^*(s)] + G_1V^*(s) + G_2^*(s) + G_3^*(s) .$$

Solving for $Y^*(s)$ yields

$$Y^*(s) = \frac{G_{ho} G_1^*(s) D(z)}{1 + G_{ho} G_1^*(s) D(z)} R^*(s) + \frac{G_1 V^*(s) + G_2^*(s) + G_3^*(s)}{1 + G_{ho} G_1^*(s) D(z)} . \quad (II-6)$$

In terms of the z-transform, $Y(z)$ is given by

$$Y(z) = \frac{G_{ho} G_1(z) D(z)}{1 + G_{ho} G_1(z) D(z)} R(z) + \frac{G_1 V(z) + G_2(z) + G_3(z)}{1 + G_{ho} G_1(z) D(z)} . \quad (II-7)$$

After determining $G_{ho} G_1(z)$, $G_2(z)$, $G_3(z)$, $G_1 V(z)$, and $R(z)$, $Y(z)$ may be obtained as a function of z . Considering first $G_{ho} G_1(z)$,

$$G_{ho} G_1(z) = \frac{(1 - \cos \omega_n T) (z + 1)}{z^2 - 2z \cos \omega_n T + 1} \quad (II-8)$$

while

$$G_2(z) = \frac{A z(z - \cos \omega_n T) + (B/\omega_n)(z \sin \omega_n T)}{z^2 - 2z \cos \omega_n T + 1} \quad (II-9)$$

and

$$G_3(z) = - \frac{\mu g \operatorname{sgn}(\dot{y})}{\omega_n^2} \frac{(1 + \cos \omega_n T) z(z + 1)}{(z - 1)(z^2 - 2z \cos \omega_n T + 1)} . \quad (II-10)$$

For the disturbance input

$$G_1 V(z) = \mathcal{Z} [G_1(s) V(s)] \quad (II-11)$$

and finally, for the unit-step input,

$$R(z) = \frac{z}{z - 1} . \quad (\text{II-12})$$

For zero disturbance input, Dq. II-7 therefore becomes

$$\begin{aligned} Y(z) = & \frac{(1 - \cos \omega_n T) z(z + 1) D(z)}{(z - 1) [z^2 - 2z \cos \omega_n T + 1 + (1 - \cos \omega_n T)(z + 1) D(z)]} \\ & + \frac{A z(z - \cos \omega_n T) + (B/\omega_n)(z \sin \omega_n T)}{z^2 - 2z \cos \omega_n T + 1 + (1 - \cos \omega_n T)(z + 1) D(z)} \\ & + \frac{(-\mu g \operatorname{sgn}(\dot{y})/\omega_n^2)(1 - \cos \omega_n T) z(z + 1)}{(z - 1) [z^2 - 2z \cos \omega_n T + 1 + (1 - \cos \omega_n T)(z + 1) D(z)]} . \end{aligned} \quad (\text{II-13})$$

Additional details of these calculations may be found in Appendix A.

B. MODERN CONTROL THEORY

In classical control theory, briefly discussed in the previous section, the equation for $Y(z)$ contains a nonlinearity, which is the $\operatorname{sgn}(\dot{y})$ term, arising from Coulomb friction. Although the $\operatorname{sgn}(\dot{y})$ term is nonlinear as well as discontinuous, it is piecewise continuous. Since y appears only in this nonlinear term as a means to determine which constant portion of the piecewise continuous curve is applicable, the equation may be solved by treating this term as a disturbance input with two distinct constant values, one for positive output velocity and the other for negative output velocity (Figure II-3). The ordinate of Figure II-3 is the magnitude of the Coulomb friction force divided by the spring stiffness.

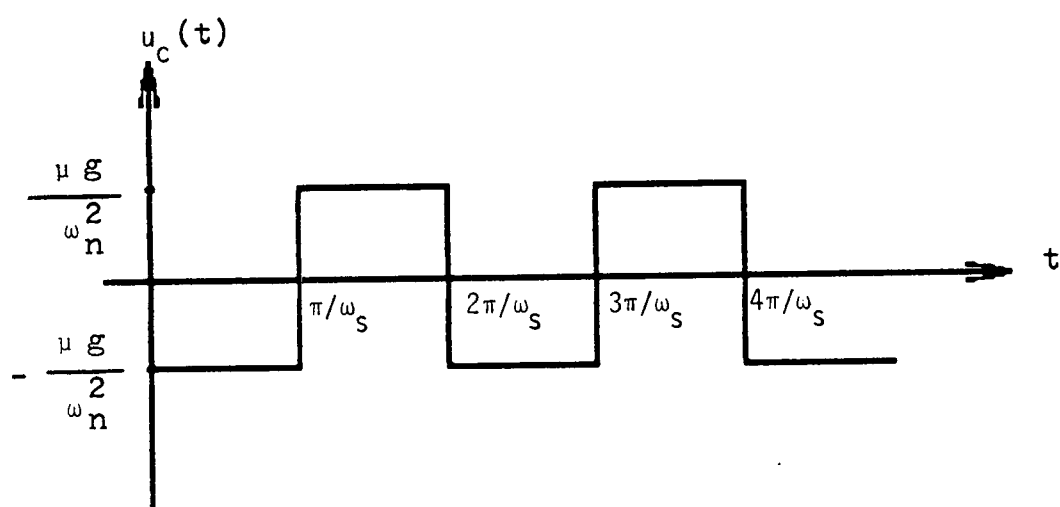


Figure II-3. The pseudo-disturbance input.

This pseudo-disturbance input is therefore a square wave of known frequency, namely the frequency of the output velocity of the system. In Figure II-3, ω_n is the natural frequency of the plant while ω_s is the oscillatory frequency of the system.

While classical control theory, which relies on the use of the z-transform, is quite useful for dealing with single-input, single-output systems, the state space method, which is synonymous with modern control theory, is more suitable for multiple-input, multiple-output systems and especially for digital computer simulation. A goal of this section is to show how to apply modern control theory to the identical problem previously discussed with classical control theory and how to design the digital controllers which perform the discrete equivalent of P-, I-, PI-, and PID-controllers. The state diagram method and direct design method will also be discussed.

By assuming the $\text{sgn}(\dot{y})$ term is a pseudo-disturbance input (U_c) and all initial conditions are zero, Eq. II-4 becomes

$$(s^2 + \omega_n^2) Y(s) = \omega_n^2 U_1(s) + \omega_n^2 U_c(s) \quad (\text{II-14})$$

where

$$U_1(s) = X(s)$$

and

$$U_c(s) = - \frac{\mu g}{\omega_n^2} \frac{\text{sgn}(\dot{y})}{s} .$$

Solving for $Y(s)$ for the general case where a disturbance input $V(s)$ may also be applied to the system yields

$$Y(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} [U_1(s) + U_c(s) + V(s)] .$$

Assuming $U_2(s) = U_c(s) + V(s)$ gives

$$Y(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} [U_1(s) + U_2(s)] \quad (\text{II-15})$$

and the corresponding block diagram is shown in Figure II-4.

To obtain the relation between the reference input R and output Y , the relationship between U_1 , U_2 , and Y has to be determined.

Using the state space method, state equations, which are represented by a set of first-order differential equations, must be developed.

From Eq. II-15, the corresponding differential equation is

$$\ddot{y} + \omega_n^2 y = \omega_n^2 u_1 + \omega_n^2 u_2 .$$

It is a well-known fact in modern control theory that if the following variables are defined as a set of state variables,

$$x_1 = y$$

$$x_2 = \dot{y} = \dot{x}_1$$

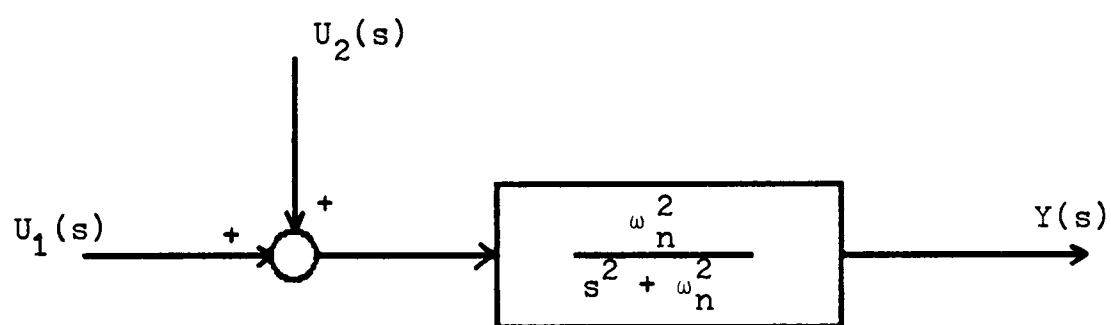


Figure II-4. The block diagram of the plant.

$$\begin{aligned}\ddot{x}_2 = \ddot{y} &= -\omega_n^2 y + \omega_n^2 u_1 + \omega_n^2 u_2 \\ &= -\omega_n^2 x_1 + \omega_n^2 u_1 + \omega_n^2 u_2\end{aligned}$$

then the solution of the state equation is guaranteed.

With the present choice of state variables, the following state equation and output equation for the system are obtained,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \omega_n^2 & \omega_n^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (\text{II-16})$$

and

$$[y] = [1 \quad 0] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (\text{II-17})$$

From Eq. II-16 and Eq. II-17, the general solution of the state equation may be obtained by using matrix algebra. The required matrix algebra and how to obtain the general solution for this discrete-time system may be found in Appendix B.

The general solution for the plant is

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} \cos \omega_n T & \frac{1}{\omega_n} \sin \omega_n T \\ -\omega_n \sin \omega_n T & \cos \omega_n T \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$$

$$+ \begin{bmatrix} 1 - \cos \omega_n T & 1 - \cos \omega_n T \\ \omega_n \sin \omega_n T & \omega_n \sin \omega_n T \end{bmatrix} \begin{bmatrix} U_1(k) \\ U_2(k) \end{bmatrix} \quad (\text{II-18})$$

and

$$[Y(k)] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} \quad (\text{II-19})$$

where $X(k)$ is the state vector, $U(k)$ is the input vector to the plant, and $Y(k)$ is the output vector, each specified at $t = kT$, $k = 0, 1, 2, \dots$, where T is the sampling period. Note that unless otherwise stated the simplified notation $X(k)$ will be used to indicate $X(kT)$. That is, $X(k)$ implies the vector $X(t)$ at $t = kT$.

Figure II-5 shows the block diagram of the system with a digital controller. The notations shown in Figure II-5 are almost the same as in Figure II-2 (p. 7) except that U_2 is a different disturbance input from V in the previous section as

$$U_2(s) = U_c(s) + V(s) .$$

The output $Y(k)$ with respect to the reference input $R(k)$ and disturbance input $U_2(k)$ can be obtained by using block diagram algebra. From the block diagram in Figure II-5,

$$U_1(s) = (\text{z.o.h.}) M^*(s)$$

$$M^*(s) = D(z) E^*(s)$$

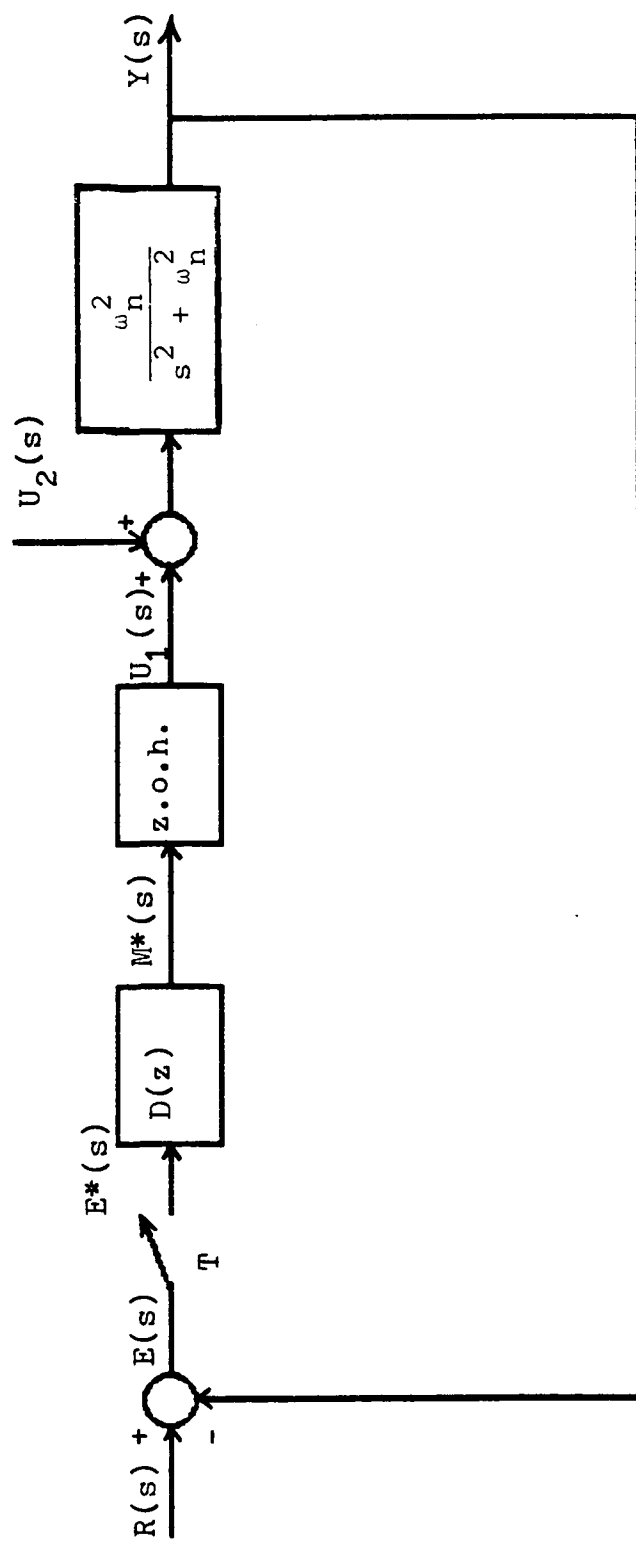


Figure II-5. The block diagram of the entire system.

and

$$E^*(s) = R^*(s) - Y^*(s) .$$

Changing the above equations to a discrete form yields

$$U_1(k) = M(k) \quad (II-20)$$

$$M(k) = D(z) E(k) \quad (II-21)$$

and

$$E(k) = R(k) - Y(k) . \quad (II-22)$$

Solving for $U_1(k)$,

$$U_1(k) = D(z) [R(k) - Y(k)]$$

and noting that

$$Y(k) = X_1(k)$$

yields

$$U_1(k) = D(z) [R(k) - X_1(k)] . \quad (II-23)$$

$U_1(k)$ is known as a function of $R(k)$ in Eq. II-23.

Substituting Eq. II-23 into Eq. II-18 enables the output $Y(k)$ to be obtained as a function of the reference input $R(k)$ and the disturbance input $U_2(k)$. The digital controller is employed in order to reduce the system error, which is the difference between the desired value of the output and the actual value of the plant output, to zero or to

a small value. The procedure of designing some digital controllers which are the digital equivalent of analog P-, I-, PI-, and PID-controllers will be introduced in the next chapter. The effect of each controller on the overall system performance will be discussed in Chapter III.

1. Proportional Controller

For the controller with proportional control action, the relationship between the output of the controller $m(t)$ and the actuating error signal $e(t)$ is

$$m(t) = K_p e(t)$$

or, in Laplace-transformed quantities,

$$\frac{M(s)}{E(s)} = K_p ,$$

while in the digital form or time sequence form,

$$\frac{M(k)}{E(k)} = K_p \quad (II-24)$$

where K_p is termed the proportional gain.

From Eq. II-21 and Eq. II-24, the $D(z)$ is K_p for the P-controller. Hence, Eq. II-23 becomes

$$U_1(k) = K_p [R(k) - X_1(k)] . \quad (II-25)$$

By substituting Eq. II-25 into Eq. II-18, the final output state equation is

$$\begin{aligned}
 \begin{pmatrix} X_1(k+1) \\ X_2(k+1) \end{pmatrix} &= \begin{pmatrix} \cos \omega_n T - K_p + K_p \cos \omega_n T & \frac{1}{\omega_n} \sin \omega_n T \\ -(1+K_p) \omega_n \sin \omega_n T & \cos \omega_n T \end{pmatrix} \begin{pmatrix} X_1(k) \\ X_2(k) \end{pmatrix} \\
 &+ \begin{pmatrix} K_p(1-\cos \omega_n T) & 1 - \cos \omega_n T \\ K_p \omega_n \sin \omega_n T & \omega_n \sin \omega_n T \end{pmatrix} \begin{pmatrix} R(k) \\ U_2(k) \end{pmatrix}
 \end{aligned} \tag{II-26}$$

and

$$[Y(k)] = [1 \quad 0] \begin{pmatrix} X_1(k) \\ X_2(k) \end{pmatrix}. \tag{II-27}$$

The detail calculations for Eqs. II-26 and II-27 can be found in Appendix C. The computer program for solving the above state equations as well as the results will be presented in Appendix G.

2. I-Controller

For the digital controller with the digital equivalent of integral control action, the value of the controller output $m(t)$ is changed at a rate proportional to the integral of the actuating error signal $e(t)$. Namely,

$$m(t) = K_i \int_0^t e(t) dt$$

or, in terms of the Laplace transform.

$$\frac{M(s)}{E(s)} = \frac{K_i}{s}$$

where K_i is an adjustable constant. The integral operation K_i/s can be approximated in the discrete domain by using Tustin's method (11) for transformation to the z-domain. Tustin's method may be summarized by the substitution as

$$H(z) = H(s) \Big|_{s = \frac{2}{T} \frac{z-1}{z+1}}$$

where $H(s)$ is a continuous transfer function and $H(z)$ is its discrete equivalent. Essentially Tustin's method is the trapezoidal rule for integration. Thus,

$$\begin{aligned} D(z) = \frac{M(z)}{E(z)} &= \frac{K_i}{s} \Big|_{s = \frac{2}{T} \frac{z-1}{z+1}} \\ &= \frac{K_i T (z+1)}{2(z-1)} \\ &= K_2 \frac{1+z^{-1}}{1-z^{-1}} \end{aligned} \tag{II-28}$$

where

$$K_2 = \frac{K_i T}{2} .$$

Rearranging Eq. II-28 to solve for the manipulated variable yields

$$M(z) - z^{-1}M(z) = K_2 [E(z) + z^{-1}E(z)] ,$$

while taking the inverse transform yields

$$M(k) = M(k-1) + K_2 E(k) + K_2 E(k-1) . \quad (II-29)$$

Because it is difficult to get the relationship between $M(k)$ and $E(k)$ from Eq. II-29, the state diagram method will be employed instead of using the same method as the one used in designing the proportional controller.

The state diagram method is similar to the signal flow graph method and leads to an easy solution of the problem. It is also a very useful method to obtain the state variable analysis of digital systems which contains digital as well as analog elements. For these reasons, the state diagram of the entire system with the digital approximation of integral control action will be constructed step by step.

First the relationship between the output $Y(k)$ and the plant input $U(k)$ (i.e., $U_1(k) + U_2(k)$) will be investigated. From Eq. II-15, substituting $U(s) = U_1(s) + U_2(s)$ yields

$$Y(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} U(s) .$$

Using the method of direct decomposition, the transfer function of the plant can be portrayed by the state diagram. Multiplying the numerator and the denominator of the right-hand side of $Y(s)/U(s)$ by a variable $X(s)$ yields

$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} \frac{X(s)}{X(s)} = \frac{s^{-2}}{\frac{1}{\omega_n^2} + s^{-2}} \frac{X(s)}{X(s)} .$$

Now equating the numerators on both sides gives

$$Y(s) = s^{-2} X(s) \quad (\text{II-30})$$

while the same operation on the denominator gives

$$U(s) = \left(\frac{1}{\omega_n^2} + s^{-2} \right) X(s) . \quad (\text{II-31})$$

In order to construct a state diagram, Eq. II-31 must first be written in a cause-and-effect relation. Therefore, solving for $X(s)$ in Eq. II-31 gives

$$X(s) = \omega_n^2 U(s) - \omega_n^2 s^{-2} X(s) .$$

The state diagram portraying Eq. II-30 and Eq. II-31 is drawn as shown in Figure II-6. The state diagram of the zero-order hold can be connected as shown in Figure II-7. Applying the direct decomposition scheme to $D(z)$ yields

$$D(z) = \frac{M(z)}{E(z)} = \frac{K_2(1 + z^{-1})}{(1 - z^{-1})} \frac{X(z)}{X(z)} .$$

Equating the numerator gives

$$M(z) = K_2 (1 + z^{-1}) X(z) \quad (\text{II-32})$$

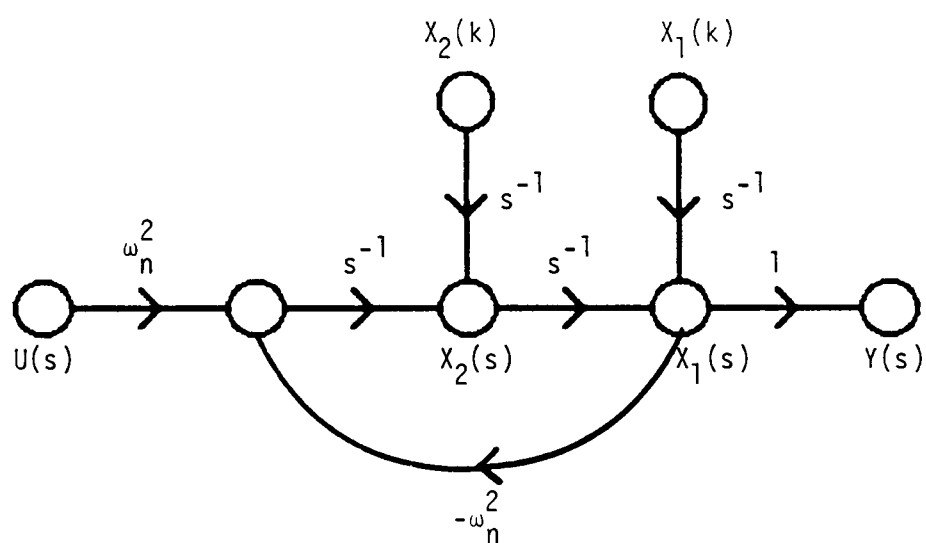


Figure II-6. The state diagram of the plant.

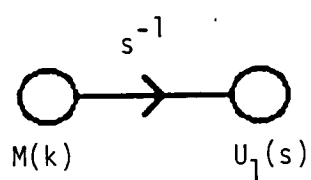


Figure II-7. The state diagram of the zero-order hold.

while equating the denominators gives

$$E(z) = (1 - z^{-1}) X(z) . \quad (\text{II-33})$$

From Eq. II-33, solving for $X(z)$ yields

$$X(z) = E(z) + z^{-1} X(z) . \quad (\text{II-34})$$

The state diagram for the digital controller is constructed as shown in Figure II-8 using Eqs. II-32 and II-34, as well as

$$E(k) = R(k) - Y(k) = R(k) - X_1(k) .$$

The state diagram for the entire system is constructed as shown in Figure II-9. In Figure II-9 the independent and dependent variables of the equations become the input nodes and output nodes, respectively. Applying Mason's gain formula to Figure II-9 with $X_1(s)$ and $X_2(s)$ as output nodes and $X_1(k)$, $X_2(k)$, $X_3(k)$, $U_2(k)$, and $R(k)$ as input nodes. the dynamic equations and the discrete state equation can all be determined directly from the state diagram. More detail and calculations can be found in Appendix D.

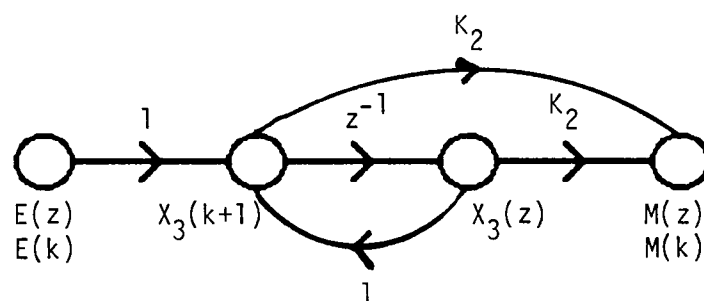


Figure II-8. The state diagram for the I-controller.

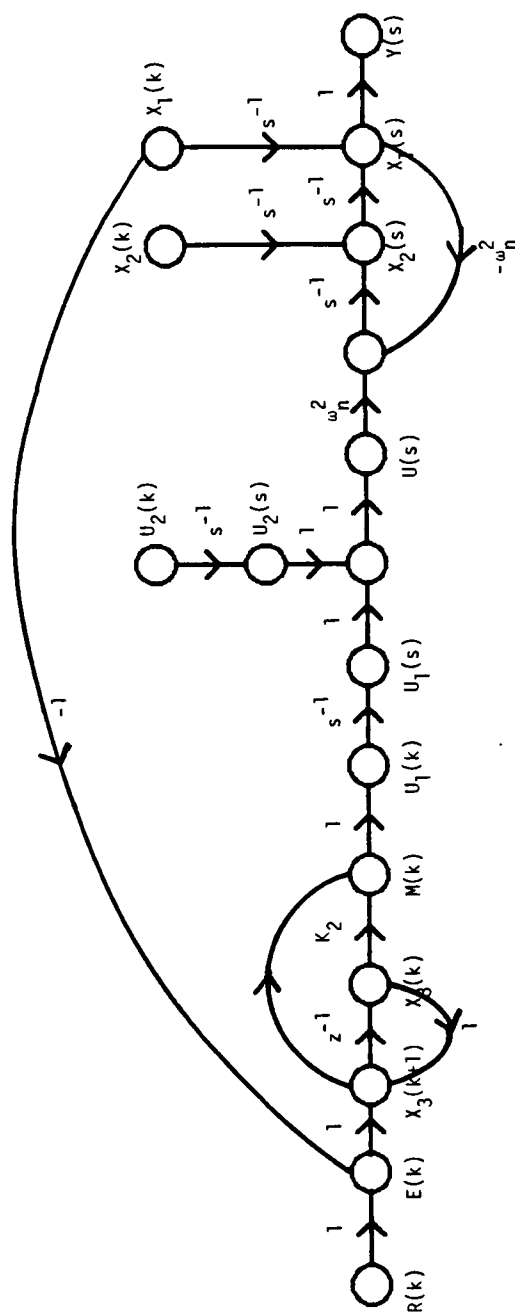


Figure II-9. The state diagram of the entire system with the I-controller.

The final discrete state equation of this system is

$$\begin{aligned}
 \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} &= \begin{pmatrix} \cos \omega_n T - K_2(1 - \cos \omega_n T) & \frac{1}{\omega_n} \sin \omega_n T & 2K_2(1 - \cos \omega_n T) \\ -(1 + K_2)\omega_n \sin \omega_n T & \cos \omega_n T & 2K_2\omega_n \sin \omega_n T \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} \\
 &+ \begin{pmatrix} K_2(1 - \cos \omega_n T) & 1 - \cos \omega_n T \\ K_2\omega_n \sin \omega_n T & \omega_n \sin \omega_n T \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R(k) \\ U_2(k) \end{pmatrix} \quad (\text{II-35})
 \end{aligned}$$

and

$$[Y(k)] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} \quad (\text{II-36})$$

3. PI-Controller

The transfer function of a PI-controller is

$$\frac{M(s)}{E(s)} = K_p + \frac{K_i}{s}$$

in terms of Laplace transforms. Using Tustin's method for obtaining the discrete equivalent of the PI-controller yields the following transfer function in the z-plane

$$\frac{M(z)}{E(z)} = \frac{\frac{2K_p + K_i T}{2} + \frac{K_i T - 2K_p}{2} z^{-1}}{1 - z^{-1}} \quad (\text{II-37})$$

where K_p represents the proportional gain and K_i represents the coefficient of the integral. Equation II-37 can be rewritten as

$$\frac{M(z)}{E(z)} = \frac{K_3 + K_4 z^{-1}}{1 - z^{-1}} \quad (\text{II-38})$$

where

$$K_3 = \frac{2K_p + K_i T}{2}$$

and

$$K_4 = \frac{K_i T - 2K_p}{2}.$$

Because the state diagram for the PI-controller is very similar to the previously discussed I-controller, only the key points will be presented. From Eq. II-38, the state diagram for the digital controller is constructed as shown in Figure II-10. Adding Figure II-10 to Figure II-6, the state diagram for the entire system is drawn as shown in Figure II-11. Comparing Figure II-11 with Figure II-9, it is noted to be almost the same except for the constants K_3 and K_4 . In the same manner as before the final form of the state equation can be found as

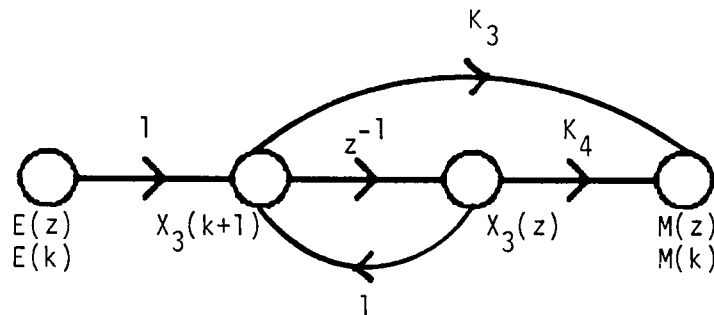
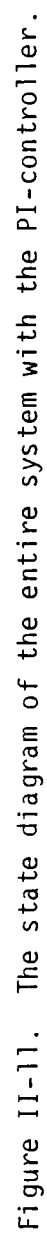


Figure II-10. The state diagram for the PI-controller.



$$\begin{aligned}
 \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} &= \begin{bmatrix} \cos \omega_n T - K_3(1 - \cos \omega_n T) & \frac{1}{\omega_n} \sin \omega_n T (K_3 + K_4)(1 - \cos \omega_n T) \\ -(1 + K_3)\omega_n \sin \omega_n T & \cos \omega_n T & (K_3 + K_4)\omega_n \sin \omega_n T \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \\
 &+ \begin{bmatrix} K_3(1 - \cos \omega_n T) & 1 - \cos \omega_n T \\ K_3 \omega_n \sin \omega_n T & \omega_n \sin \omega_n T \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R(k) \\ U_2(k) \end{bmatrix} \quad (\text{II-39})
 \end{aligned}$$

and

$$[Y(k)] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \quad (\text{II-40})$$

Additional details can be found in Appendix E.

