

To the Graduate Council:

I am submitting herewith a thesis written by Mark John Lancaster entitled "Generalizations of the Stirling, Lah, and Cycle Numbers of the First and Second Kinds." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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Generalizations of the Stirling, Lah, and Cycle Numbers of the First and Second Kinds

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DEDICATION

This thesis is dedicated to my parents

Marvin and Eva Lancaster,

who sacrificed some of their dreams

so that I could attain some of mine.

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ABSTRACT

As Stirling, Lah, and cycle numbers of the second kind satisfy very similar recurrence relations, one might expect that these numbers can be derived from one recurrence relation. This is indeed true, and this one recurrence provides one of several ways to define the *SLC numbers of the second kind*, which generalize the Stirling, Lah, and cycle numbers of the second kind. In the first chapter, five equivalent ways to define SLC numbers of the first and second kind are presented, and relationships between the numbers of the first and second kind are explored. In the second chapter, special cases of the SLC numbers of the first and second kind are considered. In particular, we find that binomial and q -binomial coefficients are SLC numbers, and that the SLC numbers provide a generalization of Louis Comtet's generalization of Stirling numbers of the first and second kind.

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Chapter 1

SLC Numbers

1.1 Stirling, Lah, and Cycle Numbers of the Second Kind

Out of the many ways to view Stirling, Lah, and cycle numbers of the second kind, we first consider the role of these numbers as transition matrices between different bases of $\mathbf{C}[x]$, the $\mathbf{C}$ -vector space of polynomials with complex coefficients. The three bases we will consider are the standard basis $\{x^i\}_{i=0}^{\infty}$, and two others: $\{x^{\bar{i}}\}_{i=0}^{\infty}$ and $\{x^{\underline{i}}\}_{i=0}^{\infty}$, where $x^{\bar{i}} := x(x+1)(x+2)\cdots(x+(i-1))$ and $x^{\underline{i}} := x(x-1)(x-2)\cdots(x-(i-1))$ for $i > 0$, and $x^{\bar{0}} = x^{\underline{0}} := 1$. A proof that the latter two sets each are a basis for $\mathbf{C}[x]$ is deferred to Section 1.2. The $x^{\bar{i}}$'s are called *rising factorial polynomials*, and the $x^{\underline{i}}$'s are called *falling factorial polynomials*. The transition matrices between the above bases, which define Stirling, Lah, and cycle numbers of the second kind, are given below.

Definition 1.1: For all $n, k \in \mathbf{N}$, we define $S(n, k)$, the Stirling number of the second kind, via

$$x^n = \sum_{k \geq 0} S(n, k) x^{\bar{k}}, \quad (1.1.1)$$

$L(n, k)$, the Lah number of the second kind, via

$$x^{\bar{n}} = \sum_{k \geq 0} L(n, k) x^{\underline{k}}, \text{ and} \quad (1.1.2)$$

$C(n, k)$, the cycle number of the second kind, via

$$x^{\bar{n}} = \sum_{k \geq 0} C(n, k) x^{\bar{k}}. \quad (1.1.3)$$

In particular, if $k > n$, then $S(n, k) = L(n, k) = C(n, k) = 0$.

From the above definition, we can show the following recurrence relations hold:

For $n, k \in \mathbf{P}$, we have

$$S(n, k) = S(n-1, k-1) + kS(n-1, k), \quad (1.1.4)$$

$$L(n, k) = L(n-1, k-1) + (n-1+k)L(n-1, k), \text{ and} \quad (1.1.5)$$

$$C(n, k) = C(n-1, k-1) + (n-1)C(n-1, k), \quad (1.1.6)$$

subject to the boundary conditions

$$S(n, 0) = L(n, 0) = C(n, 0) = \delta_{n,0} \text{ for all } n \in \mathbf{N} \text{ and}$$

$$S(0, k) = L(0, k) = C(0, k) = 0 \text{ for all } k \in \mathbf{P}.$$

It is not difficult to show that the above recurrence relations (1.1.4) - (1.1.6) imply the equations (1.1.1) - (1.1.3) respectively. Hence, at this point, we may view the Stirling, Lah, and cycle numbers of the second kind in terms of matrices or in terms of recurrence relations.

1.2 Rising/Falling Factorial Polynomials and Difference Operators

In order to generalize equations (1.1.1) - (1.1.3), we consider bases other than $\{x^i\}_{i=0}^\infty$, $\{x^{\bar{i}}\}_{i=0}^\infty$, and $\{x^{\underline{i}}\}_{i=0}^\infty$. If we take $\langle a_n \rangle_{n \geq 0}$ and $\langle b_n \rangle_{n \geq 0}$ as sequences of complex numbers, we consider two new sets of polynomials $\{\rho_n(x)\}_{n \geq 0}^{\langle a_n \rangle}$ and $\{\varphi_n(x)\}_{n \geq 0}^{\langle b_n \rangle}$, where

$$\rho_n(x) = \prod_{i=0}^{n-1} (x + a_i) \text{ and} \quad (1.2.1)$$

$$\varphi_n(x) = \prod_{i=0}^{n-1} (x - b_i). \quad (1.2.2)$$

Using the convention for empty products, we have $\rho_0(x) = \varphi_0(x) = 1$. We will usually drop the superscripted reference to the sequence to streamline notation; it will be added whenever clarity is necessary. We say that $\{\rho_n(x)\}_{n \geq 0}^{\langle a_n \rangle}$ are *rising factorial polynomials with respect to* $\langle a_n \rangle_{n \geq 0}$ and that $\{\varphi_n(x)\}_{n \geq 0}^{\langle b_n \rangle}$ are *falling factorial polynomials with respect to* $\langle b_n \rangle_{n \geq 0}$. In particular, if we let $a_n = b_n = n$ for all $n \in \mathbf{N}$, then we have $\rho_n(x) = x^{\bar{n}}$ and $\varphi_n(x) = x^{\underline{n}}$. If we take $a_n = b_n = 0$, we have $\rho_n(x) = \varphi_n(x) = x^n$. In Section 1.3, we will need to compare the coefficients of $\rho_n(x)$ or $\varphi_n(x)$, so we need that $\{\rho_n(x)\}_{n \geq 0}$ and $\{\varphi_n(x)\}_{n \geq 0}$ are bases for $\mathbf{C}[x]$. This follows from the proposition below.

Proposition 1.2: Suppose that for all $n \in \mathbf{N}$, $P_n(x)$ is polynomial of degree n with coefficients in $\mathbf{C}$. Then $\{P_n(x)\}_{n \geq 0}$ is a basis for $\mathbf{C}[x]$.

Proof: It suffices to show that each non-zero polynomial $P \in \mathbf{C}[x]$ can be written as a unique linear combination of polynomials in $\{P_n(x)\}_{n \geq 0}$, with $P_n(x)$ assumed to be monic. The proof proceeds by induction on $n = \deg P(x)$. If $n = 0$, then $P(x) = \omega = \omega P_0(x)$, where $0 \neq \omega \in \mathbf{C}$, and if $P(x) = \omega = \sum_{k=0}^n \xi_k P_k(x)$, then $\xi_0 = \omega$ and $\xi_1 = \dots = \xi_n = 0$ by comparing degrees. This establishes the induction base. So, suppose that polynomials with degree $k \leq n$ can be written uniquely using elements of $\{P_n(x)\}_{n \geq 0}$, where $n \in \mathbf{P}$, and let $P \in \mathbf{C}[x]$ have degree $n+1$. If $P(x) = \omega_{n+1}x^{n+1} + \dots + \omega_1x + \omega_0$, then $P(x) - \omega_{n+1}P_{n+1}(x) = Q(x)$, where $\deg Q \leq n$. Thus, $Q(x) = \sum_{k=0}^n \xi_k P_k(x)$ uniquely, so that $P(x) = \omega_{n+1}P_{n+1}(x) + \sum_{k=0}^n \xi_k P_k(x)$. This shows that P can be written as a linear combination of the $P_n(x)$'s. We now show that such a representation is unique. Suppose that $P(x) = \sum_{k=0}^m \xi'_k P_k(x)$ as well. By comparing degrees, we find that $m = n+1$ and $\omega_{n+1} = \xi'_{n+1}$, so that $P(x) = \omega_{n+1}P_{n+1}(x) + \sum_{k=0}^n \xi'_k P_k(x)$. Hence, $Q(x) = P(x) - \omega_{n+1}P_{n+1}(x) = \sum_{k=0}^n \xi'_k P_k(x)$. Since $Q(x)$ can be written uniquely as a linear combination of the $P_n(x)$'s, $\xi'_k = \xi_k$ for $k = 0, \dots, n$. ■

For any polynomial $P \in \mathbf{C}[x]$, we define the sequence of difference operators $\langle \Delta_i^{(b_n)} \rangle_{i \geq 0}$ for all sequences $\langle b_n \rangle_{n \geq 0}$ in $\mathbf{C}$ by

$$\Delta_0^{(b_n)} P(x) = P(x) \text{ and}$$

$$\Delta_{i+1}^{\langle b_n \rangle} P(x) = \frac{\Delta_i^{\langle b_n \rangle} P(x) - \Delta_i^{\langle b_n \rangle} P(b_i)}{x - b_i}, \text{ where}$$

$$\Delta_i^{\langle b_n \rangle} P(b_i) := \left(\Delta_i^{\langle b_n \rangle} P(x) \right) \Big|_{x=b_i}.$$

It is easy to check that for each $i \in \mathbf{N}$, $\Delta_i^{\langle b_n \rangle}$ is a linear operator on $\mathbf{C}[x]$. To keep the notation at a minimum, the superscripted $\langle b_n \rangle$ will be suppressed; it will be added whenever clarification is necessary. Note that $\langle \Delta_i \rangle_{i \geq 0}$ behaves quite nicely on the basis $\{\varphi_n(x)\}_{n \geq 0}^{\langle b_n \rangle}$. Indeed, if $n \in \mathbf{N}$, we have

$$\begin{aligned} \Delta_0 \varphi_n(x) &= \varphi_n(x), \\ \Delta_1 \varphi_n(x) &= \frac{\varphi_n(x) - \varphi_n(b_0)}{x - b_0} = \frac{\varphi_n(x) - 0}{x - b_0} = \prod_{i=1}^{n-1} (x - b_i), \\ \Delta_2 \varphi_n(x) &= \frac{\prod_{i=1}^{n-1} (x - b_i) - \left(\prod_{i=1}^{n-1} (b_1 - b_i) \right)}{x - b_1} = \frac{\prod_{i=1}^{n-1} (x - b_i) - 0}{x - b_1} = \prod_{i=2}^{n-1} (x - b_i), \\ &\vdots \\ \Delta_n \varphi_n(x) &= \frac{\prod_{i=n-1}^{n-1} (x - b_i) - \prod_{i=n-1}^{n-1} (b_{n-i} - b_i)}{x - b_{n-1}} = \frac{1 - 0}{1} = 1, \text{ and} \\ \Delta_{n+1} \varphi_n(x) &= \frac{1 - 1}{x - b_n} = 0. \end{aligned}$$

Thus, if $i > n$, we have $\Delta_i \varphi_n(x) = 0$.

