To the Graduate Council:

I am submitting herewith a dissertation written by Vagelis Stefanopoulos entitled "The Role of Critical Eigenvalues and Eigenfunctions in a Class of Singularly Perturbed Problems." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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THE ROLE OF CRITICAL EIGENVALUES AND EIGENFUNCTIONS IN A CLASS OF SINGULARLY PERTURBED PROBLEMS

A Dissertation

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ABSTRACT

We consider the eigenvalue problem $-\varepsilon^2\Delta\psi + f'(u_\varepsilon)\psi = \mu\psi$, with Neumann boundary conditions, in a smooth bounded domain Ω in $\mathbb{R}^2$, arising from the evolution problem $u_t = \varepsilon^2\Delta u - f(u)$, with the same boundary conditions. Here f is the derivative of a double well potential with two equal minima at ± 1, and u_ε is a smooth function with a sharp interface γ contained in Ω.

The goal is to characterize the eigenvalues μ that vanish in the limit $\varepsilon \to 0$ (critical), and their corresponding eigenfuntions. It is shown that these eigenvalues are related to the eigenvalues λ of a regular Sturm–Liouville problem on the curve γ by $\mu = \varepsilon^2\lambda + O(\varepsilon^3)$. By changing to local coordinates r(=signed distance from γ) and s(=arclength on γ) in a neighborhood of γ, and then stretching variables ($\eta = r/\varepsilon$), Alikakos and Fusco have shown that the critical eigenfunctions, appropriately normalized, separate in the limit $\varepsilon \to 0$ to $U'(\eta)\Theta(s)$, where U' is the derivative of the solution of $U'' - f(U) = 0$, $U(0) = 0$, $U(\pm\infty) = \pm 1$, and $\Theta(s)$ is a periodic function. Elaborating on this separation result, we prove that Θ is an eigenfuction of the above mentioned Sturm–Liouville problem. We thus establish the following characterization for the eigenpair (μ, ψ):

$$\mu_i = \varepsilon^2\lambda_i + O(\varepsilon^3)\,, \quad \psi_i = U'(r/\varepsilon)\Theta_i(s) + O(\varepsilon)\,, \quad i = 1, 2, \ldots, m\,,$$

where m is fixed but arbitrary. The proof involves among other things spectral analysis of radial operators, justification of formal asymptotic expansions, and perturbation theory.

The same analysis carried to the case where u_ε is a radial equilibrium of the corresponding evolution problem with an interface γ, shows that every such solution is unstable.

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INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, and consider the free energy functional

$$J_\varepsilon(u) = \int_\Omega \left\{ \frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right\} dx \,, \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a small parameter and F is a double–well potential with equal, nondegenerate minima at $u = -1$ and $u = 1$ (Fig. 1).[1] Two evolution problems associated to J_ε, in the sense that J_ε serves as a Liapunov functional for them, are the **Allen–Cahn** model

$$\begin{cases} u_t = \varepsilon^2 \Delta u - f(u)\,, & \text{in } \Omega \times (0,\infty)\,, \\ \dfrac{\partial u}{\partial n} = 0\,, & \text{on } \partial\Omega \times (0,\infty)\,, \\ u(x,0) = u_0(x)\,, & \text{in } \Omega\,, \end{cases} \tag{1.2}$$

and the **Cahn–Hilliard** model

$$\begin{cases} u_t = \Delta\big(-\varepsilon^2 \Delta u + f(u)\big)\,, & \text{in } \Omega \times (0,\infty)\,, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial}{\partial n}\big(-\varepsilon^2 \Delta u + f(u)\big) = 0\,, & \text{on } \partial\Omega \times (0,\infty)\,, \\ u(x,0) = u_0(x)\,, & \text{in } \Omega\,, \end{cases} \tag{1.3}$$

where Δ is the Laplacian, n is the unit outward normal to $\partial\Omega$ and $f = F'$. It is shown in [F2] that the Allen–Cahn and Cahn–Hilliard models can also be considered as the L^2 and H_0^{-1}, respectively, gradient flows for the functional J_ε. Here H_0^{-1} is the subspace of H^{-1} consisting of functions with zero average.

[1] All figures can be found in the appendix.

Equation (1.2) was proposed in [A–C] as a model for the motion of an antiphase boundary that separates two phases of a crystalline solid, whereas (1.3) was proposed in [C], [C–H] to model phase separation and coarsening in binary alloys at a fixed temperature. The function F represents the free energy per unit volume and its two wells correspond to different stable material phases.

In this work we assume $N = 2$ and we concentrate on the Allen–Cahn model. As the functional J_ε suggests, a typical initial condition for (1.2) (and for (1.3) as well) is evolved into a layered function with sharp interfaces of width $O(\varepsilon)$ [deM–S1], [F1], [F–Hs]. For the Allen–Cahn model in one dimension the motion of the developed interfaces is studied in [B–K1], [C–P1,2], [Fu], [Fu–H] where it is shown that their speed is exponentially small (in ε). In higher dimensions [B] and [B–K2] have shown that radially symmetric interfaces evolve by mean curvature (see (1.5)). In [deM–S2] the connection between the dynamics of (1.2) and motion by mean curvature is established for smooth interfaces without symmetry assumptions. The same connection is also established in [E–S–S] without requiring smoothness of the interface and also allowing the possibility of singularities. For a survey of these and related results we refer to [F3] where the Cahn–Hilliard and Phase–Field models are discussed as well.

THE PROBLEM. An important aspect of the study of the slow motion in [C–P1,2], [deM–S2], [Fu], [Fu–H] is the spectral analysis of the linearization of (1.2) about an appropriate function u_ε related to a solution of (1.2). In this thesis we consider such a linearization and study the eigenvalue problem

$$(1.4) \qquad \begin{cases} -\varepsilon^2\Delta\psi + f'(u_\varepsilon)\psi = \mu\psi\,, & \text{in } \Omega\,, \\ \dfrac{\partial\psi}{\partial n} = 0\,, & \text{on } \partial\Omega \end{cases}$$

associated to (1.2), where u_ε is a layered "step–like" function that takes values from near -1 to near $+1$ (the points where F attains its equal minima) and jumps from -1 to $+1$ in an ε–neighborhood of a smooth closed curve γ contained in Ω. The precise definition of u_ε is given in (2.9). We are interested in particular in the critical eigenvalues of (1.4) (i.e. the ones that vanish as $\varepsilon \to 0$) and their corresponding eigenfunctions, whose relevance to the motion of the interface has been brought out in the works of [A–B–F1,2], [A–F1,2], [C–P1,2], [deM–S2]. The critical eigenfunctions are localized on the interface and drive it while the critical eigenvalues are related to the speed of the interface. Therefore obtaining estimates for these eigenvalues/eigenfunctions is important in making rigorous the relationship of (1.2) and (1.3) for small ε to certain geometrical problems like **motion by mean curvature**

$$v = -\varepsilon^2 \kappa \,, \tag{1.5}$$

where v is the normal velocity of the interphase γ and κ is its mean curvature, or the **Hele–Shaw Problem**

$$\begin{cases} \Delta\mu = 0\,, & \text{in } \Omega\setminus\gamma\,, \\ \dfrac{\partial\mu}{\partial n} = 0\,, & \text{on } \partial\Omega \\ \mu = \varepsilon\alpha\kappa\,, & \text{on } \gamma\,, \end{cases} \qquad \& \qquad v = \beta\left[\frac{\partial\mu}{\partial n}\right]_\gamma , \tag{1.6}$$

where v and κ are as before, $[\partial\mu/\partial n]_\gamma$ is the jump of the normal derivative of μ across γ and α,β are constants.

MOTIVATION. In the approach of [deM–S2] an approximate solution $u_a^\varepsilon(x,t)$ of (1.2) is constructed via the method of asymptotic expansions, and a theorem is proved stating that if u^ε is the true solution, then $\|u^\varepsilon - u_a^\varepsilon\|$ stays small over a time interval of order ε^{-2}. An important step in carrying out the argument involves a lower bound on the spectrum of the linear operator

$$\mathcal{L}_t^\varepsilon w = -\varepsilon^2 \Delta w + f'(u_a^\varepsilon(\cdot,t))w \,.$$

It is shown that $\min \sigma(\mathcal{L}_t^\varepsilon) \geq -C\varepsilon^2$, where C is a constant. This allows control of $\|u^\varepsilon - u_a^\varepsilon\|$ over the desired time interval (which we notice that it has to be of length of order ε^{-2} in order to meaningfully compare (1.2) with (1.5)), and ultimately allows the proof of the rigorous relationship between (1.2) and (1.5). This is one motivation for studying the eigenvalue problem (1.4). However there is more to it. Before we get into this, we want to give a clue about the origin of the critical eigenvalues. For this purpose we consider an 1–dimensional analog of (1.4) where

$$f(u) = \frac{1}{2}(u^3 - u)\,, \qquad u_\varepsilon = \tanh\frac{x}{2\varepsilon}\,, \qquad \Omega = (-1, 1).$$

Then (1.4) becomes

$$(1.7) \qquad -\varepsilon^2\psi'' + q(x)\psi = \mu\psi\,, \qquad -1 < x < 1\,, \qquad \psi'(\pm 1) = 0\,,$$

with

$$q(x) = \frac{3}{2}\tanh^2\frac{x}{2\varepsilon} - \frac{1}{2}.$$

Hence, for ε small, q is exponentially close to 1, except for those x belonging to an interval of length of order ε about zero. The supposition that the eigenvalues μ of (1.7) are approximated by the eigenvalues $\tilde{\mu}$ of

$$-\varepsilon^2\phi'' + \phi = \tilde{\mu}\phi\,, \qquad -1 < x < 1\,, \qquad \phi'(\pm 1) = 0\,,$$

leads to the conclusion that μ_n is close to

$$\tilde{\mu}_n = 1 + \varepsilon^2 n^2 \pi^2 \qquad , \qquad n = 0, 1, 2, 3, \ldots.$$

However, this is false because (1.7) has an exponentially small eigenvalue, $\mu_1 = O(e^{-c/\varepsilon})$, ([C–P1], [Fu–H]). Thus the problem is considerably more delicate.

The example above suggests that the critical eigenvalues owe their existence to the layer of u_ε, that is, its behavior near the interface. The objective of this thesis is to make this precise for the problem in higher space dimensions. For special geometries the spectrum may be calculated explicitly. The first example where this was done is

The Solá-Morales Example : Consider the eigenvalue problem (1.4) in the case where $\Omega = \{(x,y) : -1 < x < 1, -1 < y < 1\}$ and $u_\varepsilon = u_\varepsilon(x) = U(x/\varepsilon)$ with $U = U(\eta)$, $\eta \in \mathbb{R}$ being the solution of

$$U'' - f(U) = 0\,, \qquad U(0) = 0\,, \qquad \lim_{\eta \to \pm\infty} U(\eta) = \pm 1\,. \tag{1.8}$$

The eigenfunctions ψ are of the form

$$\psi(x,y) = X(x)Y(y)\,,$$

where X and Y are solutions of the eigenvalue problems

$$-\varepsilon^2 X'' + f'(u_\varepsilon)X = \lambda X\,, \qquad -1 < x < 1\,, \qquad X'(\pm 1) = 0\,, \tag{1.9}$$

$$-\varepsilon^2 Y'' = \varepsilon^2 \nu Y\,, \qquad -1 < y < 1\,, \qquad Y'(\pm 1) = 0\,. \tag{1.10}$$

Then the eigenvalues μ of (1.4), in this setting, are given by

$$\mu = \mu_{mn} = \lambda_m + \varepsilon^2 \nu_n = \lambda_m + \varepsilon^2 n^2 \pi^2\,, \qquad m,n = 0,1,2,3,...$$

and so the spectrum of (1.4) in this setting consists of three types of eigenvalues

$$\begin{aligned}
&\mu_{00} = O(e^{-c/\varepsilon})\,, && \text{because } \lambda_0 = O(e^{-c/\varepsilon})\,, \quad \text{([C–P1], [Fu–H])}\\
&\mu_{0n} = O(\varepsilon^2)\,, && n > 0\,,\\
&\mu_{mn} \geq C\,, && m > 0\,,
\end{aligned}$$

where c and C are constants independent of ε. Let us point out some of the significant aspects of this computation. First there is only one negative eigenvalue,

μ_{00}, and the rest of the spectrum is positive. Therefore u_ε, if it were an equilibrium would be 1–dimensional unstable. The exponentially small eigenvalue suggests a superslow flow on the associated unstable manifold. Moreover the positivity of the rest of the spectrum together with its difference in the order of magnitude suggests that the unstable manifold is a stable object on which the flow evolves slower than it evolves towards it. It turns out that all these can be made rigorous by applying the 1–dimensional theory developed in [C–P1,2] and [Fu–H]. The spectrum provides important stability information and suggests persistence for extremely long times of certain special solutions.

We note that in the example above the potential and the geometry are very special and allow computation by separation of variables. It is natural to try to understand the problem for a general interface γ and general geometry Ω. A result in that direction is

The Alikakos–Fusco Estimate : Alikakos and Fusco for the purpose of making rigorous and more generally clarifying the relationship between the Cahn–Hilliard equation and the Hele–Shaw problem (1.6), considered the eigenvalue problem (1.4) with a profile u_ε that is related to the the inner expansion of a solution of (1.2) with interface γ. The relevant form of u_ε (dictated by formal asymptotics) in a δ–neighborhood of γ is given by

$$u_\varepsilon(x) = U\left(\frac{r}{\varepsilon}\right) + \varepsilon\kappa(s)V\left(\frac{r}{\varepsilon}\right) + O(\varepsilon^2)\,, \qquad -\delta \le r \le \delta\,, \tag{1.11}$$

where time is suppressed. Here $r = r(x)$ is the signed distance of x from γ, that is taken negative if x is inside γ, and positive otherwise, $s = s(x)$ is the distance along γ of the projection of x from a fixed point on γ, κ is the curvature of γ at s, U is the solution of (1.8) and V is a smooth bounded function in $\mathbb{R}$ satisfying the orthogonality condition

$$(1.12) \qquad \int_{-\infty}^{\infty} f''(U)U'^2 V \, d\eta = 0 .$$

In [A–F2], for a u_ε defined as above, they established the following uniform (in ε) estimate for the counting function of the critical eigenvalues of (1.4)

$$(1.13) \qquad \#\{\, \mu : \mu \le c_o \text{ and } \mu \le K\varepsilon^2 \,\} \le C(K+\bar{K})^{(N-1)/2}$$

for all ε small, where c_0, C, K,$\bar{K}$ are constants and N is the dimensionality of Ω. This estimate implies that the nth eigenvalue behaves like $\varepsilon^2 n^{2/(N-1)}$ ([Cou–H]) and therefore for $N = 2$ like $\varepsilon^2 n^2$. This agrees with the explicit calculation in the Solá-Morales example. In particular this result attempts to quantify the intuition that the critical eigenvalues are coming from a problem with 1–dimension less. In the present work we determine precisely the origin of these eigenvalues. In that direction we establish the

Theorem (Theorem 6.5, Corollary 6.6, Corollary 6.8). *If $\{\mu_i\}_{i=1}^m$ are critical eigenvalues of (1.4) with u_ε as in (1.11) then*

$$\lim_{\varepsilon\to 0} \frac{\mu_i}{\varepsilon^2} = \lambda_i ,$$

where λ_i is the ith $i = 1,\ldots,m$, eigenvalue of the problem

$$(1.14) \qquad \begin{cases} -\Theta''(s) + g(s)\Theta(s) = \lambda\Theta(s), & 0 < s < \ell, \\ \Theta(0) = \Theta(\ell), \quad \Theta'(0) = \Theta'(\ell), \end{cases}$$

where ℓ is the length of γ and s is arclength.

The result above makes precise the estimate (1.13) and complements it by showing where the critical eigenvalues are coming from. In addition it provides information about the asymptotic form of the critical eigenfunctions and in particular on the way they separate in the limit $\varepsilon \to 0$. In [A–F2] it is shown that if ψ is a critical eigenfunction of (1.4) with u_ε as in (1.11), then $\psi \to U'\Theta$ as $\varepsilon \to 0$ where U is the solution of (1.8) and Θ is some ℓ–periodic function of s. In this thesis we establish that Θ is an eigenfunction of (1.14). Although in dimensions more than one no general statement can be made about the multiplicity of the eigenvalues, as a consequence of the theorem above, we have that the multiplicity of the critical eigenvalues is at most two, and moreover, we establish a kind of one–to–one correspondence between the μ's and the λ's and show

Corollary (Remark 6.9). *If $\{\mu_i\}_{i=1}^{m}$ are the first $2n+1$ critical eigenvalues of (1.4) then there is an ε_0 such that*

$$\mu_1 < \mu_2 \leq \mu_3 < \mu_4 \leq \mu_5 < \cdots < \mu_{2n} \leq \mu_{2n+1}\,,$$

for all $0 < \varepsilon < \varepsilon_0$.

The potential g in (1.14) depends on the curvature κ and in general on f and the form of u_ε. However, in the case where u_ε is a radial equilibrium for the Allen–Cahn (1.2) or the Cahn–Hilliard (1.3), it is shown that $g(s) = -\kappa^2$ (Proposition 7.1), independent of f.

The present work is organized as follows. Chapter 2 contains background material that is needed in the rest of the development. In chapter 3 we study a particular case of (1.4) that plays an important role in the subsequent work. The domain Ω here is an annulus, γ is a circle of radius ρ and $u_\varepsilon(x) = U\big((r-\rho)/\varepsilon\big)$, where U is the solution of (1.6) and $r = |x|$. In the direction of studying the spectrum of the related operator, we extend the results of [A–MP–P] and [N–F] regarding the asymptotic form of the principal eigenvalue and the decomposition of the spectrum to a critical and noncritical part. In fact there is only one critical eigenvalue. The presence of the curvature term $1/r$ in the operator results in an $O(\varepsilon^2)$ eigenvalue in contrast to $O(e^{-c/\varepsilon})$ that occurs in the one-dimensional case. The rest of the spectrum is bounded away from zero. In chapter 4 we justify the formal asymptotic expansion of the principal eigenfunction of the operator introduced in chapter 3 up to any order and we show that the limit as $\varepsilon \to 0$ of the principal eigenvalue when divided by ε^2 is $-1/(4\rho^2)$, and thus it is a purely geometrical quantity (the curvature of γ) independent of the function f. In chapter 5, with the aid of the results for the radial case, we study the radial part of the general operator and we show that it has one critical eigenvalue while the rest of the spectrum is bounded away from zero uniformly in ε. After obtaining information about the asymptotic expansion of the critical eigenvalue, we show that the principal eigenvalue generates the potential $g(s)$ in (1.14) in the sense that the limit as $\varepsilon \to 0$ of the principal eigenvalue when divided by ε^2 is $g(s)$. Chapter 6 is dedicated to the proof of the main theorem. One of the tools that is used in the proof is a perturbation result from mathematical physics (Lemma 2.2) that enables us to relate the spectra of (1.4) and (1.14). A consequence of the main theorem is the characterization of the multiplicity of the critical eigenvalues. In chapter 7, we consider another radial problem in which u_ε is either an equilibrium of (1.2) or of (1.3). In this case the analysis of the spectrum provides stability information for these special solutions. The rest of this final chapter consists of some observations and speculations.

PRELIMINARIES

2.1. The Setting.

Let $f : \mathbb{R} \to \mathbb{R}$, $(f = F')$ be a C^3 function with the following properties:

(a) f has exactly three zeros $f(\pm 1) = f(0) = 0$,

(b) $f'(\pm 1) > 0$, $f'(0) < 0$,

(c) $\int_{-1}^{1} f(z)\, dz = 0$.

The hypotheses above on f imply that F has two equal minima at ± 1; therefore, there exists a unique solution U of

$$(2.1) \qquad \begin{cases} U'' - f(U) = 0\,, & U = U(z)\,, \qquad z \in \mathbb{R}\,, \\ U(0) = 0\,, & \lim_{z \to \pm\infty} U(z) = \pm 1\,, \end{cases}$$

corresponding to the heteroclinic orbit joining $(0, -1)$ to $(0, 1)$ in the phase plane. This solution U is strictly increasing and we will refer to it as the "heteroclinic" (see Fig. 2). It can be shown, using phase plane analysis, that there are positive constants β, β' such that the following estimates hold true

$$(2.2) \qquad \begin{aligned} &1 - U(z) \sim \beta' e^{-\beta z}\,, \\ &U'(z) \sim \beta\beta' e^{-\beta z}\,, \qquad \text{as } z \to \infty\,, \\ &U''(z) \sim -\beta^2 \beta' e^{-\beta z}\,, \end{aligned}$$

with similar estimates as $z \to -\infty$. An example for the considerations above is

$$F(u) = \frac{1}{8}(u^2 - 1)^2\,, \qquad f(u) = \frac{1}{2}(u^3 - u)\,,$$

for which the corresponding heteroclinic solution is

$$U(z) = \tanh \frac{z}{2}\,.$$

We call the example above the model problem and we study it in some detail in section 3.3.

Let Ω be a bounded domain in $\mathbb{R}^2$ with sufficiently smooth boundary, and let γ be a C^2 simple closed curve, parametrized by arclength $s \in [\,0, \ell\,]$, contained in Ω. If the point $\gamma(s)$ on the curve is represented by the vector $\langle \gamma_1(s), \gamma_2(s) \rangle$, then the unit (outward) normal vector to γ at s is $\bar{v}(s) = \langle \dot{\gamma}_2(s), -\dot{\gamma}_1(s) \rangle$, where $\dot{\gamma}_i$, is the derivative of γ_i with respect to s, $i = 1, 2$. Let r be the signed distance along $\bar{v}$ of the point x in Ω from γ, satisfying $r < 0$ inside γ and $r > 0$ outside. Then for δ sufficiently small, the equation

$$x = \gamma(s) + r\bar{v}(s) \tag{2.3}$$

defines a diffeomorphism from the cylinder $[-2\delta, 2\delta\,] \times [\,0, \ell\,]$, where the end points $s = 0$ and $s = \ell$ are identified, into the the 2δ–strip

$$\overline{\Omega}_{2\delta} := \{\, x : d(x, \gamma) \leq 2\delta \,\}\,, \tag{2.4}$$

and $\overline{\Omega}_{2\delta} \subset \Omega$, with $d(\cdot\,,\cdot)$ being the Euclidean distance. Recalling that the curvature $\kappa = \kappa(s)$ of γ at s is defined by

$$\kappa = \det \begin{pmatrix} \dot{\gamma}_1 & \dot{\gamma}_2 \\ \ddot{\gamma}_1 & \ddot{\gamma}_2 \end{pmatrix}\,,$$

an easy computation shows that the Jacobian of the transformation defined by (2.3) is given by

$$J(r, s) = \frac{\partial(x_1, x_2)}{\partial(r, s)} = (1 + \kappa r)\,.$$

Moreover, for the local coordinate system (r,s), it can be shown that the following geometric laws hold

$$(2.5)\qquad |\nabla r|^2 = 1\,, \qquad \Delta r = \frac{\kappa}{1+\kappa r}\,,$$

$$(2.6)\qquad |\nabla s|^2 = \frac{1}{(1+\kappa r)^2}\,, \qquad \Delta s = -\frac{\dot\kappa r}{(1+\kappa r)^3}\,,$$

where ∇ and Δ are the gradient and the Laplacian in the original variable x, and $\dot\kappa$ is the derivative of κ with respect to s.

Motivated by formal asymptotics (see chapter 1), we define

$$(2.7)\qquad \overline{u}_\varepsilon(r,s) = U\left(\frac{r}{\varepsilon}\right) + \varepsilon\kappa(s)V\left(\frac{r}{\varepsilon}\right) + \varepsilon^2 W\left(\frac{r}{\varepsilon}, s\right)\,,$$

where U is the "heteroclinic", V is a bounded $C^1(\mathbb{R})$ function satisfying the orthogonality condition

$$(2.8)\qquad \int_{-\infty}^{\infty} f''(U)U'^2 V\,dz = 0\,,$$

and W is a bounded $H^2(\mathbb{R}\times[0,\ell])$ function. If $\omega : \mathbb{R} \to \mathbb{R}$ is an even, nondecreasing in $(-\infty,0)$, C^∞ cut–off function

$$\omega(z) = 1\,, \quad \text{if } |z| \le \frac{1}{2}\,, \qquad \omega(z) = 0\,, \quad \text{if } |z| \ge 1\,,$$

we define $u_\varepsilon : \Omega \to \mathbb{R}$ by

$$(2.9)\qquad u_\varepsilon(x) = \begin{cases} \omega\left(\frac{r}{2\delta}\right)\overline{u}_\varepsilon(r,s) \pm \left(1 - \omega\left(\frac{r}{2\delta}\right)\right)\,, & \text{if } d(x,\gamma) \le 2\delta\,, \\ \pm 1\,, & \text{if } d(x,\gamma) \ge 2\delta\,, \end{cases}$$

where (r,s) are the local coordinates for $x \in \overline{\Omega}_{2\delta}$, and the minus sign corresponds to points x inside γ while the plus sign to x's outside γ.

In the present work we study the eigenvalue problem

$$(2.10)\qquad \begin{cases} \mathcal{L}^\varepsilon\psi := -\varepsilon^2\Delta\psi + f'(u_\varepsilon)\psi = \mu\psi\,, & \text{in } \Omega\,, \\ \dfrac{\partial\psi}{\partial n} = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

where u_ε is defined in (2.9).

2.2. Facts.

Our hypotheses imply that the potential $f'(u_\varepsilon)$ is strictly positive and close to $f'(-1)$, $f'(1)$ for x respectively inside or outside γ, except an $O(\varepsilon)$ strip around the interface. This suggests that all the interesting phenomena take place in a thin neighborhood of γ. This becomes precise by the estimates that follow.

Lemma 2.1 (Agmon type estimates). *Let $0 < a < \min\{ f'(\pm 1) \}$. Then there is a positive constant ε_0 such that if ψ is an L^2 normalized eigenfunction of (2.9) corresponding to an eigenvalue $\mu \le a$ the following pointwise estimate holds*

$$|\psi(x)| \le Ce^{-c|r|/\varepsilon}, \qquad x \in \Omega_{2\delta} \setminus \Omega_{2\varepsilon b}, \tag{2.11}$$

for all $0 < \varepsilon < \varepsilon_0$. Here C, c are constants depending only on a, and b is a constant independent of ε.

For the proof we refer to [H] and [H–S]. An application of the maximum principle to the elliptic equation

$$-\varepsilon^2 \Delta\psi + h(x)\psi = 0, \qquad x \in \Omega \setminus \Omega_{2\varepsilon b},$$

where $h(x) = f'(u_\varepsilon) - \mu > 0$ in $\Omega \setminus \Omega_{2\varepsilon b}$, leads to the exponential estimate away from γ:

$$|\psi(x)| \le Ce^{-c/\varepsilon}, \qquad x \in \Omega \setminus \Omega_{2\varepsilon b}, \tag{2.12}$$

for $0 < \varepsilon < \varepsilon_0$.