4. PID-Controller

One of the most widely used controllers in the design of control systems is the PID-controller. The proportional control component simply multiplies the error signal by a constant K_p , the integral control component multiplies the integral of $e(t)$ by K_i , and the derivative control component generates a signal which is proportional to the time derivative of the error signal.

In the s-plane the PID-controller is given by

$$D(s) = \frac{M(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d s$$

which may be transformed to the z -plane using Tustin's method (Figure II-12) which yields

$$\frac{M(z)}{E(z)} = \frac{\frac{(K_i T^2 + 2K_d + 2K_p T)z^2}{2T} + \frac{(K_i T^2 - 2K_p T - 4K_d)z}{2T} + \frac{K_d}{T}}{z^2 - z}$$

or

$$= \frac{K_5 + K_6 z^{-1} + K_7 z^{-2}}{1 - z^{-1}}$$

where

$$K_5 = \frac{K_i T^2 + 2K_d + 2K_p T}{2T}$$

$$K_6 = \frac{K_i T^2 - 2K_p T - 4K_d}{2T}$$

and

$$K_7 = \frac{K_d}{T}.$$

In the same manner as for the PI-controller, the state diagram of the PID-controller is drawn as shown in Figure II-13. Adding this to the state diagram of the plant (Figure II-6, p. 26) enables the state diagram for the entire system to be represented as shown in Figure II-14. Reduction of this diagram yields the final form of the state equation as

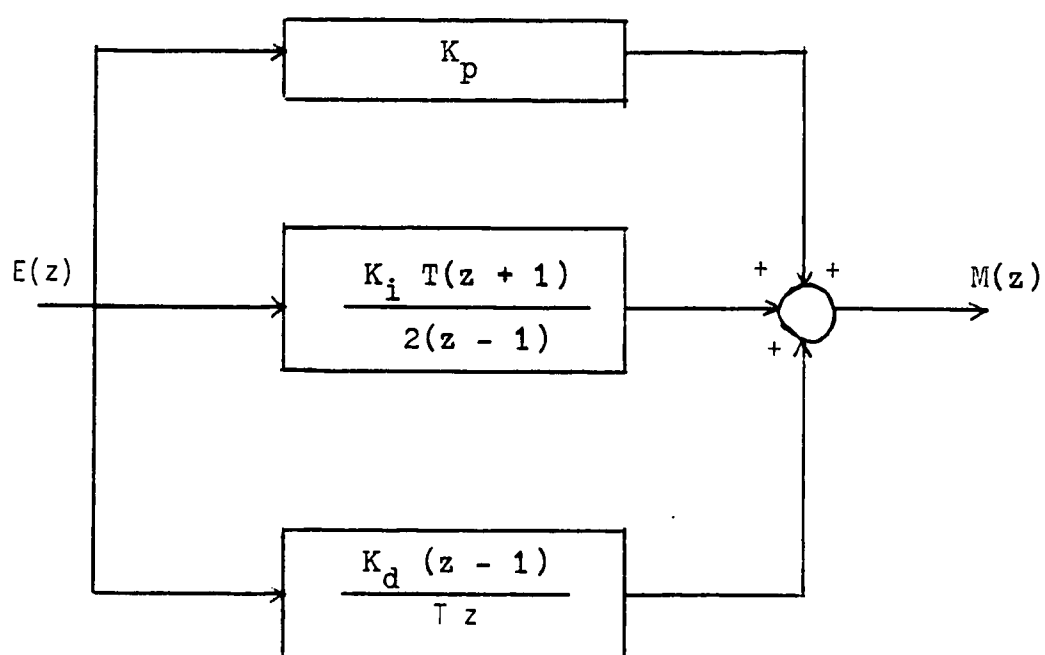


Figure II-12. A digital PID-controller.

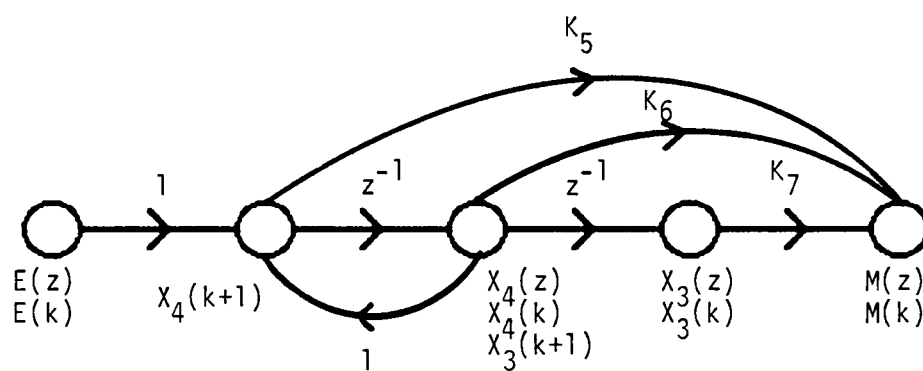


Figure II-13. The state diagram for the PID-controller.

$$\begin{aligned}
 \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} &= \begin{bmatrix} \cos \omega_n T - K_5(1 - \cos \omega_n T) & \frac{\sin \omega_n T}{\omega_n} & K_7(1 - \cos \omega_n T) & (K_5 + K_6)(1 - \cos \omega_n T) \\ -(1 + K_5)\omega_n \sin \omega_n T & \cos \omega_n T & K_7\omega_n \sin \omega_n T & (K_5 + K_6)\omega_n \sin \omega_n T \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} \\
 &+ \begin{bmatrix} K_5(1 - \cos \omega_n T) & 1 - \cos \omega_n T \\ K_5 \omega_n \sin \omega_n T & \omega_n \sin \omega_n T \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R(k) \\ U_2(k) \end{bmatrix} \quad (\text{II-41})
 \end{aligned}$$

and

$$[Y(k)] = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}. \quad (\text{II-42})$$

Additional details of these calculations may be found in Appendix E.

5. Direct Design Method

This section introduces a direct design method for designing a digital controller. In this direct design technique, an input and desired transient output is specified. Knowing the mathematical model of the plant enables one to solve for the required algorithm to obtain the desired system output.

Consider the control system shown in Figure II-5 (p. 19). From this block diagram,

$$Y(s) = G(s) [U_1(s) + U_2(s)]$$

$$U_1(s) = G_{ho}(s) M^*(s)$$

$$M^*(s) = D(z) E^*(s)$$

and

$$E^*(s) = R^*(s) - Y^*(s) ,$$

where $G(s)$ is the transfer function of the plant and $U_2(s)$ is the disturbance input

$$U_2(s) = U_c(s) + V(s) .$$

Combining the above equations yields

$$\begin{aligned} Y(s) &= G(s) [G_{ho}(s)D(z) [R^*(s) - Y^*(s)] + U_2(s)] \\ &= G(s)G_{ho}(s)D(z) [R^*(s) - Y^*(s)] + G(s)U_2(s) . \end{aligned}$$

In terms of the z-transform this equation becomes

$$Y(z) = G_{ho}G(z)D(z) [R(z) - Y(z)] + GU_2(z) .$$

Solving for $D(z)$ gives

$$D(z) = \frac{Y(z) - GU_2(z)}{G_{ho}G(z) [R(z) - Y(z)]} . \quad (II-43)$$

The input $R(z)$, disturbance input $U_2(z)$, and desired output $Y(z)$ can be specified. As $GU_2(z)$ and $G_{ho}G(z)$ are known, the

z-transform of the control algorithm of the digital controller can be found from Eq. II-43.

For example, consider a desired output to a reference unit step input as

$$y(t) = 1 - e^{-t/\tau}$$

or in the z-plane

$$\begin{aligned} Y(z) &= \frac{z}{z-1} - \frac{z}{z-e^{-T/\tau}} \\ &= \frac{z(1-e^{-T/\tau})}{(z-1)(z-e^{-T/\tau})} \end{aligned}$$

where τ is the desired time constant of the system. The z-transform of the unit step input is

$$R(z) = \frac{z}{z-1}$$

and for the plant

$$\begin{aligned} G(z) &= \mathcal{Z} \left(\frac{\omega_n^2}{s^2 + \omega_n^2} \right) \\ &= \frac{\omega_n^2 z \sin \omega_n T}{z^2 - 2z \cos \omega_n T + 1} . \end{aligned}$$

From Eq. II-8, it is known that

$$G_{ho} G(z) = \frac{(1 - \cos \omega_n T)(z + 1)}{z^2 - 2z \cos \omega_n T + 1} .$$

From Figure II-3 (p. 13), $U_c(t)$ can be written in terms of the Laplace transform as

$$U_c(s) = M_k \cdot \frac{1}{s}$$

where

$$M_k = \frac{\mu g}{\omega_n^2} \quad \text{when } X_2(k) < 0$$

and

$$M_k = - \frac{\mu g}{\omega_n^2} \quad \text{when } X_2(k) > 0 .$$

From $U_c(s)$ and the disturbance input

$$\begin{aligned} GU_2(z) &= \mathcal{Z} [G(s)U_c(s) + G(s)V(s)] \\ &= \mathcal{Z} \left[\frac{\omega_n^2}{s^2 + \omega_n^2} \frac{M_k}{s} + G(s)V(s) \right] \\ &= \frac{M_k(1 - \cos \omega_n T)z(z + 1)}{(z - 1)(z^2 - 2z \cos \omega_n T + 1)} + \mathcal{Z} [G(s)V(s)] . \end{aligned}$$

For zero disturbance input (i.e., $V(s) = 0$) combining the above equations into Eq. II-43 gives

$$\begin{aligned}
 D(z) &= \frac{[1 - e^{-T/\tau} - M_k(1 - \cos \omega_n T)] z^2}{-(1 - e^{-T/\tau}) [2 \cos \omega_n T + M_k(1 - \cos \omega_n T)] z} \\
 &\quad \frac{+ 1 - e^{-T/\tau} + M_k(1 - \cos \omega_n T)e^{-T/\tau}}{(1 - \cos \omega_n T)(z - 1)(z + 1)} \\
 &= \frac{K_8 + K_9 z^{-1} + K_{10} z^{-2}}{1 - z^{-2}} \\
 &= \frac{M(z)}{E(z)} , \tag{II-44}
 \end{aligned}$$

where

$$\begin{aligned}
 K_8 &= \frac{1 - e^{-T/\tau} - M_k(1 - \cos \omega_n T)}{(1 - \cos \omega_n T)} \\
 K_9 &= \frac{-(1 - e^{-T/\tau})[2 \cos \omega_n T + M_k(1 - \cos \omega_n T)]}{1 - \cos \omega_n T}
 \end{aligned}$$

and

$$K_{10} = \frac{1 - e^{-T/\tau} + M_k(1 - \cos \omega_n T)e^{-T/\tau}}{1 - \cos \omega_n T} .$$

Equation II-44 can be rewritten as

$$M(k) - M(k-2) = K_8 E(k) + K_9 E(k-1) + K_{10} E(k-2) .$$

Solving for $M(k)$ gives

$$M(k) = M(k-2) + K_8 E(k) + K_9 E(k-1) + K_{10} E(k-2) .$$

To use the signal flow graph method, the calculations follow the same procedure as for the PID-controller. Using the method of direct decomposition from Eq. II-44,

$$M(z) = (K_8 + K_9 z^{-1} + K_{10} z^{-2}) X(z) \quad (\text{II-45})$$

and

$$E(z) = (1 - z^{-2}) X(z)$$

or rewriting the last equation gives

$$X(z) = E(z) + z^{-2} X(z) . \quad (\text{II-46})$$

From Eqs. II-45 and II-46, the state diagram for $M(z)/E(z)$ is constructed in Figure II-15. Hence

$$\begin{aligned} E(k) &= R(k) - Y(k) \\ &= R(k) - X_1(k) . \end{aligned}$$

Adding Figure II-15 to Figure II-6 (p. 26), the state diagram of the plant, the state diagram for the entire system is shown in Figure II-16. In Figure II-16 comparing all calculations with the PID-controller yields

$$\begin{aligned} X_1(k+1) &= [\cos \omega_n T - K_8(1 - \cos \omega_n T)] X_1(k) + \frac{1}{\omega_n} \sin \omega_n T X_2(k) \\ &\quad + (K_8 + K_{10})(1 - \cos \omega_n T) X_3(k) + K_9(1 - \cos \omega_n T) X_4(k) \\ &\quad + K_8(1 - \cos \omega_n T) R(k) + (1 - \cos \omega_n T) U_2(k) , \quad (\text{II-47}) \end{aligned}$$

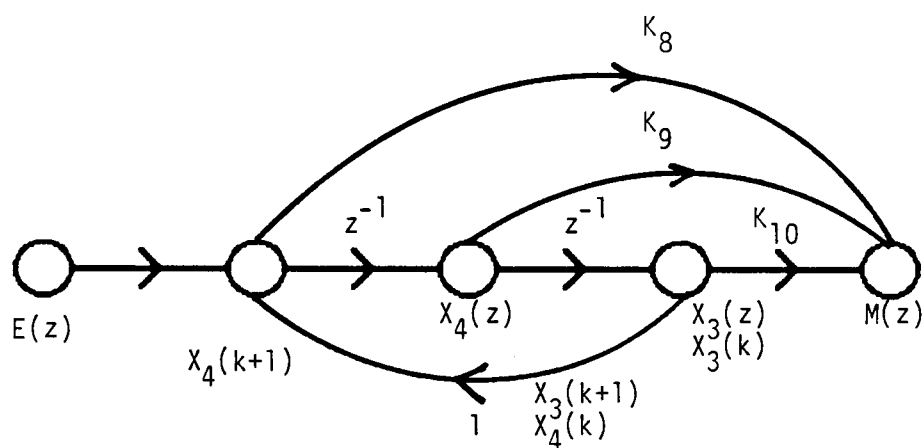


Figure II-15. The state diagram for the digital controller.

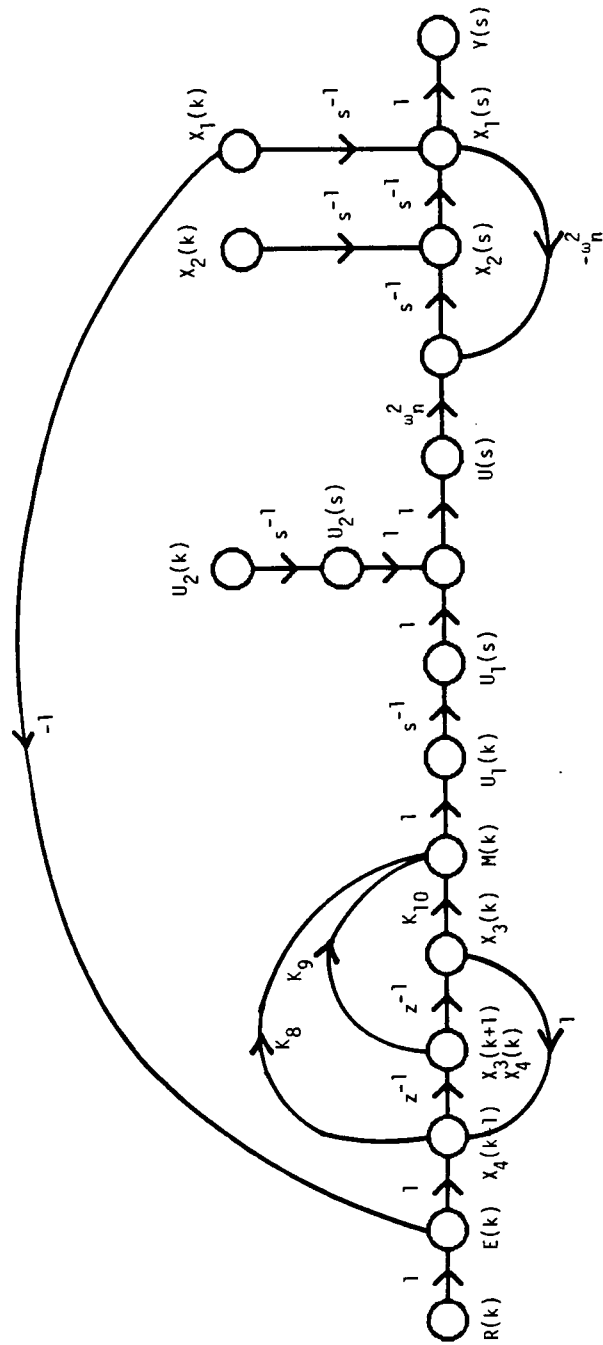


Figure II-16. The state diagram for the entire system.

$$\begin{aligned}
X_2(k+1) = & -(1 + K_8)\omega_n \sin \omega_n T X_1(k) + \cos \omega_n T X_2(k) \\
& + (K_8 + K_{10})\omega_n \sin \omega_n T X_3(k) + K_9\omega_n \sin \omega_n T X_4(k) \\
& + K_8\omega_n \sin \omega_n T R(k) + \omega_n \sin \omega_n T U_2(k) , \quad (\text{II-48})
\end{aligned}$$

$$X_3(k+1) = X_4(k) , \quad (\text{II-49})$$

and

$$X_4(k+1) = -X_1(k) + X_3(k) + R(k) . \quad (\text{II-50})$$

Considering the disturbance input as a unit step input gives

$$\begin{aligned}
GU_2(z) &= \mathcal{Z} [G(s)U_c(s) + G(s)V(s)] \\
&= \mathcal{Z} \left(\frac{\omega_n^2}{s^2 + \omega_n^2} \frac{M_k}{s} + \frac{\omega_n^2}{s^2 + \omega_n^2} \frac{1}{s} \right) \\
&= \frac{(1 + M_k)(1 - \cos \omega_n T)z(z + 1)}{(z - 1)(z^2 - 2z \cos \omega_n T + 1)} .
\end{aligned}$$

For the unit step disturbance input to the same system as one previously discussed with the same desired input and reference input gives

$$\begin{aligned}
 D(z) &= \frac{[1 - e^{-T/\tau} - (1 + M_k)(1 - \cos \omega_n T)] z^2}{-(1 - e^{-T/\tau})[2 \cos \omega_n T + (1 + M_k)(1 - \cos \omega_n T)] z} \\
 &\quad + \frac{1 - e^{-T/\tau} + (1 + M_k)(1 - \cos \omega_n T)e^{-T/\tau}}{(1 - \cos \omega_n T)(z - 1)(z + 1)} \\
 &= \frac{K_{11} + K_{12} z^{-1} + K_{13} z^{-2}}{1 - z^{-2}} \\
 &= \frac{M(z)}{E(z)}, \tag{II-51}
 \end{aligned}$$

where

$$\begin{aligned}
 K_{11} &= \frac{1 - e^{-T/\tau} - (1 + M_k)(1 - \cos \omega_n T)}{(1 - \cos \omega_n T)} \\
 K_{12} &= \frac{-(1 - e^{-T/\tau}) [2 \cos \omega_n T + (1 + M_k)(1 - \cos \omega_n T)]}{1 - \cos \omega_n T}
 \end{aligned}$$

and

$$K_{13} = \frac{1 - e^{-T/\tau} + (1 + M_k)(1 - \cos \omega_n T)e^{-T/\tau}}{1 - \cos \omega_n T}.$$

Comparing Eq. II-51 with Eq. II-44, it is noted that both equations are almost the same except for the constants. The final state equations are the same as Eqs. II-47, II-48, II-49, and II-50. The results for $\tau = 1$ and $\tau = 10$ will be presented in Chapter III and the computer programs both with and without the disturbance input may be found in Appendix H.

CHAPTER III

RESULTS AND DISCUSSION

In order to verify the results obtained from the closed-form solutions developed in Chapter II, it was decided to develop an experimental system for the simulation of the real-time digital control of a nonlinear system. The experimental system, described in detail in Appendix F, basically consisted of an analog computer simulation of the nonlinear plant with a minicomputer, A/D, and D/A boards used as the digital controller, sampler, and zero-order hold, respectively. In this analysis, the natural frequency of the plant, ω_n , the sampling period, T , and the coefficient of friction times gravity, μg , are taken arbitrarily to be $\sqrt{0.05}$ (rad/sec), 0.1 (sec), and $0.0092 \text{ (ft/s}^2\text{)}$, respectively.

In many practical cases, performance criteria for the transient response of a dynamic system are frequently stated in terms of time domain quantities. The settling time is the time required for the oscillations to stay within some specified small percentage of the final value. The most common values used are $\pm 2\%$ and $\pm 5\%$. The maximum overshoot is the maximum deviation of the output value above its steady-state value. It is expressed as a percentage of the final value in this thesis. If the final value of the response differs from some desired value, a steady-state error exists. To investigate the transient response of the system to a unit-step input, the values of

settling time (TS), maximum (percent) overshoot (OS), and steady-state error (ESS) will be plotted as functions of the parameters of the digital controllers. TS represents the settling time corresponding to a $\pm 2\%$ tolerance. For comparison purposes, the analytical results for digital controllers will be compared to the response of a second order plant with linear (i.e., viscous) friction controlled by an analog controller. This will allow qualitative comparison between a continuous linear system with a discrete nonlinear system.

For the experimental system, the output can be obtained by substituting the above chosen values into Eq. F-1. Thus Eq. F-1 becomes

$$\ddot{y} = 0.05 x - 0.05 \dot{y} - 0.0092 \operatorname{sgn}(\dot{y}) . \quad (\text{III-1})$$

Following the experimental procedure explained in Appendix F, the analog circuit for Eq. III-1 can be obtained (Figure III-1), and the output displacement (y) can be obtained in the time-domain (Figure III-2). The results from the closed-form solutions of a P-controller, using Eqs. II-26 and II-27, and using a value of K_p of unity, are presented in Figure III-2. The computer program for the output data from the experimental system and closed-form solution can be found in Appendix I.

Figure III-2 presents data from the experimental system as well as the closed-form solution. Although the general shape of the predicted and measured responses are similar, a steady-state discrepancy exists between them. The difference between the two graphs

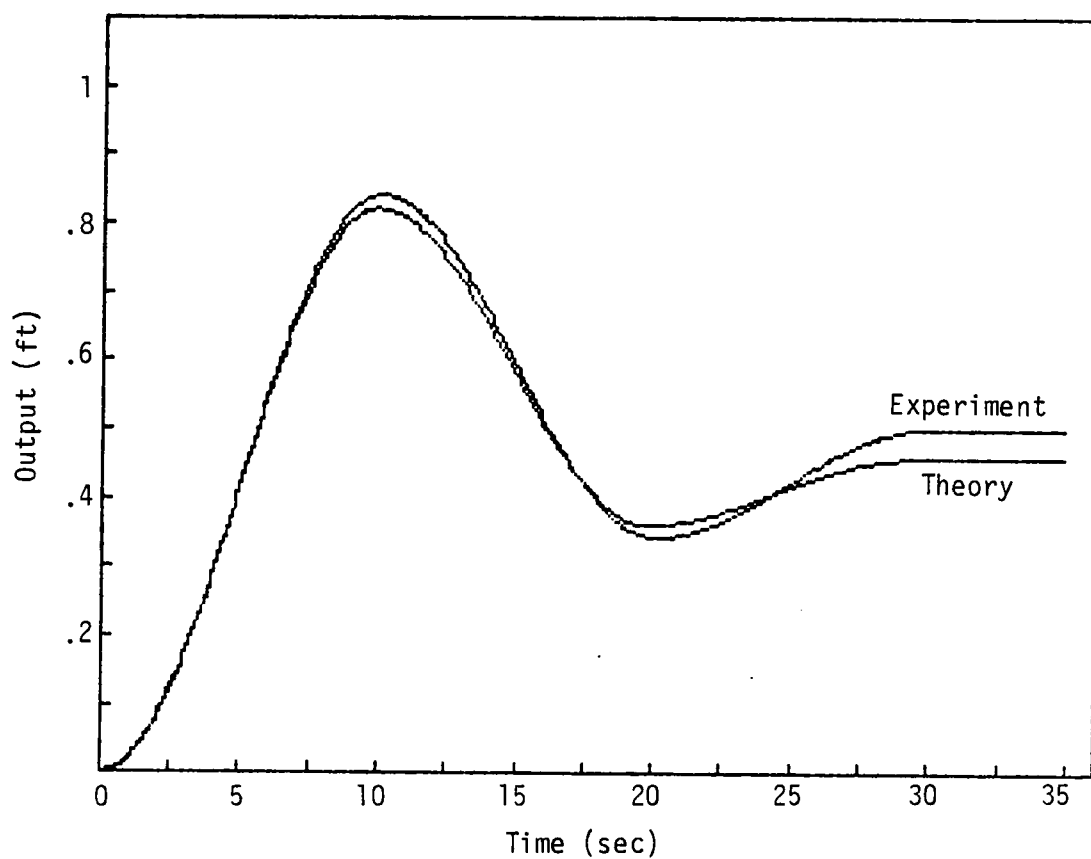


Figure III-2. The system response for the P-controller with K_p equal to unity.

may be due to the fact that it is not possible to make the computer read the output signals or voltages at the exact time it is supposed to. In other words, at the instant the computer is supposed to sample the data, it actually reads a program statement. The difference therefore likely occurs because the computer cannot read both the time and the output voltages simultaneously. The reason for the difference in the final steady-state value between theory and experiment is that it is impossible to make the zener diode circuitry correspond exactly to the change in the sign of the velocity. As can be seen from Figure III-2 the computer simulation of the closed-form solution and the experimental result agree with a reasonable amount of accuracy when the above mentioned problem is considered.

Next in order of P-, PI-, and PID-controllers, the system performance will be discussed by varying the parameters K_p , K_i , and K_d . As the P-controller has only one parameter K_p , the settling time (TS), maximum (percent) overshoot (OS), and steady-state error (ESS) can be easily obtained from Eqs. II-26 and II-27 using the general computer program in Appendix G which may be used for all digital controllers including the digital equivalent P-, I-, PI-, PD-, and PID-controller.

Figure III-3 indicates that the settling time increases as the proportional gain, K_p , increases, although the system becomes more oscillatory as K_p is increased. From this fact it can be assumed that OS will increase as K_p is increased as verified in Figure III-4. In general, as K_p of a P-controller increases, TS increases, OS

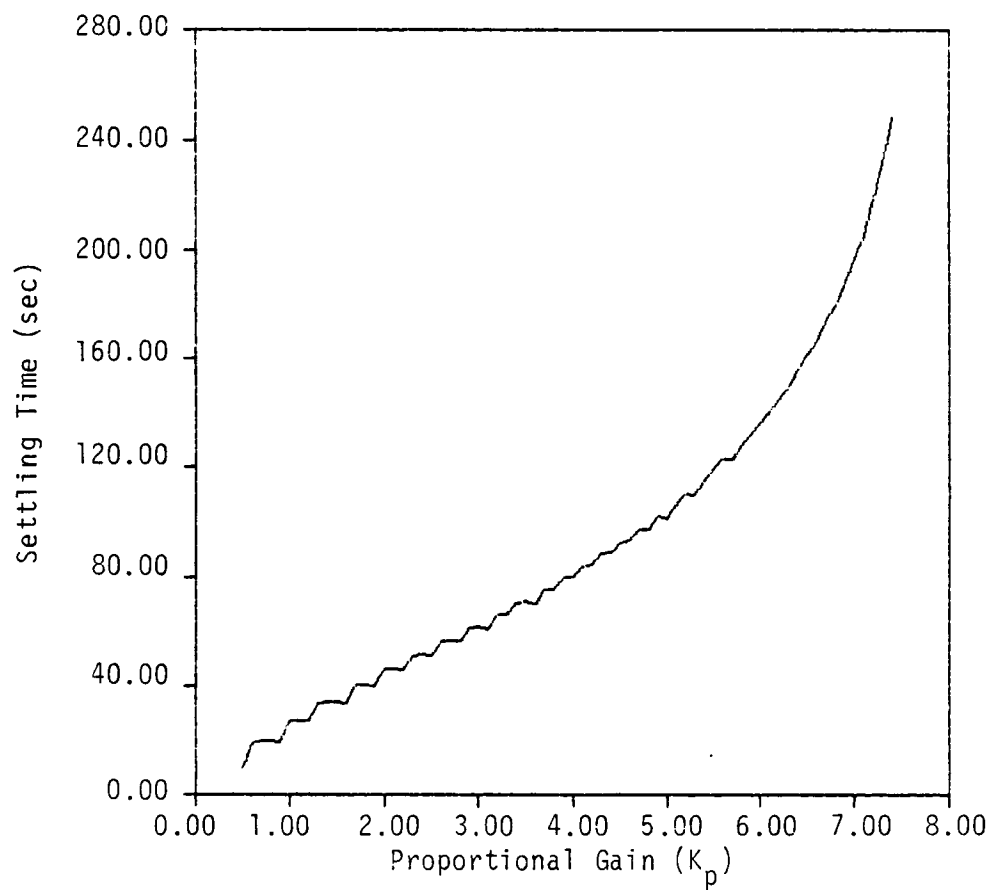


Figure III-3. Settling time for the P-controller.