In addition to $\Delta_i^{\langle b_n \rangle}$, we also have the *forward difference operator* Δ and the *backward difference operator* ∇ , where

$$\Delta P(x) = P(x+1) - P(x) \text{ and}$$

$$\nabla P(x) = P(x) - P(x-1)$$

for all $P \in \mathbf{C}[x]$. We will also define $\Delta^0 P(x) = P(x)$ and $\nabla^0 P(x) = P(x)$. Finally, if $k \in \mathbf{P}$, we define $\Delta^k P(x) = \Delta(\Delta^{k-1} P(x))$ and $\nabla^k P(x) = \nabla(\nabla^{k-1} P(x))$. Obviously, both Δ and ∇ are linear operators on $\mathbf{C}[x]$, and it is easy to check that $\Delta(x^n) = nx^{n-1}$ and $\nabla(x^n) = nx^{n-1}$ for all $n \in \mathbf{P}$. Hence, we have

$$\Delta^k \left(x^{\overline{n}} \right) = n^{\overline{k}} x^{\overline{n-k}} = \begin{cases} 0 & \text{if } k > n \\ n^{\overline{n}} = n! & \text{if } k = n \end{cases} \text{ and} \quad (1.2.3)$$

$$\nabla^k \left(x^{\overline{n}} \right) = n^{\overline{k}} x^{\overline{n-k}} = \begin{cases} 0 & \text{if } k > n \\ n^{\overline{n}} = n! & \text{if } k = n \end{cases}. \quad (1.2.4)$$

Note that $\Delta_i^{(b_n)}$ does not reduce to Δ^i when $b_n = n$ for all $n \in \mathbf{N}$. For example, if $i = 1$, $\Delta^1 P(x) = \Delta P(x) = P(x+1) - P(x)$ and $\Delta_1 P(x) = \frac{P(x) - P(0)}{x}$, so $\Delta_1 \neq \Delta^1$.

1.3 Generalizations of the Stirling, Lah, and Cycle Numbers of the First and Second Kinds

If one considers the recurrences (1.1.4) - (1.1.6), one sees that they are all of the form

$$A(n, k) = A(n-1, k-1) + \xi(n, k)A(n-1, k), \quad (1.3.1)$$

where $\xi(n, k) \in \mathbf{C}$. If we assume the boundary condition

$$A(0, k) = \delta_{0,k}, \quad (1.3.2)$$

we are able to prove that $A(n, k) = 0$ whenever $k > n \geq 0$ and $A(n, n) = 1$ for all $n \in \mathbf{N}$.

Proposition 1.3 gives a proof by induction on n .

Proposition 1.3: Given a recurrence of the form (1.3.1) subject to the boundary condition (1.3.2), then for all $n \in \mathbf{N}$

$$A(n, k) = 0 \text{ for all } k > n, \text{ and} \quad (1.3.3)$$

$$A(n, n) = 1. \quad (1.3.4)$$

Proof: We prove this result by induction on n . If $n = 0$, we see that (1.3.3) and (1.3.4) follow directly from (1.3.2). So, suppose (1.3.3) and (1.3.4) hold for some $n \in \mathbf{P}$ and let $k > n+1$. By (1.3.1), we have $A(n+1, k) = A(n, k-1) + \xi(n+1, k)A(n, k) = 0$ since $k-1 > n$ and $k > n$. This proves (1.3.3). Also, $A(n+1, n+1) = A(n, n) + \xi(n+1, n+1)A(n, n+1) = 1 + \xi(n+1, n+1)0 = 1$ by (1.3.3) since $n+1 > n$. This establishes (1.3.4). ■

With the following definition and Theorems 1.6 & 1.9, we can obtain the Stirling, Lah, and cycle numbers of the first and second kind from one recurrence relation.

Definition 1.4: For all sequences $\langle a_n \rangle_{n \geq 0}, \langle b_n \rangle_{n \geq 0}$ in $\mathbf{C}$ and all $n, k \in \mathbf{N}$, we define the array $\left(A(n, k; \langle a_n \rangle, \langle b_n \rangle) \right)_{n, k \geq 0}$ of *SLC numbers of the second kind* by $\rho_n(x) = \sum_{k \geq 0} A(n, k; \langle a_n \rangle, \langle b_n \rangle) \varphi_k(x)$, where $\rho_n(x)$ and $\varphi_k(x)$ are defined by (1.2.1) and (1.2.2) respectively.

To condense notation, we will drop the references to the sequences; they will be added whenever clarification is necessary.

Lemma 1.5: For all polynomials $P(x) \in \mathbf{C}[x]$, we have

$$(1) \quad P(x) = \sum_{k \geq 0} \Delta_k^{(b_n)} P(b_k) \varphi_k(x),$$

and if $\langle b_n \rangle_{n \geq 0}$ is a sequence of distinct complex numbers, then

$$(2) \quad P(x) = \sum_{k \geq 0} \left(\sum_{j=0}^k \left(\frac{P(b_j)}{(b_j)_k} \right) \right) \varphi_k(x), \text{ where } (b_j)_k := \prod_{\substack{0 \leq i \leq k \\ i \neq j}} (b_j - b_i)$$

$$\text{and } (b_0)_0 := 1.$$

Proof: First, we establish (1): If we write $P(x) = \sum_{i \geq 0} \xi_i \varphi_i(x)$, then by the linearity of Δ_k

we have $\Delta_k P(x) = \sum_{i \geq 0} \xi_i \Delta_k \varphi_i(x) = \sum_{i \geq k} \xi_i \Delta_k \varphi_i(x)$ (since $i < k \Rightarrow \Delta_k \varphi_i(x) = 0$).

Thus, $\Delta_k P(x) = \sum_{i \geq k} \xi_i \Delta_k \varphi_i(x) = \xi_k \Delta_k \varphi_k(x) + \sum_{i > k} \xi_i \Delta_k \varphi_i(x) =$

$\xi_k + \sum_{i > k} \xi_i (x - b_k) \cdots (x - b_{i-1})$. Hence, $\Delta_k P(b_k) = \xi_k$, as desired.

Next, the proof of (2) (due to S. Mulay [Mul]) proceeds by induction on

$n = \deg P(x)$: We begin by showing that $P(x) = \sum_{k=0}^n \left(\sum_{j=0}^k \left(\frac{P(b_j)}{(b_j)_k} \right) \right) \varphi_k(x)$. If

$\deg P(x) = 0$, then $P(x) = c \neq 0$, and $\frac{P(b_0)}{(b_0)_0} \varphi_0(x) = \frac{c}{1} \cdot 1 = c$. Thus, our

induction base is established. Next, suppose that (2) holds for all polynomials of degree strictly smaller than n , and let $\deg P(x) = n$. By Lagrange interpolation, we may write

$$P(x) = \sum_{j=0}^n P(b_j) \Lambda_j(x), \text{ where} \quad (1.3.5)$$

$$\Lambda_j(x) = \frac{(x - b_0) \cdots (x - b_n)}{(x - b_j)(b_j)_n}. \quad (1.3.6)$$

In particular,

$$\Lambda_n(x) = \frac{(x - b_0) \cdots (x - b_{n-1})}{(x - b_n)(b_n)_n} = \frac{\varphi_n(x)}{(b_n)_n}. \quad (1.3.7)$$

Now, if $j < n$, then $\Lambda_j(x) - \frac{\varphi_n(x)}{(b_j)_n} = \frac{(x - b_0) \cdots (x - b_n)}{(x - b_j)(b_j)_n} - \frac{(x - b_0) \cdots (x - b_{n-1})}{(b_j)_n}$,

which is a polynomial of degree $n - 1$ since the b_j 's are distinct. So we define the polynomial

$$\psi_j(x) = \Lambda_j(x) - \frac{\varphi_n(x)}{(b_j)_n} \text{ for } 0 \leq j \leq n - 1. \quad (1.3.8)$$

Substituting (1.3.7) and (1.3.8) into (1.3.5) yields

$$P(x) = \left\{ \sum_{j=0}^{n-1} P(b_j) \left[\frac{\varphi_n(x)}{(b_j)_n} + \psi_j(x) \right] \right\} + P(b_n) \frac{\varphi_n(x)}{(b_n)_n} \quad (1.3.9)$$

By applying the induction hypothesis to each $\psi_j(x)$, we then have

$$\psi_j(x) = \sum_{k=0}^{n-1} \left(\sum_{i=0}^k \frac{\psi_j(b_i)}{(b_i)_k} \right) \varphi_k(x) \text{ for } 0 \leq j \leq n - 1. \quad (1.3.10)$$

However, for $i < n$, we see that $\varphi_n(b_i) = 0$. By (1.3.8), we see that

$$\Lambda_j(b_i) = \frac{\varphi_n(b_i)}{(b_j)_n} + \psi_j(b_i) = \psi_j(b_i) \text{ for } i < n. \quad (1.3.11)$$

But $\Lambda_j(b_i) = \delta_{ji}$ for $i < n$, and so

$$\psi_j(b_i) = \delta_{j,i} \text{ for } i < n. \quad (1.3.12)$$

Hence, (1.3.10) simplifies to

$$\psi_j(x) = \sum_{k=j}^{n-1} \frac{\varphi_k(x)}{(b_j)_k} \text{ for } 0 \leq j \leq n - 1. \quad (1.3.13)$$

By substituting (1.3.13) into (1.3.9), we conclude that

$$P(x) = \left\{ \sum_{j=0}^{n-1} P(b_j) \left[\frac{\varphi_n(x)}{(b_j)_n} + \sum_{k=j}^{n-1} \frac{\varphi_k(x)}{(b_j)_k} \right] \right\} + P(b_n) \frac{\varphi_n(x)}{(b_n)_n} =$$

$$\sum_{j=0}^{n-1} P(b_j) \sum_{k=j}^n \frac{\varphi_k(x)}{(b_j)_k} + P(b_n) \frac{\varphi_n(x)}{(b_n)_n} = \sum_{j=0}^n P(b_j) \sum_{k=j}^n \frac{\varphi_k(x)}{(b_j)_k} = \sum_{k=0}^n \left(\sum_{j=0}^k \frac{P(b_j)}{(b_j)_k} \right) \varphi_k(x). \text{ To}$$

finish the proof of (2), we need only show that $\sum_{j=0}^k \frac{P(b_j)}{(b_j)_k} = 0$ when $k > n$. Since

$P(x) = a_n x^n + \dots + a_1 x + a_0$, it suffices to show that

$$\sum_{j=0}^k \frac{b_j^r}{(b_j)_k} = 0 \text{ for } 0 \leq r \leq n \text{ when } k > n. \quad (1.3.14)$$

Let $\lambda_j(x) = \frac{(x-b_0) \cdots (x-b_k)}{(x-b_j)(b_j)_k}$ for $0 \leq j \leq k$, and define $f_r(x) = \sum_{j=0}^k b_j^r \lambda_j(x)$.