The estimate (2.12) shows that the mass of the eigenfunctions corresponding to eigenvalues satisfying the hypothesis of Lemma 2.1. is concentrated in an $O(\varepsilon)$

neighborhood around γ. Choosing δ sufficiently small, we may assume without loss of generality that for $x \in \overline{\Omega}_\delta = \{\, x : d(x,\gamma) \le \delta \,\}$ the Jacobian J satisfies

$$0 < \delta_* < J = 1 + \kappa r \le \delta^* ,$$

with $\delta^* = 1 + \delta \max\{\, |\kappa(s)| : 0 \le s \le \ell \,\}$. Identifying the functions $\psi(x)$ and $\psi(r(x), s(x))$ and with the use of (2.5) and (2.6), we can write (2.10) in the coordinates (r,s) as

$$\mathcal{L}^{\varepsilon}_{r,s}\psi := -\frac{\varepsilon^2}{1+\kappa r}((1+\kappa r)\psi_r)_r - \frac{\varepsilon^2}{1+\kappa r}\left(\frac{\psi_s}{1+\kappa r}\right)_s + f'(\overline{u}_\varepsilon)\psi = \mu\psi , \tag{2.13}$$

where $\psi = \psi(r,s)$, $-\delta \le r \le \delta$, $0 \le s \le \ell$, is an eigenfunction of (2.10), and $\overline{u}_\varepsilon$ is given by (2.7). Observe that for $-2\delta \le r \le 2\delta$ we have

$$\psi(r,0) = \psi(r,\ell), \qquad \psi_r(r,0) = \psi_r(r,\ell), \qquad \psi_s(r,0) = \psi_s(r,\ell), \tag{2.14}$$

with the same conditions holding for the second derivatives as well. We will refer to the above as the periodicity conditions.

In terms of the **stretched variable** $\eta = r/\varepsilon$, and with

$$\tilde{\psi}(\eta,s) = \sqrt{\varepsilon}\psi(\varepsilon\eta,s), \qquad -\frac{\delta}{\varepsilon} \le \eta \le \frac{\delta}{\varepsilon}, \quad 0 \le s \le \ell, \tag{2.15}$$

we can rewrite (2.13) as

$$-\frac{1}{1+\kappa\varepsilon\eta}\left((1+\kappa\varepsilon\eta)\tilde{\psi}_\eta\right)_\eta - \frac{\varepsilon^2}{1+\kappa\varepsilon\eta}\left(\frac{\tilde{\psi}_s}{1+\kappa\varepsilon\eta}\right)_s + f'(U_\varepsilon)\tilde{\psi} = \mu\tilde{\psi} , \tag{2.16}$$

where U_ε is the stretched profile

$$U_\varepsilon(r,s) = U(\eta) + \varepsilon\kappa(s)V(\eta) + \varepsilon^2 W(\eta,s) . \tag{2.17}$$

Let $\mathcal{L}^{\varepsilon}_{\eta,s}$ be the operator defined in the left hand side of (2.16). Defining

$$\mathcal{L}^{\varepsilon}_{\eta} := -\frac{1}{1+\kappa\varepsilon\eta}\frac{\partial}{\partial\eta}\left((1+\kappa\varepsilon\eta)\frac{\partial}{\partial\eta}\right) + f'(U_\varepsilon), \tag{2.18}$$

$$\mathcal{L}^{\varepsilon}_{s} := -\frac{1}{1+\kappa\varepsilon\eta}\frac{\partial}{\partial s}\left(\frac{1}{1+\kappa\varepsilon\eta}\frac{\partial}{\partial s}\right) , \tag{2.19}$$

we can write by (2.16)

$$\mathcal{L}^{\varepsilon}_{\eta,s} = \mathcal{L}^{\varepsilon}_{\eta} + \varepsilon^2 \mathcal{L}^{\varepsilon}_{s} \,. \tag{2.20}$$

Lemma 2.2 (The Decoupling of Small Eigenfunctions). *Let $\{\tilde{\psi}_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ be a family of stretched eigenfunctions of (2.16) corresponding to $\{\mu_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$, with $\mu_\varepsilon = O\left(\varepsilon^2\right)$, for $0<\varepsilon<\varepsilon_0$, and normalized by requiring*

$$\|\tilde{\psi}_\varepsilon\|^2_{L^2(\Omega_{\delta/\varepsilon})} = \int_0^\ell \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \tilde{\psi}^2_\varepsilon(\eta,s)(1+\kappa\varepsilon\eta)\,d\eta\,ds = 1\,.$$

Then $\tilde{\psi}_\varepsilon(\eta,s)$ separates in the limit $\varepsilon\to 0$, and along a subsequence of ε's

$$\int_0^\ell \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \left(\tilde{\psi}_\varepsilon(\eta,s) - \dot{U}(\eta)\Theta(s)\right)^2 d\eta\,ds \to 0\,,$$

where U is the "heteroclinic", and Θ is an ℓ-periodic function of s.

For the proof we refer to [A–F2].

Lemma 2.3 (Approximate eigenvalues/eigenfunctions). *Let $\mathcal{A}$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and let I be a compact interval in $\mathbb{R}$. Let $\{\psi_1,\ldots,\psi_n\}$ be linearly independent normalized elements in the domain of $\mathcal{A}$ and let $\mu_1,\ldots,\mu_n$ be in I. Assume that the following conditions hold true:*

(1) $\mathcal{A}\psi_i = \mu_i\psi_i + r_i\,, \qquad \|r_i\| \le \varepsilon^*\,, \qquad i=1,\ldots,n\,,$

(2) there is a number $a>0$ such that I is a–isolated in the spectrum of $\mathcal{A}$ in the sense that

$$\left(\sigma(\mathcal{A}) \setminus I\right) \cap \left(I + (-a,a)\right) = \varnothing\,.$$

If E = span $\{\psi_1,\ldots,\psi_n\}$, and F = closed subspace associated to $\sigma(\mathcal{A})\cap I$, then

$$\bar{d}(E,F) := \sup_{\substack{\phi\in E\\ \|\phi\|=1}} d(\phi,F) \le \frac{\sqrt{n}\,\varepsilon^*}{\sqrt{\lambda_*}\,a}$$

where λ_* is the smallest eigenvalue of the matrix $\big(\langle\psi_i,\psi_j\rangle\big)$, and $d(\cdot\,,\cdot)$ is the distance induced by the norm of $\mathcal{H}$.

For the proof we refer to [H–S].

Remark 2.4. The "distance" $\bar{d}$ is not symmetric, and $\bar{d}(E,F)=0$ if and only if $E\subset F$.

Remark 2.5. Let Π_F be the orthogonal projection on the eigenspace F. If $\bar{d}(E,F)<1$ then it can be shown ([H–S]) that the restriction $\Pi_F|E$ to E is injective. Therefore if $\sigma(\mathcal{A})\cap I$ is discrete of finite multiplicity and if $\bar{d}(E,F)<1$ then Lemma 2.2 implies that $\mathcal{A}$ has at least n eigenvalues in I.

A RADIAL CASE

In this chapter we consider a radial form of the eigenvalue problem (2.10) in which the domain Ω is an annulus and the interface is circular. More precisely let $\rho > 1$ be a fixed number and let $\Omega = \{ x : 0 < \rho - 1 < r = |x| < \rho + 1 \}$. Let γ be the circle of radius ρ contained in Ω and define $u_\varepsilon(r) = U\big((r-\rho)/\varepsilon\big)$ where U is the heteroclinic. We note that this u_ε is a special case of (2.7) for which $V = W = 0$. We also note that $r - \rho$ is the signed distance of x from γ. In this setting (2.10) becomes

$$(3.1)\quad \begin{cases} \mathcal{L}^\varepsilon_\rho \phi := -\dfrac{\varepsilon^2}{r}(r\phi')' + f'\left(U\left(\dfrac{r-\rho}{\varepsilon}\right)\right)\phi = \mu^{\rho,\varepsilon}\phi\,, & \rho - 1 < r < \rho + 1\,, \\ \phi'(\rho-1) = \phi'(\rho+1) = 0\,. \end{cases}$$

Similar eigenvalue problems have been studied in [A–B–F1,2], [A–S], [A–MP–P], [C–P1,2], [Fu–H], [N–F].

3.1. The Radial Operator.

To study the eigenvalue problem (3.1), it is convenient to use the stretched variable η defined by $\eta = (r-\rho)/\varepsilon$. Then for

$$\Phi(\eta) = \sqrt{\varepsilon}\,\phi(\rho + \varepsilon\eta)\,,$$

we rewrite (3.1) as

$$(3.2)\qquad \begin{cases} \mathcal{L}^\varepsilon_\rho \Phi := -\dfrac{1}{\rho+\varepsilon\eta}\left((\rho+\varepsilon\eta)\Phi'\right)' + f'(U(\eta))\Phi = \mu^{\rho,\varepsilon}\Phi\,, & \eta \in I_\varepsilon\,, \\ \Phi'(\eta) = 0\,, & \eta \in \partial I_\varepsilon\,, \end{cases}$$

where $I_\varepsilon = (-1/\varepsilon, 1/\varepsilon)$. We observe that the eigenvalues $\mu^{\rho,\varepsilon}$ are not affected by the change of variable from r to η. Moreover defining the norm $\|\,.\,\|_{\rho,\varepsilon}$ by

$$\|h\|^2_{\rho,\varepsilon} := \int_{I_\varepsilon} h^2(\eta)\,(\rho+\varepsilon\eta)\,d\eta\,,$$

we see from

$$\int_{\rho-1}^{\rho+1} \phi^2(r)\;rdr = \int_{I_\varepsilon} \phi^2(\rho+\varepsilon\eta)\,(\rho+\varepsilon\eta)\,\varepsilon d\eta = \int_{I_\varepsilon} \Phi^2(\eta)\,(\rho+\varepsilon\eta)\,d\eta = \|\Phi\|^2_{\rho,\varepsilon}\,,$$

that if the eigenfunctions ϕ are $L^2(rdr)$ normalized, then the stretched eigenfunctions Φ are $L^2((\rho+\varepsilon\eta)d\eta)$ normalized. Moreover, observe that if h is a radial function in Ω and we define $\tilde{h}(\eta) = \sqrt{\varepsilon}\,h(\rho+\varepsilon\eta)$, $\eta \in I_\varepsilon$, then

$$\|h\|_{L^2(\Omega)} = \left(\int_0^{2\pi}\int_{\rho-1}^{\rho+1} (h^2(r)\,r\,dr\,d\theta\right)^{1/2} = \sqrt{2\pi}\|\tilde{h}\|_{\rho,\varepsilon}\,.$$

Thus the normalization $\|\tilde{h}\|_{\rho,\varepsilon} = 1$ corresponds to $\|h\|_{L^2(\Omega)} = \sqrt{2\pi}$, where $\tilde{h}$ and h are related as above. For $\eta \in I_\varepsilon$, we have $\rho - 1 < \rho + \varepsilon\eta < \rho + 1$; therefore

$$(\rho-1)\int_{I_\varepsilon} \Phi^2(\eta)\,d\eta \le \|\Phi\|^2_{\rho,\varepsilon} \le (\rho+1)\int_{I_\varepsilon} \Phi^2(\eta)\,d\eta\,.$$

Hence the norm $\|\,.\,\|_{\rho,\varepsilon}$ is equivalent to the usual L^2 norm. Denoting by

$$\|h\|^2 := \int_{-\infty}^{\infty} h^2(\eta)\,d\eta$$

the latter, we observe that if $h \in L^2(\mathbb{R})$, then

$$\lim_{\varepsilon\to 0} \|h\|_{\rho,\varepsilon} = \sqrt{\rho}\,\|h\|\,.$$

Remark 3.1. As $\varepsilon \to 0$, $I_\varepsilon \to \mathbb{R}$ and $f'(U(\pm 1/\varepsilon)) \to f'(\pm 1)$ exponentially as follows from (2.2). Thus it is natural to associate with (3.2) the following singular Sturm–Liouville problem in $L^2(\mathbb{R})$

$$-y'' + f'(U)y = \mu^* y\,, \quad y = y(\eta)\,, \quad \eta \in \mathbb{R}\,. \tag{3.3}$$

We define

$$(3.4) \qquad \mathcal{L}^* := -\frac{d^2}{d\eta^2} + f'(U), \qquad D(\mathcal{L}^*) = H^2(\mathbb{R}).$$

For the operator $\mathcal{L}^*$ we have the following

► (i) Since $f'(U)$ is bounded below, $\mathcal{L}^*$ is of limit–point type at $\pm\infty$ (see [Co–L]), and so real eigenvalues of $\mathcal{L}^*$ (if any) are simple. Moreover its eigenfunctions are characterized by the number of their zeros [W].

► (ii) From (2.1) after differentiation, we obtain

$$(3.5) \qquad -U''' + f'(U)U' = 0.$$

Thus U' is a solution of (3.3) corresponding to $\mu^* = 0$. Using the estimates (2.2) we see that U' is in $L^2(\mathbb{R})$. Therefore U' is an eigenfunction of $\mathcal{L}^*$. Moreover since it is positive (as the derivative of an increasing function), we finally conclude by (i) that U' is the principal eigenfunction of $\mathcal{L}^*$.

► (iii) The hypotheses on f imply that $f'(U) \to f'(\pm 1) > 0$, as $\eta \to \pm\infty$ and so if $f^* = \min\{ f'(\pm 1) \}$, the eigenvalues of $\mathcal{L}^*$ are bounded below f^* while the essential spectrum is $[f^*, \infty)$.

Now we deal with the spectral properties of $\mathcal{L}^\varepsilon_\rho$. We start with a Lemma which leads to obtaining pointwise estimates of the type (2.12) for the principal eigenfunction of $\mathcal{L}^\varepsilon_\rho$. Similar estimates have been obtained in [A–S].

Lemma 3.2. *Let $w > 0$ be a solution of the problem*

$$(3.6) \qquad \begin{cases} -\dfrac{1}{\rho+\varepsilon\eta}((\rho+\varepsilon\eta)w')' + g(\eta)\, w = 0, & -1/\varepsilon < \eta < 1/\varepsilon \\ w'(-1/\varepsilon) = w'(1/\varepsilon) = 0, & \end{cases}$$

where g is a bounded, continuous function satisfying

$$(3.7) \qquad g(\eta) > c^2 > 0, \quad \eta \in [-1/\varepsilon, -a) \cup (a, 1/\varepsilon]$$

for some positive constants c and a. Then the following estimates hold:

$$w(\eta) \le Ae^{-c|\eta|}, \qquad \eta \in [-1/\varepsilon, -a] \cup [a, 1/\varepsilon], \tag{3.8}$$

$$|w'(\eta)| \le CAe^{-c|\eta|}, \qquad \eta \in [-1/\varepsilon, -a] \cup [a, 1/\varepsilon], \tag{3.9}$$

$$|w''(\eta)| \le \bar{C}Ae^{-c|\eta|}, \qquad \eta \in [-1/\varepsilon, -a] \cup [a, 1/\varepsilon], \tag{3.10}$$

where $A = 2\max\{w(\pm a)\}e^{ca}$ *and* C, $\bar{C}$ *are constants independent of* ε. *Moreover the maximum of* w *is attained on* $[-a, a]$.

Proof. Let $\eta \ge a$. Then the equation in (3.6), and (3.7) yield

$$((\rho + \varepsilon\eta)w')' = (\rho + \varepsilon\eta)g(\eta)w > (\rho + \varepsilon\eta)c^2 w > 0\,.$$

Integrating over $[\eta, 1/\varepsilon]$ and using the boundary condition at $1/\varepsilon$ we have

$$-(\rho + \varepsilon\eta)w'(\eta) \ge 0\,.$$

Thus w is decreasing on $[a, 1/\varepsilon]$. Utilizing this information in the equation (3.6), we get

$$\begin{aligned} 0 &= -w'' - \frac{\varepsilon}{\rho + \varepsilon\eta}w' + g(\eta)w \\ &\ge -w'' + c^2 w\,. \end{aligned}$$

Now if $\bar{w}$ is the solution of the problem

$$\begin{cases} -\bar{w}'' + c^2\bar{w} = 0\,, & a < \eta < 1/\varepsilon \\ \bar{w}(a) = \bar{w}(a) = \alpha\,, & \bar{w}'(1/\varepsilon) = 0\,, \end{cases}$$

then by a known comparison principle (see for example [P–W]), we conclude that $w(\eta) \le \bar{w}(\eta)$, where

$$\bar{w}(\eta) = \alpha\frac{\cosh c(\eta - 1/\varepsilon)}{\cosh c(a - 1/\varepsilon)}\,.$$

Therefore

$$w(\eta) \leq 2\alpha e^{c(a-\eta)}\,, \qquad a < \eta < 1/\varepsilon\,. \tag{3.11}$$

For $\eta \in (-1/\varepsilon, 1/\varepsilon)$, define $v(\eta) = w(-\eta)$. Then v satisfies

$$\begin{cases} -v'' + \dfrac{\varepsilon}{\rho - \varepsilon\eta}\, v' + g(-\eta)\, v = 0\,, & -1/\varepsilon < \eta < 1/\varepsilon \\ v'\,(-1/\varepsilon) = v'\,(1/\varepsilon) = 0\,. \end{cases}$$

Now if $\eta \geq a,\ v$ is increasing, and since $0 < \rho - 1 < \rho - \varepsilon\eta$, we obtain

$$-v'' + c^2 v \leq 0\,.$$

Thus

$$v(\eta) \leq 2\hat{\alpha} e^{c(a-\eta)}\,, \qquad a < \eta < 1/\varepsilon\,,$$

where $\hat{\alpha} = w(-a)$. But then from the definition of v, we conclude

$$w(\eta) \leq 2\hat{\alpha} e^{c(a+\eta)}\,, \qquad -1/\varepsilon < \eta < -a\,, \tag{3.12}$$

and so the first assertion is proved.

To prove (3.9) we start from the equation

$$((\rho + \varepsilon\eta)w')' = (\rho + \varepsilon\eta)g(\eta)w\,. \tag{3.13}$$

If M is a bound for g, then from (3.13) and (3.11) we get

$$((\rho + \varepsilon\eta)w')' \leq AM(\rho + 1)e^{-c\eta}\,, \qquad \eta \geq a\,.$$

Integrating over $[\,\eta, 1/\varepsilon\,]$ and using the boundary condition at $1/\varepsilon$, we obtain

$$-(\rho + \varepsilon\eta)w'(\eta) \leq AM(\rho + 1)c^{-1}[\,e^{-c\eta} - e^{-c/\varepsilon}\,]\,.$$

Then

$$(3.14) \qquad 0 \le -w'(\eta) \le \frac{AM(\rho+1)}{c(\rho-1)}\,[\,e^{-c\eta} - e^{-c/\varepsilon}\,], \qquad \eta \ge a\,.$$

Repeating the same argument we get

$$(3.15) \qquad 0 \le w'(\eta) \le \frac{AM(\rho+1)}{c(\rho-1)}\,[\,e^{c\eta} - e^{-c/\varepsilon}\,], \qquad \eta \le -a\,.$$

Now (3.9) follows from (3.14) and (3.15).

Utilizing (3.8) and (3.9) in the equation (3.6), we obtain

$$|w''(\eta)| \le MAe^{-c\eta} + \frac{\varepsilon}{\rho-1}\,CAe^{-c\eta}\,, \quad \eta \notin [-a, a\,]$$

from where (3.10) follows.

Finally evaluating w at the (interior by (3.8)) point where it attains its maximum, we obtain

$$0 \le -w'' = -g(\eta)\,w\,,$$

which in light of (3.7) implies that this maximum is attained on $[-a, a\,]$. □

Remark. The existence of a positive solution of (3.6) imposes a structure hypothesis on g. More precisely, a necessary condition for existence of positive solutions is that g changes sign on $[\,-1/\varepsilon, 1/\varepsilon\,]$. This follows from the proof of Lemma 3.2, because if g is positive, say, then w is increasing and decreasing at the same time, and thus it is constant. But the only constant solution of (3.6) is the zero solution.

Proposition 3.3. *Let $\mu_1^{\rho,\varepsilon} < \mu_2^{\rho,\varepsilon} < \cdots \to \infty$ be the spectrum of $\mathcal{L}_\rho^\varepsilon$ with homogeneous Neumann boundary conditions. Then the following hold:*

(1) $\mu_1^{\rho,\varepsilon} < 0$, *for all ε small.*

(2) $\mu_1^{\rho,\varepsilon} \to 0$, *as $\varepsilon \to 0$.*

(3) *There exist positive constants* ε_0, *and* C_0 *independent of* ε *such that* $\mu_2^{\rho,\varepsilon} > C_0$ *for all* $0 < \varepsilon < \varepsilon_0$.

Proof. (1) The principal eigenvalue of (3.1) is given by

$$\begin{aligned}
\mu_1^{\rho,\varepsilon} &= \inf_{\substack{h\in W^{1,2}(\Omega)\\ h=h(|x|)}} \frac{\displaystyle\int_\Omega \left[|\nabla h|^2 + f'(U)h^2\right] dx}{\|h\|^2_{L^2(\Omega)}} \\
&= \inf_{\substack{h\in W^{1,2}(\Omega)\\ h=h(r)}} \frac{\displaystyle\int_0^{2\pi}\int_{\rho-1}^{\rho+1} \left[(h_r)^2 + f'(U)h^2\right] r\,dr\,d\theta}{\displaystyle\int_0^{2\pi}\int_{\rho-1}^{\rho+1} h^2\, r\,dr\,d\theta} \\
&= \inf_{\tilde h\in W^{1,2}(I_\varepsilon)} \frac{2\pi\displaystyle\int_{I_\varepsilon} \left[(\tilde h')^2 + f'(U)\tilde h^2\right](\rho+\varepsilon\eta)\,d\eta}{2\pi\displaystyle\int_{I_\varepsilon} \tilde h^2\,(\rho+\varepsilon\eta)\,d\eta},
\end{aligned}$$

where $\tilde h(\eta) = \sqrt{\varepsilon}\, h(\rho+\varepsilon\eta)$. Thus we have

$$\mu_1^{\rho,\varepsilon} = \inf_{\substack{\Phi\in W^{1,2}\\ \|\Phi\|_{\rho,\varepsilon}=1}} \int_{I_\varepsilon} \left[\Phi'^2 + f'(U)\Phi^2\right](\rho+\varepsilon\eta)\,d\eta\,,$$

where $W^{1,2} = W^{1,2}(I_\varepsilon)$. We note that the same chracterization holds for any other eigenvalue. For $\Phi = a_\varepsilon U'$, with $a_\varepsilon = \|U'\|_{\rho,\varepsilon}$, we get

$$\mu_1^{\rho,\varepsilon} \le a_\varepsilon(\rho+1)\int_{I_\varepsilon} \left[U''^2 + f'(U)U'^2\right](\rho+\varepsilon\eta)\,d\eta\,.$$

Utilizing the facts that $f'(U)$ is positive outside a compact interval independent of ε and that U' is the principal eigenfunction of $\mathcal{L}^*$, we finally obtain

$$\mu_1^{\rho,\varepsilon} < a_\varepsilon(\rho+1)\int_{-\infty}^{\infty} \left[U''^2 + f'(U)U'^2\right](\rho+\varepsilon\eta)\,d\eta = 0\,.$$

This completes the proof of (1).

(2) Let Φ_1 be the principal normalized ($\|\Phi_1\|_{\rho,\varepsilon} = 1$) eigenfunction of $\mathcal{L}_\rho^\varepsilon$ that we can take to be positive. Then

$$-\frac{1}{\rho+\varepsilon\eta}((\rho+\varepsilon\eta)\Phi_1')' + f'(U)\Phi_1 = \mu_1^{\rho,\varepsilon}\Phi_1 , \qquad \eta \in I_\varepsilon .$$

Multiplying by U' and integrating over I_ε, we get

$$-\int_{I_\varepsilon} ((\rho+\varepsilon\eta)\Phi_1')' \, U' \, d\eta + \int_{I_\varepsilon} f'(U)\Phi_1 U' \, (\rho+\varepsilon\eta) \, d\eta = \mu_1^{\rho,\varepsilon} \int_{I_\varepsilon} \Phi_1 U' \, (\rho+\varepsilon\eta) \, d\eta .$$

Integrating by parts twice and using the boundary conditions that Φ_1 satisfies and (3.5), we obtain

$$(3.16) \qquad (\rho+\varepsilon\eta)\Phi_1 U''|_{\partial I_\varepsilon} - \varepsilon \int_{I_\varepsilon} \Phi_1 U'' \, d\eta = \mu_1^{\rho,\varepsilon} \int_{I_\varepsilon} \Phi_1 U' \, (\rho+\varepsilon\eta) \, d\eta .$$

We want to show that the right hand side of (3.16) tends to zero as $\varepsilon \to 0$. This will imply that $\mu_1^{\rho,\varepsilon} \to 0$, provided that its coefficient is bounded away from zero. We do this in three steps.

Step 1. H^2 estimates.

From the identity

$$\mu_1^{\rho,\varepsilon} = \int_{I_\varepsilon} \left[\Phi_1'^2 + f'(U)\Phi_1^2 \right] (\rho+\varepsilon\eta) \, d\eta ,$$

we get

$$\mu_1^{\rho,\varepsilon} \geq \int_{I_\varepsilon} f'(U)\Phi_1^2 \, (\rho+\varepsilon\eta) \, d\eta ,$$

and moreover via (1)

$$\|\Phi_1'\|_{\rho,\varepsilon}^2 = \mu_1^{\rho,\varepsilon} - \int_{I_\varepsilon} f'(U)\Phi_1^2 \, (\rho+\varepsilon\eta) \, d\eta \leq - \int_{I_\varepsilon} f'(U)\Phi_1^2 \, (\rho+\varepsilon\eta) \, d\eta .$$

Now since $|U| \leq 1$, if $m = \min\{ f'(s) : |s| \leq 1 \}$ and $M = \max\{ |f'(s)| : |s| \leq 1 \}$ from the the first inequality above, we obtain

$$(3.17) \qquad \mu_1^{\rho,\varepsilon} \geq m ,$$

while from the second

$$(3.18) \qquad \|\Phi_1'\|_{\rho,\varepsilon} \le \sqrt{M}\,.$$

Multiplying the equation

$$-\Phi_1'' - \frac{\varepsilon}{\rho+\varepsilon\eta}\Phi_1' + f'(U)\Phi_1 = \mu_1^{\rho,\varepsilon}\Phi_1$$

by Φ_1'', integrating over I_ε, and using the boundary conditions, we obtain

$$\|\Phi_1''\|_{\rho,\varepsilon}^2 = \int_{I_\varepsilon} (f'(U) - \mu_1^{\rho,\varepsilon})\Phi_1\Phi_1''\,(\rho+\varepsilon\eta)\,d\eta\,.$$

Hence finally using the Cauchy–Schwarz inequality,

$$(3.19) \qquad \|\Phi_1''\|_{\rho,\varepsilon} \le 2M\,.$$

Step 2. L^∞ estimates.