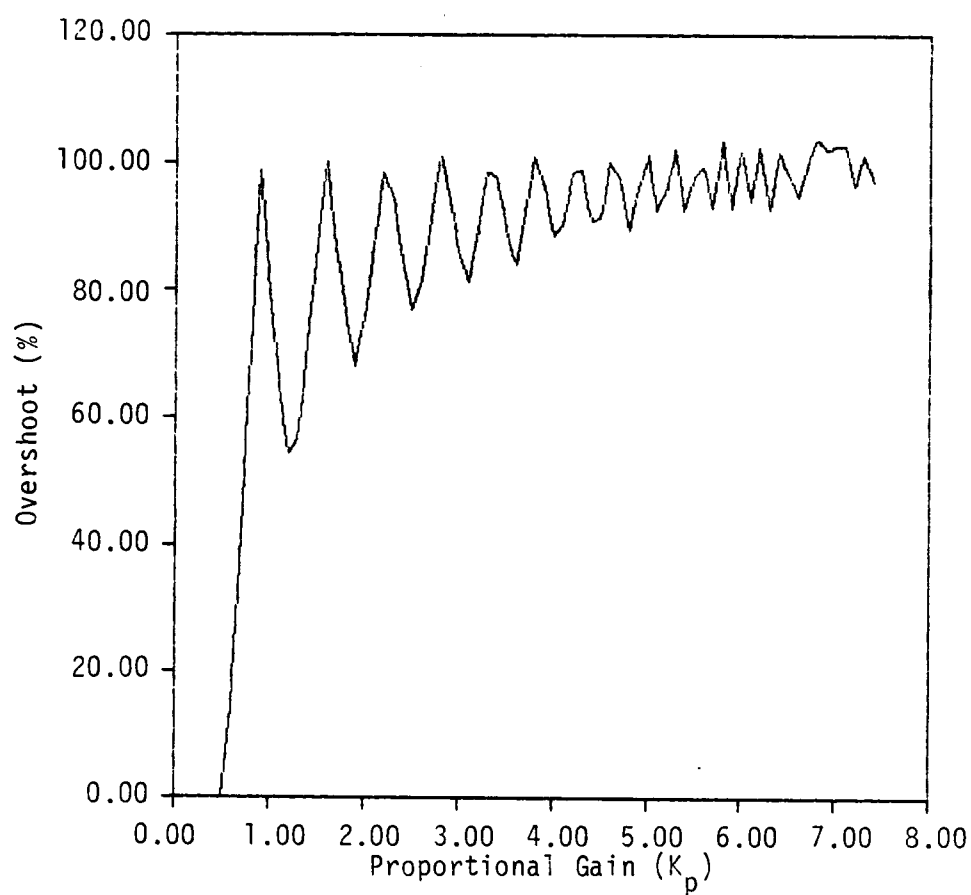


Figure III-4. Overshoot for the P-controller.

increases, while ESS decreases as indicated in Figure III-5. This same trend should be observed for a disturbance input. By comparing the above data trends with those of linear second order plant with an analog controller, it is easy to show that as K_p increases, TS increases, OS increases, while ESS decreases. It can be noted that the qualitative trends of analog controller action is similar to the digital controller action of a nonlinear plant.

In a PI-controller, TS, OS, and ESS may be obtained by varying parameters K_p and K_i from Eqs. II-39 and II-40. Because the system is not necessarily stable for all values of K_p and K_i , these parameters will be varied in the region of stability only (Figure III-6). From the general computer program in Appendix G, and using the plot package Program Surf for the IBM personal computer, TS vs. K_p and K_i (Figure III-7), OS vs. K_p and K_i (Figure III-8), and ESS vs. K_p and K_i (Figure III-9) can be plotted by varying K_p from 0.5 to 6.7 with increments of 0.2 and K_i from 0.01 to 0.105 with increments of 0.005. Even though a three dimensional plot is not as clear as a two dimensional plot, it is an acceptable way to present the trends of the data for the three parameters. To understand these figures, the following observations are useful. The following three dimensional plots show various values for each of the three parameters being considered in this study. Due to the results of some cases it is necessary to view the results from more than one perspective in order to observe all the details. If the figure has a subscript (a), the results are shown with the origin (Xmin,Ymin) being nearest to the

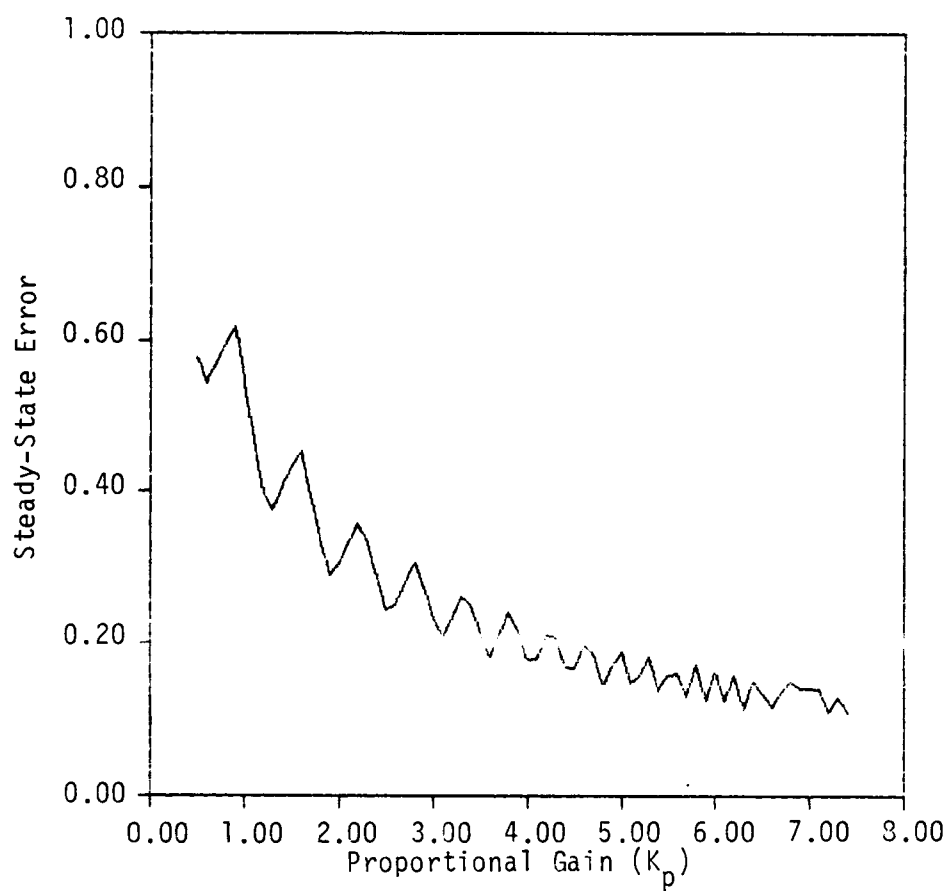


Figure III-5. Steady-state error for the P-controller.

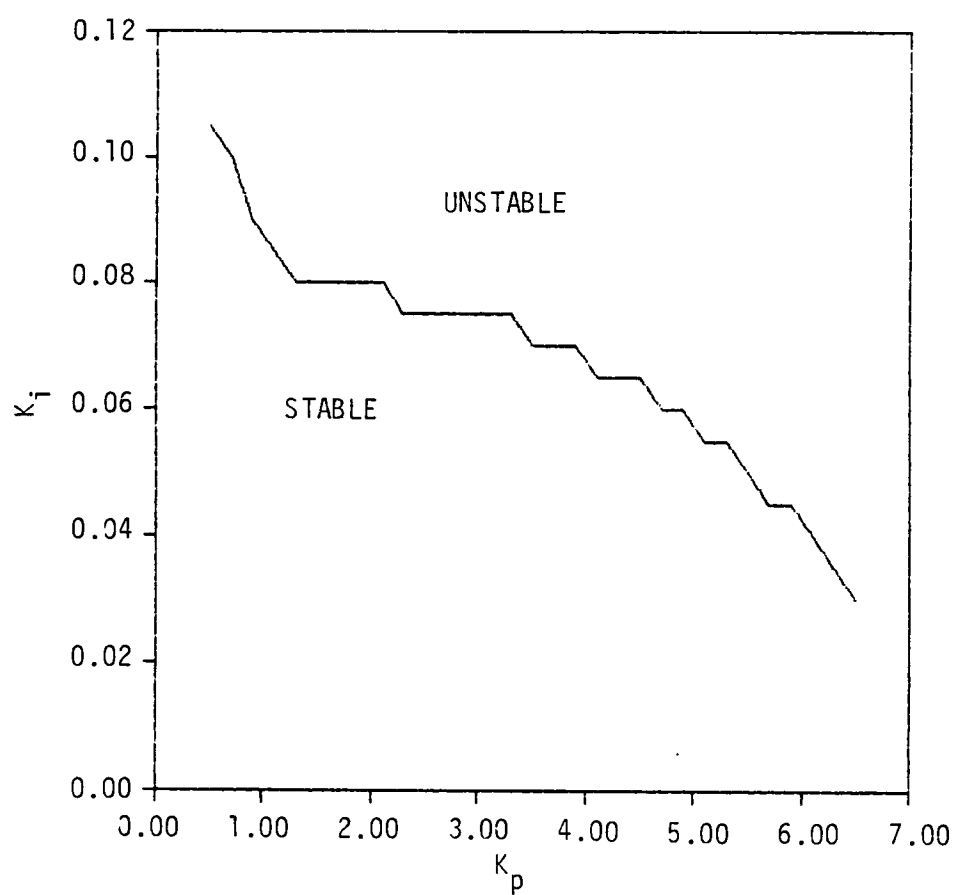


Figure III-6. The stability region for the PI-controller.

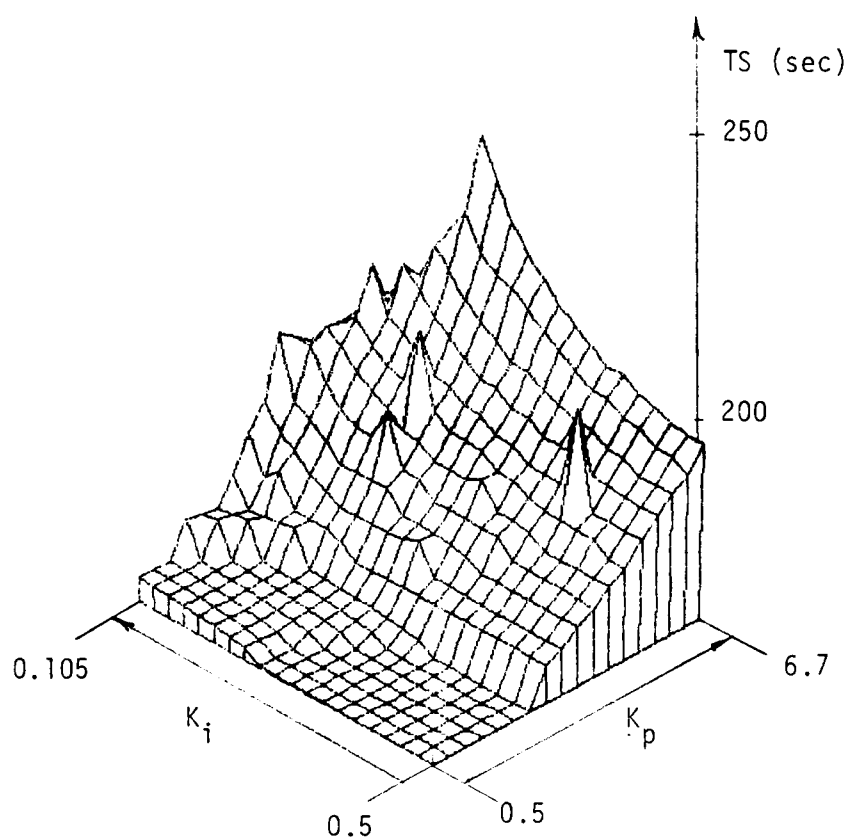


Figure III-7. Settling time for the PI-controller.

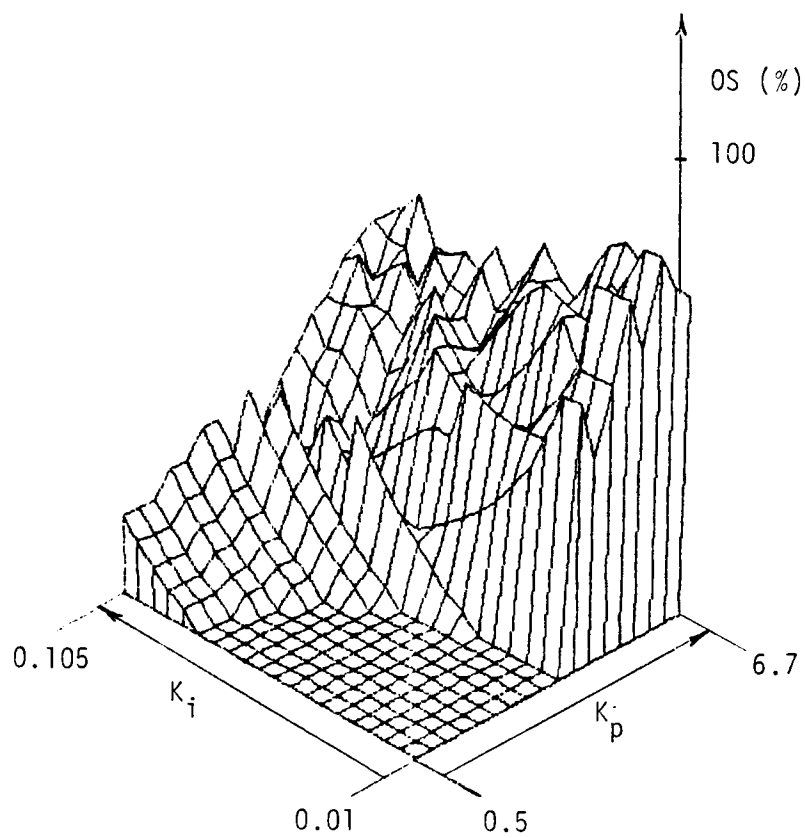


Figure III-8. Overshoot for the PI-controller.

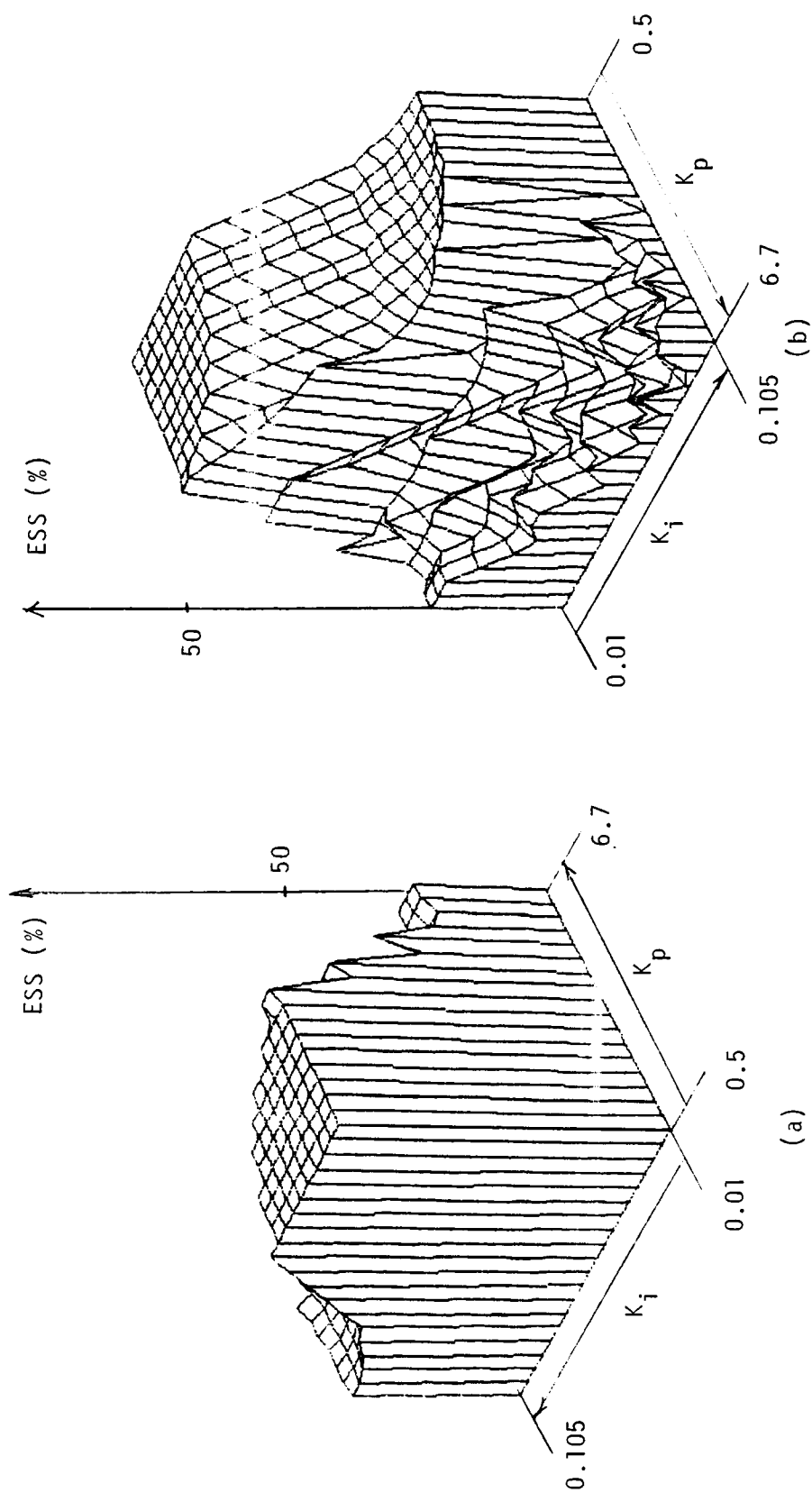


Figure III-9. Steady-state error for the PI-controller.

observer. The X-axis is the closest edge to the right, while the Y-axis is the closest edge to the left. If the subscript (b) is used, the results are shown with the figure being rotated 180 degrees (in the X-Y plane) from their representation shown in the (a) subscripted figure.

For a PI-controller, the characteristics of K_i and the relationship between K_p and K_i will be discussed. In Figures III-7, III-8, and III-9(a) the origin is at the closest corner to the observer, K_p is on the X-axis, and K_i is on the Y-axis. In Figure III-9(b) the closest corner is the X_{max} and Y_{max} point. TS increases for fixed K_p and increasing K_i as does TS for increasing K_p in the P-controller. As K_p is increased, increasing K_i makes the system more oscillatory and eventually unstable. Comparing values of OS for the P-controller with those for the PI-controller, it is noted that the maximum values of OS in the PI-controller (Figure III-8) is less than in the P-controller (Figure III-4). This is believed to be at least partly due to the fact that the PI-controller generally has a larger steady-state value than the P-controller (Figure III-9). In the response of a linear second order plant with an analog controller, it is a well known fact that from the root locus technique that as K_i and K_p increases, TS increases, OS increases, and ESS decreases, respectively. It can again be noted that the qualitative trend of an analog controller action for a linear plant is similar to the digital controller action of a nonlinear plant.

Before investigation of the PID-controller, the PD-controller will be studied in order to determine the effect of K_d , the derivative control constant, by comparing it with the P- and PI-controller. The range of K_p and K_d for which the system is stable was determined and then, within the stable range, both K_p and K_d were varied from 1 to 20 with increments of 1. It is noted that it is of primary importance to consider the ratio of K_p/K_d instead of considering what specified value K_p and K_d have when the system performance is observed. The specific value of the parameter will not be of as much use in designing the system if different ranges of other parameters are chosen.

From the same computer program as was used for the PI-controller, the output data for TS, OS, and ESS was obtained and plotted in Figures III-10, III-11, and III-12, respectively. It can be seen from Figure III-10 that the PD-controller makes TS significantly smaller than the P- or PI-controller. From this fact it follows that K_d makes the system response more rapid. It can be also noted that some ratios of K_p/K_d makes TS quite small. For example, if K_p is small and K_d is only slightly larger than K_p , TS has a relatively small value, but as K_d increases above some value, TS increases also. If K_p is large and K_d is small, TS has a large value, but as K_d increases, TS decreases.

The variation of OS may be explained in the same manner as the above discussion concerning the settling time (Figure III-11). It can be seen that Figure III-11 has almost the same shape as Figure III-10. If K_d is small, no matter what value K_p has, OS has a large value. However, by making K_d larger, OS can be decreased.

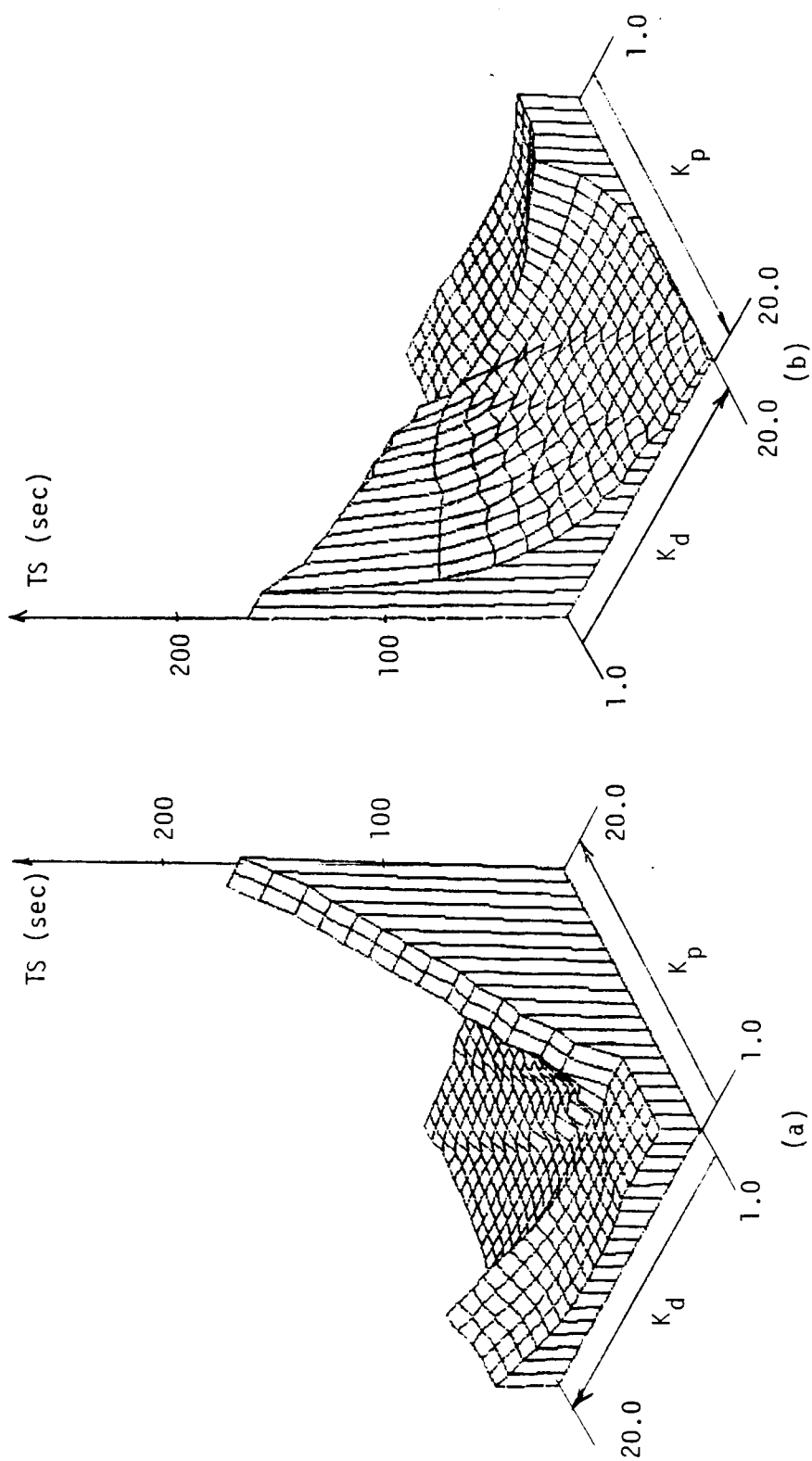


Figure III-10. Settling time for the PD-controller.

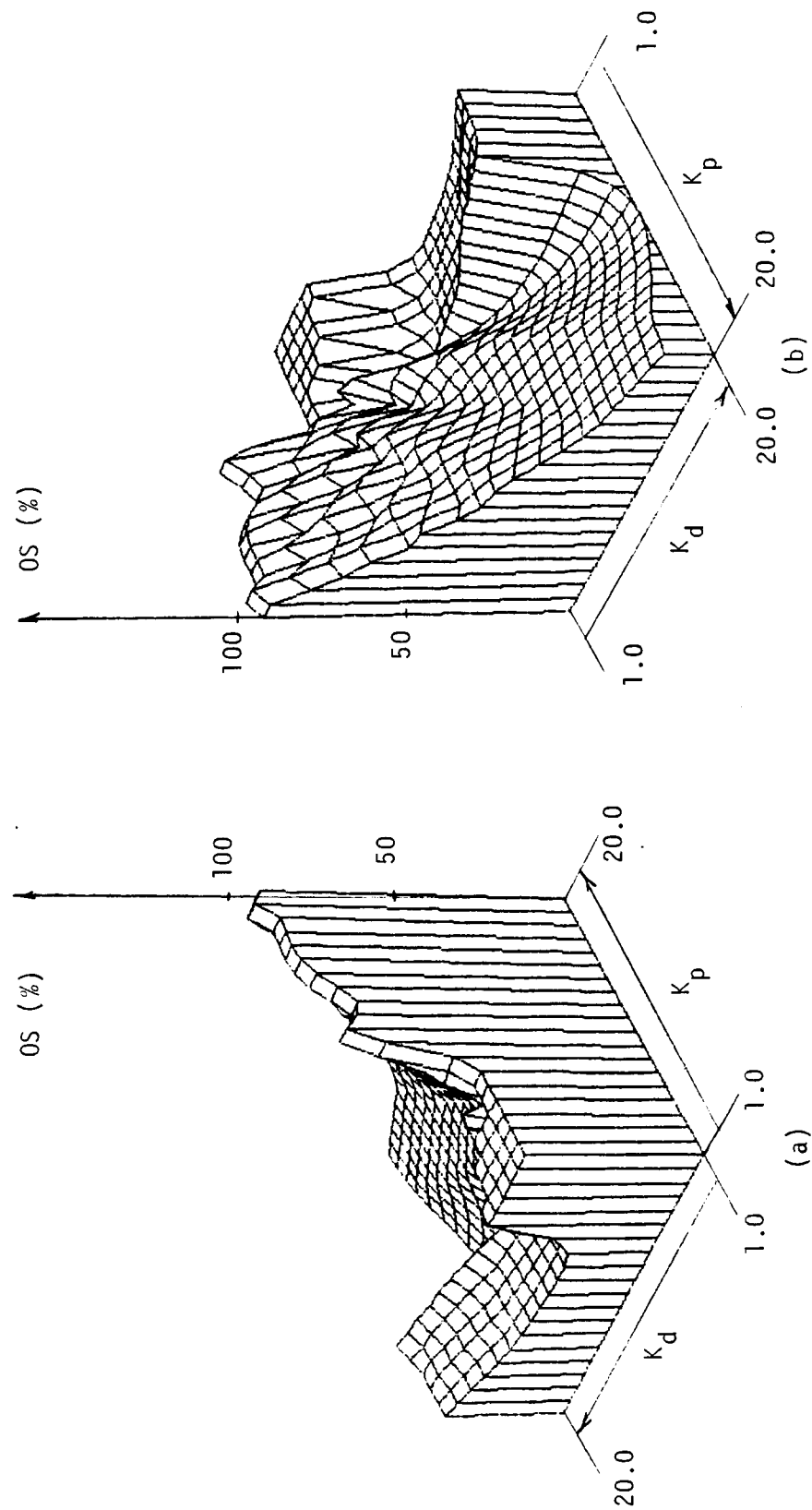


Figure III-11. Overshoot for the PD-controller.