Note that $\deg f_r(x) \leq k$ since $\deg \lambda_j(x) = k$ for $0 \leq j \leq k$. Finally, we define the polynomial

$$g_r(x) = f_r(x) - x^r \text{ for } 0 \leq r \leq n. \quad (1.3.15)$$

Note that $g_r(b_j) = 0$ for $0 \leq j \leq k$ and $\deg g_r(x) \leq k$ if $k \geq n$. Since the b_j 's are distinct, we see that $g_r(x) = 0$ for $0 \leq r \leq n$ when $k \geq n$. By (1.3.15), we then have

$$f_r(x) = x^r \text{ for } 0 \leq r \leq n \text{ when } k \geq n. \quad (1.3.16)$$

By (1.3.16), we see that the leading term coefficient of $f_r(x)$ is

$$\sum_{j=0}^k \frac{b_j^r}{(b_j)_k} = \delta_{r,k} \text{ for } 0 \leq r \leq n \text{ when } k \geq n. \quad (1.3.17)$$

In particular, when $k > n$, we have $\sum_{j=0}^k \frac{b_j^r}{(b_j)_k} = 0$, which establishes (1.3.14).

Thus, $\sum_{j=0}^k \frac{P(b_j)}{(b_j)_k} = 0$ when $k > n$. Hence, the proof of (2) is complete. ■

Theorem 1.6: Given sequences $\langle a_n \rangle_{n \geq 0}, \langle b_n \rangle_{n \geq 0}$ in $\mathbf{C}$, the following are equivalent ways to specify the array $(A(n, k))_{n, k \geq 0}$:

$$(1) \quad \forall n, k \in \mathbf{N}, \rho_n(x) = \sum_{k \geq 0} A(n, k) \varphi_k(x).$$

$$(2) \quad \forall n, k \in \mathbf{P}, A(n, k) = A(n-1, k-1) + (a_{n-1} + b_k)A(n-1, k),$$

subject to the boundary conditions $A(0, k) = \delta_{0,k}$ and

$$A(n, 0) = \prod_{i=0}^{n-1} (a_i + b_0) \text{ for all } n, k \in \mathbf{N}.$$

$$(3) \quad \forall n, k \in \mathbf{P}, A(n, k) = \sum_{j=k}^n A(j-1, k-1) \prod_{i=j}^{n-1} (a_i + b_k)$$

subject to the same boundary conditions in (2).

$$(4) \quad \forall n, k \in \mathbf{N}, A(n, k) = \Delta_k^{(b_n)} \rho_n(b_k).$$

If one further assumes $\langle b_n \rangle_{n \geq 0}$ to be a sequence of distinct complex

numbers, then (1) - (4) are equivalent to

$$(5) \quad \forall n, k \in \mathbf{N}, A(n, k) = \sum_{j=0}^k \frac{\rho_n(b_j)}{(b_j)_k}.$$

Proof: (1) $\Rightarrow$ (2): If we suppose that $n \in \mathbf{P}$, we have

$$\begin{aligned} \sum_{k \geq 0} A(n, k) \varphi_k(x) &= \rho_n(x) = \rho_{n-1}(x)(x + a_{n-1}) = \left(\sum_{k \geq 0} A(n-1, k) \varphi_k(x) \right) (x + a_{n-1}) = \\ &= \sum_{k \geq 0} A(n-1, k) \varphi_k(x) (x - b_k + b_k + a_{n-1}) = \\ &= \sum_{k \geq 0} A(n-1, k) \varphi_k(x) (x - b_k) + \sum_{k \geq 0} A(n-1, k) \varphi_k(x) (b_k + a_{n-1}) = \\ &= \sum_{k \geq 0} A(n-1, k) \varphi_{k+1}(x) + \sum_{k \geq 0} A(n-1, k) \varphi_k(x) (b_k + a_{n-1}) = \\ &= \sum_{k \geq 1} A(n-1, k-1) \varphi_k(x) + \sum_{k \geq 0} (a_{n-1} + b_k) A(n-1, k) \varphi_k(x). \end{aligned}$$

By equating coefficients of the basis elements $\varphi_k(x)$, we find that for $n, k \in \mathbf{P}$,

$$A(n, k) = A(n-1, k-1) + (a_{n-1} + b_k)A(n-1, k). \text{ If } n = 0, \text{ we have}$$

$$\rho_0(x) = 1 = \sum_{k \geq 0} A(0, k) \varphi_k(x). \text{ Hence, } A(0, 0) = 1 \text{ (since } \varphi_0(x) = 1) \text{ and}$$

$$A(0, k) = 0 \text{ for } k \in \mathbf{P}. \text{ Finally, if } n \in \mathbf{P} \text{ and } k = 0, \text{ we compare the } \varphi_0(x) \text{'s and}$$

have $A(n, 0) = (a_{n-1} + b_0)A(n-1, 0)$. By iteration, we have

$$A(n, 0) = (a_{n-1} + b_0)(a_{n-2} + b_0) \cdots (a_0 + b_0)A(0, 0) = \prod_{i=0}^{n-1} (a_i + b_0).$$

(2) $\Rightarrow$ (1): We induct on n . If $n = 0$, we have that $A(0, k) = \delta_{0,k}$ by our boundary conditions, so that $\rho_0(x) = 1 = \sum_{k \geq 0} A(0, k)\varphi_k(x)$. Hence, we assume

that (1) is true for row $n-1$. Note that by Proposition 1.3, we may write

$$\begin{aligned} \sum_{k=0}^n A(n, k)\varphi_k(x) &= \sum_{k \geq 0} A(n, k)\varphi_k(x). \text{ Thus, } \rho_n(x) = \rho_{n-1}(x)(x + a_{n-1}) = \\ &= \sum_{k \geq 0} A(n-1, k)\varphi_k(x)(x - b_k + a_{n-1} + b_k) = \\ &= \sum_{k \geq 0} A(n-1, k)\varphi_k(x)(x - b_k) + \sum_{k \geq 0} A(n-1, k)\varphi_k(x)(a_{n-1} + b_k) = \\ &= \sum_{k \geq 0} A(n-1, k)\varphi_{k+1}(x) + \sum_{k \geq 0} (a_{n-1} + b_k)A(n-1, k)\varphi_k(x) = \\ &= \sum_{k \geq 1} A(n-1, k-1)\varphi_k(x) + \sum_{k \geq 1} (a_{n-1} + b_k)A(n-1, k)\varphi_k(x) + A(n-1, 0)(a_{n-1} + b_0) = \\ &= \sum_{k \geq 1} A(n, k)\varphi_k(x) + A(n, 0) = \sum_{k \geq 0} A(n, k)\varphi_k(x). \end{aligned}$$

(2) $\Rightarrow$ (3): Since the boundary conditions are the same, we assume $n, k \geq 1$.

Thus,

$$\begin{aligned} \sum_{j=k}^n A(j, k) \prod_{i=j}^{n-1} (a_i + b_k) &= \sum_{j=k}^n \left(A(j-1, k-1) + (a_{j-1} + b_k)A(j-1, k) \right) \prod_{i=j}^{n-1} (a_i + b_k) = \\ &= \sum_{j=k}^n A(j-1, k-1) \prod_{i=j}^{n-1} (a_i + b_k) + \sum_{j=k}^n A(j-1, k) \prod_{i=j-1}^{n-1} (a_i + b_k) = \\ &= \sum_{j=k}^n A(j-1, k-1) \prod_{i=j}^{n-1} (a_i + b_k) + \sum_{j=k-1}^{n-1} A(j, k) \prod_{i=j}^{n-1} (a_i + b_k) = \\ &= \sum_{j=k}^n A(j-1, k-1) \prod_{i=j}^{n-1} (a_i + b_k) + \sum_{j=k}^{n-1} A(j, k) \prod_{i=j}^{n-1} (a_i + b_k) \text{ since } A(k-1, k) = 0 \text{ by} \end{aligned}$$

Proposition 1.3. Hence, $A(n, k) = \sum_{j=k}^n A(j-1, k-1) \prod_{i=j}^{n-1} (a_i + b_k)$.

(3) $\Rightarrow$ (2): Again we assume $n, k \geq 1$. Hence,

$$\begin{aligned} A(n, k) &= \sum_{j=k}^n A(j-1, k-1) \prod_{i=j}^{n-1} (a_i + b_k) = \\ &= A(n-1, k-1) + \sum_{j=k}^{n-1} A(j-1, k-1) \prod_{i=j}^{n-1} (a_i + b_k) = \end{aligned}$$

$$A(n-1, k-1) + (a_{n-1} + b_k) \sum_{j=k}^{n-1} A(j-1, k-1) \prod_{i=j}^{n-2} (a_i + b_k) =$$

$A(n-1, k-1) + (a_{n-1} + b_k)A(n-1, k)$ by (3).

(1) $\Rightarrow$ (4): Applying (1) of Lemma 1.5 to the polynomial $\rho_n(x)$, we have

$$\rho_n(x) = \sum_{k \geq 0} \Delta_k \rho_n(b_k) \varphi_k(x). \text{ Since we have assumed } \rho_n(x) = \sum_{k=0}^n A(n, k) \varphi_k(x),$$

we conclude $A(n, k) = \Delta_k \rho_n(b_k)$ by Proposition 1.2.