Let $[-a,a]$, $a > 1$ be a fixed interval outside of which $f'(U) > c^2$, where c is a positive constant. Then for $g = f'(U) - \mu_1^{\rho,\varepsilon}$ we have $m \le g(\eta) \le M - m \le 2M$, $\eta \in I_\varepsilon$. Moreover since $\mu_1^{\rho,\varepsilon}$ is nonpositive $g > c^2$ outside $[-a,a]$. Thus the hypotheses of Lemma 3.2 are satisfied; therefore, the maximum of Φ_1 is attained at some point in $[-a,a]$, and moreover,

$$(3.20) \qquad \Phi_1(\eta) \le 2\,e^{ca}\max\{\,\Phi_1(\pm a)\,\}\,e^{-c|\eta|}\,, \qquad \eta \notin [-a,a]\,,$$

by (3.8). We now show that we can obtain a bound in the right hand side of (3.20) that is independent of ε. Let $\eta^* \in [-a,a]$ be the point at which Φ_1 attains its maximum and let $\eta \in [-a,a]$. Then

$$\Phi_1(\eta^*) = \Phi_1(\eta) + \int_\eta^{\eta^*} \Phi_1'(s)\,ds \le \Phi_1(\eta) + \int_{-a}^{a} |\Phi_1'(s)|\,ds\,.$$

Integrating with respect to η over $[-a,a]$, we get

$$2a\Phi_1(\eta^*) \le \int_{-a}^{a} \Phi_1(s)\,ds + 2a\int_{-a}^{a} |\Phi_1'(s)|\,ds$$
$$\le \sqrt{2a}\left(\int_{-a}^{a} |\Phi_1(s)|^2\,ds\right)^{1/2} + 2a\sqrt{2a}\left(\int_{-a}^{a} |\Phi_1'(s)|^2\,ds\right)^{1/2}.$$

Now using that

$$\left(\int_{-a}^{a} |h(\eta)|^2\,d\eta\right)^{1/2} \le \frac{1}{\sqrt{\rho-1}}\left(\int_{-a}^{a} |h(\eta)|^2(\rho+\varepsilon\eta)\,d\eta\right)^{1/2} \le \frac{1}{\sqrt{\rho-1}}\|h\|_{\rho,\varepsilon},$$

from the inequality above, via (3.18), we obtain

$$\Phi_1(\eta^*) \le C(a,\rho,M), \tag{3.21}$$

where $C(a,\rho,M)$ is a constant. The same argument applied to Φ_1', via (3.19), leads to

$$\max_{-a\le\eta\le a} |\Phi_1'(\eta)| \le C(a,\rho,M). \tag{3.22}$$

Using the equation that Φ_1 satisfies, (3.21), and (3.22), we have

$$\max_{-a\le\eta\le a} |\Phi_1''(\eta)| \le C(a,\rho,M). \tag{3.23}$$

Now via (3.21), (3.20) becomes

$$\Phi_1(\eta) \le Ce^{-c|\eta|}, \qquad \eta \notin [-a,a]. \tag{3.24}$$

Utilizing (3.24) in (3.9) and (3.10), we obtain

$$|\Phi_1'(\eta)| \le Ce^{-c|\eta|}, \qquad \eta \notin [-a,a], \tag{3.25}$$

$$|\Phi_1''(\eta)| \le Ce^{-c|\eta|}, \qquad \eta \notin [-a,a], \tag{3.26}$$

where C is a generic constant independent of ε. By (3.24) choosing b sufficiently large, we get

$$\int_{b\leq|\eta|\leq 1/\varepsilon} \Phi_1^2\,(\rho+\varepsilon\eta)\,d\eta < \frac{1}{2}\,.$$

Hence

$$\int_{-b}^{b} \Phi_1^2\,(\rho+\varepsilon\eta)\,d\eta \geq \frac{1}{2}\,. \tag{3.27}$$

This via

$$\Phi_1^2(\eta^*)\int_{-b}^{b}(\rho+\varepsilon\eta)\,d\eta \geq \int_{-b}^{b}\Phi_1^2\,(\rho+\varepsilon\eta)\,d\eta$$

yields

$$\Phi_1(\eta^*) \geq \frac{1}{2\sqrt{b\rho}}\,. \tag{3.28}$$

Step 3. Estimates of the terms in (3.16).

• $\int_{I_\varepsilon} \Phi_1 U'(\rho+\varepsilon\eta)\,d\eta$ is bounded away from zero. We argue by contradiction. Assume that there is a sequence of ε's along which the integral above converges to zero. Since the integrand is positive and U' is independent of ε this would imply that Φ_1 converges to zero a.e. along a subsequence of ε's. But this contradicts (3.21) and (3.27).

• The first term in (3.16) is exponentially small as follows by (2.2) and (3.21), while the integral in the second is bounded, because Φ_1 and U'' are in L^2. Thus the left hand side of (3.16) is $O(\varepsilon)$. This combined with the fact that the coefficient of $\mu_1^{\rho,\varepsilon}$ is bounded away from zero imply that $\mu_1^{\rho,\varepsilon}\to 0$ as $\varepsilon\to 0$. The proof of (2) is complete.

(3) If $\underline{\lim}\,\mu_2^{\rho,\varepsilon} \geq \min\{\, f'(\pm 1)\,\}$ as $\varepsilon \to 0$, we have nothing to prove because $\min\{\, f'(\pm1)\,\} > 0$ by hypothesis (b) on f. So without loss of generality, we assume

that $\mu_2^{\rho,\varepsilon}$ remains bounded, below $\min\{ f'(\pm 1) \}$, for $0 < \varepsilon < \varepsilon_0$. In this case, there is a fixed interval $[-a, a]$, and a positive constant c, such that

$$\mu_2^{\rho,\varepsilon} < f'(U) - c^2 \qquad \eta \notin [-a, a], \tag{3.29}$$

uniformly in ε for $0 < \varepsilon < \varepsilon_0$ (see Fig. 3). Because of the nodal characterization of the eigenfunctions, we know that Φ_2 has one zero $\eta_0 = \eta_0^\varepsilon$. Thus we may assume without loss of generality that $\Phi_2(-1/\varepsilon) < 0$ and $\Phi_2(1/\varepsilon) > 0$ (see Fig. 4). We claim that $\eta_0 \in [-a, a]$. Evaluating Φ_2 at the points where it attains its extrema, we have

$$-\Phi_2'' = \big(\mu_2^{\rho,\varepsilon} - f'(U)\big)\Phi_2 \,.$$

It then follows that the negative minimum and the positive maximum are attained in th place where $\mu_2^{\rho,\varepsilon} - f'(U) \geq 0$, namely in $[-a, a]$. Therefore $\eta_0 \in [-a, a]$. We note that Φ_2 behaves like Φ_1, for $\eta \geq a$ (by (3.29)). Moreover $\Phi_2 < 0$, for $\eta \leq -a$ therefore Φ_2 is decreasing. Hence applying the Comparison Principle to $-\Phi_2$, we obtain $0 \leq -\Phi_2(\eta) \leq -2\Phi_2(-a)e^{c(a+\eta)}$, $-1/\varepsilon \leq \eta \leq -a$ as in (3.12). Thus finally $2\Phi_2(-a)e^{c(a+\eta)} \leq \Phi_2(\eta) \leq 0$, $-1/\varepsilon \leq \eta \leq -a$. Therefore pointwise estimates like (3.21), (3.22), (3.23), (3.24), (3.27), (3.28) hold for Φ_2 as well. To show that $\mu_2^{\rho,\varepsilon}$ is bounded away from zero, we start with

$$\begin{aligned}
-\frac{1}{\rho+\varepsilon\eta}\left((\rho+\varepsilon\eta)\Phi_1'\right)' + f'(U)\Phi_1 &= \mu_1^{\rho,\varepsilon}\Phi_1\,, \qquad \eta \in I_\varepsilon\,,\\
-\frac{1}{\rho+\varepsilon\eta}\left((\rho+\varepsilon\eta)\Phi_2'\right)' + f'(U)\Phi_2 &= \mu_2^{\rho,\varepsilon}\Phi_2\,, \qquad \eta \in I_\varepsilon\,.
\end{aligned}$$

Multiplying the first equation by Φ_2, and the second by Φ_1, subtracting, and integrating the resulting equation over $[\eta_0, 1/\varepsilon]$, we obtain

$$\int_{\eta_0}^{1/\varepsilon} \left[-\left((\rho+\varepsilon\eta)\Phi_1'\right)'\Phi_2 + \left((\rho+\varepsilon\eta)\Phi_2'\right)'\Phi_1\right] d\eta$$
$$= \left(\mu_1^{\rho,\varepsilon} - \mu_2^{\rho,\varepsilon}\right)\int_{\eta_0}^{1/\varepsilon} \Phi_1\Phi_2\,(\rho+\varepsilon\eta)\,d\eta\,.$$

Integrating by parts and using the boundary conditions ($\Phi_2(\eta_0) = 0, \Phi_1'(1/\varepsilon) = \Phi_2'(1/\varepsilon) = 0$), we have

$$-\Phi_1(\eta_0)\Phi_2'(\eta_0)(\rho+\varepsilon\eta_0) = (\,\mu_1^{\rho,\varepsilon} - \mu_2^{\rho,\varepsilon}\,) \int_{\eta_0}^{1/\varepsilon} \Phi_1\Phi_2\,(\rho+\varepsilon\eta)\,d\eta\,. \tag{3.30}$$

From our assumptions on Φ_2, it follows that $-\Phi_2'(\eta_0) < 0$ so the left hand side of (3.30) is negative, while the integral term in the same equation is positive. To obtain the lower bound of $\mu_2^{\rho,\varepsilon}$ we solve (3.30).

Claim 1. There is a positive constant C_1 idependent of ε such that

$$C_1 \le \int_{\eta_0}^{1/\varepsilon} \Phi_1\Phi_2(\rho+\varepsilon\eta)\,d\eta \le 1\,, \tag{3.31}$$

for all ε sufficiently small.

Proof of the Claim. We start with the left inequality. Via orthogonality of the eigenfunctions we have

$$\int_{\eta_0}^{1/\varepsilon} \Phi_1\Phi_2\,(\rho+\varepsilon\eta)\,d\eta = -\int_{-1/\varepsilon}^{\eta_0} \Phi_1\Phi_2\,(\rho+\varepsilon\eta)\,d\eta = \int_{-1/\varepsilon}^{\eta_0} \Phi_1|\Phi_2|\,(\rho+\varepsilon\eta)\,d\eta\,,$$

by the sign assumption on Φ_2, thus

$$\begin{aligned}\int_{\eta_0}^{1/\varepsilon} \Phi_1\Phi_2\,(\rho+\varepsilon\eta)\,d\eta &= \frac{1}{2}\int_{-1/\varepsilon}^{1/\varepsilon} \Phi_1|\Phi_2|\,(\rho+\varepsilon\eta)\,d\eta \\ &\ge \frac{1}{2}\int_{-a}^{a} \Phi_1|\Phi_2|\,(\rho+\varepsilon\eta)\,d\eta\,.\end{aligned}$$

In light of Harnack's inequality, applied to the equation $(\mathcal{L}_\rho^\varepsilon - \mu_2^{\rho,\varepsilon})\Phi_1 = 0$,

$$\min_{|\eta|\le a} \Phi_1 \ge C' \max_{|\eta|\le a} \Phi_1\,, \tag{3.32}$$

where C' is a positive constant independent of ε, and (3.28) the inequality above becomes

$$\int_{\eta_0}^{1/\varepsilon} \Phi_1\Phi_2\,(\rho+\varepsilon\eta)\,d\eta \ge C'' \int_{-a}^{a} |\Phi_2|\,(\rho+\varepsilon\eta)\,d\eta\,.$$

Supposing that along a sequence of ε's $\int_{\eta_0}^{1/\varepsilon} \Phi_1\Phi_2\,(\rho+\varepsilon\eta)\,d\eta \to 0$, we are led to the conclusion, via the inequality above, that $|\Phi_2| \to 0$ a.e. in $[-a,a]$. This, however, contradicts the fact that Φ_2 is a normalized function attaining its maximum on $[-a,a]$. The right inequality follows from Cauchy–Schwarz because Φ_1 and Φ_2 are normalized. The proof of the Claim is complete.

Claim 2. There is a positive constant C_2 idependent of ε such that

$$\Phi_1(\eta_0)\Phi_2'(\eta_0)(\rho+\varepsilon\eta_0) \geq C_2\,, \tag{3.33}$$

for all sufficiently small ε.

Proof of the Claim. We argue by contradiction and assume that there is a sequence $\varepsilon_n \to 0$ along which $\Phi_1(\eta_0)\Phi_2'(\eta_0)(\rho+\varepsilon\eta_0) \to 0$. This via (3.32) would imply that ${\Phi_2^{\varepsilon_n}}'(\eta_0^{\varepsilon_n}) \to 0$. Because of (3.21), (3.22), (3.23) the sequence $\{\Phi_2^{\varepsilon_n}\}$ is bounded in the $C^2[-a,a]$ topology. Recalling that $\mu_2^{\rho,\varepsilon}$ and η_0 are bounded, and using the Arzela–Ascoli theorem and a diagonal argument we can choose a subsequence of $\{\varepsilon_n\}$, that we denote again by $\{\varepsilon_n\}$, along which the following hold

$$\Phi_2^{\varepsilon_n} \xrightarrow{C^2} \Phi_2^*\,, \qquad \mu_2^{\rho,\varepsilon_n} \to \mu_2^{\rho,*}\,, \qquad \eta_0^{\varepsilon_n} \to \eta_0^*\,.$$

We then have that $\Phi_2^{\varepsilon_n}$ converges uniformly on $[-a,a]$ to the solution of the initial value problem[2]

$$\begin{cases} -\Phi_2^{*\prime\prime} + f'(U)\Phi_2^* = \mu_2^\rho \Phi_2^*\,, & -a<\eta<a\,, \\ \Phi_2^*(\eta_0^*) = \Phi_2^{*\prime}(\eta_0^*) = 0\,. \end{cases}$$

By uniqueness Φ_2^* is identically zero. But this again contradicts the fact that Φ_2 is positive in a set of positive measure, uniformly for all small ε. This completes

[2] It turns out that the hypothesis that $\mu_2^{\rho,\varepsilon} < \min f'(\pm 1)$ is equivalent to the fact that the limiting operator $\mathcal{L}^*$ (see (Remark 3.1)) has a second eigenvalue. In this case $\mu_2^{\rho,*}$ is this eigenvalue and Φ_2^* is its corresponding eigenfunction (cf. section 3.2).

the proof of the Claim.

Utilizing (3.31) and (3.33) in (3.30), we obtain

$$\begin{aligned}\mu_2^{\rho,\varepsilon} &= \mu_1^{\rho,\varepsilon} + \frac{\Phi_1(\eta_0)\Phi_2'(\eta_0)(\rho+\varepsilon\eta_0)}{\displaystyle\int_{\eta_0}^{1/\varepsilon} \Phi_1\Phi_2\,(\rho+\varepsilon\eta)\,d\eta} \\ &\geq \mu_1^{\rho,\varepsilon} + C_3\,.\end{aligned}$$

The conclusion then follows from the fact that $\mu_1^{\rho,\varepsilon} = O(\varepsilon^2)$. □

We collect the estimates obtained for Φ_1 in the proof of Proposition 3.3, in the

Lemma 3.4 *Let Φ_1 be the principal eigenfunction of $\mathcal{L}_\rho^\varepsilon$. Then there are constants a, $C = C(a,\rho,f')$, $C' = Ce^{ca}$ and ε_0 such that the following pointwise estimates hold true*

$$\begin{aligned}
&\max_{-a\leq\eta\leq a} \Phi_1 \leq C, \qquad && \Phi_1(\eta) \leq C'e^{-c|\eta|}, \quad && \eta \notin (-a,a)\,, \\
&\max_{-a\leq\eta\leq a} |\Phi_1'| \leq C, && |\Phi_1'(\eta)| \leq C'e^{-c|\eta|}, && \eta \notin (-a,a)\,, \\
&\max_{-a\leq\eta\leq a} |\Phi_1''| \leq C, && |\Phi_1''(\eta)| \leq C'e^{-c|\eta|}, && \eta \notin (-a,a)\,,
\end{aligned}$$

for all $0 < \varepsilon < \varepsilon_0$, where c is a fixed constant.

Similar results to those of Proposition 3.3 have been obtained in [A–B–F1,2], [C–P1] and [N–F]. Our method in the proof of parts (2) and (3) is that of [N–F].

3.2. Some Remarks.

► The estimate (3.24) is the analog of (2.12) in the stretched variable setting under the appropriate normalization, $\Phi = \sqrt{\varepsilon}\phi$. From the proof of Proposition 3.3 it follows that the estimates above hold true for the second as well as any other eigenfunction of $\mathcal{L}_\rho^\varepsilon$, corresponding to an eigenvalue that remains below $\min\{\, f'(\pm 1)\,\}$, for all small ε.

▶ Using the estimates of Lemma 3.4 and the Arzela–Ascoli Theorem it can be shown (as in [A–MP–P] or [N–F]) that Φ_1 converges in the C^2_{loc} sense[3] to $(\sqrt{\rho}\,\|U'\|)^{-1}U'$ (the principal eigenfunction of the $\mathcal{L}^*$). Moreover if $\mathcal{L}^*$, the limiting operator, has a second eigenvalue, then it can be shown (using for example the variational characterization of eigenvalues and an estimate like (3.24)) that $\mu_2^{\rho,\varepsilon}$ converges to that eigenvalue, and Φ_2 converges to the corresponding eigenfunction in the above mentioned sense. This is, in particular, the case in the model problem (see section 3.3), where the limiting operator $\mathcal{L}^*$ has two eigenvalues. If on the other hand $\mathcal{L}^*$ has no other eigenvalue exept 0, then $\mu_2^{\rho,\varepsilon}$ converges to a point in the essential spectrum of $\mathcal{L}^*$. Hence, in both cases, it is bounded below by a positive constant.

▶ We expect $\mu_1^{\rho,\varepsilon}$ to be of order $O(\varepsilon^2)$. To prove this it is natural to use (3.16). So dividing through by $\varepsilon^2 \int_{I_\varepsilon} \Phi_1 U'(\rho+\varepsilon\eta)\,d\eta$ and using the information we have for each term in (3.16), we obtain

$$\frac{\mu_1^{\rho,\varepsilon}}{\varepsilon^2} = -\frac{\displaystyle\int_{I_\varepsilon} \Phi_1 U''\,d\eta}{\varepsilon \displaystyle\int_{I_\varepsilon} \Phi_1 U'\,(\rho+\varepsilon\eta)\,d\eta} + O(\varepsilon)\,.$$

Assuming that Φ_1 converges in the C^2_{loc} sense to $(\sqrt{\rho}\,\|U'\|)^{-1}U'$, and because of the estimates provided by Lemma 3.5 it follows that as $\varepsilon \to 0$, the term

$$\int_{I_\varepsilon} \Phi_1 U''\,d\eta$$

---

[3] By this we mean that if K is any compact set in $\mathbb{R}$ then

$$\sup_K |\Phi_1 - aU'| + \sup_K |(\Phi_1 - aU')'| \to 0\,,$$

as $\varepsilon \to 0$, where $a = \sqrt{\rho}\,\|U'\|^{-1}$. Notice that even though Φ_1 is not defined in the whole real line, the above makes sense because $K \subset I_\varepsilon$, for all $\varepsilon < \varepsilon_0$ ($I_\varepsilon \to \mathbb{R}$, as $\varepsilon \to 0$).

tends to zero. Hence the first term in the right hand side of the equality above is of the form $0/0$, as $\varepsilon \to 0$, and thus to be able to evaluate the limit we need a "L'Hôpital" type argument.Information about the "derivative" of Φ_1 with respect to ε is provided by the asymptotic expansion in ε of Φ_1. In the next chapter we obtain such an asymptotic expansion, and using it we prove that

$$\lim_{\varepsilon \to 0} \frac{\mu_1^{\rho,\varepsilon}}{\varepsilon^2} = -\frac{1}{4\rho^2}.$$

3.3. Appendix: The Model Problem.

The objective of this section is to compute the spectrum and the eigenfunctions of the model bistable operator defined by

$$\text{(A.1)} \qquad \mathcal{A}h := -h'' + f'(\phi)h\,, \qquad D(\mathcal{A}) = H^2(\mathbb{R})\,,$$

where

$$f(\phi) = \frac{1}{2}(\phi^3 - \phi)\,, \quad \text{and} \quad \phi(x) = \tanh\frac{x}{2}\,, \quad x \in \mathbb{R}\,.$$

We note that ϕ is the unique (heteroclinic) solution of the problem

$$\text{(A.2)} \qquad \begin{cases} y'' - f(y) = 0 \\ y(0) = 0\,, \qquad y(\pm\infty) = \pm 1\,. \end{cases}$$

Since the potential

$$f'(\phi) = \frac{3}{2}\tanh^2\frac{x}{2} - \frac{1}{2}$$

is bounded, the operator $\mathcal{A}$ is self–adjoint in L^2, and moreover of limit–point type at $\pm\infty$. Therefore for λ real, the equation

$$\text{(A.3)} \qquad \mathcal{A}h = \lambda h$$

has at most one L^2 solution (see [Co–L]). In particular the eigenvalues of $\mathcal{A}$ are simple.

Since $f'(\phi) \to 1$ as $x \to \pm\infty$, it follows that if $\sigma_e(\mathcal{A})$ is the essential spectrum of $\mathcal{A}$, then

$$\sigma_e(\mathcal{A}) = [1, \infty).$$

Now if $\sigma_d(\mathcal{A})$ is the set of the eigenvalues (discrete spectrum), it follows

$$\sigma_d(\mathcal{A}) \subset [-1/2, 1).$$

To compute the eigenvalues of $\mathcal{A}$, we follow a standard method that can be found for example in [L–L], where similar eigenvalue problems for the Schrödinger operator are studied.

Claim 1. $\sigma_d(\mathcal{A}) = \{0, 3/4\}$

Proof of the Claim. We write (A.3) as

$$h'' + \left(\lambda - \frac{3}{2}\tanh^2\frac{x}{2} + \frac{1}{2}\right) h = 0, \qquad x \in \mathbb{R}. \tag{A.4}$$

Using the change of variable $\xi = \tanh \frac{x}{2}$, we obtain

$$\frac{dh}{dx} = \frac{1-\xi^2}{2}\frac{dh}{d\xi}, \qquad \frac{d^2h}{dx^2} = \frac{1-\xi^2}{2}\frac{d}{d\xi}\left(\frac{1-\xi^2}{2}\frac{dh}{d\xi}\right),$$

hence (A.6) becomes

$$\frac{1-\xi^2}{2}\frac{d}{d\xi}\left(\frac{1-\xi^2}{2}\frac{dh}{d\xi}\right) + (\lambda - \frac{3}{2}\xi^2 + \frac{1}{2})h = 0, \quad -1 < \xi < 1$$

or equivalently

$$\frac{d}{d\xi}\left((1-\xi^2)\frac{dh}{d\xi}\right) + \left(6 - \frac{4(1-\lambda)}{1-\xi^2}\right) h = 0, \quad -1 < \xi < 1.$$

For λ an eigenvalue of $\mathcal{A}$, we have $1-\lambda>0$. Setting $a=2\sqrt{1-\lambda}$ and $s=2$ the equation above becomes

$$\frac{d}{d\xi}\left((1-\xi^2)\frac{dh}{d\xi}\right)+\left(s(s+1)-\frac{a^2}{1-\xi^2}\right)h=0$$

which is the Legendre equation. Its solution h that is finite for $\xi=1$ ($x=\infty$) is given by

$$h(\xi)=(1-\xi^2)^{a/2}\,F(a-s,\,a+s+1,\,a+1,\,(1-\xi)/2),$$

where F is the hypergeometric function . For h to be finite at $\xi=-1$ we require that $a-s=-n$ where $n=0,1,2,\ldots$; then F is a Legendre polynomial of degree n. Thus the eigenvalues are determined from the equation

$$a^2=(s-n)^2\,.$$

Recalling that $a=2\sqrt{1-\lambda}$, we obtain

$$\lambda=\lambda_n=1-\frac{(s-n)^2}{4},\qquad n=0,1,2,\ldots.$$

Moreover the requirement $a>0$ determines the number of the eigenvalues via $a=s-n$. Hence $n<s=2$, therefore

$$\begin{aligned}\lambda_0&=1-\frac{(2-0)^2}{4}=0\\ \lambda_1&=1-\frac{(2-1)^2}{4}=\frac{3}{4}.\end{aligned}$$

The proof of the claim is complete. $\square$

Next we compute the corresponding eigenfunctions.

Claim 2. The eigenfunctions of $\mathcal{A}$ are given by

$$h_0(x)=3\,\mathrm{sech}^2\frac{x}{2},\qquad h_1(x)=3\,\mathrm{sech}\frac{x}{2}\tanh\frac{x}{2}\,.$$

Proof of the Claim. We start from the fact that the general solution of the equation

$$\frac{d}{d\xi}\left((1-\xi^2)\,\frac{dh}{d\xi}\right)+\left(k(k+1)-\frac{m^2}{1-\xi^2}\right)h=0\,,\qquad -1<\xi<1$$

where $k,m=0,1,2,...$, that is bounded at $\xi=\pm 1$ is given by

$$h(\xi)=(1-\xi^2)^{m/2}\,\frac{d^m}{d\xi^m}\,P_k(\xi)\,,$$

where

$$P_k(\xi)=\frac{1}{2^k\,k!}\,\frac{d^k}{d\xi^k}\,(\xi^2-1)^k$$

is the Legendre polynomial of degree k. In the case we discuss we have

$$k=s=2\,,\qquad m^2=a^2=4\,(1-\lambda)\,,$$

and so the eigenfunctions of (A.4) are given by

$$h_n(\xi)=(1-\xi^2)^{m_n/2}\,\frac{d^{m_n}}{d\xi^{m_n}}\,P_2(\xi)\,,\qquad n=0,1$$

with

$$m_0=2\sqrt{1-\lambda_0}=2\,,\qquad m_1=2\,\sqrt{1-\lambda_1}=1\,.$$

Since

$$P_2(\xi)=\frac{1}{2}\,(3\,\xi^2-1)$$

we obtain

$$h_0(\xi)=(1-\xi)P_2''(\xi)=3\,(1-\xi^2)\,,$$
$$h_1(\xi)=(1-\xi)^{1/2}P_2'(\xi)=3\,\xi\,(1-\xi^2)^{1/2}\,.$$

Changing to the x variable, we finally have

$$h_0(x)=3\,\left(1-\tanh^2\frac{x}{2}\right)=3\,\mathrm{sech}^2\frac{x}{2}\,,$$
$$h_1(x)=3\,\tanh\frac{x}{2}\left(1-\tanh^2\frac{x}{2}\right)^{1/2}=3\,\mathrm{sech}\frac{x}{2}\,\tanh\frac{x}{2}\,.$$

The proof of the claim is complete. $\square$

ASYMPTOTIC EXPANSIONS

In chapter 3 we proved that the spectrum of $\mathcal{L}^\rho$ is decomposed to a critical part, consisting of the principal eigenvalue, and the noncritical one. We now use this decomposition to obtain an asymptotic expansion of the principal eigenfunction. We deal only with this eigenfunction, hence we omit subscripts.