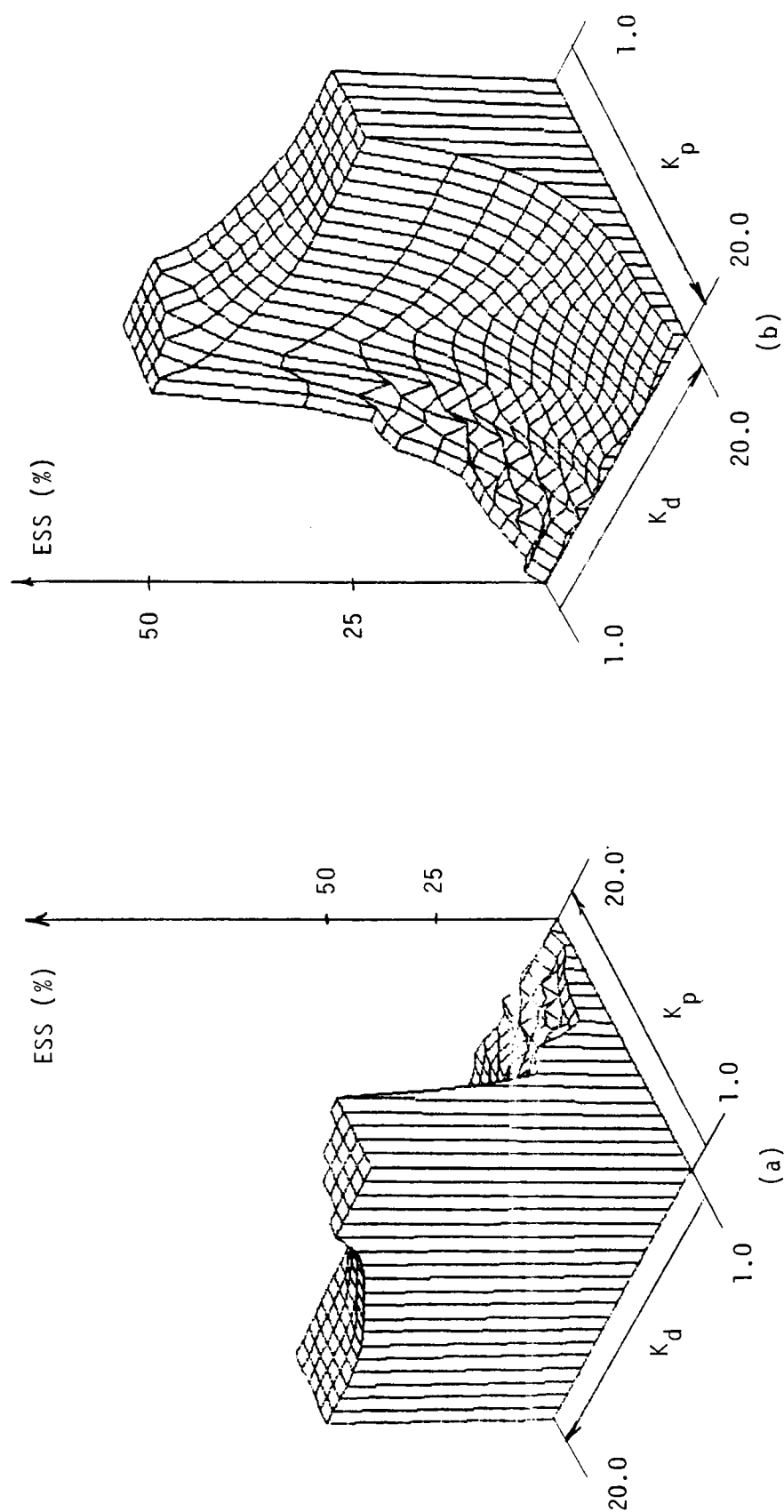


Figure III-12. Steady-state error for the PD-controller.

Concerning the steady-state error, K_d does not as strongly affect it as K_i does. It can be observed in Figure III-12 that the K_p -axis has a steeper slope than the K_d -axis. It is therefore indicated that the parameter K_d enables the system to respond more rapidly and reduces the overshoot but does not strongly reduce the steady-state error. This is as might be expected since K_d has no effect on a nonchanging (i.e., steady) output since it is derivative in nature. It is apparent from the known response of a linear second order plant with an analog PD-controller that the derivative control action adds damping to the system, thereby reducing OS and TS.

The effects on the response of the system of the parameters K_p , K_i , and K_d of the P-, PI-, and PD-controller have been discussed. Using this information, the PID-controller will be investigated in order to obtain a better system response. In order to properly investigate the PID-controller, a larger range of parameters than the PD-controller is used. The system performance will be discussed while letting K_p vary from 20 to 100 with increments of 20 and both K_i and K_d from 5 to 100 with increments of 5. Generally, it can be shown that the digital equivalent of a PID-controller enables a system to have the best response among digital controllers approximating analog control actions by choosing an appropriate combination of values of the three parameters.

Figure III-13 shows the stable region of the PID-controller. Each line on the figure separates the stable from the unstable region. The region of stability for each value of K_p is above and to the left

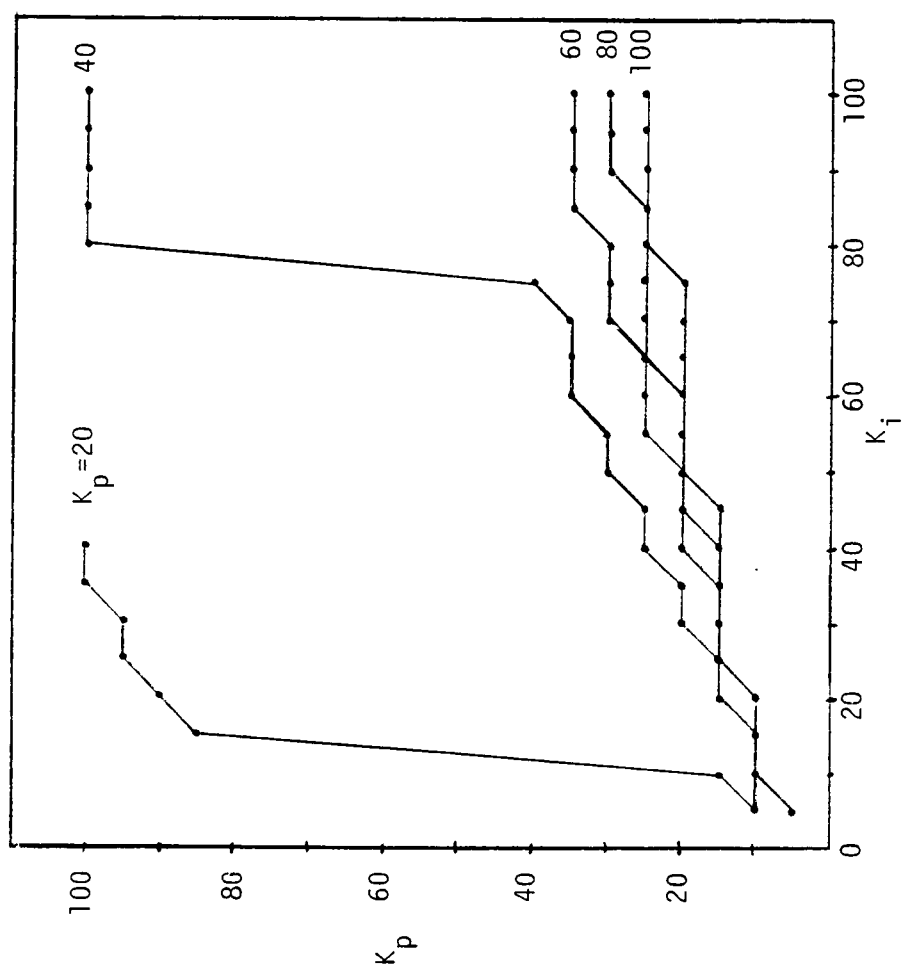


Figure III-13. The stability region for the PID-controller.

of its respective line. The stable region is enlarged for increasing values of K_p . From Figures III-14 to III-28 the X-axis presents K_i and the Y-axis presents K_d . When K_p is small, the ratio of K_d/K_i has to be large in order to make the system stable. This means that within these specified ranges of the three parameters, larger K_p enables the system to be stable for a wider range of K_i and K_d . As discussed before, for fixed K_p and increasing K_d/K_i , TS decreases. From Figures III-14 to III-18, it is observed that as K_p is increased without changing K_i and K_d , TS usually tends to decrease.

From Figures III-19 to III-23, it is observed that when K_p is fixed, OS decreases as K_d increases, although increasing K_i still makes OS increase. In other words, OS increases as K_d/K_i decreases. These facts indicate that K_d may force OS to be small, that is, as the ratio of K_d/K_i is increasing OS is decreasing. It can also be seen that as K_p is increased, OS is increased.

Concerning ESS, it should be true that ESS is more dependent on K_i than K_d although it is difficult to say that this is always true. Generally a PID-controller has a significantly smaller steady-state error than other controllers, but it is difficult to determine some kind of general rule to obtain the smallest ESS. From Figures III-24 to III-28, it may be observed that as K_p increases, ESS is small when the ratio of K_d/K_i is small. A root locus plot of a linear second order plant with an analog PID-controller shows that decreasing K_d/K_i adds damping to the system, reducing TS and OS while increasing K_i reduces damping, increasing TS and OS.

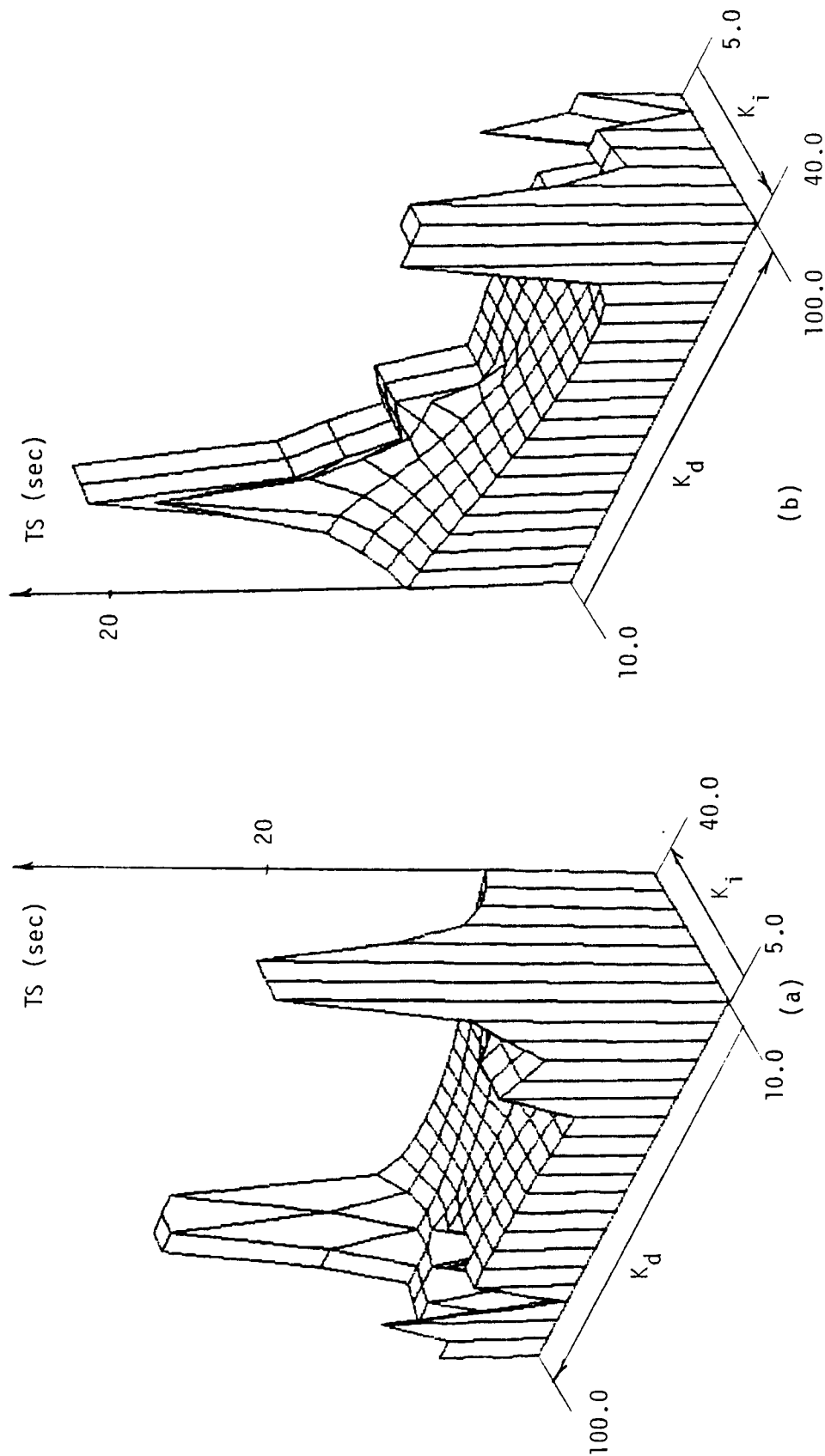


Figure III-14. Settling time for the PID-controller when $K_p = 20$.

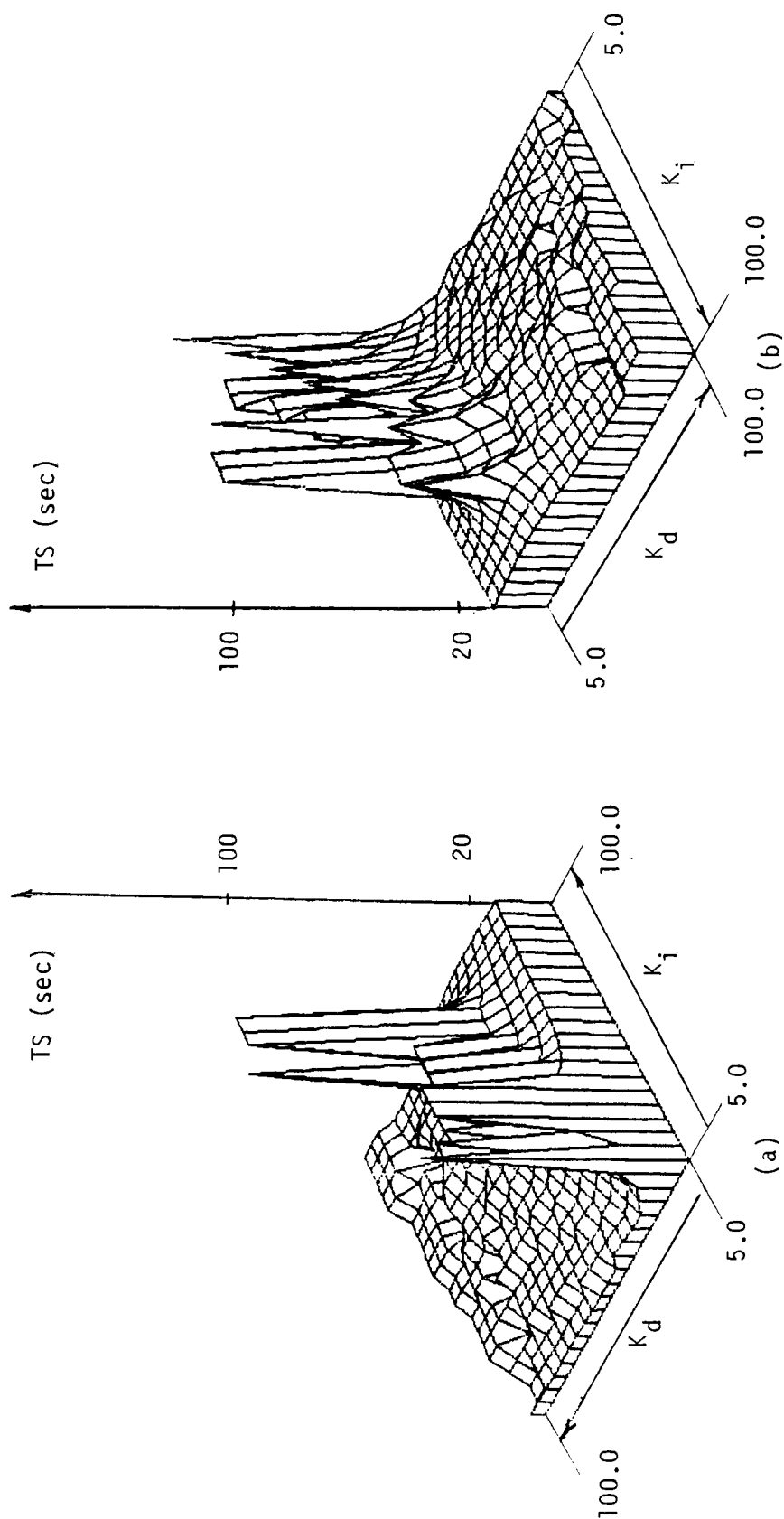


Figure III-15. Settling time for the PID-controller when $K_p = 40$.

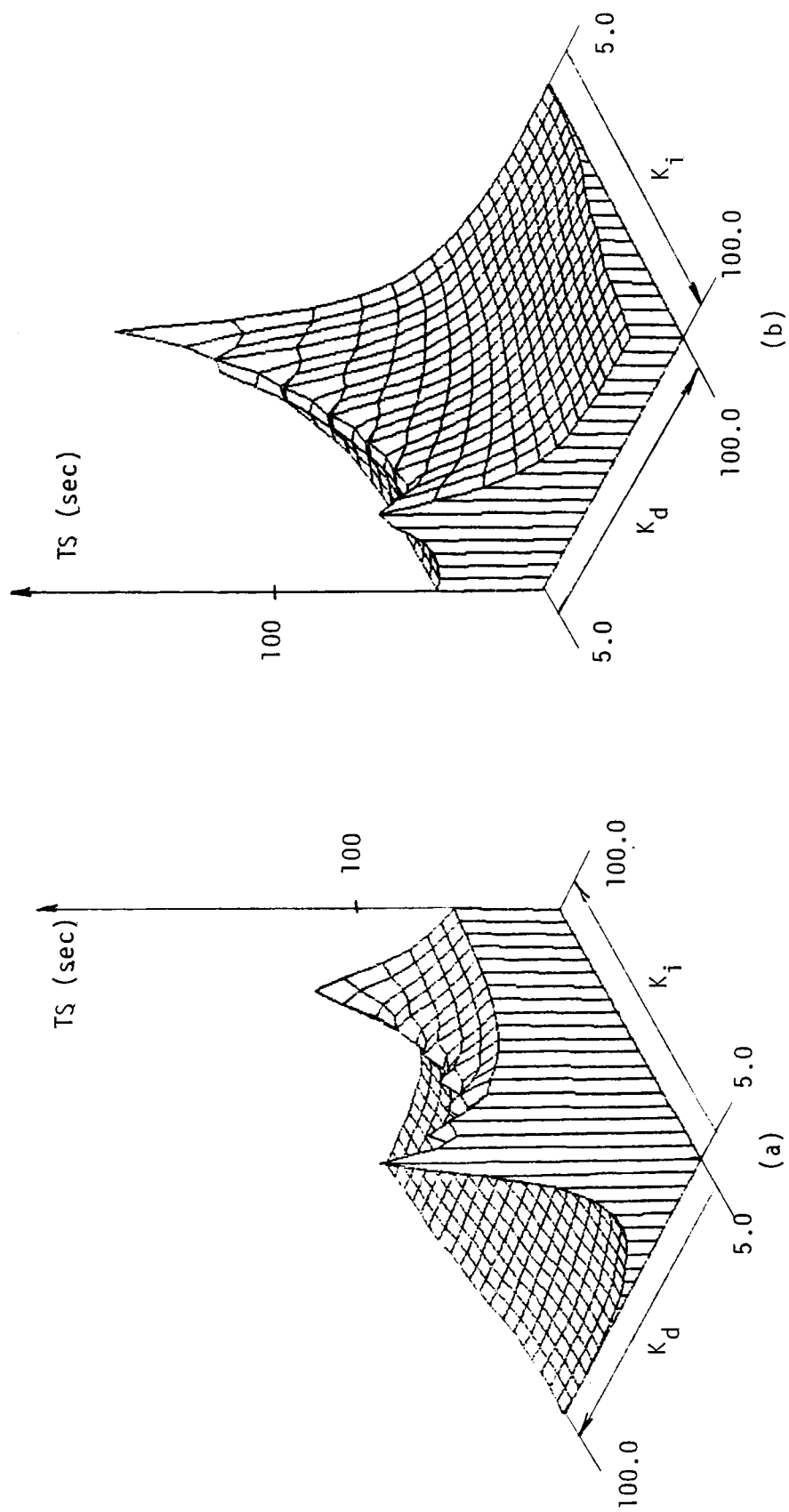


Figure III-16. Settling time for the PID-controller when $K_p = 60$.

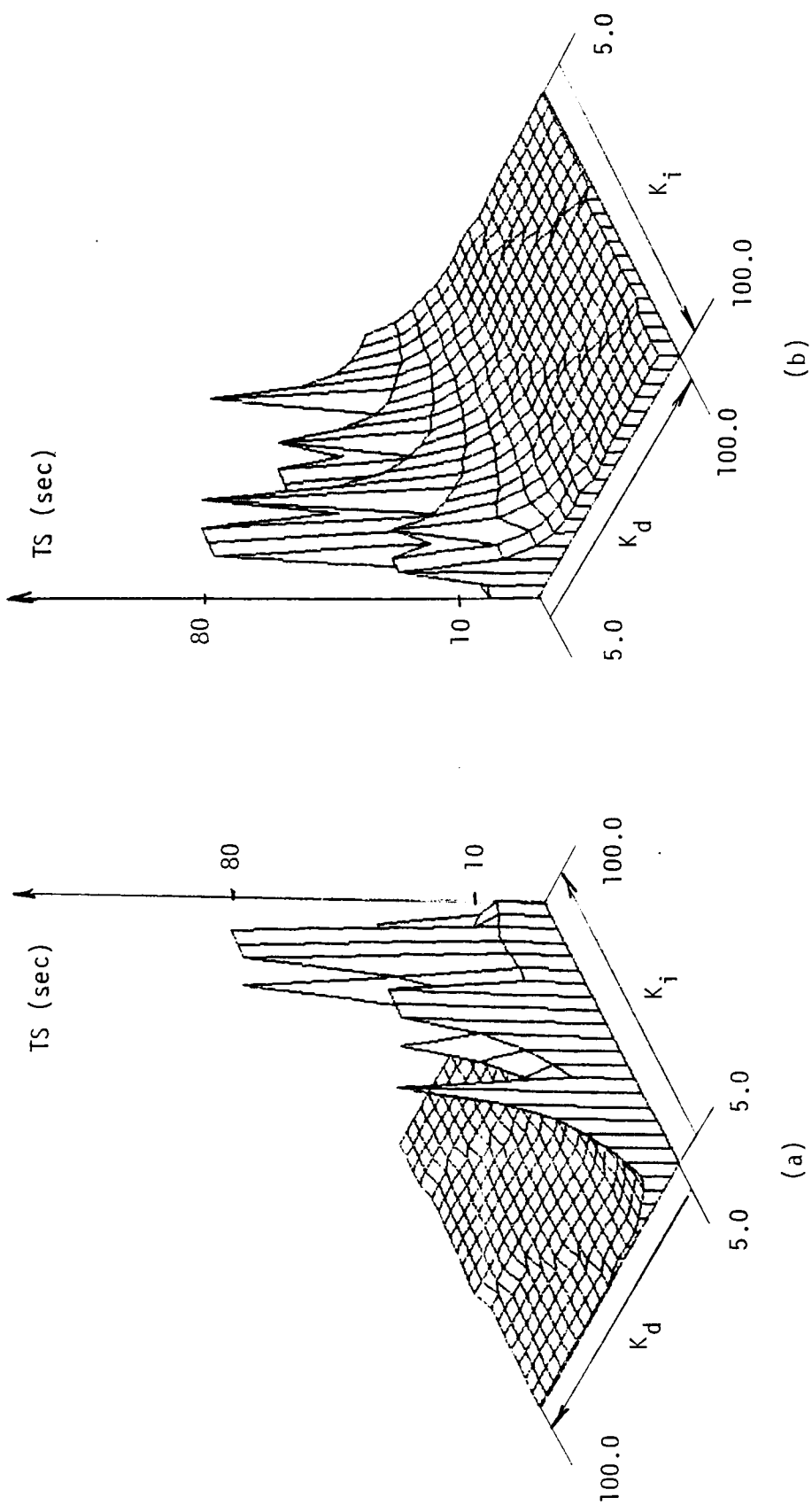


Figure III-17. Settling time for the PID-controller when $K_p = 80$.

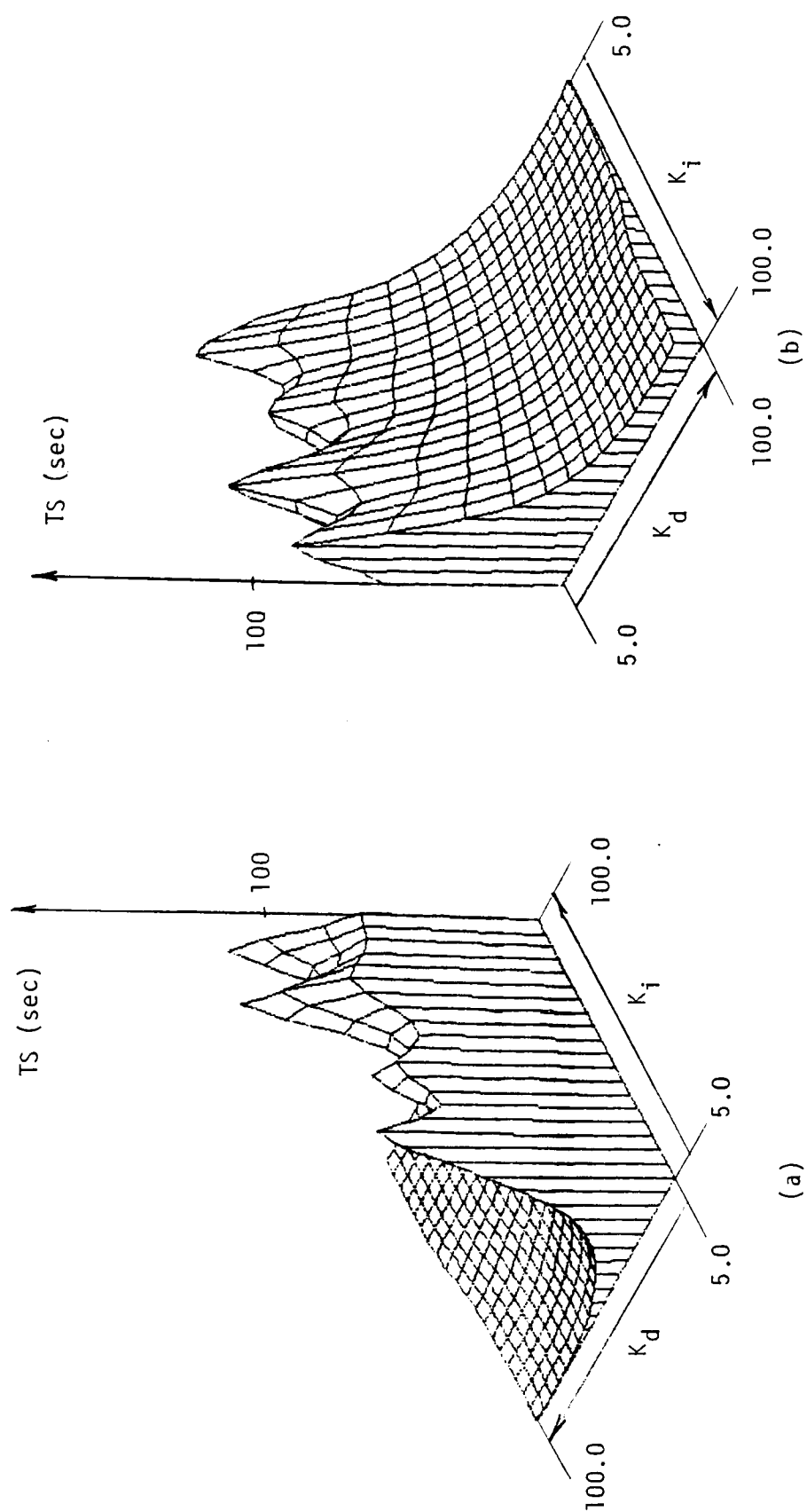


Figure III-18. Settling time for the PID-controller when $K_p = 100$.