(4) $\Rightarrow$ (1): If we assume $A(n, k) = \Delta_k \rho_n(b_k)$, then we immediately obtain from (1) of Lemma 1.5 that $\rho_n(x) = \sum_{k \geq 0} \Delta_k \rho_n(b_k) \varphi_k(x)$. Hence,

$$\rho_n(x) = \sum_{k=0}^n A(n, k) \varphi_k(x).$$

(1) $\Rightarrow$ (5): Applying (2) of Lemma 1.5 to polynomial $\rho_n(x)$, we find that

$$\rho_n(x) = \sum_{k \geq 0} \left(\sum_{j=0}^k \left(\frac{\rho_n(b_j)}{(b_j)_k} \right) \right) \varphi_k(x). \text{ Since we have assumed that}$$

$$\rho_n(x) = \sum_{k \geq 0} A(n, k) \varphi_k(x), \text{ we conclude that } A(n, k) = \sum_{j=0}^k \frac{\rho_n(b_j)}{(b_j)_k}$$

by Proposition 1.2.

(5) $\Rightarrow$ (1): If we define $A(n, k) = \sum_{j=0}^k \frac{\rho_n(b_j)}{(b_j)_k}$, then since by (2) of Lemma 1.5

$$\rho_n(x) = \sum_{k \geq 0} \left(\sum_{j=0}^k \left(\frac{\rho_n(b_j)}{(b_j)_k} \right) \right) \varphi_k(x), \text{ it follows that } \rho_n(x) = \sum_{k \geq 0} A(n, k) \varphi_k(x). \blacksquare$$

Remark 1.7: If we know that the term $a_{n-1} + b_k > 0$ in (2) of Theorem 1.6 for all

$n-1, k \in \mathbb{N}$ and $2b_k - b_{k-1} - b_{k+1} \geq 0$ for $1 < k < n$ ($n = 1, 2, 3, \dots$), then

by Theorem 2 in [Kur], we can conclude that the triangular array formed

by the $A(n, k)$'s has the strong logarithmic concavity property - i.e.,

$$A(n, k)^2 - A(n, k-1)A(n, k+1) > 0 \text{ for } 1 \leq k \leq n \text{ } (n = 1, 2, 3, \dots).$$

Definition 1.8: For all sequences $\langle a_n \rangle_{n \geq 0}, \langle b_n \rangle_{n \geq 0}$ in $\mathbf{C}$ and all $n, k \in \mathbf{N}$, we define the array $(a(n, k; \langle a_n \rangle, \langle b_n \rangle))_{n, k \geq 0}$ of SLC numbers of the first kind by $\varphi_n(x) = \sum_{k \geq 0} a(n, k; \langle a_n \rangle, \langle b_n \rangle) \rho_k(x)$, where $\rho_k(x)$ and $\varphi_n(x)$ are defined by (1.2.1) and (1.2.2) respectively.

Theorem 1.9: Given sequences $\langle a_n \rangle_{n \geq 0}, \langle b_n \rangle_{n \geq 0}$ in $\mathbf{C}$, the following are equivalent ways to specify the array $(a(n, k))_{n, k \geq 0}$:

$$(1) \quad \forall n, k \in \mathbf{N}, \quad \varphi_n(x) = \sum_{k \geq 0} a(n, k) \rho_k(x).$$

$$(2) \quad \forall n, k \in \mathbf{P}, \quad a(n, k) = a(n-1, k-1) - (b_{n-1} + a_k) a(n-1, k),$$

subject to the boundary conditions $a(0, k) = \delta_{0, k}$ and

$$a(n, 0) = (-1)^n \prod_{i=0}^{n-1} (b_i + a_0) \text{ for all } n, k \in \mathbf{N}.$$

$$(3) \quad \forall n, k \in \mathbf{P}, \quad a(n, k) = \sum_{j=k}^n (-1)^{n-j} a(j-1, k-1) \prod_{i=j}^{n-1} (b_i + a_k)$$

subject to the same boundary conditions in (2).

$$(4) \quad \forall n, k \in \mathbf{N}, \quad a(n, k) = \Delta_k^{(-a_n)} \varphi_n(-a_k).$$

If one further assumes $\langle a_n \rangle_{n \geq 0}$ to be a sequence of distinct complex numbers, then (1) - (4) are equivalent to

$$(5) \quad \forall n, k \in \mathbf{N}, \quad a(n, k) = \sum_{j=0}^k \frac{\varphi_n(-a_j)}{(-a_j)_k}.$$

Proof: Apply Theorem 1.6 using the fact that $\{\rho_n(x)\}_{n \geq 0}^{(-b_n)} = \{\varphi_n(x)\}_{n \geq 0}^{(b_n)}$ and

$$\{\varphi_n(x)\}_{n \geq 0}^{(-a_n)} = \{\rho_n(x)\}_{n \geq 0}^{(a_n)}, \text{ so that } a(n, k; \langle a_n \rangle, \langle b_n \rangle) = A(n, k; \langle -b_n \rangle, \langle -a_n \rangle). \quad \blacksquare$$

The next theorem shows that the SLC numbers of the first and second kind are inverses of each other as matrices.

Theorem 1.10: For $0 \leq j \leq n$, we have

$$\sum_{k=j}^n a(n, k)A(k, j) = \begin{cases} 0 & j < n \\ 1 & j = n \end{cases} \text{ and}$$

$$\sum_{k=j}^n A(n, k)a(k, j) = \begin{cases} 0 & j < n \\ 1 & j = n \end{cases}.$$

Proof: We will show the first equality; the second is similar. Note that $\varphi_n(x) =$

$$\sum_{k=0}^n a(n, k)\rho_k(x) = \sum_{k=0}^n \sum_{j=0}^k a(n, k)A(k, j)\varphi_k(x) = \sum_{j=0}^n \sum_{k=j}^n a(n, k)A(k, j)\varphi_k(x).$$

Comparing coefficients gives the desired result since $a(n, n) = A(n, n) = 1$ by

Proposition 1.3. ■

Using Theorem 1.10, we may now prove that the SLC numbers have an inversion principle.

Theorem 1.11: If $\langle c_n \rangle_{n \geq 0}$, $\langle d_n \rangle_{n \geq 0}$ are sequences in $\mathbf{C}$, then

$$\forall n \in \mathbf{N}, c_n = \sum_{k \geq 0} a(n, k)d_k \Leftrightarrow \forall n \in \mathbf{N}, d_n = \sum_{k \geq 0} A(n, k)c_k.$$

Proof: We will prove the $\Leftarrow$ direction; the $\Rightarrow$ direction is similar. Let $n \in \mathbf{N}$. Then

$$\sum_{k \geq 0} a(n, k)d_k = \sum_{k \geq 0} a(n, k) \sum_{j \geq 0} A(k, j)c_j.$$

Since both sums are finite, we can exchange the order of summation. Hence, $\sum_{k \geq 0} a(n, k)d_k =$

$$\sum_{j \geq 0} c_j \sum_{k \geq 0} a(n, k)A(k, j) = c_n \text{ by Theorem 1.10.} \quad \blacksquare$$

Chapter 2

Applications

2.1 Comtet Numbers of the Second Kind

Depending on $\langle b_n \rangle_{n \geq 0}$, Theorem 1.6 can yield up to five equivalent ways to define the array $(A(n, k))_{n, k \geq 0}$. If $a_n = 0$ for all $n \in \mathbf{N}$, we see that (1) of Theorem 1.6 reduces to

$$\forall n, k \in \mathbf{N}, \quad x^n = \sum_{k \geq 0} A(n, k) \varphi_k(x). \quad (2.1.1)$$

The array $(A(n, k))_{n, k \geq 0}$ defined by (2.1.1) is L. Comtet's generalization for Stirling numbers of the second kind, which appears in [Com]. In the case where $a_n = 0$ for all $n \in \mathbf{N}$, we refer to $A(n, k)$ as the *Comtet number of the second kind with respect to* $\langle b_n \rangle_{n \geq 0}$. From (2.1.1), Comtet then derives recurrences, generating functions, etc. that generalize classical formulas relating to Stirling numbers of the second kind.

Unfortunately, it is not mentioned in [Com] that these recurrences also produce equivalent ways to define $(A(n, k))_{n, k \geq 0}$. In the case where $a_n = 0$ for all $n \in \mathbf{N}$, we add three equivalent ways to define $(A(n, k))_{n, k \geq 0}$ to the five equivalences given by Theorem 1.6.

Theorem 2.1: For all sequences $\langle b_n \rangle_{n \geq 0}$ in $\mathbf{C}$, the following are equivalent ways to define the array $(A(n, k))_{n, k \geq 0}$:

- (1) $\forall n, k \in \mathbf{N}, \quad x^n = \sum_{k \geq 0} A(n, k) \varphi_k(x).$
- (2) $\forall n, k \in \mathbf{P}, \quad A(n, k) = A(n-1, k-1) + b_k A(n-1, k),$
subject to the boundary conditions $A(0, k) = \delta_{0, k}$ and
 $A(n, 0) = b_0^n$ for all $n, k \in \mathbf{N}.$
- (3) $\forall n, k \in \mathbf{P}, \quad A(n, k) = \sum_{j=k}^n A(j-1, k-1) b_k^{n-j}$

subject to the same boundary conditions in (2).

- (4) $\forall n, k \in \mathbf{N}, \quad A(n, k) = \Delta_k^{(b_n)}(x^n) \Big|_{x=b_k}.$

$$(5) \quad \forall n, k \in \mathbf{N}, \quad A(n, k) = \sum_{j=0}^k \frac{b_j^n}{(b_j)_k}, \text{ provided that } \langle b_n \rangle_{n \geq 0} \text{ is a}$$

sequence of distinct complex numbers.

$$(6) \quad \forall k \in \mathbf{N}, \quad \sum_{n \geq 0} A(n, k) \frac{x^n}{n!} = \sum_{j=0}^k \frac{e^{b_j x}}{(b_j)_k}, \text{ provided that } \langle b_n \rangle_{n \geq 0} \text{ is a}$$

sequence of distinct complex numbers.

$$(7) \quad \forall n, k \in \mathbf{N}, \quad A(n, k) = \sum_{\substack{d_0 + d_1 + \dots + d_k = n-k \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \dots b_k^{d_k}.$$

$$(8) \quad \forall k \in \mathbf{N}, \quad \sum_{n \geq 0} A(n, k) x^n = \frac{x^k}{(1 - b_0 x)(1 - b_1 x) \dots (1 - b_k x)}.$$

Proof: Substituting $a_n = 0$ for all $n \in \mathbf{N}$, we have $\rho_n(x) = x^n$ for all $n \in \mathbf{N}$. Theorem 1.6 then yields that (1) - (5) are equivalent.