4.1. The Principal Eigenfunction.

We recall that if $(\mu^{\rho,\varepsilon}, \Phi)$ is the principal eigenpair, then

$$\text{(4.1)} \qquad \begin{cases} \mathcal{L}_\rho^\varepsilon \Phi := -\Phi'' - \dfrac{\varepsilon}{\rho + \varepsilon\eta}\Phi' + f'(U)\Phi = \mu^{\rho,\varepsilon}\Phi\,, & \eta \in I_\varepsilon \\ \Phi'(\eta) = 0\,, & \eta \in \partial I_\varepsilon\,, \end{cases}$$

where $I_\varepsilon = (-1/\varepsilon, 1/\varepsilon)$. Substituting the formal expressions

$$\Phi = \bar{\Phi}_0 + \varepsilon\bar{\Phi}_1 + \varepsilon^2\bar{\Phi}_2 + \cdots\,, \qquad \mu^{\rho,\varepsilon} = \bar{\mu}_0 + \varepsilon\bar{\mu}_1 + \varepsilon^2\bar{\mu}_2 + \cdots$$

into (4.1), we obtain, after equating like powers of ε,

$$\text{(4.2)} \qquad \begin{cases} -\bar{\Phi}_0'' + f'(U)\bar{\Phi}_0 = \bar{\mu}_0\bar{\Phi}_0\,, \\ -\bar{\Phi}_k'' + f'(U)\bar{\Phi}_k = \displaystyle\sum_{i=0}^{k} \bar{\mu}_i\bar{\Phi}_{k-i} + \frac{1}{\rho}\sum_{i=0}^{k-1}\Big(-\frac{\eta}{\rho}\Big)^i \bar{\Phi}_{k-1-i}''\,, & k \geq 1 \end{cases}$$

Instead of solving (4.2) in I_ε with homogeneous Neumann conditions at ∂I_ε, we solve them in the whole real line and we seek solutions that are in L^2. This choice is suggested by the following observation. Since we already know that $\mu^{\rho,\varepsilon} \to 0$ as $\varepsilon \to 0$, we choose $\bar{\mu}_0 = 0$ in the first equation in (4.2). The corresponding $\bar{\Phi}_0$ then is U', which by (2.2) almost satisfies the boundary conditions, for ε small. We thus have what we have already noted, namely, that $\Phi \to \xi^0 U'$ in some sense, where ξ^0 is some normalization constant. At first, it seems that there is some freedom in choosing the function $\bar{\Phi}_0$ (and hence the rest of the $\bar{\Phi}_k$'s), but there is not. This follows from the fact that the left hand sides of (4.2) are of the form $\mathcal{L}^*\bar{\Phi}$, where $\mathcal{L}^*$ was defined in (3.4), and so the equation $\mathcal{L}^*\bar{\Phi}_0 = 0$ has (up to a multiplicative constant) only one $L^2(\mathbb{R})$ solution. Dealing with the rest of the $\bar{\Phi}_k$'s we need the following:

Lemma 4.1. *Let φ be a continuous function satisfying the conditions*

$$(i)\ \ |\varphi(\eta)| \le Ce^{-\alpha|\eta|}, \quad |\eta| \ge M, \qquad \text{and} \ \ (ii)\ \ \int_{-\infty}^{\infty} U'(\eta)\varphi(\eta)\, d\eta = 0,$$

where C, α, and M are positive constants. Then the problem

$$-h'' + f'(U)h = \varphi, \qquad \int_{-\infty}^{\infty} U'(\eta)h(\eta)\, d\eta = 0, \tag{4.3}$$

has a unique exponentially decaying solution h_0. Moreover h_0' decays exponentially as $|\eta| \to \infty$.

Proof. We know that U' is the unique (up to a multiplicative constant) L^2 solution of the homogeneous equation in (4.3). Seeking a second solution in the form wU', we see that w satisfies

$$w(\eta) = \int_0^{\eta} \frac{c}{U'^2(s)}\, ds + w_0\,, \tag{4.4}$$

where $c \neq 0$ is a constant. Then a particular solution h_p of the nonhomogeneous equation is given by

$$h_p(\eta) = -\frac{1}{c}\int_0^{\eta} U'(\eta)U'(s)\left[\, w(\eta) - w(s)\,\right]\varphi(s)\, ds\,.$$

Therefore, the general solution of the nonhomogeneous equation is $aU' + bwU' + h_p$, where a, b are arbitrary constants. Collecting terms, we obtain

$$h(\eta) = AU'(\eta) + U'(\eta)\left[B - \int_0^\eta U'(s)\varphi(s)\,ds\right]\int_0^\eta \frac{ds}{U'^2(s)} \\ + U'(\eta)\int_0^\eta U'(s)\varphi(s)\int_0^s \frac{dt}{U'^2(t)}\,ds\,,$$

where A, B are new constants. Defining

$$Z(\eta) = \int_{-\infty}^\eta U'(s)\varphi(s)\,ds\,,$$

and observing that $Z(\infty) = 0$, by the orthogonality condition (ii), we rewrite the representation of h as

$$h(\eta) = AU'(\eta) + \left[B + Z(0)\right]U'(\eta)\int_0^\eta \frac{ds}{U'^2(s)} \\ + U'(\eta)\left[-Z(\eta)\int_0^\eta \frac{ds}{U'^2(s)} + \int_0^\eta Z'(s)\int_0^s \frac{dt}{U'^2(t)}\,ds\right].$$

Integrating by parts in the bracket, we finally obtain

$$h(\eta) = AU'(\eta) + \left[B + Z(0)\right]U'(\eta)\int_0^\eta \frac{ds}{U'^2(s)} - U'(\eta)\int_0^\eta \frac{Z(s)}{U'^2(s)}\,ds\,. \tag{4.5}$$

We estimate each term in the representation of h separately. We start with Z. From (2.2) we have that

$$C_1 e^{-\beta\eta} \le U'(\eta) \le C_2 e^{-\beta\eta}\,, \quad \text{as } \eta \to \infty\,. \tag{4.6}$$

Then observing that

$$Z(\eta) = -\int_\eta^\infty U'(s)\varphi(s)\,ds\,,$$

and using hypothesis (i) on φ, we obtain

$$|Z(\eta)| \le C_3 e^{-(\alpha+\beta)\eta}\,, \quad \text{as } \eta \to \infty\,, \tag{4.7}$$

with a similar estimate for $\eta \to -\infty$.

Claim 1. $I(\eta) := U'(\eta) \int_0^\eta Z(s) U'^{-2}(s)\, ds$ decays exponentially as $|\eta| \to \infty$.

Proof of the Claim. We choose $M > 0$ so that for $\eta \geq M$ (4.6) and (4.7) are satisfied. Then

$$I(\eta) = U'(\eta) \left[\int_0^M \frac{Z(s)}{U'^2(s)}\, ds + \int_M^\eta \frac{Z(s)}{U'^2(s)}\, ds \right],$$

from where it follows that

$$\begin{aligned} |I(\eta)| &\leq U'(\eta) \left[C' + C'' \int_M^\eta e^{-(\alpha+\beta)s} e^{2\beta s}\, ds \right] \\ &\leq C \left(e^{-\beta\eta} + e^{-\beta\eta} \int_M^\eta e^{(\beta-\alpha)s}\, ds \right), \end{aligned}$$

where C', C'', C are constants. The inequality above implies that $|I(\eta)|$ decays exponentially. The proof for $\eta \to -\infty$ is similar.

Now the term $U'(\eta) \int_0^\eta U'^{-2}(s)\, ds$ by (4.4) is a second solution of the homogeneous equation and as such it is not in $L^2(\mathbb{R})$. Thus for h to be in L^2, we must choose $B = -Z(0)$, in (4.5). Hence we define

$$h_0(\eta) = AU'(\eta) - U'(\eta) \int_0^\eta \frac{Z(s)}{U'^2(s)}\, ds. \tag{4.8}$$

Then by Claim 1, h_0 is exponentially decaying as $|\eta| \to \infty$. The orthogonality condition in (4.3) determines A uniquely.

Claim 2. h_0' decays exponentially as $|\eta| \to \infty$.

Proof of the Claim. Differentiating (4.8), we get

$$h_0'(\eta) = AU''(\eta) - U''(\eta) \int_0^\eta \frac{Z(s)}{U'^2(s)}\, ds - \frac{Z(\eta)}{U'(\eta)}.$$

Since U'' satisfies exponential estimates of the same order as U', the second term in the expression above behaves like I in Claim 1, and so for $|\eta|$ big enough it is exponentially small. Now from (4.7) and (4.8) follows that Z/U' is exponentially decaying. Therefore h_0' decays exponentially as $|\eta| \to \infty$. This completes the proof of the Lemma. $\square$

Remark. From (4.5) we see that any bounded solution h of (4.3), with φ satisfying the hypotheses of Lemma 4.1 is given by (4.8). Hence such an h is necessarily in $L^2(\mathbb{R})$, and moreover h and h' decay exponentially as $|\eta| \to \infty$.

We now return to the formal asymptotic expansion of Φ. For solvability in (4.2), we choose the $\bar{\mu}_k$'s so that the right hand sides are orthogonal to U' (the kernel of the $\mathcal{L}^*$ operator). We write a few of the equations (4.2)

$$\begin{aligned}
-\bar{\Phi}_0'' + f'(U)\bar{\Phi}_0 &= \bar{\mu}_0\bar{\Phi}_0 \\
-\bar{\Phi}_1'' + f'(U)\bar{\Phi}_1 &= \bar{\mu}_0\bar{\Phi}_1 + \bar{\mu}_1\bar{\Phi}_0 + \frac{1}{\rho}\bar{\Phi}_0' \\
-\bar{\Phi}_2'' + f'(U)\bar{\Phi}_2 &= \bar{\mu}_0\bar{\Phi}_2 + \bar{\mu}_1\bar{\Phi}_1 + \bar{\mu}_2\bar{\Phi}_0 + \frac{1}{\rho}\left[\bar{\Phi}_1' - \frac{\eta}{\rho}\bar{\Phi}_0'\right] \\
&\vdots
\end{aligned}$$

Since $\mu^{\rho,\varepsilon} \to 0$ as $\varepsilon \to 0$, we choose $\bar{\mu}_0 = 0$ and so $\bar{\Phi}_0 = U'$. Then $\bar{\Phi}_1$ satisfies

$$-\bar{\Phi}_1'' + f'(U)\bar{\Phi}_1 = \bar{\mu}_1 U' + \frac{1}{\rho}U''.$$

For the right hand side to be orthogonal to U' we require $\bar{\mu}_1 = 0$. Then as can be easily verified, $\bar{\Phi}_1 = -(1/2\rho)U'\eta + a_1U'$, where a_1 is appropriately chosen so $\bar{\Phi}_1$ and U' are orthogonal. Utilizing this in the the equation for $\bar{\Phi}_2$, we see that

$$-\bar{\Phi}_2'' + f'(U)\bar{\Phi}_2 = \bar{\mu}_2 U' - \frac{3}{2\rho^2}U''\eta - \frac{1}{2\rho^2}U' + \frac{a_1}{\rho}U''.$$

For solvability we need $\bar{\mu}_2 = -1/(4\rho^2)$. Then $\bar{\Phi}_2 = -(3/8\rho^2)U'\eta^2 - (a_1/2\rho)U'\eta + a_2U'$, where a_2 is an appropriate constant. Continuing this way and with the aid of Lemma 4.1, we obtain up to any order, the unique $\bar{\Phi}_k$'s that are orthogonal to U', and decay exponentially. The next Proposition justifies the formal expansion.

Proposition 4.2. *Let Φ be the principal eigenfunction of (4.1) normalized by $\|\Phi\|_{\rho,\varepsilon} = 1$ and let $\bar{\Phi}_k$, $k = 0,1,2,\ldots$ be the solutions of (4.2) provided by Lemma 4.1. Then, given n*

$$\left\| \Phi - \xi_n^\varepsilon \sum_{m=0}^{k} \varepsilon^m \bar{\Phi}_m \right\|_{\rho,\varepsilon} = O(\varepsilon^{k+1}), \qquad 0 \leq k \leq n, \tag{4.9}$$

$$\left\| \left(\Phi - \xi_n^\varepsilon \sum_{m=0}^{k} \varepsilon^m \bar{\Phi}_m \right)' \right\|_{\rho,\varepsilon} = O(\varepsilon^{k+1}), \qquad 0 \leq k \leq n, \tag{4.10}$$

for all ε small, where ξ_n^ε is a function of ε satisfying

$$\lim_{\varepsilon \to 0} \xi_n^\varepsilon = \left(\sqrt{\rho}\, \|U'\|\right)^{-1} := \xi_* .$$

The proof is given in the next three Lemmas. But first we introduce some notation. Writing

$$\mathcal{L}_\rho^\varepsilon \bar{\Phi}_k = -\frac{\varepsilon}{\rho + \varepsilon\eta} \bar{\Phi}_k' + \mathcal{L}^* \bar{\Phi}_k , \qquad k = 0,1,2,\ldots ,$$

where $\mathcal{L}^* \bar{\Phi}_k$ is the left hand side of (4.2), and defining

$$S_k = \sum_{m=0}^{k} \varepsilon^m \bar{\Phi}_m , \qquad k = 0,1,2,\ldots , \tag{4.11}$$

we compute

$$\mathcal{L}_\rho^\varepsilon S_k = \sum_{m=0}^{k} \varepsilon^m \bar{\mu}_m S_{k-m} - \varepsilon^{k+1} A_k , \qquad k = 0,1,2,\ldots , \tag{4.12}$$

where

$$A_k = \frac{1}{\rho + \varepsilon\eta} \sum_{m=0}^{k} \left(-\frac{\eta}{\rho} \right)^m \bar{\Phi}_{k-m}' , \qquad k = 0,1,2,\ldots . \tag{4.13}$$

Lemma 4.3. *For each $k = 0, 1, 2, \dots$ there is a function ξ_k^ε given by*

$$\xi_k^\varepsilon = \left(\sum_{m=0}^{k} \varepsilon^m \left\langle \bar{\Phi}_m , \Phi \right\rangle_{\rho,\varepsilon} + O\big(\varepsilon^{-c/\varepsilon}\big) \right)^{-1}, \qquad k = 0, 1, 2, \dots \tag{4.14}$$

so that the following holds

$$\left\| \Phi - \xi_k^\varepsilon \sum_{m=0}^{k} \varepsilon^m \bar{\Phi}_m \right\|_{\rho,\varepsilon} = O\big(\varepsilon^{k+1}\big), \qquad k = 0, 1, 2, \dots \tag{4.15}$$

where $\langle \, , \, \rangle_{\rho,\varepsilon}$ is the inner product in $L^2\left((\rho + \varepsilon\eta) d\eta \, , \, I_\varepsilon\right)$ and c is a positive constant.

Proof. The proof is by induction.

We show that $\|\Phi - \xi_0^\varepsilon \bar{\Phi}_0\|_{\rho,\varepsilon} = O(\varepsilon)$, where $\xi_0^{\varepsilon\,-1} = \left\langle \bar{\Phi}_0 , \Phi \right\rangle_{\rho,\varepsilon} + O\big(e^{-c/\varepsilon}\big)$. Since $\bar{\Phi}_0$ does not satisfy the boundary conditions on ∂I_ε, we define $\tilde{\bar{\Phi}}_0 = \bar{\Phi}_0 + a\eta^2 + b\eta$ so that $\tilde{\bar{\Phi}}_0' = 0$ on ∂I_ε. This is accomplished if

$$a = -\frac{\varepsilon}{4} \left[U''(1/\varepsilon) - U''(-1/\varepsilon) \right] , \quad \text{and } b = -\frac{1}{2} \left[U''(1/\varepsilon) - U''(-1/\varepsilon) \right] .$$

Setting $\sigma_0 = a\eta^2 + b\eta$, and $w_0 = \mathcal{L}_\rho^\varepsilon \sigma_0$ we have

$$\mathcal{L}_\rho^\varepsilon \tilde{\bar{\Phi}}_0 = -\frac{\varepsilon U''}{\rho + \varepsilon\eta} + w_0 \, , \qquad \tilde{\bar{\Phi}}_0' = 0 \quad \text{on } \partial I_\varepsilon \, ,$$

Using as in [A–B–F1,2] the decomposition

$$\Phi - \tilde{\bar{\Phi}}_0 = c_0 \Phi + Q_0 \, , \qquad \text{with } \left\langle \Phi, Q_0 \right\rangle_{\rho,\varepsilon} = 0 \, , \tag{4.16}$$

we see that Q_0 satisfies

$$\mathcal{L}_\rho^\varepsilon Q_0 = (1 - c_0) \mu^{\rho,\varepsilon} \Phi + \frac{\varepsilon U''}{\rho + \varepsilon\eta} - w_0$$
$$Q_0'(-1/\varepsilon) = Q_0'(1/\varepsilon) = 0$$

We estimate Q_0. Taking inner products with Q_0, and using the orthogonality condition $\langle \Phi, Q_0 \rangle_{\rho,\varepsilon} = 0$, we obtain

$$(4.17) \qquad \langle \mathcal{L}^\varepsilon_\rho Q_0, Q_0 \rangle_{\rho,\varepsilon} = \int_{I_\varepsilon} U'' Q_0 \, d\eta + \int_{I_\varepsilon} w_0 Q_0 (\rho + \varepsilon\eta) \, d\eta \, .$$

On the other hand, Proposition 3.3 yields

$$\langle \mathcal{L}^\varepsilon_\rho Q_0, Q_0 \rangle_{\rho,\varepsilon} \geq C_0 \|Q_0\|^2_{\rho,\varepsilon}$$

and so from (4.17), we obtain

$$C_0 \|Q_0\|^2_{\rho,\varepsilon} \leq \varepsilon \left(\int_{I_\varepsilon} \frac{U''^2}{\rho + \varepsilon\eta} \, d\eta \right)^{1/2} \|Q_0\|_{\rho,\varepsilon} + \|w_0\|_{\rho,\varepsilon} \|Q_0\|_{\rho,\varepsilon} \, .$$

Clearly, $\|w_0\|_{\rho,\varepsilon} = O(e^{-c/\varepsilon})$, for some positive constant c, so finally

$$(4.18) \qquad \|Q_0\|_{\rho,\varepsilon} \leq \text{Const.}\, \varepsilon \, .$$

Now writing (4.16) in terms of the original function $\bar{\Phi}_0$, we have

$$(4.19) \qquad \Phi(1 - c_0) = \bar{\Phi}_0 + Q_0 + \sigma_0 \, .$$

Taking inner products with Φ we get

$$(4.20) \qquad 1 - c_0 = \langle \bar{\Phi}_0, \Phi \rangle_{\rho,\varepsilon} + O(e^{-c/\varepsilon}) \, .$$

On the other hand by taking norms in (4.19) and using that $\bar{\Phi}_0 = U'$, $1 - c_0 > 0$ (by (4.20)), and (4.18), we have

$$(4.21) \qquad 1 - c_0 = \left(\int_{I_\varepsilon} U'^2 (\rho + \varepsilon\eta) \, d\eta \right)^{1/2} + O(\varepsilon) \, ,$$

and so

$$\lim_{\varepsilon \to 0} (1 - c_0) = \sqrt{\rho} \, \|U'\| \neq 0 \, .$$

If we set $\xi_0^\varepsilon = (1-c_0)^{-1}$, from (4.19) we get

$$\|\Phi - \xi_0^\varepsilon \bar{\Phi}_0\|_{\rho,\varepsilon} = O(\varepsilon)\,.$$

From the definition of ξ_0^ε and (4.20), it then follows

$$\xi_0^{\varepsilon\,-1} = \left\langle \bar{\Phi}_0, \Phi \right\rangle_{\rho,\varepsilon} + O\left(e^{-c/\varepsilon}\right).$$

Hence the first step of the induction is complete.

We now assume that with ξ_i^ε as in (4.14) we have $\|\Phi - \xi_i^\varepsilon S_i\|_{\rho,\varepsilon} = O\left(\varepsilon^{i+1}\right)$, for every $0 \le i \le k$, and show that $\|\Phi - \xi_{k+1}^\varepsilon S_{k+1}\|_{\rho,\varepsilon} = O\left(\varepsilon^{k+2}\right)$, where ${\xi_{k+1}^\varepsilon}^{-1} = \sum_{m=0}^{k+1} \varepsilon^m \left\langle \bar{\Phi}_m, \Phi \right\rangle_{\rho,\varepsilon} + O\left(e^{-c/\varepsilon}\right)$. Defining $\tilde{S}_{k+1} = S_{k+1} + \sigma_{k+1}$, and $w_{k+1} = \mathcal{L}_\rho^\varepsilon \sigma_{k+1}$, so that

$$\mathcal{L}_\rho^\varepsilon \tilde{S}_{k+1} = \mathcal{L}_\rho^\varepsilon S_{k+1} + w_{k+1}\,, \qquad \tilde{S}'_{k+1} = 0 \quad \text{on } \partial I_\varepsilon\,,$$

we have that $\|\sigma_{k+1}\|_{\rho,\varepsilon}$ and $\|w_{k+1}\|_{\rho,\varepsilon}$ are exponentially small quantities. Using the same decomposition as before we write

$$\Phi - \tilde{S}_{k+1} = c_{k+1}\Phi + Q_{k+1}\,, \qquad \text{with } \left\langle \Phi, Q_{k+1} \right\rangle_{\rho,\varepsilon} = 0\,. \tag{4.22}$$

Hence Q_{k+1} satisfies

$$\begin{aligned} &\mathcal{L}_\rho^\varepsilon Q_{k+1} = (1-c_{k+1})\mu^{\rho,\varepsilon}\Phi - \mathcal{L}_\rho^\varepsilon S_{k+1} - w_{k+1} \\ &Q'_{k+1}(-1/\varepsilon) = Q'_{k+1}(1/\varepsilon) = 0 \end{aligned}$$

Using (4.12), we obtain

$$\mathcal{L}_\rho^\varepsilon Q_{k+1} = (1-c_{k+1})\mu^{\rho,\varepsilon}\Phi - \sum_{m=0}^{k+1} \varepsilon^m \bar{\mu}_m S_{k+1-m} + \varepsilon^{k+2} A_{k+1} - w_{k+1}\,. \tag{4.23}$$

Now

$$\sum_{m=0}^{k+1} \varepsilon^m \bar{\mu}_m S_{k+1-m} = \sum_{m=0}^{k+1} \frac{\varepsilon^m \bar{\mu}_m}{\xi_{k+1-m}^\varepsilon} \left[\xi_{k+1-m}^\varepsilon S_{k+1-m} + \Phi - \Phi\right].$$

To simplify the notation, we set

$$\frac{\bar{\mu}_m}{\xi^\varepsilon_{k+1-m}} = q_m\,, \quad \sum_{m=0}^{k+1} \frac{\varepsilon^m \bar{\mu}_m}{\xi^\varepsilon_{k+1-m}} = M_{k+1}\,, \quad \Phi - \xi^\varepsilon_{k+1-m} S_{k+1-m} = T_m$$

Then

$$\sum_{m=0}^{k+1} \varepsilon^m \bar{\mu}_m S_{k+1-m} = M_{k+1}\Phi - \sum_{m=0}^{k+1} q_m \varepsilon^m T_m\,. \tag{4.24}$$

We note that the sum in the right hand side of (4.23) starts from $m = 2$, because $q_0 = q_1 = 0$ ($\mu_0^{\rho,\varepsilon} = \mu_1^{\rho,\varepsilon} = 0$) and thus by the induction hypothesis, we have the estimates for T_m, $0 \le m \le k-1$. From (4.23) (4.24) we have

$$\mathcal{L}^\varepsilon_\rho Q_{k+1} = (1 - c_{k+1})\mu^{\rho,\varepsilon}\Phi - M_{k+1}\Phi + \sum_{m=0}^{k+1} q_m \varepsilon^m T_m + \varepsilon^{k+2} A_{k+1} - w_{k+1}\,.$$

By taking inner products with Q_{k+1} and using the Proposition 3.3, we obtain

$$C_0 \|Q_{k+1}\|^2_{\rho,\varepsilon} \le \left(\sum_{m=0}^{k+1} \varepsilon^m |q_m|\, \|T_m\|_{\rho,\varepsilon} + \varepsilon^{k+2} \|A_{k+1}\|_{\rho,\varepsilon} + \|w_{k+1}\|_{\rho,\varepsilon} \right) \|Q_{k+1}\|_{\rho,\varepsilon}\,.$$

By the induction hypothesis, we have $\|T_m\|_{\rho,\varepsilon} = O(\varepsilon^{k+2-m})$, $0 \le m \le k+1$, and by Lemma 4.1 $\|A_{k+1}\|_{\rho,\varepsilon} < \infty$. Therefore the inequality above becomes

$$\|Q_{k+1}\|_{\rho,\varepsilon} \le \text{Const.}\, \varepsilon^{k+2}\,.$$

From (4.22), we have

$$\Phi(1 - c_{k+1}) = S_{k+1} + Q_{k+1} + \sigma_{k+1}\,, \tag{4.25}$$

and thus

$$1 - c_{k+1} = \langle S_{k+1}, \Phi \rangle + O(e^{-c/\varepsilon})\,.$$

Taking norms as before in (4.25) we obtain

$$|1 - c_{k+1}| = \|S_{k+1}\|_{\rho,\varepsilon} + O(e^{k+2})\,.$$

Now

$$\lim_{\varepsilon\to 0}\|S_{k+1}\|_{\rho,\varepsilon} = \lim_{\varepsilon\to 0}\left[\int_{I_\varepsilon}\left(\sum_{m=0}^{k+1}\varepsilon^m\bar{\Phi}_m\right)^2(\rho+\varepsilon\eta)\,d\eta\right]^{1/2} = \sqrt{\rho}\,\|U'\| \neq 0$$

Setting $\xi_{k+1}^\varepsilon = (1-c_{k+1})^{-1}$, we obtain

$$\|\Phi - \xi_{k+1}^\varepsilon S_{k+1}\|_{\rho,\varepsilon} = O\big(\varepsilon^{k+2}\big)\,,$$

where

$${\xi_{k+1}^\varepsilon}^{-1} = \big\langle\, S_{k+1}, \Phi\,\big\rangle_{\rho,\varepsilon} + O\big(e^{-c/\varepsilon}\big)\,.$$

This completes the proof of the Lemma. □

Lemma 4.4. *Let n be fixed. If ξ_n^ε is as in Lemma 4.3 then (4.9) holds, i.e.*

$$\left\|\Phi - \xi_n^\varepsilon\sum_{m=0}^{k}\varepsilon^m\bar{\Phi}_m\right\|_{\rho,\varepsilon} = O\big(\varepsilon^{k+1}\big)\,, \qquad 0\le k\le n\,.$$

Proof. Let k be fixed. Then from

$$\|\Phi - \xi_k^\varepsilon S_k\|_{\rho,\varepsilon} = O\big(\varepsilon^{k+1}\big)\,, \qquad 0\le k\le n\,,$$

we obtain

$$\|\Phi - \xi_n^\varepsilon S_k\|_{\rho,\varepsilon} = O\big(\varepsilon^{k+1}\big) + |\xi_k^\varepsilon - \xi_n^\varepsilon|\,\|S_k\|_{\rho,\varepsilon}\,. \tag{4.26}$$

Now

$$\begin{aligned}
\xi_n^\varepsilon - \xi_k^\varepsilon &= \frac{1}{\big\langle\, S_n, \Phi\,\big\rangle_{\rho,\varepsilon} + O\big(e^{-c/\varepsilon}\big)} - \frac{1}{\big\langle\, S_k, \Phi\,\big\rangle_{\rho,\varepsilon} + O\big(e^{-c/\varepsilon}\big)}\\
&= \frac{-\sum_{m=k+1}^{n}\varepsilon^m\big\langle\,\bar{\Phi}_m, \Phi\,\big\rangle_{\rho,\varepsilon} + O\big(e^{-c/\varepsilon}\big)}{\Big(\big\langle\, S_n, \Phi\,\big\rangle_{\rho,\varepsilon} + O\big(e^{-c/\varepsilon}\big)\Big)\Big(\big\langle\, S_k, \Phi\,\big\rangle_{\rho,\varepsilon} + O\big(e^{-c/\varepsilon}\big)\Big)}
\end{aligned}$$

Since the denominator is bounded away from zero it follows

$$|\xi_n^\varepsilon - \xi_k^\varepsilon| \leq O(\varepsilon^{k+1}) .$$

Utilizing this in (4.26), we finally obtain (4.9). □

Lemma 4.5. *Let n be fixed. If ξ_n^ε is as in Lemma 4.3, then (4.10) holds, i.e.*

$$\left\| \left(\Phi - \xi_n^\varepsilon \sum_{m=0}^{k} \varepsilon^m \bar{\Phi}_m \right)' \right\|_{\rho,\varepsilon} = O(\varepsilon^{k+1}) , \qquad 0 \leq k \leq n .$$