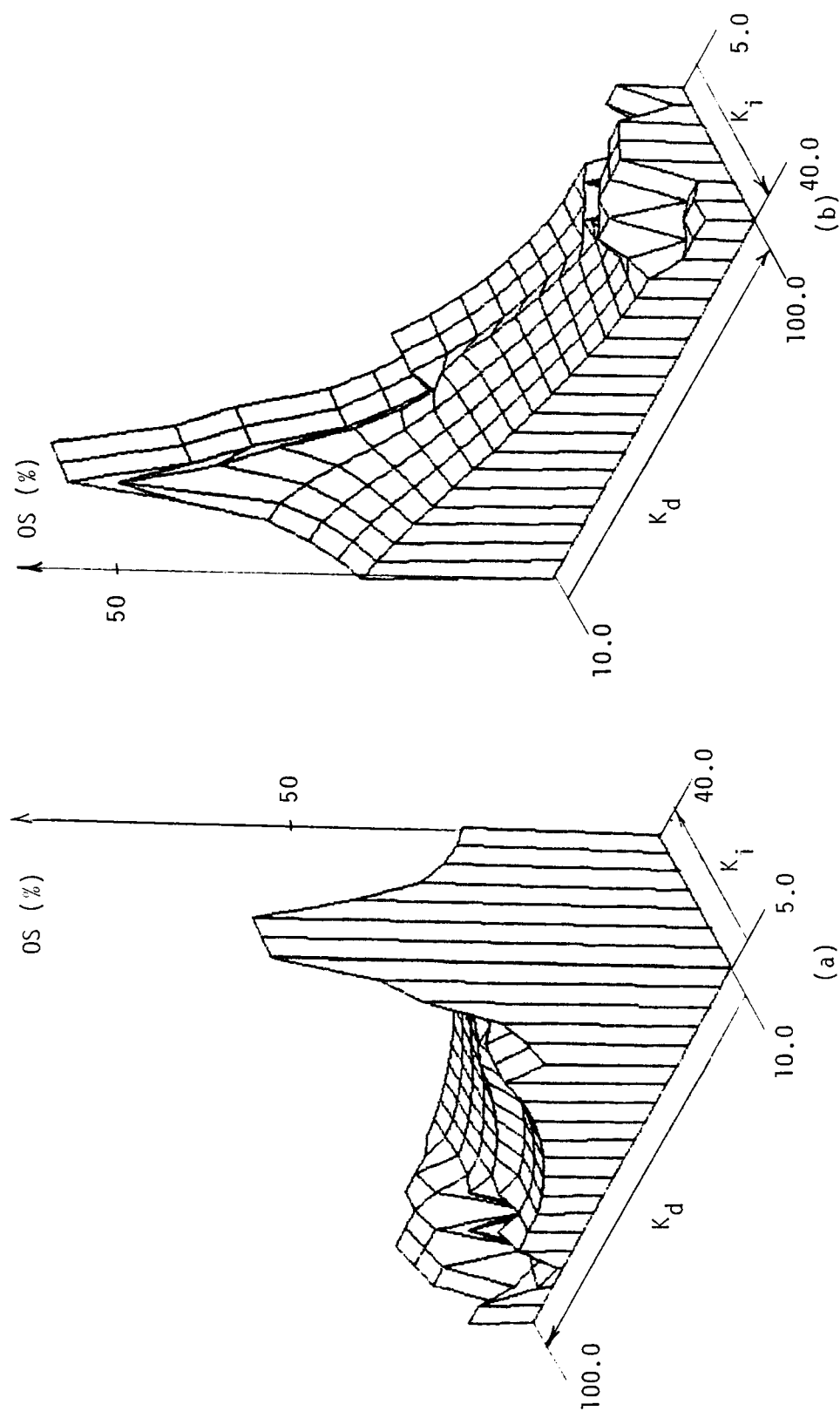


Figure III-19. Overshoot for the PID-controller when $K_p = 20$.

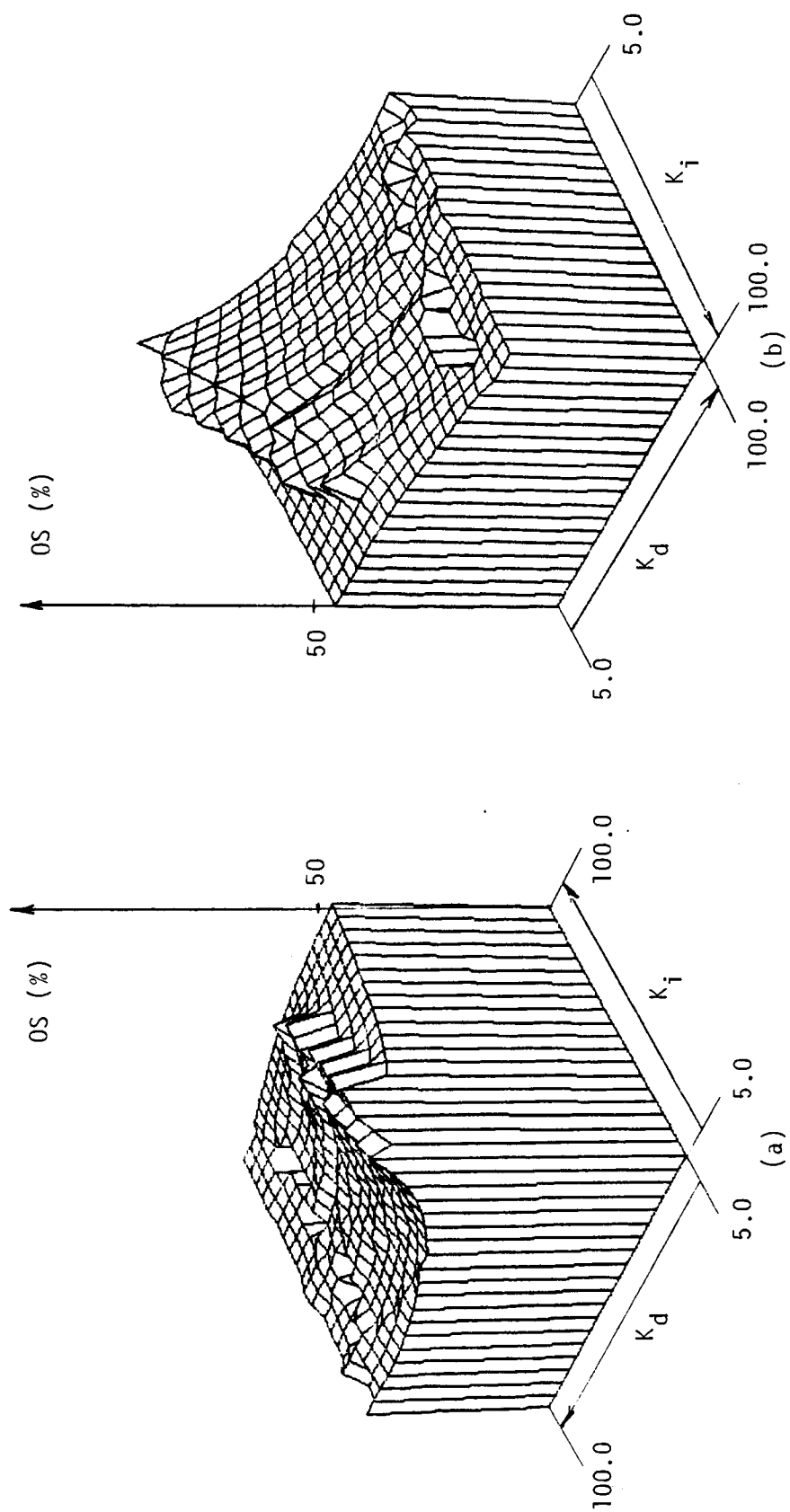


Figure III-20. Overshoot for the PID-controller when $K_p = 40$.

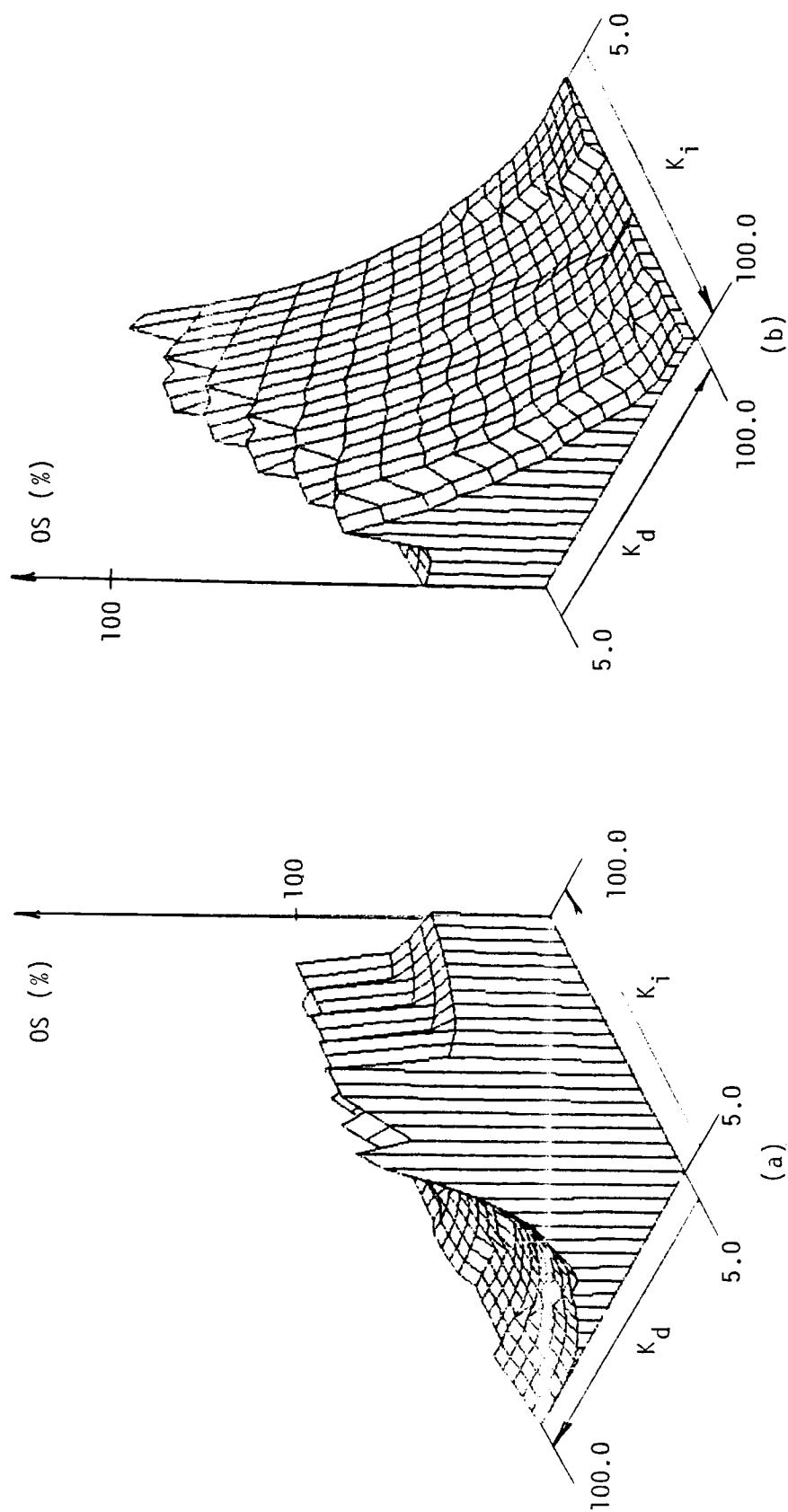


Figure III-21. Overshoot for the PID-controller when $K_p = 60$.

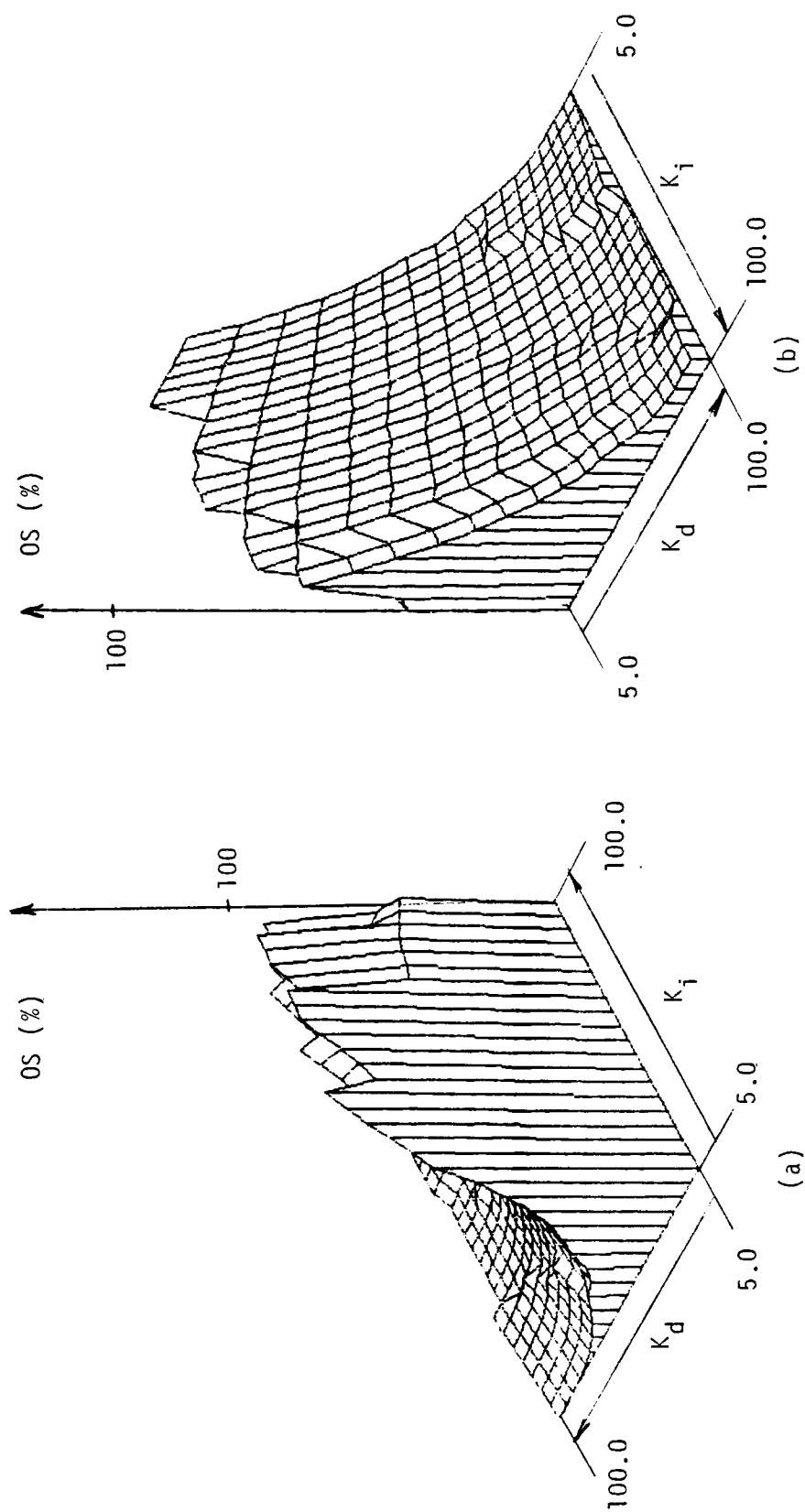


Figure III-22. Overshoot for the PID-controller when $K_p = 80$.

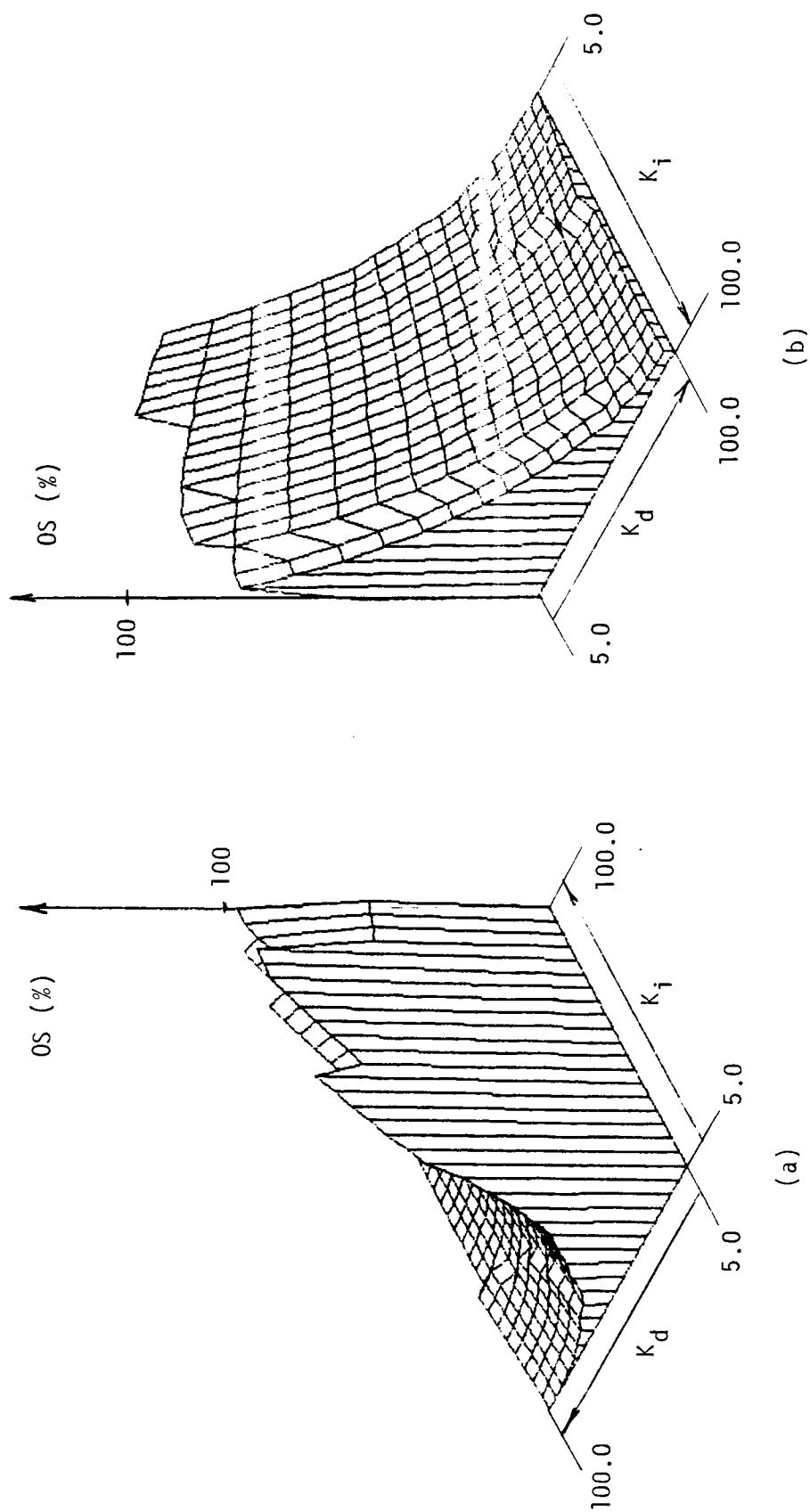


Figure III-23. Overshoot for the PID-controller when $K_p = 100$.

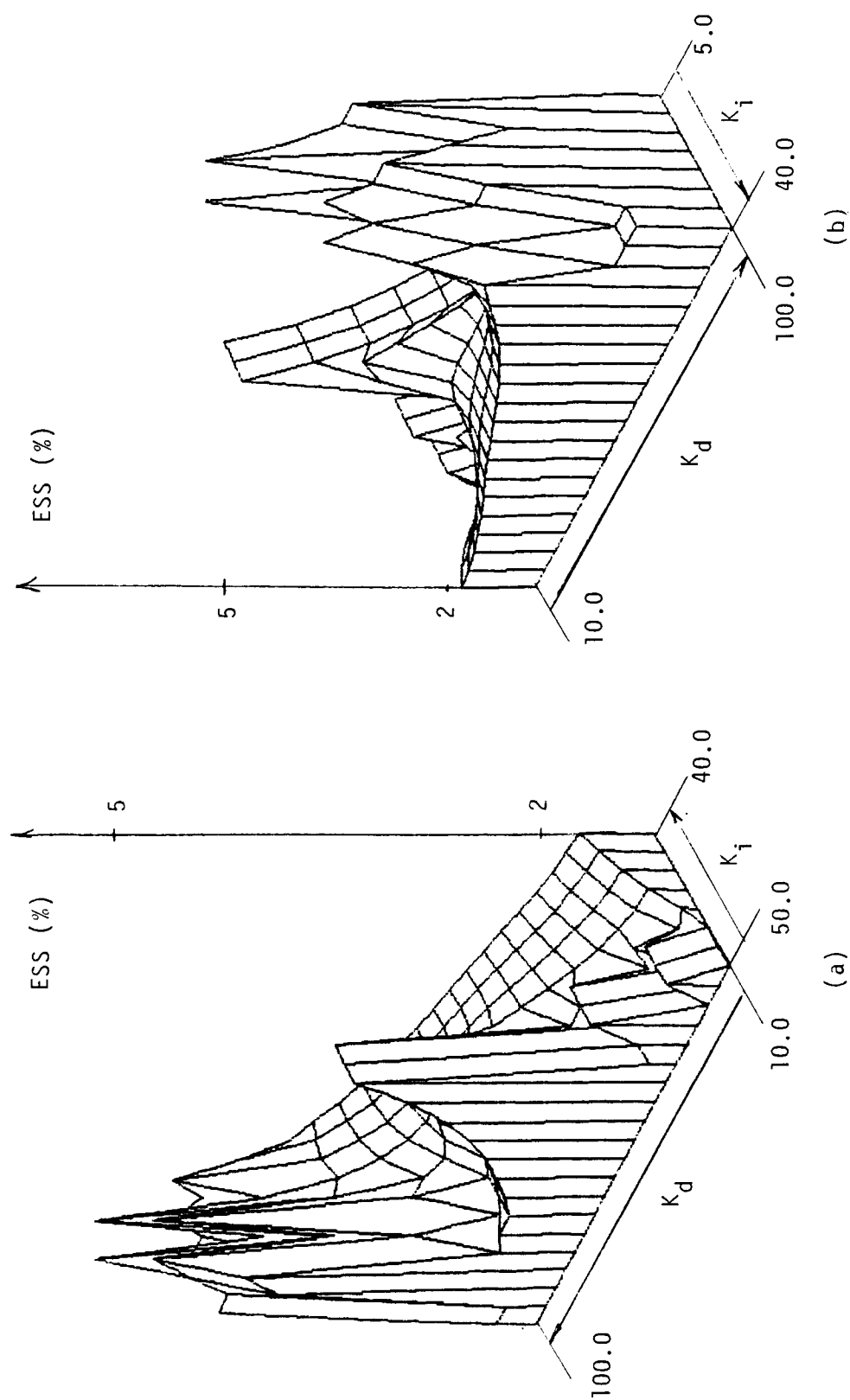


Figure III-24. Steady-state error for the PID-controller when $K_p = 20$.

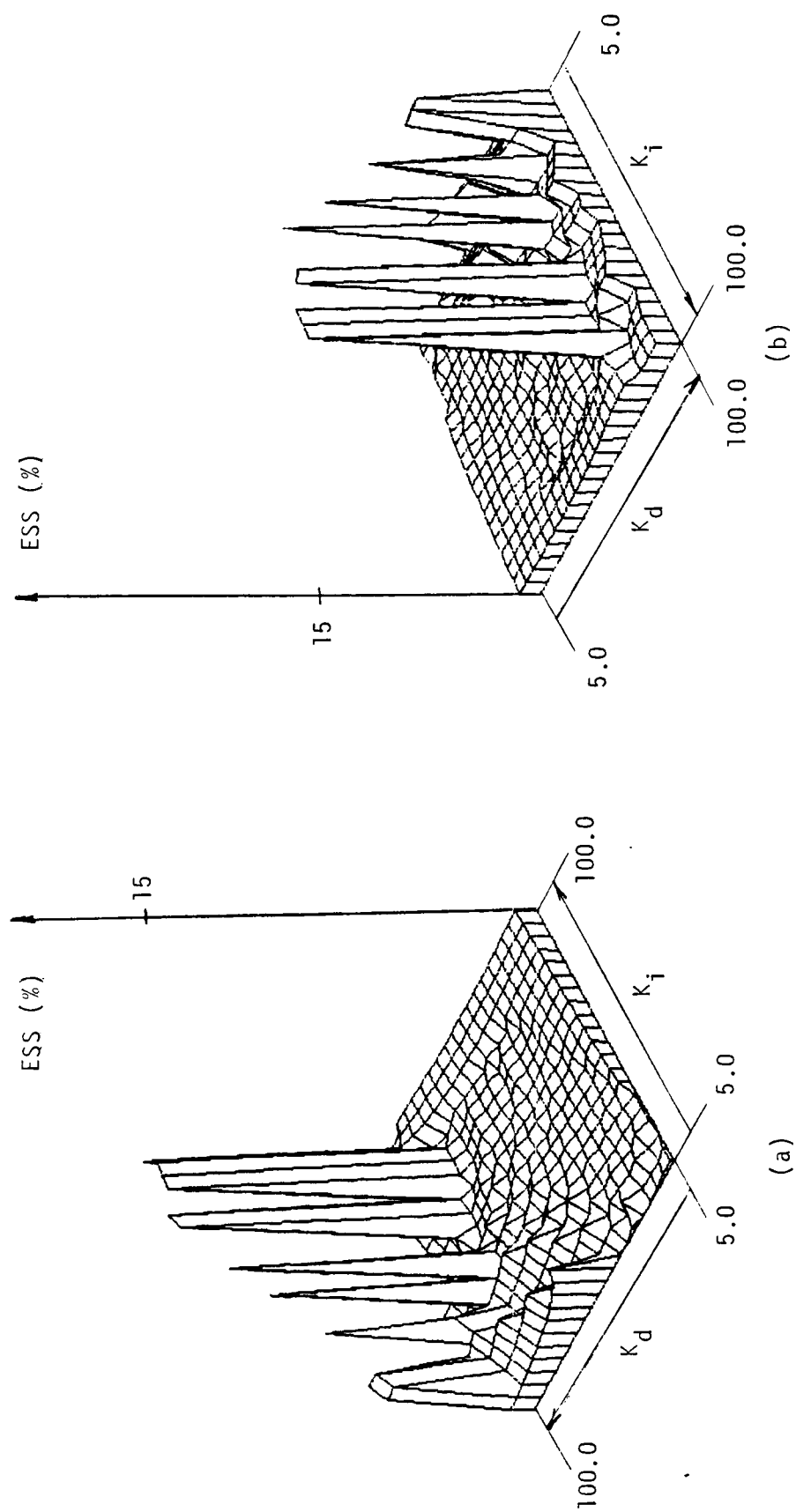


Figure III-25. Steady-state error for the PID-controller when $K_p = 40$.