$$(5) \Rightarrow (6): \text{ For } k \in \mathbf{N}, \quad \sum_{n \geq 0} A(n, k) \frac{x^n}{n!} = \sum_{n \geq 0} \left(\sum_{j=0}^k \frac{b_j^n}{(b_j)_k} \right) \frac{x^n}{n!} = \sum_{n \geq 0} \sum_{j=0}^k \frac{(b_j x)^n}{n! (b_j)_k} =$$

$$\sum_{j=0}^k \frac{1}{(b_j)_k} \sum_{n \geq 0} \frac{(b_j x)^n}{n!} = \sum_{j=0}^k \frac{1}{(b_j)_k} e^{b_j x}.$$

$$(6) \Rightarrow (5): \text{ For } k \in \mathbf{N}, \text{ we have } \sum_{n \geq 0} A(n, k) \frac{x^n}{n!} = \sum_{j=0}^k \frac{1}{(b_j)_k} e^{b_j x} =$$

$$\sum_{j=0}^k \frac{1}{(b_j)_k} \sum_{n \geq 0} \frac{(b_j x)^n}{n!} = \sum_{n \geq 0} \left(\sum_{j=0}^k \frac{b_j^n}{(b_j)_k} \right) \frac{x^n}{n!}. \text{ By comparing the coefficients of the } x^n \text{'s, we have } A(n, k) = \sum_{j=0}^k \frac{b_j^n}{(b_j)_k}.$$

(2) $\Rightarrow$ (8): We proceed by induction on k . For $k = 0$, we have

$$\sum_{n \geq 0} A(n, 0) x^n = \sum_{n \geq 0} (b_0 x)^n = \frac{1}{1 - b_0 x} = \frac{x^0}{1 - b_0 x}. \text{ Thus, we suppose that}$$

$$\sum_{n \geq 0} A(n, k-1) x^n = \frac{x^{k-1}}{(1 - b_0 x)(1 - b_1 x) \dots (1 - b_{k-1} x)} \quad (2.1.2)$$

Since $A(0, k) = 0$ for $k > 0$, we have $\sum_{n \geq 0} A(n, k) x^n = \sum_{n \geq 1} A(n, k) x^n =$

$$\sum_{n \geq 1} (A(n-1, k-1) + b_k A(n-1, k)) x^n = \sum_{n \geq 0} A(n, k-1) x^{n+1} + \sum_{n \geq 0} b_k A(n, k) x^{n+1} =$$

$$x \sum_{n \geq 0} A(n, k-1) x^n + b_k \sum_{n \geq 0} A(n, k) x^{n+1}. \text{ By (2.1.2), we now have}$$

$$\sum_{n \geq 0} A(n, k) x^n = x \cdot \frac{x^{k-1}}{(1-b_0 x)(1-b_1 x) \cdots (1-b_{k-1} x)} + b_k x \sum_{n \geq 0} A(n, k) x^n, \text{ so that}$$

$$\sum_{n \geq 0} A(n, k) x^n = \frac{x^k}{(1-b_0 x)(1-b_1 x) \cdots (1-b_k x)}.$$

$$(8) \Rightarrow (7): \text{ By (8), we have } \sum_{n \geq 0} A(n, k) x^n = \frac{x^k}{(1-b_0 x)(1-b_1 x) \cdots (1-b_k x)} =$$

$$x^k \left(\sum_{j \geq 0} (b_0 x)^j \sum_{j \geq 0} (b_1 x)^j \cdots \sum_{j \geq 0} (b_k x)^j \right) = x^k \sum_{j \geq 0} \left(\sum_{\substack{d_0 + d_1 + \cdots + d_k = j \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k} \right) x^j =$$

$$\sum_{j \geq 0} \left(\sum_{\substack{d_0 + d_1 + \cdots + d_k = j \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k} \right) x^{j+k} = \sum_{n \geq k} \left(\sum_{\substack{d_0 + d_1 + \cdots + d_k = n-k \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k} \right) x^n =$$

$$\sum_{n \geq 0} \left(\sum_{\substack{d_0 + d_1 + \cdots + d_k = n-k \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k} \right) x^n \text{ by the empty sum convention. Comparing}$$

$$\text{coefficients gives } A(n, k) = \sum_{\substack{d_0 + d_1 + \cdots + d_k = n-k \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k} \text{ for all } n, k \in \mathbf{N}.$$

$$(7) \Rightarrow (2): \text{ We begin by establishing the boundary conditions of (2). First, } A(0, 0) = \sum_{d_0=0} b_0^{d_0} = b_0^0 = 1. \text{ If } k \in \mathbf{P}, \text{ then } A(0, k) = \sum_{\substack{d_0 + d_1 + \cdots + d_k = -k \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k} = 0 \text{ by}$$

$$\text{the empty sum convention. If } n \in \mathbf{P}, \text{ then } A(n, 0) = \sum_{d_0=n} b_0^{d_0} = b_0^n. \text{ If we now}$$

$$\text{assume } n, k \in \mathbf{P}, \text{ we have } A(n, k) = \sum_{\substack{d_0 + d_1 + \cdots + d_k = n-k \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k} =$$

$$\sum_{\substack{d_0 + d_1 + \cdots + d_k = n-k \\ d_i \in \mathbf{N}, d_i = 0}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k} + \sum_{\substack{d_0 + d_1 + \cdots + d_k = n-k \\ d_i \in \mathbf{N}, d_i > 0}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k} =$$

$$\sum_{\substack{d_0 + d_1 + \cdots + d_{k-1} = n-k \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \cdots b_{k-1}^{d_{k-1}} + b_k \left(\sum_{\substack{d_0 + d_1 + \cdots + d_k = n-k \\ d_i \in \mathbf{N}, d_k > 0}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k-1} \right) =$$

$$\sum_{\substack{d_0 + d_1 + \cdots + d_{k-1} = (n-1)-(k-1) \\ d_i \in \mathbf{N}}} b_0^{d_0} b_1^{d_1} \cdots b_{k-1}^{d_{k-1}} + b_k \left(\sum_{\substack{d_0 + d_1 + \cdots + d_k = (n-1)-k \\ d_i \in \mathbf{N}, d_k' \geq 0}} b_0^{d_0} b_1^{d_1} \cdots b_k^{d_k'} \right) =$$

$$A(n-1, k-1) + b_k A(n-1, k). \blacksquare$$

In the sections that follow, we will examine special cases of Theorems 1.6 and 2.1. In particular, we will find that binomial coefficients, q -binomial coefficients, and the Stirling number of the second kind are Comtet numbers.

2.2 Binomial Coefficients

When we choose $a_n = 0$ and $b_n = 1$ for all $n \in \mathbf{N}$, (2) of Theorem 2.1 reduces to the following:

$$\begin{aligned} \forall n, k \in \mathbf{P}, \quad A(n, k) &= A(n-1, k-1) + A(n-1, k) \\ \text{subject to the boundary conditions } A(0, k) &= \delta_{0,k} \text{ and} \\ A(n, 0) &= 1 \text{ for all } n, k \in \mathbf{N}. \end{aligned}$$

Since this is the recurrence relation that binomial coefficients satisfy, we find that $A(n, k) = \binom{n}{k}$, which counts the number of k -element subsets of a set with n elements. Since the b_n 's are not distinct, (5) and (6) of Theorem 2.1 do not apply.

Corollary 2.2: The following are equivalent ways to define the array $\left(\binom{n}{k}\right)_{n, k \geq 0}$:

- (1) $\forall n, k \in \mathbf{N}, \quad x^n = \sum_{k \geq 0} \binom{n}{k} (x-1)^k.$
- (2) $\forall n, k \in \mathbf{P}, \quad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$
subject to the boundary conditions $\binom{0}{k} = \delta_{0,k}$ and $\binom{n}{0} = 1$
for all $n, k \in \mathbf{N}.$
- (3) $\forall n, k \in \mathbf{P}, \quad \binom{n}{k} = \sum_{j=k}^n \binom{j-1}{k-1}$
subject to the same boundary conditions in (2).
- (4) $\forall n, k \in \mathbf{N}, \quad \binom{n}{k} = \Delta_k (x^n) \Big|_{x=1}.$
- (5) $\forall n, k \in \mathbf{N}, \quad \binom{n}{k} = \sum_{\substack{d_0 + d_1 + \dots + d_k = n-k \\ d_i \in \mathbf{N}}} 1.$
- (6) $\forall k \in \mathbf{N}, \quad \sum_{n \geq 0} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}.$

Proof: Apply Theorem 2.1 with $b_n = 1$ for all $n \in \mathbf{N}$; (5) and (6) of Corollary 2.2

correspond to (7) and (8) of Theorem 2.1, respectively. Alternatively, one could also recover the binomial coefficients by using Theorem 1.6 with $a_n = 1$ and $b_n = 0$ for all $n \in \mathbf{N}$. ■

By (5) of Corollary 2.2, we find that $\binom{n}{k}$ is number of weak compositions of $n - k$ into k parts - i.e., the number of ways to write $n - k$ as a sum of k non-negative integers.

2.3 q -Binomial Coefficients

In this section, we choose $a_n = 0$ and $b_n = q^n$ for all $n \in \mathbf{N}$. Thus, (2) of Theorem 2.1 reduces to the following:

$$\begin{aligned} \forall n, k \in \mathbf{P}, \quad A(n, k) &= A(n-1, k-1) + q^k A(n-1, k) \\ \text{subject to the boundary conditions } A(0, k) &= \delta_{0,k} \text{ and} \\ A(n, 0) &= 1 \text{ for all } n, k \in \mathbf{N}. \end{aligned}$$

Since q -binomial coefficients satisfy the above recurrence relation, we see that $A(n, k) = \binom{n}{k}_q$. If q is a prime number raised to some positive power, $\binom{n}{k}_q$ gives the number of k -dimensional subspaces of an n -dimensional vector space whose underlying field is $\mathbf{F}_q$.

