To the Graduate Council:

I am submitting herewith a dissertation written by Ruth Margaret Bisgrove Hofstra entitled "Multidimensional Assignment and Scheduling with Application to a Tourism Convention Scheduling Problem." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Management Science.

Kenneth C. Gilbert

Kenneth C. Gilbert, Major Professor

We have read this dissertation and recommend its acceptance:

Robert Garfinkel

Oscar S Fowler

Richard E. Rosenthal

Robert A. McLean

Accepted for the Council:

CWMinkel

The Graduate School

MULTIDIMENSIONAL ASSIGNMENT AND SCHEDULING WITH APPLICATION TO A TOURISM CONVENTION SCHEDULING PROBLEM

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Ruth Margaret Bisgrove Hofstra

December 1983

To My Two Sons,

John and Charles Hofstra

ACKNOWLEDGMENTS

I would like to express my deep gratitude and appreciation to Dr. Kenneth C. Gilbert, my dissertation director, for his expert help, encouragement, and patience in the completion of my research and dissertation. I would also like to thank sincerely each of the rest of my doctoral committee for his continued interest and help: Dr. Robert S. Garfinkel, Dr. Richard E. Rosenthal, Dr. Oscar F. Fowler, and Dr. Robert A. McLean. Very special thanks to Dr. Garfinkel, for his counsel and valuable suggestions throughout my degree program.

I acknowledge with pleasure the help received by kind suggestions through correspondence, as well as from the published articles and books, of Dr. Alan M. Frieze, Dr. Victor Klee, Dr. William P. Pierskalla, and Dr. Herbert John Ryser. Sincere appreciation also to Dr. John N. Hooker, Jr., my supervisor and friend at Oak Ridge National Laboratory, for his help and inspiring example.

Other people have helped in particular ways. Mrs. Maxine Martin did an excellent job of typing and helping to prepare the dissertation manuscript to be published in final form, and I thank her very much for that.

I would especially like to thank my two sons, John and Charles, for their wonderful support and encouragement in my studies. And I thank God for providing all my needs of every kind enabling me to finish the work: "In all thy ways acknowledge Him, and He shall direct thy paths." Proverbs 3:6.

ABSTRACT

The dissertation makes four major contributions: (1) An applied problem has been formulated, solved, and implemented along with computational tests. (2) An in-depth literature review has been done for the multidimensional assignment problem, with special results derived for one, and a summary of others. (3) A solvable case of the NP-hard three-dimensional planar assignment problem has been solved with a polynomial-time solution. (4) A new multiple traveling salesman problem has been introduced, and a heuristic developed and analyzed for a class of random problems.

The source problem is a tourism industry convention scheduling problem in which interviews among buyers and sellers are to be scheduled on a one-to-one basis during available time periods. Preference lists provided by both sellers and buyers are to be used to get mutual preference coefficients for optimization. A secondary goal is overall distance minimization for buyers moving between meetings.

The dissertation contains an exposition of several different types of model formulations of higher dimensional assignment problems and related problems which appear in the literature. The focus is on the three-dimensional planar assignment problem, which is the basic model of the tourism convention application. The general problem is shown to be NP-hard in computational complexity.

The primary tourism convention problem is shown to belong to a polynomial bounded special class of the three-dimensional planar assignment problem. A two phase optimization solution is presented which consists of a two-dimensional capacitated transportation problem and a succession of two-dimensional classical assignment problems. In the first phase, optimal interviews overall are found for buyers and sellers, and in the second phase, nonconflicting time schedules are made for those optimal meetings. An application of the "marriage theorem" is used to derive the nonconflicting schedules.

A new multiple traveling salesman problem is defined which is a simultaneous, overlapping, nonconflicting problem. It appears in consideration of the secondary objective of the tourism problem, and is referred to as the "musical chairs traveling salesman problem" (MCTSP). An heuristic method of solution is presented which is a succession of distance minimizing two-dimensional classical assignment problems, after a judicious starting assignment is found by the first one. When the secondary objective is included in the tourism convention problem, the second phase of the three-dimensional planar assignment solution is modified to include the MCTSP heuristic.

A probabilistic analysis is done for the general MCTSP heuristic, using for weights independent random variables from a gamma $g(1/n,\beta)$ distribution, where n is the number of time periods, salesmen, and cities (assumed equal for the analysis). The ratio of the expected objective function value of the heuristic solution to that of a randomly chosen solution is shown to be less than $(1 + \ln(n-1))/(n-1)$.

The solution method is computer implemented using four mutual preference objective coefficients with data from a recent tourism convention. A logarithmic cost is found to give the largest percentages of high preferences to both buyers and sellers. Computational tests are reported for problems of varying size using random independent preferences and random competitive preferences. The algorithm demonstrates the ability to resolve the conflicts and give almost as good solutions for data with strong competition and similar preferences as for random preferences, as indicated by ranks of preferences for resulting assignments. The average distance traveled when the MCTSP heuristic is not used in Phase II is greater than when it is used by a factor of 2.5 or more in all random problems, based on a sample grid supplied for the problems. With the number of time periods held constant, and numbers of buyers and sellers proportional, the CPU time is almost linear with problem size as depicted by the number of sellers.

Several potentially fruitful research areas are presented with some conjectures.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

OVERVIEW

1-1. INTRODUCTION

This dissertation is based on a travel convention application, which is modeled as a three-dimensional planar assignment problem. A secondary objective requires the solution of a variation of the m-salesmen traveling salesman problem, which we have coined the "musical chairs traveling salesman problem."

A polynomial-time algorithm for our case of the three-dimensional planar assignment problem is presented. A heuristic for the musical chairs traveling salesman problem is given, and its expected performance is analyzed. The methods are implemented with data from a recent travel convention.

A literature survey of assignment problems and m-salesmen traveling salesman problems presents numerous problems related to those solved here. The conclusion of the dissertation describes related prospective research problems.

1-2. THE SOURCE PROBLEM: A TOURISM INDUSTRY CONVENTION

1-2.1. Description of the Source Problem

This research originated in a scheduling problem often encountered in the annual conventions of several trade associations, particularly those in the travel and tourism industry (Fowler, 1976).

Due to the nature of the business conducted, the convention trade shows for these organizations do not use the typical drop-in display booths. Instead, individual meetings between the sales representatives and the prospective buyers are scheduled. The prospective buyers are tour brokers. The sellers are representatives of motels, hotels, and various tourist attractions.

The typical situation involves an unequal number of buyers and sellers, a series of time periods in which all visits must be concluded, and a requirement that these meetings are on a one buyer, one seller basis.

When the tourism convention is in the planning stage, each seller who expects to set up a booth at the convention submits to those who are planning the meeting schedules, a preference list of buyers he would like to meet. Each prospective buyer, on the other hand, who expects to attend the convention, submits a preference list of sellers whose booths he would like to visit. All lists are in order of preference, lower numbers in sequence showing more favored choices. These preferences are used, in some way, to measure the desirability of buyer j meeting seller k. This coefficient is denoted by c_{jk}.

There are fewer buyers than sellers. Hence each buyer will be assigned a meeting in each time period available for meetings at the convention, while some sellers will have fewer meetings than time periods available. An additional requirement will be imposed in the planning to balance to the extent possible the idle time periods among the sellers.

The tourism convention scheduling problem is to arrange all buyer-seller meetings, buyer j with seller k, in such a way as to minimize the sum of the c_{jk} over all scheduled meetings. The meetings take place in a large arena, and so there is a secondary objective of minimizing travel distance of the "traveling buyers," as they go from booth to booth in successive time periods.

1-2.2. Mathematical Formulation

An integer programming formulation may be made for the time-buyer-seller scheduling process for the tourism industry convention. It constitutes a three-index assignment problem of the three planar sum type, with a secondary objective which resembles the m-salesmen traveling salesman problem. These problems will be explored in later chapters.

Let positive integers q, r, and s be the number of time periods, buyers, and sellers, respectively, with $q < r < s$. The "cost" of assigning buyer j to seller k is c_{jk}. (The derivation of these costs from the ordinal ranking specified by the buyers and sellers is discussed later.) The distance at the convention from the booth of seller k to the booth of seller h is given by d_{kh}. Both c_{jk} and d_{kh} are assumed nonnegative. Define $N = [qr/s]$, the integral part of the average number of buyers a seller will see. The decision variable $x_{ijk} = 1$, if in time period i buyer j meets seller k; otherwise, $x_{ijk} = 0$.

The source problem is stated:

$$\text{Minimize} \sum_{i=1}^{q} \sum_{j=1}^{r} \sum_{k=1}^{s} c_{jk}x_{ijk} \qquad \text{(First objective)} \qquad (1.1)$$

$$\text{Minimize} \sum_{k=1}^{s} \sum_{h=1}^{s} d_{kh} \left(\sum_{i=1}^{q-1} \sum_{j=1}^{r} x_{ijk}\, x_{i+1,jh} \right) \qquad \text{(Secondary objective)} \qquad (1.2)$$

Subject to:

$$\sum_{i=1}^{q} x_{ijk} \leq 1 \qquad (j=1,\ldots,r;k=1,\ldots,s) \qquad (1.3)$$

$$\sum_{j=1}^{r} x_{ijk} \leq 1 \qquad (i=1,\ldots,q;k=1,\ldots,s) \qquad (1.4)$$

$$\sum_{k=1}^{s} x_{ijk} = 1 \qquad (i=1,\ldots,q;j=1,\ldots,r) \qquad (1.5)$$

$$\sum_{i=1}^{q} \sum_{j=1}^{r} x_{ijk} \geq N \qquad (k=1,\ldots,s) \qquad (1.6)$$

$$x_{ijk} = 0,1 \qquad (i=1,\ldots,q;j=1,\ldots,r;k=1,\ldots,s) \qquad (1.7)$$

The objective (1.1), with (1.3), (1.4), (1.5) and (1.7) are a three-dimensional assignment problem: each buyer-seller pair can meet at most once (1.3); each seller in each time meets at most one buyer (1.4); each buyer in each time meets exactly one seller (1.5). The objective (1.2) resembles a traveling salesman problem in that it minimizes total distance traveled by all buyers. Constraints (1.6) are an added optional restriction which require each seller to have approximately an average number of meetings at the least.

1-3. SUMMARY

There are a large number of practical problems which could be viewed from the perspective of multidimensional traveling salesman, transportation and/or assignment problems. Scheduling a timetable of meetings between tour brokers and vendors is a ranked multi-objective problem as well.

The literature survey of multidimensional assignment problems reveals a vast array of formulations. The travel convention problem belongs to the class of irregular three-dimensional planar assignment problems, using the first objective. The three-dimensional planar assignment problem is not a minimum weight three-dimensional matching problem, belongs to computational complexity class NP-hard, and because of the lack of total unimodularity of the constraint matrix cannot be solved as a relaxed LP to get integer solutions as required.

Although the general three-dimensional planar assignment problem is a NP-hard problem, we show that our problem is in class P, by virtue of the fact that the desirability of an assignment is time invariant. We solve this problem optimally by first assigning buyers to sellers in Phase I, ignoring the time dimension, using a capacitated transportation algorithm. The solution of this problem gives the set of booths which each buyer will visit. Corresponding to this solution, in Phase II we use the "marriage theorem" to prove the existence of nonconflicting time schedules for the buyers, which are found by solving a sequence of two-dimensional assignment or matching problems. Thus, because of the time invariant nature of the stated preferences of the buyers and sellers, we define two phases for arriving at the scheduled meetings.

The secondary objective of the tourism problem is to minimize the total distance traveled by the buyers in proceeding to the scheduled meetings with sellers during the successive time periods. When the secondary objective is included, Phase I of the solution is unchanged, since it determines the optimal meetings according to the first objective, which takes precedence over the second. Phase II becomes the "musical chairs traveling salesman problem," an NP-hard problem. We solve this problem heuristically by adding a distance objective to the series of assignment problems of Phase II, after an auspicious start for each buyer in the first time period. The heuristic is essentially a global nearest neighbor approach.

The heuristic which we develop for the musical chairs traveling salesman problem is of more general significance, since it can be applied to the general m-salesmen traveling salesman problem.

For a general probabilistic analysis, the admissible "distances," or weights, are assumed to be independent random variables from a gamma $g(1/n,\beta)$ distribution, where n is the number of time periods, buyers, and sellers (assumed equal for the analysis). The ratio of the expected objective function value to that of a randomly chosen solution is shown to approach zero as n approaches infinity.

CHAPTER II

SURVEY OF MULTIDIMENSIONAL ASSIGNMENT PROBLEMS

2-1. OVERVIEW OF TWO-DIMENSIONAL ASSIGNMENT PROBLEMS

As a part of the survey of multidimensional assignment problems, the two-dimensional assignment problem is considered first as a backdrop. The classic simple case will be seen from several viewpoints, and a variety of other related problems, called "assignment problems," will be described, some of which are linear programs and some of which are not. Each model has several higher-dimensional counterparts.

2-1.1. Classical Simple Assignment Problems (2DAP)

2-1.1.1. Regular.

(a) Definition. Suppose that there are n positions to be filled by n men, and costs a_{ij} for the ith man in the jth position. Any assignment is specified by a permutation

$$\pi = \begin{pmatrix} 1,2,\ldots,n \\ j_1,j_2,\ldots,j_n \end{pmatrix},$$

in which the ith man is assigned to position $j_i = \pi(i)$. An optimal assignment is one that minimizes $\sum_{i=1}^{n} a_{i\pi(i)}$ (Hall, 1967).

Looking only at the "cost" matrix, the 2DAP may be briefly defined as the problem of finding a subset of the elements of the nxn matrix, (a_{ij}), with exactly one element from each row and column, such that the sum of the elements is minimized (Lawler, 1976). The problem is also that of a permutation of the columns of the matrix which minimizes the trace of the matrix (the sum of the elements on the main diagonal).

Returning to the context of assigning men to positions, in order to obtain another mathematical model, let $x_{ij}=1$ if man i is assigned to position j; otherwise, $x_{ij}=0$. Then the 2DAP is stated:

$$\text{Minimize } \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_{ij} \tag{2.1}$$

Subject to:

$$\sum_{i=1}^{n} x_{ij}=1 \quad (j=1,\ldots,n)$$

$$\sum_{j=1}^{n} x_{ij}=1 \quad (i=1,\ldots,n)$$

$$x_{ij}=0,1 \quad (\text{all } i,j) \;.$$

The two-dimensional matching problem (2DM) (Garey and Johnson, 1979) affords another introduction to 2DAP. Let A and B be disjoint sets having the same number n of elements. Let subset $M\subseteq A\times B$ be a collection of admissible ordered pairs, called cells, $\{(a,b)\}$. A matching, 2DM, is a subset $M'\subseteq M$, where M' has exactly n elements and no two elements of M' agree in any coordinate. The 2DAP is the weighted matching problem in which each cell has a weight (cost) and the objective is to find a matching having the minimum total weight. If $M\subset A\times B$, a matching may be considered always possible for the sake of the assignments, by giving very unfavorable costs to "inadmissible" cells, effecting M=AxB.

Like many combinatorial optimization problems, the 2DAP is associated with a network (Klee, 1980). The minimum weight matching problem can be formulated as a problem on a complete bipartite graph K_{nn} with n nodes in each class. The number a_{ij} becomes a weight on the edge from node i of the first class to node j of the second, an assignment becomes a matching (a set of n disjoint edges spanning the 2n nodes of K_{nn}), and the optimal solution gives the value of the minimum weight matching.

In addition to the foregoing descriptions of the assignment problem 2DAP, of order n, it can also be expressed as a 2n x n^2 zero-sum, two-person game (von Neumann, 1953).

(b) Exact methods of solution. The 2DAP has n! feasible assignments. In practice, if n is of even moderate size, this is a prohibitive number for solution by total enumeration. The 2DAP is readily seen to be a special case of the two-dimensional transportation problem, with equal numbers of sources and destinations, and all supplies and demands equal to one. Both the 2DAP and the two-dimensional transportation problem may be formulated as integer linear programming (ILP) problems defined on a graph. They are included among those ILP's on graphs which are really LP's, in the sense that every basic solution to the corresponding LP is integer (Garfinkel and Nemhauser, 1972). The extreme points of the polytope formed by the feasible region have integer coordinates. This is guaranteed by a constraint matrix which is known to be totally unimodular, a property which will be studied in relation to some three-dimensional assignment problems in Section 2-3. With the capacitated two-dimensional transportation problem, the 2DAP belongs to the class of problems called minimum cost capacitated flow problems. Although they could be solved by the simplex method, special purpose, graph-oriented algorithms are considerably more efficient. The special transportation simplex method can be further streamlined for the 2DAP solution (Spivey and Thrall, 1970).

In regard to the network algorithms for the exact solution of the 2DAP, the dual to the ILP formulation provides an important ingredient. The dual to the corresponding LP is formulated:

$$\text{Maximize } W = \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \tag{2.2}$$

$$\text{Subject to: } u_i + v_j \leq a_{ij} \text{ (all } i,j)$$

The dual decision variables, u_i and v_j, are unrestricted in sign. Complementary slackness conditions are: $x_{ij}(u_i + v_j - a_{ij}) = 0$ (all i,j). A variety of network algorithms have been devised. They share the idea that at each iteration the trial values of x_{ij}, u_i, and v_j, always satisfy two of the three conditions: primal feasibility, dual feasibility, or complementary slackness. They differ principally in which two of the three conditions are simultaneously satisfied.

(c) Time complexity. The 2DAP is solvable in polynomial time, i.e., polynomial bounded number of computational steps. The required number of elementary computational steps is bounded by a polynomial in the size of the problem (Lawler, 1976). This problem therefore belongs to the class known as P (Garey and Johnson, 1979). The 2DAP can be reduced to a minimum cost flow problem, which can be solved with exactly n flow augmentations. Each flow augmentation can be determined by a shortest path computation of $O(n^2)$ complexity. Thus the 2DAP is at worst $O(n^3)$ in complexity (Lawler, 1976).

2-1.1.2. Irregular. Suppose sets A and B in the matching problem have unequal cardinality; let $|A| = p$ and $|B| = q$, and $p < q$. Then some members of B will fail to be matched. Some positions will fail to be filled in the assignment. When the two index sets have unequal cardinality, the assigned problem is called irregular (Shamma, 1971). The irregular 2DAP is stated:

$$\text{Minimize} \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij}x_{ij} \tag{2.3}$$

Subject to:

$$\sum_{i=1}^{p} x_{ij} \leq 1 \qquad (j=1,\ldots,q)$$

$$\sum_{j=1}^{q} x_{ij} = 1 \qquad (i=1,\ldots,p)$$

$$x_{ij} = 0,1 \qquad (\text{all } i,j)\ .$$

This is a weighted, incomplete matching problem (Lawler, 1976).

The same situation may occur in the two-dimensional transportation problem, when total supply is not equal to total demand.

In order to solve the irregular 2DAP by the same method used for the regular 2DAP, dummies are supplied to make the index sets equal in cardinality, and suitable costs, usually zero, are given to the resulting potential matches. The best available algorithms are of complexity $O(pq^2)$ (Klee, 1980).

References to the best available algorithms for the 2DAP, as well as other recent survey material regarding combinatorial optimization, appear in Klee (1980). Other references to exact solution methods for the 2DAP are given in Kurzberg (1962), and also in Hung and Rom (1980). The standard algorithm for the classical two-dimensional assignment problem, 2DAP, is still considered to be the Hungarian method [Kuhn (1955,1956); Lawler (1976); McGinnis (1983)].

2-1.2. Other ILP Two-Dimensional Assignment Problems

2-1.2.1. Frieze (AP2). Another version of a two-dimensional assignment problem appears in Frieze (1974). Consider m projects and n students, usually $m < n$. The maximum number of students which can be assigned to project j is given by b_j $(j=1,\ldots,m)$. Each student must be assigned to exactly one project. AP2 is formulated:

$$\text{Minimize (or maximize)} \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}x_{ij} \tag{2.4}$$

Subject to:

$$\sum_{i=1}^{n} x_{ij} \leq b_j \quad (j=1,\ldots,m)$$

$$\sum_{j=1}^{m} x_{ij} = 1 \quad (i=1,\ldots,n)$$

$$x_{ij} = 0,1 \quad (\text{all } i,j)$$

The a_{ij} are appropriate cost (or profit) coefficients. This problem is considered to be solved efficiently by well-known methods. This version will be seen in higher-dimensional problems of the same pattern. AP2 does not reduce to the irregular 2DAP when $b(j) = 1$, $(j=1,\ldots,m)$, unless $m > n$.