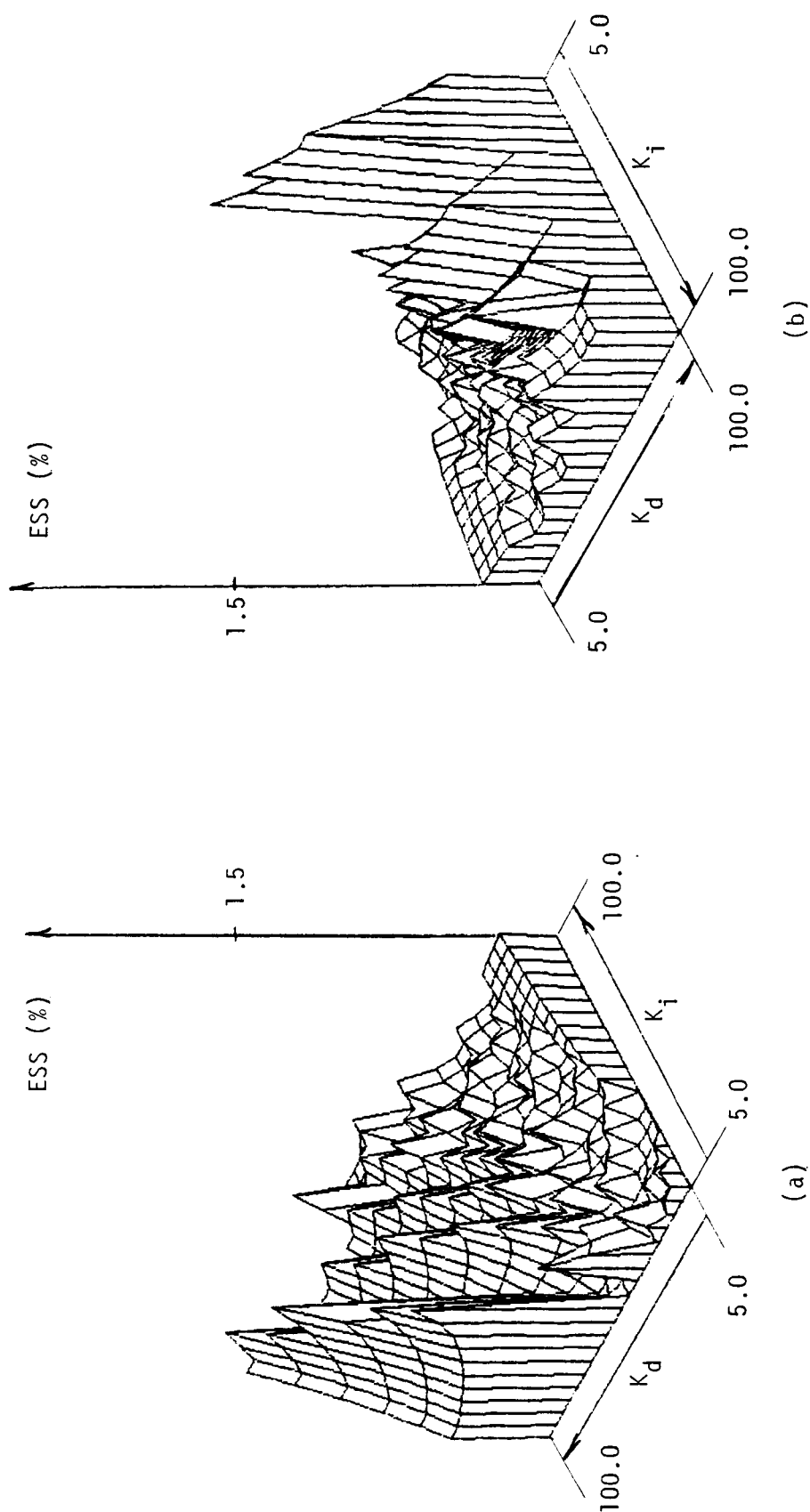


Figure III-26. Steady-state error for the PID-controller when $K_p = 60$.

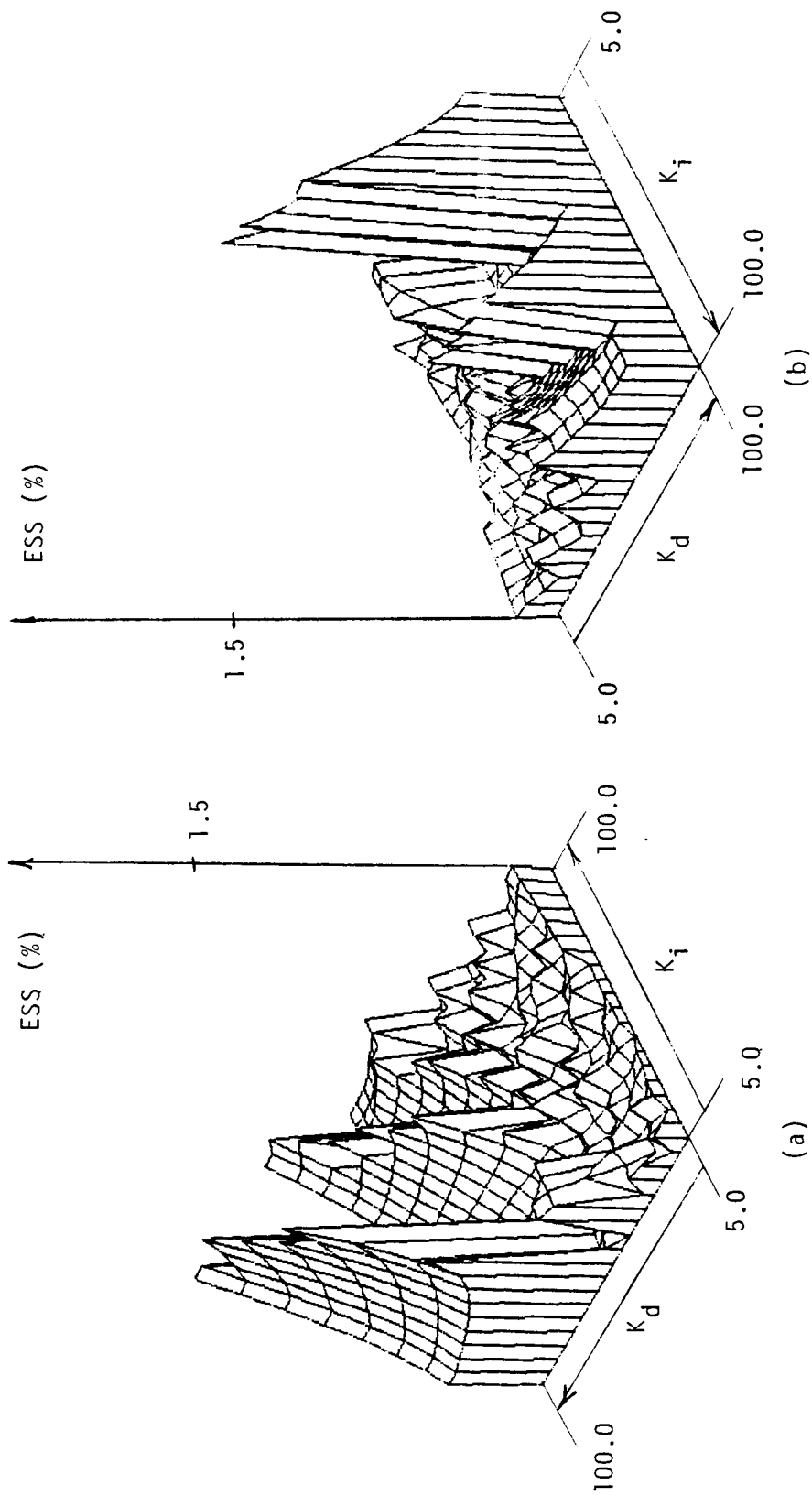


Figure III-27. Steady-state error for the PID-controller when $K_p = 80$.

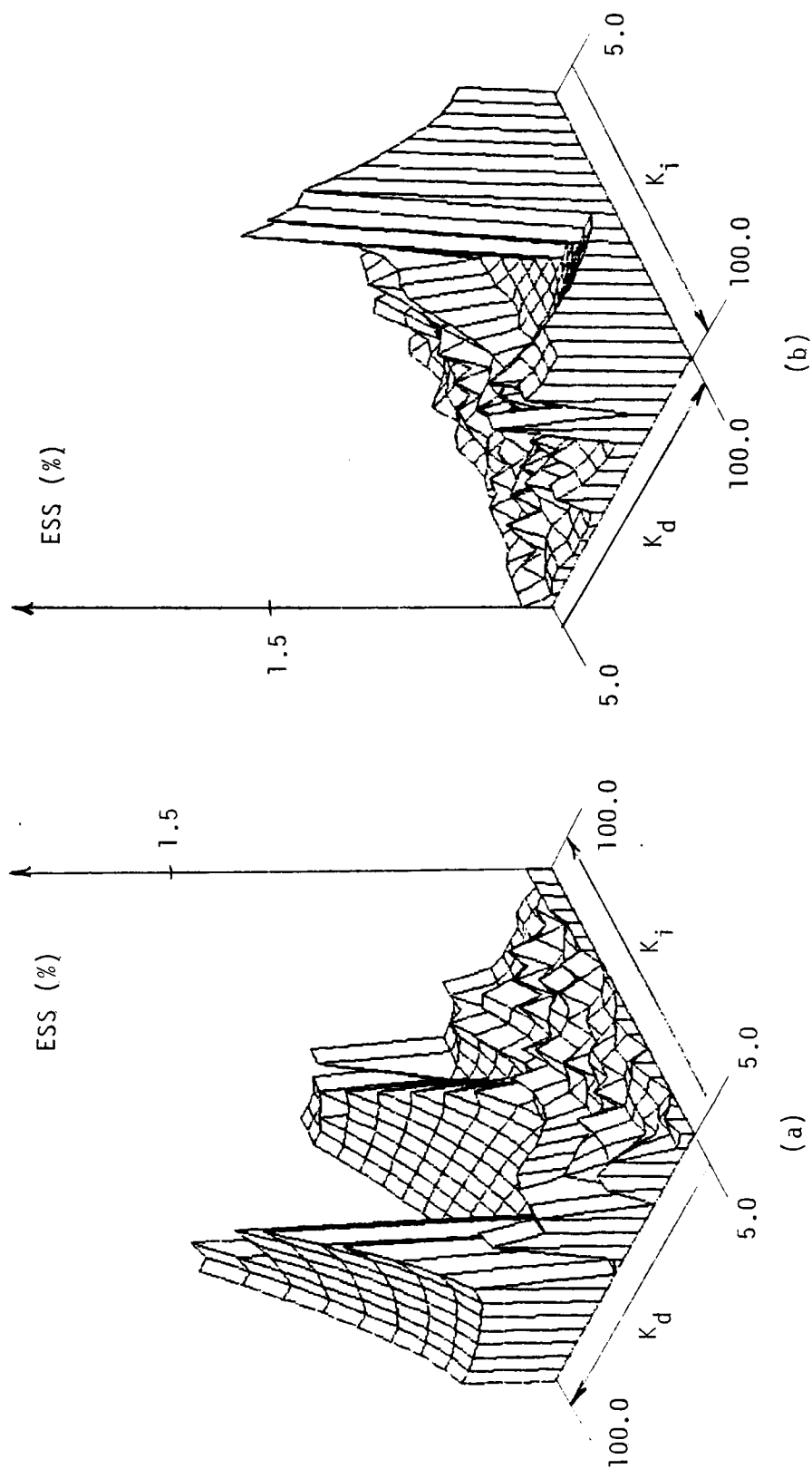


Figure III-28. Steady-state error for the PID-controller when $K_p = 100$.

Generally speaking, for fast response the ratio of K_d/K_i has to be large, while for small OS the ratio of K_d/K_i has to be large, and for small ESS, K_p and K_i have to be large. If one wants to determine good digital controller parameter values from this information, it can be suggested that one may first choose the value of K_p and then vary the ratio of K_d/K_i to obtain acceptable performance characteristics.

Another technique to determine the value of the parameters in designing a digital controller was presented in Chapter II. Thus far the output of the system has been determined by choosing the parameters of the digital controller, but the direct design method enables one to choose an algorithm for a specified output. Using this method, the desired output can be obtained from the known input by determining the algorithm of the digital controller. A technique to find the control algorithm of a digital controller, $D(z)$, with and without a disturbance input was presented in Chapter II by forcing the output to, arbitrarily, be $y(t) = 1 - e^{-t/\tau}$ where τ is the time constant of the system. In the absence of a disturbance input, the parameters K_8 , K_9 , and K_{10} can be found as a function of ω_n , T , and τ . The results are presented in Figure III-29 for $\tau = 1$ and $\tau = 10$. This figure also indicates that TS is 4.0 and 39.2 (seconds) for time constants of 1 and 10 seconds, respectively, and OS and ESS are both zero percent. The results with the disturbance input are exactly the same as without the disturbance input indicating that the direct design method can also control the disturbance input. These results

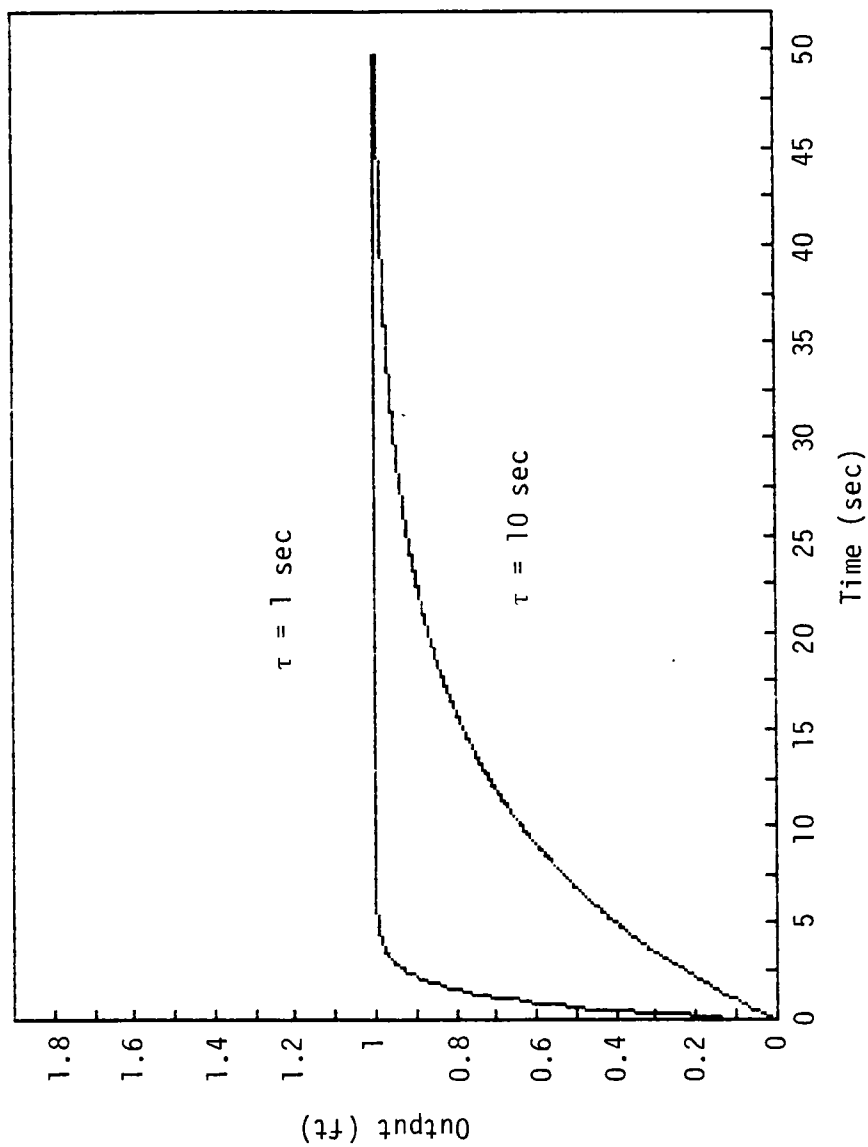


Figure III-29. System response using a direct design method.

show that it is possible to specify the desired response and by an inverse technique find the required control algorithm.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

This thesis presents information on the modeling, control, and design of a nonlinear discrete control system. Because of the nonlinearity arising from Coulomb friction, the closed-loop transfer function does not exist.

Two methods have been presented for the development of the closed-form solution of the system. It has been shown that the determination of the output is more complicated using classical control theory than it is using modern control theory. In the classical method the solutions can be obtained by alternating between two linear equations. In modern control theory, however, the output can be obtained easily from state equations without switching between equations due to the inclusion of a pseudo disturbance input for the Coulomb friction. Modern control theory also has a good basis for developing general algorithms that are suited for computer simulation.

For designing the values of the parameters in various controllers, the state-space method in modern control theory is more powerful and simpler than the classical method. The state diagram method provides a simple tool to assist in calculating the output of the system.

Among controllers, the PI-controller is better than the PD-controller in reducing the steady-state error, but the PD-controller

is able to reduce the maximum overshoot better than the PI-controller. In general it may be noted that in regard to the various parameters in controllers that K_i primarily provides action to reduce the steady-state error, where K_d generally provides an anticipatory action to make the system response more rapid and to reduce the overshoot in the response. Because of these reasons, the PID-controller generally has the best response of the system among controllers.

Comparing the response of a linear continuous system with the digital control of a nonlinear system, it was observed that the effects of the design parameters K_p , K_i , and K_d of the analog and digital controllers on linear and nonlinear plants, respectively, were qualitatively similar. This is not too surprising since if the sampling period of the sampled data system is small, the digital controller behaves similar to an analog controller. It should also be noted that linear viscous friction is a first order approximation to nonlinear Coulomb friction. It is important to note, however, that the design engineer may easily obtain qualitative information on the trends of system performance with the design parameters of digital controllers from linear theory for analog systems.

Additional areas which should be studied in more detail are the effects of the sampling period T on the nonlinear system performance as well as additional analysis of the response due to a disturbance input. In this thesis the sampling period was held constant although it is a well-known fact that the sampling period had a strong

influence on system performance. It is felt that this area should be explored in more detail.

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APPENDIXES

APPENDIX A

DETAILS FOR EQS. II-8, II-9, II-10, II-11, II-12, AND II-13

In this section the detail development of Eqs. II-8, II-9, II-10, II-11, II-12, and II-13 in Chapter II will be presented. Equation II-8 can be obtained through the use of the z-transformation. That is,

$$G_{ho} G_1(z) = \mathcal{Z} [G_{ho}(s) G_1(s)]$$

where

$$G_{ho}(s) = \frac{1 - e^{-Ts}}{s}$$

and

$$G_1(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}.$$

Calculating $G_{ho} G_1(z)$ yields

$$\begin{aligned} G_{ho} G_1(z) &= \mathcal{Z} \left(\frac{1 - e^{-Ts}}{s} \frac{\omega_n^2}{s^2 + \omega_n^2} \right) \\ &= \frac{z - 1}{z} \mathcal{Z} \left(\frac{\omega_n^2}{s(s^2 + \omega_n^2)} \right). \end{aligned}$$

Performing a partial fraction expansion of the function of s yields

$$G_{ho} G_1(z) = \frac{z - 1}{z} \mathcal{Z} \left(\frac{1}{s} - \frac{s}{s^2 + \omega_n^2} \right)$$

Finding the z-transform from Table 1 gives

$$\begin{aligned}
 G_{ho} G_1(z) &= \frac{z-1}{z} \left(\frac{z}{z-1} - \frac{z(z - \cos \omega_n T)}{z^2 - 2z \cos \omega_n T + 1} \right) \\
 &= 1 - \frac{(z-1)(z - \cos \omega_n T)}{z^2 - 2z \cos \omega_n T + 1} \\
 &= \frac{z^2 - 2z \cos \omega_n T + 1 - (z^2 - z \cos \omega_n T - z + \cos \omega_n T)}{z^2 - 2z \cos \omega_n T + 1} \\
 &= \frac{z^2 - 2z \cos \omega_n T + 1 - z^2 + z \cos \omega_n T + z - \cos \omega_n T}{z^2 - 2z \cos \omega_n T + 1} \\
 &= \frac{z - z \cos \omega_n T + 1 - \cos \omega_n T}{z^2 - 2z \cos \omega_n T + 1} \\
 &= \frac{z(1 - \cos \omega_n T) + 1 - \cos \omega_n T}{z^2 - 2z \cos \omega_n T + 1},
 \end{aligned}$$

which may be expressed as

$$G_{ho} G_1(z) = \frac{(1 - \cos \omega_n T)(z + 1)}{z^2 - 2z \cos \omega_n T + 1}.$$

Equation II-9 can be obtained by taking the z-transform of $G_2(s)$.

Table 1. Table of Laplace transforms and z-transforms

Laplace Transform	Time Function	z-Transform
$F(s)$	$f(t)$	$F(z)$
1	$\delta(t)$	1
e^{-kTs}	$\delta(t - kT)$	z^{-k}
$\frac{1}{s}$	$U_s(t)$	$\frac{z}{z - 1}$
$\frac{\omega_n}{s^2 + \omega_n^2}$	$\sin \omega_n t$	$\frac{z \sin \omega_n T}{z^2 - 2z \cos \omega_n T + 1}$
$\frac{s}{s^2 + \omega_n^2}$	$\cos \omega_n t$	$\frac{z(z - \cos \omega_n T)}{z^2 - 2z \cos \omega_n T + 1}$

$$\begin{aligned}
 G_2(z) &= \mathcal{Z} [G_2(s)] \\
 &= \mathcal{Z} \left(\frac{As + B}{s^2 + \omega_n^2} \right) \\
 &= \mathcal{Z} \left(\frac{As}{s^2 + \omega_n^2} + \frac{B}{s^2 + \omega_n^2} \right) \\
 &= \mathcal{Z} \left(\frac{As}{s^2 + \omega_n^2} \right) + \frac{B}{\omega_n} \mathcal{Z} \left(\frac{\omega_n}{s^2 + \omega_n^2} \right) .
 \end{aligned}$$

Finding the z-transform from Table 1 gives

$$\begin{aligned}
 G_2(z) &= A \frac{z(z - \cos \omega_n T)}{z^2 - 2z \cos \omega_n T + 1} + \frac{B}{\omega_n} \frac{z \sin \omega_n T}{z^2 - 2z \cos \omega_n T + 1} \\
 &= \frac{Az(z - \cos \omega_n T) + (B/\omega_n)z \sin \omega_n T}{z^2 - 2z \cos \omega_n T + 1} .
 \end{aligned}$$

Equation II-10 can be obtained from $G_3(s)$ as

$$\begin{aligned}
 G_3(z) &= \mathcal{Z} [G_3(s)] \\
 &= \mathcal{Z} \left(\frac{-\mu g \operatorname{sgn}(\dot{y})}{s(s^2 + \omega_n^2)} \right) .
 \end{aligned}$$

Factoring out the constants and performing a partial fraction expansion yields

$$G_3(z) = \frac{-\mu g \operatorname{sgn}(\dot{y})}{\omega_n^2} \mathcal{Z} \left(\frac{1}{s} - \frac{s}{s^2 + \omega_n^2} \right) .$$

$$\begin{aligned}
G_3(z) &= \frac{-\mu g \operatorname{sgn}(\dot{y})}{\omega_n^2} \left(\frac{z}{z-1} - \frac{z(z - \cos \omega_n T)}{z^2 - 2z \cos \omega_n T + 1} \right) \\
&= \frac{-\mu g \operatorname{sgn}(\dot{y})}{\omega_n^2} \frac{z(z^2 - 2z \cos \omega_n T + 1) - z^2 + z \cos \omega_n T + z - \cos \omega_n T}{(z-1)(z^2 - 2z \cos \omega_n T + 1)} \\
&= \frac{-\mu g \operatorname{sgn}(\dot{y})}{\omega_n^2} \frac{(1 - \cos \omega_n T)z(z+1)}{(z-1)(z^2 - 2z \cos \omega_n T + 1)}
\end{aligned}$$

Substituting Eqs. II-8, II-9, II-10, and II-12 into Eq. II-7, II-13 is obtained as

$$\begin{aligned}
Y(z) &= \frac{\frac{(1 - \cos \omega_n T)(z+1)D(z)}{z^2 - 2z \cos \omega_n T + 1}}{1 + \frac{(1 - \cos \omega_n T)(z+1)D(z)}{z^2 - 2z \cos \omega_n T + 1}} \cdot \frac{z}{z-1} \\
&\quad + \frac{\frac{Az(z - \cos \omega_n T) + (B/\omega_n)z \sin \omega_n T}{z^2 - 2z \cos \omega_n T + 1}}{1 + \frac{(1 - \cos \omega_n T)(z+1)D(z)}{z^2 - 2z \cos \omega_n T + 1}} \\
&\quad + \frac{\frac{-\mu g \operatorname{sgn}(\dot{y})(1 - \cos \omega_n T)z(z+1)}{\omega_n^2 (z-1)(z^2 - 2z \cos \omega_n T + 1)}}{1 + \frac{(1 - \cos \omega_n T)(z+1)D(z)}{z^2 - 2z \cos \omega_n T + 1}}.
\end{aligned}$$

Rearranging this equation yields

$$\begin{aligned}
 Y(z) = & \frac{(1 - \cos \omega_n T) z(z + 1) D(z)}{(z - 1)[z^2 - 2z \cos \omega_n T + 1 + (1 - \cos \omega_n T)(z + 1)D(z)]} \\
 & + \frac{Az(z - \cos \omega_n T) + (B/\omega_n)z \sin \omega_n T}{z^2 - 2z \cos \omega_n T + 1 + (1 - \cos \omega_n T)(z + 1)D(z)} \\
 & + \frac{-\mu g \operatorname{sgn}(\dot{y})}{\omega_n^2} \frac{(1 - \cos \omega_n T) z(z + 1)}{(z - 1)[z^2 - 2z \cos \omega_n T + 1 + (1 - \cos \omega_n T)(z + 1)D(z)]} .
 \end{aligned}$$

APPENDIX B

DETAILS FOR EQS. II-18 AND II-19

This section shows how to obtain Eqs. II-18 and II-19 from Eqs. II-16 and II-17. From Eq. II-16

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \omega_n^2 & \omega_n^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} .$$

Rewriting this equation in the general form of the state space equation yields

$$\dot{x}(t) = A x(t) + B u(t) . \quad (B-1)$$

For a typical interval, the last differential equation can be evaluated at t_{k+1} as

$$x(t_{k+1}) = \Phi(t_{k+1}, t_k) x(t_k) + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau) B(\tau) d\tau u(t_k) \quad (B-2)$$

where

$$\Phi(t_{k+1}, t_k) = e^{A(t_k) \Delta t_k}$$

and

$$\Delta t_k = t_{k+1} - t_k .$$

Because Eq. B-1 has the constant coefficient matrix A and B , Eq. B-2 can be simply rewritten as

$$x(t_{k+1}) = \Phi(t_{k+1}, t_k) x(t_k) + B_1(t_k) u(t_k) \quad (B-3)$$

where

$$B_1(t_k) = [\Phi(t_{k+1}, t_k) - I] A^{-1} B .$$

$\Phi(t_{k+1}, t_k)$ can be obtained by finding the transition matrix $\Phi(t, 0)$ as

$$\begin{aligned} \Phi(t, 0) &= \mathcal{L}^{-1} \{ [sI - A^{-1}] \} \\ &= \mathcal{L}^{-1} \left(\begin{bmatrix} s & -1 \\ \omega_n^2 & s \end{bmatrix}^{-1} \right) \\ &= \mathcal{L}^{-1} \left(\begin{bmatrix} \frac{s}{s^2 + \omega_n^2} & \frac{1}{s^2 + \omega_n^2} \\ \frac{-\omega_n^2}{s^2 + \omega_n^2} & \frac{s}{s^2 + \omega_n^2} \end{bmatrix} \right) . \end{aligned}$$

Taking the inverse Laplace transform, using Table 1 (p. 95), gives

$$\Phi(t, 0) = \begin{bmatrix} \cos \omega_n t & \frac{1}{\omega_n} \sin \omega_n t \\ -\omega_n \sin \omega_n t & \cos \omega_n t \end{bmatrix} . \quad (B-4)$$

Defining $T = \Delta t_k = t_{k+1} - t_k$ and by using methods of computing the transition matrix yields

$$\begin{aligned} \Phi(t_{k+1}, t_k) &= \Phi(t_{k+1} - t_k, 0) \\ &= \Phi(T, 0) . \end{aligned}$$

From Eq. B-4, changing t to T gives

$$\begin{aligned}\Phi(T, 0) &= \Phi(t_{k+1}, t_k) \\ &= \begin{pmatrix} \cos \omega_n T & \frac{1}{\omega_n} \sin \omega_n T \\ -\omega_n \sin \omega_n T & \cos \omega_n T \end{pmatrix}.\end{aligned}$$

$B_1(t_k)$ can be found as

$$\begin{aligned}B_1(t_k) &= [\Phi(t_{k+1}, t_k) - I] A^{-1} B \\ &= \begin{pmatrix} \cos \omega_n T - 1 & \frac{1}{\omega_n} \sin \omega_n T \\ -\omega_n \sin \omega_n T & \cos \omega_n T - 1 \end{pmatrix} \frac{1}{\omega_n^2} \begin{pmatrix} 0 & -1 \\ \omega_n^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \omega_n^2 & \omega_n^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \cos \omega_n T & 1 - \cos \omega_n T \\ \omega_n \sin \omega_n T & \omega_n \sin \omega_n T \end{pmatrix}.\end{aligned}$$

The discrete-time state equation is therefore

$$X(t_{k+1}) = \begin{pmatrix} \cos \omega_n T & \frac{1}{\omega_n} \sin \omega_n T \\ -\omega_n \sin \omega_n T & \cos \omega_n T \end{pmatrix} X(t_k) + \begin{pmatrix} 1 - \cos \omega_n T & 1 - \cos \omega_n T \\ \omega_n \sin \omega_n T & \omega_n \sin \omega_n T \end{pmatrix} U(t_k).$$

By notating $X(t_{k+1}) = X(k+1)T = X(k+1)$, the last equation becomes

$$X(k+1) = \begin{pmatrix} \cos \omega_n T & \frac{1}{\omega_n} \sin \omega_n T \\ -\omega_n \sin \omega_n T & \cos \omega_n T \end{pmatrix} X(k) + \begin{pmatrix} 1 - \cos \omega_n T & 1 - \cos \omega_n T \\ \omega_n \sin \omega_n T & \omega_n \sin \omega_n T \end{pmatrix} U(k).$$

APPENDIX C

DETAILS FOR EQ. II-26

This section shows how to obtain Eq. II-26 from Eq. II-18.