Corollary 2.3: The following are equivalent ways to define the array $\left(\binom{n}{k}_q\right)_{n, k \geq 0}$:

- (1) $\forall n, k \in \mathbf{N}, \quad x^n = \sum_{k \geq 0} \binom{n}{k}_q (x-1)(x-q) \cdots (x-q^{k-1}).$
- (2) $\forall n, k \in \mathbf{P}, \quad \binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q,$
subject to the boundary conditions $\binom{0}{k}_q = \delta_{0,k}$ and $\binom{n}{0}_q = 1$ for all $n, k \in \mathbf{N}$.

$$(3) \quad \forall n, k \in \mathbf{P}, \binom{n}{k}_q = \sum_{j=k}^n \binom{j-1}{k-1}_q (q^k)^{n-j}$$

subject to the same boundary conditions in (2).

$$(4) \quad \forall n, k \in \mathbf{N}, \binom{n}{k}_q = \Delta_k(x^n) \Big|_{x=q^k}.$$

$$(5) \quad \forall n, k \in \mathbf{N}, \binom{n}{k}_q = \sum_{j=0}^k \frac{(q^j)^n}{(q^j)_k}.$$

$$(6) \quad \forall k \in \mathbf{N}, \sum_{n \geq 0} \binom{n}{k}_q \frac{x^n}{n!} = \sum_{j=0}^k \frac{e^{q^j x}}{(q^j)_k}.$$

$$(7) \quad \forall n, k \in \mathbf{N}, \binom{n}{k}_q = \sum_{\substack{d_0 + d_1 + \dots + d_k = n-k \\ d_i \in \mathbf{N}}} q^{0d_0} q^{1d_1} \dots q^{kd_k}.$$

$$(8) \quad \forall k \in \mathbf{N}, \sum_{n \geq 0} \binom{n}{k}_q x^n = \frac{x^k}{(1-x)(1-qx) \dots (1-q^k x)}.$$

Proof: Apply Theorem 2.1 with $b_n = q^n$ for all $n \in \mathbf{N}$. ■

$$\text{From (7) of Corollary 2.3, we have } \binom{n}{k}_q = \sum_{\substack{d_0 + d_1 + \dots + d_k = n-k \\ d_i \in \mathbf{N}}} q^{0d_0 + 1d_1 + \dots + kd_k} = \sum_{t \geq 0} p(k, n-k, t) q^t,$$

where $p(k, n-k, t)$ denotes the number of partitions of t with at most $n-k$ parts, each no larger than k (cf. (1.20) of [Wag]).

2.4 Stirling Numbers of the Second Kind

When we let $a_n = 0$ and $b_n = n$ for all $n \in \mathbf{N}$, (1) of Theorem 2.1 simplifies to the following:

$$\forall n, k \in \mathbf{N}, x^n = \sum_{k \geq 0} A(n, k) x^{\underline{k}}.$$

Comparing the above to (1.1.1), we note that $A(n, k) = S(n, k)$, the Stirling number of the second kind. One interpretation of $S(n, k)$ is that it counts the number of ways to distribute n labeled balls among k unlabeled urns with no urn empty. A second interpretation, due to S. Milne in [Mil], is in terms of restricted growth functions. A surjective function

$f: [n] \rightarrow [k]$ is said to be of *restricted growth* if and only if

- (i) $f(1) = 1$, and
- (ii) $f(v+1) \leq 1 + \max_{1 \leq s \leq v} f(s)$.

In terms of the above functions, $S(n, k)$ counts the number of restricted growth functions $f: [n] \rightarrow [k]$. In the appendix, we give combinatorial arguments for (1) and (2) of Corollary 2.4 using the first interpretation of $S(n, k)$.

Corollary 2.4: The following are equivalent ways to define the array $(S(n, k))_{n, k \geq 0}$:

$$(1) \quad \forall n, k \in \mathbf{N}, \quad x^n = \sum_{k \geq 0} S(n, k) x^k.$$

$$(2) \quad \forall n, k \in \mathbf{P}, \quad S(n, k) = S(n-1, k-1) + kS(n-1, k),$$

subject to the boundary conditions $S(0, k) = \delta_{0,k}$ for all $k \in \mathbf{N}$

and $S(n, 0) = 0$ for all $n \in \mathbf{P}$.

$$(3) \quad \forall n, k \in \mathbf{P}, \quad S(n, k) = \sum_{j=k}^n S(j-1, k-1) k^{n-j}$$

subject to the same boundary conditions in (2).

$$(4) \quad \forall n, k \in \mathbf{N}, \quad S(n, k) = \Delta_k(x^n) \Big|_{x=k}.$$

$$(5) \quad \forall n, k \in \mathbf{N}, \quad S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n.$$

$$(6) \quad \forall k \in \mathbf{N}, \quad \sum_{n \geq 0} S(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}.$$

$$(7) \quad \forall n, k \in \mathbf{N}, \quad S(n, k) = \sum_{\substack{d_1 + \dots + d_k = n-k \\ d_i \in \mathbf{N}}} 1^{d_1} 2^{d_2} \dots k^{d_k}.$$

$$(8) \quad \forall k \in \mathbf{N}, \quad \sum_{n \geq 0} S(n, k) x^n = \frac{x^k}{(1-x)(1-2x) \dots (1-kx)}.$$

Proof: Apply Theorem 2.1 with $b_n = n$ for all $n \in \mathbf{N}$. Note that (5) of Theorem 2.1

simplifies to $S(n, k) = \sum_{j=0}^k \frac{j^n}{(j)_k}$. Further, $\sum_{j=0}^k \frac{j^n}{(j)_k} = \sum_{j=0}^k \frac{j^n}{(-1)^{k-j} j! (k-j)!} =$

$\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \frac{j^n}{k!} = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n$, which establishes (5) of Corollary 2.4.

Also, (6) of Theorem 2.1 gives $\sum_{n \geq 0} S(n, k) \frac{x^n}{n!} = \sum_{j=0}^k \frac{e^{jx}}{(j)_k}$, and $\sum_{j=0}^k \frac{e^{jx}}{(j)_k} = \sum_{j=0}^k \frac{e^{jx}}{(-1)^{k-j} j! (k-j)!} = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{e^{jx}}{k!} = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (e^x)^j (-1)^{k-j} = \frac{(e^x - 1)^k}{k!}$, which proves (6). ■

Note that from (1) of Corollary 2.5 and (1.2.3) it follows that $S(n, k) = \frac{\Delta^k (x^n) \Big|_{x=0}}{k!}$ for all $n, k \in \mathbf{N}$.

2.5 Stirling Numbers of the First Kind

The Stirling number of the first kind, denoted by $s(n, k)$, is usually defined by (1) of Corollary 2.5. Since (1) of Corollary 2.5 can be retrieved from Theorem 1.6 by choosing $a_n = -n$ and $b_n = 0$ for all $n \in \mathbf{N}$, we have $A(n, k) = s(n, k)$. Since the b_n 's are not distinct, (5) of Theorem 1.6 is not applicable.

Corollary 2.5: The following are equivalent ways to define the array $(s(n, k))_{n, k \geq 0}$:

- (1) $\forall n, k \in \mathbf{N}, x^n = \sum_{k \geq 0} s(n, k) x^k$.
- (2) $\forall n, k \in \mathbf{P}, s(n, k) = s(n-1, k-1) - (n-1)s(n-1, k)$, subject to the boundary conditions $s(0, k) = \delta_{0,k}$ for all $k \in \mathbf{N}$ and $s(n, 0) = 0$ for all $n \in \mathbf{P}$.
- (3) $\forall n, k \in \mathbf{P}, s(n, k) = \sum_{j=k}^n s(j-1, k-1) (-n)^{n-j}$
subject to the same boundary conditions in (2).
- (4) $\forall n, k \in \mathbf{N}, s(n, k) = \Delta_k \left(x^n \right) \Big|_{x=0}$.

Proof: Apply Theorem 1.6 with $a_n = -n$ and $b_n = 0$ for all $n \in \mathbf{N}$. Alternatively, one could also use Theorem 1.9 with $a_n = 0$ and $b_n = n$ for all $n \in \mathbf{N}$. ■

2.6 Cycle Numbers of the Second Kind

By putting $a_n = n$ and $b_n = 0$ for all $n \in \mathbf{N}$, (1) of Theorem 1.6 simplifies to the following:

$$\forall n, k \in \mathbf{N}, \quad x^{\overline{n}} = \sum_{k \geq 0} A(n, k) x^k.$$

Comparing the above to (1.1.3), we see that $A(n, k) = C(n, k)$, the cycle number of the second kind. Again, since the b_n 's are not distinct, (5) of Theorem 1.6 cannot be applied. We also note that cycle number of the second kind is usually referred to in the literature as the *signless Stirling number of the first kind*. One can prove that $C(n, k) = |s(n, k)|$ by verifying that $|s(n, k)|$ has the same recurrence relation and boundary conditions as $C(n, k)$ as in (2) below. An interpretation of $C(n, k)$ is that it counts the number of distributions of n labeled balls into k contents-ordered unlabeled urns with no urn empty, such that within each urn the smallest labeled ball occurs first. Combinatorial arguments for (1) and (2) are presented in the appendix.

Corollary 2.6: The following are equivalent ways to define the array $(C(n, k))_{n, k \geq 0}$:

- (1) $\forall n, k \in \mathbf{N}, \quad x^{\overline{n}} = \sum_{k \geq 0} C(n, k) x^k.$
- (2) $\forall n, k \in \mathbf{P}, \quad C(n, k) = C(n-1, k-1) + (n-1)C(n-1, k)$, subject to the boundary conditions $C(0, k) = \delta_{0,k}$ for all $k \in \mathbf{N}$ and $C(n, 0) = 0$ for all $n \in \mathbf{P}$.
- (3) $\forall n, k \in \mathbf{P}, \quad C(n, k) = \sum_{j=k}^n C(j-1, k-1) n^{n-j}$
subject to the same boundary conditions in (2).
- (4) $\forall n, k \in \mathbf{N}, \quad C(n, k) = \Delta_k \left(x^{\overline{n}} \right) \Big|_{x=0}.$

Proof: Apply Theorem 1.6 with $a_n = n$ and $b_n = 0$ for all $n \in \mathbf{N}$. ■

2.7 Lah Numbers of the Second Kind

When we let $a_n = n$ and $b_n = n$ for all $n \in \mathbf{N}$, (1) of Theorem 1.6 simplifies to the following:

$$\forall n, k \in \mathbf{N}, \quad x^{\overline{n}} = \sum_{k \geq 0} A(n, k) x^{\frac{n}{k}}.$$

Comparing the above to (1.1.2), we find that $A(n, k) = L(n, k)$, the Lah number of the second kind. These numbers were first introduced in [Lah] by I. Lah in the 1950's. We give combinatorial proofs for (1) and (2) of Corollary 2.7 in the appendix, with the interpretation that $L(n, k)$ counts the number of distributions of n labeled balls into k contents-ordered unlabeled urns with no urn empty.