Proof. From (4.25) we get

$$\Phi - \xi_k^\varepsilon S_k = \xi_k^\varepsilon Q_k + \xi_k^\varepsilon \sigma_k , \tag{4.27}$$

where $Q_k = O(\varepsilon^{k+1})$, and $\sigma_k = O(\varepsilon^{k+1})$. For an estimate on Q_k', we start from

$$\langle \mathcal{L}_\rho^\varepsilon Q_k, Q_k \rangle_{\rho,\varepsilon} = \int_{I_\varepsilon} [\, Q_k'^2 + f'(U) Q_k^2 \,](\rho + \varepsilon\eta)\, d\eta$$

from where we obtain

$$\|Q_k'\|_{\rho,\varepsilon}^2 \leq \langle \mathcal{L}_\rho^\varepsilon Q_k, Q_k \rangle_{\rho,\varepsilon} + \text{Const.} \|Q_k\|_{\rho,\varepsilon}^2$$

By the proof of the second step of induction in Lemma 4.3, we have

$$\langle \mathcal{L}_\rho^\varepsilon Q_k, Q_k \rangle_{\rho,\varepsilon} \leq \left(\sum_{m=0}^{k} \varepsilon^m |q_m| \, \|T_m\|_{\rho,\varepsilon} + \varepsilon^{k+1} \|A_k\|_{\rho,\varepsilon} + \|w_k\|_{\rho,\varepsilon} \right) \|Q_k\|_{\rho,\varepsilon} .$$

Therefore

$$\|Q_k'\|_{\rho,\varepsilon}^2 \leq \text{Const.} \varepsilon^{k+1} O(\varepsilon^{k+1}) + O(\varepsilon^{k+1})^2 .$$

Hence finally $\|Q_k'\|_{\rho,\varepsilon} \leq \text{Const.} \varepsilon^{k+1}$. Differentiating (4.27) and using the inequality above, we obtain

$$\|(\Phi - \xi_k^\varepsilon S_k)'\|_{\rho,\varepsilon} = O(\varepsilon^{k+1}) .$$

But then as in Lemma 4.4, (4.10) follows. This completes the proof. $\square$

Remark 4.6. As in the proof of Proposition 3.3 (see (3.21)) using (4.9) and (4.10), we obtain the pointwise estimate

$$\max_{\eta\in I_\varepsilon}\left|\Phi(\eta)-\xi_n^\varepsilon\sum_{m=0}^{k}\varepsilon^m\bar\Phi_m(\eta)\right| = O(\varepsilon^{k+1}), \qquad 0\le k\le n. \tag{4.28}$$

4.2. The Principal Eigenvalue.

If $\Phi = \bar\Phi_0+\varepsilon\bar\Phi_1+\varepsilon^2\bar\Phi_2+\cdots$ is the asymptotic expansion of the principal eigenfunction of $\mathcal{L}_\rho^\varepsilon$, and $\mu^{\rho,\varepsilon}=\bar\mu_0+\varepsilon\bar\mu_1+\varepsilon^2\bar\mu_2+\cdots$ is the formal expansion of the principal eigenvalue, we saw in section 4.1 that

$$\bar\Phi_0=U'\,,\qquad \bar\Phi_1=-\frac{1}{2\rho}U'\eta+a_1U'\,,\qquad \bar\Phi_2=-\frac{3}{8\rho^2}U'\eta^2-\frac{a_1}{2\rho}U'\eta+a_2U'\,,$$

where a_1, and a_2 are chosen so that $\bar\Phi_1$, and $\bar\Phi_2$ are orthogonal to U', and $\bar\mu_0=\bar\mu_1=0$, and $\bar\mu_2=-1/(4\rho^2)$, were chosen that way to have solvability in (4.2). In this section we justify the expansion of $\mu^{\rho,\varepsilon}$. The fact that $\bar\mu_0=0$, follows from Proposition 3.3 ($\bar\mu_0=\lim_{\varepsilon\to0}\mu^{\rho,\varepsilon}$). Rewriting (3.16) as

$$O(e^{-c/\varepsilon})-\varepsilon\int_{I_\varepsilon}\Phi U''\,d\eta=\mu^{\rho,\varepsilon}\int_{I_\varepsilon}\Phi U'\,(\rho+\varepsilon\eta)\,d\eta\,, \tag{4.29}$$

and using that $\Phi=\xi_0^\varepsilon U'+O(\varepsilon)$, then Proposition 4.2 for $n=0$ implies

$$\lim_{\varepsilon\to0}\int_{I_\varepsilon}\Phi U''\,d\eta=\xi_*\int_{-\infty}^{\infty}U'U''\,d\eta=0\,, \tag{4.30}$$

$$\lim_{\varepsilon\to0}\int_{I_\varepsilon}\Phi U'\,(\rho+\varepsilon\eta)\,d\eta=\xi_*\,\rho\,\|U'\|^2=\sqrt{\rho}\,\|U'\|\,, \tag{4.31}$$

by recalling that $\xi_*=(\sqrt{\rho}\,\|U'\|)^{-1}$. Thus dividing both sides of (4.29) by ε and taking limits as $\varepsilon\to0$, we get $\bar\mu_1=\lim_{\varepsilon\to0}(\mu^{\rho,\varepsilon}/\varepsilon)=0$. Next we deal with the third term.

Proposition 4.7. *If $\mu^{\rho,\varepsilon}$ is the principal eigenvalue of $\mathcal{L}_\rho^\varepsilon$ then*

$$\lim_{\varepsilon\to 0}\frac{\mu^{\rho,\varepsilon}}{\varepsilon^2} = -\frac{1}{4\rho^2}\,.$$

Proof. Dividing (4.29) by ε^2, we obtain

$$O(\varepsilon) - \frac{1}{\varepsilon}\int_{I_\varepsilon}\Phi U''\,d\eta = \frac{\mu^{\rho,\varepsilon}}{\varepsilon^2}\int_{I_\varepsilon}\Phi U'\,(\rho+\varepsilon\eta)\,d\eta\,. \tag{4.32}$$

Writing

$$\begin{aligned}\frac{1}{\varepsilon}\int_{I_\varepsilon}\Phi U''\,d\eta &= \int_{I_\varepsilon}\frac{\Phi-\xi_1^\varepsilon U'}{\varepsilon}U''\,d\eta + \frac{\xi_1^\varepsilon}{\varepsilon}\int_{I_\varepsilon}U'U''\,d\eta \\ &= \int_{I_\varepsilon}\frac{\Phi-\xi_1^\varepsilon U'}{\varepsilon}U''\,d\eta + \frac{\xi_1^\varepsilon}{\varepsilon}\left.\frac{U'^2}{2}\right|_{\partial I_\varepsilon}\,,\end{aligned}$$

and using that $\Phi = \xi_1^\varepsilon\left(U' - \varepsilon\frac{\eta}{2\rho}U' + \varepsilon a_1 U'\right) + O(\varepsilon^2)$, we obtain

$$\frac{1}{\varepsilon}\int_{I_\varepsilon}\Phi U''\,d\eta = \xi_1^\varepsilon\int_{I_\varepsilon} -\frac{\eta U'U''}{2\rho}\,d\eta + \xi_1^\varepsilon\left(a_1+\frac{1}{\varepsilon}\right)\left.\frac{U'^2}{2}\right|_{\partial I_\varepsilon} + O(\varepsilon^2)\,.$$

Taking limits we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{I_\varepsilon}\Phi U''\,d\eta &= \frac{\xi_*}{2\rho}\int_{-\infty}^{\infty} -U'U''\eta\,d\eta && (\text{by (2.2)}) \\ &= \frac{\xi_*}{4\rho}\int_{-\infty}^{\infty} U'^2\,d\eta && \left(\begin{array}{l}\text{after integrating by parts}\\ \text{and using (2.2)}\end{array}\right) \\ &= \frac{\|U'\|}{4\rho\sqrt{\rho}} && (\text{by the definition of } \xi_*)\end{aligned}$$

Taking limits in (4.32) and utilizing the result above and (4.31) we finally get

$$\lim_{\varepsilon\to 0}\frac{\mu^{\rho,\varepsilon}}{\varepsilon^2} = -\frac{1}{4\rho^2}\,.$$

Thus the proof is complete. $\square$

Chapter 5

THE RADIAL PART OF THE GENERAL OPERATOR

In this chapter we consider the radial part $\mathcal{L}^\varepsilon_\eta$ of the $\mathcal{L}^\varepsilon_{\eta,s}$ operator and we show that its spectrum is decomposed to a critical part consisting of one eigenvalue and a noncritical part. We also study the asymptotic forms of its principal eigenvalue and its corresponding eigenfunction. In this and in the next chapter we will denote the derivatives of U by $\dot{U}$, $\ddot{U}$, etc.

5.1. A Radial Operator.

We study the eigenvalue problem

$$(5.1)\qquad \begin{cases} \mathcal{L}^\varepsilon_\eta R := -\dfrac{1}{1+\kappa\varepsilon\eta}\left((1+\kappa\varepsilon\eta)R_\eta\right)_\eta + f'(U_\varepsilon)R = \nu R\,, & \eta \in I^\delta_\varepsilon \\ R_\eta(\eta) = 0\,, & \eta \in \partial I^\delta_\varepsilon \end{cases}$$

where $\kappa = \kappa(s)$ is the curvature of γ at s, and $I^\delta_\varepsilon = (-\delta/\varepsilon, \delta/\varepsilon)$. Hence R and ν depend on s as well. We recall that (see (2.17))

$$(5.2)\qquad U_\varepsilon(r,s) = U(\eta) + \varepsilon\kappa(s)V(\eta) + \varepsilon^2 W(\eta,s)\,.$$

Via (5.2) and the fact that

$$(5.3)\qquad f'(U_\varepsilon) = f'(U) + \varepsilon\kappa V f''(U) + \varepsilon^2\left[f''(U)W + \frac{\kappa^2}{2}f'''(U)V^2\right] + O(\varepsilon^3)\,,$$

it is seen that $\mathcal{L}^\varepsilon_\eta$ is a perturbation of the operator $\tilde{\mathcal{L}}^\varepsilon_\eta$ defined by

$$(5.4)\qquad \begin{cases} \tilde{\mathcal{L}}^\varepsilon_\eta \tilde{R} := -\dfrac{1}{1+\kappa\varepsilon\eta}\left((1+\kappa\varepsilon\eta)\tilde{R}_\eta\right)_\eta + f'(U)\tilde{R} = \tilde{\nu}\tilde{R}\,, & \eta \in I^\delta_\varepsilon \\ \tilde{R}_\eta(\eta) = 0\,, & \eta \in \partial I^\delta_\varepsilon \end{cases}$$

Regarding $\tilde{\mathcal{L}}_\eta^\varepsilon$ we have the following

Proposition 5.1. *With s fixed, let $\tilde{\nu}_1 < \tilde{\nu}_2 < \cdots \to \infty$ be the spectrum of $\tilde{\mathcal{L}}_\eta^\varepsilon$ with homogeneous Neumann boundary conditions. Then the following hold:*

(1) $\tilde{\nu}_1 < 0$, *for all ε small.*

(2) $\tilde{\nu}_1 \to 0$ *as $\varepsilon \to 0$; more precisely*

$$\lim_{\varepsilon\to 0} \frac{\tilde{\nu}_1}{\varepsilon^2} = -\frac{\kappa^2}{4}.$$

(3) *There exist positive constants ε_0, and C_s independent of ε such that $\tilde{\nu}_2 > C_s$ for $0 < \varepsilon < \varepsilon_0$.*

The proof of Proposition 5.1, being similar to that of Propositions 3.3, and 4.7 is omitted. We point out that (5.4) and (3.2) (and (5.1) as well) have the same limiting problem (as $\varepsilon \to 0$), namely (3.3), and so we expect a similar behavior of the two spectra which we indeed have. We note that if γ is a circle of radius ρ then

$$\lim_{\varepsilon\to 0} \frac{\tilde{\nu}_1}{\varepsilon^2} = -\frac{1}{4\rho^2} = \lim_{\varepsilon\to 0} \frac{\mu_1^\rho}{\varepsilon^2}.$$

Thus the result of Proposition 4.7 is recovered.

Remark. Because of the continuous dependence on parameters and since s lies in a compact interval, the results (1)–(3) of the Proposition 5.1 hold uniformly in s. In particular there is a positive constant $C_0 = \min\{ C_s : 0 \le s \le \ell\}$ such that $\tilde{\nu}_2 > C_0$ for all s and $0 < \varepsilon < \varepsilon_0$.

In what follows we use the notation

$$\langle u,v \rangle_\varepsilon := \int_{I_\varepsilon^\delta} uv\,(1+\kappa\varepsilon\eta)\,d\eta\,, \qquad \|u\|_\varepsilon^2 := \langle u,u \rangle_\varepsilon\,,$$

that can be extended to $\varepsilon = 0$ as

$$\langle u,v \rangle := \langle u,v \rangle_0 = \int_{-\infty}^{\infty} uv\,d\eta\,, \qquad \|u\|^2 := \|u\|_0^2 = \langle u,u \rangle_0\,.$$

If $\tilde{R}_1$ is the principal eigenfunction of (5.4) normalized by

$$\|\tilde{R}_1\|_\varepsilon = \left(\int_{I_\varepsilon^\delta} \tilde{R}_1^2 \, (1+\kappa\varepsilon\eta)\, d\eta \right)^{1/2} = 1\,, \tag{5.5}$$

then using Proposition 5.1 it can be shown the analog of Proposition 4.2 which reads as

$$\left\| \tilde{R}_1 - \tilde{\xi}_1^\varepsilon \left(\dot{U} - \varepsilon \frac{\kappa}{2} \dot{U}\eta + \varepsilon\alpha\kappa\dot{U} \right) \right\|_{H^1} = O(\varepsilon^2)\,. \tag{5.6}$$

Here $\kappa = \kappa(s)$ is the curvature, α is a constant given by

$$\alpha = \frac{\int_{-\infty}^{\infty} \dot{U}^2 \eta \, d\eta}{2\|\dot{U}\|^2}\,, \tag{5.7}$$

so chosen to have $\dot{U} \perp (-1/2)\dot{U}\eta + \alpha\dot{U}$,[4] $\lim_{\varepsilon\to 0} \tilde{\xi}_1^\varepsilon = \|\dot{U}\|^{-1}$, and the H^1 norm is compatible with the $\|\,.\,\|_\varepsilon$ norm. As we have pointed out, $\tilde{R}_1$ depends on s, (s is a parameter) but since it lies in a compact interval, the estimate (5.11) is uniform in s. Therefore,

$$\left\| \tilde{R}_1 - \tilde{\xi}_1^\varepsilon \left(\dot{U} - \varepsilon \frac{\kappa}{2} \dot{U}\eta + \varepsilon\alpha\kappa\dot{U} \right) \right\|_{L^\infty} = O(\varepsilon^2) \tag{5.8}$$

holds uniformly in s. We now study the spectrum $\{\nu_i : i = 1,2,3,\dots\}$ of the $\mathcal{L}_\eta^\varepsilon$ operator by using the information we already have for the spectrum $\{\tilde{\nu}_i : i = 1,2,3,\dots\}$ of the $\tilde{\mathcal{L}}_\eta^\varepsilon$ operator. We fix s. If R_i and $\tilde{R}_i$ are the corresponding normalized eigenfunctions, then by the variational characterization of the eigenvalues, we have

$$\begin{aligned}
\nu_1 &= \int_{I_\varepsilon^\delta} \left[R_{1\eta}^2 + f'(U_\varepsilon) R_1^2 \right] (1+\kappa\varepsilon\eta)\, d\eta \\
&\le \int_{I_\varepsilon^\delta} \left[\tilde{R}_{1\eta}^2 + f'(U_\varepsilon) \tilde{R}_1^2 \right] (1+\kappa\varepsilon\eta)\, d\eta \\
&= \tilde{\nu}_1 + \int_{I_\varepsilon^\delta} \left[f'(U_\varepsilon) - f'(U) \right] \tilde{R}_1^2 \, (1+\kappa\varepsilon\eta)\, d\eta\,.
\end{aligned}$$

[4] If f is odd then $\dot{U}$ is even, hence $\alpha = 0$.

Similarly

$$\tilde{\nu}_1 \le \nu_1 - \int_{I_\varepsilon^\delta} [\, f'(U_\varepsilon) - f'(U) \,]\, R_1^2 \,(1+\kappa\varepsilon\eta)\, d\eta \,.$$

So defining

$$\Delta_f^\varepsilon(\varphi,\psi) := \int_{I_\varepsilon^\delta} [\, f'(U_\varepsilon) - f'(U) \,]\, \varphi\psi \,(1+\kappa\varepsilon\eta)\, d\eta \,. \tag{5.9}$$

we have

$$\tilde{\nu}_1 + \Delta_f^\varepsilon(R_1,R_1) \le \nu_1 \le \tilde{\nu}_1 + \Delta_f^\varepsilon(\tilde{R}_1,\tilde{R}_1)\,. \tag{5.10}$$

Moreover

$$\begin{aligned}
\nu_2 &= \max_{v\in H^1} \min_{\substack{\varphi\perp v \\ \|\varphi\|_\varepsilon=1}} \int_{I_\varepsilon^\delta} \left[\, \varphi_\eta^2 + f'(U_\varepsilon)\varphi^2 \,\right] (1+\kappa\varepsilon\eta)\, d\eta \\
&\ge \min_{\substack{\varphi\perp \tilde{R}_1 \\ \|\varphi\|_\varepsilon=1}} \int_{I_\varepsilon^\delta} \left[\, \varphi_\eta^2 + f'(U_\varepsilon)\varphi^2 \,\right] (1+\kappa\varepsilon\eta)\, d\eta \\
&= \min_{\substack{\varphi\perp \tilde{R}_1 \\ \|\varphi\|_\varepsilon=1}} \left\{ \int_{I_\varepsilon^\delta} \left[\, \varphi_\eta^2 + f'(U)\varphi^2 \,\right] (1+\kappa\varepsilon\eta)\, d\eta + \Delta_f^\varepsilon(\varphi,\varphi) \right\} .
\end{aligned}$$

Since $\{\, \tilde{R}_i : i = 1,2,3,\dots \}$ is a complete orthonormal set in $L^2\left((1+\kappa\varepsilon\eta)d\eta, I_\varepsilon^\delta\right)$, if $\varphi \perp \tilde{R}_1$, we have

$$\varphi = \sum_{i=2}^\infty a_i \tilde{R}_i \,, \qquad a_i = \langle\, \varphi, \tilde{R}_i \,\rangle_\varepsilon = \int_{I_\varepsilon^\delta} \varphi \tilde{R}_i (1+\kappa\varepsilon\eta)\, d\eta$$

and so

$$\begin{aligned}
\langle\, \tilde{\mathcal{L}}_\eta^\varepsilon \varphi, \varphi \,\rangle_\varepsilon &= \int_{I_\varepsilon^\delta} \left[\, \varphi_\eta^2 + f'(U)\varphi^2 \,\right] (1+\kappa\varepsilon\eta)\, d\eta \\
&= \sum_{i,j=2}^\infty \langle\, a_i \tilde{\nu}_i \tilde{R}_i, \tilde{R}_j \,\rangle_\varepsilon \\
&= \sum_{i=2}^\infty a_i^2 \tilde{\nu}_i \\
&\ge \tilde{\nu}_2 \sum_{i=2}^\infty a_i^2 \,, && (\text{by Proposition 5.1(3)}) \\
&= \tilde{\nu}_2 \,, && (1 = \|\varphi\|_\varepsilon \,, \varphi \perp \tilde{R}_1)
\end{aligned}$$

Thus we finally have

$$\nu_2 \geq \tilde{\nu}_2 + \min_{\substack{\varphi \perp \tilde{R}_1 \\ \|\varphi\|_\epsilon = 1}} \Delta_f^\varepsilon(\varphi, \varphi)\,. \tag{5.11}$$

Now if φ is an L^2 normalized function by applying the mean value theorem in (5.13) and using (5.2), we get

$$|\Delta_f^\varepsilon(\varphi, \varphi)| \leq C\varepsilon\,,$$

where C is a constant independent of ε. Utilizing the inequality above in (5.10) and (5.11), we obtain

$$\nu_1 - \tilde{\nu}_1 = O(\varepsilon)^{5} \tag{5.12}$$

$$\nu_2 \geq \tilde{\nu}_2 + O(\varepsilon)\,. \tag{5.13}$$

Now (5.12) and Proposition 5.1(2) imply that $\nu_1 \to 0$ as $\varepsilon \to 0$, while (5.13) and Proposition 5.1(3) imply that ν_2 is bounded below by a positive constant independent of ε. We summarize these results in

Lemma 5.2. *Let ν_1 be the principal eigenvalue of $\mathcal{L}_\eta^\varepsilon$, and let ν_2 be its second in order. Then there are constants C_0 and ε_0 such that*

(1) $\nu_1 \to 0$ *as* $\varepsilon \to 0$ *for all* $s \in [0, \ell]$,

(2) $\nu_2 \geq C_0$ *for all* $0 < \varepsilon < \varepsilon_0$, *and for all* $s \in [0, \ell]$.

The next Lemma gives information about the asymptotic expansion of the principal eigenfunction R_1 of $\mathcal{L}_\eta^\varepsilon$.

Lemma 5.3. *Let R_1 be the normalized principal eigenfunction of $\mathcal{L}_\eta^\varepsilon$, and let V_0 be the unique solution of the problem*

$$-h'' + f'(U)h = -f''(U)V\dot{U}\,, \qquad \int_{-\infty}^{\infty} \dot{U}(\eta)h(\eta)\,d\eta = 0\,. \tag{5.14}$$

[5] In the proof of Proposition 5.4 it is shown that actually $\nu_1 - \tilde{\nu}_1 = O(\varepsilon^2)$.

Then

$$(5.15) \qquad \left\| R_1 - \xi_1^\varepsilon \left(\dot{U} - \varepsilon \frac{\kappa}{2} \dot{U} \eta + \varepsilon \alpha \kappa \dot{U} + \varepsilon \kappa V_0 \right) \right\|_{H^1} = O\left(\varepsilon^2\right) ,$$

where κ is the curvature at s, α is as in (5.7), and $\xi_1^\varepsilon \to \|\dot{U}\|^{-1}$ as $\varepsilon \to 0$.

Before the proof we make the following remarks: V in (5.14) is the second term of the expansion of U_ε, hence by (2.8) the problem (5.14) is solvable. Moreover, by Lemma 4.1 it is uniquely solvable in $L^2(\mathbb{R})$, and its solution V_0, as well as its derivative V_0' decay exponentially as $|\eta| \to \infty$.

Proof. Since $\dot{U} - \varepsilon(\kappa/2)\dot{U}\eta + \varepsilon\alpha\kappa\dot{U} + \varepsilon\kappa V_0$ does not satisfy the boundary conditions, we define

$$w = \dot{U} - \varepsilon \frac{\kappa}{2} \dot{U} \eta + \varepsilon \alpha \kappa \dot{U} + \varepsilon \kappa V_0 + a\eta^2 + b\eta ,$$

where a and b are chosen so that $w'(\pm\delta/\varepsilon) = 0$. We note that a and b are functions of s. However, because of the exponential estimates that U and V_0 satisfy a and b are exponentially small for all s. As in Proposition 4.2 we write

$$(5.16) \qquad R_1 - w = zR_1 + Q ,$$

with $Q \perp R_1$. Then Q satisfies

$$\mathcal{L}_\eta^\varepsilon Q = (1-z)\nu_1 R_1 - \mathcal{L}_\eta^\varepsilon w$$

$$Q'(-\delta/\varepsilon) = Q'(\delta/\varepsilon) = 0$$

Observe that

$$\mathcal{L}_\eta^\varepsilon w = \left(\tilde{\mathcal{L}}_\eta^\varepsilon + [\, f'(U_\varepsilon) - f'(U)] \right) \left(\dot{U} - \varepsilon \frac{\kappa}{2} \dot{U} \eta + \varepsilon \alpha \kappa \dot{U} + \varepsilon \kappa V_0 \right) + g$$

where $g = \mathcal{L}^\varepsilon_\eta(a\eta^2 + b\eta)$, and so $\|g\|_\varepsilon = O(e^{-c/\varepsilon})$, for some positive constant c. Now using (5.3), we have

$$\begin{aligned}
&\left(\tilde{\mathcal{L}}^\varepsilon_\eta + [f'(U_\varepsilon) - f'(U)]\right)\left(\dot U - \varepsilon\frac{\kappa}{2}\dot U\eta + \varepsilon\alpha\kappa\dot U + \varepsilon\kappa V_0\right) + g \\
&= \frac{\varepsilon^2\kappa^2}{1+\kappa\varepsilon\eta}\left(\ddot U(1-\alpha) + \frac{\ddot U\eta + \dot U}{2} - V_0'\right) - \varepsilon\kappa f''(U)V\dot U + \varepsilon\kappa f''(U)V\dot U \\
&\quad + \varepsilon^2 f''(U)W\dot U + \frac{\varepsilon^2\kappa^2}{2}\left(-f''(U)V\dot U\eta + f'''(U)V^2\dot U\right) \\
&\quad + \varepsilon^2\kappa^2\left(f''(U)VV_0 + \alpha f''(U)V\dot U\right) + O(\varepsilon^3) \\
&:= \varepsilon^2 Y + O(\varepsilon^3)\,.
\end{aligned}$$

Therefore we can write

$$\mathcal{L}^\varepsilon_\eta Q = (1-z)\nu_1 R_1 - \varepsilon^2 Y + Z_\varepsilon\,,$$

with $Z_\varepsilon = O(\varepsilon^3)$, from where we obtain

$$\left\langle \mathcal{L}^\varepsilon_\eta Q, Q\right\rangle_\varepsilon = -\varepsilon^2\left\langle Y, Q\right\rangle_\varepsilon + \left\langle Z_\varepsilon, Q\right\rangle_\varepsilon.$$

Since Q is orthogonal to R_1, via Lemma 5.2(2), we get

$$C_0\|Q\|^2_\varepsilon \le -\varepsilon^2\left\langle Y, Q\right\rangle_\varepsilon + \left\langle Z_\varepsilon, Q\right\rangle_\varepsilon.$$

Hence using that Y is in L^2, and $\|Z_\varepsilon\|_\varepsilon = O(\varepsilon^3)$, we obtain

$$\|Q\|_\varepsilon \le \text{Const.}\,\varepsilon^2\,.$$

Returning to (5.16) and writing it in the original variables, we get

$$R_1(1-z) = \dot U - \varepsilon\frac{\kappa}{2}\dot U\eta + \varepsilon\alpha\kappa\dot U + \varepsilon\kappa V_0 + g + a\eta^2 + b\eta\,. \tag{5.17}$$

Taking norms we have

$$|1-z| = \left\|\dot U - \varepsilon\frac{\kappa}{2}\dot U\eta + \varepsilon\alpha\kappa\dot U + \varepsilon\kappa V_0\right\|_\varepsilon + O(\varepsilon^2)\,,$$

and taking limits as $\varepsilon \to 0$, we see that the right hand side converges to $\|\dot{U}\|$. Thus setting $\xi_1^\varepsilon = (1-z)^{-1}$ from (5.17), we finally get

$$\left\| R_1 - \xi_1^\varepsilon \left(\dot{U} - \varepsilon \frac{\kappa}{2} \dot{U} \eta + \varepsilon \alpha \kappa \dot{U} + \varepsilon \kappa V_0 \right) \right\|_\varepsilon = O(\varepsilon^2) .$$

To obtain the estimate for the derivatives it suffices to show that $\|Q_\eta\|_\varepsilon \leq C\varepsilon^2$, because then the result will follow from (5.16). Starting from

$$\langle \mathcal{L}_\eta^\varepsilon Q, Q \rangle_\varepsilon = \int_{I_\varepsilon^\delta} [Q_\eta^2 + f'(U) Q^2](1 + \kappa \varepsilon \eta) \, d\eta ,$$

and since for ε small $f'(U_\varepsilon)$ is bounded, utilizing $\|Q\|_\varepsilon \leq C\varepsilon^2$, we have

$$\begin{aligned}
\|Q_\eta\|_\varepsilon^2 &\leq \langle \mathcal{L}_\eta^\varepsilon Q, Q \rangle_\varepsilon + C_1 \|Q\|_\varepsilon^2 \\
&= -\varepsilon^2 \langle Y, Q \rangle_\varepsilon + \langle Z_\varepsilon, Q \rangle_\varepsilon + C_1 \|Q\|_\varepsilon^2 \\
&\leq \varepsilon^2 \|Y\|_\varepsilon \|Q\|_\varepsilon + \|Z_\varepsilon\|_\varepsilon \|Q\|_\varepsilon + C_1 \|Q\|_\varepsilon^2 \\
&\leq C_2 \varepsilon^4 + O(e^{-c/\varepsilon}) .
\end{aligned}$$

Hence $\|Q_\eta\|_\varepsilon \leq C\varepsilon^2$. The proof of the Lemma is complete. □

Proposition 5.4. *Let ν_1 be the principal eigenvalue of $\mathcal{L}_\eta^\varepsilon$. Then*

$$\lim_{\varepsilon \to 0} \frac{\nu_1}{\varepsilon^2} = -\frac{\kappa^2}{4} + q(s) := \hat{\boldsymbol{\lambda}} , \tag{5.18}$$

where $\kappa = \kappa(s)$ is the curvature at s, and

$$q(s) := \frac{1}{\|\dot{U}\|^2} \int_{-\infty}^{\infty} \left\{ f''(U) \left(\kappa^2 V V_0 + W \dot{U} \right) \dot{U} + \frac{\kappa^2}{2} f'''(U) V^2 \dot{U}^2 \right\} d\eta \tag{5.19}$$

with V_0 being as in Lemma 5.3.