2-1.2.2. Generalized assignment problem (2DGAP). The generalized assignment problem is a 0-1 programming model in which it is desired to minimize the cost of assigning n tasks to a subset of m agents. Each task must be assigned to one agent, but each agent is limited only by the amount of resource, e.g., time, available to him and the fact that the amount of resource required by a task depends on both the task and the agent performing it. Coefficient c_{ij} is the cost incurred if task j is assigned to agent i, $r_{ij} > 0$ is the amount of a resource required by agent i to perform task j, and $a_i \geq 0$ and $b_i > 0$ are, respectively, the minimum and maximum amounts of the resource that may be expended by agent i. The formulation is:

$$\text{Minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} \tag{2.5}$$

Subject to:

$$a_i \leq \sum_{j=1}^{n} r_{ij}x_{ij} \leq b_i \quad (i=1,\ldots,m)$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad (j=1,\ldots,n)$$

$$x_{ij} = 0,1 \quad (\text{all } i,j)$$

It is readily observed that for certain choices of the parameters r_{ij}, a_i, and b_i, that 2DGAP includes regular and irregular, 2DAP and AP2. However, unlike those problems, the corresponding LP solution is not guaranteed to be integer, since the constraint matrix is not in general totally unimodular. However, an efficient branch and bound algorithm is known to exist for 2DGAP (Ross and Soland, 1975,1977).

2-1.3. Non-ILP Two-Dimensional Assignment Problems

2-1.3.1. Bottleneck. The bottleneck assignment problem with $a_{ij} \geq 0$ is obtained by replacing the linear objective of the 2DAP with: $\min \max_{i,j} (a_{ij} \cdot x_{ij})$. If a_{ij} represents the time required by worker i to do job j on an assembly line, the bottleneck assignment problem can be interpreted as that of minimizing the cycle time of the assembly line. The bottleneck assignment problem is also used to obtain equitable solutions, in the sense that the least desirable assignment is made as good as possible. In this type of application

a_{ij} is inversely related to the desirability of assigning man i to job j. Although the bottleneck problem can be reformulated as a linear assignment problem, more efficient methods are available. Two methods of solution are discussed in Garfinkel (1971), including the threshold method.

2-1.3.2. Quadratic. The quadratic assignment problem may be defined in the context of a location problem. There is a set of n locations in which m plants must be constructed, where $m < n$. The unit cost of shipping from location i to location j is c_{ij}. Also, the volume required to be shipped from plant k to plant ℓ is $d_{k\ell}$. Let $x_{ik}=1$ if plant k is assigned to location i; otherwise $x_{ik}=0$. Then the constraints are: $\sum_{k=1}^{m} x_{ik} \leq 1$, $(i=1,\ldots,n)$, since each location can have at most one plant; $\sum_{i=1}^{n} x_{ik} = 1$, $(k=1,\ldots,m)$, since each plant must have exactly one location. If plant k is assigned to location i and plant ℓ to location j, the shipping cost is $c_{ij}d_{k\ell} + c_{ji}d_{\ell k}$. Therefore, the total shipping cost is

$$Z = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sum_{\substack{k,\ell=1 \\ k\neq \ell}}^{m} c_{ij}d_{k\ell}x_{ik}x_{j\ell} .$$

Thus the two-dimensional quadratic assignment problem is stated:

$$\text{Minimize} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sum_{\substack{k,\ell=1 \\ k\neq \ell}}^{m} c_{ij}d_{k\ell}\, x_{ik}x_{j\ell} \tag{2.6}$$

Subject to:

$$\sum_{k=1}^{m} x_{ik} \leq 1 \qquad (i=1,\ldots,n)$$

$$\sum_{i=1}^{n} x_{ik} = 1 \qquad (k=1,\ldots,m)$$

$$x_{ik} = 0,1 \qquad (\text{all } i,k)$$

The book by Garfinkel and Nemhauser (1972) contains extensive material, algorithms, and references relative to the two-dimensional quadratic assignment problem and the other different types of assignment problems with integer programming formulations. A perturbation method has been developed for solving the quadratic assignment as well as quadratic bottleneck problems (Burkard, 1976).

2-1.3.3. Algebraic approach. Again the descriptive word, "generalized," is applied to a class of assignment problems, in the Generalized Linear Assignment Problem (2DGLAP) (Burkard, Hahn, and Zimmerman, 1977). In this case, the generalization is to a class of objective functions which is studied by algebraic methods and characterized in terms of an axiomatic system. It presents a different type of formulation from those described thus far in the two-dimensional assignment problem overview. This algebraic approach to network flow problems provides a unifying framework within which otherwise distinct problems can be tackled by similar methods; this idea is extended to more general linear programming type problems in Frieze (1982).

For 2DGLAP, the coefficients of the objective function can be chosen from a totally ordered commutative semigroup, which obeys a divisibility axiom. Special cases of the general model are the linear assignment problem (2DAP), the linear bottleneck problem, the p-norm assignment problem, and the lexicographic multi-criteria assignment problem. The algorithm solving the 2DGLAP is based on admissible transformations, and is polynomially bounded.

The methods and philosophy used for 2DGLAP can be used to create generalized traveling salesmen problems, quadratic assignment problems and quadratic bottleneck assignment problems, transportation problems, and certain other network flow problems.

2-2. OVERVIEW OF HIGHER DIMENSIONAL ASSIGNMENT PROBLEMS

In moving to a survey of higher dimensional assignment problems, from two-dimensional, many new formulations arise. There is added flexibility in assignment problems with new dimensions, such as time or space. The three dimensions in the special source problem of this dissertation are time periods, buyers, and sellers at a tourism industry convention. Other models will also be described in this section.

When the formulation of a three-dimensional assignment problem is as an integer linear program, the constraint matrix may not be totally unimodular, and so the corresponding LP does not have an integer solution guaranteed. Two-dimensional problems in complexity class P may give way to NP-hard or NP-complete higher dimensional problems. Clever theorems and algorithms which have been developed for three-dimensional assignment problems may generalize in an obvious way to multiindex assignment problems, and yet not be very efficient for these higher dimensions. For example, a tree-search technique has been devised by Pierskalla for one three-index assignment problem; but in generalizing to a multiindex problem, the dimensionality of the problem increases extremely fast, namely by products of factorials—and so it is very doubtful that even small multidimensional problems (dimension ≥ 5) could be solved in any reasonable amount of time. To gain a thorough understanding of these models, some stronger combinatorial concepts or topological concepts may need to be applied (Pierskalla, 1967,1968,1982).

2-2.1. A Generalized Classical Assignment Problem

The first model to be discussed is one which is called generalized in that is intended to encompass most multidimensional classical type assignments. Rather than exactly two sets of elements assigned optimally relative to each other, let ξ = the number of sets of elements that are assigned optimally relative to each other, subject to specific constraints. This generalized problem is called a "ξ-index AP," $\xi \geq 2$ (Shamma, 1971). This model is described in detail here in order to show how many different separate formulations may arise.

The following definitions summarize the different parameters encountered in the generalized assignment problem, ξ-index AP:

Dimension ξ. The number of subscripts on the decision variable x.

Index set I_j. The set of values that the jth subscript can assume $(j=1,\ldots,\xi)$.

Index range n_j. The number of elements in the jth index set, i.e., $I_j=\{1,2,\ldots,n_j\}$, $(j=1,\ldots,\xi)$.

Constraint set. The set of all constraints imposed in the ξ-index AP.

Condition. The constraints may be grouped according to the indices which are kept fixed. Such a grouping is called a condition. The number of constraints in a condition is the product of the index ranges of the indices which are held fixed to form one constraint.

Order t_j. The number of indices held fixed in each equality or inequality constraint in the jth condition. If the order is the same for all conditions, the common value t is the order of ξ-index AP.

Order set T_j. The set of the fixed indices in jth condition; hence $|T_j| = t_j$. Define $\overline{t}_j = \xi - t_j = \xi - |T_j|$.

Order complement set $\overline{T}_j$. The set of indices over which the sum is taken in the jth condition. It is the complement of the order set T_j; $|\overline{T}_j| = \overline{t}_j$.

Maximum number of conditions M. The maximum number of possible conditions. If ξ-index AP is of order t, $M = \binom{\xi}{t}$.

The parameters which have been defined will be used to formulate the generalized ξ-index AP, in the following way:

Given a class of index sets $I = \{I_1, I_2, \ldots, I_j, \ldots, I_\xi\}$, an assignment is a selection of elements of the sets in I, one element from each index set. An assignment, $\{i_1, i_2, \ldots, i_j, \ldots, i_\xi\}$, for example, might be $\{2,3,1,\ldots,7\}$. A rating matrix is assumed known, and $c_{i_1,i_2,\ldots,i_\xi}$ is the value of the matrix corresponding to the assignment above. Let the decision variable $x_{i_1,i_2,\ldots,i_k,\ldots,i_\xi}=1$ if the corresponding assignment (selection) is made, $i_k \in I_k$, over the class of index sets; otherwise, let it be zero. Now, the ξ-index AP is formulated:

$$\text{Minimize} \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_\xi=1}^{n_\xi} c_{i_1,i_2,\ldots,i} \, x_{i_1,i_2,\ldots,i_\xi} \tag{2.7}$$

Subject to:

$$\text{(Condition 1)} \quad \sum_{\text{all } i_j, j \in \overline{T}_1} \cdots \sum x_{i_1,i_2,\ldots,i_\xi} \leq 1 \quad (i_j, j \in T_1)$$

$$\text{(Condition 2)} \quad \sum_{\text{all } i_j, j \in \overline{T}_2} \cdots \sum x_{i_1,i_2,\ldots,i_\xi} \leq 1 \quad (i_j, j \in T_2)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\text{(Condition k)} \quad \sum_{\text{all } i_j, j \in \overline{T}_k} \cdots \sum x_{i_1,i_2,\ldots,i_\xi} \leq 1 \quad (i_j, j \in T_k)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\text{(Condition M)} \quad \sum_{\text{all } i_j, j \in \overline{T}_M} \cdots \sum x_{i_1,i_2,\ldots,i_\xi} = 1 \quad (i_j, j \in T_M)$$

$$x_{i_1,i_2,\ldots,i_\xi} = 0,1 \quad (i_j, j = 1,\ldots,\xi) \; .$$

Equality constraints are used when they can be attained by some feasible solution. Take at least one condition to be equality, for nontriviality of the optimal solution. These equality conditions are written last by convention. If there are less than the maximum number M of possible conditions, use $T_k=\emptyset$ for unused possible conditions, beyond the highest used condition index. For a further definition, when each index has the same range n, and each condition has the same order t, then the ξ-index AP is called regular, and may be denoted (ξ,t,n). The simple assignment problem, 2DAP, becomes $(2,1,n)$.

A method of solution is described (Shamma, 1971) for the regular ξ-index AP. The algorithm to solve (ξ,t,n) is a combinatorial technique which has not been considered before. It uses objective values of certain subsets of feasible solutions in a 2DAP whose optimal solution corresponds to an optimal solution of (ξ,t,n) of the given subset of feasible solutions. The pattern of feasible solutions is analyzed. If the index range n is odd, feasible solutions are generated based on congruence relations. If n is even, two orthogonal Latin squares are used. The optimal solution is found by solving a sequence of simple 2DAP's.

It is of further interest to note the size of ξ-index AP, in terms of the number of decision variables $(\prod_{j=1}^{\xi} n_j)$, the number of constraints in Condition i $(\prod_{j\in T_i} n_j)$, and the total number of constraints $(\sum_{i=1}^{M} \prod_{j\in T_i} n_j)$. For a regular ξ-index AP, denoted (ξ,t,n), the number of decision variables is n^{ξ}, the number of constraints in condition i is n^t, and the total number of constraints is Mn^t, with $M = \binom{\xi}{t}$

usually. Thus 2DAP, or (2,1,n) has n^2 variables and 2n constraints in all. Among three-dimensional assignment problems, (3,1,n) has n^3 decision variables and 3n constraints, while (3,2,n) has n^3 decisions and $3n^2$ constraints.

2-2.2. The Axial Three-Dimensional Assignment Problem (3DAP-Axial)

2-2.2.1. Definition. The regular 3-index AP of order 1, using the generalized terminology of Section 2-2.1, is often referred to in the literature as "the 3-dimensional assignment problem" (Klee, 1980). It may be described in concepts analogous to 2DAP. In the scheduling of men to positions, there could be an added dimension, such as time period, piece of equipment, or some other factor perhaps qualitative in nature, to be considered in making an optimal allocation. Suppose (a_{ijk}) is the cost tensor for assignment of man i to position j and equipment k. Then the 3DAP-axial is the problem of assigning men to positions and equipment on a one-one-one basis so as to minimize the overall cost.

The 3DAP-axial may be briefly defined as finding a subset of n elements of the elements of the nxnxn array (a_{ijk}), or box, with exactly one element from each row, one from each column, and one from each value on the third axis, such that the sum of the elements is minimized. Or, among all sets S of n triples (i,j,k) such that no two members of S agree in any coordinate, find one for which

$\sum_{(i,j,k)\in S} a_{ijk}$ is a minimum.

In order to obtain a mathematical model, let $x_{ijk}=1$ if the decision is to assign man i to position j and equipment k; otherwise, $x_{ijk}=0$. Then, 3DAP-axial is stated:

$$\text{Minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ijk} x_{ijk} \tag{2.8}$$

Subject to:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ijk} = 1, \quad (k=1,\ldots,n)$$

$$\sum_{i=1}^{n} \sum_{k=1}^{n} x_{ijk} = 1, \quad (j=1,\ldots,n)$$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} x_{ijk} = 1, \quad (i=1,\ldots,n)$$

$$x_{ijk} = 0,1 \qquad (\text{all } i,j,k)$$

It can be seen that 3DAP-axial, like 2DAP, is a minimum weight matching problem. A three-dimensional matching (3DM) may be described in the following way (Garey and Johnson, 1979). Let A, B, C be three pairwise disjoint sets having the same number n of elements. Let subset $M \subseteq AxBxC$ be a collection of admissible triplets, called elements, $\{(a,b,c)\}$. A matching is a subset $M' \subseteq M$, where M' has exactly n elements and no two elements of M' agree in any coordinate. For example, if A is n men, B is n positions, and C is n pieces of equipment, then the men, positions, and equipment would be matched in a one-one-one fashion. If a_{ijk} is the cost of assigning man i to position j with equipment k, then 3DAP-axial is to match men, positions, and equipment in a fashion that minimizes overall cost.

The similarities between 2DAP and 3DAP-axial are not extensive. The 3DM problem is included in the "basic core" of six known NP-Complete problems (Garey and Johnson, 1979), whereas 2DM, the "marriage problem," can be solved in polynomial time. Moving on to the minimum weight matching problems, 2DAP and 3DAP-axial: 3DAP-axial is NP-Complete, while 2DAP admits a polynomial algorithm (Klee, 1980).

Another important dissimilarity between 2DAP and 3DAP-axial is seen when considering their ILP formulations. The tripartite graphical representation cannot adequately portray 3DAP-axial, since costs are assigned to triples of nodes rather than to pairs of nodes, or arcs. Furthermore, unlike 2DAP, the solution of the corresponding relaxed LP is not guaranteed to give the required integer solutions for 3DAP-axial, since in general the constraints define a polyhedron with a great number of noninteger extreme points (Pierskalla, 1967).

A further geometric interpretation may be seen for 3DAP-axial. Let A, B, and C be sets of equal cardinality n, and (i,j,k) a point in AxBxC. Since $\sum_{j=1}^{n} \sum_{k=1}^{n} x_{ijk}=1$ $(i=1,\ldots,n)$, no other point in AxBxC can be included in a feasible solution which is on the plane perpendicular to the A-axis, through point (i,0,0). Similar statements hold for j and k. Since all constraints are equalities, each plane perpendicular to an axis (at an integer less than or equal to n) contains exactly one point. See Figure 1.

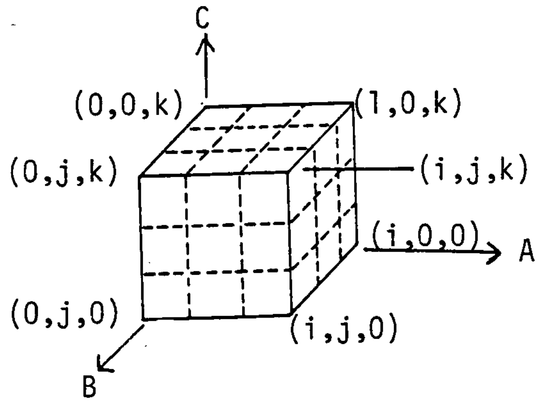


Figure 1. Representation of selection (i,j,k) from AxBxC.

The irregular 3DAP-axial occurs when the ranges of the index sets are unequal. If the index sets, A, B, and C, have cardinality p, q, and r, respectively, and $p \leq q \leq r$, then some members of B and C may fail to be matched. The formulation requires some inequality constraints in order to get feasible solutions. Then the formulation of the irregular 3DAP-axial is:

$$\text{Minimize} \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} a_{ijk} x_{ijk} \tag{2.9}$$

Subject to:

$$\sum_{i=1}^{p} \sum_{j=1}^{q} x_{ijk} \leq 1, \quad (k=1,\ldots,r)$$

$$\sum_{i=1}^{p} \sum_{k=1}^{r} x_{ijk} \leq 1, \quad (j=1,\ldots,q)$$

$$\sum_{j=1}^{q} \sum_{k=1}^{r} x_{ijk} = 1, \quad (i=1,\ldots,p)$$

$$x_{ijk} = 0,1 \qquad (\text{all } i,j,k)$$

Equality constraints are used whenever they can be attained by some feasible solution.

As usual, if a regular problem is desired for a solution method, the smaller index sets may be augmented by dummy members to give equal ranges, making the problem regular.

The irregular 3DAP-axial is then a minimum weight, incomplete matching problem, in concept.

2-2.2.2. Solution methods. An algorithm for the irregular 3DAP-axial was given by Pierskalla (1967,1968), which is based on a tree search technique of the branch and bound variety. The method is a form of implicit enumeration of all basic feasible points. Dual subproblems are used to provide easily computed but not necessarily tight bounds for the primal assignment problem.

Although the branch and bound solution by Pierskalla (1967,1968) gave the optimal solution to the axial multiindex assignment problem, it was not considered efficient for larger multidimensional problems. He presented a tri-substitution method for obtaining close-to-optimal solutions for the irregular 3DAP-axial (Pierskalla, 1966). The tri-substitution method works on a principle similar to the simplex method of linear programming; however, it differs from the simplex method in that three new variables are inserted and three removed at each step. Furthermore, it is an additive algorithm in the sense that at any step only additions and subtractions are performed.

The 3DAP-axial is included in the discussions and solutions reviewed by Leue (1972), Burkard and Frohlich (1980), and Kaufman (1975).

A two-dimensional transportation problem formulation was used by Swanson (1976) to solve a special irregular 3DAP-axial. He considers the simultaneous assignment of man-product-equipment, with the simplifying assumption that the cost of man-product assignment, d_{ij}, plus the cost of product-equipment assignment, e_{jk}, is equal to the cost of man-product-equipment assignment, c_{ijk}. That is, $c_{ijk} = d_{ij} + e_{jk}$. The method described by Swanson can be extended to as many dimensions as desired as long as the additive cost feature is kept intact. Thus an axial three-dimensional assignment problem with this special objective function can be solved by the simpler methods of a two-dimensional transportation problem, in spite of the fact that the 3DAP-axial in general is NP-complete.

The special irregular 3DAP-axial solved by Swanson's representation of it as a two-dimensional transportation problem, can be generalized not only to higher-dimensional assignment problems, but also to restricted three-dimensional transportation problems with interpretations of assignment availabilities. This is an area of possible future investigation.