Corollary 2.7: The following are equivalent ways to define the array $(L(n, k))_{n, k \geq 0}$:

$$(1) \quad \forall n, k \in \mathbf{N}, \quad x^{\overline{n}} = \sum_{k \geq 0} L(n, k) x^{\frac{n}{k}}.$$

$$(2) \quad \forall n, k \in \mathbf{P}, \quad L(n, k) = L(n-1, k-1) + (n-1+k)L(n-1, k),$$

subject to the boundary conditions $L(0, k) = \delta_{0,k}$ for all $k \in \mathbf{N}$
and $L(n, 0) = 0$ for all $n \in \mathbf{P}$.

$$(3) \quad \forall n, k \in \mathbf{P}, \quad L(n, k) = \sum_{j=k}^n L(j-1, k-1) \prod_{i=j}^{n-1} (i+k)$$

subject to the same boundary conditions in (2).

$$(4) \quad \forall n, k \in \mathbf{N}, \quad L(n, k) = \Delta_k \left(x^{\overline{n}} \right) \Big|_{x=n}.$$

$$(5) \quad \forall n, k \in \mathbf{N}, \quad L(n, k) = \frac{n!}{k!} \binom{n-1}{k-1}.$$

Proof: Apply Theorem 1.6 with $a_n = n$ and $b_n = n$ for all $n \in \mathbf{N}$. Note that (5) of

Theorem 1.6 gives $L(n, k) = \sum_{j=0}^k \frac{j^{\overline{n}}}{(j)_k}$, and $\sum_{j=0}^k \frac{j^{\overline{n}}}{(j)_k} = \frac{j(j+1) \cdots (j+n-1)}{(-1)^{k-j} j! (k-j)!} =$

$$\sum_{j=0}^k (-1)^{k-j} \frac{(n+j-1)^{\overline{n}}}{j! (k-j)!} = \frac{1}{k!} \Delta^k (x+n-1)^{\overline{n}} \Big|_{x=0} = \frac{1}{k!} n^{\overline{k}} (x+n-1)^{\overline{n-k}} \Big|_{x=0} =$$

$$\frac{1}{k!} n^{\overline{k}} (n-1)^{\overline{n-k}} = \frac{n!}{k!} \binom{n-1}{k-1}. \quad \blacksquare$$

Note that from (1) of Corollary 2.7 and (1.2.3) it follows that $L(n, k) = \frac{\Delta^k(x^n)}{k!} \Big|_{x=0}$ for all $n, k \in \mathbf{N}$.

2.8 q -Stirling Numbers of the Second Kind

We begin this section by defining $0_q = 0$ and $n_q = 1 + q + q^2 + \cdots + q^{n-1}$ for all $n \in \mathbf{P}$, where q is a prime number raised to a positive power. Further, we define $n_q! = n_q(n-1)_q \cdots 1_q$ for all $n \in \mathbf{N}$. When we let $a_n = 0$ and $b_n = n_q$ for all $n \in \mathbf{N}$, (1) of Theorem 2.1 reduces to the following:

$$\forall n, k \in \mathbf{N}, \quad x^n = \sum_{k \geq 0} A(n, k) x(x-1_q) \cdots (x-(k-1)_q).$$

Since L. Carlitz defined the array $(\tilde{S}_q(n, k))_{n, k \geq 0}$ in a manner equivalent to the above equation in [Car2], we note that $A(n, k) = \tilde{S}_q(n, k)$, a q -Stirling number of the second kind. $\tilde{S}_q(n, k)$ first appeared (although not in this notation) for q an odd prime in [Car1], a paper by Carlitz concerning abelian fields. Later, in [Car2], Carlitz defined $\tilde{S}_q(n, k)$ via

$$([x])^n = \sum_{k=0}^n q^{\binom{n}{k}} \tilde{S}_q(n, k) [x]_k, \text{ where} \quad (2.8.1)$$

$$[x] := \frac{q^x - 1}{q - 1} \text{ and } [x]_k := [x][x-1] \cdots [x-(k-1)] \text{ for all } k \in \mathbf{N}.$$

Note that (2.8.1) is (1) of Corollary 2.8 below with x replaced by $[x]$.

Corollary 2.8: The following are equivalent ways to define the array $(\tilde{S}_q(n, k))_{n, k \geq 0}$:

$$(1) \quad \forall n, k \in \mathbf{N}, \quad x^n = \sum_{k \geq 0} \tilde{S}_q(n, k) x(x-1_q) \cdots (x-(k-1)_q).$$

$$(2) \quad \forall n, k \in \mathbf{P}, \quad \tilde{S}_q(n, k) = \tilde{S}_q(n-1, k-1) + k_q \tilde{S}_q(n-1, k),$$

subject to the boundary conditions $\tilde{S}_q(0, k) = \delta_{0, k}$ for all $k \in \mathbf{N}$ and

$$\tilde{S}_q(n, 0) = 0 \text{ for all } n \in \mathbf{P}.$$

$$(3) \quad \forall n, k \in \mathbf{P}, \quad \tilde{S}_q(n, k) = \sum_{j=k}^n \tilde{S}_q(j-1, k-1) k_q^{n-j}$$

subject to the same boundary conditions in (2).

$$(4) \quad \forall n, k \in \mathbf{N}, \quad \tilde{S}_q(n, k) = \Delta_k(x^n) \Big|_{x=k_q}.$$

$$(5) \quad \forall n, k \in \mathbf{N}, \quad \tilde{S}_q(n, k) = \frac{1}{q^{\binom{k}{2}} k_q!} \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \binom{k}{j}_q j_q^n.$$

$$(6) \quad \forall k \in \mathbf{N}, \quad \sum_{n \geq 0} \tilde{S}_q(n, k) \frac{x^n}{n!} = \sum_{j=0}^k \frac{e^{j_q x}}{\binom{j}{k}_q}.$$

$$(7) \quad \forall n, k \in \mathbf{N}, \quad \tilde{S}_q(n, k) = \sum_{\substack{d_1 + \dots + d_k = n-k \\ d_i \in \mathbf{N}}} 1_q^{d_1} \dots k_q^{d_k}.$$

$$(8) \quad \forall k \in \mathbf{N}, \quad \sum_{n \geq 0} \tilde{S}_q(n, k) x^n = \frac{x^k}{(1 - 1_q x)(1 - 2_q x) \dots (1 - k_q x)}.$$

Proof: Apply Theorem 2.1 with $b_n = n_q$ for all $n \in \mathbf{N}$. Thus, (5) of Theorem 2.1

simplifies to $\tilde{S}_q(n, k) = \sum_{j=0}^k \frac{j_q^n}{\binom{j}{k}_q}$. Note that $\sum_{j=0}^k \frac{j_q^n}{\binom{j}{k}_q} =$

$$\begin{aligned} & \sum_{j=0}^k \frac{j_q^n}{(-1)^{k-j} j_q (j_q - 1_q) \dots (j_q - (j-1)_q) (k_q - j_q) ((k-1)_q - j_q) \dots ((j+1)_q - j_q)} = \\ & \sum_{j=0}^k (-1)^{k-j} \frac{j_q^n}{q^{\binom{j}{2}} j_q (j-1)_q \dots 1_q q^{j \binom{k-j}{2}} (k-j)_q (k-j-1)_q \dots 1_q} = \\ & \frac{1}{q^{\binom{k}{2}}} \sum_{j=0}^k (-1)^{k-j} q^{\binom{k}{2}} q^{-\binom{j}{2}} q^{-j \binom{k-j}{2}} \frac{j_q^n}{j_q! (k-j)_q!} = \frac{1}{q^{\binom{k}{2}}} \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \frac{j_q^n}{j_q! (k-j)_q!} = \\ & \frac{1}{q^{\binom{k}{2}} k_q!} \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \binom{k}{j}_q j_q^n, \text{ since } \binom{n}{k}_q = \frac{n_q!}{k_q! (n-k)_q!}. \text{ Thus, we have (5) of} \end{aligned}$$

Corollary 2.8. ■

By using (1) of Corollary 2.8 along with R. Davis' q -difference operator

$$\Delta_q P(x) = \frac{P(qx+1) - P(x)}{(q-1)x+1} \text{ in [Dav], one can prove (5.14) in [Wag]:}$$

$$\tilde{S}_q(n, k) = \frac{\Delta_q^k(x^n) \Big|_{x=0}}{k_q!}.$$

Note that $\Delta_q = \Delta$, the forward difference operator, when $q = 1$.

In [Mil], Milne used q -analogues of restricted growth functions in order to gain a vector space analogue of $S(n, k)$. His q -Stirling number of the second kind is denoted $S_q(n, k)$ and is defined by

$$([x])^n = \sum_{k=0}^n S_q(n, k)[x]_k. \tag{2.8.2}$$

In view of (2.8.1), we see that the two different q -Stirling numbers are related by

$$S_q(n, k) = q^{\binom{k}{2}} \tilde{S}_q(n, k). \tag{2.8.3}$$

Further generalizations can be found in [Wag], where C. Wagner generalizes $S_q(n, k)$ & $\tilde{S}_q(n, k)$ from $\mathbf{F}_q$ -vector spaces to modular binomial lattices and also gives q -analogues for the Lah numbers of the second kind.

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APPENDIX

Appendix

A.1 Combinatorial Proofs for Stirling, Lah, and Cycle Numbers of the Second Kind

We begin this section by examining three classic ways to distribute n labeled balls (labeled $1, 2, \dots, n$) among k urns. One of the more familiar ways is to distribute n labeled balls into k unlabeled urns with no urn empty. The number of such distributions is denoted by $S(n, k)$, the Stirling number of the second kind. If we add the restriction that the balls in each urn are to be linearly ordered, then the number of distributions is $L(n, k)$, the Lah number of the second kind. The cycle number of the second kind, denoted $C(n, k)$, is the last variant on distributions of n labeled balls that we will consider. The value $C(n, k)$ enumerates those distributions of n labeled balls into k unlabeled contents-ordered urns with no urn empty, such that within each urn the smallest labeled ball occurs first. Algebraically speaking, $C(n, k)$ enumerates those permutations on $[n]$ with k cycles, with the canonical representation that each cycle begins with its smallest member. The following proposition establishes the recurrence relations for (1.1.4) - (1.1.6) by a simple combinatorial proof.