From Eq. II-18, solving for $X_1(k+1)$ and $X_2(k+1)$ gives

$$\begin{aligned} X_1(k+1) = & \cos \omega_n T X_1(k) + \frac{1}{\omega_n} \sin \omega_n T X_2(k) + (1 - \cos \omega_n T) U_1(k) \\ & + (1 - \cos \omega_n T) U_2(k) \end{aligned} \quad (C-1)$$

and

$$\begin{aligned} X_2(k+1) = & -\omega_n \sin \omega_n T X_1(k) + \cos \omega_n T X_2(k) + \omega_n \sin \omega_n T U_1(k) \\ & + \omega_n \sin \omega_n T U_2(k) . \end{aligned} \quad (C-2)$$

Substituting Eq. II-25 into Eqs. C-1 and C-2 yields

$$\begin{aligned} X_1(k+1) = & (\cos \omega_n T - K_p + K_p \cos \omega_n T) X_1(k) + \frac{1}{\omega_n} \sin \omega_n T X_2(k) \\ & + K_p (1 - \cos \omega_n T) R(k) + (1 - \cos \omega_n T) U_2(k) \end{aligned}$$

and

$$\begin{aligned} X_2(k+1) = & (-\omega_n \sin \omega_n T - K_p \omega_n \sin \omega_n T) X_1(k) + \cos \omega_n T X_2(k) \\ & + K_p \omega_n \sin \omega_n T R(k) + \omega_n \sin \omega_n T U_2(k) . \end{aligned}$$

Rearranging the above equations in the matrix form gives

$$\begin{aligned}
 \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} &= \begin{pmatrix} \cos \omega_n T - K_p + K_p \cos \omega_n T & \frac{1}{\omega_n} \sin \omega_n T \\ -(1 + K_p) \omega_n \sin \omega_n T & \cos \omega_n T \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \\
 &+ \begin{pmatrix} K_p(1 - \cos \omega_n T) & 1 - \cos \omega_n T \\ K_p \omega_n \sin \omega_n T & \omega_n \sin \omega_n T \end{pmatrix} \begin{pmatrix} R(k) \\ U_2(k) \end{pmatrix}.
 \end{aligned}$$

APPENDIX D

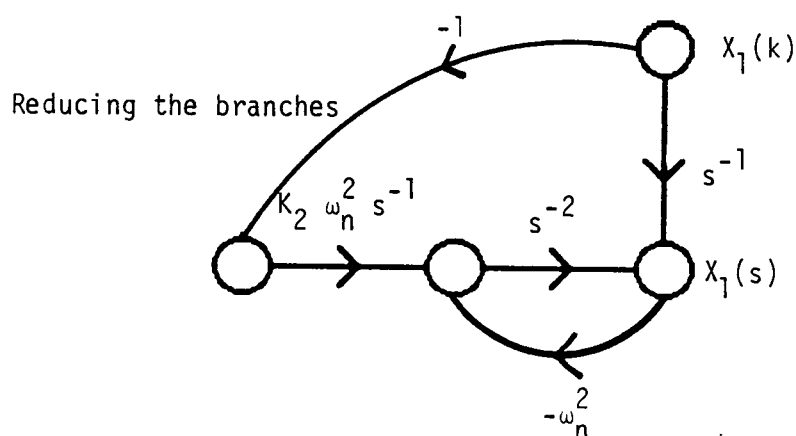
DETAILS FOR EQ. II-35

This section shows how to obtain the discrete state equation from the state diagram for the I-controller.

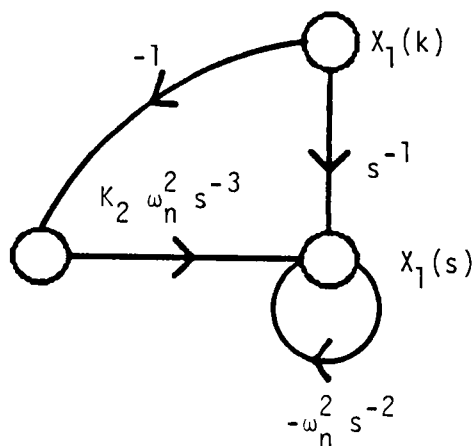
Applying Mason's gain formula (9) to Figure II-9 (p. 28) with $X_1(s)$ and $X_2(s)$ as output nodes and $X_1(k)$, $X_2(k)$, $X_3(k)$, $U_2(k)$, and $R(k)$ as inputs, the state equation can be obtained as

(1) $X_1(s)$

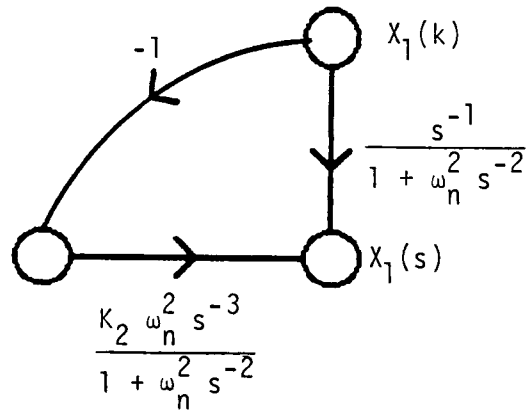
(i) $X_1(s) \dashrightarrow X_1(k)$



or



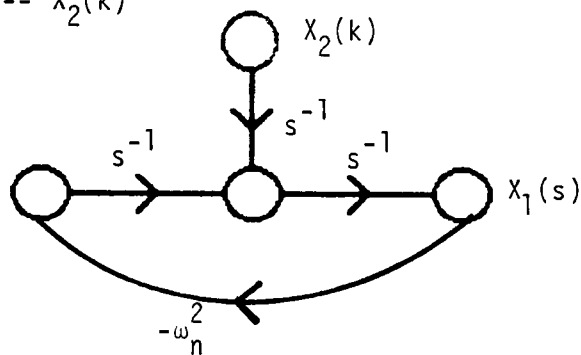
or



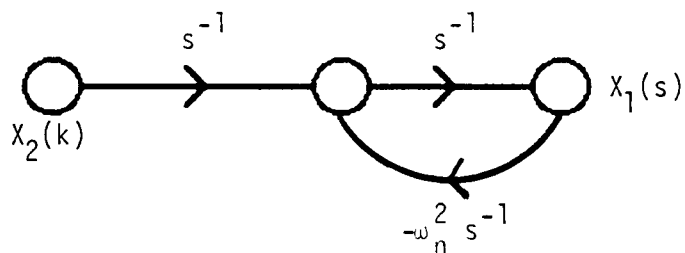
The relationship between output node $X_1(s)$ and input node $X_1(k)$ gives

$$\begin{aligned}
 X_1(s) &= \left[\frac{s^{-1}}{1 + \omega_n^2 s^{-2}} - \frac{K_2 \omega_n^2 s^{-3}}{1 + \omega_n^2 s^{-2}} \right] X_1(k) \\
 &= \left[\frac{s}{s^2 + \omega_n^2} - \frac{K_2 \omega_n^2}{s(s^2 + \omega_n^2)} \right] X_1(k)
 \end{aligned}$$

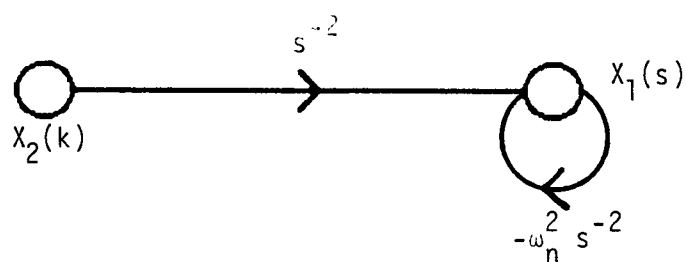
(ii) $X_1(s) \dashrightarrow X_2(k)$



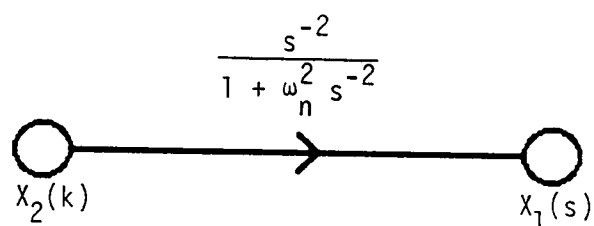
Reducing the branches yields



or



or

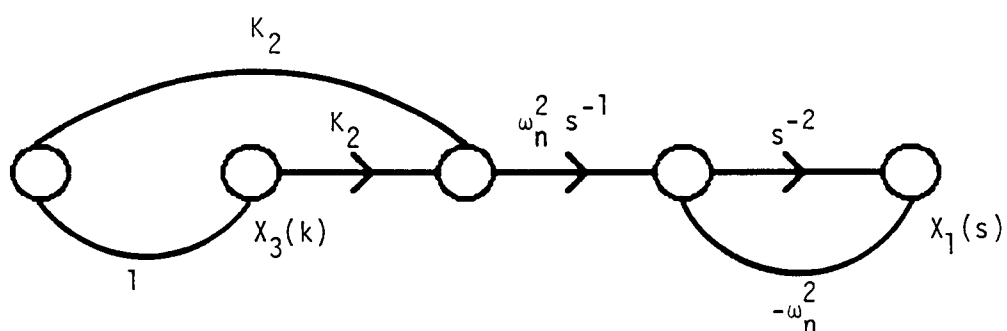


The relationship between output node $x_1(s)$ and input node $x_2(k)$ gives

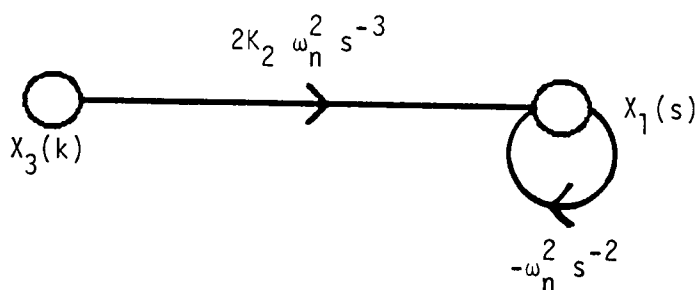
$$x_1(s) = \frac{s^{-2}}{1 + \omega_n^2 s^{-2}} x_2(k)$$

$$= \frac{1}{s^2 + \omega_n^2} x_2(k) .$$

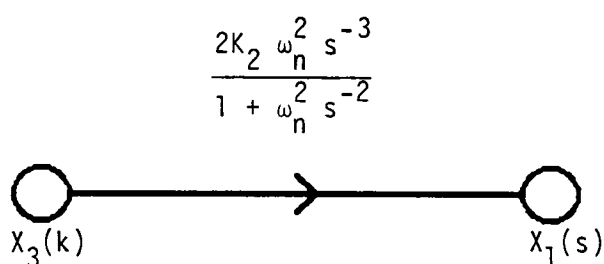
(iii) $x_1(s) \dashrightarrow x_3(k)$



or



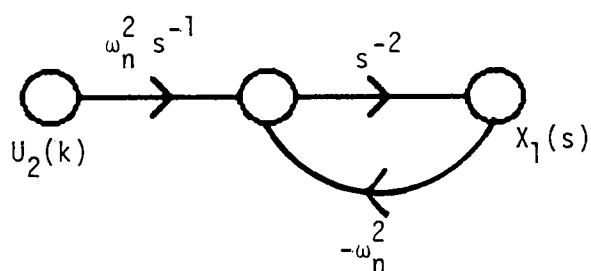
or



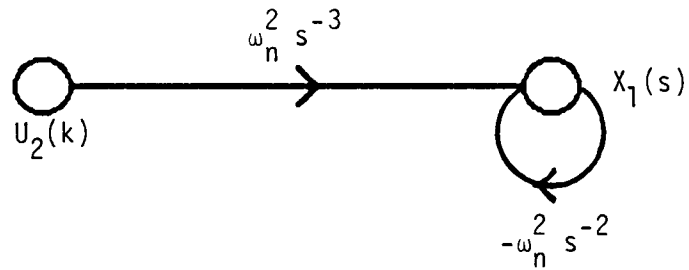
The relationship between output node $X_1(s)$ and input node $X_3(k)$ gives

$$\begin{aligned}
 X_1(s) &= \frac{2K_2 \omega_n^2 s^{-3}}{1 + \omega_n^2 s^{-2}} X_3(k) \\
 &= \frac{2K_2 \omega_n^2}{s(s^2 + \omega_n^2)} X_3(k)
 \end{aligned}$$

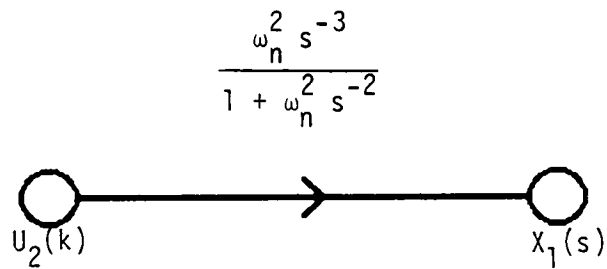
(iv) $X_1(s) \dashrightarrow U_2(k)$



or



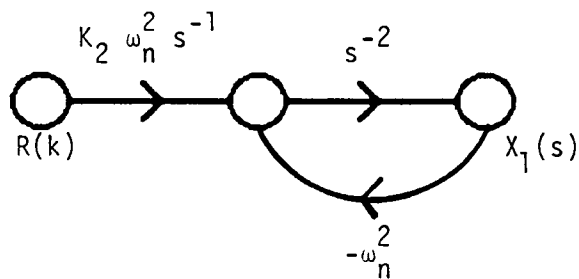
or

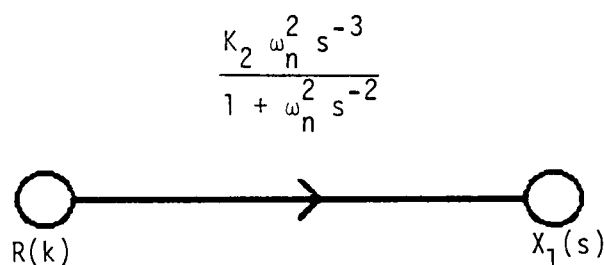


The relationship between output node $X_1(s)$ and input node $U_2(k)$ gives

$$\begin{aligned} X_1(s) &= \frac{\omega_n^2 s^{-3}}{1 + \omega_n^2 s^{-2}} U_2(k) \\ &= \frac{\omega_n^2}{s(s^2 + \omega_n^2)} U_2(k) \end{aligned}$$

(v) $X_1(s) \dashrightarrow R(k)$





The relationship between output node $X_1(s)$ and input node $R(k)$ gives

$$\begin{aligned} X_1(s) &= \frac{K_2 \omega_n^2 s^{-3}}{1 + \omega_n^2 s^{-2}} R(k) \\ &= \frac{K_2 \omega_n^2}{s(s^2 + \omega_n^2)} R(k) \end{aligned}$$

Adding the relations between output node $X_1(s)$ and all inputs, which are $X_1(k)$, $X_2(k)$, $X_3(k)$, $U_2(k)$, and $R(k)$ yields

$$\begin{aligned} X_1(s) &= \left(\frac{s}{s^2 + \omega_n^2} - \frac{K_2 \omega_n^2}{s(s^2 + \omega_n^2)} \right) X_1(k) + \frac{1}{s^2 + \omega_n^2} X_2(k) \\ &\quad + \frac{2K_2 \omega_n^2}{s(s^2 + \omega_n^2)} X_3(k) + \frac{\omega_n^2}{s(s^2 + \omega_n^2)} U_2(k) \\ &\quad + \frac{K_2 \omega_n^2}{s(s^2 + \omega_n^2)} R(k) . \end{aligned}$$

Using Table 1 (p. 95), the inverse transform with $t = (k+1)T$ enables the last equation to be written as

$$\begin{aligned}
 X_1(k+1) = & \{ \cos \omega_n T - K_2(1 - \cos \omega_n T) \} X_1(k) + \frac{1}{\omega_n} \sin \omega_n T X_2(k) \\
 & + 2K_2(1 - \cos \omega_n T) X_3(k) + (1 - \cos \omega_n T) U_2(k) \\
 & + K_2(1 - \cos \omega_n T) R(k) .
 \end{aligned} \tag{D-1}$$

Because other outputs, such as $X_2(s)$ and $X_3(k+1)$, may be solved by the same method as described in the solution of $X_1(s)$, the algebra of other state diagrams will be omitted.

(2) $X_2(s)$

(i) $X_2(s) \text{ --- } X_1(k)$

$$\begin{aligned}
 X_2(s) &= - \frac{\omega_n^2}{s^2 + \omega_n^2} X_1(k) - \frac{K_2 \omega_n^2}{s^2 + \omega_n^2} X_1(k) \\
 &= \frac{-(1 + K_2)\omega_n^2}{s^2 + \omega_n^2} X_1(k)
 \end{aligned}$$

(ii) $X_2(s) \text{ --- } X_2(k)$

$$X_2(s) = \frac{s}{s^2 + \omega_n^2} X_2(k)$$

(iii) $X_2(s) \text{ --- } X_3(k)$

$$X_2(s) = \frac{2K_2 \omega_n^2}{s^2 + \omega_n^2} X_3(k)$$

$$(iv) X_2(s) \dashrightarrow U_2(k)$$

$$X_2(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} U_2(k)$$

$$(v) X_2(s) \dashrightarrow R(k)$$

$$X_2(s) = \frac{K_2 \omega_n^2}{s^2 + \omega_n^2} R(k) .$$

Adding the relations between output node $X_2(s)$ and all inputs yields

$$\begin{aligned} X_2(s) = & \frac{-(1 + K_2)\omega_n^2}{s^2 + \omega_n^2} X_1(k) + \frac{s}{s^2 + \omega_n^2} X_2(k) \\ & + \frac{2K_2 \omega_n^2}{s^2 + \omega_n^2} X_3(k) + \frac{\omega_n^2}{s^2 + \omega_n^2} U_2(k) + \frac{K_2 \omega_n^2}{s^2 + \omega_n^2} R(k) \end{aligned}$$

or, in discrete form,

$$\begin{aligned} X_2(k+1) = & -(1 + K_2)\omega_n \sin \omega_n T X_1(k) + \cos \omega_n T X_2(k) \\ & + 2K_2 \omega_n \sin \omega_n T X_3(k) + \omega_n \sin \omega_n T U_2(k) \\ & + K_2 \omega_n \sin \omega_n T R(k) \end{aligned} \quad (D-2)$$

$$(3) X_3(k+1)$$

$$(i) X_3(k+1) \dashrightarrow X_1(k)$$

$$x_3(k+1) = -x_1(k)$$

$$(ii) \ x_3(k+1) \text{ --- } x_2(k)$$

$$x_3(k+1) = 0 \cdot x_2(k)$$

$$(iii) \ x_3(k+1) \text{ --- } x_3(k)$$

$$x_3(k+1) = x_3(k)$$

$$(iv) \ x_3(k+1) \text{ --- } u_2(k)$$

$$x_3(k+1) = 0 \cdot u_2(k)$$

$$(v) \ x_3(k+1) \text{ --- } R(k)$$

$$x_3(k+1) = R(k)$$

Adding the relations between output node $x_3(k+1)$ and all inputs yields

$$x_3(k+1) = -x_1(k) + x_3(k) + R(k) \quad (D-3)$$

Finally, forming the discrete state equations of the system from Eqs. D-1, D-2, and D-3, Eq. II-35 can be obtained.

APPENDIX E

DETAILS FOR EQS. II-39 AND II-41

This section shows how to obtain Eqs. II-39 and II-41. From Figure II-11 (p. 31) changing K_2 to K_3 and K_4 gives

$$\begin{aligned} X_1(s) = & \left(\frac{s}{s^2 + \omega_n^2} - \frac{K_3 \omega_n^2}{s(s^2 + \omega_n^2)} \right) X_1(k) + \frac{1}{s^2 + \omega_n^2} X_2(k) \\ & + \frac{(K_3 + K_4) \omega_n^2}{s(s^2 + \omega_n^2)} X_3(k) + \frac{\omega_n^2}{s(s^2 + \omega_n^2)} U_2(k) \\ & + \frac{K_3 \omega_n^2}{s(s^2 + \omega_n^2)} R(k) \end{aligned}$$

or in the discrete form

$$\begin{aligned} X_1(k+1) = & \{ \cos \omega_n T - K_3(1 - \cos \omega_n T) \} X_1(k) + \frac{1}{\omega_n} \sin \omega_n T X_2(k) \\ & + (K_3 + K_4)(1 - \cos \omega_n T) X_3(k) + (1 - \cos \omega_n T) U_2(k) \\ & + K_3(1 - \cos \omega_n T) R(k) . \end{aligned} \quad (E-1)$$

$X_2(s)$ therefore becomes

$$\begin{aligned} X_2(s) = & \frac{-(1 + K_3) \omega_n^2}{s^2 + \omega_n^2} X_1(k) + \frac{s}{s^2 + \omega_n^2} X_2(k) \\ & + \frac{(K_3 + K_4) \omega_n^2}{s^2 + \omega_n^2} X_3(k) + \frac{\omega_n^2}{s^2 + \omega_n^2} U_2(k) \end{aligned}$$

$$+ \frac{K_3 \omega_n^2}{s^2 + \omega_n^2} R(k) .$$

In the discrete form,

$$\begin{aligned} X_2(k+1) = & -(1 + K_3) \omega_n \sin \omega_n T X_1(k) + \cos \omega_n T X_2(k) \\ & + (K_3 + K_4) \omega_n \sin \omega_n T X_3(k) + \omega_n \sin \omega_n T U_2(k) \\ & + K_3 \omega_n \sin \omega_n T R(k) , \end{aligned} \quad (E-2)$$

and $X_3(k+1)$ becomes

$$X_3(k+1) = -X_1(k) + X_3(k) + R(k) . \quad (E-3)$$

Adding Eqs. E-1, E-2, and E-3, Eq. II-39 can be obtained.

Because the system with a PID-controller has three constants, K_5 , K_6 , and K_7 , one more output node, $X_4(k+1)$, is needed. By the same procedure as a PI-controller $X_1(s)$, from Figure II-14 (p. 36), becomes

$$\begin{aligned} X_1(s) = & \left(\frac{s}{s^2 + \omega_n^2} - \frac{K_5 \omega_n^2}{s(s^2 + \omega_n^2)} \right) X_1(k) + \frac{1}{s^2 + \omega_n^2} X_2(k) \\ & + \frac{K_7 \omega_n^2}{s(s^2 + \omega_n^2)} X_3(k) + \frac{(K_5 + K_6) \omega_n^2}{s(s^2 + \omega_n^2)} X_4(k) \\ & + \frac{\omega_n^2}{s(s^2 + \omega_n^2)} U_2(k) + \frac{K_5 \omega_n^2}{s(s^2 + \omega_n^2)} R(k) . \end{aligned}$$

In the discrete form this equation becomes

$$\begin{aligned} X_1(k+1) = & \{ \cos \omega_n T - K_5(1 - \cos \omega_n T) \} X_1(k) + \frac{1}{\omega_n} \sin \omega_n T X_2(k) \\ & + K_7(1 - \cos \omega_n T) X_3(k) + (K_5 + K_6)(1 - \cos \omega_n T) X_4(k) \\ & + (1 - \cos \omega_n T) U_2(k) + K_5(1 - \cos \omega_n T) R(k) , \quad (E-4) \end{aligned}$$

$X_2(s)$, also from Figure II-14 (p. 36), becomes

$$\begin{aligned} X_2(s) = & \frac{-(1 + K_5)\omega_n^2}{s^2 + \omega_n^2} X_1(k) + \frac{s}{s^2 + \omega_n^2} X_2(k) + \frac{K_7 \omega_n^2}{s^2 + \omega_n^2} X_3(k) \\ & + \frac{(K_5 + K_6)\omega_n^2}{s^2 + \omega_n^2} X_4(k) + \frac{\omega_n^2}{s^2 + \omega_n^2} U_2(k) \\ & + \frac{K_5 \omega_n^2}{s^2 + \omega_n^2} R(k) . \end{aligned}$$

While in the discrete form this equation becomes

$$\begin{aligned} X_2(k+1) = & -(1 + K_5)\omega_n \sin \omega_n T X_1(k) + \cos \omega_n T X_2(k) \\ & + K_7 \omega_n \sin \omega_n T X_3(k) + (K_5 + K_6)\omega_n \sin \omega_n T X_4(k) \\ & + \omega_n \sin \omega_n T U_2(k) + K_5 \omega_n \sin \omega_n T R(k) . \quad (E-5) \end{aligned}$$

In the same manner $X_3(k+1)$ becomes

$$X_3(k+1) = X_4(k) \quad (E-6)$$

and $x_4(k+1)$ becomes

$$x_4(k+1) = -x_1(k) + x_4(k) + R(k) . \quad (E-7)$$

Adding Eqs. E-4, E-5, E-6, and E-7, Eq. II-41 can be obtained.

APPENDIX F

EXPERIMENTAL SYSTEM

To study the feedback control system experimentally, the analog computer is used to simulate the nonlinear plant while the digital computer, A/D converter, and D/A converter are used to control the plant. This appendix describes how a circuit was constructed to solve a nonlinear differential equation of motion on the analog computer and then shows how the digital control unit was combined with this analog simulation.

For the plant, consider the same system as the one discussed in Chapter II (Figure II-1, p. 6). To construct an equivalent analog circuit, Eq. II-3 may be rearranged with the highest-ordered derivative on the left of the equality sign,

$$\ddot{y} = \omega_n^2 x - \omega_n^2 y - \mu g \operatorname{sgn}(\dot{y}) . \quad (\text{F-1})$$

Next, by cascading two integrators as shown in Figure F-1a, the terms y and $\dot{y}$ are obtained. Then the addition of constant multipliers provide the term $-\omega_n^2 y$ (Figure F-1b), while the term $\omega_n^2 x$ is obtained by a given input.

From Figure F-1, the circuit for an analog computer can be constructed by basic analog computer elements. The algebraic summation of several variables is accomplished by using resistors as the input and feedback impedances of an operational amplifier (Figure F-2).

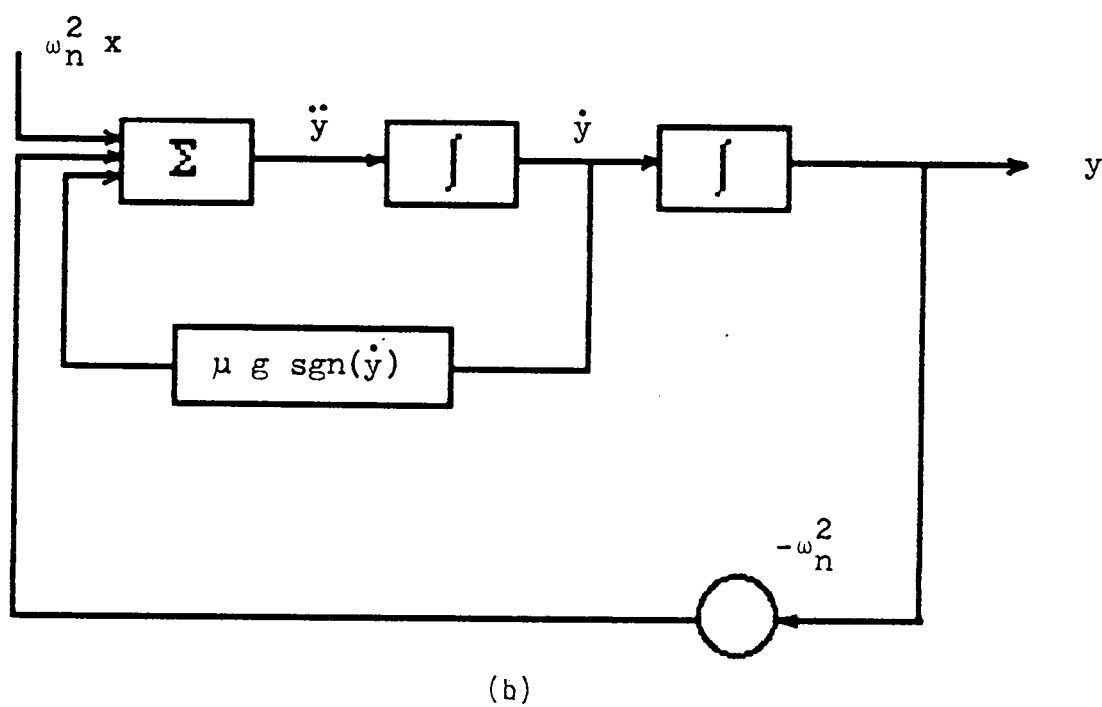
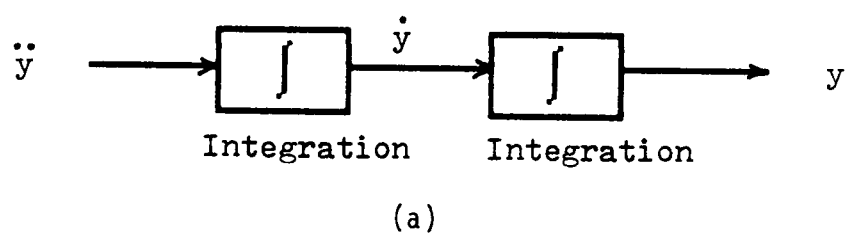
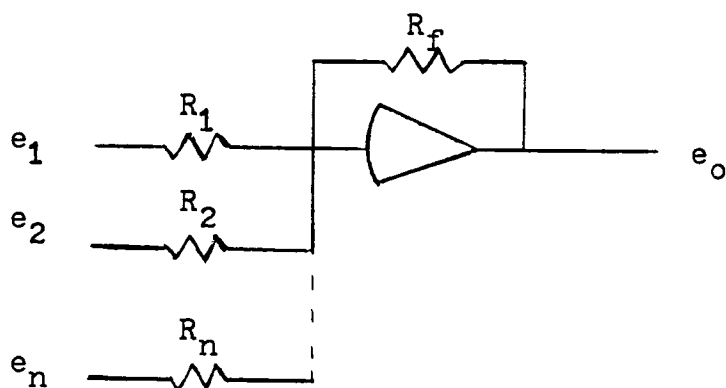


Figure F-1. Computer block diagram for Eq. F-1.



$$e_o = -\left(\frac{R_f}{R_1} e_1 + \frac{R_f}{R_2} e_2 + \dots + \frac{R_f}{R_n} e_n \right)$$

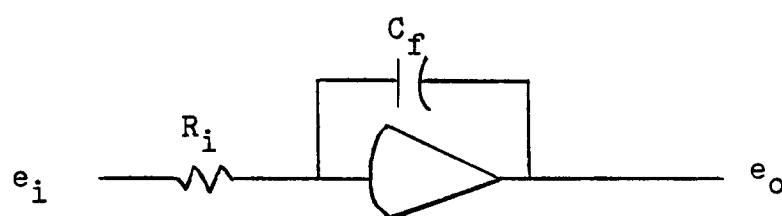
Figure F-2. Algebraic summation with the operational amplifier.