2-2.3. The Planar Three-Dimensional Assignment Problem (3DAP-Planar)

2-2.3.1. Definition. The regular 3-index AP of order 1, (3,1,n), has been called 3DAP-axial, with a maximum of 3n constraints in all. The regular 3-index AP of order 2, (3,2,n), is known as the planar three-dimensional assignment problem, or 3DAP-planar, with a maximum of $3n^2$ constraints in all [Burkhard and Frohlich (1980); Leue (1972); Shamma (1971)]. The planar problem does not involve a one-one-one matching type of assignment and so is not as similar in concept to the classical 2DAP as is the axial problem.

The nxnxn cost tensor, (a_{ijk}), is given. The mathematical formulation of 3DAP-planar is:

$$\text{Minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ijk} x_{ijk} \tag{2.10}$$

Subject to:

$$\sum_{i=1}^{n} x_{ijk} = 1, \quad (j,k=1,\ldots,n)$$

$$\sum_{j=1}^{n} x_{ijk} = 1, \quad (i,k=1,\ldots,n)$$

$$\sum_{k=1}^{n} x_{ijk} = 1, \quad (i,j=1,\ldots,n)$$

$$x_{ijk} = 0,1, \quad (\text{all } i,j,k)$$

The irregular 3-index AP of order 2 has unequal index set ranges, necessitating some inequality constraints. Let A, B, and C be the index sets with cardinality p, q, and r, respectively, with $p \leq q \leq r$. Then the formulation of the irregular 3DAP-planar is:

$$\text{Minimize} \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} a_{ijk}x_{ijk} \tag{2.11}$$

Subject to:

$$\sum_{i=1}^{p} x_{ijk} \leq 1, \quad (j=1,\ldots,q;k=1,\ldots,r)$$

$$\sum_{j=1}^{q} x_{ijk} \leq 1, \quad (i=1,\ldots,p;k=1,\ldots,r)$$

$$\sum_{k=1}^{r} x_{ijk} = 1, \quad (i=1,\ldots,p;j=1,\ldots,q)$$

$$x_{ijk} = 0,1, \qquad (\text{for all } i,j,k)$$

As usual, equality constraints are used whenever they can be attained by some feasible solution; and, if desired, the smaller index sets may be augmented with artificial elements to give a regular model formulation.

An application of the irregular 3DAP-planar is to schedule client j to meet with agent k at time i, optimally overall, so that each client meets one agent at one time, each agent meets at most one client at one time, and each client and agent pair have no more than one time of meeting scheduled. It is immediately recognized that the irregular 3DAP-planar is the skeleton model for the tourism convention problem, which is the source problem for this dissertation.

In order to get a geometric interpretation of 3DAP-planar, let (i,j,k) be an element selected from AxBxC as shown in Figure 1. Since $\sum_{k=1}^{n} x_{ijk}=1$, $(i,j=1,\ldots,n)$, no other point of AxBxC is in the same feasible solution with (i,j,k), if that point is on a line perpendicular to the AxB plane at (i,j,0). Since all constraints are equalities, each line which is perpendicular to a coordinate plane, AxB, BxC, or CxA, at a point with integer feasible coordinates, has exactly one point.

2-2.3.2. Solution methods. The primary reference for solution of the planar three-dimensional assignment problem is still the work of Vlach (1967), in much the same way that Pierskalla's work (1966, 1967,1968) is most often referenced for the axial model. Both planar and axial models may be considered to be included in the solution of the generalized ξ-index AP (Shamma, 1971) and a matroid intersection problem solution (Burkhard and Frohlich, 1980).

Vlach (1967) devised an algorithm specific to the 3DAP-planar. It is based on the branch and bound technique as described by Little, Murty, Sweeney, and Karel (1963) for TSP, but using ideas frequently employed in solving assignment problems. Instead of solving the irregular 3DAP-planar directly, Vlach normalizes the coefficients of the objective function so that $c_{ijk} = c_{ijk}{}' + k_{ijk}$, where the k_{ijk}'s are constants which make all objective function coefficients non-negative with at least one zero in every row, column, and diagonal in the objective coefficients tensor. The bounds obtained by this solution are used in a branch and bound algorithm.

Some discussion and further work on solution methods for both the planar and axial three-dimensional assignment problems are given in a paper in German by Leue (1972).

2-2.4. Three-Dimensional Transportation Problems

Transportation problems are considered to be combinatorially equivalent to assignment problems, and so the literature on higher dimensional transportation problems and their solution is of interest. Like 2DAP, the two-dimensional transportation problem can be extended in several ways, depending on the type of index summations of the constraints. Schell considered four cases: (1) three planar sums; (2) two planar sums; (3) one planar and one axial sum; and (4) three axial sums (Schell, 1955). The three planar sum case of Schell's work is called the solid transportation problem, and the multiindex problem, in a series of articles by Haley (1962, 1963,1965,1967), Vlach and Moravek (1967a,1967b), Vlach (1965), and Svabova and Marunova (1967). Quite a bit of study has gone into the determination of feasible solutions, as well as into their very existence.

The three-planar-sums solid transportation problem has for a restricted case 3DAP-planar, and is formulated in the following way:

$$\text{Minimize} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{p} c_{ijk}x_{ijk} \qquad (2.12)$$

Subject to:

$$\sum_{i=1}^{n} x_{ijk} = A_{jk}$$

$$\sum_{j=1}^{m} x_{ijk} = B_{ki}$$

$$\sum_{k=1}^{p} x_{ijk} = E_{ij}$$

$$x_{ijk} \geq 0$$

All constants are assumed given nonnegative, and the constraints hold for indices ranging over their values. It is also required that the constants satisfy:

$$\sum_{j=1}^{m} A_{jk} = \sum_{i=1}^{n} B_{ki} \qquad (k=1,\ldots,p)$$

$$\sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{m} E_{ij} \qquad (i=1,\ldots,n)$$

$$\sum_{i=1}^{n} E_{ij} = \sum_{k=1}^{p} A_{jk} \qquad (j=1,\ldots,m)$$

One application of the multiindex problem is described as minimizing the cost of moving a set of p different commodities, from n origins, to m destinations. The same set of restrictions arise when a single commodity has to be moved by different methods, e.g., road, rail, etc. This may also be a capacitated transportation problem, with upper bounds on the decision variables.

Haley (1962,1963,1965) starts a discussion of necessary and sufficient conditions for the existence of a solution, and gives an algorithm which is essentially an extension of the modi-method for the two-dimensional transportation problem. He also shows that the three-axial-sums transportation problem can be transformed into the three-planar-sums case.

It is worth noting that, although the planar and axial regular 3-index AP which have been introduced, along with some other ILP problems which are classified as assignment problems, can be considered as special cases of three-dimensional transportation problems, the methods developed for these latter problems cannot all be expected to guarantee integer solutions when used for assignment problems. In particular, Pierskalla (1968) warned that the polyhedron defined by the constraints has in general a great number of noninteger extreme points and the approach taken by Schell (1955) and Haley (1962,1963, 1965), as well as the simplex method, will most often not yield integer solutions.

2-2.5. Other ILP Three-Dimensional Assignment Problems

2-2.5.1. Frieze (AP3). Another assignment problem used by Frieze does not fit into the generalized 3-index AP described previously. The following model is used by Frieze (1974) and Frieze and Yadegar (1981):

$$\text{Maximize} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ijk} x_{ijk} \tag{2.13}$$

Subject to:

$$\sum_{i=1}^{m} \sum_{k=1}^{p} x_{ijk} \leq b_j, \quad (j=1,\ldots,n)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leq c_k, \quad (k=1,\ldots,p)$$

$$\sum_{j=1}^{n} \sum_{k=1}^{p} x_{ijk} = 1, \quad (i=1,\ldots,m)$$

$$x_{ijk} = 0,1, \quad (\text{all } i.j,k)$$

The above problem would arise if each student i had to be assigned a project j and a supervisor k, and a_{ijk} was the suitability of this assignment. Assume without loss of generality that $b_1,\ldots,b_n$, $c_1,\ldots,c_p$ are positive integers. Supervisor k is to be assigned no more than c_k student-project pairs, and project j is to be assigned no more than b_j student-supervisor pairs. Thus, AP3 is not a one-one-one assignment like the 3DAP-axial unless all b_j and c_k are equal to one.

AP3 is known to be an NP-hard problem (Frieze and Yadegar, 1981). AP3 was formulated as a bilinear problem and solved heuristically, using a series of solutions to AP2 (Frieze, 1974). In the same paper, Frieze suggests replacing the objective function of AP3 by a convex function which agrees with the original one at integer solutions and applying an algorithm by Tuy (1964). A third approach is given by

Frieze and Yadegar (1981) using the idea of Lagrangian relaxation introduced by Held and Karp (1971) and a subgradient algorithm.

2-2.5.2. Weighted (3DGAP). Another ILP three-dimensional assignment problem model (Ross and Zoltners, 1979) resembles very closely what has been named the two-dimensional generalized assignment problem (Ross and Soland, 1975,1977). A third dimension of task-completion levels is introduced into the earlier model. The formulation is:

$$\text{Minimize } \sum_{i \in I} \sum_{j \in J} \sum_{k \in K_{ij}} c_{ijk} x_{ijk}, \tag{2.14}$$

Subject to:

$$a_i \leq \sum_{j \in J} \sum_{k \in K_{ij}} r_{ijk} x_{ijk} \leq b_i \quad (\text{for all } i \in I)$$

$$\sum_{i \in I} \sum_{k \in K_{ij}} x_{ijk} = 1 \quad (\text{for all } j \in J)$$

$$\text{and } x_{ijk} = 0,1 \quad (\text{for all } i \in I,\ j \in J,\ k \in K_{ij})\ . \tag{2.12}$$

This is related to the two axial sums models, since there are two conditions for one index set each.

The weighted three-dimensional assignment problem may be viewed as a model then for determining an assignment of tasks to agents. Each task j must be completed by only one agent i at one of several admissible levels indexed by a discrete set K_{ij}. The levels may correspond to alternative methods for alternative amounts of effort the agent might employ to complete a task. Thus, the decision variable, x_{ijk}, reflects the option of choosing an agent i and a performance level k for each task j. As before, c_{ijk} is some coefficient of disutility, and r_{ijk} is a coefficient of resources used, with a_i and b_i the lower and upper bounds, respectively, for the amount of resource available to agent i.

This model is shown by Ross and Zoltners (1979) to encompass a variety of integer programming models, including knapsack models, transignment models (Caswell, 1972), the 2DGAP, and facility location. Specialized algorithms have been devised for various applications to exploit the model structure more efficiently than would be accomplished using general purpose integer programming algorithms. The most commonly employed solution is branch and bound. Other methods used include extreme point ranking, decomposition, and heuristics, with many references found in Ross and Zoltners (1979).

2-2.6. Non-ILP Three-Dimensional Assignment Problems

2-2.6.1. Bottleneck. All of the three dimensional assignment problems which have been considered up to this point have a linear objective function. The corresponding bottleneck problems are to minimize $\{\max_{i,j,k} (a_{ijk}x_{ijk})\}$.

2-2.6.2. Quadratic. Another important model can be considered in the realm of zero-one integer problems which classify as three-dimensional assignment problems, namely the three-dimensional quadratic assignment problem. Any of the linear models which have been described might for some applications better use the quadratic objective function Z, where $Z = \sum_i \sum_j \sum_k \sum_\ell \sum_m \sum_n a_{ijk\ell mn} x_{ijk} x_{\ell mn}$, accompanied by the standard definitions and explanations of variables and constants.

Pierskalla (1967) used the same tree search, branch and bound algorithm which he designed for the axial multidimensional (three and higher) assignment problem, with appropriate modification, to find an optimal solution for the multidimensional (three and higher) axial quadratic assignment problem.

2-2.6.3. Algebraic approach. Burkard and Frohlich (1980) present a matroid intersection problem which includes as special cases both the axial (3DAP-axial) and planar (3DAP-planar) three-dimensional assignment problems. They describe two approaches at solution, one using admissible transformations and another using subgradients. They report that the computational tests showed that the subgradient approach was considerably faster than the transformation method. Lawler (1976) in the chapter on "Matroid Intersections" assigns the problem to formulate the three-dimensional assignment problem as a problem involving the intersection of three matroids.

This suggests again the need to search for optimization models which may include other problem formulations which were thought to be different or even unrelated.

Algebraically, the study of combinatorial properties of multidimensional matrices, including zero-one matrices, has important meaning for assignment problems and other classical combinatorial problems (Jurkat and Ryser, 1968). Some important two-dimensional theorems such as Konig's theorem (Ryser, 1963) do not generalize trivially. Some references suggested by Ryser (1982) are by Csima (1970), and Brualdi and Csima (1975).

2-2.7. Source Problem as a Three-Dimensional Assignment Problem

It is seen that many different types of model formulations may be interpreted as higher dimensional assignment problems. At least one of the summation equality conditions must have equality to one in order to select an "assignment."

In regard to the special research problem of this investigation, the first objective (1.1) with constraints, forms a restricted irregular planar three-dimensional assignment problem, 3DAP-planar. The restriction is brought about by the last constraint (1.6), which causes the elements of the largest set to have approximately equal "lack" of assignments. The source problem is classified as irregular because of unequal index ranges for time, buyers, and sellers. The objective function coefficients are of the form, $c_{ijk} = c_{hjk}$ (for all $i,h=1,\ldots,q$).

The secondary objective of the source problem is seen to be related to the multiple traveling salesmen problem.

2-3. THE PLANAR THREE-DIMENSIONAL ASSIGNMENT PROBLEM

The tourism industry convention problem which is of special interest in this study belongs to the general class of planar three-dimensional assignment problems (irregular 3DAP-planar). Before proceeding to a discussion of special solution methods for the source problem, the general problem deserves to be recognized for its difficulty. It will be shown to be NP-hard in computational complexity, and to possess a constraint matrix which is not totally unimodular.

2-3.1. Absence of Total Unimodularity

2-3.1.1. Definition. The mathematical formulation of any version of assignment problem, if it has a linear objective function, places the problem in the category of integer linear programming. The classical two-dimensional assignment problem, 2DAP, deals with two sets of elements to be assigned relative to each other optimally. In the field of integer programming, these problems, along with two-index transportation problems, are included among those with a special mathematical structure, total unimodularity of constraint matrix, which guarantees an optimal integer solution. Therefore, these two-index assignment problems may be solved using any extreme point linear programming method, such as the simplex method. In practice, special purpose, graph-oriented algorithms are usually considerably more efficient (Garfinkel and Nemhauser, 1972).

A square, integer matrix B is defined to be unimodular if the absolute value of the determinant, $D = |\det B| = 1$. An integer mxn matrix A is called totally unimodular if every square, nonsingular submatrix B of A is unimodular. It is clear that A_{mxn} is totally unimodular if and only if the transpose $A_{mxn}{}^T$ is.

2-3.1.2. Two theorems. The following two theorems are given with proof by Garfinkel and Nemhauser (1972). They also appear in corresponding material in Lawler's book (1976). The first theorem guarantees integer extreme point optimal solution(s) for an integer linear programming problem with totally unimodular constraint matrix A and integral right-hand side vector b. The second theorem will be used to prove total unimodularity of any three-dimensional planar assignment problem if the range of each index set is set at most two.

Theorem 1: Let A be an integer matrix. The following statements are equivalent:

1. A is totally unimodular.

2. The extreme points (if any) of $S(b) = \{x | Ax \leq b, x \geq 0\}$ are integer for arbitrary integer b.

3. Every square nonsingular submatrix of A has an integer inverse.

Theorem 2: An integer matrix A with $a_{ij} = 0,1,-1$ for all i and j, is totally unimodular if:

1. No more than two nonzero elements appear in each column.

2. The rows can be partitioned into two subsets Q_1 and Q_2 such that

   (a) If a column contains two nonzero elements with the same sign, one element is in each of the subsets.

   (b) If a column contains two nonzero elements of opposite sign, both elements are in the same subset.

2-3.1.3. A totally unimodular 3DAP-planar. Let $n = 2$ for the regular three-dimensional planar assignment problem, which may then be abbreviated as (3,2,2). The eight decision variables are portrayed in Table 1.

TABLE 1

DECISION VARIABLES DISPLAYED FOR 2x2x2 3DAP-PLANAR

$i \searrow j$ \ k	1	2
1	x_{111} x_{211}	x_{112} x_{212}
2	x_{121} x_{221}	x_{122} x_{222}

The constraints for the 2x2x2 3DAP-planar are easily seen to give the corresponding constraint matrix A_{12x8}, as shown in Table 2.

TABLE 2

CONSTRAINTS AND MATRIX FOR 2x2x2 3DAP-PLANAR

$$\sum_{i=1}^{2} x_{ijk} = 1 \ (j=1,2;\ k=1,2)$$

$$\sum_{j=1}^{2} x_{ijk} = 1 \ (i=1,2;\ k=1,2)$$

$$\sum_{k=1}^{2} x_{ijk} = 1 \ (i=1,2;\ j=1,2)$$

$$A = \begin{pmatrix} 1&1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&1\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&1\\ 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&1 \end{pmatrix}$$

Theorem 3: A is totally unimodular.

Proof: A is totally unimodular if A^T is, and so it will be shown that A^T is totally unimodular using Theorem 2.

Partition the rows of A^T into $Q_1 = \{1,4,6,7\}$ and $Q_2 = \{2,3,5,8\}$, as shown on the matrix, with the Q_1 rows emphasized by "*". See Table 3.

TABLE 3

PARTITIONED TRANSPOSE OF CONSTRAINT MATRIX
OF 2x2x2 3DAP-PLANAR

$$A^T = \begin{array}{c} * \\ \\ \\ * \\ \\ * \\ * \\ \\ \end{array} \begin{pmatrix} 1&0&0&0&1&0&0&0&1&0&0&0 \\ 1&0&0&0&0&1&0&0&0&1&0&0 \\ 0&1&0&0&1&0&0&0&0&0&1&0 \\ 0&1&0&0&0&1&0&0&0&0&0&1 \\ 0&0&1&0&0&0&1&0&1&0&0&0 \\ 0&0&1&0&0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1&0&0&0&1 \end{pmatrix} \begin{array}{l} *\ Q_1 \\ \ Q_2 \\ \ Q_2 \\ *\ Q_1 \\ \ Q_2 \\ *\ Q_1 \\ *\ Q_1 \\ \ Q_2 \end{array}$$

Next, check that matrix A^T and sets Q_1 and Q_2 satisfy the requirements for Theorem 2:

1. Each column of A^T has exactly two nonzero elements.

2. (a) In each column of A^T, it is verified that one nonzero element is in Q_1 and the other is in Q_2.

(b) Matrix A^T contains only positive or zero elements, and thus there are no nonzero elements of opposite sign in any column.

By Theorem 2, matrix A^T is totally unimodular, and hence also matrix A is totally unimodular. (QED)

Note that Theorem 2 cannot be used to prove total unimodularity of A^T for a three-dimensional planar assignment problem if the range of any index is more than two, since in that case A^T has some columns with more than two nonzero elements.

A counterexample will be given next to show that larger versions of 3DAP-planar are not in general totally unimodular.

2-3.1.4. Counterexample. Although it has been proved above that a 2x2x2 3DAP-planar does have total unimodularity of constraint matrix, the following counterexample denies that property to higher index ranges. Let the 3x3x3 3DAP-planar have the 27 decision variables, (x_{ijk}), arranged in the chart in Table 4.

Define objective function coefficients, (c_{ijk}), by the chart in Table 5.

When this particular three-dimensional planar assignment problem, which is an integer linear programming problem in itself, is relaxed to a linear programming problem, the optimal LP solution yields minimum objective value equal to 0.5. The noninteger decision variables for an optimal solution are shown in Table 6 with row totals formed for calculation of the corresponding value of the objective function.