Proposition A.1: $S(n, k) = S(n-1, k-1) + kS(n-1, k),$ (A.1.1)

$$L(n, k) = L(n-1, k-1) + (n-1+k)L(n-1, k), \text{ and} \quad (\text{A.1.2})$$

$$C(n, k) = C(n-1, k-1) + (n-1)C(n-1, k), \quad (\text{A.1.3})$$

subject to the boundary conditions

$$S(n, 0) = L(n, 0) = C(n, 0) = \delta_{n,0} \text{ for all } n \in \mathbf{N} \text{ and}$$

$$S(0, k) = L(0, k) = C(0, k) = 0 \text{ for all } k \in \mathbf{P}.$$

Proof: As the boundary conditions are obvious from the definitions, we assume $n, k \in \mathbf{P}$. To establish each recurrence, we may fill our k urns in such a way that either ball n is alone in an urn, or ball n is in an urn with other balls. If ball n is

alone in an urn, then we may distribute the remaining $n - 1$ balls into $k - 1$ non-empty urns in $S(n - 1, k - 1)$, $L(n - 1, k - 1)$, or $C(n - 1, k - 1)$ ways, depending on what order condition we impose on the balls within the urns. On the other hand, if we do not want n to be alone in an urn, we may first distribute the remaining $n - 1$ balls into k non-empty urns in $S(n - 1, k)$, $L(n - 1, k)$, or $C(n - 1, k)$ ways. Thus, we only need to determine the placement of ball n . If the contents of each urn are unordered, then ball n can be placed in any urn. This can be done in k ways, establishing (A.1.1). If the contents are ordered, we may place ball n in any of the urns in $n - 1 + k$ ways, which verifies (A.1.2). If the contents are to be ordered with the smallest ball first, then we have $n - 1$ ways to place ball n (since ball n can only appear after the first ball of each urn). This shows that (A.1.3) is valid. ■

Remark A.2: We note that if $k > n$, then $S(n, k) = L(n, k) = C(n, k) = 0$. This can be seen directly from the definitions (cf. Proposition 1.3).

In addition to being distribution enumerators, we have already seen that the Stirling, Lah, and cycle numbers of the second kind play a role as transition matrices between different bases of $\mathbf{C}[x]$. Rather than defining $S(n, k)$, $L(n, k)$, and $C(n, k)$ as we did in (1.1.1) - (1.1.3), we prove (1.1.1) - (1.1.3) using only what each of $S(n, k)$, $L(n, k)$, and $C(n, k)$ count. The proof of Proposition A.3 will make use of the following counting principle: Suppose that $f: X \rightarrow Y$ with $|X|, |Y| < \infty$. If there exists an ordered partition $(Y_1, Y_2, \dots, Y_k)$ of Y such that $\forall i \in [k]$ and $\forall y \in Y_i$ $|f^{-1}(\{y\})| = n_i$, then $|X| = \sum_{i=1}^k n_i |Y_i|$.

Proposition A.3: For all $n \in \mathbf{N}$, and for all $x \in \mathbf{C}$, we have

$$x^n = \sum_{k \geq 0} S(n, k) x^k, \quad (\text{A.1.4})$$

$$x^{\overline{n}} = \sum_{k \geq 0} L(n, k) x^{\frac{k}{r}}, \text{ and} \quad (\text{A.1.5})$$

$$x^{\overline{n}} = \sum_{k \geq 0} C(n, k) x^{\frac{k}{r}}. \quad (\text{A.1.6})$$

Proof: First note that the above sums are truly finite, for if $k > n$, Remark A.2 shows us that $S(n, k) = L(n, k) = C(n, k) = 0$. We will establish the validity of (A.1.4) - (A.1.6) by showing that they each hold for all values $r \in \mathbf{N}$. It will then follow that the above equations are valid for all $x \in \mathbf{C}$, since two polynomials each of degree at most n that agree at $n+1$ distinct places must be identical. To show that (A.1.4) is valid, let $n, r \in \mathbf{N}$ and consider

$$r^n = \sum_{k=0}^r \binom{r}{k} k! S(n, k) \quad (\text{A.1.7})$$

The LHS of (A.1.7) counts all distributions of n labeled balls (labeled 1, 2, ..., n) among r labeled urns (labeled 1, 2, ..., r) with the possibility of empty urns. The RHS of (A.1.7) enumerates these distributions according to the number of non-empty urns. In particular, we may choose the urns that are non-empty in $\binom{r}{k}$ ways and fill them with n labeled balls in $k! S(n, k)$ ways. Note that (A.1.7) simplifies to

$$r^n = \sum_{k=0}^n S(n, k) r^{\frac{k}{r}}, \quad (\text{A.1.8})$$

for if $n \leq r$ we have

$$r^n = \sum_{k=0}^n S(n, k) r^{\frac{k}{r}} + \sum_{k=n+1}^r S(n, k) r^{\frac{k}{r}} = \sum_{k=0}^n S(n, k) r^{\frac{k}{r}} + 0 \quad (\text{A.1.9})$$

by Remark A.2. If $n > r$, then we see that

$$r^n = \sum_{k=0}^r \binom{r}{k} k! S(n, k) = \sum_{k=0}^r \binom{r}{k} k! S(n, k) + 0 = \sum_{k=0}^n \binom{r}{k} k! S(n, k) \quad (\text{A.1.10})$$

since $\binom{r}{k} = 0$ when $k > r$. Thus, the index in (A.1.7) can terminate with $k=n$.

Hence, equation (A.1.8) shows that the degree n polynomials $x^{\overline{n}}$ and

$\sum_{k=0}^n S(n, k) x^{\frac{k}{r}}$ agree for all $r \in \mathbf{N}$, and hence must be equal. This

establishes (A.1.4).

Equation (A.1.5) is handled in the same manner. We note that both sides of

$$r^{\overline{n}} = \sum_{k=0}^r \binom{r}{k} k! L(n, k) \quad (\text{A.1.11})$$

equal $|R|$, where $R := \{\text{distributions of } n \text{ labeled balls among } r \text{ labeled contents-ordered urns and the possibility of empty urns}\}$. Indeed, R can be mapped onto the set of permutations of $[n]$ by simply removing the urns. Each permutation of $[n]$ has $\binom{n+r-1}{r-1}$ pre-images under this mapping. Hence, $|R| = \binom{n+r-1}{r-1} n! = r^{\overline{n}}$. The RHS of (A.1.11) also equals $|R|$, for there are $\binom{r}{k}$ ways to choose k non-empty urns and $k! L(n, k)$ ways to fill the urns with n labeled balls. By the same argument as in (A.1.8), we may extend the upper limit of the sum in (A.1.11)

from r to n . Hence we have

$$r^{\overline{n}} = \sum_{k=0}^n L(n, k) r^k, \quad (\text{A.1.12})$$

which establishes (A.1.5).

To establish (1.1.6), consider

$$r^{\overline{n}} = \sum_{k=0}^n C(n, k) r^k. \quad (\text{A.1.13})$$

We may assume $n > 0$, for the case $n = 0$ is trivial. We consider a mapping φ which takes R (as defined earlier) to $S := \{\text{distributions of } n \text{ labeled balls among } n \text{ unlabeled contents-ordered urns (possibly empty), such that within each urn the smallest labeled ball occurs first}\}$ defined as follows: reading left to right, place a "/" before the first ball of each non-empty urn, and do so as well to the left of each record low. Applying φ to an element of R gives rise to an element $s \in S$, for each urn created by "/"'s has its smallest labeled ball first, and we choose to write the urns of s in *decreasing* order of the first balls of each urn. As an example, if we consider $(\{416\}, \{7523\}, \{\})$ (seven balls distributed among three urns, one of which is empty) we note that $\varphi((\{416\}, \{7523\}, \{\})) = /4 / 16 / 7 / 5 / 23 = (7)(5)(4)(23)(16)$. If $s \in S$ has k non-empty urns (there are

$C(n, k)$ such s 's), then s has r^k pre-images under φ , for we may place each of the k urns of s into any of the r labeled urns provided that we preserve the decreasing order. For instance, if (23) and (7) comprise the first urn of s , then that urn must be $\{723\}$. By summing over the number of non-empty urns, we establish (A.1.13) and hence have (A.1.6). ■

By exploiting the relationships between Stirling, Lah, and cycle numbers of the second kind and the bases $\{x^i\}_{i=0}^\infty$, $\{x^{\bar{i}}\}_{i=0}^\infty$, & $\{x^{\underline{i}}\}_{i=0}^\infty$ that is given by Proposition A.3, we now show that the Stirling, Lah, and cycle numbers are related by the following proposition.

Proposition A.4: For all $n, k \in \mathbf{N}$, $L(n, k) = \sum_{j=k}^n C(n, j)S(j, k)$.

Proof: We first give an algebraic proof. By Proposition A.3, $\sum_{k \geq 0} L(n, k)x^{\underline{k}} = x^{\bar{n}} = \sum_{j \geq 0} C(n, j)x^j = \sum_{j \geq 0} C(n, j) \sum_{k \geq 0} S(j, k)x^{\underline{k}} = \sum_{k \geq 0} \sum_{j \geq 0} C(n, j)S(j, k)x^{\underline{k}}$. By comparing coefficients of the $x^{\underline{k}}$'s, we have $L(n, k) = \sum_{j \geq 0} C(n, j)S(j, k)$. Since $C(n, j) = 0$ if $j > n$ and $S(j, k) = 0$ if $k > j$, we have $L(n, k) = \sum_{j=k}^n C(n, j)S(j, k)$.

Our second proof is combinatorial. As Proposition A.4 is trivial in the case $n = 0$, we assume that $n > 0$. We define a mapping ψ from $T := \{\text{distributions of } n \text{ labeled balls into } k \text{ unlabeled contents-ordered urns with no urn empty}\}$ to S (as defined in the proof of Proposition A.3) as follows: reading left to right, place a "/" before the first ball in each urn, and do so as well to the left of each record low. Applying ψ to an element of T will yield an element $s \in S$ with m urns (note that $k \leq m \leq n$) that we will choose to write in *decreasing* order of the first balls of each urn. If $s \in S$ has j non-empty urns (there are $C(n, j)$ such s 's), then s has $S(j, k)$ pre-images under ψ , since we may place each of the j urns of s into any of the k unlabeled urns in $S(j, k)$ ways, provided that we

maintain the decreasing order. By summing over the number of non-empty urns j with $k \leq j \leq n$, we establish the proposition. ■

VITA

Mark John Lancaster was born in Fort Smith, Arkansas on June 14, 1969. In June 1987, he graduated from Northside High School in Fort Smith. The following September, he entered Hendrix College in Conway, Arkansas. As an undergraduate, he participated in the Research Experience for Undergraduates program at the University of Tennessee working with Dr. David E. Dobbs. In June 1991, he received the degree of Bachelor of Arts *summa cum laude*, with Distinction in Mathematics from Hendrix College. In August 1991, he entered the University of Tennessee as a graduate student and was given a fellowship by the university. During his tenure as a graduate student, he also worked as a graduate teaching associate for the Department of Mathematics. He received his Master of Science degree in Mathematics from the University of Tennessee in May 1996.