Proof. Starting from

$$\begin{aligned}
\tilde{R}_1 &= \tilde{\xi}_1^\varepsilon \left(\dot{U} - \varepsilon \frac{\kappa}{2} \dot{U} \eta + \varepsilon \alpha \kappa \dot{U} \right) + O(\varepsilon^2) , \\
R_1 &= \xi_1^\varepsilon \left(\dot{U} - \varepsilon \frac{\kappa}{2} \dot{U} \eta + \varepsilon \alpha \kappa \dot{U} + \varepsilon \kappa V_0 \right) + O(\varepsilon^2) ,
\end{aligned}$$

we obtain

$$\tilde{R}_1 R_1 = \tilde{\xi}_1^\varepsilon \xi_1^\varepsilon \Big\{ \dot{U}^2 + \varepsilon\kappa\big(-\dot{U}^2\eta + 2\alpha\dot{U}^2 + \dot{U}V_0\big)\Big\} + O(\varepsilon^2)\,.$$

Then

$$\begin{aligned}\Delta_f^\varepsilon(\tilde{R}_1, R_1) &= \int_{I_\varepsilon^\delta} [\,f'(U_\varepsilon) - f'(U)\,]\,\tilde{R}_1 R_1\,(1+\kappa\varepsilon\eta)\,d\eta \\ &= \tilde{\xi}_1^\varepsilon \xi_1^\varepsilon \Big(\varepsilon^2 p^\varepsilon + \big(\varepsilon\kappa + 2\varepsilon^2\alpha\kappa^2\big)\int_{I_\varepsilon^\delta} f''(U)V\dot{U}^2\,d\eta\Big) + O(\varepsilon^3)\,,\end{aligned}$$

where we have used (5.3), with

$$p^\varepsilon = \int_{I_\varepsilon^\delta} \Big\{ f''(U)\Big(\kappa^2 V V_0 + W\dot{U}\Big)\dot{U} + \frac{\kappa^2}{2} f'''(U)V^2\dot{U}^2 \Big\}\,d\eta\,.$$

Using (2.8) and because of the exponential estimates on $\dot{U}$, we get

$$\int_{I_\varepsilon^\delta} f''(U)V\dot{U}^2\,d\eta = -\int_{|\eta|\geq\delta/\varepsilon} f''(U)V\dot{U}^2\,d\eta = O\big(e^{-c/\varepsilon}\big)\,.$$

Thus

$$\Delta_f^\varepsilon(\tilde{R}_1, R_1) = \varepsilon^2 \tilde{\xi}_1^\varepsilon\,\xi_1^\varepsilon\,p^\varepsilon + O(\varepsilon^3)\,.$$

Therefore from the definition of p^ε, q, and the properties of $\tilde{\xi}_1^\varepsilon$ and ξ_1^ε, we have

$$\lim_{\varepsilon\to 0} \frac{\Delta_f^\varepsilon(\tilde{R}_1, R_1)}{\varepsilon^2} = \lim_{\varepsilon\to 0} \tilde{\xi}_1^\varepsilon\,\xi_1^\varepsilon\,p^\varepsilon = q\,. \tag{5.20}$$

Moreover from (5.6) and Lemma 5.3, we get

$$\int_{I_\varepsilon^\delta} \tilde{R}_1 R_1\,(1+\kappa\varepsilon\eta)\,d\eta = \tilde{\xi}_1^\varepsilon \xi_1^\varepsilon \int_{I_\varepsilon^\delta} \dot{U}^2\,(1+\kappa\varepsilon\eta)\,d\eta + O(\varepsilon)\,.$$

Thus

$$\lim_{\varepsilon\to 0} \int_{I_\varepsilon^\delta} \tilde{R}_1 R_1\,(1+\kappa\varepsilon\eta)\,d\eta = 1\,. \tag{5.21}$$

Now from (5.1) and (5.4), we obtain

$$\Delta_f^\varepsilon(R_1, \tilde{R}_1) = (\nu_1 - \tilde{\nu}_1)\langle R_1, \tilde{R}_1 \rangle_\varepsilon ,$$

therefore

$$\nu_1 = \tilde{\nu}_1 + \frac{\Delta_f^\varepsilon(R_1, \tilde{R}_1)}{\langle R_1, \tilde{R}_1 \rangle_\varepsilon} .$$

Finally dividing by ε^2 and using Proposition 5.1(2), (5.20) and (5.21), we obtain

$$\lim_{\varepsilon \to 0} \frac{\nu_1}{\varepsilon^2} = -\frac{\kappa^2}{4} + q$$

The proof is now complete. □

We conclude this section with a Lemma to be used in the next chapter. For the proof we refer to [A–F2].

Lemma 5.5. *Let R_1 be the principal eigenfunction of $\mathcal{L}_\eta^\varepsilon$ normalized by requiring $\|R_1\|_\varepsilon = 1$. Then*

$$\|R_{1s}\|_\varepsilon = O(\varepsilon) ,$$

where R_{1s} is the partial derivative of R_1 with respect to (the parameter) s.

5.2. Asymptotic Expansions.

This section deals with the principal eigenfunction and eigenvalue, so we omit subscripts. As in the radial case we consider the formal expressions

$$R = \bar{R}_0 + \varepsilon \bar{R}_1 + \varepsilon^2 \bar{R}_2 + \cdots , \qquad \nu = \bar{\nu}_0 + \varepsilon \bar{\nu}_1 + \varepsilon^2 \bar{\nu}_2 + \cdots$$

Substituting these expressions in (5.1), we obtain after equating like powers of ε

$$
\begin{aligned}
-\bar{R}_0'' + f'(U)\bar{R}_0 &= \bar{\nu}_0 \bar{R}_0 \\
-\bar{R}_1'' + f'(U)\bar{R}_1 &= \bar{\nu}_1 \bar{R}_0 + \bar{\nu}_0 \bar{R}_1 + \kappa \left(\bar{R}_0' - f''(U)V\bar{R}_0 \right) \\
-\bar{R}_2'' + f'(U)\bar{R}_2 &= \bar{\nu}_2 \bar{R}_0 + \bar{\nu}_1 \bar{R}_1 + \bar{\nu}_0 \bar{R}_2 + \kappa \left(\bar{R}_1' - f''(U)V\bar{R}_1 \right) \\
&\quad - \kappa^2 \eta \bar{R}_0' - \left(f''(U)W + \frac{\kappa^2}{2} f'''(U)V^2 \right) \bar{R}_0 \\
&\ \vdots
\end{aligned}
$$

Choosing $\bar{\nu}_0 = 0$ ($\nu_1 \to 0$ as $\varepsilon \to 0$), we get $\bar{R}_0 = \dot{U}$. For solvability in the second equation we require $\bar{\nu}_1 = 0$ (we recall that $\int_{-\infty}^{\infty} \dot{U}\ddot{U}\, d\eta = 0$, and $\int_{-\infty}^{\infty} f''(U)V\dot{U}^2\, d\eta = 0$), hence $\bar{R}_1 = \kappa \hat{R}_1$, where $\hat{R}_1$ is a solution of the equation

$$
-\hat{R}_1'' + f'(U)\hat{R}_1 = \ddot{U} - f''(U)V\dot{U}\,. \tag{5.22}
$$

For solvability in the third equation we need

$$
\begin{aligned}
\bar{\nu}_2 = \frac{1}{\|\dot{U}\|^2} \int_{-\infty}^{\infty} &\left\{ -\kappa \left(\bar{R}_1' - f''(U)V\bar{R}_1 \right) \dot{U} + \kappa^2 \eta \bar{R}_0' \dot{U} \right\} d\eta \\
&+ \frac{1}{\|\dot{U}\|^2} \int_{-\infty}^{\infty} \left(f''(U)W + \frac{\kappa^2}{2} f'''(U)V^2 \right) \bar{R}_0 \dot{U}\, d\eta\,,
\end{aligned}
$$

or equivalently after using that $\bar{R}_0 = \dot{U}$ and $\bar{R}_1(\eta, s) = \kappa(s)\hat{R}_1(\eta)$, and integrating by parts

$$
\begin{aligned}
\bar{\nu}_2 = -\frac{\kappa^2}{2} + \frac{\kappa^2}{\|\dot{U}\|^2} \int_{-\infty}^{\infty} &\left(\hat{R}_1 \ddot{U} + f''(U)V\hat{R}_1 \dot{U} \right) d\eta \\
&+ \frac{1}{\|\dot{U}\|^2} \int_{-\infty}^{\infty} \left(f''(U)W + \frac{\kappa^2}{2} f'''(U)V^2 \right) \dot{U}^2\, d\eta\,.
\end{aligned}
$$

Utilizing the fact that the unique L^2 solution of (5.22) that is orthogonal to $\dot{U}$ is given by $\hat{R}_1 = (-1/2)\dot{U}\eta + \alpha\dot{U} + V_0$ and the orthogonality condition (2.8) the first integral in the expression of $\bar{\nu}_2$ becomes

$$
\frac{\|\dot{U}\|^2}{4} + \int_{-\infty}^{\infty} \left(V_0 \ddot{U} - \frac{1}{2} f''(U)V\dot{U}^2 \eta + f''(U)VV_0\dot{U} \right) d\eta\,.
$$

Hence for q as in (5.19) we have

$$\bar{\nu}_2 = -\frac{\kappa^2}{4} + q + \frac{\kappa^2}{\|\dot{U}\|^2}\int_{-\infty}^{\infty}\left(V_0\ddot{U} - \frac{1}{2}f''(U)V\dot{U}^2\eta\right)d\eta\,. \tag{5.23}$$

Next we show that

$$\int_{-\infty}^{\infty}\left(V_0\ddot{U} - \frac{1}{2}f''(U)V\dot{U}^2\eta\right)d\eta = 0\,.$$

Indeed, multiplying by $\dot{U}\eta$ the equation (5.14) that V_0 satisfies, and integrating we get

$$\int_{-\infty}^{\infty} -V_0''\dot{U}\eta\,d\eta + \int_{-\infty}^{\infty} f'(U)V_0\dot{U}\eta\,d\eta = -\int_{-\infty}^{\infty} f''(U)V\dot{U}^2\eta\,d\eta\,.$$

Integrating by parts twice in the first integral, and using the exponential estimates that $\dot{U}$ and V_0 satisfy, we obtain

$$\int_{-\infty}^{\infty}(-\dddot{U} + f'(U)\dot{U})V_0\eta\,d\eta + \int_{-\infty}^{\infty}(-V_0\ddot{U} + V_0'\dot{U})\,d\eta = -\int_{-\infty}^{\infty} f''(U)V\dot{U}^2\eta\,d\eta\,,$$

which via (2.1) leads to

$$2\int_{-\infty}^{\infty} V_0\ddot{U}\,d\eta = \int_{-\infty}^{\infty} f''(U)V\dot{U}^2\eta\,d\eta\,.$$

Thus (5.23) becomes

$$\bar{\nu}_2 = -\frac{\kappa^2}{4} + q = \hat{\boldsymbol{\lambda}}\,,$$

as in Proposition 5.4. Hence Lemma 5.3 and Proposition 5.4 justify the formal results above, and thus

$$R(\eta, s) = \xi_1^\varepsilon\dot{U}(\eta) + \varepsilon\xi_1^\varepsilon\kappa(s)\hat{R}_1(\eta) + O(\varepsilon^2)\,,$$

where $\hat{R}_1$ is the unique solution of (5.22) that is orthogonal to $\dot{U}$. Moreover, as in Lemma 5.3 it can be shown that

$$R(\eta, s) = \xi_2^\varepsilon\dot{U}(\eta) + \varepsilon\xi_2^\varepsilon\kappa(s)\hat{R}_1(\eta) + \varepsilon^2\xi_2^\varepsilon\hat{R}_2(\eta, s) + O(\varepsilon^3)\,, \tag{5.24}$$

where $\hat{R}_2(\eta, s)$ is the unique orthogonal to $\dot{U}$ solution of

$$
(5.25) \qquad -\hat{R}_2'' + f'(U)\hat{R}_2 = \boldsymbol{\lambda}\dot{U} + \kappa^2\left(\hat{R}_1' - f''(U)V\hat{R}_1\right) \\
- \kappa^2\ddot{U}\eta - \left(f''(U)W + \frac{\kappa^2}{2}f'''(U)V^2\right)\dot{U}\,,
$$

with $\hat{R}_1$ as above, and $\xi_2^\varepsilon \to \|\dot{U}\|^{-1}$, as $\varepsilon \to 0$.

Chapter 6

THE GENERAL OPERATOR

In this chapter we consider the general operator defined in (2.10). We show that its critical eigenvalues originate from the eigenvalues of an 1–dimensional Sturm–Liouville problem. As a consequence of that connection we show that the multiplicity of the critical eigenvalues is at most two. Moreover, we show that there is a kind of one–to–one correspondence between the two spectra.

As we have noticed, if ψ is an eigenfunction of (2.10) corresponding to the eigenvalue μ, then the stretched function

$$\tilde{\psi}(\eta, s) = \sqrt{\varepsilon}\psi(\varepsilon\eta, s) \tag{6.1}$$

satisfies the equation

$$\mathcal{L}^{\varepsilon}_{\eta,s}\tilde{\psi} := \mathcal{L}^{\varepsilon}_{\eta}\tilde{\psi} + \varepsilon^2\mathcal{L}^{\varepsilon}_{s}\tilde{\psi} = \mu\tilde{\psi} \tag{6.2}$$

in the strip $\Omega_{\delta/\varepsilon}$, where $\mathcal{L}^{\varepsilon}_{\eta}$ and $\mathcal{L}^{\varepsilon}_{s}$ are defined in (2.18) and (2.19) respectively. If μ is an eigenvalue of (2.10) satisfying the hypothesis of Lemma 2.1, and $\tilde{\psi}$ is a corresponding stretched eigenfunction, then from (6.1) and (2.11) we obtain

$$|\tilde{\psi}(\eta, s)| \leq C\sqrt{\varepsilon}e^{-c|\eta|}\,, \qquad 2b \leq |\eta| < 2\delta/\varepsilon\,, \qquad 0 \leq s \leq \ell\,, \tag{6.3}$$

for $0 < \varepsilon < \varepsilon_0$, where C, c, and b are positive constants independent of ε. If ψ and $\tilde{\psi}$ are as above then

$$\int_{\Omega} \psi^2(x)\,dx = \int_{\Omega_\delta} \psi^2(x)\,dx + \int_{\Omega\setminus\Omega_\delta} \psi^2(x)\,dx\,.$$

Now changing to r, s variables and then stretching, we get after using (2.12)

$$\int_{\Omega} \psi^2(x)\,dx = \int_0^{\ell} \int_{I_\varepsilon^\delta} \tilde{\psi}^2(\eta, s)\,(1+\kappa\varepsilon\eta)\,d\eta\,ds + O\big(e^{-c/\varepsilon}\big)\,, \tag{6.4}$$

where $I_\varepsilon^\delta = (-\delta/\varepsilon, \delta/\varepsilon)$. This fact motivates the following normalization

$$\|\tilde{\psi}\|^2_{L^2(\Omega_{\delta/\varepsilon})} = \int_0^{\ell} \int_{I_\varepsilon^\delta} \tilde{\psi}^2(\eta, s)\,(1+\kappa\varepsilon\eta)\,d\eta\,ds = 1\,. \tag{6.5}$$

In addition to the notation

$$\big\langle\, u, v \,\big\rangle_\varepsilon := \int_{I_\varepsilon^\delta} uv\,(1+\kappa\varepsilon\eta)\,d\eta\,, \qquad \|u\|_\varepsilon^2 := \big\langle\, u, u \,\big\rangle_\varepsilon\,,$$

$$\big\langle\, u, v \,\big\rangle := \big\langle\, u, v \,\big\rangle_0 = \int_{-\infty}^{\infty} uv\,d\eta\,, \qquad \|u\|^2 := \|u\|_0^2 = \big\langle\, u, u \,\big\rangle_0\,,$$

used in the previous chapter we will also use

$$\big\langle\, u, v \,\big\rangle_{\varepsilon s} := \int_0^{\ell} \int_{I_\varepsilon^\delta} uv\,(1+\kappa\varepsilon\eta)\,d\eta\,ds\,, \qquad \|u\|_{\varepsilon s}^2 := \big\langle\, u, u \,\big\rangle_{\varepsilon s}\,,$$

$$\big\langle\, u, v \,\big\rangle_s := \big\langle\, u, v \,\big\rangle_{0s} = \int_0^{\ell} \int_{-\infty}^{\infty} uv\,d\eta\,ds\,, \qquad \|u\|_s^2 := \|u\|_{0s}^2 = \big\langle\, u, u \,\big\rangle_{0s}\,.$$

If $\tilde{\psi}$ satisfies (6.2), and R is the principal eigenfunction of the $\mathcal{L}_\eta^\varepsilon$ operator normalized by $\|R\|_\varepsilon = 1$, we define

$$\theta(s) := \big\langle\, \tilde{\psi}(\cdot, s), R(\cdot, s) \,\big\rangle_\varepsilon\,. \tag{6.6}$$

Then as in [A–F2] we consider the decomposition

$$\tilde{\psi}(\eta, s) = \phi(\eta, s) + \theta(s) R(\eta, s)\,, \tag{6.7}$$

where

$$\big\langle\, \phi(\cdot, s), R(\cdot, s) \,\big\rangle_\varepsilon = 0\,. \tag{6.8}$$

Lemma 6.1. *Let $\tilde{\psi}$ be a stretched eigenfunction corresponding to the eigenvalue $\mu \le C\varepsilon^2$, with C independent of ε, and normalized by $\|\tilde{\psi}\|_{\varepsilon s} = 1$. Then there exist constants ε_0, and C_1 independent of ε, such that*

$$\|\tilde{\psi} - \theta R\|_{\varepsilon s} \le C_1 \varepsilon\,, \tag{6.9}$$

$$\|\tilde{\psi}_s\|_{\varepsilon s} \le C_1\,. \tag{6.10}$$

for all $0 < \varepsilon < \varepsilon_0$.

Proof. From (6.7) since (6.8) holds, we obtain

$$\|\tilde{\psi}\|_\varepsilon^2 = \|\phi\|_\varepsilon^2 + \theta^2(s)\|R\|_\varepsilon^2 = \|\phi\|_\varepsilon^2 + \theta^2(s)\,.$$

Integrating with respect to s, we get

$$\|\tilde{\psi}\|_{\varepsilon s}^2 = \|\phi\|_{\varepsilon s}^2 + \int_0^\ell \theta^2(s)\,ds\,, \tag{6.11}$$

from where we obtain

$$\int_0^\ell \theta^2(s)\,ds \le 1\,. \tag{6.12}$$

Now if ν is the principal eigenfunction of the $\mathcal{L}_\eta^\varepsilon$ operator from (6.7) we get

$$\mathcal{L}_\eta^\varepsilon \tilde{\psi} = \mathcal{L}_\eta^\varepsilon \phi + \nu\theta R\,,$$

and so taking inner products with $\tilde{\psi}$, we obtain

$$\begin{aligned}\langle \mathcal{L}_\eta^\varepsilon \tilde{\psi}, \tilde{\psi} \rangle_\varepsilon &= \langle \mathcal{L}_\eta^\varepsilon \phi, \tilde{\psi} \rangle_\varepsilon + \nu\theta \langle R, \tilde{\psi} \rangle_\varepsilon \\ &= \langle \mathcal{L}_\eta^\varepsilon \phi, \phi \rangle_\varepsilon + \theta \langle \mathcal{L}_\eta^\varepsilon \phi, R \rangle_\varepsilon + \nu\theta^2\,.\end{aligned}$$

Hence by Lemma 5.2

$$\langle \mathcal{L}_\eta^\varepsilon \tilde{\psi}, \tilde{\psi} \rangle_\varepsilon \ge C_0 \|\phi\|_\varepsilon^2 + \theta \langle \mathcal{L}_\eta^\varepsilon \phi, R \rangle_\varepsilon + \nu\theta^2\,. \tag{6.13}$$

Integrating by parts twice in the left hand side of (6.13), and observing that $\phi_\eta(\pm\delta/\varepsilon, s) = \tilde{\psi}_\eta(\pm\delta/\varepsilon, s)$, since R satisfies the boundary conditions, we obtain

$$
\begin{aligned}
\langle \mathcal{L}_\eta^\varepsilon \phi, R \rangle_\varepsilon &= \nu \langle \phi, R \rangle_\varepsilon - \tilde{\psi}_\eta(\eta, s)(1+\kappa\varepsilon\eta)R(\eta, s)\big|_{\eta=\pm\delta/\varepsilon} \\
&= -\tilde{\psi}_\eta(\eta, s)(1+\kappa\varepsilon\eta)R(\eta, s)\big|_{\eta=\pm\delta/\varepsilon} \qquad \text{(by (6.8))}
\end{aligned}
$$

By (6.4) and a simple regularity argument the boundary term above is of order $O(e^{-c/\varepsilon})$. Utilizing this fact in (6.13), we obtain

$$
\langle \mathcal{L}_\eta^\varepsilon \tilde{\psi}, \tilde{\psi} \rangle_\varepsilon \geq C_0 \|\phi\|_\varepsilon^2 + \theta O(e^{-c/\varepsilon}) + \nu\theta^2 .
$$

Integrating with respect to s, we have

$$
\langle \mathcal{L}_\eta^\varepsilon \tilde{\psi}, \tilde{\psi} \rangle_{\varepsilon s} \geq C_0 \|\phi\|_{\varepsilon s}^2 + O(e^{-c/\varepsilon}) \int_0^\ell \theta(s)\, ds + \nu \int_0^\ell \theta^2(s)\, ds ,
$$

and since

$$
\begin{aligned}
\langle \mathcal{L}_\eta^\varepsilon \tilde{\psi}, \tilde{\psi} \rangle_{\varepsilon s} &= \langle \mathcal{L}_{\eta,s}^\varepsilon \tilde{\psi}, \tilde{\psi} \rangle_{\varepsilon s} - \varepsilon^2 \langle \mathcal{L}_s^\varepsilon \tilde{\psi}, \tilde{\psi} \rangle_{\varepsilon s} \\
&= \mu \|\tilde{\psi}\|_{\varepsilon s}^2 + \varepsilon^2 \int_0^\ell \int_{I_\varepsilon^\delta} \left(\frac{\tilde{\psi}_s}{1+\kappa\varepsilon\eta} \right)_s \tilde{\psi}\, d\eta\, ds \qquad \text{(by (6.2))} \\
&= \mu - \varepsilon^2 \int_0^\ell \int_{I_\varepsilon^\delta} \frac{\tilde{\psi}_s^2}{1+\kappa\varepsilon\eta}\, d\eta\, ds \qquad \text{(by (2.14))}
\end{aligned}
$$

we get

$$
\begin{aligned}
\text{(6.14)} \qquad C_0 \|\phi\|_{\varepsilon s}^2 &+ \varepsilon^2 \int_0^\ell \int_{I_\varepsilon^\delta} \frac{\tilde{\psi}_s^2}{1+\kappa\varepsilon\eta}\, d\eta\, ds \\
&\leq \mu - \nu \int_0^\ell \theta^2(s)\, ds - O(e^{-c/\varepsilon}) \int_0^\ell \theta(s)\, ds .
\end{aligned}
$$

Now utilizing

$$
\begin{aligned}
&\mu \leq C\varepsilon^2 , && \text{(by hypothesis)} \\
&-\nu \leq C\varepsilon^2 , && \text{(by Proposition 5.4)} \\
&\left| \int_0^\ell \theta(s)\, ds \right| \leq \sqrt{\ell} \left(\int_0^\ell \theta^2(s)\, ds \right)^{1/2} \leq \sqrt{\ell} , && \text{(by (6.12))}
\end{aligned}
$$

with C being a generic constant, in (6.14), we obtain

$$C_0\|\phi\|_{\varepsilon s}^2 + \varepsilon^2 \int_0^\ell \int_{I_\varepsilon^\delta} \frac{\tilde{\psi}_s^2}{1+\kappa\varepsilon\eta}\, d\eta\, ds \le C\varepsilon^2 . \tag{6.15}$$

Hence

$$\|\phi\|_{\varepsilon s} \le C\varepsilon , \tag{6.16}$$

$$\int_0^\ell \int_{I_\varepsilon^\delta} \frac{\tilde{\psi}_s^2}{1+\kappa\varepsilon\eta}\, d\eta ds \le C . \tag{6.17}$$

Now (6.9) follows from (6.7) and (6.16), while (6.10) follows from (6.17) and the assumption $0 < \delta_* < 1 + \kappa\varepsilon\eta$. The proof is complete. □

Remark 6.2. We already know that $\|R - \xi^\varepsilon \dot{U}\|_\varepsilon \to 0$, as $\varepsilon \to 0$, where $\xi^\varepsilon \to \|\dot{U}\|^{-1} := \xi_*$, by Lemma 5.3, and so Lemma 6.1 in light of the separation result of Lemma 2.2, says that $\theta \to \xi_*^{-1}\Theta$, in L^2, along a sequence of ε's converging to 0.

Remark 6.3. From (6.11) and (6.16), we obtain

$$1 - C^2\varepsilon^2 \le \int_0^\ell \theta^2(s)\, ds \le 1 . \tag{6.18}$$

Moreover since the left hand side of (6.14) is nonnegative we get

$$\mu \ge \nu \int_0^\ell \theta^2(s)\, ds + O(e^{-c/\varepsilon}) ,$$

which because of (6.18) and (5.18) becomes

$$\mu \ge \nu - O(\varepsilon^4) \ge -\overline{C}\varepsilon^2 , \tag{6.19}$$

with $\overline{C}$ independent of ε. The estimate (6.19) was first obtained in [deM–S2] in the case where $\Omega = \mathbb{R}^N$ and $V = 0$ in the definition of u_ε (see (2.7), (2.9)). In

[A–F1] it is proved that the same estimate holds for a bounded domain Ω, in the presence of a nonzero V satisfying the orthogonality condition (2.8).