TABLE 4

DECISION VARIABLES DISPLAYED FOR 3x3x3 3DAP-PLANAR

i \ j \ k	1	2	3
1	x_{111} x_{211} x_{311}	x_{112} x_{212} x_{312}	x_{113} x_{213} x_{313}
2	x_{121} x_{221} x_{321}	x_{122} x_{222} x_{322}	x_{123} x_{223} x_{323}
3	x_{131} x_{231} x_{331}	x_{132} x_{232} x_{332}	x_{133} x_{233} x_{333}

TABLE 5

OBJECTIVE FUNCTION COEFFICIENTS FOR 3x3x3 3DAP-PLANAR

i	j	k = 1	k = 2	k = 3
1	1	1	0	0
1	2	0	1	0
1	3	0	0	0
2	1	0	0	1
2	2	0	1	1
2	3	0	0	0
3	1	0	1	0
3	2	1	0	1
3	3	0	0	0

TABLE 6

NONINTEGER OPTIMAL LP SOLUTION FOR COUNTEREXAMPLE 3DAP-PLANAR

i j k	1	2	3	Obj.
	0	.5	.5	0
1	.5	0	.5	0
	.5	.5	0	0
	.5	.5	0	0
2	.5	0	.5	.5
	0	.5	.5	0
	.5	0	.5	0
3	0	1	0	0
	.5	0	.5	0
				.5

It is clear that with all objective function coefficients c_{ijk} = 0 or 1, no integer solution could give an objective value of 0.5. Furthermore, there are only twelve feasible 0 or 1 solutions, and these also can easily be checked to see that none gives an objective value of zero.

Therefore, linear programming solution methods are not adequate to produce guaranteed optimal integer solutions for three-dimensional planar assignment problems, and usually other techniques must be employed.

2-3.2. Computational Complexity NP-Hard

A total assignment problem (3DA), which is a special case of 3DAP'-planar, is proved NP-complete in a paper by Frieze (1981). Since the problem 3DA is reducible to the irregular 3DAP-planar, the latter is of computational complexity NP-hard (or NP-complete).

The problem 3DA, shown by Frieze to be NP-complete, is the existence of a total assignment of the regular planar form, in a subset of PxQxR, where P, Q, R are finite disjoint sets of equal size. One application of 3DA is the following timetabling problem: given m teachers, m classes and m time periods and a set S of triples, $\{(i,j,k)\}$, where we can schedule teacher i to meet class j in time period k, we ask if it is possible, using only triples from S, to arrange that in each time period, each teacher is assigned to exactly one class and vice-versa, so that after m periods every teacher has taught every class.

The mathematical formulation is derived for $a_{ijk} = 1$ if (i,j,k) is in the set S; otherwise $a_{ijk} = 0$. The 3DA is stated:

$$\text{Maximize} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} a_{ijk} x_{ijk} \tag{2.15}$$

Subject to:

$$\sum_{i=1}^{m} x_{ijk} = 1 \quad (j,k=1,2,\ldots,m)$$

$$\sum_{j=1}^{m} x_{ijk} = 1 \quad (i,k=1,2,\ldots,m)$$

$$\sum_{k=1}^{m} x_{ijk} = 1 \quad (i,j=1,2,\ldots,m)$$

$$x_{ijk} = 0 \text{ or } 1 \text{ (all } i,j,k)$$

If the optimal solution to this 3DAP-planar is m^2, then a feasible solution exists for a problem with the cells of S admissible; that is, a total assignment exists.

Although complexity has been mentioned quite a bit earlier in this dissertation, we give a brief discussion here because of the application to 3DAP-planar, the central problem. Standard references for detailed analysis of the concepts of complexity are Garey and Johnson (1979), Aho et al. (1974), and Klee (1980). A problem X is said to be (polynomially) reducible to Y, written $X \propto Y$, if each polynomially bounded algorithm for solving Y can be used to produce a polynomially bounded algorithm for solving X. This does not require that X or Y actually admits a good algorithm, but only if Y does so does X. Note that reducibility is transitive. A problem Y is called NP-hard if every problem in NP is reducible to Y; if, in addition, Y itself belongs to NP, then Y is NP-complete (Klee, 1980).

Since the NP-complete problem 3DA is reducible to 3DAP-planar, then 3DAP-planar is NP-hard.

CHAPTER III

A SURVEY OF MULTIPLE TRAVELING SALESMAN PROBLEMS

It has been noted in Chapter I that scheduling a timetable of meetings between tour brokers and vendors is a multiobjective problem. The primary objective is to arrange meetings in harmony with the preferences specified by the participants. The secondary objective is to minimize the travel time of the clients as they move from booth to booth for the meetings in successive time periods. The previous chapter was a survey of multidimensional assignment problems and a study of the three-dimensional planar assignment problem in particular, in order to gain an appreciation of the primary objective of the tourism industry convention scheduling problem. In this chapter, the familiar single traveling salesman problem (TSP) and m-salesmen traveling salesman problem (m-TSP) will be surveyed, followed by the definition of a new type, which we have coined the musical chairs traveling salesman problem (MCTSP). This review will give a stronger background for the presentation of solution methods and heuristics in later chapters.

3-1. OVERVIEW OF SINGLE TRAVELING SALESMAN PROBLEM (TSP)

The traveling salesman problem (TSP) is that of finding a least-distance route for a salesman who must visit each of n cities, starting and ending his journey at City 1.

3-1.1. Mathematical Formulation

Let $N = \{1,2,\ldots,n\}$ be the set of cities, or nodes. Let a_{ij} = distance from i to j ($a_{ii} = \infty$ for all i). In many applications, it is true that $a_{ij} = a_{ji}$ (symmetric case), but sometimes not (asymmetric). Let $x_{ij} = 1$ if the salesman goes directly from i to j; otherwise $x_{ij} = 0$. Then the TSP is:

$$\text{Minimize } \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_{ij} , \tag{3.1}$$

Subject to:

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j=1,2,\ldots,n) \tag{3.2}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i=1,2,\ldots,n) \tag{3.3}$$

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} \geq 1 \quad \text{(all nonempty proper subsets } S \text{ of } N) \tag{3.4}$$

$$x_{ij} = 0,1 \quad \text{(all } i,j) \tag{3.5}$$

Constraints (3.2), (3.3), and (3.5) are recognized to be the 2DAP, and (3.4) is a set of 2^n-2 loop constraints to eliminate subtours from feasible solutions. The TSP is of computational complexity NP-hard (Klee, 1980). There are alternative constraints to (3.4) for subtour elimination in the TSP formulation (Wagner, 1975; Svestka and Huckfeldt, 1973; Christofides, 1979).

An alternative formulation of TSP, called the Miller-Tucker-Zemlin formulation (1960), is given by Svestka and Huckfeldt (1973).

This formulation also appears in the survey article by Bellmore and Nemhauser (1968). Additional decision variables y_i, for i=1,2,...,n, are arbitrary real numbers which satisfy constraints (3.4') given below, instead of (3.4) from previous formulation.

$$y_i - y_j + nx_{ij} \leq n - 1, \text{ for all } i \neq j \text{ and } i \neq 1,\ j \neq 1. \qquad (3.4')$$

Thus there are (n-1) (n-2) constraints of type (3.4'), and n constraints for each of types (3.2) and (3.3).

3-1.2. Graph Theoretic Formulation

The TSP is included in five basic network optimization problems discussed by Klee (1980) in a survey of combinatorial optimization. The assumptions which are used in the definition of TSP are that a loopless directed or undirected finite graph G = (N,A) is given, as foundation of a network (G,α), with function $\alpha:A \Rightarrow [0,\infty]$. A tour is a spanning simple circuit, or Hamiltonian cycle, and TSP is the minimum tour problem, that of finding a shortest tour for a complete G. In many applications, the triangle inequality for α is assumed, especially if the interpretation is some kind of distance.

3-1.3. Algorithms

One exact solution technique is to solve relaxation 2DAP's and use branch and bound to break up subtours (Garfinkel, 1982a; Bellmore and Malone, 1971).

Bellmore and Nemhauser (1968) compiled a survey of algorithms that had been developed for either exact or approximate (heuristic) solutions to TSP. They state minimum assumption, fundamental definitions, followed by several theorems, solution techniques, and computational results. Another important algorithm is branch and bound, based on 1-trees (Held and Karp, 1970,1971). Effective solution techniques for the Bottleneck TSP are reported by Garfinkel and Gilbert (1978).

3-1.4. Related Problems

The applications of TSP include computer wiring, vehicle routing, clustering of data, job shop scheduling, archaeological seriation, and minimizing wallpaper waste (Garfinkel, 1977; Klee, 1980).

The TSP can be formulated as a problem calling for a maximum weight intersection of three matroids (Lawler, 1976). The TSP is also a special case of the quadratic assignment problem (Garfinkel and Nemhauser, 1972).

In the case that one node is not distinguished as the home city, then the salesman need not traverse a Hamiltonian cycle, so that the problem is to find a minimum Hamiltonian chain. To transform the shortest Hamiltonian chain problem to TSP, it is only necessary to add a "dummy" node * as well as arcs (i,*) and (*,i) for all $i \in N$, with distances $a_{i*} = a_{*i} = 0$ (Garfinkel, 1982b).

3-2. OVERVIEW OF M-SALESMEN TRAVELING SALESMAN PROBLEM (M-TSP)

One important extension of TSP allows a fixed number m of salesmen. Then the m-salesmen traveling salesman problem (m-TSP) may be stated as follows (Svestka and Huckfeldt, 1973): Given m salesmen and n cities, find m sorties such that every city (except the home city) is visited exactly once by exactly one salesman, so that the total distance traveled by all the salesmen is minimum. Two other important scheduling and sequencing applications include bank messenger scheduling (Svestka and Huckfeldt, 1973) and printing press scheduling for multiedition periodicals (Gorenstein, 1970).

3-2.1. Mathematical Formulation

A new distance coefficient matrix (d_{ij}) is formed from the original (a_{ij}) for one salesman. Matrix (a_{ij}) is augmented by m-1 new rows and m-1 new columns, where each new row and column is a duplicate of the first row and column of the matrix (a_{ij}). It is assumed the first row and column correspond to the home city. Set all other new elements of the augmented matrix to infinity. See Table 7.

TABLE 7

NEW COEFFICIENT MATRIX FOR M-TSP

$$(d_{k\ell}) = \begin{array}{|c|c|l} \hline \infty & a_{1j} & \left.\right\} m-1 \\ \hline a_{i1} & (a_{ij}) & \left.\right\} n \\ \hline \multicolumn{1}{c}{\underbrace{\qquad\qquad}_{m-1}} & \multicolumn{1}{c}{\underbrace{\qquad\qquad}_{n}} & \end{array}$$

All other terms have the same definition as for TSP formulation: n, x_{ij}, y_i. Define $r = m + n - 1$, and then $i,j=1,2,\ldots,r$. Then m-TSP is (Svestka and Huckfeldt, 1973):

$$\text{Minimize} \quad \sum_{i=1}^{r} \sum_{j=1}^{r} d_{ij} x_{ij} , \tag{3.6}$$

Subject to:

$$\sum_{i=1}^{r} x_{ij} = 1 \quad (j=1,2,\ldots,r) \tag{3.7}$$

$$\sum_{j=1}^{r} x_{ij} = 1 \quad (i=1,2,\ldots,r) \tag{3.8}$$

$$y_i - y_j + \left\lceil \frac{n}{m} \right\rceil x_{ij} \leq \left\lceil \frac{n}{m} \right\rceil + m - 2 \quad (\text{all } i \neq j, \text{ and } i,j=1,2,\ldots m) \tag{3.9}$$

$$x_{ij} = 0,1 \text{ and } y_i \text{ are arbitrary real numbers (all } i,j) \tag{3.10}$$

This formulation of m-TSP reduces to the TSP when $m = 1$. Constraints (3.7), (3.8), and (3.10) are the 2DAP, for r cities. Cities $2,\ldots,m-1$ are dummy nodes. Constraints (3.9) block the formation of infeasible tours or subtours, those containing no dummy nodes. There are $(n-1)(n-2)$ constraints, independent of m, of type (3.9), and r constraints for each of types (3.7) and (3.8).

Svestka and Huckfeldt (1973) developed a branch and bound algorithm for this formulation based on the 2DAP relaxation. Another formulation for m-TSP appears in Fox et al. (1980).

3-2.2. Related Problems

A graph-theoretic formulation of m-TSP, which reduces to TSP when m=1, is given by Frieze (1980), along with a worst-case analysis of an approximation algorithm.

Other variations on m-TSP are being studied as to applications, formulations, other restrictions, and algorithms (Ali and Kennington, 1980; Gavish and Srikanth, 1980; Rao, 1980).

3-3. FORMULATION OF THE MUSICAL CHAIRS TRAVELING SALESMAN PROBLEM

The second objective of the tourism scheduling problem gives rise to a new type of multiple traveling salesman problem which we have coined the musical chairs traveling salesman problem (MCTSP). This section gives the mathematical formulation of this problem.

3-3.1. Mathematical Formulation

Let $N = \{1,2,...,n\}$ represent a set of n cities, and $M = \{1,2,...,m\}$ a set of m traveling salesmen. Suppose each salesman $j \in M$ has a specified list $N_j \subseteq N$ of q cities which he must visit in a sequence of q periods, one city per period. The trips must be nonconflicting, i.e., at most one salesman can visit a given city in a given time period. Assume $0 < q < m < n$. This implies that each city $k \in N$ is to be visited by at most q traveling salesmen, whose identities are known in advance.

Let $M_k \subseteq M$ be the set of salesmen who will visit city k, for each $k \in N$. The distance from city k to city h, for any two cities, k and h, is denoted by d_{kh}. A trip of salesman j must be a Hamiltonian chain on the complete subgraph whose nodes are N_j. A feasible solution is one in which no two of these Hamiltonian chains agree in any component. The optimal solution minimizes the sum of all the salesmen's trip lengths, i.e., the total distance traveled by all the salesmen.

Define $x_{ijk} = 1$ if during time period i salesman j visits city k; otherwise, $x_{ijk} = 0$. The MCTSP can now be formulated as:

$$\text{Minimize} \sum_{k=1}^{n} \sum_{h=1}^{n} d_{kh} \left(\sum_{i=1}^{q-1} \sum_{j=1}^{m} x_{ijk} x_{i+1,jh} \right) \tag{3.11}$$

Subject to:

$$\sum_{k \in N_j} x_{ijk} = 1 \quad (j=1,\ldots,m;\ i=1,\ldots q) \tag{3.12}$$

$$\sum_{j=1}^{m} x_{ijk} \leq 1 \quad (k=1,\ldots,n;\ i=1,\ldots,q) \tag{3.13}$$

$$x_{ijk} = 0,1 \quad (\text{all } i,j,k) \tag{3.14}$$

Constraints set (3.12) requires that each salesman j must visit each city in N_j exactly once. Constraint set (3.13) is the nonconflicting constraint: at most one salesman is at any city at any one time. Constraints (3.14) require the 0,1 decisions.

Objective function (3.11) reflects the fact that each salesman's trip is a Hamiltonian chain. This corresponds to our tourism industry application. If a Hamiltonian ciruit (rather than a chain) were required, the objective function would include for each salesman the distance from the city visited at time q to the city visited at time 1. The objective function could be similarly modified for the case when the salesman must start and end at home.

Thus MCTSP is a simultaneous, overlapping, nonconflicting multiple salesman problem. If $m = 1$ and $n = q$, MCTSP reduces to TSP; and since TSP has been seen to be NP-hard in computational complexity, then the same is true of MCTSP.

CHAPTER IV

POLYNOMIAL-TIME SOLUTION FOR FIRST OBJECTIVE OF TOURISM PROBLEM

The tourism convention scheduling problem, using the primary objective only, corresponds to the 3DAP-planar in which the costs are time invariant ($c_{tjk} = c_{t'jk}$, for all t,t'). For the first time, we present here a polynomial-time optimal solution of complexity $O(r^3s^3)$ for this important class of problems.

Using a capacitated transportation model, we first assign the buyers to sellers, ignoring the time dimension (Phase I). By application of the marriage theorem, we prove the existence of a schedule which is feasible over time corresponding to the Phase I solution. This feasible solution can be found by solving a succession of 2DAP's in Phase II.

The Phase I capacitated transportation problem is equivalent to a 2DAP of complexity at most $O(r^3s^3)$. Phase II is composed of q 2DAP's each of complexity $O(s^3)$. Therefore, the solution method is in class P of complexity $O(r^3s^3) + qO(s^3) = O(r^3s^3)$. See Lawler (1976) for techniques.

4-1. SOLUTION METHOD—PHASE I

Suppose in the tourism industry problem, (1.1) and (1.3) through (1.7) in Chapter I, we ignore the times and consider the problem of assigning buyers to sellers. Assume for discussion purposes the appropriate dummy buyers have been added so that $r = s$. The problem can be formulated as a capacitated transportation problem. Each buyer supplies q time periods, each seller demands q time periods, and each buyer-seller pair can meet at most one time. Let $z_{jk} = 1$ if buyer j meets seller k, and 0 otherwise. The "cost" of a meeting between buyer j and seller k may be designated by c_{jk}, with lower values indicating greater mutual desirability of a meeting.

The formulation is:

$$\text{Minimize} \sum_{j=1}^{s} \sum_{k=1}^{s} c_{jk} z_{jk} \qquad (4.1)$$

Subject to:

$$\sum_{j=1}^{s} z_{jk} = q \qquad (k=1,\ldots,s)$$

$$\sum_{k=1}^{s} z_{jk} = q \qquad (j=1,\ldots,s)$$

$$z_{jk} = 0,1 \qquad (j=1,\ldots,s;\ k=1,\ldots,s)\ .$$

The integrality constraints on z_{jk} may be replaced by $0 \leq z_{jk} \leq 1$, giving a capacitated transportation problem which will produce integer solutions.

Problem (4.1) is a relaxation of the original 3DAP-planar of Chapter I. Suppose that the qr constraints of (1.5) are replaced by the r surrogate constraints obtained by summing over i. Then suppose the qs constraints in (1.4) are replaced by the s constraints obtained by summing over i. Then letting $z_{jk} = \sum_{i=1}^{q} x_{ijk}$, for all j,k gives formulation (4.1) when r = s.

We refer to the solution of (4.1) as Phase I.

At the conclusion of the Phase I solution, it is known which buyer-seller pairs optimally are to have meetings, based on their mutual preferences. It remains for Phase II to schedule the times for the meetings in a nonconflicting fashion. One might question the existence of a feasible nonconflicting scheduling solution in the light of the complicated situation. However, we present an algorithm for the completion of the polynomial-time solution.

4-2. SOLUTION METHOD—PHASE II

4-2.1. The Marriage Theorem

The method of determining a feasible time schedule corresponding to the optimal solution to (4.1) is based on the Marriage Theorem (Ryser, 1963), which is:

"Let A be a (0,1)-matrix of order n such that each row and column sum of A is equal to the positive integer k. Then

$$A = P_1 + P_2 + \ldots + P_k$$

where the P_i are permutation matrices."

A graph theoretic version of the theorem (Bondy and Murty, 1976, p. 73) is:

"If G is a k-regular bipartite graph with k > 0, then G has a perfect matching." Thus if a two-dimensional assignment problem has exactly k > 0 admissible cells in each row and column, it has a feasible solution. (Note that the theorem does not imply that k or more admissible cells in each row and column guarantee a feasible solution. However, if the marriage theorem conditions are satisfied, there are k different solutions having no arcs in common.)

4-2.2. The Time Schedule

The solution to (4.1) in Phase I gives a set of rq buyer-seller pairs such that each buyer and seller is in exactly q pairs. Consider first the 2DAP of assigning buyers to sellers in time period 1, considering only the rq cells corresponding to pairs obtained in Phase I as admissible. By the Marriage Theorem this problem has a feasible solution. After the first period assignments are made, each buyer and seller has exactly q-1 feasible cells remaining. So again in the second time period a feasible set of assignments can be found by solving a 2DAP. In general, in time period i, $i \leq i \leq q$, there are exactly $q + 1 - i$ feasible cells corresponding to each buyer and seller, so that a feasible assignment exists.

The process of solving this sequence of 2DAP's is referred to as Phase II.

Thus we see that Phase I and Phase II together provide a new polynomial-time solution for the primary objective of the tourism convention scheduling problem, an important special class in the class of NP-hard three-dimensional planar assignment problems.

4-3. FORMULATION FOR IMPLEMENTATION

In this section we formulate the model as implemented for the tourism problem. This formulation takes into account the fact that the number of sellers exceeds the number of buyers and that the number of visits of each seller is to be at least the integral part of the average number. We present a small example to illustrate the procedure. The computational results for the implementation for the tourist problem are given in Chapter VII.