For an integration, it is necessary to choose a capacitor as the feedback impedance and a resistor for the input impedance, as illustrated in Figure F-3.

The $\text{sgn}(\dot{y})$ is a nonlinear element caused by Coulomb friction. Coulomb friction is a retarding force that has a constant amplitude with respect to velocity, but the sign of the frictional force changes with the reversal of the direction of velocity. The functional description of the friction to velocity is represented in Figure F-4a. For the simulation of such a discontinuity on the analog computer, the diode circuitry shown in Figure 4-4b may be used. The combined circuit is shown in Figure F-5.

A digital computer is used as a controller or processor to perform computations necessary for control. Since the controlled process (i.e., the plant) contains analog elements, the signals found in the control system are both analog and digital. Therefore, the process of signal conversion is essential so that the digital and analog components can be interfaced in the same system. More precisely speaking, the output signals of the plant (analog device) must first undergo an analog-to-digital (A/D) conversion before they can be processed by a digital controller, and similarly the signal from a digital controller or computer must first undergo a digital-to-analog (D/A) conversion before it can be sent to a plant for processing.

The experimental system will use the HP 9826 digital computer as a controller and HP 3497A (DATA ACQUISITION/CONTROL UNIT) as A/D



$$\frac{E_o(s)}{E_i(s)} = - \frac{K_c}{s}$$

where

$$K_c = \frac{1}{R_i C_f}$$

Figure F-3. Integration with the operational amplifier.

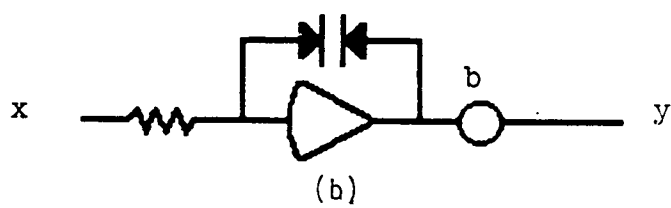
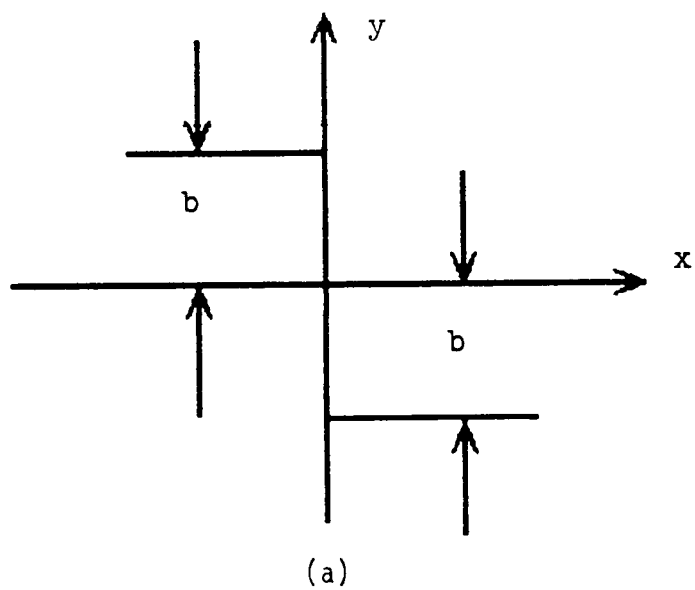


Figure F-4. Bang-bang or flip-flop circuit.

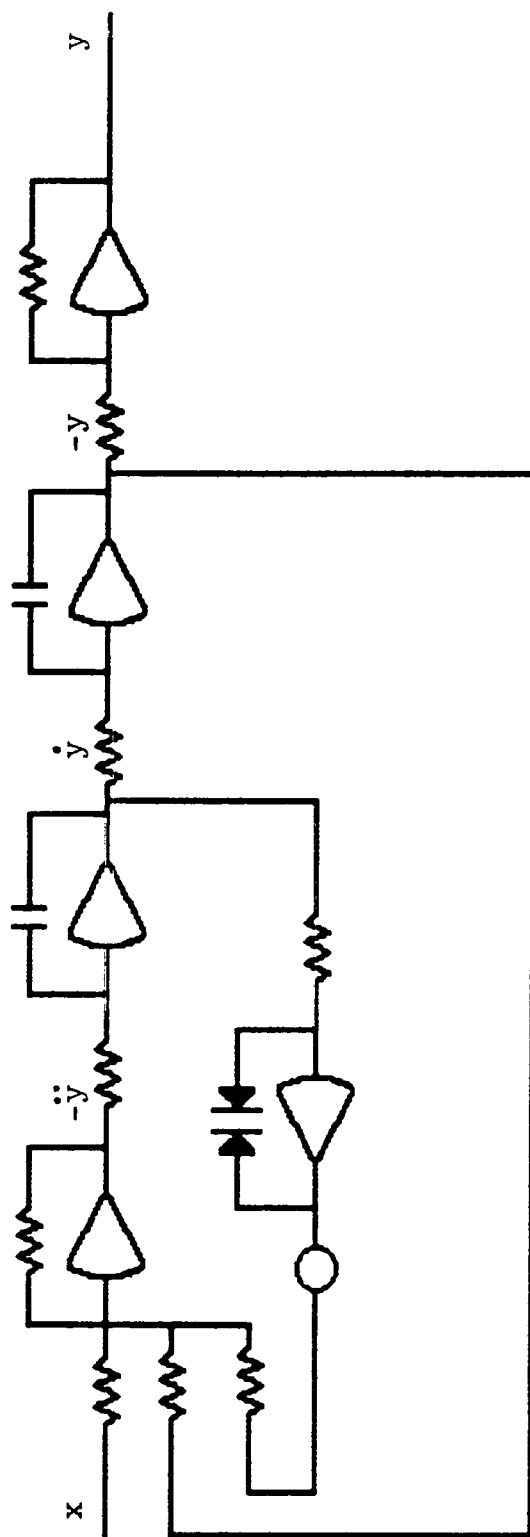


Figure F-5. Computer diagram corresponding to Figure F-1.

and D/A. For the D/A converter, the HP 3497A provides 0 ± 10 V programmable voltage sources for control applications. For the A/D converter, the HP 3497A receives inputs from the plant, measures the input data and outputs the measurement over a communication interface to the controller (HP 9826 computer). The controller processes the data for display, storage, as well as for real time control purposes.

Figure F-6 shows the diagram of the overall control system which combines the plant (analog device), digital controller, and interface devices. In Figure F-6 the interface at the input of the computer is an A/D converter and is required to convert the error signal which is a continuous-time signal, into a form that can be readily processed by the computer. At the computer output a D/A converter is required to convert the digital command signals of the computer into a form necessary to drive the plant.

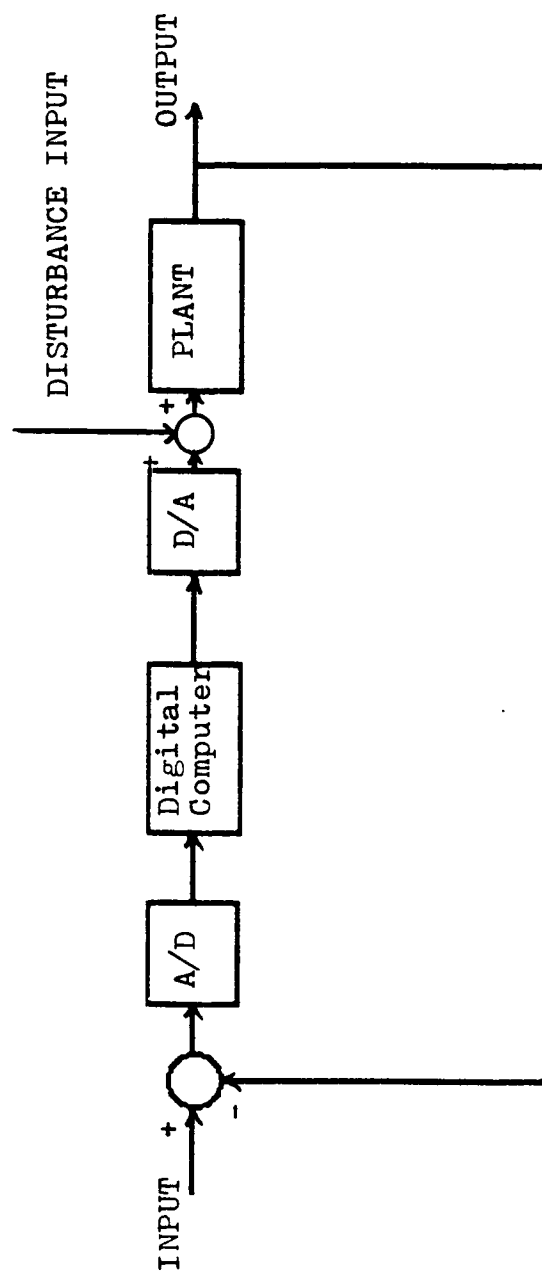


Figure F-6. Digital control system block diagram.

APPENDIX G

COMPUTER PROGRAM FOR THE STATE EQUATION WITH DIGITAL CONTROLLERS

```

10  '
20  '
30  '   THIS PROGRAM IS TO SOLVE THE STATE EQUATION FOR ENTIRE
40  '   SYSTEM USING STATE SPACE METHOD.
50  '   AS CHANGING PARAMETERS, WHICH ARE Kp, Ki, AND Kd,
60  '   THIS IS FIT FOR P, I, PI, PID DIGITAL CONTROLLERS.
70  '
80  '
90  '   VARIABLES :
100 '
110 '
120 '       KP      : DKP(*) - ARRAY FOR Kp
130 '       KI      : DKI(*) - ARRAY FOR Ki
140 '       KD      : DKD(*) - ARRAY FOR Kd
150 '       T        - SAMPLING PERIOD
160 '       X1(*), X2K1,
170 '       X3K1, X4K1 - STATE VARIABLES
180 '       OS(*) : DOS(*) - ARRAY FOR OVERSHOOT
190 '       TS(*) - ARRAY FOR SETTLING TIME
200 '       TP(*) - ARRAY FOR PEAK TIME
210 '       SS(*) - ARRAY FOR STEADY-STATE VALUE
220 '       SERR(*) - ARRAY FOR STEADY-STATE ERROR
230 '
240 '
250 '
260 '
270 ' READ Kp, Ki, Kd, T
280 '
290 INPUT "Kp,Ki,Kd,T ? ",KP,KI,KD,T
300 DIM X1(4000)
310 DIM DKP(25,25),DKD(25,25),DOS(25,25)
320 DIM OS(100),TP(25,25),SS(25,25),TS(25,25),SERR(25,25)
330 DIM CHECK(4000),TP1(100),CHANGE(100)
340 DIM CHINT(100)
350 '
360 ' DEFINE INITIAL CONDITIONS
370 '
380 WN=SQR(.05)
390 C=.0092
400 X1(0)=0
410 RK=1
420 SN=SIN(WN*T)
430 CS=COS(WN*T)
440 SP=1
450 SD=1
460 DKDO=KD
470 '
480 ' MAKE DATA FILES
490 '
500 OPEN "DAT7" FOR OUTPUT AS #1
510 OPEN "DAT8" FOR OUTPUT AS #2
520 OPEN "DAT9" FOR OUTPUT AS #3
530 OPEN "DATD" FOR OUTPUT AS #4
540 '
550 ' CHANGE PARAMETERS
560 '
570 FOR JK=1 TO 20
580 KD=DKDO
590 FOR I1=1 TO 20
600 PRINT KP,KD
610 DKP(JK,I1)=KP

```

```

620 DKD(JK,I1)=KD
630 '
640 ' DECIDE COEFFICIENTS OF THE CONTROLLER
650 '
660 K5=(KI*T+2*KD+2*KP*T)/(2*T)
670 K6=(KI*T+2*KP*T-4*KD)/(2*T)
680 K7=KD/T
690 NU=1
700 J1=1
710 CHANGE(0)=0!
720 '
730 ' CALCULATE THE FINAL STATE EQUATION
740 '
750 X2K0=0
760 X3K0=0
770 X4K0=0
780 FOR K=0 TO 3998
790 U2=C/(WN*WN)
800 IF X2K0>=0 THEN U2=-U2
810 X1(K+1)=(CS-K5+K5*CS)*X1(K)+SN*X2K0/WN+(1-CS)*(K7*X3K0+(K5+K6)*X4K0+K5*RK+
U2)
820 X2K1=-(1+K5)*WN*SN*X1(K)+CS*X2K0+K7*WN*SN*X3K0+(K5+K6)*WN*SN*X4K0+K5*WN*SN
*RK+WN*SN*U2
830 X3K1=X4K0
840 X4K1=-X1(K)+X4K0+RK
850 '
860 ' FIND MAXIMUM OVERSHOOT AND PEAK TIME
870 '
880 IF K=0 THEN GOTO 1000
890 IF X1(K)>X1(K-1) AND X1(K)>X1(K+1) THEN PRINT NU: OS(NU)=X1(K): TP1(NU)=K:
NU=NU+1
900 '
910 ' FIND THE STABILITY
920 '
930 IF OS(2)>OS(1) THEN GOTO 1420
940 IF ABS(X2K1)<.0001 THEN: IF X2K1=X2K0 OR X1(K+1)=X1(K) THEN SS(JK,I1)=X1(K
): GOTO 1050
950 '
960 ' FIND A STEADY STATE VALUE
970 '
980 IF X2K1<0! THEN CHECK(K+1)=1 ELSE CHECK(K+1)=2
990 IF X2K0*X2K>=0! THEN GOTO 1000 ELSE CHANGE(J1)=K-1: CHINT(J1)=CHANGE(J1)-C
HANGE(J1-1): J1=J1+1: IF CHECK(K+1)<>CHECK(K-1) THEN GOTO 1000 ELSE SS(JK,I1)=X1
(K-1): GOTO 1050
1000 X2K=X2K0
1010 X2K0=X2K1
1020 X3K=X3K1
1030 X4K=X4K1
1040 NEXT K
1050 PRINT K
1060 '
1070 ' CHECK S.S. VALUE AGAIN
1080 '
1090 J2=J1-1
1100 IF CHINT(J2)<TP1(1)/2 THEN J3=CHANGE(J2-1): SS(JK,I1)=X1(J3): J2=J2-1: GO
TO 1100
1110 FK=K+1
1120 FK1=K+1
1130 '
1140 ' FIND A SETTLING TIME
1150 '
1160 IF X1(FK1)>1.02*SS(JK,I1) OR X1(FK1)<.98*SS(JK,I1) THEN TS(JK,I1)=FK1+1 EL

```

```

SE FK1=FK1-1: GOTO 1160
1170 '
1180 ' CALCULATE A STEADY STATE ERROR
1190 '
1200 SERR(JK,I1)=ABS(1-SS(JK,I1))
1210 '
1220 ' CHANGE COEFFICIENTS
1230 '
1240 IF KI=0 THEN GOTO 1250 ELSE KI=KI+SI
1250 '
1260 ' FIND PERCENTAGE OF OVERSHOOT AND PEAK TIME
1270 '
1280 TP(JK,I1)=TP1(I1)
1290 DOS(JK,I1)=(OS(1)-SS(JK,I1))*100/SS(JK,I1)
1300 '
1310 ' WRITE DATA ON FILES
1320 '
1330 PRINT JK,I1,TP(JK,I1),DOS(JK,I1),SS(JK,I1),TS(JK,I1)
1340 PRINT #1, DKP(JK,I1),DKD(JK,I1),TS(JK,I1)*T
1350 PRINT #2, DKP(JK,I1),DKD(JK,I1),DOS(JK,I1)
1360 PRINT #3, DKP(JK,I1),DKD(JK,I1),SERR(JK,I1)
1370 KD=KD+SD
1380 FOR IO=1 TO 2
1390 OS(IO)=0
1400 NEXT IO
1410 NEXT I1
1420 KP=KP+SP
1430 FOR IK=1 TO 2
1440 OS(IK)=0
1450 NEXT IK
1460 PRINT #4,DKP(JK,I1-1),DKD(JK,I1-1)
1470 NEXT JK
1480 LPRINT " COEFFICIENT VS. SETTLING TIME "
1490 FOR J2=1 TO 20
1500 FOR I2=1 TO 20
1510 LPRINT DKP(J2,I2),DKD(J2,I2),TS(J2,I2)*T
1520 NEXT I2
1530 NEXT J2
1540 LPRINT " COEFFICIENT VS. OVERSHOOT(%) "
1550 FOR J3=1 TO 20
1560 FOR I3=1 TO 20
1570 LPRINT DKP(J3,I3),DKD(J3,I3),DOS(J3,I3)
1580 NEXT I3
1590 NEXT J3
1600 LPRINT " COEFFICIENT VS. S.S. ERROR "
1610 FOR J4=1 TO 20
1620 FOR I4=1 TO 20
1630 LPRINT DKP(J4,I4),DKD(J4,I4),SERR(J4,I4)
1640 NEXT I4
1650 NEXT J4
1660 LPRINT " COEFFICIENT VS. PEAK TIME "
1670 FOR J5=1 TO 20
1680 FOR I5=1 TO 20
1690 LPRINT DKP(J5,I5),DKD(J5,I5),TP(J5,I5)
1700 NEXT I5
1710 NEXT J5
1720 END

```

APPENDIX H

COMPUTER PROGRAM FOR THE DIRECT DESIGN METHOD

```

10  ! SSMDD
20  ! USING STATE SPACE METHOD WITH THE
30  ! DIRECT DESIGN OF DIGITAL CONTROLLER.
40  INPUT "T,Tau ? ".T,Tau
50  GINIT
60  GRAPHICS ON
70  FRAME
80  WINDOW 0,133,-50,50
90  DIM X1(900),X2(900),X3(900),X4(900)
100 RAD
110 Wn=SQR(.05)
120 C=.0092
130 Mk=C/Wn^2
140 U2=C/(Wn*Wn)
150 X1(0)=0
160 X2(0)=0
170 X3(0)=0
180 X4(0)=0
190 Rk=1
200 Et=EXP(-T/Tau)
210 Sn=SIN(Wn*T)
220 Cs=COS(Wn*T)
230 FOR K=0 TO 399
240 IF X2(K)>=0 THEN U2=-U2
250 IF X2(K)>=0 THEN Mk=-Mk
260 K8=(1-Et-Mk*(1-Cs))/(1-Cs)
270 K9=-(1-Et)*(2*Cs+Mk*(1-Cs))/(1-Cs)
280 K10=(1-Et+Mk*(1-Cs)*Et)/(1-Cs)
290 X1(K+1)=(Cs-K8+K8*Cs)*X1(K)+Sn*X2(K)/Wn+(1-Cs)*((K8+K10)*X3(K)+K9*X4(K)+K8
    *Rk+U2)
300 X2(K+1)=-((1+K8)*Wn*Sn*X1(K)+Cs*X2(K)+(K8+K10)*Wn*Sn*X3(K)+K9*Wn*Sn*X4(K)+K
    8*Wn*Sn*Rk+Wn*Sn*U2
310 X3(K+1)=X4(K)
320 X4(K+1)=-X1(K)+X3(K)+Rk
330 NEXT K
340 FOR I=0 TO 399
350 PRINT I,X1(I),X2(I)
360 DRAW I=133/400,X1(I)*40
370 NEXT I
380 END

```

```

10  ! SSDDD
20  ! FOR A DISTURBANCE INPUT
30  ! USING STATE SPACE METHOD WITH THE
40  ! DIRECT DESIGN OF DIGITAL CONTROLLER.
50  INPUT "T,Tau ? ",T,Tau
60  GINIT
70  GRAPHICS ON
80  FRAME
90  WINDOW 0,133,-50,50
100 DIM X1(499),X2(499),X3(499),X4(499),T1(499)
110 DIM X21(499),X22(499)
120 INPUT "FILE NAME",A$
130 CREATE BDAT A$,1000,8
140 ASSIGN @Path1 TO A$
150 RAD
160 Wn=SQR(.05)
170 C=.0092
180 Mk=C/Wn^2
190 Uc=C/(Wn*Wn)
200 X1(0)=0
210 X2(0)=0
220 X21(0)=0
230 X22(0)=0
240 X3(0)=0
250 X4(0)=0
260 Rk=1
270 Vk=1
280 Et=EXP(-T/Tau)
290 Sn=SIN(Wn*T)
300 Cs=COS(Wn*T)
310 FOR K=0 TO 499
320 IF X2(K)>=0 THEN Uc=-Uc
330 IF X2(K)>=0 THEN Mk=-Mk
340 U2=Uc+Vk
350 K11=(1-Et-(1+Mk)*(1-Cs))/(1-Cs)
360 K12=-(1-Et)*(2*Cs+(1+Mk)*(1-Cs))/(1-Cs)
370 K13=(1-Et+(1+Mk)*(1-Cs)*Et)/(1-Cs)
380 X1(K+1)=(Cs-K11+K11*Cs)*X1(K)+Sn*X2(K)/Wn+(1-Cs)*((K11+K13)*X3(K)+K12*X4(K)
)+K11*Rk+U2)
390 X21(K+1)=-(1+K11)*Wn*Sn*X1(K)+Cs*X2(K)+(K11+K13)*Wn*Sn*X3(K)
400 X22(K+1)=K12*Wn*Sn*X4(K)+K11*Wn*Sn*Rk+Wn*Sn*U2
410 X2(K+1)=X21(K+1)+X22(K+1)
420 X3(K+1)=X4(K)
430 X4(K+1)=-X1(K)+X3(K)+Rk
440 T1(K+1)=K*T
450 NEXT K
460 FOR I=0 TO 499
470 PRINT I,X1(I),X2(I)
480 DRAW I*133/400,X1(I)*40
490 NEXT I
500 OUTPUT @Path1:T1(*),X1(*)
510 END

```

APPENDIX I

COMPUTER PROGRAM FOR THE EXPERIMENTAL SYSTEM AND CLOSED-FORM
SOLUTION WITHOUT DIGITAL CONTROLLERS

```

10 ! FINAL
20 ! THIS PROGRAM IS FOR DIGITAL CONTROLLER
30 ! VARIABLES :
40 !       Y : OUTPUT Y(Z)
50 !       E : R - Y E(Z)
60 !       M : D(Z)*E(Z)=M(Z)
70 !       P : TIME SAMPLING
80 CLEAR 709
90 GINIT
100 GRAPHICS ON
110 OPTION BASE 0
120 INPUT "FILE NAME",A$
130 CREATE BDAT A$,700,8
140 ASSIGN @Path1 TO A$
150 DIM Yz(349),T(349)
160 INTEGER I,J,M
170 Y=0.
180 P=.1
190 OUTPUT 709:"AC41"
200 FOR I=1 TO 349
210 E=1000-Y
220 M=E
230 M$=VAL$(M)
240 OUTPUT 709:"A04.0."&M$
250 IF I=1 THEN T0=TIMEDATE
260 T(I)=TIMEDATE
270 IF T(I)-T0<P*I THEN GOTO 260
280 ENTER 709:Y
290 T(I)=T(I)-T0
300 Yz(I)=Y
310 Y=Y*1000
320 NEXT I
330 Yz(0)=0
340 T(0)=0
350 PRINT USING "15X,4A,11X,6A";"TIME","OUTPUT"
360 FRAME
370 FOR J=0 TO 349
380 PRINT USING "2X,ZZZ,10X,DDD.DD,10X,DD.DDDD";J,T(J),Yz(J)
390 DRAW 133/400*J,Yz(J)*50
400 NEXT J
410 OUTPUT @Path1:T(*),Yz(*)
420 END

```

```

10  .....
20  '
30  '   THIS PROGRAM IS TO PLOT THE SYSTEM RESPONSE WITHOUT
40  '   DIGITAL CONTROLLERS.
50  '
60  '
70  '   VARIABLES :
80  '
90  '           KP   : PROPORTIONAL GAIN
100  '           T    : SAMPLING PERIOD
110  '           X1(*) : DISPLACEMENT
120  '           X2(*) : VELOCITY
130  '
140  ' .....
150  '
160  ' READ Kp, Ki, Kd, T
170  '
180  INPUT "Kp,Ki,Kd,T ? ",KP,KI,KD,T
190  DIM X1(4000)
200  DIM DKP(100),DKI(100),DKD(100),DOS(100)
210  DIM OS(100),TP(100),SS(100),TS(100),SERR(100)
220  DIM CHECK(4000),TP1(100),CHANGE(100)
230  DIM CHINT(100)
240  '
250  ' DEFINE INITIAL CONDITIONS
260  '
270  WN=SQR(.05)
280  C=.0092
290  X1(0)=0
300  X2K0=0
310  X3K0=0
320  X4K0=0
330  RK=1
340  SN=SIN(WN*T)
350  CS=COS(WN*T)
360  '
370  ' MAKE DATA FILES
380  '
390  OPEN "SYS1.DAT" FOR OUTPUT AS #1
400  '
410  ' DECIDE COEFFICIENTS OF THE CONTROLLER
420  '
430  K5=(KI*T*T+2*KD+2*KP*T)/(2*T)
440  K6=(KI*T*T-2*KP*T-4*KD)/(2*T)
450  K7=KD/T
460  NU=1
470  J1=1
480  CHANGE(0)=0!
490  '
500  ' CALCULATE THE FINAL STATE EQUATION
510  '
520  FOR K=0 TO 3998
530  U2=C/(WN*WN)
540  IF X2K0>=0 THEN U2=-U2
550  X1(K+1)=(CS-K5+K5*CS)*X1(K)+SN*X2K0/WN+(1-CS)*(K7*X3K0+(K5+K6)*X4K0+K5*RK+
560  U2)
570  X2K1=- (1+K5)*WN*SN*X1(K)+CS*X2K0+K7*WN*SN*X3K0+(K5+K6)*WN*SN*X4K0+K5*WN*SN
580  X3K1=X4K0
590  X4K1=-X1(K)+X4K0+RK
600  PRINT K+1,X1(K+1),X2K1
    IF K=0 THEN GOTO 670

```

```

610 IF X1(K)>X1(K-1) AND X1(K)>X1(K+1) THEN PRINT NU: OS(NU)=X1(K): TP1(NU)=K:
    NU=NU+1
620 '
630 ' FIND A STEADY STATE VALUE
640 '
650 IF X2K1<0! THEN CHECK(K+1)=1 ELSE CHECK(K+1)=2
660 IF X2K0*X2K>=0! THEN GOTO 670 ELSE CHANGE(J1)=K-1: CHINT(J1)=CHANGE(J1)-CH
    ANGE(J1-1): J1=J1+1: IF CHECK(K+1)<>CHECK(K-1) THEN GOTO 670 ELSE SS(I1)=X1(K-1)
    : GOTO 720
670 X2K=X2K0
680 X2K0=X2K1
690 X3K0=X3K1
700 X4K0=X4K1
710 NEXT K
720 PRINT K
730 FOR I1=K-1 TO K+10
740 X1(I1+1)=X1(I1)
750 PRINT X1(I1)
760 NEXT I1
770 FOR I2=0 TO K+10
780 PRINT #1,I2*T,X1(I2)
790 PRINT I2,X1(I2)
800 NEXT I2
810 END

```

VITA

Jungkeun Rhee was born in Seoul, Korea, on January 23, 1959. He attended elementary school and middle school in that city and graduated from Seoul High School in February 1977.

He entered Hanyang University, Seoul, Korea, in March 1978 and in February 1982 he received a Bachelor of Science degree in Mechanical Engineering.

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