Remark 6.4. Let ψ_i, ψ_j be critical eigenfunctions of (2.10), and let $\tilde{\psi}_i$, $\tilde{\psi}_j$ be as in (6.1). Then starting from

$$\int_\Omega \psi_i \psi_j \, dx = \int_{\Omega_\delta} \psi_i \psi_j \, dx + \int_{\Omega \setminus \Omega_\delta} \psi_i \psi_j \, dx ,$$

and changing variables in the middle integral, we obtain via (2.12)

$$\int_\Omega \psi_i \psi_j \, dx = \int_0^\ell \int_{I_\varepsilon^\delta} \tilde{\psi}_i \tilde{\psi}_j \,(1 + \kappa \varepsilon \eta)\, d\eta \, ds + O\big(e^{-c/\varepsilon}\big) .$$

Now using the decomposition (6.7) we get from the equality above

$$\int_\Omega \psi_i \psi_j \, dx = \int_0^\ell \int_{I_\varepsilon^\delta} \phi_i \phi_j \,(1 + \kappa \varepsilon \eta)\, d\eta \, ds + \int_0^\ell \theta_i \theta_j \, ds + O\big(e^{-c/\varepsilon}\big) .$$

Finally utilizing (6.16) in the last equality, we obtain

$$\int_\Omega \psi_i \psi_j \, dx = \int_0^\ell \theta_i \theta_j \, ds + O\big(\varepsilon^2\big) . \tag{6.20}$$

Theorem 6.5 *Let $\{\psi_i\}_{i=1}^m$ be eigenfunctions of (2.10) with $\|\psi_i\|_{L^2(\Omega_\delta)} = 1$, corresponding to the eigenvalues $\{\mu_i\}_{i=1}^m$, satisfying $\mu_i \le C\varepsilon^2$, $i = 1, 2, \ldots, m$, and let $\{\tilde{\psi}_i\}_{i=1}^m$ be the associated stretched eigenfunctions normalized by $\|\tilde{\psi}_i\|_{\varepsilon s} = 1$, $i = 1, 2, \ldots, m$. Then the following hold : for each $i = 1, 2, \ldots, m$*

$$\|\tilde{\psi}_i - \dot{U} \Theta_{j_i}\|_{\varepsilon s} \to 0 , \tag{6.21}$$

$$\frac{\mu_i}{\varepsilon^2} \to \lambda_{j_i} , \tag{6.22}$$

as $\varepsilon \to 0$, where Θ_{j_i} and λ_{j_i} are eigenfunctions and eigenvalues of the problem

$$\begin{cases} -\,\Theta''(s) + \hat{\lambda}(s)\Theta(s) = \lambda \Theta(s) , & 0 < s < \ell \\ \Theta(0) = \Theta(\ell) , \quad \Theta'(0) = \Theta'(\ell) & \end{cases} \tag{6.23}$$

where $\hat{\lambda}(s)$ is defined in (5.18). Moreover if $\{\lambda_i\}_{i=1}^m$ are eigenvalues of (6.23), then there is an $\varepsilon_0 > 0$ and there are eigenvalues $\{\mu_{j_i}\}_{i=1}^m$ of (2.10) such that

$$\mu_{j_i} = \varepsilon^2 \lambda_i + O(\varepsilon^3), \qquad i = 1, 2, 3, ..., m \tag{6.24}$$

for all $0 < \varepsilon < \varepsilon_0$.

Proof. Let $(\mu, \tilde{\psi})$ be an eigenpair satisfying the hypothesis.

Step 1. We show that along a sequence of ε's $\mu/\varepsilon^2 \to \lambda$, and $\tilde{\psi} \to \dot{U}\Theta$, where λ is an eigenvalue of (6.23) and Θ is one of its corresponding eigenfunctions.

Proof of Step 1. Starting as in Lemma 6.1 from the decomposition $\tilde{\psi} = \phi + \theta R$, where $\theta = \left\langle \tilde{\psi}, R \right\rangle_\varepsilon$, and $\left\langle \phi, R \right\rangle_\varepsilon = 0$, and (6.2), we get

$$\mathcal{L}_\eta^\varepsilon \phi - \nu \phi = \frac{\varepsilon^2}{1 + \kappa\varepsilon\eta} \left(\frac{\tilde{\psi}_s}{1 + \kappa\varepsilon\eta} \right)_s - \nu \tilde{\psi} + \mu \tilde{\psi} .$$

Multiplying by the test function $h(s)R(\eta, s)$ and integrating with respect to η and s, we have

$$\begin{aligned} \int_0^\ell \left\langle \mathcal{L}_\eta^\varepsilon \phi - \nu\phi, R \right\rangle_\varepsilon h(s)\, ds &= \varepsilon^2 \int_0^\ell \int_{I_\varepsilon^\delta} \left(\frac{\tilde{\psi}_s}{1 + \kappa\varepsilon\eta} \right)_s R(\eta, s) h(s)\, d\eta\, ds \\ &\quad - \int_0^\ell \nu \left\langle \tilde{\psi}, R \right\rangle_\varepsilon h(s)\, ds + \mu \int_0^\ell \left\langle \tilde{\psi}, R \right\rangle_\varepsilon h(s)\, ds . \end{aligned} \tag{6.25}$$

Integrating by parts twice in the left hand side of (6.25) and using on the one hand the orthogonality condition $\left\langle \phi, R \right\rangle_\varepsilon = 0$, and on the other the fact that $\tilde{\psi}_\eta(\pm\delta/\varepsilon) = \phi_\eta(\pm\delta/\varepsilon) = O(e^{-c/\varepsilon})$, the left hand side becomes

$$O(e^{-c/\varepsilon}) \int_0^\ell h(s)\, ds .$$

Moreover, since $\tilde{\psi}_s(\cdot, 0) = \tilde{\psi}_s(\cdot, \ell)$, the first term of the right hand side equals

$$-\varepsilon^2 \int_0^\ell \int_{I_\varepsilon^\delta} \frac{\tilde{\psi}_s}{1 + \kappa\varepsilon\eta} \Big(R_s(\eta, s) h(s) + R(\eta, s) h_s(s) \Big)\, d\eta\, ds .$$

Hence (6.25) takes the form

$$O(e^{-c/\varepsilon})\int_0^\ell h(s)\,ds = -\varepsilon^2\int_0^\ell\int_{I_\varepsilon^\delta}\frac{\tilde{\psi}_s}{1+\kappa\varepsilon\eta}R_s(\eta,s)h(s)\,d\eta\,ds$$
$$-\varepsilon^2\int_0^\ell\int_{I_\varepsilon^\delta}\frac{\tilde{\psi}_s}{1+\kappa\varepsilon\eta}R(\eta,s)h_s(s)\,d\eta\,ds - \int_0^\ell \nu\theta(s)h(s)\,ds + \mu\int_0^\ell \theta(s)h(s)\,ds\,.$$

Dividing by ε^2 and utilizing the facts

$$\lim_{\varepsilon\to 0}\|R-\xi^\varepsilon\dot{U}\|_\varepsilon \to 0\,, \qquad \text{(Lemma 5.3)}$$
$$\lim_{\varepsilon\to 0}\frac{\nu}{\varepsilon^2} = \hat{\boldsymbol{\lambda}}(s) \qquad \text{(Proposition 5.4)}$$
$$\|R_s\|_\varepsilon \le C\varepsilon\,, \qquad \text{(Lemma 5.5)}$$
$$\|\tilde{\psi}_s\|_{\varepsilon s} \le C\,, \qquad \text{(by (6.10))}$$
$$\theta \to \xi_*^{-1}\Theta \quad \text{along a sequence of } \varepsilon\text{'s} \qquad \text{(Remark 6.2)}$$
$$-\overline{C} \le \frac{\mu}{\varepsilon^2} \le C\,, \qquad \text{(by hypothesis and (6.19))}$$

we obtain by passing to the limit along a sequence of ε's for which $R_s \to 0$, $\theta \to \xi_*^{-1}\Theta$, and $\mu/\varepsilon^2 \to \lambda$,

$$\int_0^\ell \left[\Theta_{ss} + \left(\lambda - \hat{\boldsymbol{\lambda}}(s)\right)\Theta\right]h(s)\,ds = 0\,.$$

Hence μ/ε^2 converges along a sequence of ε's to an eigenvalue of the problem (6.23) while $\tilde{\psi}$ converges to $\dot{U}\Theta$, where Θ is a corresponding eigenfunction. This completes the proof of Step 1.

In general the eigenvalues of (6.23) are not simple and their multiplicity is at most two. Let $\lambda_1 < \lambda_2 < \ldots$ be the sequence of the distinct eigenvalues. Let λ_n be such an eigenvalue and let Θ_n be one of its (possibly two) corresponding eigenfunctions.

Step 2. We show that there is an ε_n and an eigenvalue μ_{j_n} of (2.10) such that

$$|\varepsilon^2\lambda_n - \mu_{j_n}| \le \varepsilon^3(\tilde{c}_n + 1)$$

for $0 < \varepsilon < \varepsilon_n$, where $\tilde{c}_n$ is a constant independent of ε.

Proof of Step 2. With $\omega : \mathbb{R} \to \mathbb{R}$ an even C^∞ cut–off function as in the definition of u_ε (see (2.9)), and $x \in \Omega$, we define

$$\hat{\psi}_n(x) = \sigma_n^\varepsilon \omega\left(\frac{r}{2\delta}\right)\left[\dot{U}\left(\frac{r}{\varepsilon}\right) + \varepsilon\kappa(s)\hat{R}_1\left(\frac{r}{\varepsilon}\right) + \varepsilon^2\hat{R}_2\left(\frac{r}{\varepsilon}, s\right)\right]\Theta_n(s), \tag{6.26}$$

where r and s are the local coordinates of x in $\overline{\Omega}_{2\delta}$ (see (2.4)), $\hat{R}_1$ and $\hat{R}_2$ are the unique orthogonal to $\dot{U}$ solutions of (5.22) and (5.25) respectively and σ_n^ε a normalization constant so that $\|\hat{\psi}_n\|_{L^2(\Omega)} = 1$. We point out that the definition of $\hat{\psi}_n$ makes sense for all $x \in \Omega$ because outside $\overline{\Omega}_{2\delta}$ it is $\hat{\psi}_n \equiv 0$. If $\mathcal{L}^\varepsilon$ is the operator defined in (2.10) we will show that $\hat{\psi}_n$ is an approximate eigenfunction of $\mathcal{L}^\varepsilon$ corresponding to the approximate eigenvalue $\varepsilon^2\lambda_n$, in the sense that

$$\begin{cases} \mathcal{L}^\varepsilon\hat{\psi}_n = \varepsilon^2\lambda_n\hat{\psi}_n + r_n\,, & \text{in } \Omega \\ \dfrac{\partial\hat{\psi}_n}{\partial n} = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{6.27}$$

where the error term r_n satisfies

$$\|r_n\|_{L^2(\Omega)} \le \tilde{c}_n\varepsilon^3\,. \tag{6.28}$$

- *Verification of (6.27) and (6.28).* We omit the subscript n. Noting that

$$\mathcal{L}^\varepsilon\hat{\psi} = \begin{cases} \mathcal{L}^\varepsilon_{r,s}\hat{\psi}\,, & x \in \overline{\Omega}_{2\delta} \\ 0\,, & x \in \Omega \setminus \overline{\Omega}_{2\delta} \end{cases}$$

where $\mathcal{L}^\varepsilon_{r,s}$ is as in (2.13), we consider the cases: $x \in \overline{\Omega}_\delta$, and $x \in \overline{\Omega}_{2\delta} \setminus \overline{\Omega}_\delta$. First recall that

$$\mathcal{L}^\varepsilon_{r,s}\hat{\psi} = \mathcal{L}^\varepsilon_r\hat{\psi} + \varepsilon^2\mathcal{L}^\varepsilon_s\hat{\psi}\,,$$

with

$$\mathcal{L}^\varepsilon_r\hat{\psi} = -\frac{\varepsilon^2}{1+\kappa r}\left((1+\kappa r)\hat{\psi}_r\right)_r + f'(u_\varepsilon)\hat{\psi}\,, \qquad \mathcal{L}^\varepsilon_s\hat{\psi} = -\frac{1}{1+\kappa r}\left(\frac{\hat{\psi}_s}{1+\kappa r}\right)_s\,,$$

where u_ε is as in (2.9). For $x \in \overline{\Omega}_\delta$ it is $\omega \equiv 1$, hence

$$\hat{\psi} = \sigma^\varepsilon \left[\dot{U}\left(\frac{r}{\varepsilon}\right) + \varepsilon\kappa(s)\hat{R}_1\left(\frac{r}{\varepsilon}\right) + \varepsilon^2\hat{R}_2\left(\frac{r}{\varepsilon}, s\right)\right]\Theta(s),$$
$$u_\varepsilon = U\left(\frac{r}{\varepsilon}\right) + \varepsilon\kappa(s)V\left(\frac{r}{\varepsilon}\right) + \varepsilon^2 W\left(\frac{r}{\varepsilon}, s\right).$$

Starting from

$$\mathcal{L}_r^\varepsilon\hat{\psi} = -\varepsilon^2\hat{\psi}_{rr} + f'(U)\hat{\psi} - \frac{\varepsilon^2\kappa}{1+\kappa r}\hat{\psi}_r + [f'(U_\varepsilon) - f'(U)]\hat{\psi},$$

and using (5.3), (5.22), and (5.25) we compute

$$\frac{1}{\sigma^\varepsilon\Theta}\mathcal{L}_r^\varepsilon\hat{\psi} = \varepsilon^2\hat{\boldsymbol{\lambda}}\dot{U} + \left(\varepsilon\kappa - \varepsilon\kappa^2 r - \frac{\varepsilon\kappa}{1+\kappa r}\right)\ddot{U} + \left(\varepsilon^2\kappa^2 - \frac{\varepsilon^2\kappa^2}{1+\kappa r}\right)\hat{R}_1' + O(\varepsilon^3).$$

Therefore

$$\mathcal{L}_r^\varepsilon\hat{\psi} = \sigma^\varepsilon\left[\varepsilon^2\hat{\boldsymbol{\lambda}}\dot{U} - \frac{\varepsilon\kappa^3 r^2}{1+\kappa r}\ddot{U} + \frac{\varepsilon^2\kappa^3 r}{1+\kappa r}\hat{R}_1'\right]\Theta + O(\varepsilon^3).$$

Now since

$$\int_0^\ell\int_{-\delta}^\delta\left\{\frac{\varepsilon\kappa^3 r^2}{1+\kappa r}\ddot{U}\left(\frac{r}{\varepsilon}\right)\Theta\right\}^2(1+\kappa r)\,dr\,ds$$
$$\overset{(r=\varepsilon\eta)}{=} \varepsilon^7\int_0^\ell\int_{I_\varepsilon^\delta}\frac{(\kappa^3\eta^2)^2}{1+\kappa\varepsilon\eta}\ddot{U}^2(\eta)\Theta^2\,d\eta\,ds = O(\varepsilon^7)$$

and

$$\int_0^\ell\int_{-\delta}^\delta\left\{\frac{\varepsilon^2\kappa^3 r}{1+\kappa r}\hat{R}_1'\left(\frac{r}{\varepsilon}\right)\Theta\right\}^2(1+\kappa r)\,dr\,ds$$
$$\overset{(r=\varepsilon\eta)}{=} \varepsilon^7\int_0^\ell\int_{I_\varepsilon^\delta}\frac{(\kappa^3\eta)^2}{1+\kappa\varepsilon\eta}\hat{R}_1'^2(\eta)\Theta^2\,d\eta\,ds = O(\varepsilon^7)$$

we finally get

$$\mathcal{L}_r^\varepsilon\hat{\psi} = \sigma^\varepsilon\varepsilon^2\hat{\boldsymbol{\lambda}}\dot{U}\Theta + O(\varepsilon^3).$$

Moreover

$$\varepsilon^2 \mathcal{L}_s^\varepsilon \hat{\psi} = -\varepsilon^2 \hat{\psi}_{ss} - \varepsilon^2 r \left[\frac{\kappa^2 r - 2\kappa}{(1+\kappa r)^2} \hat{\psi}_{ss} - \frac{\kappa_s}{(1+\kappa r)^3} \hat{\psi}_s \right],$$

where κ_s is the derivative of the curvature κ. Now using the definition of $\hat{\psi}$, we find

$$\int_0^\ell \int_{-\delta}^{\delta} \varepsilon^4 r^2 \left\{ \frac{\kappa^2 r - 2\kappa}{(1+\kappa r)^2} \hat{\psi}_{ss} - \frac{\kappa_s}{(1+\kappa r)^3} \hat{\psi}_s \right\}^2 (1+\kappa r)\, dr\, ds = O(\varepsilon^7)$$

by changing to stretched variables. Therefore we can write

$$\varepsilon^2 \mathcal{L}_s^\varepsilon \hat{\psi} = -\varepsilon^2 \sigma^\varepsilon \dot{U} \Theta_{ss} + O(\varepsilon^3).$$

Now

$$\begin{aligned}
\mathcal{L}_{r,s}^\varepsilon \hat{\psi} &= \mathcal{L}_r^\varepsilon \hat{\psi} + \varepsilon^2 \mathcal{L}_s^\varepsilon \hat{\psi} \\
&= \varepsilon^2 \sigma^\varepsilon \dot{U} \left(-\Theta_{ss} + \hat{\lambda} \Theta\right) + O(\varepsilon^3) \\
&= \varepsilon^2 \sigma^\varepsilon \dot{U} \lambda \Theta + O(\varepsilon^3) \\
&= \varepsilon^2 \lambda \sigma^\varepsilon \left[\dot{U} + \varepsilon \kappa \hat{R}_1 + \varepsilon^2 \hat{R}_2\right] \Theta + O(\varepsilon^3) \\
&= \varepsilon^2 \lambda \hat{\psi} + A^\varepsilon,
\end{aligned}$$

with $A^\varepsilon = O(\varepsilon^3)$. Similarly for $x \in \overline{\Omega}_{2\delta} \setminus \overline{\Omega}_\delta$ we obtain

$$\mathcal{L}_{r,s}^\varepsilon \hat{\psi} = \varepsilon^2 \lambda \hat{\psi} + B^\varepsilon,$$

with $B^\varepsilon = O(\varepsilon^3)$, by observing that for ε small and $|r| > \delta$, $\dot{U}$, $\hat{R}_1$, and $\hat{R}_2$ are exponentially small, and that since ω does not depend on s the s part of the operator is not affected by the presence of ω and so the eigenvalue λ. Thus (6.27) and (6.28) are proved.

Choose ε_n so that

$$\varepsilon_n(\tilde{c}_n + 1) < \min\{\lambda_{n+1} - \lambda_n,\ \lambda_n - \lambda_{n-1}\}.$$

By our assumption the right hand side is positive and so such an ε_n exists. Then the interval

$$I_n^\varepsilon := \big[\lambda_n - \varepsilon(\tilde{c}_n + 1),\ \lambda_n + \varepsilon(\tilde{c}_n + 1)\big]$$

contains no other eigenvalue of (6.23) for all $0 < \varepsilon < \varepsilon_n$. For $k > \tilde{c}_n + 1$ define

$$I_{n,k}^\varepsilon := \left[\varepsilon^2\lambda_n - \varepsilon^3\frac{\tilde{c}_n + 1}{k},\ \varepsilon^2\lambda_n + \varepsilon^3\frac{\tilde{c}_n + 1}{k}\right].$$

Then either

$$\left(I_{n,k}^\varepsilon + \left(-\varepsilon^3\frac{k-1}{k}(\tilde{c}_n + 1),\ \varepsilon^3\frac{k-1}{k}(\tilde{c}_n + 1)\right)\right) \cap \left(\sigma(\mathcal{L}^\varepsilon) \setminus I_{n,k}^\varepsilon\right) \neq \varnothing,$$

or

$$\left(I_{n,k}^\varepsilon + \left(-\varepsilon^3\frac{k-1}{k}(\tilde{c}_n + 1),\ \varepsilon^3\frac{k-1}{k}(\tilde{c}_n + 1)\right)\right) \cap \left(\sigma(\mathcal{L}^\varepsilon) \setminus I_{n,k}^\varepsilon\right) = \varnothing.$$

In the first case there is an eigenvalue μ_{j_n} of $\mathcal{L}^\varepsilon$ such that

$$|\varepsilon^2\lambda_n - \mu_{j_n}| < \varepsilon^3(\tilde{c}_n + 1).$$

In the second case Lemma 2.3 applies with $\varepsilon^* = \tilde{c}_n\varepsilon^3$, $a = ((k-1)/k)(\tilde{c}_n + 1)\varepsilon^3$, $E = \operatorname{span}\{\hat{\psi}_n\}$, $F =$ closed subspase associated to $\sigma(\mathcal{L}^\varepsilon) \cap I_{n,k}^\varepsilon$, and $\lambda_* = 1$, therefore

$$\bar{d}(E, F) = d(\hat{\psi}_n, F) \le \frac{\tilde{c}_n\varepsilon^3}{[(k-1)/k](\tilde{c}_n + 1)\varepsilon^3} = \frac{\tilde{c}_n}{\tilde{c}_n + 1}\frac{k}{k-1} < 1,$$

by the choice of k. Hence $I_{n,k}^\varepsilon$ contains an eigenvalue μ_{j_n} of $\mathcal{L}^\varepsilon$. Thus in both cases there is an eigenvalue μ_{j_n} of $\mathcal{L}^\varepsilon$ such that

$$|\varepsilon^2\lambda_n - \mu_{j_n}| \le \varepsilon^3(\tilde{c}_n + 1), \tag{6.29}$$

for $0 < \varepsilon < \varepsilon_n$. In particular

$$\lim_{\varepsilon \to 0} \frac{\mu_{j_n}}{\varepsilon^2} = \lambda_n \, .$$

The proof of Step 2 is complete.

From (6.29) we draw the conclusion that the convergence $\mu/\varepsilon^2 \to \lambda$, in Step 1, happens in a continuous manner as $\varepsilon \to 0$ and not just along a sequence of ε's. Moreover since Θ in the limit $\tilde{\psi} \to \dot{U}\Theta$ is the eigenfunction corresponding to λ, the limit above is independent of the sequence. Thus $\tilde{\psi} \to \dot{U}\Theta$, as $\varepsilon \to 0$.

Now given $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ distinct eigenvalues of (6.23) (multiplicities are not counted), with corresponding eigenfunctions $\Theta_1, \Theta_2, \ldots, \Theta_m$ we form the approximate eigenfunctions $\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_m$ as in (6.26). Then for

$$c_m = \max\{\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_m\}$$

$$\varepsilon_0 \le \frac{1}{c_m + 1} \min_{1 \le i \le m-1} \{\lambda_{i+1} - \lambda_i\}$$

we have that there are eigenvalues $\mu_{j_1}, \mu_{j_2}, \ldots, \mu_{j_m}$ of $\mathcal{L}^\varepsilon$ such that

$$|\varepsilon^2 \lambda_i - \mu_{j_i}| \le (c_m + 1)\varepsilon^3 \, ,$$

$i = 1, 2, \ldots, m$, for all $0 < \varepsilon < \varepsilon_0$. $\square$

Despite the fact that in more than one dimensions no general statement about the multiplicity of the eigenvalues can be made, because of the connection of the μ's with the λ's we have the following

Corollary 6.6. *Let $\mu_i \le \mu_{i+1} \le \mu_{i+2}$, be three consecutive critical eigenvalues of $\mathcal{L}^\varepsilon$. Then μ_k/ε^2, $k = i, i+1, i+2$ cannot converge to the same λ. Hence the multiplicity of the critical eigenvalues is at most two, for all ε small.*

Proof. From the Theorem above we have that

$$\frac{\mu_i}{\varepsilon^2} \le \frac{\mu_{i+1}}{\varepsilon^2} \le \frac{\mu_{i+2}}{\varepsilon^2}$$

converge, as $\varepsilon \to 0$, to some $\lambda_{j_i} \leq \lambda_{j_{i+1}} \leq \lambda_{j_{i+2}}$. Now if ψ_i, ψ_{i+1}, ψ_{i+2} are the orthogonal to each other critical eigenfunctions corresponding to the μ's above, and if Θ_{j_i}, $\Theta_{j_{i+1}}$,$\Theta_{j_{i+2}}$ are associated to the ψ's in the sense of (6.21) (i.e., $\psi \sim \tilde{\psi} \to \dot{U}\Theta$), then from (6.20), we obtain by passing to the limit $\varepsilon \to 0$

$$\delta_{kl} = \|\dot{U}\|^2 \int_0^\ell \Theta_{j_k}\Theta_{j_l}\, ds\,, \qquad k,l \in \{i, i+1, i+2\}\,. \tag{6.30}$$

Since the Θ's are the corresponding eigenfunctions to λ's the assumption that $\lambda_{j_i} = \lambda_{j_{i+1}} = \lambda_{j_{i+2}} = \lambda$ leads to the conclusion that Θ_{j_i}, $\Theta_{j_{i+1}}$,$\Theta_{j_{i+2}}$, orthogonal to each other by (6.30), are eigenfunctions corresponding to λ. This is a contradiction because the multiplicity of the λ's is at most two. $\square$

Remark 6.7. From the proof of Corollary 6.6 we see that if λ is a simple eigenvalue of (6.23) and $\mu/\varepsilon^2 \to \lambda$, then μ is simple as well. Indeed if we assume that μ has multiplicity two and ψ_i, ψ_j are the orthogonal to each other eigenfunctions corresponding to μ, then from (6.30) we see that

$$0 = \int_0^\ell \Theta_i\Theta_j\, ds\,,$$

where Θ_i and Θ_j are eigenfunctions of (6.23) corresponding to λ. But this contradicts the simplicity of λ.

Corollary 6.8. *Let $\mu_1 < \mu_2 \leq \cdots \leq \mu_m$ be the first m (counting multiplicities) eigenvalues of the $\mathcal{L}^\varepsilon$ operator. Then*

$$\lim_{\varepsilon \to 0} \frac{\mu_i}{\varepsilon^2} = \lambda_i\,,$$

$i = 1, 2, \ldots, m$, where $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_m$ are the first m eigenvalues of (6.23).

Proof. First we show that

$$\lim_{\varepsilon \to 0} \frac{\mu_1}{\varepsilon^2} = \lambda_1\,.$$

From the Theorem we have that there exists an eigenvalue of (6.23) λ^* such that $\mu_1/\varepsilon^2 \to \lambda^*$. On the other hand Step 2 of the proof of the Theorem yields an eigenvalue μ° such that $\mu^\circ/\varepsilon^2 \to \lambda_1$. Then it follows that $\mu^\circ \leq \mu_1$, for all ε sufficiently small. Since the principal eigenvalue is simple it follows that $\mu^\circ = \mu_1$, which in turn implies that $\lambda_1 = \lambda^*$.

Next we show that if

$$\lim_{\varepsilon \to 0} \frac{\mu_i}{\varepsilon^2} = \lambda_i\,, \qquad i = 1, 2, \ldots, k\,,$$

then

$$\lim_{\varepsilon \to 0} \frac{\mu_{k+1}}{\varepsilon^2} = \lambda_{k+1}\,.$$

Clearly

$$\lim_{\varepsilon \to 0} \frac{\mu_{k+1}}{\varepsilon^2} \geq \lim_{\varepsilon \to 0} \frac{\mu_k}{\varepsilon^2} = \lambda_k\,. \tag{6.31}$$

We argue by contradiction. First assume that

$$\lim_{\varepsilon \to 0} \frac{\mu_{k+1}}{\varepsilon^2} = \lambda_l > \lambda_{k+1}$$

Then there exists a $\mu^\circ \neq \mu_{k+1}$ such that $\mu^\circ/\varepsilon^2 \to \lambda_{k+1}$.