4-3.1. Phase I

In the tourism problem, $q < r < s$, so that some, perhaps all, of the sellers will have less than q visits. Thus in Phase I a single dummy buyer is added to (4.1) which supplies q(s-r) time periods. The value of $z_{r+1,k}$ is the number of time periods that seller k is idle. We want the number of visits of the individual sellers to be at least the integral part of the average number. Thus we add the constraint:

$$0 \leq z_{r+1,k} \leq q - [qr/s] .$$

This insures that each seller has at least [qr/s] visits, as required by the tourism problem. Assign "cost," $c_{r+1,k} = 0$ for the dummer buyer r+1.

The formulation of the Phase I capacitated transportation problem now is:

$$\text{Minimize} \sum_{j=1}^{r+1} \sum_{k=1}^{s} c_{jk} z_{jk} \tag{4.2}$$

Subject to:

$$\sum_{j=1}^{r+1} z_{jk} = q \qquad (k=1,\ldots,s)$$

$$\sum_{k=1}^{s} z_{jk} = q \qquad (j=1,\ldots,r+1)$$

$$0 \leq z_{jk} \leq 1 \qquad (j=1,\ldots,r;\ k=1,\ldots,s)$$

$$0 \leq z_{r+1,k} \leq q - [qr/s] \quad (k=1,\ldots,s)\ .$$

$$z_{jk} \text{ integer } (j=1,\ldots,r+1;\ k=1,\ldots,s)$$

4-3.2. Phase II

Since $r < s$, there will be $s - r$ idle sellers during each time period. Thus in each time period, one dummy buyer $r + 1$ capable of meeting $s - r$ sellers is added (rather than $s - r$ individual dummies), thus transforming the assignment problems for scheduling meetings in each time period into capacitated transportation problems. In time period i, cell $(r+1,k)$, for $i \leq k \leq s$, is admissible if and only if seller k has been idle in less than $z_{r+1,k}$ of the first $i - 1$ time periods. For $1 \leq j \leq r$, $1 \leq k \leq s$, cell (j,k) is admissible in time period i if and only if $z_{jk} = 1$ and buyer j has not been assigned to seller k in any of the previous time periods. (This formulation is equivalent to solving a sequence of 2DAP's with $s - r$ dummies satisfying the Marriage Theorem conditions. Hence feasible solutions are still guaranteed.)

The formulation of the capacitated transportation problem to be solved in time period i is now given.

Define decision variables

$$x_{jk}^{(i)} = \begin{cases} 1 & \text{if in time period i buyer j visits seller k} \\ 0 & \text{otherwise} \end{cases}$$

$$(j=1,\ldots,r+1;\ k=1,\ldots,s)$$

For the cost matrix let

$$a_{jk} = \begin{cases} 0 & \text{if } z_{jk} = 1 \\ & \text{and } x_{jk}^{(i')} = 0 \text{ for all } i',\ \ 0 \leq i' \leq i-1 \\ \infty & \text{otherwise} \end{cases}$$

$$(j=1,\ldots,r;\ k=1,\ldots,s)$$

For the dummer buyer r+1, define

$$a_{r+1,k} = \begin{cases} 0 & \text{if } \sum_{i'=1}^{i-1} x_{r+1,k}^{(i')} \leq z_{r+1,k} \\ \infty & \text{otherwise} \end{cases}$$

$$(k=1,\ldots,s)$$

The problem is formulated for period i:

$$\text{Minimize} \sum_{j=1}^{r+1} \sum_{k=1}^{s} a_{jk}\ x_{jk}^{(i)} \tag{4.3}$$

Subject to:

$$\sum_{j=1}^{r} x_{jk}^{(i)} = 1 \qquad (k=1,\ldots,s)$$

$$\sum_{k=1}^{s} x_{jk}^{(i)} = 1 \qquad (j=1,\ldots,r)$$

$$\sum_{k=1}^{s} x_{r+1,k}^{(i)} = s - r$$

$$y_{jk}^{(i)} \geq 0 \text{ integer} \qquad (j=1,\ldots,r+1;\ k=1,\ldots,s)$$

For a given value of i, (4.3) can be seen to be a capacitated transportation problem.

Note that the decision variable x_{ijk} of the original problem formulation in Chapter I, (1.1) through (1.7), corresponds to $x_{jk}^{(i)}$ of the above formulation.

4-3.3. Example

Let q = 3, r = 6, and s = 10. Then [qr/s] = 1, so that each seller must have at least one meeting during the three time periods. Costs c_{jk} appear in Table 8 for the Phase I capacitated transportation problem.

The Phase I solution matrix, (z_{jk}), overall is shown in Table 9.

TABLE 8

PHASE I COST AND REQUIREMENTS

Buyer \ Seller	1	2	3	4	5	6	7	8	9	10	Supplies
1	59	3	56	4	8	5	60	76	76	60	3
2	4	6	24	76	25	59	56	57	56	58	3
3	57	6	76	56	7	5	57	58	24	4	3
4	76	58	5	4	3	58	59	56	58	56	3
5	76	24	76	23	22	76	76	76	76	25	3
6	56	56	58	5	8	25	58	59	4	7	3
7	2	2	2	2	2	2	2	2	2	2	12
Demands	3	3	3	3	3	3	3	3	3	3	30 Total

TABLE 9

PHASE I SOLUTION MATRIX

Buyer \ Seller	1	2	3	4	5	6	7	8	9	10
1		1		1		1				
2	1						1	1		
3		1				1				1
4			1	1	1					
5		1			1					1
6				1					1	1
7	2	0	2	0	1	1	2	2	2	0

A Phase II feasible scheduling of meetings for the buyers is shown in Table 10. Each column contains the sellers a buyer will visit in the three time periods.

TABLE 10

PHASE II MEETING SCHEDULES

Time \ Buyer	1	2	3	4	5	6
1	2	1	10	3	5	4
2	6	7	2	4	10	9
3	4	8	6	5	2	10

The secondary objective of minimizing the total distance traveled by the buyers is treated heuristically in an alternative Phase II (when the secondary objective is to be included). This heuristic is described in Chapter V.

CHAPTER V

HEURISTIC SOLUTION FOR MUSICAL CHAIRS TRAVELING SALESMAN PROBLEM

In the preceding chapter, a polynomial-time solution was described in two phases for scheduling optimal interviews at a tourism industry convention. These meetings were arranged on the basis of mutual preferences of buyers and sellers, that is, clients and tour brokers. Nonconflicting meetings were scheduled in the available time periods, but distances between sellers' booths were not considered in that solution to the tourism convention problem.

In this chapter, we develop a heuristic solution method for the MCTSP, which arises from the secondary objective. The MCTSP is the problem of finding a solution which minimizes the total distance traveled by the buyers. A mathematical formulation for the MCTSP was given in Chapter III. With one buyer, this reduces to the TSP, which is NP-hard (Klee, 1980).

5-1. OVERVIEW OF HEURISTIC

This heuristic is a modification of Phase II of the assignment problem solution method. Phase I of the solution which was described in the preceding chapter will be unchanged, since it determines optimal interviews on the basis of the first objective, which takes precedence over the secondary. Phase II becomes a variation of the m-salesmen traveling salesman problem: the minimization of the total distance traveled by all the buyers in proceeding to scheduled meetings with sellers during the available successive time periods. We solve this problem heuristically by adding a distance objective to the series of assignment problems of Phase II used for scheduling. In the capacitated transportation problem solved for time period i, $2 \leq i \leq q$, the cost of an admissible cell (j,k) is the distance from the seller which buyer j visited in time period i - 1 to seller k.

The first time period is handled differently. There is not a "home" booth for each buyer in the sense of the traveling salesman problem, and each buyer's route is a Hamiltonian chain rather than a Hamiltonian cycle. For the first time period, the buyers are assigned auspiciously for their starting sellers, in the following way: Each buyer j has a set N_j of q sellers. For each seller k in N_j let b_{jk} denote the distance from seller k to his nearest neighbor in N_j. An auspicious start for buyer j would be a seller k for which b_{jk} is large. He then would only travel this distance once since he does not return "home." For the first time period, then, we solve the capacitated transportation problem as a maximization using a reward matrix (b_{jk}) in the admissible cells.

5-2. FORMULATION

In this section the formulation of the MCTSP heuristic is given. The example from the preceding chapter will be continued in the next section. Chapter VI contains an analysis of the performance of this polynomial-time heuristic for a certain class of random MCTSP's.

The formulation of the capacitated transportation problem solved each time period i has constraints identical to those given in (4.3). Let d_{kh} = the distance from seller k to seller h. The objective in the first time period is

$$\text{Maximize} \sum_{j=1}^{r+1} \sum_{k=1}^{s} b_{jk}^{(1)} y_{jk}^{(1)} \tag{5.1}$$

$$b_{jk}^{(1)} = \begin{cases} \min\{d_{kh} \mid z_{jh} = 1, h \neq k \\ \qquad\qquad \text{if } z_{jk} = 1 \\ -\infty \text{ otherwise} \end{cases} \tag{5.2}$$

$$(j=1,\ldots,r;\ k=1,\ldots,s)$$

$$b_{r+1,k} = 0 \qquad\qquad (k=1,\ldots,s) \tag{5.3}$$

The equation (5.2) defines the reward of an admissible cell in period 1. Thus $b_{jk}^{(1)}$ is the minimum distance from seller k to another seller to be visited by buyer j. Otherwise, the cell is inadmissible. Equation (5.3) states that the cost of assigning a dummy buyer to any seller is zero.

Let $p^i(j)$ indicate the seller, for any i,j,h, $(1 \leq i \leq q;$ $1 \leq j \leq r;\ 1 \leq h \leq s)$, to which buyer j is assigned in time period i: $p^i(j) = h$ if and only if $x_{jh}^{(i)} = 1$. For time period i, $(2 \leq i \leq q)$, the Phase II objective is:

$$\text{Minimize} \sum_{j=1}^{r+1} \sum_{k=1}^{s} b_{jk}^{(i)} x_{jk}^{(i)} \tag{5.4}$$

where

$$b_{jk}^{(i)} = \begin{cases} d_{p^{i-1}(j),k} \\ \qquad \text{if } z_{jk} = 1 \text{ and } x_{jk}^{(i')} = 0, \ (0 \leq i' \leq i-1) \\ \infty \text{ otherwise} \end{cases} \tag{5.5}$$

$$(j=1,\ldots,r;\ k=1,\ldots,s)$$

For the dummy buyer r+1 define

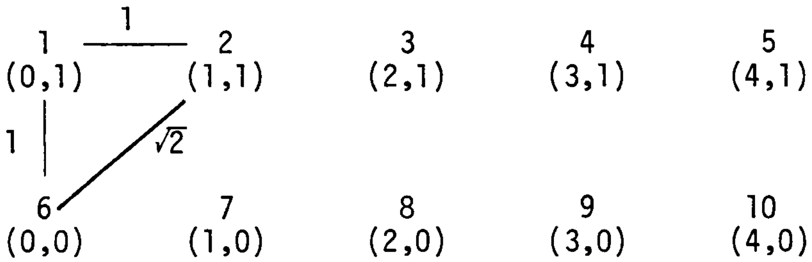
$$b_{r+1,k}^{(i)} = \begin{cases} 0 \text{ if } \sum_{i'=1}^{i-1} x_{r+1,k}^{(i')} \leq z_{r+1,k} \\ \infty \text{ otherwise} \end{cases} \tag{5.6}$$

$$(k=1,\ldots,s)$$

Equation (5.5) defines the cost of cell (j,k) as the distance from the current location of buyer j to seller k if he is to visit seller k and has not yet done so. Otherwise, this cell is inadmissible. Equation (5.6) says that the cost of assigning a dummy to a seller k is zero if seller k has not yet had his quota of idle periods, $z_{r+1,k}$. Otherwise, this cell is inadmissible.

5-3. EXAMPLE

Continuing the small example, suppose the ten seller booths are set up in the convention arena in two rows of five, with distances as shown, on a coordinate system (at the real conventions, it is often not possible to travel a straight route between two booths, because of arrangement in aisles, different rooms, or other setups):



Thus, dist(1,6)=dist(6,1)=1; dist(3,7)=dist(7,3)=$\sqrt{2}$ = 1.414. Other distances between any two booths are easily calculated, using $\sqrt{5}$ = 2.236; $\sqrt{10}$ = 3.162; $\sqrt{17}$ = 4.123. The seller to seller distance matrix (d_{kh}) is seen in Table 11.

TABLE 11

SELLER TO SELLER DISTANCES

Seller \ Seller	1	2	3	4	5	6	7	8	9	10
1	0	1	2	3	4	1	1.414	2.346	3.162	4.123
2	1	0	1	2	3	1.414	1	1.414	2.236	3.162
3	2	1	0	1	2	2.236	1.414	1	1.414	2.236
4	3	2	1	0	1	3.162	2.236	1.414	1	1.414
5	4	3	2	1	0	4.123	3.162	2.236	1.414	1
6	1	1.414	2.236	3.162	4.123	0	1	2	3	4
7	1.414	1	1.414	2.236	3.162	1	0	1	2	3
8	2.236	1.414	1	1.414	2.236	2	1	0	1	2
9	3.162	2.236	1.414	1	1.414	3	2	1	0	1
10	4.123	3.162	2.236	1.414	1	4	3	2	1	0

The nearest neighbor matrix for the first time period maximization is given in Table 12.

TABLE 12

NEAREST NEIGHBOR DISTANCES

Buyer \ Seller	1	2	3	4	5	6	7	8	9	10
1	$-\infty$	1.414	$-\infty$	2	$-\infty$	1.414	$-\infty$	$-\infty$	$-\infty$	$-\infty$
2	1.414	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	1	1	$-\infty$	$-\infty$
3	$-\infty$	1.414	$-\infty$	$-\infty$	$-\infty$	1.414	$-\infty$	$-\infty$	$-\infty$	3.162
4	$-\infty$	$-\infty$	1	1	1	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
5	$-\infty$	3	$-\infty$	$-\infty$	1	$-\infty$	$-\infty$	$-\infty$	$-\infty$	1
6	$-\infty$	$-\infty$	$-\infty$	1	$-\infty$	$-\infty$	$-\infty$	$-\infty$	1	1
7	0	0	0	0	0	0	0	0	0	0

Table 13 gives the Phase II scheduling for the meetings with sellers as determined by the MCTSP heuristic for the three time periods.

TABLE 13

MCTSP MEETING SCHEDULES

Time \ Buyer	1	2	3	4	5	6
1	4	1	10	3	2	9
2	2	7	6	4	5	10
3	6	8	2	5	10	4

CHAPTER VI

PROBABILISTIC ANALYSIS OF THE MCTSP HEURISTIC

6-1. INTRODUCTION

The results presented in this section are primarily of theoretical interest, since the assumptions required for the analysis are very limiting. For a certain class of random problems it is shown that the ratio of the expected objective function value of the heuristic solution to that of a randomly chosen solution is less than or equal to $\frac{1 + \ln(n-1)}{n-1}$ where n is the number of time periods, buyers, and sellers (assumed to be equal). This result does not necessarily apply to the Euclidian distance problems which we solve in the tourism scheduling problem. It only offers the reassurance that, for a class of problems which we are able to study analytically, the heuristic can be shown to perform quite well.

6-2. ASSUMPTIONS

For purposes of the analysis assume that there are n cities, n salesmen, and n time periods. (Although similar results can be obtained without this assumption, its use greatly simplifies the arguments.) We also assume that the distances between cities are given by an nxn asymmetric distance matrix (d_{hk}). It is assumed that the distances d_{hk}, $h \neq k$, are independent observations of a random variable x having a gamma probability density function:

$$g(x:\alpha, \beta) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha} \Gamma(\alpha)}, \quad \text{for } x \geq 0 .$$

For our analysis we assume $\alpha = \frac{1}{n}$, and β can be an arbitrarily defined positive number. The expected value of x is $E(x) = \frac{\beta}{n}$. (Hence if we wish to have E(x) equal to some constant c for all n, we could define $\beta = cn$.) The variance of x is $V(x) = \frac{1}{n} \beta^2$.

We assume that the initial placement of the salesman (i.e., the choice of a feasible assignment problem solution for time period 1) is arbitrary, e.g., salesman j is assigned to city j, j=1,...,n. The assignment problem solved for time period i, $2 \leq i \leq n$, has n - i + 1 admissible cells in each row and column. Cell (j,k) is admissible if salesman j has not yet visited city k. The cost of an admissible cell (j,k) is d_{hk} where h is the city occupied by salesman j at time i-1. A cell (j,k) which is admissible at time i-1, is still admissible at time i, unless salesman j was assigned city k at time i-1. However, the cost of an admissible cell (j,k) changes in each time period because salesman j is starting from a different city.

The total distance D traveled by the salesmen is the sum of the optimal objective function values of the assignment problems solved in periods 2 through n:

$$D = \sum_{i=2}^{n} Z_i^* \tag{6.1}$$

where Z_i^* is the optimal objective function value for the ith assignment problem solved in period i. D and the Z_i^* are functions of the random variables d_{hk}, and are therefore random variables. An upper bound on the expected value of D can be obtained from upper bounds on the expected values of the Z_i^*.

6-3. ANALYSIS

Let $\overline{Z}_i$ denote an upper bound on the expected value of Z_i^*. In estimating $\overline{Z}_2$ we can assume that the costs of the n(n-1) admissible cells are independent observations of x, since these costs are simply the n(n-1) nondiagonal elements in (d_{hk}). In estimating $\overline{Z}_i$ $3 \leq i \leq n$, we are in fact being conservative in assuming that the costs of admissible cells are independent observations of x. We show below that the expected cost of an admissible cell in any time period is less than or equal to E(x).

The unconditional probability that a randomly selected distance d_{hk}, $h \neq k$ will appear as the cost of an admissible cell in time period i, $2 \leq i \leq n$, is the probability that cell (j,k) is admissible where j is the salesman in city h at time i-1. Cell (j,k) is admissible if salesman j has not yet visited city k. There are n-1 cities other than h to be visited by j. He has already visited i-1 of these and has n-i+1 yet to visit. The probability that k is among the cities yet to be visited is therefore $\frac{n-i+1}{n-1}$.

Suppose, on the other hand, that in p, $1 \leq p \leq i-1$, of the first i-1 periods, the optimal assignment problem solution included the cell which for that period had a given distance d_{hk}, $h \neq k$, as its cost. During these p time periods, the salesman at city h was assigned to city k. We can conclude salesman i who is currently at city h, was not assigned to city k in any of those p time periods. Thus he will visit city k in one of the remaining n-p time periods (or n-p-1 if period i-1 is not one of the p periods). He has n-i+1 cities yet to visit. The probability that k is among these cities yet to be visited is at least $\frac{n-i+1}{n-p}$.

We have shown that in time period i the probability that a given distance will appear as the cost of an admissible cell, given that it is the cost of a cell appearing in an optimal assignment in one or more previous periods, exceeds the probability that a randomly selected distance will appear as the cost of an admissible cell. Furthermore, the expected cost of a cell used in an optimal solution is less than E(x). Hence the expected value of admissible cells in any time period is less than or equal to E(x). Thus in computing $\overline{Z}_i$, we are being conservative in assuming that f(x) is the pdf of the costs.

We now compute $\overline{Z}_i$, an upper bound on the expected value of the optimal assignment problem solution at period i using this assumption.

At stage i each row and column has k = n-i+1 admissible cells. Each of the k permutation matrices in the marriage theorem gives a feasible assignment. None of these solutions have cells in common. The objective function value Z for each of these solutions is the sum of n independent random variables each having probability density function $g(x: \frac{1}{n}, \beta)$. Thus the pdf of Z is $g(Z: 1, \beta)$, an exponential with $g(Z) = \frac{1}{\beta} e^{-Z/\beta}$, $Z > 0$, and $E(Z) = \beta$. The cumulative distribution of Z is:

$G(Z) = 1 - e^{-Z/\beta}$, $Z \geq 0$. The distribution of the objective function value W of the best of these n-i+1 solutions is the first order statistic of n-i+1 observation of Z.

The cumulative distribution of W, H(W), is given by:

$$H(W) = 1 - [1 - G(W)]^{n-i+1}, \; W \geq 0 \tag{6.2}$$
$$= 1 - e^{-\left(\frac{n-i+1}{\beta}\right)W} .$$

H(W) is the cumulative exponential distribution with $E(W) = \frac{\beta}{n-i+1}$. Thus from (6.2) we have an upper bound $\overline{Z}_i = \frac{\beta}{n-i+1}$

From (6.1) it follows:

$$D \leq \sum_{i=2}^{n} \frac{\beta}{n-i+1}$$

$$= \beta \sum_{i=1}^{n-1} \frac{1}{n-i}$$

$$= \beta \sum_{i=1}^{n-1} \frac{1}{i} \tag{6.3}$$

The following steps provide a bound on the summation in (6.3).

$$\sum_{i=1}^{n-1} (1/i)$$

$$= 1 + \sum_{i=2}^{n-1} (1/i)$$

$$< 1 + \sum_{i=2}^{n-1} \int_{i-1}^{i} (i/x)dx$$

$$= 1 + \int_{1}^{n-1} (1/x)dx$$

$$= 1 + \ln(n-1).$$

The summation, $\sum_{i=2}^{n-1} (1/i)$, is a rectangular approximation of $\int_1^{n-1} (1/x)dx = \ln(n-1)$ as shown in Figure 2. From Figure 2 it can also be observed that $\sum_{i=2}^{n-1} (1/i) < \ln(n-1)$. Hence from (6.3) we obtain:

$$D < \beta(1 + \ln(n-1)).$$

A solution which is randomly chosen from the set of all feasible MCTSP solutions has an objective function value v which is the sum of (n-1) n distances. Its expected value is therefore $(n-1)n\ E(x) = (n-1)\beta$. Hence the ratio of the expected heuristic objective function value to that of a randomly chosen solution is less than or equal to $\frac{1 + \ln(n-1)}{n-1}$. This ratio is near zero for large n. The $g(x:1/n,\beta)$ distribution was chosen to make the analysis mathematically more tractable. We can only speculate whether the results would be more or less optimistic, if we could analyze other classes of distributions. We know of no reason why the choice of $g(x:1/n,\beta)$ would bias the results. The gamma distribution is a very flexible model which has been used in a variety of applications.

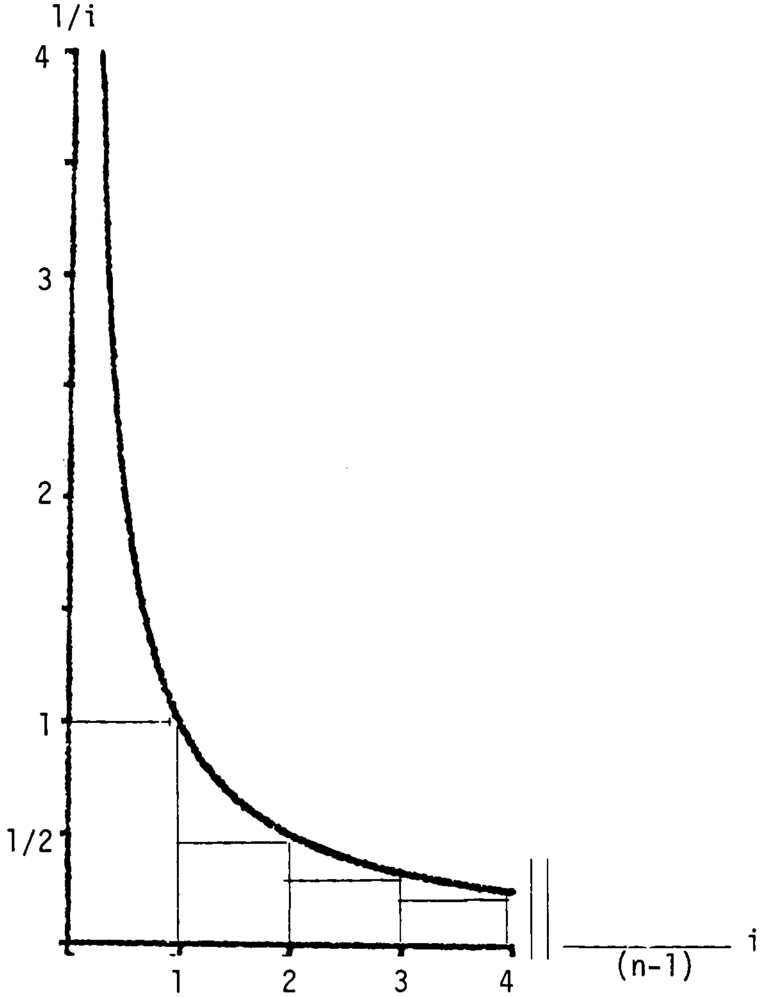


Figure 2. Rectangular estimate of integral.

CHAPTER VII

COMPUTER IMPLEMENTATION

The algorithms described in Chapters IV and V were implemented on the DEC-10 computer at The University of Tennessee. The FORTRAN program uses the primal network code, "GNET," of Bradley, Brown and Graves (1975,1977) to solve the capacitated transportation subproblems. This program was applied to data taken from a recent convention in the tourism industry, using four possible mutual preference coefficients for the primary objective function. We have supplied a 15 x 14 grid to locate the sellers, in order to obtain distance data to be used in the MCTSP heuristic. The heuristic which is currently being used was also applied, using the same data. In Section 1, the four cost coefficients are defined for the 3DAP-planar algorithm, and the currently used heuristic is described.

Since the true objective in these assignment problems is not well defined, the purpose of these tests is to study the characteristics of the solution produced by each of five methods (as opposed to demonstrating that one method dominates the others). These results are presented in Section 2.

In addition, we solved with our algorithm eight problems in which the buyer and seller preference lists were generated randomly. The problems were used to estimate: (1) the effect of problem size on computation times and the quality of solutions; (2) the effect of the preference structure on the quality of the solutions. With number of time periods held constant, and numbers of buyers and sellers proportional, the CPU time is almost linear with problem size as depicted by number of sellers. Furthermore, even for problems in which certain buyers and sellers are much more popular than others, the 3DAP-planar algorithm is able to resolve the conflict and come up with solutions which are almost as good as when the preferences are uncorrelated. The problems with random data are reported in Section 3.

7-1. THE TOURISM CONVENTION EXAMPLE

There were $s = 204$ sellers, $r = 142$ buyers, and $q = 25$ time periods, in these representative convention data. Preference lists, identification codes, and names were provided for buyers and sellers. Each buyer's list had between zero and 108 requests for sellers, with an average of 27.1. Each seller's list had between one and 107 requests for buyers, with an average of 27.4.

7-1.1. Cost Coefficients for the 3DAP-Planar Algorithm

In order to use the 3DAP-planar algorithm, preference "cost" coefficients, c_{jk}, must be supplied for each pair of buyer j and seller k. Initially, in the case of the tourism convention examples, each buyer (seller) had ranked approximately 27 choices of sellers (buyers), in order of preference, a low ranking signifying a high preference. If a seller was not given a place on a buyer's list, then he was assigned a rank equal to the total of all possible ranks. Thus, sellers not given a rank were assigned a rank of 20910, derived by:

$$\sum_{k=1}^{s} k = s(s+1)/2 \ .$$

The essentially infinite rank causes the scheduling of these unrequested meetings to be denied if possible. Similarly, each seller had ranked approximately 27 choices of buyers, and buyers not given a rank were assigned a rank of 10153:

$$\sum_{j=1}^{r} j = r(r+1)/2 \ .$$

In representing the desirability of a meeting, four cost coefficients were tested. A sum cost:

$$c_{jk} = \alpha_{jk} + \beta_{kj} \tag{7.1}$$

a logarithmic cost:

$$c_{jk} = \ln(\alpha_{jk}) + \ln(\beta_{kj}) = \ln(\alpha_{jk}\beta_{kj}) \tag{7.2}$$

a buyer preference cost:

$$c_{jk} = \alpha_{jk} \tag{7.3}$$

and a seller preference cost:

$$c_{jk} = \beta_{kj} \tag{7.4}$$

where α_{jk} = the rank assigned to seller k by buyer j, and β_{kj} = the rank assigned to buyer j by seller k, $(j=1,\ldots,r;k=1,\ldots,s)$.

The sum cost gives a balanced combined ranking, based directly on the two ordinal rankings in the preference lists. The logarithmic cost is designed to give smaller differences among higher ranked choices since the log function has diminishing slope. (The opinion expressed by the schedulers was that the buyers and sellers would, for example, be relatively indifferent in choosing between their 25th and 30th preferences, but not between their 1st and 2nd preferences.)

The buyer preference cost and the seller preference cost were used primarily to provide a basis of comparison for the other schedules. However, from a practical standpoint, the buyer preference cost is reasonable, since a seller would be interested in being visited by any prospective buyer who requested him.

7-1.2. Currently Used Scheduling Method

The scheduling method currently used is a three phase heuristic developed by Dr. Oscar Fowler of The University of Tennessee and Dr. John Plotnicki of Colorado State University. The phases use different rules to define admissible buyer-seller pairs and to define the order in which the pairs are considered. When an admissible pair is considered, a visit will be arranged between the given buyer and seller if they have a free time period in common. Once a visit is arranged, it is permanently fixed, as is the time period in which it takes place.

The first two phases of the heuristic are buyer oriented. Each phase makes multiple passes over the list of buyers. Each pass attempts to schedule one additional visit for each buyer. In scheduling a visit for a buyer j, the sellers are considered in the order which they occur on buyer j's list of preferences. Phase I only considers perfect matches, i.e., visits between a buyer j and a seller k where j has requested k and vice versa. Phase II considers visits in which only the buyer has requested the seller. Phase III is seller oriented. This phase makes multiple passes over the list of sellers attempting to schedule one additional visit for each seller during each pass. In scheduling a visit for a seller, the buyers are considered in the order of the seller's preference. In this phase, a buyer-seller pair is admissible if the seller requested the buyer. The three phases are described below.

Phase I—Perfect Matches

This phase makes multiple passes through the list of buyers. Each pass attempts to schedule one additional visit for each buyer. Only buyer-seller pairs in which the buyer and seller choose each other are admissible. The buyer's preference determines the order in which these pairs are considered. In each pass each buyer is assigned a visit with the most preferred seller k on his request list satisfying the following conditions (if such a seller exists):

(1). A meeting between j and k has not already been scheduled.

(2). Buyer j is on the requested list of seller k.

(3). Buyer j and seller k have a free time period in common.

If, in a given pass, no seller on buyer j's preference list satisfies these conditions, no visit will be scheduled for j. Then buyer j will be skipped on subsequent passes. The passes stop when no buyer's preference list contains sellers satisfying (1), (2), and (3). At this point the second phase of the heuristic begins.

Phase II—Buyer Requests

This phase is identical to Phase I except that condition (2) is dropped. A pair (j,k) is admissible if buyer j requested seller k. As before, the buyers are considered in rotation and each buyer's preference list determines the order in which the sellers are considered.

Phase III—Seller Requests

This phase is identical to Phase II except the roles of buyers and sellers are requested. Multiple passes are made over the list of sellers. Each pass attempts to schedule one additional visit for each seller. For each seller, the buyers are considered in the order in which they occur on the seller's list of preferences, and a visit is scheduled when a buyer having a free time period with the seller is found. A seller is eliminated once his preference list is exhausted and the passes stop when all sellers have been eliminated.

The steps of Phase I and Phase II of the heuristic are given below. Phases I and II differ only in the criteria used to determine admissibility in step 6. Phase III is the same as Phase II with the roles of buyers and sellers reversed. The variables used are as follows.

j = index of buyer being considered for scheduling.

p_j = position on buyer j's preference list currently being considered, i.e., the seller k to be considered is the p_jth choice of buyer j.

q_j = total number of sellers on buyer j's list of preferences.

r = number of buyers.

The steps are as follows:

1. Set $p_j = 0$, $j=1,\ldots,r$
   Set $j = 0$. Go to Step 2.

2. Set $j = (j+1) \text{ Mod } (r+1)$. If $j = 0$, set $j = 1$.
   Go to Step 3.

3. If buyer j has been eliminated, go to step 2. Otherwise go to Step 4.

4. Set $p_j = p_j + 1$. If $p_j > q_j$, go to Step 7.

5. Let k be the seller having position p_j on buyer j's list of preferences, i.e., $\alpha_{jk} = p_j$. If (Phase I only: j is on k's preference list and) j and k have a free time period i in common, go to Step 6. Otherwise, go to Step 4.

6. Schedule a visit between j and k in period i. Go to Step 2.

7. Eliminate buyer j. If all buyers have now been eliminated, stop. Otherwise, go to Step 2.

7-2. TESTS USING TOURISM DATA

Five schedules were derived for the sample tourism convention data. These schedules were derived using the current heuristic, Method I, and the 3DAP-planar algorithm with the four different cost coefficients (7.1) through (7.4), Methods II through V, respectively.

The current heuristic does not consider distances, but we have calculated them for the buyers in that schedule. The MCTSP heuristic which was described in Chapter V was implemented in conjunction with Methods II through V.

7-2.1. Remarks about Methods

In terms of the preference rankings of buyers' and sellers' assignments, the heuristic and the other methods are not directly comparable, and are biased in favor of the heuristic for the following reasons.

(1). The 3DAP-planar algorithm methods were constrained to give each seller at least 17 visits (the integral part of the average number of visits per seller).

(2). The heuristic had added flexibility in scheduling the times of meetings. In actuality, some buyers and sellers were represented by more than one agent. The 204 sellers were represented by 300 agents, and the 142 buyers were represented by 242 agents. In our program we ignored the extra agents completely. In the heuristic, they were ignored in the logic used matching buyer-seller pairs except in determining if a buyer and seller had a free time period in common. A visit could be scheduled between a buyer j and a seller k if any agent of j had a free time period in common with an agent of k. The result was the sellers and buyers having multiple agents could get more visits and also more of their high preferences.

The 3DAP-planar algorithm could handle multiple agents in an ad hoc fashion, simply by treating them as separate buyers or sellers.

In this case, the preference lists of the agents representing a seller(buyer) must be mutually exclusive in order to prevent duplicate assignments. However, this was not done in our test runs, nor were the balancing constraints relaxed. A rigorous study of the case of multiple agents is an area of future research.

7-2.2. Results

A summary of the results from the five methods is given in Table 14. The total number of meetings scheduled by the heuristic is 3539, almost the same as the 3550 meetings scheduled for each test of the 3DAP-planar algorithm. This algorithm gives 25 meetings to each individual buyer and at least 17 to each seller.

Column 1 of Table 14 is the CPU time consumed by the entire run. Column 2 is the number of requested visits granted per seller. These numbers do not include visits scheduled with buyers who were not ranked by the seller. Column 3 is the rank which the seller had assigned to the buyers with whom the requested visits were scheduled. Columns 4 and 5 give the corresponding information for buyers. Column 6 is the distance traveled per visit for the buyers. No unit is given for the distance, but on the grid used for sellers' booths, the distance between any two adjacent booths is one, and the maximum distance between any two booths is about 19.

Method I used 47 seconds of CPU time, and methods II through V each used about 6.5 minutes. CPU time is not a major consideration in the light of other expenditures and the infrequency of the use of the system. The average distance traveled per buyer is 7.40 for the

TABLE 14

SUMMARY OF TESTS ON TOURISM EXAMPLE

Method Used	(1) DEC-10 CPU Time	Seller						Buyer						(6) Distance for Buyer		
		(2) No. Visits			(3) Rank of Visit			(4) No. Visits			(5) Rank of Visit					
		Min.	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.
(I) Heuristic	47 s	1	11.8	20	1	14.0	78	0	16.8	78	1	25.5	108	1	7.40	17.8
(II) Sum Costs	6.5 m	0	11.1	18	1	18.6	93	0	16.5	25	1	18.6	25	1	3.10	15
(III) Logarithmic Costs	6.5 m	1	13.5	23	1	14.9	93	0	13.3	25	1	17.5	107	1	3.10	18.4
(IV) Buyer Preference Costs	6.5 m	0	5.3	14	1	22.0	106	0	16.5	25	1	13.3	47	1	3.05	17.7
(V) Seller Preference Costs	6.2 m	1	16.6	23	1	11.3	103	0	4.0	17	1	23.0	102	1	3.09	16.4

heuristic, versus approximately 3.10 for the 3DAP-planar methods in conjunction with the MCTSP heuristic.

Predictably, Method IV gives a good solution from the standpoint of buyers' preferences, and Method V gives a good solution from the standpoint of sellers' preferences. The heuristic performed well in terms of the total number of requested visits granted to buyers, with an average slightly higher than Method IV. This high average was achieved by granting some buyers as many as 78 of their requested visits (the result of multiple agents and no constraints on the number of buyers' or sellers' idle periods). Method II appears to be biased toward buyer requests, while Method III is biased toward seller requests.

Figure 3 shows the percentage of buyers who got visits with their i-th choice for i=1,...,25. Predictably, Method IV dominates the other methods. Method III does well in giving buyers their first 5 or so choices, and then drops rapidly. Method II performs well throughout the first 25 choices.

Figure 4 shows similar data for the sellers. Method V dominates the other methods. Methods I and III appear to be tied as the second best. From Figures 3 and 4 it can be seen that Method III (logarithmic costs) is the only method which gives both buyers and sellers a large percentage of their high preferences.

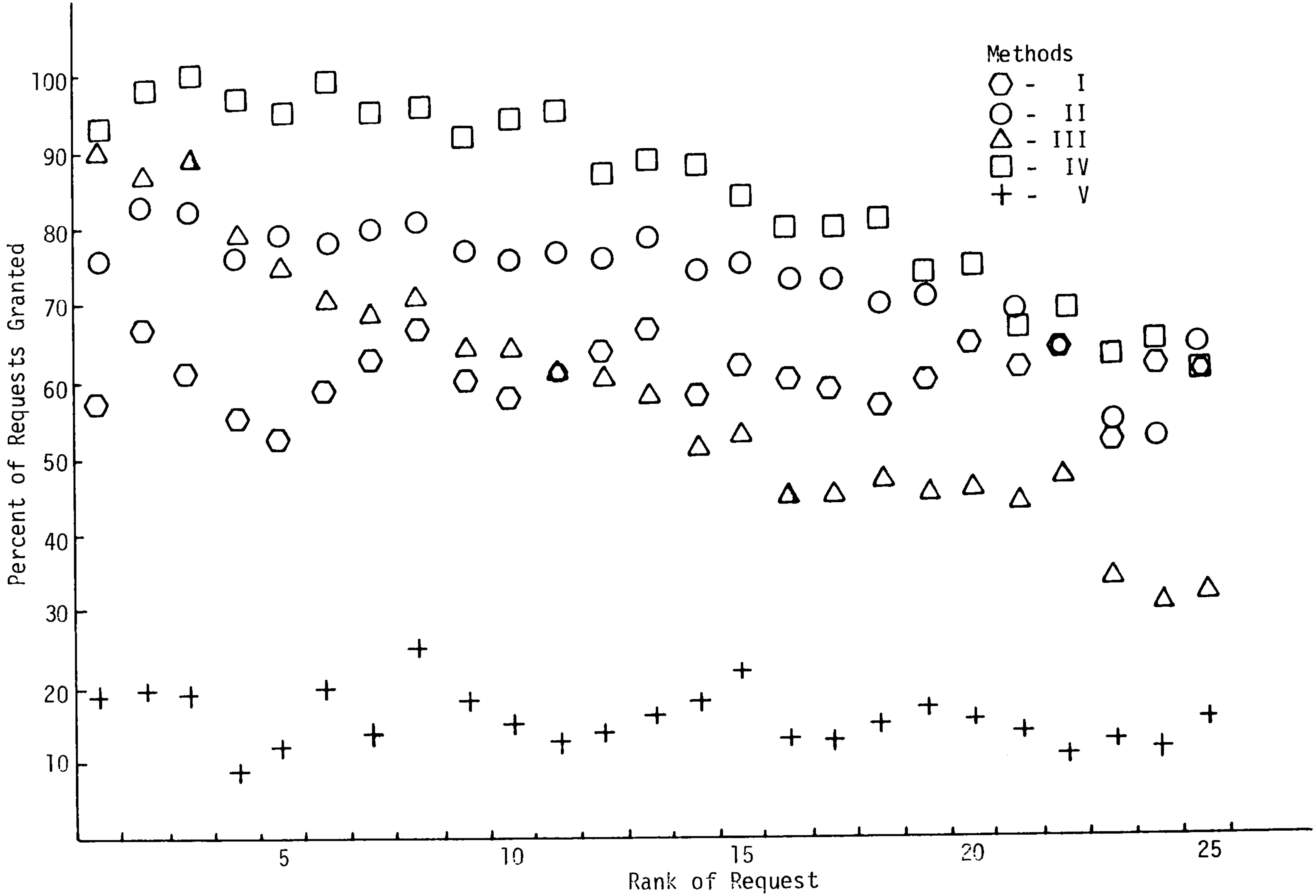


Figure 3. Percentage of buyer requests scheduled.