<u>Case 1.</u> $\lambda_k < \lambda_{k+1}$.

Then $\mu_k < \mu^\circ < \mu_{k+1}$, for all ε sufficiently small.

<u>Case 2.</u> $\lambda_k = \lambda_{k+1}$.

Since the multiplicity of the μ's is at most two (Corollary 6.6), we conclude that $\mu^\circ \notin \{\mu_1, \mu_2, \ldots, \mu_{k-1}\}$. Thus $\mu_k \leq \mu^\circ < \mu_{k+1}$, for all ε sufficiently small. In both cases the conclusion is that μ° is the $k+1$st term in the sequence of the eigenvalues. Hence $\mu^\circ = \mu_{k+1}$, a contradiction. This contradiction shows that $\lambda_l \leq \lambda_{k+1}$. On the other hand if we assume that

$$\lim_{\varepsilon \to 0} \frac{\mu_{k+1}}{\varepsilon^2} = \lambda_l < \lambda_{k+1}\,,$$

then $l = k$, so we consider

<u>Case 1.</u> $\lambda_{k-1} < \lambda_k < \lambda_{k+1}$.

In this case we have that λ_k is the limit of μ_k/ε^2 and μ_{k+1}/ε^2, which is a contradiction by Remark 6.7.

<u>Case 2.</u> $\lambda_{k-1} = \lambda_k < \lambda_{k+1}$.

If this happens then we have that μ_{k-1}/ε^2, μ_k/ε^2 and μ_{k+1}/ε^2 converge to λ_k, which is a contradiction by Corollary 6.6. Thus $\lambda_l = \lambda_{k+1}$. We show that $l = k+1$. Again we argue by contradiction. So assume that $l \neq k+1$.

<u>Case 1.</u> $l = k$.

Then since the multiplicity of λ's is at most two we get $\lambda_k = \lambda_{k+1} < \lambda_{k+2}$. Now by Corollary 6.6 we conclude that $l = k+1$ which is a contradiction.

<u>Case 2.</u> $l = k+2$.

Then $\lambda_k < \lambda_{k+1} = \lambda_{k+2} < \lambda_{k+3}$. Set $\lambda = \lambda_{k+1} = \lambda_{k+2}$. Then applying the same argument as in Step 2 of the proof of Theorem 6.5, we conclude that λ is the limit of exactly two (Remark 2.5 and Corollary 6.6) μ "eigenvalues," namely μ_{k+1}/ε^2 and μ_{k+2}/ε^2. This would imply that $l = k+1$ which is a contradiction. The proof of the Corollary is complete. $\square$

Remark 6.9. It is known, see for example [Co–L], that the eigenvalues λ of (6.23) satisfy

$$\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \lambda_5 < \cdots < \lambda_{2n} \leq \lambda_{2n+1} < \cdots .$$

Thus from Corollary 6.8 we conclude that given m, there exists an $\varepsilon_m > 0$, such that the first m eigenvalues μ of $\mathcal{L}^\varepsilon$ satisfy the same order relation, namely

$$\mu_1 < \mu_2 \leq \mu_3 < \mu_4 \leq \mu_5 < \cdots < \mu_{2n} \leq \mu_{2n+1} ,$$

for all $0 < \varepsilon < \varepsilon_m$ and $2n+1 \leq m$.

DISCUSSION

We conclude this work with some remarks, observations and speculations originating from the already mentioned results. These topics will serve as departure points for further research.

▶ The results regarding the asymptotic form of the critical eigenvalues, (Propositions 4.7 and 5.4 and Theorem 6.5) can be viewed as an attempt to study the variation of these eigenvalues with respect to the variation of the domain (stretched variables), or with respect to the diffusion coefficient ε^2. In that direction the interpretation of these results is

$$\left.\frac{d\mu}{d\varepsilon^2}\right|_{\varepsilon^2=0} = \lambda\,,$$

i.e. the derivative of $\mu = \mu(\varepsilon^2)$ at $\varepsilon^2 = 0$ exists. Here μ is a critical eigenvalue and λ is either an eigenvalue of the Sturm–Liouville problem (6.23), or $\lambda = \hat{\boldsymbol{\lambda}}$, in the case where μ is the principal eigenvalue of the radial part of the operator (Proposition 4.7). From this point of view these results are related to the ones about continuity properties of the eigenvalues with respect to the coefficients of the equation or the domain (see [Cou–H]).

▶ The eigenvalues of the linearized Cahn–Hilliard operator (see (1.3))

$$\text{(7.1)}\qquad \begin{cases} -\Delta(-\varepsilon^2\Delta\phi + f'(u_\varepsilon)\phi = \mu\phi\,, & \text{in } \Omega \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\Delta\phi}{\partial n} = 0\,, & \text{on } \partial\Omega \end{cases}$$

are characterized variationally ([B–F]) by

$$\mu_n = \max_{M_n} \min_{\substack{H \perp M_n \\ h \in \overline{W}_N^{2,2}}} = \frac{\displaystyle\int_\Omega \left[\varepsilon^2 |\nabla h|^2 + f'(u_\varepsilon) h^2\right] dx}{\displaystyle\int_\Omega H^2\, dx},$$

where

$$\overline{W}_N^{2,2} := W^{2,2}(\Omega) \cap \left\{ h : \int_\Omega h\, dx = 0\,, \ \frac{\partial h}{\partial n} = 0 \quad \text{on } \partial\Omega \right\},$$

$$\|h\|_{L^2(\Omega)} = 1\,, \qquad H = \left(-\Delta_N\right)^{-1/2} h\,,$$

$$M_n \subset L^2(\Omega)\,, \qquad \dim M_n = n - 1\,.$$

We note that Δ_N is the Laplacian with homogeneous Neumann conditions, and H is orthogonal to M_n in the usual L^2 sense. Now observe that the numerator in the Rayleigh quotient for μ_n coincides with the one for the eigenvalues of the Allen–Cahn operator which we have studied. This suggests that information about the spectrum of the Cahn–Hilliard operator can be obtained through the spectrum of the second order operator. In fact there are results in this direction ([A–B–F1], [A–F1,2], [B–F]).

▶ A radial eigenvalue problem similar to (3.1), that is relevant to the study of the slow motion of special solutions (bubbles) of the Cahn–Hilliard problem, is

$$(7.2) \qquad \begin{cases} \hat{\mathcal{L}} h := -\dfrac{\varepsilon^2}{r}(r h')' + f'(u_\varepsilon) h = \hat{\mu} h\,, & \rho - 1 < r < \rho + 1\,, \\ h'(\rho - 1) = h'(\rho + 1) = 0\,, \end{cases}$$

where now u_ε is a radial (increasing) equilibrium of (1.2) of constant mass m^*, with a layer in a small neighborhood of a circle of radius ρ ([A–F3], [A–McK], [A–S]). Hence u_ε satisfies

$$(7.3) \qquad \begin{cases} -\dfrac{\varepsilon^2}{r}(r u_\varepsilon')' + f(u_\varepsilon) = \sigma_\varepsilon\,, & \rho - 1 < r < \rho + 1\,, \\ u_\varepsilon'(\rho - 1) = u_\varepsilon'(\rho + 1) = 0\,, & \displaystyle\int_{\rho-1}^{\rho+1} u_\varepsilon(r)\, r\, dr = m^*, \end{cases}$$

where $\sigma_\varepsilon \to 0$, as $\varepsilon \to 0$. This u_ε is characterized variationally as a minimizer of the functional (1.1) subject to the mass constraint. For this u_ε it can be shown ([A–McK], [A–S]) that there are points r_1^ε and r_2^ε in $(\rho-1,\rho+1)$ with $r_1^\varepsilon < \rho < r_2^\varepsilon$ satisfying

$$|r_1^\varepsilon - r_2^\varepsilon| \le C\varepsilon \tag{7.4}$$

and such that the following estimates hold

$$\begin{aligned}
&\left.\begin{aligned}
&0 \le 1 - u_\varepsilon(r) \le Ce^{-(c/\varepsilon)(r-r_2^\varepsilon)}\,,\\
&0 \le u_\varepsilon'(r) \le (1/\varepsilon)Ce^{-(c/\varepsilon)(r-r_2^\varepsilon)}\,,\\
&|u_\varepsilon''(r)| \le (1/\varepsilon^2)Ce^{-(c/\varepsilon)(r-r_2^\varepsilon)}\,,
\end{aligned}\right\}\quad r_2^\varepsilon \le r \le \rho+1\\
&\left.\begin{aligned}
&0 \le u_\varepsilon(r) - 1 \le Ce^{-(c/\varepsilon)(r_1^\varepsilon-r)}\,,\\
&0 \le u_\varepsilon'(r) \le (1/\varepsilon)Ce^{-(c/\varepsilon)(r_1^\varepsilon-r)}\,,\\
&|u_\varepsilon''(r)| \le (1/\varepsilon^2)Ce^{-(c/\varepsilon)(r_1^\varepsilon-r)}\,,
\end{aligned}\right\}\quad \rho-1 \le r \le r_1^\varepsilon
\end{aligned}\tag{7.5}$$

where C, c are constants independent of ε. We note that the same estimates hold if we replace σ_ε with 0 and discard the mass constraint. In this case u_ε is an equilibrium of the Allen–Cahn problem (1.2), and the estimates above are like the ones provided by Lemma 3.2. In particular the results we are about to mention also hold in this case.

For the spectrum of (7.1) we have

Proposition 7.1. *There exist positive constants ε_0, and $\hat{C}$ independent of ε such that the spectrum of $\hat{\mathcal{L}}$ with homogeneous Neumann boundary conditions has the following structure:*

$$\hat{\mu}_1 < 0 < \hat{C} < \hat{\mu}_2 \le \hat{\mu}_3 \le \cdots \to \infty\,,$$

for $0 < \varepsilon < \varepsilon_0$. *Moreover*

$$\lim_{\varepsilon \to 0} \frac{\hat{\mu}_1}{\varepsilon^2} = -\frac{1}{\rho^2}.$$

The proof of the Proposition above is similar to the ones of Propositions 3.3 and 4.7 and so we only give a sketch of it.

Sketch of the Proof. Differentiating (7.2), and setting $w = u_\varepsilon'$, we obtain

$$\begin{cases} -\dfrac{\varepsilon^2}{r}(rw')' + f'(u_\varepsilon)w = -\dfrac{\varepsilon^2}{r^2}w, & \rho - 1 < r < \rho + 1, \\ w(\rho-1) = w(\rho+1) = 0. \end{cases} \tag{7.6}$$

From the variational characterization of the eigenvalues we obtain

$$\hat{\mu}_1 \le \frac{\displaystyle\int_{\rho-1}^{\rho+1} \left(\varepsilon^2 (w')^2 + f'(u_\varepsilon) w^2\right) r\, dr}{\displaystyle\int_{\rho-1}^{\rho+1} w^2\, r\, dr} + O\big(e^{-c/\varepsilon}\big),$$

and so via (7.6)

$$\hat{\mu}_1 \le -\varepsilon^2 \frac{\displaystyle\int_{\rho-1}^{\rho+1} w^2 r^{-2}\, r\, dr}{\displaystyle\int_{\rho-1}^{\rho+1} w^2\, r\, dr} + O\big(e^{-c/\varepsilon}\big) \le -\frac{\varepsilon^2}{(\rho-1)^2} + O\big(e^{-c/\varepsilon}\big) < 0,$$

for ε small.

Next we show that $\hat{\mu}_1/\varepsilon^2 \to -1/\rho^2$, as $\varepsilon \to 0$. Let h_1 be the principal eigenfunction of (7.2) that we can take to be positive, and normalized by

$$\int_{\rho-1}^{\rho+1} h_1^2\, r\, dr = 1.$$

Multiplying (7.2) by w and (7.6) by h, subtracting the resulting equations and integrating, we get

$$\hat{\mu}_1 \int_{\rho-1}^{\rho+1} h_1 w\, r\, dr = -\varepsilon^2 \int_{\rho-1}^{\rho+1} \frac{1}{r} h_1 w\, dr - \varepsilon^2 \int_{\rho-1}^{\rho+1} \big(h_1' w r - h_1 w' r\big)'\, dr.$$

Now using on the one hand that $h_1' = w = 0$ at $r = \rho \pm 1$, and on the other that $h_1 w'$ is exponentially small, by the above mentioned estimates, we get

$$\hat{\mu}_1 \int_{\rho-1}^{\rho+1} h_1 w \, r \, dr = -\varepsilon^2 \int_{\rho-1}^{\rho+1} \frac{1}{r} h_1 w \, dr - \varepsilon^2 O\big(e^{-c/\varepsilon}\big) . \tag{7.7}$$

We estimate the integrals in (7.7).

(i) Let ρ^ε be the point defined by $u_\varepsilon(\rho^\varepsilon) = 0$. Then $\rho^\varepsilon \in (r_1^\varepsilon, r_2^\varepsilon)$, and so $\rho^\varepsilon \to \rho$, as $\varepsilon \to 0$ by (7.4).

(ii) Defining $U_\varepsilon(\eta) = u_\varepsilon(\rho^\varepsilon + \varepsilon\eta)$, where $\eta = (r - \rho^\varepsilon)/\varepsilon$, we obtain from (7.3)

$$-U_\varepsilon'' - \frac{\varepsilon}{\rho_\varepsilon + \varepsilon\eta} U_\varepsilon' + f(U_\varepsilon) = \sigma_\varepsilon \,, \quad \eta \in I_\varepsilon \,, \quad U_\varepsilon'(\eta) = 0 \,, \quad \eta \in \partial I_\varepsilon \,, \tag{7.8}$$

where $I_\varepsilon = \big((\rho - \rho^\varepsilon - 1)/\varepsilon \,,\ (\rho - \rho^\varepsilon + 1)/\varepsilon\big)$. Recalling that u_ε is increasing, from the estimates (7.5) follows that $-1 \le u_\varepsilon \le 1$. Thus U_ε is bounded independently of ε. So using a diagonal argument we can find a sequence $\varepsilon_n \to 0$ such that the corresponding solutions U_{ε_n} converge in the C^2_{loc} topology i.e. in C^2 sense over compact sets in $\mathbb{R}$ (see Section 3.2), to the unique increasing solution of

$$-U'' + f(U) = 0 \,, \quad U(0) = 0 \,, \quad U(\pm\infty) = \pm 1 \,,$$

i.e. to the "heteroclinic." The uniqueness of such a U implies that the limit is independent of the sequence ε_n, thus U_ε converges to U, in the above mentioned sense, as $\varepsilon \to 0$.

(iii) Let V_ε be defined by $V_\varepsilon(\eta) = \sqrt{\varepsilon} h_1(\rho^\varepsilon + \varepsilon\eta)$. Then V_ε satisfies

$$\begin{cases} - V_\varepsilon'' - \dfrac{\varepsilon}{\rho^\varepsilon + \varepsilon\eta} V_\varepsilon' + f'(U_\varepsilon) V_\varepsilon = \hat{\mu}_1 V_\varepsilon \,, & \eta \in I_\varepsilon \,, \\ V_\varepsilon'(\eta) = 0 \,, \quad \eta \in \partial I_\varepsilon \,, & \displaystyle\int_{\partial I_\varepsilon} V_\varepsilon^2(\eta)\,(\rho^\varepsilon + \varepsilon\eta)\, d\eta = 1 \,. \end{cases} \tag{7.9}$$

Utilizing that $f'(U_\varepsilon)$ converges to $f'(U)$ as $\varepsilon \to 0$, uniformly on compact sets in $\mathbb{R}$, it follows via the maximum principle that V_ε satisfies estimates as in Lemma

3.4. Hence passing to the limit $\varepsilon \to 0$ in (7.9) we obtain that along a sequence of ε's V_ε converges to the solution of

$$(7.10) \qquad -V'' + f'(U)V = \mu^* V\,, \quad V = V(\eta)\,, \quad \eta \in \mathbb{R}\,, \quad \sqrt{\rho}\,\|V\|_{L^2(\mathbb{R})} = 1$$

Now $V \geq 0$ and since the maximum of the V_ε's is attained in a compact interval independent of ε, the maximum of V is attained in the same interval. Thus V is nontrivial and thus it is the principal eigenfunction of (7.10). Therefore $\mu^* = 0$ (see Remark 3.1), and because of simplicity $V = aU'$, where $a = \rho^{-1/2}\|U\|^{-1}_{L^2(\mathbb{R})}$. Again because of the independency of the limit from the sequence of ε's, we conclude that V_ε converges to aU', and $\hat{\mu}_1 \to 0$, as $\varepsilon \to 0$.

From the definition of w and u_ε follows that $U'_\varepsilon(\eta) = \varepsilon w(\rho^\varepsilon + \varepsilon\eta)$, hence

$$\begin{aligned}\int_{\rho-1}^{\rho+1} h_1 w\, r\, dr &= \int_{I_\varepsilon} h_1(\rho^\varepsilon + \varepsilon\eta) w(\rho^\varepsilon + \varepsilon\eta)\,(\rho^\varepsilon + \varepsilon\eta)\,\varepsilon d\eta \\ &= \frac{1}{\sqrt{\varepsilon}} \int_{I_\varepsilon} V_\varepsilon(\eta) U'_\varepsilon(\eta)\,(\rho^\varepsilon + \varepsilon\eta)\, d\eta\,.\end{aligned}$$

Similarly

$$\int_{\rho-1}^{\rho+1} \frac{1}{r} h_1 w\, dr = \frac{1}{\sqrt{\varepsilon}} \int_{I_\varepsilon} V_\varepsilon(\eta) U'_\varepsilon(\eta) \frac{1}{\rho^\varepsilon + \varepsilon\eta}\, d\eta\,.$$

Utilizing these expressions in (7.7) we obtain

$$\frac{\hat{\mu}_1}{\varepsilon^2} \int_{I_\varepsilon} V_\varepsilon(\eta) U'_\varepsilon(\eta)\,(\rho^\varepsilon + \varepsilon\eta)\, d\eta = -\int_{I_\varepsilon} V_\varepsilon(\eta) U'_\varepsilon(\eta) \frac{1}{\rho^\varepsilon + \varepsilon\eta}\, d\eta - \sqrt{\varepsilon}\, O\big(e^{-c/\varepsilon}\big)\,.$$

Because of the estimates (7.5) (in the stretched form), and the results of (i)–(iii) Lebesgue's Theorem applies and so taking limits as $\varepsilon \to 0$ in the equality above we have

$$\lim_{\varepsilon \to 0} \frac{\hat{\mu}_1}{\varepsilon^2} \rho\, a\, \|U\|^2_{L^2(\mathbb{R})} = -\frac{1}{\rho}\, a\, \|U\|^2_{L^2(\mathbb{R})}\,,$$

or equivalently

$$\lim_{\varepsilon \to 0} \frac{\hat{\mu}_1}{\varepsilon^2} = -\frac{1}{\rho^2}\,.$$

Finally the assertion that $\hat{\mu}_2 > \hat{C} > 0$ is proved in the same way as in the Proposition 3.4.

Remark. In the proof above we used an alternative way in showing that $\hat{\mu}_1 \to 0$ and $V_\varepsilon \to aU'$, or equivalently that the first term in the asymptotic expansion of the principal eigenfunction is U'. This method has been used in [A–B–F1] and [N–F].

▶ As we mentioned the result above also holds in the case where u_ε is an equilibrium of the Allen–Cahn problem (1.2). We thus conclude that there is no equilibrium of (1.2) with interior interface that is stable. For this type of circular interfaces equation (1.14) takes the form

$$\begin{cases} -\Theta''(s) - \dfrac{1}{\rho^2}\Theta(s) = \lambda\Theta(s), & 0 < s < \ell \\ \Theta(0) = \Theta(\ell), \quad \Theta'(0) = \Theta'(\ell) \end{cases}$$

with $\ell = 2\pi\rho$. We note that the first three, counting multiplicities, eigenvalues of the problem above are

$$\lambda_1 = -\frac{1}{\rho^2}, \qquad \text{and} \quad \lambda_2 = \lambda_3 = 0\,.$$

This suggests that for a general curve γ (1.14) has at least two negative eigenvalues with the extreme case being the circular interface. Apparently there is some interesting relationship between the spectrum and the isoperimetric inequality. A consequence of such result would be that any equilibrium of (1.2) with interior interface is at least 2–d unstable, and that any equilibrium of the Cahn–Hilliard with interior interface is unstable. These results would complement well the work on the complexity of stable equilibria for (1.4) done by Casten and Holland [Ca–H], Matano [M], and others, and will also provide information for the constrained problem (solutions of the Cahn–Hilliard with specified mass), and for systems.

▶ Our main theorem clarifies (1.13) but it does not include it, since the convergence

$$\frac{\mu_i}{\varepsilon^2} \to \lambda_i$$

can not be assumed uniform over all critical eigenvalues without justification. We expect such a uniform correspondence to hold; it would add significance to the Θ–equation. This in turn could be used for building the tangent space to an invariant manifold in L^2 over which the flow would be approximately determined by (1.5). That would be the dynamical systems approach for relating (1.2) to (1.5).

▶ We expect these results to have analogs in the Cahn–Hilliard setting.

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APPENDIX–FIGURES

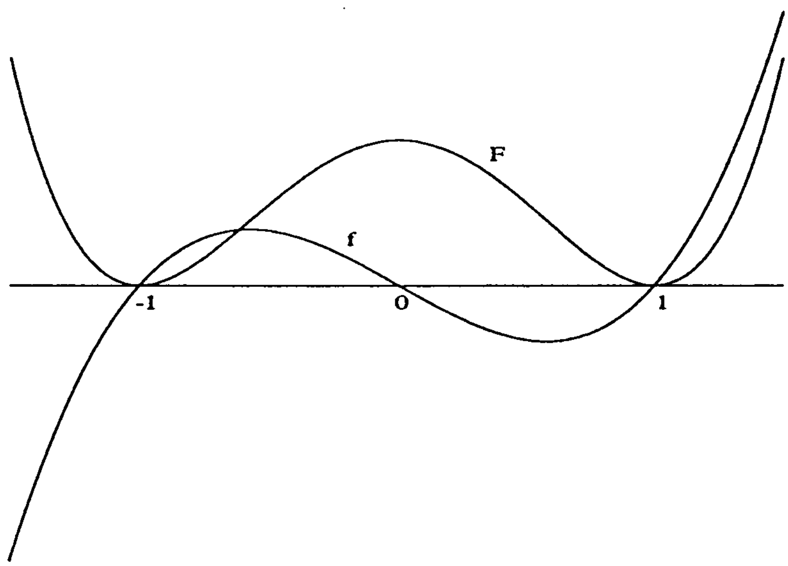

Figure 1. The functions F and $f = F'$.

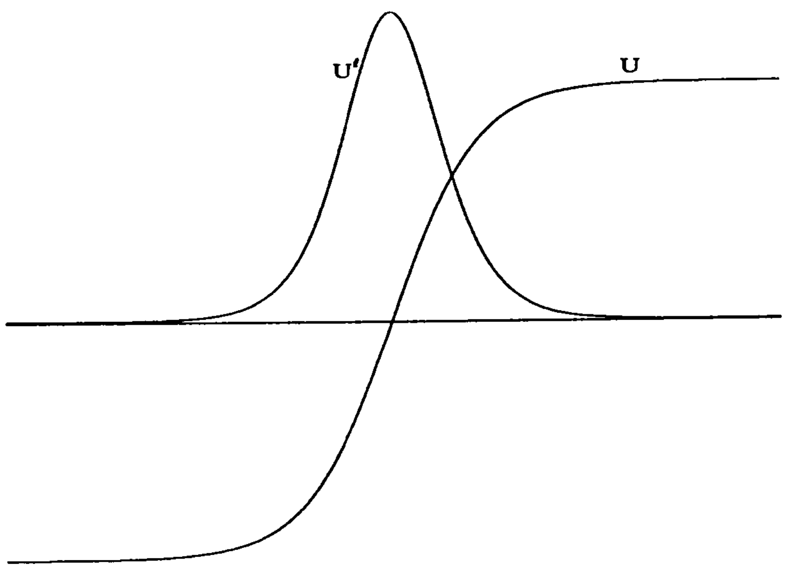

Figure 2. The heteroclinic U and its derivative U'.

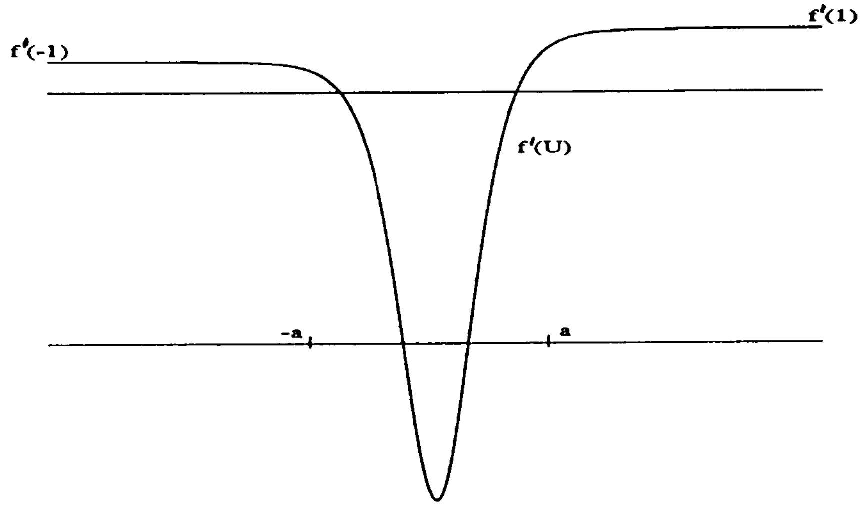

Figure 3. The potential $f'(U)$.

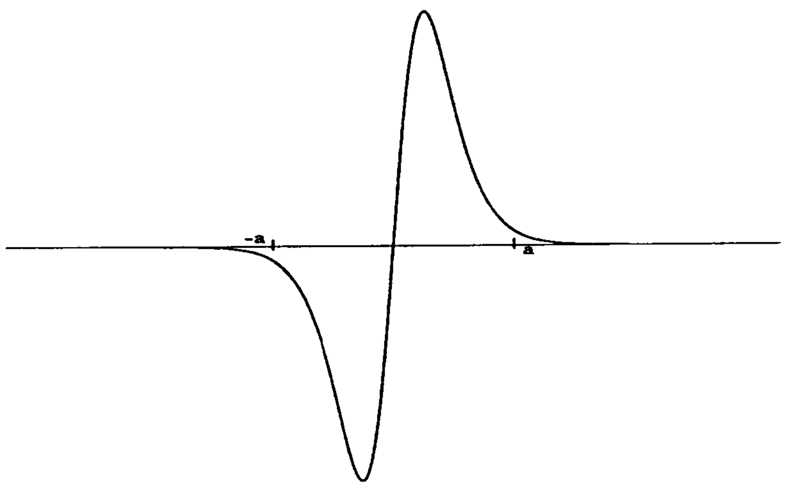

Figure 4. The second eigenfunction of $\mathcal{L}_{\rho,\varepsilon}$.

VITA

Vagelis Stefanopoulos was born in Athens, Greece, on February 25, 1958. He attended elementary and high school in that city. After graduating from high school he enrolled in the University of Patras, Greece, from where in 1984 he received a Bachelor of Science degree in Mathematics. In the same year Vagelis went as a graduate teaching assistant in the Department of Mathematics at Purdue University, Indiana, from where he received a Masters of Science degree, in 1986. In the Fall of the same year he came to Knoxville and started his study towards a Doctor of Philosophy degree in Mathematics.

Upon graduation from the University of Tennessee he will take the position of visiting assistant professor at the University of Maryland at Baltimore County.