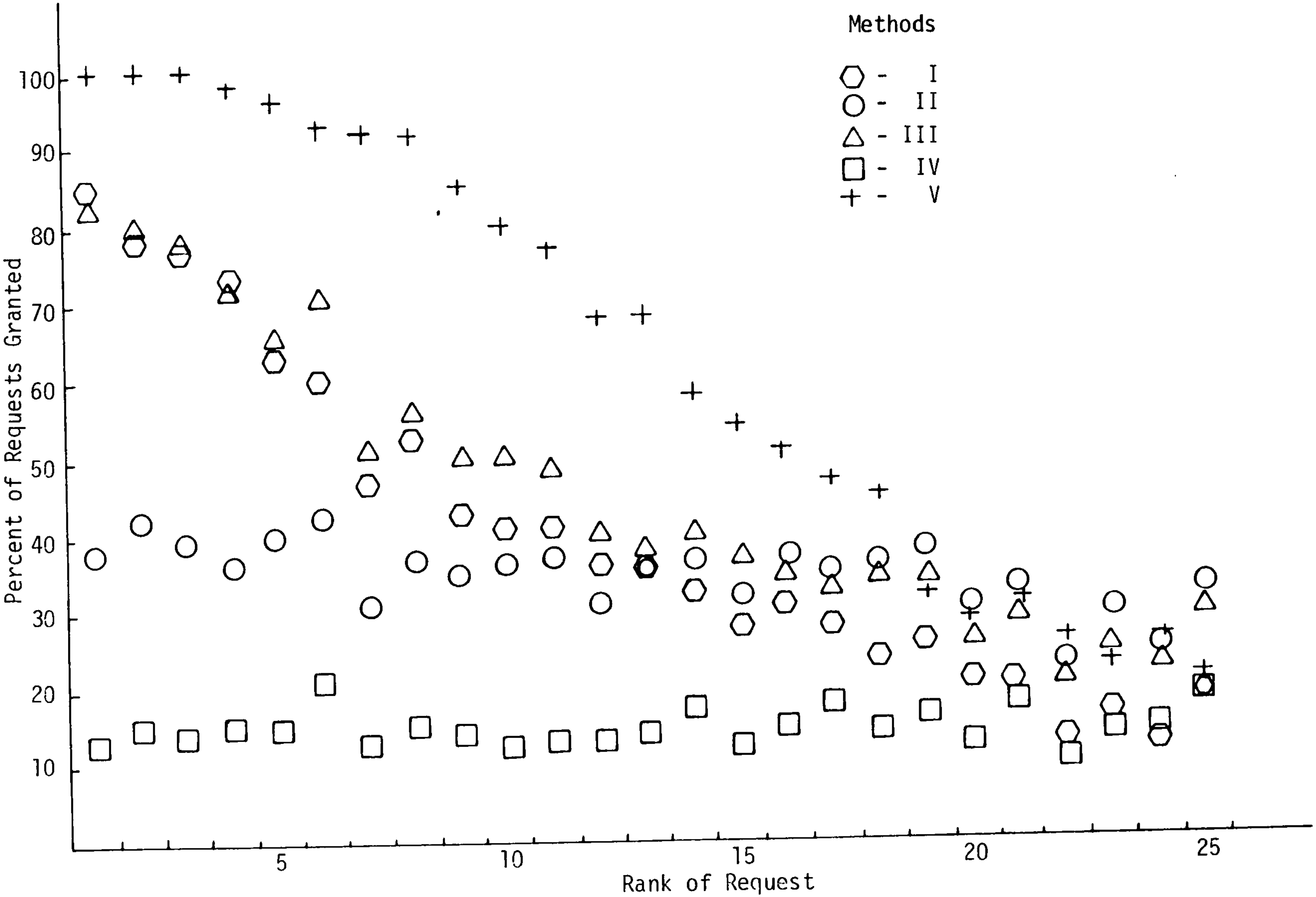


Figure 4. Percentage of seller requests scheduled.

7-3. PROBLEMS USING RANDOM DATA

7-3.1. Description of Problems

We ran two sets of problems, using the 3DAP-planar algorithm with sum costs and four sizes of problems in each set. The four sizes each used q = 25 time periods. The numbers of buyers and sellers (rxs) was 25x50, 50x100, 75x100, and 100x200 for Problems I through IV, respectively. The sum cost objective was chosen arbitrarily as a representative one which has been considered for practical use, and for ease of interpretation.

Each of the eight problems was run with and without the implementation of the MCTSP heuristic for distance minimization in Phase II of the solution. With the use of the same distance grid for all problems, the distance between adjacent sellers is one. The maximum possible distance between sellers with 50, 100, 150, and 200 sellers is 14.3, 15.2, 16.6, and 19.1, respectively.

The first set of problems gave to buyers preference lists which were independent random permutations of s sellers, and to the sellers preference lists which were independent random permutations of r buyers. Since every buyer (seller) was given a rank by every seller (buyer), no missing ranks were supplied to the data.

In the second set of problems, some competition and conflict were introduced by making some buyers (sellers) generally more highly preferred than others. A buyer's preference list is a random permutation of s sellers, subject to the following requirements: First place on the list is filled by any seller k with probability proportional to k. Second place on the list is then filled by any seller k, given that he was not first on the list, with probability proportional to k. In general,the i-th place on the list (i = 2 through s) is filled by any seller k, given that he was not one of the first i - 1 on the list, with probability proportional to k. Each seller's preference list is constructed in the same fashion, relative to the buyers.

7-3.2. Results

Tables 15 and 16 give summaries of the results of the two sets of random problems. In each table, Column 1 gives the CPU time with and without use of the MCTSP heuristic for distance minimization in Phase II of the solution. Columns 2 and 3 give the information concerning ranks of the visits granted to a seller and a buyer. Columns 4 and 5 give the distance results with and without the MCTSP heuristic.

Note that the numbers of requested visits granted are not reported in Tables 15 and 16. All visits are requested, and each buyer was assigned 25 meetings for each time period. The average number of meetings per seller was 12.5 in all problems. Therefore, each seller was constrained to have at least 12 meetings. In each problem, this minimum was attained by the schedules. In the first set of independent random problems, the maximum attained number of meetings for a seller was 17, 18, 17, 18 for the problems I through IV, respectively. In the second set of competitive random problems, the corresponding numbers were 17, 20, 20, 20. This indicates a

TABLE 15

SUMMARY OF INDEPENDENT RANDOM TESTS

Problem Size (Buyers x Sellers)	(1) DEC-10 CPU Time		(2) Seller Rank of Visit			(3) Buyer Rank of Visit			Distance for Buyer (4) MCTSP			Distance for Buyer (5) No MCTSP		
	MCTSP	No MCTSP	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.
(I) 25x50	54 s	49 s	1	10.7	25	1	14.5	45	1	1.71	14.04	1	5.40	14.04
(II) 50x100	2.0 m	1.8 m	1	16.9	49	1	18.0	58	1	2.14	13.04	1	5.82	14.87
(III) 75x150	3.2 m	2.9 m	1	21.3	75	1	21.4	76	1	2.58	14.04	1	6.52	15.63
(IV) 100x200	4.7 m	4.2 m	1	24.5	81	1	24.8	94	1	2.98	17.03	1	7.46	18.39

TABLE 16

SUMMARY OF COMPETITIVE RANDOM TESTS

Problem Size (Buyers x Sellers)	(1) DEC-10 CPU Time		(2) Seller Rank of Visit			(3) Buyer Rank of Visit			Distance for Buyer (4) MCTSP			Distance for Buyer (5) No MCTSP		
	MCTSP	No MCTSP	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.
(I) 25x50	56 s	50 s	1	10.9	25	1	16.8	50	1	1.75	11.40	1	5.62	14.32
(II) 50x100	2 m	1.8 m	1	19.1	50	1	22.1	100	1	2.23	14.32	1	5.92	14.87
(III) 75x150	3.3 m	3.0 m	1	24.4	75	1	27.9	150	1	2.56	15.81	1	6.68	16.12
(IV) 100x200	4.9 m	4.4 m	1	31.1	100	1	31.3	200	1	3.02	17.69	1	7.48	17.69

small amount more imbalance in the number of meetings granted to sellers when there is more competition for meetings. However, the 3DAP-planar algorithm appears to be able to resolve the conflicts and produce schedules which have almost the same maximum numbers of meetings for sellers in the four problems, as for independent random data in the first set of problems.

Figure 5 shows the relation between CPU time and the size of the random problem as measured by the number of sellers. The largest CPU time is shown, i.e., for the random competitive problems using the MCTSP heuristic. It appears to have a slight upward curvature. This applies to problems with $q = 25$, and number of buyers proportional (half) to the number of sellers.

As to the differences of quality of solution obtained for the two sets of problems, the ranks of scheduled visits for the random competitive preferences were somewhat higher. In all cases shown in Table 16 the maximum possible rank (least favored preference) was attained, but somewhat lower maximum attained ranks are seen in Table 15. Average ranks between the two sets of problems show higher figures for the random competitive data than for the independent random data. However, the difference is not great.

The average distances traveled when the MCTSP heuristic is not used is greater than when it is used in all problems by a factor of 2.5 or more. The four competitive random problems with and without MCTSP each have slightly higher distances than the corresponding four independent random problems. This may be explained by the fact that

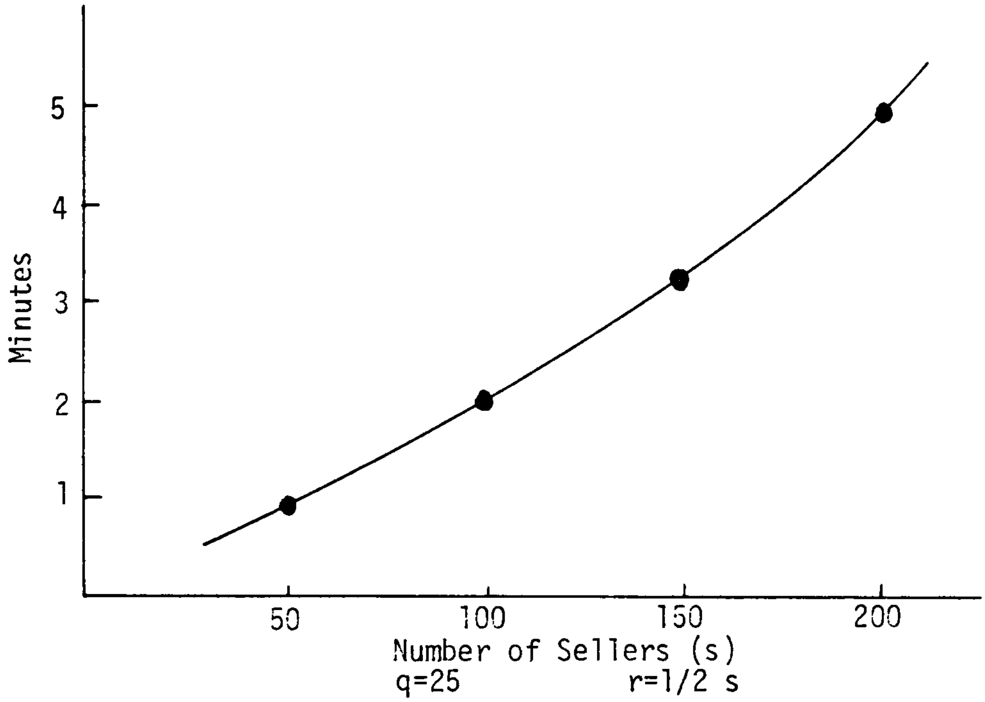


Figure 5. CPU time for random problems.

a few of the sellers have a relatively large number of visits when the competitive random preference data is used, in spite of the constrained minimum number of visits for all sellers.

We have seen that for the problems reviewed, those which were designed with differences of desirability among buyers and sellers were given poorer quality solutions as to ranks of assigned visits and distances traveled. A higher maximum number of visits attained by any seller in the second set of problems also indicates the differences among the sellers' "popularity" with buyers.

The reduction in the quality of the solutions for the competitive random data is not very marked. The solutions are almost as good as those for the independent random problems, which gives a good recommendation to the 3DAP-planar algorithm's ability to resolve the conflicts, even in the presence of strong competition and similar preferences.

CHAPTER VIII

CONCLUSION

"It is the large generalization, limited by the happy particularity, which is the fruitful conception" (Whitehead, 1925). In this closing chapter, we look back on what has been accomplished, and look ahead at other prospective research.

8-1. ACCOMPLISHMENTS

One of the earliest optimization problems to be studied in the field of operations research was the assignment problem (Lawler, 1976). We have studied a special class of three-dimensional planar assignment problems, with a very realistic and reasonable restriction on the objective function ($c_{ijk} = c_{i'jk}$, for all i,i'). We have developed the first polynomial-time solution, $O(n^6)$, which we have seen for an important class of problems, represented in this study by a scheduling problem arising in the tourism industry conventions.

Furthermore, we have defined an important new class of m-salesmen traveling salesman problems, called the musical chairs traveling salesman problem. It arises as a secondary problem in the tourism scheduling, and a heuristic is devised which is very useful in its own right for other applications. This heuristic is analyzed probabilistically for a class of cost distributions.

The current method which is being used in the tour convention scheduling was described. Recent tour convention data were used with the current heuristic, and also with the two phase 3DAP-planar algorithm for four different cost coefficients, with and without the MCTSP heuristic. In addition, eight problems of differing sizes and differing types of random data were analyzed. The solution method was found to perform well.

We have indicated some areas of prospective research throughout the dissertation. In the following sections are some other research directions.

8-2. MCTSP ALGORITHMS

A topic for further research is the development of other algorithms for the MCTSP. A rather special feature of this problem is the fact that, independent of the distance matrix, certain feasible solutions are much more likely to be optimal than others. The fact that an optimal solution can be characterized a priori could be exploited in a solution method.

An optimal solution is likely to be one in which the different salesmen's Hamiltonian chains have many arcs in common. For example, if we were to modify the problem so that each salesman starts at a different city and then makes a Hamiltonian tour of all the cities, then an optimal solution would exist in which all of the salesmen's tours contain the same set of arcs. This problem could be solved by solving the standard traveling salesman problem.

An examination of random problems verifies that it is true, in general, that in an optimal solution the different salesmen's chains will have many arcs in common. Consider a random problem (either symmetric or asymmetric) in which the distances are independent observations of a random variable X with expected value E(X) and variance V(X). (The independence assumption is for simplicity; a similar argument holds without this assumption.)

Assume that there are n salesmen, n cities, and q times ($q \leq n$). Consider an arbitrarily chosen feasible solution. Let Y_i denote the length of the chain of salesman i. Then,

$$E(Y_i) = qE(X) \text{ and}$$
$$V(Y_i) = qV(X).$$

Let Z denote the total length of all the chains of all the salesmen. Then:

$$E(Z) = nE(Y_i) = nqE(X)$$

and

$$V(Z) = \sum_{i=1}^{n} V(Y_i) + 2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} \rho_{ij} \sqrt{V(Y_i)\, V(Y_j)}$$

$$= nqV(X) + 2qV(X) \sum_{i=1}^{n} \sum_{j=1}^{i-1} \rho_{ij} \qquad (8.1)$$

where ρ_{ij} is the correlation coefficient between Y_i and Y_j:

$$\rho_{ij} = P_{ij}/q \qquad (8.2)$$

in which p_{ij} denotes the number of arcs the chains of salesman i and salesman j have in common.

From (8.1) and (8.2) it can be seen that the larger number of arcs the chains comprising a solution have in common, the larger V(Z) will be. The variance of a given solution is between nqV(X), (corresponding to none of the chains having arcs in common), and $n^2qV(X)$, (corresponding to each of the chains having the same set of arcs).

Since E(Z) is the same for all of the feasible solutions, those solutions having the larger values of V(Z) are more likely to have very small (and also very large) values of Z. Hence solutions in which the chains have many arcs in common are more likely to be optimal.

The heuristic method presented in Chapter V is biased toward producing the type of solution most likely to be optimal. As was shown in Chapter VI, the arcs most likely to be admissible for a salesman at a given time period are those which have been used in previous time periods by other salesmen.

This idea could be further exploited by first finding an optimal traveling salesman tour of all the cities (assuming this is computationally feasible). Then a MCTSP which gives preference to the arcs in this tour could be applied. For example, the heuristic of Chapter V could be modified by assigning a cost of zero in the assignment problems to the arcs in the optimal traveling salesman tour. This would tend to give our essentially greedy algorithm some "look ahead" capability.

8-3. OBJECTIVE FUNCTIONS

8-3.1. Efficient Versus Equitable

The linear (efficient) and bottleneck (equitable) objective functions are included among the general class of ℓ_p and ℓ_∞ norms which are defined on a real vector space. Let $c_{jk}x_{ijk}$ be nonnegative components of a vector, $\underline{cx}$, in qrs-dimensional vector space. The ℓ_p norm is defined by (Ortega, 1972):

$$\|cs\|_p = \left\{ \sum_{i=1}^{q} \sum_{j=1}^{r} \sum_{k=1}^{s} (c_{jk}x_{ijk})^p \right\}^{1/p} .$$

The ℓ_∞ norm is the limiting case of the ℓ_p norm as $p \to \infty$ (Rudin, 1974).

A multiobjective algorithm which treats the linear and bottleneck objectives would be easy to develop. Another interesting possibility would be the development of an algorithm which handles the general ℓ_p norm.

8-3.2. Time Preferences

This section provides some very stimulating ideas for future research. It consists of conjectures rather than rigorously proved results.

8-3.2.1. Conjectures. The tourism scheduling problem has been described under assumptions that some of the sellers at the convention will necessarily have some idle periods. Suppose each seller k can specify a degree of preference c'_{ik} for each time period

i. Assume c''_{jk} is the combined mutual preference of buyer j and seller k, as before. Suppose also that the cost of buyer j meeting seller k at time i is:

$$c_{ijk} = c'_{ik} + c''_{jk} \qquad (i=1,\ldots,q;\ j=1,\ldots,r;\ k=1,\ldots,s) \tag{8.3}$$

where as before q, r, and s are respectively the number of time periods, buyers, and sellers.

A similar problem is one in which the buyers do not necessarily have q visits, are therefore idle in some time periods, and have stated time preferences c'''_{ij}. The cost coefficients in (8.3) are replaced by:

$$c_{ijk} = c'_{ik} + c''_{jk} + c'''_{ij} \ . \tag{8.4}$$

We conjecture that the problems having cost coefficients given by (8.3) and (8.4) can be solved optimally by a polynomial-time algorithm.

We give below a formulation of a capacitated transportation problem which is the basis of our conjecture. The formulation relates to the problem associated with (8.3), but can be easily modifed for (8.4). The solution of this capacitated transportation problem provides each buyer with a set of sellers to visit. It also provides each seller with not only the set of buyers who will visit, but also the set of time periods during which he will be idle. The problem is formulated so that associated with any feasible solution there appears to exist a nonconflicting time schedule in which the specified visits are made and the buyers are idle during the specified time periods. We have not yet been able to develop an algorithm to find the nonconflicting time schedule (if one exists).

8-3.2.2. Formulation. The general idea of the formulation is as follows: Suppose we were to simultaneously solve three capacitated transportation problems, assigning (1) buyers to sellers, (2) sellers to time periods, and (3) time periods to buyers. We add an artificial or slack variable (i.e., a dummy cell) to each row and column of all the problems. However, each artificial or slack variable is both a row dummy for one capacitated problem and a column dummy for another. Each of the three possible combinations of capacitated transportation is linked by a set of artificial or slack variables. This linkage is made in such a way so as to guarantee that if buyer j is assigned to seller k, then seller k will be assigned to some time period i, and time period i will be assigned to buyer j.

The problem which results after adding the linking artificial or slack variables is a $(q + r + s)$ by $(q + r + s)$ capacitated transportation problem. Its solution provides a solution to the three assignment problems (1), (2), and (3). Furthermore, it appears that there is a nonconflicting time schedule of visits corresponding to this solution. However, the solution does not directly provide a time schedule and we have been unable to develop a method for determining one. Table 17 illustrates the capacitated transportation

TABLE 17

INITIAL ARTIFICIAL SOLUTION

	q times			s sellers						r buyers				
	r	x	x	x	x	x	x	x	x	0	0	0	0	r
q times	x	r	x	x	x	x	x	x	x	0	0	0	0	. . .
	x	x	r	x	x	x	x	x	x	0	0	0	0	r
	x	x	x	0	0	0	0	0	0	q	x	x	x	q
r buyers	x	x	x	0	0	0	0	0	0	x	q	x	x	.
	x	x	x	0	0	0	0	0	0	x	x	q	x	. .
	x	x	x	0	0	0	0	0	0	x	x	x	q	q
	0	0	0	q	x	x	x	x	x	x	x	x	x	q
	0	0	0	x	q	x	x	x	x	x	x	x	x	. . .
	0	0	0	x	x	q	x	x	x	x	x	x	x	
s sellers	0	0	0	x	x	x	q	x	x	x	x	x	x	
	0	0	0	x	x	x	x	q	x	x	x	x	x	
	0	0	0	x	x	x	x	x	q	x	x	x	x	q
	r . . . r			q . . .					q	q . . . q				

formulation. The numbers in the cells represent an all artificial-slack solution. We assume $q \leq r \leq s$. The numbers on the right margin are supplies and the numbers on the bottom margins are demands.

The rs admissible cells in the center rectangle (buyer-seller) represent assignment of buyers to sellers. Each buyer supplies q visits and each seller demands q visits. The cost of the cell corresponding to buyer j and seller k is c''_{jk}. The capacity of each cell is one.

The qr admissible cells in the upper right rectangle (time-buyer) represent assignments of time periods to buyers. Each time period supplies r visits and each buyer demands q visits. (Since each buyer is to make q visits, all of the rq assignments are used in the final solution.) The cost of each of these cells is zero. (If buyers were to make less than q visits and had time preference, as in (8.4), these costs would reflect the buyers' preferences.

The remaining admissible cells are designed to assure that if buyer j is assigned to seller k, seller k will be assigned to some time period i and time period i will be assigned to buyer j.

The sq admissible cells in the lower left rectangle (seller-time) represent the assignment of time periods for visits to sellers. Each seller supplies q visits and each time period demands r visits. The cost of assigning seller k to time period i is c'_{ik}. The capacity of each cell is one.

The q diagonal cells in the diagonal of the upper left square (time-time) represent unused visits in time period i. The variable corresponding to cell (i,i) in the square represents the number of idle buyers in time period i. Since these cells represent artificial variables, their cost is some large number M.

The diagonal cells of the center right square (buyer-buyer) represent idle periods for each buyer. The artificial variable corresponding to a cell (j,j) in the square is the number of unassigned time periods buyer j has. The cost of each cell is also M.

8-3.2.3. Linkage. The linkage provided by the artificial and surplus variables is most easily seen by observing the stepping stone pattern of a simplex pivot. The number of the cells in Table 18 illustrate how the values of the variables might change in the most simple type of pivot. In this pivot, when seller-buyer assignment (k,j) is made, a seller-time assignment (k,i) and a time-buyer assignment (i,j) is effected by the linking variables. Table 19 illustrates a more complex pattern which might occur when reassignments are being made inside each of the three subproblems.

8-3.2.4. Solution. An inductive proof of the existence of a nonconflicting time schedule corresponding to the optimal solution seems possible. Obviously, such a schedule exists for the original artificial solution (nobody going anywhere) shown in Table 17. Assuming you have a solution with a nonconflicting time schedule, you would need to show that no unresolvable conflicts are introduced by a pivot of the primal algorithm. Even if a nonconflicting schedule does exist, the solution does not directly provide it.

TABLE 18

A SIMPLE PIVOT

	times			sellers						buyers			
times		x	x	x	x	x	x	x	x				
	x	-1	x	x	x	x	x	x	x		+1		
	x	x		x	x	x	x	x	x				
buyers	x	x	x								x	x	x
	x	x	x		+1					x	-1	x	x
	x	x	x							x	x		x
	x	x	x							x	x	x	
sellers					x	x	x	x	x	x	x	x	x
		+1		x	-1	x	x	x	x	x	x	x	x
				x	x		x	x	x	x	x	x	x
				x	x	x		x	x	x	x	x	x
				x	x	x	x		x	x	x	x	x
				x	x	x	x	x		x	x	x	x

TABLE 19

A MORE COMPLEX PIVOT

	times			sellers						buyers			
times	-1	x	x	x	x	x	x	x	x		+1		
	x		x	x	x	x	x	x	x		-1		+1
	x	x		x	x	x	x	x	x				
buyers	x	x	x								x	x	x
	x	x	x			+1		-1		x		x	x
	x	x	x							x	x		x
	x	x	x					+1		x	x	x	-1
sellers					x	x	x	x	x				
				x		x	x	x	x				
		+1		x	x	-1	x	x	x				
	+1	-1		x	x	x		x	x				
				x	x	x	x		x				
				x	x	x	x	x					

It may be possible to start with the time schedule corresponding to Table 17, and at each pivot update the schedule. Or it may be possible to further "jury rig" the formulation to provide this schedule directly. It is more likely that a feasible time schedule can be found in a Phase II application of the marriage theorem as in our original heuristic, using the entire $(r + s + q)$ by $(r + s + q)$ tableau shown in Table 17. At each stage i, in the seller-time, time-time, and time-buyer subproblems, only the cells corresponding to time period i would be feasible.

REFERENCES

REFERENCES

Aho, Alfred V., John E. Hopcroft and Jeffrey D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, MA (1974).

Ali, A. Iqbal and Jeff L. Kennington, "The m-Travelling Salesmen Problem," Technical Report OR 80016, Southern Methodist University, Dallas, TX (July 1980).

Austin, A. Keith, "An Elementary Approach to NP-Completeness," Amer. Math. Monthly 90, 398-399 (1983).

Bellmore, M. and G. L. Nemhauser, "The Traveling Salesman Problem: A Survey," Oper. Res. 16, 538-558 (1968).

Bellmore, Mandell and John C. Malone, "Pathology of Traveling-Salesman Subtour-Elimination Algorithms," Oper. Res. 19, 278-307 (1971).

Bondy, J. A. and U. S. R. Murty, Graph Theory with Applications, American Elsevier, New York (1976).

Bradley, G. H., G. G. Brown and G. W. Graves, "Program GNET," Western Management Science Institute, Graduate School of Management, University of California, Los Angeles (1975).

Bradley, G. H., G. G. Brown and G. W. Graves, "Design and Implementation of Large Scale Primal Transshipment Algorithms," Man. Sci. 24, 1-34 (1977).

Brualdi, R. A. and J. Csima, "Extremal Plane Stochastic Matrices of Dimension Three," Linear Algebra and its Applic. 11, 105-133 (1975).

Burkard, Rainer E., "The Perturbation Method for Solving Quadratic Assignment and Bottleneck Problems," Rept. 1976-2, Mathematical Institute, University of Cologne, W. Germany (February 1976).

Burkard, Rainer E. and K. Frohlich, "Some Remarks on 3-Dimensional Assignment Problems," Meth. of Oper. Res. 36, 31-36 (1980).

Burkard, Rainer E., Willi Hahn and Uwe Zimmermann, "An Algebraic Approach to Assignment Problems," Math. Prog. 12, 318-327 (1977).

Caswell, W., "The Transignment Problem," Ph.D. Thesis, Rennselaer Polytechnic Institute, Troy, NY (1972).

Christofides, Nicos, "The Travelling Salesman Problem," in Christofides, N., A. Mingozzi, P. Toth and G. Sandi (eds.), Combinatorial Optimization, Wiley, New York (1979).

Csima, J., "Multidimensional Stochastic Matrices and Patterns," J. Algebra 14, 194-202 (1970).

DeWitt, Henry Koch, "The Theory of Random Graphs with Application to the Probabilistic Analysis of Optimization Algorithms," Ph.D. Thesis, Dept. of Comp. Sci., University of California, Los Angeles (1977).

Fisher, Marshall L. and Ramchandran Jaikumar, "A Generalized Assignment Heuristic for Vehicle Routing," Networks 11, 109-124 (1981).

Fox, K. R., B. Gavish, and S. C. Graves, "An n-Constraint Formulation of the (Time-Dependent) Traveling Salesman Problem," Oper. Res. 28, 1018-1021 (1980).

Fowler, Oscar S., "The Time Constrained Competing Traveling Salesmen Problem," Management Department, The University of Tennessee, Knoxville, TN (1976).

Frieze, Alan M., "A Bilinear Programming Formulation of the 3-Dimensional Assignment Problem," Math. Prog. 7, 376-379 (1974).

Frieze, Alan M., "An Extension of Christofides Heuristic to the k-Person Travelling Salesman Problem," Dept. of Computer Science and Statistics, Queen Mary College, University of London (November 1980).

Frieze, Alan M., "Complexity of a 3-Dimensional Assignment Problem," Dept. of Computer Science and Statistics, Queen Mary College, University of London (July 1981).

Frieze, Alan M., "Algebraic Linear Programming," Math. of Oper. Res. 7, 172-182 (1982).

Frieze, Alan M., personal correspondence, Dept. of Computer Science and Statistics, Queen Mary College, University of London (1983).

Frieze, Alan M. and J. Yadegar, "An Algorithm for Solving 3-Dimensional Assignment Problems with Application to Scheduling a Teaching Practice," J. Opnal. Res. Soc. 32, 989-995 (1981).

Gale, David, "Optimal Assignments in an Ordered Set: An Application of Matroid Theory," J. Comb. Theory 4, 176-180 (1968).

Garey, Michael R. and David S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, San Francisco, CA (1979).

Garfinkel, Robert S., "An Improved Algorithm for the Bottleneck Assignment Problem," Oper. Res. 19, 1747-1751 (1971).

Garfinkel, Robert S., "Minimizing Wallpaper Waste, Part I: A Class of Traveling Salesman Problems," Oper. Res. 25, 741-751 (1977).

Garfinkel, Robert S., "Selected Topics in Combinatorial Optimization," Management Science Seminar, The University of Tennessee, Knoxville, TN (1982).

Garfinkel, Robert S., "The Traveling Salesman Problem: Motivation and Modeling," Working Paper No. 146, College of Business Administration, The University of Tennessee, Knoxville, TN (May 1982).

Garfinkel, Robert S. and Kenneth C. Gilbert, "The Bottleneck Traveling Salesman Problem: Algorithms and Probabilistic Analysis," J. A. C. M. 25, 435-448 (1978).

Garfinkel, Robert S. and George L. Nemhauser, Integer Programming, Wiley, New York (1972).

Gavish, Bezalel and Kizhanatham Srikanth, "An Optimal Solution Method for the Multiple Travelling Salesman Problem," Working Paper No. 8027, The Graduate School of Management, The University of Rochester, Rochester, NY (November 1980).

Gorenstein, Samuel, "Printing Press Scheduling for Multiedition Periodicals," Man. Sci. 16, 373-383 (1970).

Haley, K. B., "The Solid Transportation Problem," Oper. Res. 10, 448-463 (1962).

Haley, K. B., "The Multi-Index Problem," Oper. Res. 11, 368-379 (1963).

Haley, K. B., "The Existence of a Solution to the Multiindex Problem," Opnal. Res. Quart. 16, 471-474 (1965).

Haley, K. B., "Note on the Letter by Moravek and Vlach," Oper. Res. 15, 545-546 (1967).

Hall, Marshall, Jr., Combinatorial Theory, Blaisdell, Waltham, MA (1967).

Harary, Frank, "Graph Theoretic Methods in the Management Sciences," invited address at the Annual Meeting of TIMS, Philadelphia (October 1958).

Held, Michael and Richard M. Karp, "The Traveling Salesman Problem and Minimum Spanning Trees," Oper. Res. 18, 1138-1167 (1970).

Held, Michael and Richard M. Karp, "The Traveling Salesman Problem and Minimum Spanning Trees: Part II," Math. Prog. 1, 6-25 (1971).

Hung, Ming S. and Walter O. Rom, "Solving the Assignment Problem by Relaxation," Oper. Res. 28, 969-982 (1980).

Jurkat, W. B. and H. J. Ryser, "Extremal Configurations and Decomposition Theorems. I," J. Algebra 8, 194-222 (1968).

Karonski, Michal, "A Review of Random Graphs," J. Graph Theory 6, 349-389 (1982).

Kaufman, L., "The Location of Economic Activities by 0-1 Programming," Dissertation, University of Bruxelles (1975).

Kise, Hiroshi and Toshihide Ibarak, "On Balut's Algorithm and NP-Completeness for a Chance-Constrained Scheduling Problem," Man. Sci. 29, 384-388 (1983).

Klee, Victor, "Combinatorial Optimization: What is the State of the Art," Math. of Oper. Res. 5, 1-26 (1980).

Kuhn, H. W., "Hungarian Method for the Assignment Problem," Naval Res. Log. Quart. 2, 83-97 (1955).

Kuhn, H. W., "Variants of Hungarian Method for Assignment Problems," Naval Res. Log. Quart. 3, 253-258 (1956).

Kurzberg, Jerome M., "On Approximation Methods for the Assignment Problem," J.A.C.M. 9, 419-439 (1962).

Lawler, Eugene L., Combinatorial Optimization: Networks and Matroids, Holt, Rinehart and Winston, New York (1976).

Leue, O., "Methoden zur Losung Dreidimensionaler Zuordnungsprobleme," Angewandte Informatik, 154-162 (1972).

Little, John D. C., Katta G. Murty, Dura W. Sweeney and Caroline Karel, "An Algorithm for the Traveling Salesman Problem," Oper. Res. 11, 972-989 (1963).

Liu, C. L., Introduction to Combinatorial Mathematics, McGraw-Hill, New York (1968).

McGinnis, Leon F., "Implementation and Testing of a Primal-Dual Algorithm for the Assignment Problem," Oper. Res. 31, 277-291 (1983).

Miller, C. E., A. W. Tucker and R. A. Zemlin, "Integer Programming Formulations and Traveling Salesman Problems," J.A.C.M. 7, 326-329 (1960).

Motzkin, T. S., "The Assignment Problem," Proceedings of Symposia in Applied Mathematics 6 (1956).

Mycielski, Jan, "The Meaning of the Conjecture P $\neq$ NP for Mathematical Logic," Amer. Math. Monthly 90, 129-130 (1983).

Ortega, James M., Numerical Analysis: A Second Course, Academic Press, New York (1972).

Pierskalla, William P., "The Tri-Substitution Method for Obtaining Near-Optimal Solutions to the Three-Dimensional Assignment Problem," Tech. Memo No. 71, Operations Research Group, Case Institute of Technology, Cleveland, OH (October 1966).

Pierskalla, William P., "The Multidimensional Assignment and Quadratic Assignment Problems," Tech. Memo No. 93, Operations Research Group, Case Institute of Technology, Cleveland, OH (September 1967).

Pierskalla, William P., "The Multidimensional Assignment Problem," Oper. Res. 16, 422-431 (1968).

Pierskalla, William P., personal correspondence, Leonard Davis Institute of Health Economics, University of Pennsylvania, Philadelphia, PA (1982).

Rao, M. R., "A Note on the Multiple Traveling Salesman Problem," Oper. Res. 28, 628-632 (1980).

Ross, G. Terry and Richard M. Soland, "A Branch and Bound Algorithm for the Generalized Assignment Problem," Math. Prog. 8, 91-103 (1975).

Ross, G. Terry and Richard M. Soland, "Modeling Facility Location Problems as Generalized Assignment Problems," Man. Sci. 24, No. 3 (1977).

Ross, G. Terry and Andris A. Zoltners, "Weighted Assignment Models and their Application," Man. Sci. 25, 683-696 (1979).

Rudin, Walter, Real and Complex Analysis, McGraw-Hill, New York (1974).

Ryser, Herbert John, Combinatorial Mathematics, Carus Math. Monograph No. 14, Math. Assoc. Am., Wiley, New York (1963).

Ryser, Herbert John, personal correspondence, Mathematics Dept., California Institute of Technology, Pasadena, CA (1982).

Sahni, S. and T. Gonzalez, "P-Complete Approximation Problems," J.A.C.M. 23, 555-565 (1976).

Schell, Emil D., "Distribution of a Product by Several Properties," Procs. 2nd. Symposium in Linear Programming, DCS/Comptroller, H. Q. U. S. Air Force, Washington, DC, 615-642 (January 1955).

Shamma, M. M., "A Generalized Assignment Problem," Ph.D. Thesis, Technological Institute, CS/OR Center, Southern Methodist University, Dallas, TX (1971).

Spivey, W. Allen and Robert M. Thrall, Linear Optimization, Holt, Rinehart and Winston, New York (1970).

Svabova, Milada and Eva Marunova, "An n-Dimensional Transport Problem," Ekonomicko-Matematicky 3, 308-323 (1967).

Svestka, Joseph A. and Vaughn E. Huckfeldt, "Computational Experience with an m-Salesman Traveling Salesman Algorithm," Man. Sci. 19, 790-799 (1973).

Swanson, Leonard W., "N-Dimensional Scheduling," Graduate School of Management, Northwestern University, Evanston, IL (1976).

Tuy, Hoang, "Concave Programming under Linear Constraints," Soviet Mathematics, 1437-1440 (1964).

Vlach, Milan, "Determination of a Feasible Solution of the Three-Index Transport Problem," Ekonomicko-Matematicky Obzor 1, 155-163 (1965).

Vlach, Milan, "Branch and Bound Method for the Three-Index Assignment Problem," Ekonomicko-Matematicky Obzor 3, 181-191 (1967).

Vlach, Milan and Jaroslav Moravek, "Comments upon the Three-Index Transport Problem," Ekonomicko-Matematicky Obzor 3, 75-77 (1967).

Vlach, Milan and Jaroslav Moravek, "On the Necessary Conditions for the Existence of the Solution of the Multiindex Transportation Problem," Oper. Res. 15, 542-545 (1967).

von Neumann, J., "A Certain-Zero-Sum Two-Person Game Equivalent to the Optimal Assignment Problem," in H. Kuhn and A. Tucker (eds.), Contributions to the Theory of Games II (Ann. Math. Study No. 28), Princeton University Press, 5-12 (1953).

Wagner, Harvey M., Principles of Operations Research, Prentice-Hall, Englewood Cliffs, NJ (1975).

Walkup, David W., "On the Expected Value of a Random Assignment Problem," SIAM J. Comput. 8, 440-442 (1979).

Walkup, David W., "Matchings in Random Regular Bipartite Digraphs," Discrete Math. 31, 59-64 (1980).

Whitehead, Alfred North, Science and the Modern World, Macmillan, New York (1925).

VITA

The author was born Ruth Margaret Bisgrove in Utica, New York, on December 10, 1924.

She was graduated as class valedictorian in 1942 from Whitesboro Central School, Whitesboro, New York. She holds the A.B. (1947) and A.M. (1949) degrees from Syracuse University, Syracuse, New York. The Bachelor's degree has a double major in Mathematics and Education, and a minor in Latin. The Master's degree has a major in Mathematics. She will receive the Doctor of Philosophy degree at The University of Tennessee, Knoxville, in 1983 with a major in Management Science and supporting area in Statistics.

At Syracuse University, she was a member of Alpha Phi social sorority, and was President of Women's Chapel Association. As an undergraduate, she became a member of Phi Beta Kappa, Pi Mu Epsilon, Pi Lambda Theta, and Eta Pi Upsilon. As a graduate, she became a member of Sigma Xi (associate) and Sigma Delta Epsilon. She is a member of The Institute of Management Science, The Operations Research Society of America, The American Mathematical Society, and The Mathematical Association of America.

The author has held university teaching positions in mathematics, and has been employed as computer applications mathematician and analyst. She will be joining the faculty of the Management Department, The University of Tennessee at Chattanooga in September, 1983.

She was married to Donald J. Hofstra and has two sons, John and Charles.